Stochastic Leader-Following for Heterogeneous Linear Agents with Communication Delays

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***Abstract*—We study the leader-following problem for linear stochastic multi-agent systems with uniform and constant communication delays on directed or undirected graphs. We consider both the state feedback and output feedback solutions. In the latter case, the agents can be a set of heterogeneous linear systems. By resorting to a new approach based on the scalar Lambert equation we obtain a constructive design with less conservative closed-form delay bounds. In particular, it is possible to compensate arbitrarily large delays if the agents are not unstable.**

***Index Terms*—Multi-agent systems, Stochastic systems, Delay systems**

I. Introduction

The cooperative control of a group of agents is a topic of growing interest in recent years due to its high potential in many applications such as vehicle formation [1], autonomous vehicles [2], robotic systems [3], sensor networks [4], target tracking [5], and synchronization [6], [7]. A central task in this area is to design a distributed network protocol, based on interactions between neighbors, that drives the group of agents to agree on certain variables of interest as time goes on. In the case of leader-following, the consensus trajectory is determined in advance by the leader. In this context it is customary to assume to have no control on the leader and its dynamics. The leader-following problem consists in designing a control law local to each agent that ensures that the followers converge in an appropriate sense to the trajectory of the leader. The problem has been solved for linear agent dynamics with both fixed or switching network topology, [8], [9], [10], [11], and has been extended to output-feedback consensus [12], [13], nonlinear systems [14], [15], [16], systems with additive disturbances [17], [18], [19] and input delay [20], [21].

The presence of communication delays, which is common in a network of agents, is challenging since it may undermine the stability of the consensus dynamics (see [22], [23], [24], [25], [26], [27], [28] and the references therein). The solutions proposed in the literature either refer to special cases such as scalar systems or integrators [11], or are based on predictors with integral terms that are expensive to implement [26]. Truncated predictor approaches are limited by the requirement that the agents are deterministic and not exponentially unstable [27]. Moreover, other approaches are based on LMIs to design the consensus gain [25], [29]. Besides being more complex to compute the LMI approach does not allow for an easy estimation of the delay bound.

In this paper we propose a distributed consensus protocol based on output feedback for stochastic and heterogeneous multi-agent systems affected by disturbances and communication delays on directed or undirected network topologies. Our design improves over [26], [27] since it is not based on distributed terms and it works also for unstable agents. The approach includes heterogeneous agents and it is thus more general than the recent paper [30]. With respect to [29] and [25] the advantage is that the design is constructive and it allows for an easy computation of a sufficient and not concervative upper bound for the delay. The features of the proposed approach can be summarized as follows.

- A novel constructive design based on the scalar Lambert equation that yields less conservative delay bounds without resorting to LMIs. It is possible to compensate arbitrarily large delays if the agents are not unstable and to include heterogeneous agents.
- The consensus gain is locally designed and it may be different for each agent, thus adding flexibility.
- The approach is well suited for agents with limited computational power: the parameters are cheap to compute and there are no distrbuted terms in the controller.

The paper is organized as follows. Section II introduces the problem and some notions on directed and undirected graphs. In Section III the leader following problem with communication delays is addressed for both full and partial state information. In Section IV the same problem is addressed and solved for heterogeneous systems. Section V presents two examples while concluding remarks are given in Section VI.

Notation. $\|x\|$ denotes the Euclidean norm for $x \in \mathbb{R}^n$, and $\|M\|$ is the operator norm when $M \in \mathbb{R}^{n\times m}$; $|\mathcal{S}|$ denotes the cardinality of the set $\mathcal{S}$. I_n is the identity matrix in $\mathbb{R}^n$. $\mathbf{1}_N$ is the vector in $\mathbb{R}^N$ with entries 1. Given $M \in \mathbb{R}^{n\times n}$, $\mathbf{sp}(M)$ is its spectrum, $\mathbf{Tr}\{M\}$ its trace, and $\mu(M) := \max_{\lambda\in\mathbf{sp}(M)} \mathbf{Re}\{\lambda\}$ its spectral abscissa; When $\mu(M) < 0$, M is said to be Hurwitz stable. $M > 0$ denotes a positive definite matrix; if $M = M^\top > 0$, $x^\top M x \geq 0$ will be denoted $\|x\|_M^2$. $A \otimes B$ denotes the Kronecker product of matrices A and B; we recall that $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$ when AC and BD are defined; $\text{col}_{k=1}^n(M_k)$ and $\text{row}_{k=1}^n(M_k)$ denote respectively the vertical and horizontal composition of the n vectors or matrices M_k, and $\text{diag}(v)$ denotes a diagonal matrix with the vector v on the diagonal. $\mathbf{E}[\cdot]$ denotes the expectation;

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on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$, $\mathcal{L}_2(\Omega; \mathbb{R}^n)$ denotes the linear space of square integrable random vectors X of $\mathbb{R}^n$ endowed with the norm $\|X\|_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} := (\mathbf{E}[\|X\|^2])^{1/2}$.

II. Preliminaries and problem statement

A. The network and its graph representation

In the present paper we will consider a network of N agents connected to a leader. Depending on the kind of connection among the agents the network may be represented through a directed or undirected graph as better explained below.

The leader is described by the dynamics

$$\mathrm{d}X_0(t) = A_0 X_0(t)\mathrm{d}t + P_0 dN_0(t), \tag{1a}$$
$$Y_0(t) = C_0 X_0(t), \tag{1b}$$
$$\mathrm{d}M_0(t) = D_0 X_0(t)\mathrm{d}t + Q_0 dN_0(t), \tag{1c}$$

where $X_0(t) \in \mathbb{R}^{n_0}$ is the leader state, $Y_0(t) \in \mathbb{R}^q$ is the consensus output, $M_0(t) \in \mathbb{R}^m$ denotes the measured output and $N_0(t) \in \mathbb{R}^{s_0}$ is a standard Wiener process that includes state and measurement noises. The dynamics of the N agents, are given for $k = 1, \cdots, N$, by

$$\mathrm{d}X_k(t) = (A_k X_k(t) + B_k U_k(t))\mathrm{d}t + P_k dN_k(t) \tag{2a}$$
$$Y_k(t) = C_k X_k(t), \tag{2b}$$
$$\mathrm{d}M_k(t) = D_k X_k(t)\mathrm{d}t + Q_k dN_k(t), \tag{2c}$$

where $X_k(t) \in \mathbb{R}^{n_k}$ denotes the state of the k-th agent, $Y_k(t) \in \mathbb{R}^q$ is the associated output, $M_k(t) \in \mathbb{R}^m$ denotes the measured output and $N_k(t) \in \mathbb{R}^{s_k}$ is a standard Brownian motion, that includes state and measurement noises, independent from $N_0(t)$ and any other $N_j(t)$, $j \neq k$. Moreover, for all $k = 0, 1, \ldots, N$, the norm $\|X_k(0)\|_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} := (\mathbf{E}[\|X_k(0)\|^2])^{1/2} < \infty$.

Remark 1: The dimensions n_k, $k = 0, 1, \ldots, N$, of the state spaces can be different for the agents (and the leader). In this heterogeneous context we are interested in reaching consensus on the variables $Y_k(t)$, $k = 0, 1, \ldots, N$. For this reason we consider consensus variables $Y_k(t)$, $k = 0, 1, \ldots, N$, with equal dimension q, not affected by noise and not even measured. On the other hand, the measured outputs M_0 and M_k, $k = 0, 1, \ldots, N$, are realistically affected by noise and with different dimensions.

