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Tensors Associated with Mean Quadratic Differences Explaining the Riskiness of Portfolios of Financial Assets

Pierpaolo Angelini [1,*] and Fabrizio Maturo [2]

[1] Dipartimento di Scienze Statistiche, Università La Sapienza, 00185 Rome, Italy
[2] Faculty of Economics, Universitas Mercatorum, 00186 Rome, Italy; fabrizio.maturo@unimercatorum.it
* Correspondence: pier.angelini@uniroma1.it

Abstract: Bound choices such as portfolio choices are studied in an aggregate fashion using an extension of the notion of barycenter of masses. This paper answers the question of whether such an extension is a natural fashion of studying bound choices or not. Given n risky assets, the question of why it is appropriate to treat only two risky assets at a time inside the budget set of the decision-maker is handled in this paper. Two risky assets are two goods. They are two marginal goods. The question of why they always give rise to a joint good inside the budget set of the decision-maker is addressed by this research work. A single risky asset is viewed as a double one using four nonparametric joint distributions of probability. The variability of a joint distribution of probability always depends on the state of information and knowledge associated with a given decision-maker. For this reason, two variability tensors are defined to identify the riskiness of the same risky asset. A multilinear version of the Sharpe ratio is shown. It is based on tensors. After computing the expected return on an n-risky asset portfolio, its riskiness is obtained using mean quadratic differences developed through tensors.

Keywords: utility; quadratic metric; multilinear relationship; α-product; α-norm; rational behavior

MSC: 60A05; 60B05; 91B24; 91B16; 91B06; 91B08

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1. Introduction

In this paper, bound choices such as portfolio choices are studied without adding new axiomatic constructions or using known ones. Previous studies tend to add or use such formalistically abstract constructions (Echenique 2020; Chambers et al. 2017; Nishimura et al. 2017). Such constructions are exact (Halevy et al. 2018). Nevertheless, in our opinion, they are empty, and this characteristic is perhaps inevitable. Conversely, we propose an approach of an operational nature based on metric measures (choices being made by a given decision-maker and expressed by means of specific measures put forward by Corrado Gini are dealt with by Wang et al. (2018)). The advantages of this approach are essentially two. First, such measures indicating rational choices can be introduced without a problem. This is because they comply with any reasonable axiomatic construction (Cassese et al. 2020). Second, such measures are in accordance with one of the fundamental needs of science, which must work with notions of ascertained validity in a pragmatic sense. In our opinion, science must not take combinations of axioms as indefectible concepts, but it must be based on actual experiences, which are at least conceptually possible. Such experiences are subjected to a measure. A remarkable point of this research work is the following. Such a point is connected with how a measure can be obtained. In our opinion, bound choices must be studied under conditions of uncertainty and riskiness (Angelini and Maturo 2021b). They are real and unavoidable conditions (Chudjakow and Riedel 2013; Machina 1987). It follows that we focus on the notion of probability and its properties. This notion is intrinsically subjective (Pfanzagl 1967). A theorem enunciating the notion of utility to be a metric measure is shown by us. Hence, prevision (probability) and utility are two metric measures (that which has been made by Viscusi and Evans (2006) and Abdellaoui et al. (2013) is enlarged by us in this paper). Prevision (probability) and utility are innovatively discovered to be two sides of the same coin.

In this study, probability judgments always depend on the state of information and knowledge associated with a given individual. We focus on a specific interpretation referred to Bayes' theorem. It is essential to our purposes to explain why we focus on the following geometric interpretation referred to Bayes' theorem, where two stages are distinguished. Every bound choice is intrinsically a barycenter of masses subjectively distributed over a finite set of alternatives. In the first stage, all the barycenters of masses are considered. Their number is infinite. All of them give rise to a convex set. It is the budget set of the decision-maker. In the second stage, one of the barycenters is chosen, so a probabilistic but convergent element is associated with a rational choice.