The agents are connected through a network described by an unweighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is the set of vertices representing the N agents and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges of the graph. Edge (i, j) indicates that agent i can send information to agent j. In this case i is a neighbor of j. The set of neighbors of node j is denoted by $\mathcal{N}^j$. The connections graph is represented through the $N \times N$ adjacency matrix $\mathcal{A} = [a_{ij}]$, whose (i, j)-th entry $a_{ij} \neq 0$ if $(i, j) \in \mathcal{E}$ and 0 if $(i, j) \notin \mathcal{E}$. A path is a sequence of connected edges in a graph. The graph is *connected* if there is a path (namely, a sequence of connected edges) between every pair of vertices.

The leader is introduced by extending the underlying graph with a graph $\overline{\mathcal{G}}$ which has the vertex 0 representing the leader, and edges between the leader 0 and its neighbors. Depending on whether or not the graph $\mathcal{G}$ (or $\bar{\mathcal{G}}$) is oriented we have two cases.

Undirected graphs: The graph $\mathcal{G}$ is *undirected* if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. The degree matrix $\mathcal{D}$ is a diagonal matrix with $\mathcal{D}_{ii} = |\mathcal{N}^i|$. The Laplacian $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{N \times N}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. For undirected graphs $\mathcal{L}$ is symmetric; the spectrum of $\mathcal{L}$ is in the closed right half plane with one and only one eigenvalue at zero if and only if the graph is connected. In this case the right eigenvector associated to the 0 eigenvalue is $\mathbf{1}_N$ (*i.e.* $\mathcal{L}\mathbf{1}_N = 0$). If the leader is taken into account, then one has to consider the diagonal matrix $\mathcal{L}_0 = \mathrm{diag}(\ell_{0,1}, \ldots, \ell_{0,N})$ where $\ell_{0,j} = 1$ if $0 \in \mathcal{N}^j$ and 0 otherwise. The undirected graph $\overline{\mathcal{G}}$ is connected if and only if $\overline{\mathcal{L}} = [\bar{\ell}_{ij}] = \mathcal{L}_0 + \mathcal{L}$ is positive definite ([11], [24], [31]). The following assumption ensures that each follower can receive information from the leader.

Assumption 1: The undirected graph $\overline{\mathcal{G}}$ is connected.

Directed graphs: The graph is said to be *directed* if $(i, j) \in \mathcal{E}$ does not necessarily imply $(j, i) \in \mathcal{E}$. We will assume that the graph is *simple*, i.e. $a_{i,i} = 0$ for all $i \in \mathcal{V}$. A directed path from node i_1 to node i_l is a sequence of edges (i_k, i_{k+1}), $k = 1, 2, \ldots, l-1$. A directed graph $\mathcal{G}$ is *strongly connected* if between any pair of distinct nodes i and j in $\mathcal{G}$, there exists a directed path from i to j, $i, j \in \mathcal{N}$. A directed graph $\mathcal{G}$ *contains a directed spanning tree* if there exists a root node that has directed paths to all other nodes. The Laplacian $\mathcal{L} \in \mathbb{R}^{N \times N}$ is defined as $\mathcal{L} := [\ell_{i,j}] = \mathcal{M} - \mathcal{A}$ where the i-th diagonal entry of the diagonal matrix $\mathcal{M}$ is given by $m_i = \sum_{j=1}^{N} a_{i,j}$. By construction $\mathcal{L}$ has a zero eigenvalue with an associated eigenvector $\mathbf{1}_N$ (i.e. such that $\mathcal{L}\mathbf{1}_N = 0$) and if the graph is strongly connected all the other eigenvalues lie in the open right-half complex plane. When also the leader is considered one refers, as above, to the extended directed graph $\overline{\mathcal{G}}$ and to $\mathcal{L}_0 = \mathrm{diag}(\ell_{0,1}, \ldots, \ell_{0,N}\}$ where $\ell_{0,j} = 1$ if $0 \in \mathcal{N}^j$ and 0 otherwise. The Laplacian is $\overline{\mathcal{L}} = [\bar{\ell}_{i,j}] = \mathcal{L}_0 + \mathcal{L}$. In the case of directed graphs, the following assumption replaces Assumption 1 to guarantee that each follower can receive information from the leader.

Assumption 2: A directed spanning tree is contained in $\overline{\mathcal{G}}$ with the leader as the root node.

Lemma 1: [32] Under Assumption 2 there exists a positive definite matrix $D := \mathrm{diag}\ \{d_1, \ldots, d_N\}$ such that $(D\mathcal{L} + \mathcal{L}^\top D)\mathbf{1}_N = 0$ and $\hat{\mathcal{L}} := D\overline{\mathcal{L}} + \overline{\mathcal{L}}^\top D > 0$. □

The matrix $\hat{\mathcal{L}}$ is the Laplacian of the undirected graph $\hat{\mathcal{G}}$ obtained by taking the union of the edges and their reversed edges in the balanced graph $D\overline{\mathcal{L}}$. $\hat{\mathcal{G}}$ is called the *mirror* of $\overline{\mathcal{G}}$.

B. Problem statement

The consensus problem is stated in terms of the definitions given below of noise-to-state exponential (NSE) $\mathcal{L}_2$-stability and noise-input-to-state exponential (NISE) $\mathcal{L}_2$-stability, exponential versions of noise-to-state stability (NSS) [33], a stochastic analog of input-to-state stability.

Definition 1: Given

$$\mathrm{d}Z(t) = (AZ(t) + BU(t))\mathrm{d}t + P\mathrm{d}N(t) \tag{3}$$

where $Z(t) \in \mathbb{R}^n$, $N(t)$ is a standard Brownian motion, $\|Z(0)\|_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} < \infty$ and $U(t) \in \mathbb{R}^m$ an exogenous stochastic input such that $\|U(t)\|_{\mathcal{L}_2(\Omega;\mathbb{R}^m)} < \infty$ for each $t \geq 0$, we say that (3) or $Z(t)$ is noise-input-to-state exponentially $\mathcal{L}_2$-stable (NISE $\mathcal{L}_2$-stable) if there exists $a, k, p, b > 0$ such that $\forall t \geq 0$

$$\|Z(t)\|^2_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} \leq ae^{-kt}\|Z(0)\|^2_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} + b\sup_{\tau\leq t}\|U(\tau)\|^2_{\mathcal{L}_2(\Omega;\mathbb{R}^m)} + p \quad (4)$$

If $b = 0$ in (4) we say that (3) or $Z(t)$ is noise-to-state exponentially $\mathcal{L}_2$-stable (NSE $\mathcal{L}_2$-stable).

Remark 2: NISE $\mathcal{L}_2$-stability guarantees that the $\mathcal{L}_2$-norm of a stochastic process $Z(t)$ at each time t is bounded by the sup of the $\mathcal{L}_2$-norm of the input $U(\tau)$ over the interval $[0, t]$ plus an exponentially decreasing term (depending on the $\mathcal{L}_2$-norm of $Z(0)$) and a number p. In absence of input one gets noise-to-state exponential (NSE) $\mathcal{L}_2$-stability, and in absence of noise we recover the standard $\mathcal{L}_2$ exponential stability. The number p in (4) is proportional to the noise variance $\mathbf{Tr}\{PP^\top\}$.[1]

Due to the type of consensus variables considered and the eventual presence of communication delays the problem can be formulated as follows

Definition 2: Stochastic Leader Output Following Problem (SLOFP): Given a graph topology $\overline{\mathcal{G}}$ associated to (1)-(2), find $U_k(t)$ for each agent k so that $Y_0(t) - Y_k(t)$ is NSE $\mathcal{L}_2$-stable.

Definition 3: Stochastic Leader Output Following Problem with Communication delays (SLOFPD): Given a graph topology $\overline{\mathcal{G}}$ associated to (1)-(2), and a uniform, constant communication delay $\delta > 0$ among all the nodes, find $U_k(t)$ for each agent k so that $Y_0(t) - Y_k(t)$ is NSE $\mathcal{L}_2$-stable.

In the homogeneous agents case ($A_k = A_0 = A$, $B_k = B$, $C_k = C_0 = C$), it is natural to look at the $\mathcal{L}_2$-consensus on the agents' and leader's states. Moreover, the consensus problem with homogeneous agents is instrumental to understand and solve the consensus problem with heterogeneous agents.

Definition 4: Stochastic Leader Following Problem (SLFP): Given a graph topology $\overline{\mathcal{G}}$ associated to (1)-(2), find $U_k(t)$ for each agent k so that $X_0(t) - X_k(t)$ is NSE $\mathcal{L}_2$-stable.

Definition 5: Stochastic Leader Following Problem with Communication delays (SLFPD): Given a graph topology $\overline{\mathcal{G}}$ associated to (1)-(2), and a uniform, constant communication delay $\delta > 0$ among all the nodes, find $U_k(t)$ for each agent k so that $X_0(t) - X_k(t)$ is NSE $\mathcal{L}_2$-stable.

Remark 3: Clearly, a solution to SLOFPD and SLFPD may include a bound on the maximum allowable δ. Notice that any solution to SLOFPD can be extended to the case of distinct, time-varying continuous delays $\delta_{ij}(t) \leq \delta$ by using a local buffer to store incoming information for $\tau \in [t - \delta, t]$.

III. The Stochastic Leader Following Problem with Communication delays and homogeneous agents

In this section we consider the case of homogeneous agents, that is $A_k = A_0 = A$, $B_k = B$, $C_k = C_0 = C$ (and consequently $n_k = n_0 = n$), $k = 1, \ldots, N$. Section III-A introduces the basic controller for the simple delay-free case. The result for networks with delays is in Section III-B.

[1]See the proof of Lemma 7 and, in particular, eq. (43) for a bound on p.

A. SLFP for homogeneous agents

Let us first consider homogeneous agents on an undirected graph with no communication delay. When the leader's state is accessible to all agents the solution of SLFP is trivial (Lemma 2 below) and the control gain can be computed locally at each agent. Theorem 1 extends the solution when "good" estimates of X_0 are available to agents.

Lemma 2: If Assumption 1 holds, (A, B) is a stabilizable pair, and F_k, $k = 1, \ldots, N$, is such that $A + BF_k$ is Hurwitz stable, the leader's state X_0 is available at each agent k then the control

$$U_k(t) = F_k(X_k(t) - X_0(t)),\ k = 1, \ldots, N \quad (5)$$

solves SLFP.

Proof. Let $\varepsilon_k(t) = X_k(t) - X_0(t)$, $k = 1, \ldots, N$. We have

$$\mathrm{d}\varepsilon_k(t) = (A + BF_k)\varepsilon_k(t)\mathrm{d}t + P^E_k\mathrm{d}N^E_k(t), \quad (6)$$

where $P^E_k = \mathrm{row}(P_k, -P_0)$ and $N^E_k(t) = \mathrm{col}(N_k(t), N_0(t))$ is a standard Brownian motion. Since $A + BF_k$ is Hurwitz stable, it follows from Lemma 7 in Appendix A that $\varepsilon_k(t)$ is NSE $\mathcal{L}_2$-stable and therefore (5) solves SLFP. □

The control (5) is not distributed as each agent needs the leader's state X_0. If X_0 is available only to its neighbors then one can still solve the problem with a local strategy similar to (5) as long as a distributed estimate of $X_0(t)$ is available.

Theorem 1: If Assumption 1 holds, (A, B) is a stabilizable pair, and F_k, $k = 1, \ldots, N$, is such that $A + BF_k$ is Hurwitz, $\hat{X}_{0,k}(t)$, is an estimate of $X_0(t)$ available at agent k such that

$$\sup_{t\geq 0}\|\hat{X}_{0,k}(t) - X_0(t)\|^2_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} = c < \infty \quad (7)$$

then the control

$$U_k(t) = F_k(X_k(t) - \hat{X}_{0,k}(t)),\ k = 1, \ldots, N, \quad (8)$$

solves SLFP.

Proof. We have

$$\mathrm{d}\varepsilon_k(t) = ((A + BF_k)\varepsilon_k(t) - BF_k(\hat{X}_{0,k}(t) - X_0(t)))\mathrm{d}t + P^E_k\mathrm{d}N^E_k(t), \quad (9)$$

where $P^E_k = \mathrm{row}(P_k, -P_0)$ and $N^E_k(t) = \mathrm{col}(N_k(t), N_0(t))$ is a standard Brownian motion. Hence, $\varepsilon_k(t)$ is NISE $\mathcal{L}_2$-stable. Since $A + BF_k$ is Hurwitz stable, by (7) and Lemma 7 in Appendix A it follows that $\varepsilon_k(t)$ is NSE $\mathcal{L}_2$-stable and therefore (5) solves SLFP with homogeneous agents. □

The previous Theorem assumes that at each node an estimate $\hat{X}_{0,k}(t)$ of X_0 that satisfies (7) can be computed and the agent state $X_k(t)$ is accessible to each agent k. We notice that if $\varepsilon_k(t)$ is NSE $\mathcal{L}_2$-stable then (7) is satisfied. In the next section we show how to obtain such a distributed estimate in presence of a communication delay $\delta > 0$.

B. SLFPD for homogeneous agents

In this section, we look at the solution of SLFPD keeping an eye to the solution of SLFP. We first introduce in Section III-B1 the solution to the problem of reaching consensus on Y_0 when the leader's state is available to its neighboring agents with a delay $\delta > 0$ (*complete leader's and agents' state* *information*). We address the case when the leader's output $M_0(t)$ only is available to its neighboring agents with a delay $\delta > 0$ in Section III-B2 (*partial leader's state and complete agents' state information*). In this case the idea is to use M_0 to estimate X_0, with a distributed consensus algorithm, and then generate a control to achieve consensus on Y_0. Finally, in Section III-B3, we consider the case where each agent cannot access its own state X_k. In this situation, the measurement M_k is used to (locally) estimate X_k (*partial leader's state and partial agents' state information*).

1) Complete leader's and agents' state information - homogeneous agents: Assume that the leader's state $X_0(t)$ is available to the neighboring agents with a delay $\delta > 0$. Let

$$\begin{aligned}\dot{\hat{X}}_{0,k}(t) =& A\hat{X}_{0,k}(t) - \frac{\gamma}{\delta}e^{(A-\frac{\gamma}{\delta}I_n)\delta}\Big(\sum_{j=1}^{N}\ell_{k,j}\hat{X}_{0,j}(t-\delta) \\ &+ \ell^0_{k,k}(\hat{X}_{0,k}(t-\delta) - X_0(t-\delta))\Big) \end{aligned} \tag{10}$$

be the estimator of $X_0(t)$ for the agent k, where $\gamma > 0$ is a design parameter. Note that (10) requires at each agent k the estimates $\hat{X}_{0,j}$ only for $\ell_{k,j} \neq 0$ and X_0 *only if* the agent k is connected to the leader (this is modeled by $\ell^0_{k,k}$).