This paper fills a conceptual and mathematical gap existing in the current literature. It is possible to enlarge the notion of rational behavior. It follows that the optimization principle can be enlarged as well. How is this possible? Given two goods denoted by ${}_1X$ and ${}_2X$, their possible values meant as pure numbers are expressed by the sets $I({}_1X)$ and $I({}_2X)$. A given decision-maker chooses $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ inside his or her budget set. This means that a given decision-maker is indifferent to the exchange of ${}_1X$, identified with $I({}_1X)$, for $\mathbf{P}({}_1X)$ and of ${}_2X$, identified with $I({}_2X)$, for $\mathbf{P}({}_2X)$. Also, this implies that he or she is indifferent to the exchange of X_{12} for $\mathbf{P}(X_{12})$, where $\mathbf{P}(X_{12})$ extends the notion of barycenter of masses. This is because $\mathbf{P}(X_{12})$ is the determinant based on four measures denoted by $\mathbf{P}({}_1X\,{}_1X)$, $\mathbf{P}({}_1X\,{}_2X)$, $\mathbf{P}({}_2X\,{}_1X)$, and $\mathbf{P}({}_2X\,{}_2X)$. This means that four nonparametric joint distributions of probability are considered. X_{12} is a multiple good of order 2, whose elements are ${}_1X$ and ${}_2X$. The possible values for X_{12} coincide with the components of a tensor. $\mathbf{P}(X_{12})$ is a multiple choice associated with a multiple good. What will be said in this paper is more general than one might think at first. This is because the mathematical notion of α-product on which $\mathbf{P}({}_1X\,{}_1X)$, $\mathbf{P}({}_1X\,{}_2X)$, $\mathbf{P}({}_2X\,{}_1X)$, and $\mathbf{P}({}_2X\,{}_2X)$ are based is discovered. Such a notion uses subjective probabilities intrinsically connected with exchangeable or symmetric events. Such a notion does not only explain bound choices but also can treat multilinear issues of statistical inference. This makes explicit where the results of this paper can be applied. Every bound choice is studied using subjective tools, probability, and utility, inside a subset of a linear space over $\mathbb{R}$. Linear spaces over $\mathbb{R}$ with a different dimension are here handled. A specific element is held fixed: possible and objective alternatives whose number is finite are always summarized. Possible and objective alternatives are real data. They can be viewed as sampling data. This makes explicit how the results of this paper can be applied. It is then possible to find out a strict connection between how bound choices are dealt with within this context and the least-squares model (as an alternative, a connection between economics and mathematics based on differential equations could be developed by examining Oderinu et al. (2023) as well).

1.1. Bound Choices Made by the Decision-Maker under Claimed Conditions of Certainty

We do not study more than two goods at a time inside the budget set of the decision-maker. This is because we use mathematical methods via a quadratic metric. Every bound choice being made by a given decision-maker inside his or her budget set is a measure obtained using a quadratic metric. It is not convenient to use a non-quadratic metric. For instance, in statistics, variance, standard deviation, and the covariance of two variables are indices obtained using a quadratic metric. It is certainly possible to study n goods, with $n > 2$ which is an integer. Nevertheless, whenever we want to obtain a measure, it is not convenient to study more than two goods at a time. Another remarkable issue developed in this paper is the following. Conceptually, the conditions of certainty referred to nonrandom goods[1] have to be understood as intrinsically fictitious. Given two nonrandom goods with downward-sloping demand curves, that which is chosen by a given decision-maker is denoted by (x_1, x_2) (primordial and fundamental aspects about revealed preference theory studying bound choices are dealt with by Samuelson (1948)). In our opinion, the objects of decision-maker choice are explicitly bilinear and disaggregate measures (an analysis based on the two-good assumption is made by Cherchye et al. (2018)). Hence, there exists a one-to-one correspondence between two-dimensional points of the budget set of the decision-maker and bilinear and disaggregate measures. Each measure is decomposed into two linear measures. Each of them is a one-dimensional point. We establish the following:

Definition 1. *Given two nonrandom goods with downward-sloping demand curves, that which is demanded for each of them under claimed conditions of certainty by a given decision-maker is an average quantity. We write*

$$x_1 = x_1^1\, p_1^1 + \ldots + x_1^m\, p_1^m \tag{1}$$

and

$$x_2 = x_2^1\, p_2^1 + \ldots + x_2^m\, p_2^m, \tag{2}$$

where $\{p_1^i\}$ *and* $\{p_2^j\}$ *are two sets of* m *nonnegative masses. Their sum is always equal to* 1 *with regard to each of them. Each mass of them is always between* 0 *and* 1*, endpoints included. The possible quantities which can be demanded for good* 1 *are expressed by* $\{x_1^1, \ldots, x_1^m\}$*, whereas the possible quantities which can be demanded for good* 2 *are given by* $\{x_2^1, \ldots, x_2^m\}$*.*

The possible quantities which can be demanded for good 1 and good 2 are pure numbers. They are possible events. Their nature is objective. They are not directly observed, but they are estimated. What is directly observed is given by (x_1, x_2). The possible quantities which can be demanded for good 1 and good 2 are the components of two vectors of E^m, where E^m is an m-dimensional linear space over $\mathbb{R}$ with a Euclidean structure. Given an orthonormal basis of E^m, any vector whatsoever of E^m is always expressed as a linear combination of basis vectors. The real coefficients of this linear combination are its components. One and only one set of components of a vector of E^m uniquely identifies it. In this paper, good 1 and good 2 are jointly considered, so the weighted average of m^2 possible quantities which can be demanded for good 1 and good 2 is also studied. Such quantities are obtained by taking the Cartesian product given by $\{x_1^1, \ldots, x_1^m\} \times \{x_2^1, \ldots, x_2^m\}$ into account. Such quantities are the components of an affine tensor. The notion of event is always subdivisible, so m^2 possible alternatives can be studied. This means that a nonparametric joint distribution of probability is dealt with[2]. Every weighted average of m possible quantities which can be demanded for good 1 and good 2 is always found between the lowest possible quantity and the highest possible one (the rationality of the behaviors associated with decision-makers viewed to be as consumers is dealt with by Varian (1983)). The same is true regarding $m \times m = m^2$ possible quantities (that which is demanded by a given decision-maker being faced with his or her budget constraint is studied by Varian (1982)). All coherent weighted averages of m^2 possible alternatives identify a two-dimensional convex set. It is a continuous subset of $\mathbb{R} \times \mathbb{R}$. It is the budget set of the decision-maker. Two one-dimensional convex sets coinciding with two closed line segments appear as well. They belong to two mutually orthogonal axes of a two-dimensional Cartesian coordinate system. Strictly speaking, we deal with two half-lines, where each of them extends indefinitely from zero toward positive real numbers before being restricted. At the first stage, all coherent weighted averages of m^2 possible alternatives are handled. Their number is infinite. All coherent weighted averages of m possible alternatives for good 1 and good 2 are also dealt with. Their number is infinite. At a second stage, (x_1, x_2) is chosen. This choice depends on further hypotheses of an empirical nature. Boundary points that are found on each axis of a two-dimensional Cartesian coordinate system identify degenerate averages[3]. The budget line identifying the budget set of the decision-maker is a hyperplane embedded in a two-dimensional Cartesian coordinate system. Its negative slope depends on the prices of good 1 and good 2. We write

$$b_1\, x_1 + b_2\, x_2 \leq b, \tag{3}$$

where b_1, b_2, and b are positive real numbers, to identify his or her budget constraint. The slope of (3) is given by $-\frac{b_1}{b_2}$. Its horizontal intercept is given by $\frac{b}{b_1}$, whereas its vertical one is given by $\frac{b}{b_2}$. A specific pair of known and objective prices is denoted by (b_1, b_2), whereas the objective amount of money the decision-maker has to spend is expressed by b. Conditions of certainty are fictitious. This is because actual situations are uncertain. In particular, variations in the total amount of money the decision-maker has to spend could happen. Also, risks of external origin determining variations in his or her income could occur as well. This means that if (b_1, b_2, b) represents the decision-maker budget, then b must be assumed of an uncertain nature at the time of choice. The same b can appear even when the state of information and knowledge associated with a given decision-maker is assumed to have become complete later. On the other hand, if there is no ignorance anymore because further information is later acquired, then it is also possible to observe a parallel shift outward or inward of the budget line. Given (x_1, x_2), the weighted average of m^2 possible alternatives is a summarized element of the Fréchet class. According to our approach, the decision-maker also chooses this summarized element in addition to (x_1, x_2)[4].

We establish the following:

Definition 2. *The set of all weighted averages of m^2 possible alternatives, with the same given marginal weighted averages of m possible quantities which can be demanded for good 1 and m possible quantities which can be demanded for good 2, constitutes the Fréchet class.*

We note the following:

Remark 1. *The possible quantities which can be demanded for each nonrandom good taken into account are possible points (pure numbers) belonging to sets whose elements are finite in number. By definition, a hyperplane embedded in a two-dimensional Cartesian coordinate system never separates a coherent summary of possible points from their sets. In other terms, the budget line never separates (x_1, x_2) from $\{x_1^1, \ldots, x_1^m\}$, $\{x_2^1, \ldots, x_2^m\}$, and $\{x_1^1, \ldots, x_1^m\} \times \{x_2^1, \ldots, x_2^m\}$. This characterizes the points of the convex set. The budget set of the decision-maker is a convex set.*