Let w_k, $k = 1\dots,N$, denote a solution of the Lambert-type equation

$$w_k = -\lambda_k\gamma e^{-\gamma}e^{-w_k}, \tag{11}$$

where $\lambda_1,\dots,\lambda_N$ are the eigenvalues of $\overline{\mathcal{L}}$. If

$$\frac{e^{\gamma-1}}{\gamma} \geq \lambda_k,\ k = 1,\dots,N, \tag{12}$$

then there is a real negative solution $w_k \in [-1, 0)$ (see Appendix B). The inequality (12) can always be satisfied by choosing γ sufficiently large, and an upper bound for the eigenvalues of $\overline{\mathcal{L}}$ can be computed in a distributed way, since it is equivalent to finding the maximum out-degree, $\max_k |\{j : k \in \mathcal{N}^j\}|$, over the graph (see Theorem 2 in [24]). We always refer in the sequel to the negative solutions $w_k < 0$.

Theorem 2: If Assumption 1 holds, (A, B) is a stabilizable pair, $F_k, k = 1,\dots,N$, are such that $A+BF_k$ is Hurwitz stable, $\gamma > 0$ satisfies (12) and for $k = 1,\dots,N$ the w_k, negative solutions of (11), are such that the matrices $\overline{W}_k = A + \frac{w_k}{\delta}I_n$ are Hurwitz stable and satisfy

$$\frac{1}{\delta}\int_0^\delta \|e^{\overline{W}_k\theta}\|\,\mathrm{d}\theta \,<\, e^{w_k+1},\ k = 1,\dots,N, \tag{13}$$

then the control

$$U_k(t) = F_k(X_k(t) - \hat{X}_{0,k}(t)),\ k = 1,\dots,N, \tag{14}$$

where $\hat{X}_{0,k}(t)$ is the solution of (10), solves the SLFPD and the estimation error $\hat{\varepsilon}_k(t) = \hat{X}_{0,k}(t) - X_0(t)$ is NSE $\mathcal{L}_2$-stable.

Proof. Since $\sum_{j=1}^{N}\ell_{k,j} = 0$, (10) is equivalent to

$$\begin{aligned}\dot{\hat{X}}_{0,k}(t) =& A\hat{X}_{0,k}(t) - \frac{\gamma}{\delta}e^{(A-\frac{\gamma}{\delta}I_n)\delta}\Big(\sum_{j=1}^{N}\ell_{k,j}(\hat{X}_{0,j}(t-\delta) \\ &- X_0(t-\delta)) + \ell^0_{k,k}(\hat{X}_{0,k}(t-\delta) - X_0(t-\delta))\Big). \end{aligned} \tag{15}$$

Let $\hat{\varepsilon}_k(t) = \hat{X}_{0,k}(t) - X_0(t)$ be the error and $\hat{\varepsilon}(t) = \mathrm{col}^N_{k=1}\hat{\varepsilon}_k(t) \in \mathbb{R}^{nN}$ the total estimation error. Standard computations show that the total estimation error dynamics is

$$\begin{aligned}\mathrm{d}\hat{\varepsilon}(t) =& ((I_N \otimes A)\hat{\varepsilon}(t) - (\overline{\mathcal{L}} \otimes (\frac{\gamma}{\delta}e^{(A-\frac{\gamma}{\delta}I_n)\delta}))\hat{\varepsilon}(t-\delta))\mathrm{d}t \\ &- (\mathbf{1}_N \otimes I_n)P_0\mathrm{d}N_0(t). \end{aligned} \tag{16}$$

Consider the change of coordinates, $\tilde{\varepsilon} = (\mathrm{col}^N_{k=1}(v_k^\top \otimes I_n))\hat{\varepsilon}$, with $v_k^\top$ such that $v_k^\top\overline{\mathcal{L}} = \lambda_k v_k^\top$ is the left eigenvector of $\lambda_k \in \mathbf{sp}(\overline{\mathcal{L}})$. We recall that λ_k are positive real and $T = \mathrm{col}^N_{k=1}(v_k^\top \otimes I_n)$ is non singular with inverse $T^{-1} = \mathrm{row}^N_{k=1}(u_k \otimes I_n)$, u_k being the right eigenvector of λ_k, $v_k^\top u_j = \delta_{ij}$. Denoting $\tilde{\varepsilon}_k = (v_k^\top \otimes I_n)\hat{\varepsilon}$ and $A_\gamma = A - \frac{\gamma}{\delta}I_n$ we obtain from (16)

$$\begin{aligned}\mathrm{d}\tilde{\varepsilon}_k(t) =& (v_k^\top \otimes I_n)\Big[(I_N \otimes A)\varepsilon(t) - \overline{\mathcal{L}} \otimes (\frac{\gamma}{\delta}e^{A_\gamma\delta})\varepsilon(t-\delta)\Big]\mathrm{d}t \\ &- ((v_k^\top \otimes I_n)(\mathbf{1}_N \otimes I_n)P_0\mathrm{d}N_0(t) \\ =& (A(v_k^\top \otimes I_n)\varepsilon(t) - \lambda_k\frac{\gamma}{\delta}e^{A_\gamma\delta}(v_k^\top \otimes I_n)\varepsilon(t-\delta))\mathrm{d}t \\ &- (v_k^\top\mathbf{1}_N)P_0\mathrm{d}N_0(t) \\ =& (A\tilde{\varepsilon}_k(t) - \lambda_k\frac{\gamma}{\delta}e^{A_\gamma\delta}\tilde{\varepsilon}_k(t-\delta))\mathrm{d}t - (v_k^\top\mathbf{1}_N)P_0\mathrm{d}N_0(t) \\ =& (\overline{W}_k\tilde{\varepsilon}_k(t) + \lambda_k\frac{\gamma}{\delta}e^{A_\gamma\delta}(e^{-\overline{W}_k\delta}\tilde{\varepsilon}_k(t) - \tilde{\varepsilon}_k(t-\delta)))\mathrm{d}t \\ &- (v_k^\top\mathbf{1}_N)P_0\mathrm{d}N_0(t) \end{aligned} \tag{17}$$

where we used the equality $(a^\top \otimes M) = M(a^\top \otimes I_n)$, that holds for any $a \in \mathbb{R}^N$, $M \in \mathbb{R}^{n\times n}$ and the crucial property

$$e^{A_\gamma\delta}e^{-\overline{W}_k\delta} = e^{A\delta-\gamma I_n}e^{-A\delta-w_kI_n} = e^{-(\gamma+w_k)}I_n, \tag{18}$$

and by assumption w_k is such that $\overline{W}_k$ is Hurwitz stable and satisfies (13). Let us now consider

$$\sigma_k(t) = e^{-\overline{W}_k\delta}\tilde{\varepsilon}_k(t) - \tilde{\varepsilon}_k(t-\delta). \tag{19}$$