1.2. A Random Good: Logical and Probabilistic Aspects

Assets are goods that provide a flow of services over time. A flow of consumption services can be provided by assets. A flow of money that can be used to purchase consumption can also be provided by assets. Financial assets provide a monetary flow. For instance, the flow of services provided by financial assets can be the flow of interest payments. In this paper, we focus on the future return provided by financial assets under conditions of uncertainty and riskiness. This future or expected return must be estimated by a given individual with respect to observed returns in the past. One of the observed returns can be the actual return. Financial assets such as risky assets are studied under conditions of uncertainty and riskiness. They are random goods. A random good is a random quantity[5] viewed as a specification of what will be chosen in each different outcome of a random process. The different outcomes of a random process are different random events. A random good is intrinsically characterized by a nonparametric probability distribution consisting of a list of different outcomes and the probability associated with each outcome (Gilio and Sanfilippo (2014)). The decision-maker chooses a nonparametric probability distribution of obtaining different random events. We establish the following:

Definition 3. *Let $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ be the identity function on $\mathbb{R}$, where $\mathbb{R}$ is a linear space over itself. Given m incompatible and exhaustive events, a random good denoted by X is the restriction of $id_{\mathbb{R}}$ to $I(X) = \{x^1, x^2, \ldots, x^m\} \subset \mathbb{R}$ such that we write $id_{\mathbb{R}|I(X)} : I(X) \to \mathbb{R}$.*

A random good is nothing but a random variable X on a sample space denoted by Ω. It is a function from Ω into the set $\mathbb{R}$ of real numbers such that the pre-image of any interval of $\mathbb{R}$ is an event in Ω. Our intervals are: $[x^1, x^1], \ldots, [x^m, x^m]$. The points in Ω are real numbers only. Given an orthonormal basis of E^m, a random good is represented by a vector whose contravariant components coincide with the elements of $I(X) = \Omega$ before transferring them on a one-dimensional straight line, on which an origin, a unit of length, and an orientation are established. We write

$$I(X) = \{x^1, x^2, \ldots, x^m\}, \tag{4}$$

with $x^1 < x^2 < \ldots < x^m$ without loss of generality. It is clear that we have $\inf I(X) = x^1$ and $\sup I(X) = x^m$. A located vector at the origin of E^m is completely established by its endpoint. An ordered m-tuple of real numbers can be either a point of an affine space denoted by $\mathcal{E}^m$ or a vector of E^m. Accordingly, $\mathcal{E}^m$ and E^m are isomorphic. Each event is generically denoted by E_i, $i = 1, \ldots, m$. We write

$$X = x^1|E_1| + x^2|E_2| + \ldots + x^m|E_m|, \tag{5}$$

where we have

$$|E_i| = \begin{cases} 1, & \text{if } E_i \text{ is true} \\ 0, & \text{if } E_i \text{ is false} \end{cases} \tag{6}$$

for every $i = 1, \ldots, m$. Regarding a given set of information and knowledge, we consider the finest possible partition of X into elementary events. The nature of this partition is always relative, arbitrary, and temporary. That alternative which will turn out to be verified a posteriori is nothing but a random point contained in $I(X)$ (von Neumann 1936). This point contained in $I(X)$ is a real number. It expresses everything there is to be said whenever uncertainty ceases. Each possible value for X could uniquely be expressed by

$$\{x^1 + a,\ x^2 + a,\ \ldots,\ x^m + a\}, \tag{7}$$

where $a \in \mathbb{R}$ is an arbitrary constant. We consider infinite changes of origin in this way (Angelini and Maturo 2021a).

We deal with ordered m-tuples of real numbers (that which is objectively possible is dealt with by Coletti et al. (2016)). All possible values for X are uncertain, so it makes sense that the decision-maker attributes to each of them a probability. $I(X)$ with the assignment of probabilities is a probability space denoted by $(\Omega, \mathcal{F}, \mathbf{P})$. The set of all possible outcomes is denoted by Ω. This set is embedded in a larger space with a linear structure. We write $\mathcal{F} = \{\emptyset, \Omega\}$ to denote a set of events[6], whereas $\mathbf{P}$ is a function of probability or prevision defined as an expression of the subjective opinion of a given decision-maker[7]. We think of probability as being a mass. It is always a nonnegative and additive function. Its value is equal to 1 on the whole space of the possible values for the random good taken into account. The notion of probability is not undefined within this context (Anscombe and Aumann 1963). It is the degree of belief in the occurrence of a single event attributed by a given decision-maker at a given instant and with a given set of information and knowledge (Schmeidler 1989). Uncertainty about an event depends on the existence of imperfect information and knowledge by the decision-maker (Capotorti et al. 2014). We speak about uncertainty in the simple sense of ignorance (Jurado et al. 2015). Uncertainty consists of two different aspects. Possibility and probability are the two aspects of it. In this paper, they are studied inside linear spaces over $\mathbb{R}$. Possibility and probability are expressed by two vectors of E^m used to obtain $\mathbf{P}(X)$, where $\mathbf{P}(X)$ is viewed to be as a scalar or inner product written in the form

$$\mathbf{P}(X) = x^1\, p_1 + x^2\, p_2 + \ldots + x^m\, p_m. \tag{8}$$

We write

$$\mathbf{x} = (x^1, x^2, \ldots, x^m) \tag{9}$$

to denote what is objectively possible, whereas we write

$$\mathbf{p} = (p_1, p_2, \ldots, p_m) \tag{10}$$

to denote what is subjectively probable. Since we have

$$p_1 + p_2 + \ldots + p_m = 1, \tag{11}$$

with $0 \leq p_i \leq 1$, $i = 1, \ldots, m$, all those evaluations such that (11) holds are coherent. Their number is equal to ∞^{m-1}. Given $x^1, x^2, \ldots, x^m$, a random process consists of ∞^{m-1} possible choices of masses such that a weighted average of m values given by x^1, x^2, …, x^m takes place. If X is viewed to be as a vector of E^m, then it is a linear combination of m incompatible and exhaustive events expressed by

$$X = x^1\,|E_1|\,\mathbf{e}_1 + x^2\,|E_2|\,\mathbf{e}_2 + \ldots + x^m\,|E_m|\,\mathbf{e}_m, \tag{12}$$

where $\mathcal{B}_m = \{\mathbf{e}_i\}$, $i = 1, \ldots, m$, is an orthonormal basis of E^m. Regarding $\mathcal{B}_m$, we write

$$\mathbf{x} = x^1\,\mathbf{e}_1 + x^2\,\mathbf{e}_2 + \ldots + x^m\,\mathbf{e}_m. \tag{13}$$

If the Einstein summation convention is used, then it gives

$$\mathbf{x} = x^i\,\mathbf{e}_i. \tag{14}$$

1.3. The Objectives of the Paper

All the objectives of this paper are innovative. Bound choices are based on possible alternatives. Every choice is a barycenter of masses distributed over a finite set of possible alternatives. The latter is embedded in a larger and more manageable space. Regarding choices being made under conditions of uncertainty and riskiness, possible alternatives are not estimated, but they are observed. What is chosen by a given decision-maker inside his or her budget set coincides with a coherent summary of a nonparametric joint distribution of mass. This summary is a bilinear measure. It is always decomposed into two linear measures. The budget set of the decision-maker consists of points such that each point of it has two Cartesian coordinates. Each of them is a summary of a nonparametric marginal distribution of mass related to a marginal good. Given the two marginal distributions of mass, all possible joint distributions of mass constitute the Fréchet class. We admit that it is useful to compare a concrete (nonparametric) probability distribution with a model which is not a continuous function such as the density function of a continuous random variable, but it is itself a distribution of mass. The latter is characterized by probabilities that are finitely but not countably additive. All possible joint distributions of mass are summarized by a given decision-maker. He or she chooses one of these summaries according to his or her variable state of information and knowledge. He or she can choose a coherent summary such that there is no linear correlation between good 1 and good 2. He or she could also choose a coherent summary such that there is an inverse or direct linear relationship between good 1 and good 2. Regarding the Fréchet class, two extreme limit cases together with an intermediate case are accordingly taken into account. They are paradigmatic cases. Each of them identifies the above model. If good 1 and good 2 are two risky assets, then it is methodologically possible to validate that the notion of risk is intrinsically subjective. We develop the notion of mean quadratic difference put forward by Corrado Gini. We develop it via a tensorial approach. The variability of a distribution of mass always depends on how the decision-maker estimates all the masses under consideration. It follows that the origin of this variability is not random within this context. It is not standardized because the decision-maker makes explicit, from time to time, the knowledge hypothesis underlying it. The origin of the variability of a distribution of mass is not connected with the theory of measurement errors, where such errors are random. Regarding the Sharpe ratio, after computing the expected return on an n-risky asset portfolio, its riskiness is obtained using the notion of mean quadratic difference. The decision-maker always maximizes his or her subjective utility connected with average quantities. In this paper, the notion of utility is a metric measure as well. What is said in this paper can be extended. Multilinear relationships between variables are discovered and handled, so an extension of the least-squares model can be made. In economics, it is frequent that there is one-way causation in the sense that given variables influence another variable, but there is no feedback in the opposite direction. This means that a specific variable does not influence other variables. Conversely, all the multilinear indices we propose in this paper allow the studying of relationships between variables in such a way that there is a two-way causation. Hence, it is possible to study variables influencing each other.