Since $\overline{W}_k$ is Hurwitz stable and $\|\sigma_k(t)\|_{\mathcal{L}_2(\Omega;\mathbb{R}^n)}$ is finite for finite t because $\tilde{\varepsilon}_k(t)$ obeys a linear equation, Lemma 7 in Appendix A implies that $\tilde{\varepsilon}_k(t)$ in (17) is NISE $\mathcal{L}_2$-stable with input $\sigma_k(t)$. If in addition we prove that

$$\sup_{t\geq 0}\|\sigma_k(t)\|_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} < \infty \tag{20}$$

then Lemma 6 in Appendix A guarantees that $\tilde{\varepsilon}_k$ is also NSE $\mathcal{L}_2$-stable. Indeed, by using the variation of constants formula in $[t-\delta, t]$ we obtain from (17) that

$$\begin{aligned}\sigma_k(t) =& e^{-\overline{W}_k\delta}\tilde{\varepsilon}_k(t) - \tilde{\varepsilon}_k(t-\delta) \\ =& \int_{t-\delta}^{t}\frac{\lambda_k\gamma}{\delta}e^{\overline{W}_k(t-\tau)}e^{-(\gamma+w_k)}\sigma_k(\tau)\,\mathrm{d}\tau \\ &-(v_k^\top\mathbf{1}_N)\int_{t-\delta}^{t}e^{\overline{W}_k(t-\tau-\delta)}P_0\mathrm{d}N_0(\tau) \end{aligned} \tag{21}$$

where the last passage is accounted for by $A_\gamma \overline{W}_k = \overline{W}_k A_\gamma$ and $e^{-(\gamma+w_k)I_n} = e^{-(\gamma+w_k)}I_n$. By using standard properties of norms and the Itô isometry, we get that

$$
\begin{aligned}
\|\sigma_k(t)\|_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} &\le \int_{t-\delta}^{t} \frac{\lambda_k\gamma}{\delta} e^{-(\gamma+w_k)} \mathbf{E}[\|e^{\overline{W}_k(t-\tau)}\sigma_k(\tau)\|]\,\mathrm{d}\tau \\
&\quad + |v_k^\top \mathbf{1}_N| \left(\int_{t-\delta}^{t} \|e^{\overline{W}_k(t-\tau-\delta)}P_0\|^2 \mathrm{d}\tau\right)^{1/2} \\
&\le \int_0^\delta \frac{\lambda_k\gamma}{\delta} e^{-(w_k+\gamma)} \|e^{\overline{W}_k\theta}\|\,\mathrm{d}\theta \sup_{\tau\in[t-\delta,t]} \|\sigma_k(\tau)\|_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} \\
&\quad + |v_k^\top \mathbf{1}_N| \left(\int_0^\delta \|e^{\overline{W}_k(\theta-\delta)}P_0\|^2 d\theta\right)^{1/2}.
\end{aligned}
$$

Moreover, by using (12) and (13)

$$
\int_0^\delta \frac{\lambda_k\gamma}{\delta} e^{-(w_k+\gamma)} \|e^{\overline{W}_k\theta}\|\,\mathrm{d}\theta < \int_0^\delta \frac{e^{-(w_k+1)}}{\delta} \|e^{\overline{W}_k\theta}\|\,\mathrm{d}\theta < 1
$$

which implies (20). Since each $\tilde{\varepsilon}_k(t)$, $k = 1, \ldots, N$, is NSE $\mathcal{L}_2$-stable then $\tilde{\varepsilon}(t)$ and also $\hat{\varepsilon}(t)$ (which is an invertible linear transformation of $\tilde{\varepsilon}(t)$) is NSE $\mathcal{L}_2$-stable, hence (7) is satisfied for all $k = 1, \ldots, N$. By Theorem 1, the SLFPD is thus solved with the control (14) with $\hat{X}_{0,k}(t)$ solution of (10). □

At this point, one wonders whether the conditions of Theorem 2 are restrictive or not. In the next two lemmas we prove that they hold for any delay when $\mu(A)$ satisfies an upper bound, and for any $\mu(A)$ when the delay satisfies an upper bound.

Lemma 3: Let $\delta > 0$. *If*

$$
\mu(A) < \min\left\{\frac{|w_k|}{\delta}, \frac{1-\ln(e-1)}{\delta}\right\},\; k = 1, \ldots, n, \quad (22)
$$

and

$$
\|e^{As}\| \le e^{\mu(A)s},\; \forall s \ge 0, \quad (23)
$$

then, the matrices $\overline{W}_k$, *are all Hurwitz stable and satisfy* (13).

Proof. Indeed, $\mu(A) + \frac{w_k}{\delta} < 0$ by (22) and $\overline{W}_k$ is Hurwitz stable. Moreover, $\|e^{\overline{W}_k s}\| \le e^{(\mu(A)+\frac{w_k}{\delta})s}$ for all $s \ge 0$ by (23). Hence,

$$
\int_0^\delta \|e^{\overline{W}_k\theta}\|\,\mathrm{d}\theta \le \int_0^\delta e^{(\mu(A)+\frac{w_k}{\delta})\theta}\,\mathrm{d}\theta = \frac{\delta(1-e^{\mu(A)\delta+w_k})}{|\mu(A)\delta+w_k|}.
$$

Let $x = |\mu(A)\delta + w_k|$. When $x > 1$ it is immediate to see that the right-hand side of the last equation satisfies (13). If $x \in [0,1]$, using the fact that the function $\frac{1-e^{-x}}{xe^{-x}}$, $x \in [0,1]$, takes its maximum $e-1$ at $x = 1$,

$$
\begin{aligned}
\frac{1}{\delta}\int_0^\delta \|e^{\overline{W}_k\theta}\|\,\mathrm{d}\theta &\le \frac{(1-e^{\mu(A)\delta+w_k})}{|\mu(A)\delta+w_k|} = \frac{1-e^{-x}}{xe^{-x}}e^{-x} \\
&\le (e-1)e^{-x} = (e-1)e^{\mu(A)\delta}e^{w_k} < e^{w_k+1}, \quad (24)
\end{aligned}
$$

where the last passage descends from the fact that (22) implies $(e-1)e^{\mu(A)\delta} < e$. Hence, condition (13) is satisfied. □

It should be noticed that (22) is satisfied whenever $\mu(A) \le 0$, while in general $\|e^{As}\| \le he^{\mu(A)s}$ for all $s \ge 0$ and for some $h \ge 1$. Hence, in (23) we require a bit more than $\mu(A) \le 0$.

Lemma 4: Let $\mu(A)$ *be given. There exists* $\delta^* > 0$ *such that for all* $\delta \le \delta^*$ *the matrices* $\overline{W}_k$, $k = 1, \ldots, n$, *are all Hurwitz stable and satisfy (13).*

Proof. Notice that

$$
\delta \to 0 \Rightarrow \frac{1}{\delta}\int_0^\delta \|e^{\overline{W}_k\theta}\|\,\mathrm{d}\theta \to e^{w_k} < e^{w_k+1}. \quad (25)
$$

Hence, there exists $\overline{\delta} > 0$ such that condition (13) is satisfied for all $\delta \le \overline{\delta}$. Moreover, when $\delta \to 0$ one can pick $\gamma := \overline{\gamma}\delta$, with $\overline{\gamma} > 0$, so that $\frac{\gamma}{\delta}$ in (10) is bounded and equal to $\overline{\gamma}$ and $\overline{W}_k \to A - \lambda_k\overline{\gamma}I_n$ (since $w_k \to 0$ and $\frac{w_k}{\delta} \to -\lambda_k\overline{\gamma}$ as $\delta \to 0$), with $\overline{\gamma}$ selected so that $A - \lambda_k\overline{\gamma}I_n$ is Hurwitz stable. Hence, there exists $\delta^* \le \overline{\delta}$ such $\overline{W}_k$, $k = 1, \ldots, n$, is Hurwitz stable and satisfies (13) for all $\delta \le \delta^*$. □

2) Partial leader's state and complete agents' state information - homogeneous agents: In the present subsection we weaken the hypotheses by assuming that only the leader's output is available to its neighboring agents (with communication delay δ). In this case we have to modify each local estimator (10) which is thus given by

$$
\begin{aligned}
\dot{\hat{X}}_{0,k}(t) &= A\hat{X}_{0,k}(t) - \frac{\gamma}{\delta}e^{A_\gamma\delta}\Big(\sum_{j=1}^N \ell_{k,j}\hat{X}_{0,j}(t-\delta) \\
&\quad + \ell_{k,k}^0(\hat{X}_{0,k}(t-\delta) - \hat{X}_{0,k}^0(t))\Big) \quad (26) \\
\mathrm{d}\hat{X}_{0,k}^0(t) &= (A - L_kD_0)\hat{X}_{0,k}^0(t)\mathrm{d}t + L_k\mathrm{d}M_k(t-\delta), \quad (27)
\end{aligned}
$$

if $\ell_{k,k}^0 \ne 0$, with L_k a matrix of suitable dimensions and $\hat{X}_{0,k}^0(t)$ representing an estimate of $X_0(t-\delta)$. Therefore, each neighbor of the leader computes also an estimate $\hat{X}_{0,k}^0(t)$ of $X^0(t-\delta)$ (compare with (10)). Upon this observation, Theorem 2 is modified as follows.

Theorem 3: In the same hypotheses of Theorem 2, if (D_0, A) *is detectable and* L_k, $k = 1, \ldots, N$, *are such that* $A - L_kD_0$ *is Hurwitz stable, then the control*

$$
U_k(t) = F_k(X_k(t) - \hat{X}_{0,k}(t)), \quad (28)
$$

where $\hat{X}_{0,k}(t)$ *is the solution of* (26), *solves the SLFPD and the estimation error* $\hat{\varepsilon}_k(t) = \hat{X}_{0,k}(t) - X_0(t)$ *is NSE* $\mathcal{L}_2$*-stable.*

The proof of Theorem 3 follows the one of Theorem 2, but in addition we use the Kalman-Bucy filter (27) to obtain the estimate $\hat{X}_{0,k}^0$ of $X_0(t-\delta)$ having $\sup_{t\ge 0}\|\hat{X}_{0,k}^0(t) - X_0(t-\delta)\|_{\mathcal{L}_2(\Omega;\mathbb{R}^n)} < \infty$ for $k = 1, \ldots, N$.

3) Partial leader's state and partial agents' state information - homogeneous agents: In this case, in (28) the state $X_k(t)$ is reconstructed by the agent k with the estimate $\hat{X}_{k,k}(t)$ using the Kalman-Bucy filter with the measurement $M_k(t)$,

$$
\begin{aligned}
\mathrm{d}\hat{X}_{k,k}(t) &= ((A - G_kD_k)\hat{X}_{k,k}(t) + BU_k(t))\mathrm{d}t \\
&\quad + G_k\mathrm{d}M_k(t), \quad (29)
\end{aligned}
$$

with

$$
U_k(t) = F_k(\hat{X}_{k,k}(t) - \hat{X}_{0,k}(t)). \quad (30)
$$

Theorem 4: In the same hypotheses of Theorem 2, if (D_k, A), $k = 0, \ldots, N$ *is detectable and* L_k, G_k, $k = 1, \ldots, N$, *are such that* $A - L_kD_0$ *and* $A - G_kD_k$ *are Hurwitz stable, then the control* (30) *with* $\hat{X}_{0,k}(t)$ *and* $\hat{X}_{k,k}(t)$ *solutions of* (26), (29) *respectively, solves the SLFPD and the estimation errors* $\hat{\varepsilon}_k(t) = \hat{X}_{0,k}(t) - X_0(t)$ *and* $\hat{\varepsilon}_{k,k}(t) = \hat{X}_{k,k}(t) - X_k(t)$ *are NSE* $\mathcal{L}_2$*-stable.*

The proof follows easily from Theorem 3 and the linearity of the dynamics.

IV. Stochastic Leader Output Following Problem with communication delays and heterogeneous agents

In this section we consider the case of heterogeneous agents (2) and the more general SLOFPD. First, for the delay–free case, we modify the key Lemma 2 as follows.

Lemma 5: If Assumption 1 holds, (A_k, B_k), $k = 1, \dots, N$, *is stabilizable,* F_k *is such that* $A_k + B_k F_k$ *is Hurwitz stable and* Γ_k, Π_k, $k = 1, \dots, N$, *are matrices such that*

$$\Pi_k A_0 = A_k \Pi_k + B_k \Gamma_k, \tag{31}$$

$$C_0 = C_k \Pi_k, \tag{32}$$

then SLOFP is solved by the control

$$U_k(t) = F_k(X_k(t) - \Pi_k X_0(t)) + \Gamma_k X_0(t). \tag{33}$$

Proof. Let $\varepsilon_k(t) = X_k(t) - \Pi_k X_0(t)$, $k = 1, \dots, N$. We have

$$\mathrm{d}\varepsilon_k(t) = (A_k + B_k F_k)\varepsilon_k(t)\mathrm{d}t + P_k^E \mathrm{d}N_k^E(t), \tag{34}$$

where $P_k^E = \mathrm{row}(P_k, -\Pi_k P_0)$ and $N_k^E(t) = \mathrm{col}(N_k(t), N_0(t))$ is a standard Brownian motion. Since $A_k + B_k F_k$ is Hurwitz stable, it follows from Lemma 7 in the Appendix that $\varepsilon_k(t)$ is NSE $\mathcal{L}_2$-stable. Hence, also

$$Y_k(t) - Y_0(t) = C_k(X_k(t) - \Pi_k X_0(t)) = C_k \varepsilon_k(t) \tag{35}$$

is NSE $\mathcal{L}_2$-stable and it follows that (33) solves SLOFP. □

Remark 4: The matrix equations (31)-(32) are well-known in the output regulation literature [34]. Equation (31) is a Sylvester-type equation and the matrices Π_k and Γ_k can be given the following interpretation (with absence of noise): $u_k(t) = \Gamma_k X_0(t)$ is the control the agent k must apply to force its state evolution $X_k(t)$ to remain for all times inside the hyper-plane $\mathcal{H}_k := \{(x_k, x_0) \in \mathbb{R}^{n_k} \times \mathbb{R}^{n_0} : x_k = \Pi_k x_0\}$. It is known that a Sylvester equation $AX + XB = C$ has a (unique) solution for any matrix C if $\mathbf{sp}(A) \cap \mathbf{sp}(-B) = \{\emptyset\}$. Equation (32) guarantees that $\mathcal{H}_k \subset \mathcal{Y}_k := \{(x_k, x_0) \in \mathbb{R}^{n_k} \times \mathbb{R}^{n_0} : C_k x_k = C_0 x_0\}$, i.e. $u_k(t) = \Gamma_k X_0(t)$ is the control the agent k must apply to force $Y_k(t) = Y_0(t)$ for all times. The consensus law (35) and, in particular, the matrix F_k is designed in such a way to make attractive the hyper-plane $\mathcal{H}_k \subset \mathcal{Y}_k$.

Hence for solving SLOFPD it is sufficient to design an estimate $\hat{X}_{0,k}(t)$ of $X_0(t)$ for each agent such that $\hat{X}_{0,k}(t) - X_0(t)$ is NSE $\mathcal{L}_2$-stable and taking into account the communication delay $\delta > 0$ and the underlying network. Moreover, as in the case of homogeneous agents, an estimate $\hat{X}_{k,k}(t)$ of $X_k(t)$ for each agent k can be computed from the measured output M_k. Theorem 3 is thus modified as follows.

Theorem 5: In the same hypotheses of Theorem 2, if (D_0, A_0) *is detectable and* L_k, $k = 1, \dots, N$, *are such that* $A_0 - L_k D_0$ *are Hurwitz stable, then the control*

$$U_k(t) = F_k(X_k(t) - \Pi_k \hat{X}_{0,k}(t)) + \Gamma_k \hat{X}_{0,k}(t), \tag{36}$$

with

$$\dot{\hat{X}}_{0,k}(t) = A_0 \hat{X}_{0,k}(t) - \frac{\gamma}{\delta} e^{A_\gamma \delta} \Big(\sum_{j=1}^{N} \ell_{k,j} \hat{X}_{0,j}(t-\delta) + \ell_{k,k}^0 (\hat{X}_{0,k}(t-\delta) - \hat{X}_{0,k}^0(t)) \Big), \tag{37}$$

and

$$\mathrm{d}\hat{X}_{0,k}^0(t) = (A_0 - L_k D_0)\hat{X}_{0,k}^0(t)\mathrm{d}t + L_k \mathrm{d}M_0(t-\delta) \tag{38}$$

if $\ell_{k,k}^0 \neq 0$, *where* $A_\gamma = A_0 - \frac{\gamma}{\delta} I_n$ *solves the SLOFPD and the estimation error* $\hat{\varepsilon}_k(t) = \hat{X}_{0,k}(t) - X_0(t)$ *is NSE* $\mathcal{L}_2$*-stable.*

If the underlying topology is that of a directed graph, the agent control which solves SLOFPD is (36) with

$$\dot{\hat{X}}_{0,k}(t) = A_0 \hat{X}_{0,k}(t) - \frac{\gamma}{\delta} e^{A_\gamma \delta} \Big(\sum_{j=1}^{N} (d_k \ell_{k,j} + d_j \ell_{j,k}) \cdot \cdot \hat{X}_{0,j}(t-\delta) + 2\ell_{k,k}^0 (\hat{X}_{0,k}(t-\delta) - \hat{X}_{0,k}^0(t)) \Big) \tag{39}$$

and $\hat{X}_{0,k}^0(t)$, when $\ell_{k,k}^0 \neq 0$, solution of (38). The main result for directed graphs (containing a spanning tree) is the following and it is proved exactly as Theorem 5 (in practice, the Laplacian $\overline{\mathcal{L}}$ of the undirected graph is replaced by the Laplacian $\hat{\mathcal{L}}$ of the directed graph: see introductory section II-A on undirected and directed graphs).

Theorem 6: Let $\overline{\mathcal{G}}$ *be a directed graph. In the same hypotheses of Theorem 2, with Assumption 1 replaced by Assumption 2, if* (D_0, A_0) *is detectable and* L_k, $k = 1, \dots, N$, *are such that* $A_0 - L_k D_0$ *are Hurwitz stable, then the control*(36), *where* $\hat{X}_{0,k}(t)$ *is the solution of* (39), *and* $\hat{\mathcal{L}}$ *replaces* $\overline{\mathcal{L}}$, *solves the SLOFPD and the estimation error* $\hat{\varepsilon}_k(t) = \hat{X}_{0,k}(t) - X_0(t)$ *is NSE* $\mathcal{L}_2$*-stable.*

Notice that (39) requires for each agent k the estimates $\hat{X}_{0,k}$ of X_0 computed at the neighboring nodes $j \neq k$ (in the mirror graph) and the leader's output information when the agent k is connected to the leader (the connection is modeled by $\ell_{k,k}^0$). The communication delay is modeled by the second term, delayed by δ, in the left-hand part of (39).

V. Examples

In this Section, we present an example with homogeneous agents which is intended to be a comparison with the paper [25], and another example with heterogeneous agents taken from [35]. In both cases we consider the case of partial leader's state information (but locally accessible state for each agent) and similarly to [25], we consider a network of five agents. The communication topology is the directed cyclic graph $0 \to 1 \to 2 \to 3 \to 4 \to 1$, $\mathcal{V} = \{1, 2, 3, 4\}$. Only agent 1 has access to the leader and the graph satisfies Assumption 2.

A. Example 1 (SLFPD - Homogeneous agents)

In [25], the leader and the agents are described by the model (1)–(2), where for all $k \in \mathcal{V}$

$$A_k = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B_k = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with $P_k = 0.2 \cdot I_2$ and $Q_k = 0.15$. We do not need the matrices C_k of (1b) and (2b) since we solve SLFPD (*i.e.* state consensus). In this case $\mu(A) = 0$ and (23) is satisfied, thus we have the remarkable result that SLFPD is solvable for any delay. The solution is given by the control law (8) with

$$F_k = \frac{1}{3} \begin{bmatrix} -2 & 8 \\ -2 & 10 \end{bmatrix}, \quad \mathbf{sp}(A + BF_k) = \{-1, -2\}, \tag{40}$$

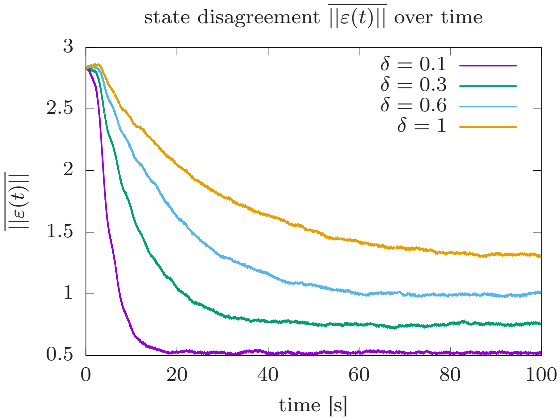

Fig. 1. Average $\overline{||\varepsilon(t)||}$ over the four agents of the empirical computation of $\mathbb{E}[||\varepsilon_k(t)||]$, with $\varepsilon_k(t) = X_0(t) - X_k(t)$ for $k = 1, 2, 3, 4$.

and the filter (26), where we set $\gamma = 4$ to satisfy (12). In fact, the Laplacian $\hat{\mathcal{L}}$ of the mirror $\hat{\mathcal{G}}$ of $\overline{\mathcal{G}}$ can be obtained as in Section II-A by choosing $D = I_4$ to obtain

$$\hat{\mathcal{L}} = \overline{\mathcal{L}} + \overline{\mathcal{L}}^\top = \begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \tag{41}$$

and $\gamma = 4$ satisfies (12) for all the eigenvalues $\mathbf{sp}(\hat{\mathcal{L}}) = \{0.29,\ 2,\ 2.81,\ 4.9\}$. The corresponding solutions of the Lambert-type equation (11) are $w_1 \approx -0.020$, $w_2 \approx -0.175$, $w_3 \approx -0.269$, $w_4 \approx -0.797$. In Fig. 1 we plot, for $\delta \in \{0.1, 0.3, 0.6, 1.0\}$, the arithmetic mean on the four agents $\overline{||\varepsilon(t)||}$ of $\varepsilon_k(t) = X_0(t) - X_k(t)$ for $k \in \mathcal{V}$ over 10^3 Monte Carlo runs. Smaller delays yield smaller values of $||\varepsilon_k(t)||$ and faster convergence towards a steady-state of its expectation. We notice that the state noise in [25] is modeled as a deterministic disturbance while in our paper it is modeled as a Wiener process with comparable amplitude and richer frequency content. Moreover, in the example of [25] the total delay is 0.1s (0.05s of input delay, 0.05s of output delay), and the steady state is reached in about 100s, whilst in our case it is reached in less than 20s for larger delays. Moreover our approach deals with arbitrary large delay, whereas this is not guaranteed in [25].

B. Example 2 (SLOFPD - Heterogeneous agents)

We solve SLOFPD with the model of [35],

$$A_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & c_k \\ 0 & -d_k & -a_k \end{bmatrix},\ B_k = \begin{bmatrix} 0 \\ 0 \\ b_k \end{bmatrix},\ C_k = D_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The parameters $\{a_k, b_k, c_k, d_k\}$ are set as $\{10, 2, 1, 0\}$, $\{1, 1, 1, 0\}$, $\{2, 1, 1, 10\}$ and $\{2, 1, 1, 1\}$ respectively. For the leader we set

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_0 = C_0.$$

The noise intensities are $P_0 = 0.2 \cdot I_2$, $P_k = 0.2 \cdot I_3$, $Q_k = 0.15$ for $k \in \mathcal{V}$. In Fig. 2 (top), we plot the sample second-order moment of the disagreement among the four agents

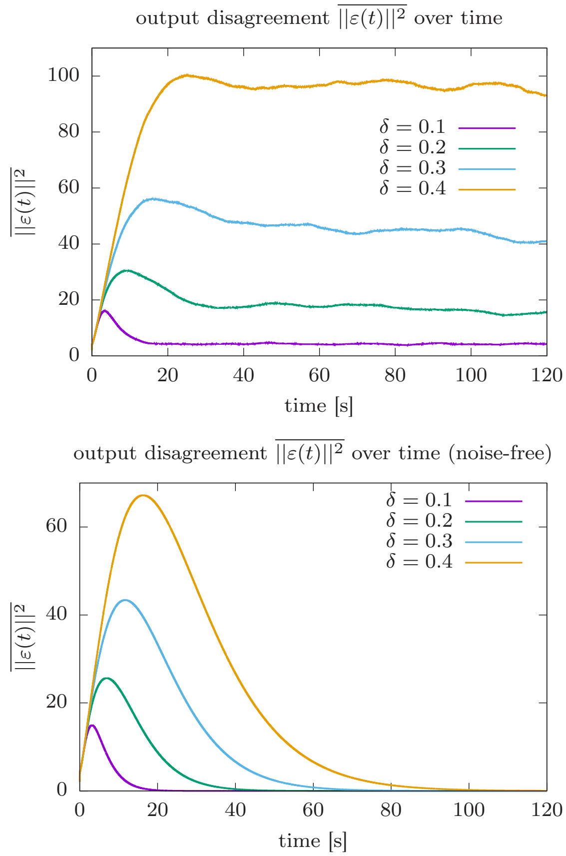

Fig. 2. Average $\overline{||\varepsilon(t)||^2}$ with $\varepsilon_k(t) = Y_0(t) - Y_k(t)$ for $k \in \mathcal{V}$ with noises (top). The noise-less case is shown in the bottom plot.

for $\delta \in \{0.1, 0.2, 0.3, 0.4\}$, that is, the arithmetic mean of $\mathbf{E}[||\varepsilon_k(t)||^2]$, with $\varepsilon_k(t) = Y_0(t) - Y_k(t)$ for $k \in \mathcal{V}$, where the expectation $\mathbf{E}[\cdot]$ is obtained through 10^3 Monte Carlo runs. For completeness, the noise-free case is represented in Fig. 2 (bottom), with $P_k = P_0 = 0$ and $Q_k = Q_0 = 0$, to show the convergence to zero of the square of the disagreement.

VI. Conclusions

In this paper we propose a solution of the leader-following problem for linear stochastic agents with a common uniform constant communication delay. The solution in presence of generic communication delays remains a challenging problem. The future developments will be aimed at dealing with distinct and possibly time-varying delays for which the techniques presented in this paper will need substantial developments.

Appendix

A. Noise-input-to-state and noise-to-state $\mathcal{L}_2$-stability

Lemma 6: Given (3), if $Z(t)$ is NISE $\mathcal{L}_2$-stable and $\sup_{t \geq 0} \|U(t)\|^2_{\mathcal{L}_2(\Omega;\mathbb{R}^m)} < \infty$ then $Z(t)$ is NSE $\mathcal{L}_2$-stable.

Lemma 7: Given (3), if A is Hurwitz stable and $\|U(t)\|^2_{\mathcal{L}_2(\Omega;\mathbb{R}^m)}$ is finite for all t, then $Z(t)$ is NISE $\mathcal{L}_2$-stable.

Proof. Let $\Pi, Q > 0$ be symmetric positive definite matrices such that $\Pi A + A^\top \Pi = -Q < 0$. By Itô rule, with $T_\Pi = \mathbf{Tr}\{P^\top \Pi P\}$ and $\mathbf{E}_Z(t) = \mathbf{E}[\|Z(t)\|^2_\Pi]$,

$$\begin{aligned} \mathrm{d}\|Z(t)\|^2_\Pi = (-\|Z(t)\|^2_Q + 2Z^\top(t)\Pi U(t) + T_\Pi)\mathrm{d}t \\ +2Z^\top(t)\Pi \mathrm{d}N(t) \end{aligned} \tag{42}$$

so that $Z(t)$ is NISE $\mathcal{L}_2$-stable since by Dynkin's formula, for any $0 < \mu < \rho = \frac{\lambda_{min}(Q)}{\lambda_{max}(\Pi)}$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_Z(t) \le -(\rho-\mu)\mathbf{E}_Z(t) + \frac{\lambda_{max}(\Pi)}{\mu}\mathbf{E}[\|U(t)\|^2] + T_\Pi, \\ &\|Z(t)\|^2_{\mathcal{L}_2} \le \frac{\mathbf{E}_Z(t)}{\lambda_{min}(\Pi)} \le \frac{\lambda_{max}(\Pi)}{\lambda_{min}(\Pi)}\|Z(0)\|^2_{\mathcal{L}_2} e^{-(\rho-\mu)t} \\ &+\frac{(\rho-\mu)^{-1}}{\lambda_{min}(\Pi)}\lambda_{max}(\Pi)\Big[\frac{\sup_{\tau\le t}\|U(\tau)\|^2_{\mathcal{L}_2}}{\mu} + \mathbf{Tr}\{PP^\top\}\Big] \end{aligned} \tag{43}$$

B. Some elementary facts on scalar Lambert equations

Consider the scalar equation $w = be^{-w}$. Any solution $w(b)$ is denoted by $w^{(k)}(b)$, the k-th branch of the Lambert W-function. If $b = -e^{-1}$ there is a double root $w(b) = w^{(0)}(b) = w^{(-1)}(b) = -1$ together with a countable infinity of simple complex roots for $k \ne 0, -1$. If $b \ne -e^{-1}$ there is a countable infinity of simple complex roots only. Moreover, if b is real then for $-\frac{1}{e} \le b < 0$ there are two possible real values of $w(b)$, more precisely $w(b) = w^{(0)}(b) \in [-1, 0)$ and $w(b) = w^{(-1)}(b) \in (-\infty, -1]$. Finally, $w(b) = w^{(0)}(b) \to 0^-$ and $w(b) = w^{(-1)}(b) \to -\infty$ as $b \to 0$. For $b > 0$ there is one real value of $w(b)$, and $w(b) = w^{(0)}(b) > 0$.

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