In Section 2 of the paper, antisymmetric tensors identifying multilinear indices are handled. In Section 3, risky assets viewed to be as random goods are studied. In Section 4, analytic conditions allowing the studying of a single risky asset as a double one are developed. Section 5 shows a variability tensor. Section 6 shows another variability tensor. In Section 7, a multilinear approach to the Sharpe ratio is dealt with. In Section 8, future perspectives of our research are outlined after discussing the main results contained in the paper.

2. Two Random Goods That Are Jointly Considered: From Disaggregate Choices to Aggregate Ones

2.1. Bound Choices Made by a Given Decision-Maker under Conditions of Uncertainty and Riskiness: Their Decomposition Inside His or Her Budget Set

Two random goods which are jointly considered inside the budget set of the decision-maker can be handled through the same framework characterizing bound choices being made by him or her under claimed conditions of certainty (portfolio choices with transient price impact are studied by Ekren and Muhle-Karbe (2019)). Given two marginal random goods denoted by ${}_1X$ and ${}_2X$, the number of the possible values for each of them is first equal to m. ${}_1X$ and ${}_2X$ are linearly independent. Hence, we consider two mutually orthogonal axes of a two-dimensional Cartesian coordinate system, on which an origin, the same unit of length, and an orientation are established. The possible values for each random good taken into account are transferred on a one-dimensional straight line. Thus, we do not consider an m-dimensional point, but we deal with m one-dimensional points on a one-dimensional straight line. We pass from E^m to a linear space over $\mathbb{R}$ with its dimension which is equal to 1. There exists a one-to-one correspondence between a one-dimensional linear subspace of E^m and a one-dimensional straight line, on which an origin, a unit of length, and an orientation are chosen. A one-dimensional linear subspace of E^m contains all collinear vectors[8] regarding one of the two vectors belonging to E^m. Its contravariant components coincide with the possible values for a marginal good. Two one-dimensional linear subspaces of E^m are dealt with. These subspaces identify two one-dimensional straight lines, on which an origin, the same unit of length, and an orientation are chosen. They establish the budget set of the decision-maker. They establish an uncountable subset of the direct product of $\mathbb{R}$ and $\mathbb{R}$ denoted by $\mathbb{R} \times \mathbb{R}$. Its dimension is equal to 2. All the m^2 possible values for two random goods which are jointly considered give rise to ${}_1X\,{}_2X$. All the m^2 possible values for ${}_1X\,{}_2X$ identify, together with m^2 probabilities, $\mathbf{P}({}_1X\,{}_2X)$. If $\mathbf{P}({}_1X\,{}_2X)$ is bilinear, where $\mathbf{P}$ stands for prevision or mathematical expectation of a joint random good denoted by ${}_1X\,{}_2X$, then $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ are linear. We write $\mathbf{P}({}_1X\,{}_2X) \equiv (\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ because we deal with a bilinear measure coinciding with a two-dimensional point. If $\mathbf{P}$ is linear, then ${}_1X$ must always be a random good with its possible values which are all nonnegative. The same is true by considering the possible values for ${}_2X$ on the vertical axis. If $\mathbf{P}$ is linear, then it is first additive and convex.

Given two random goods, $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ tell us how much the decision-maker is choosing to demand for one of the two random goods taken into account and how much he or she is choosing to demand for the other. His or her budget set is established by the negative slope of the budget line coinciding with a hyperplane embedded in a two-dimensional Cartesian coordinate system. His or her budget set is also established by the two mutually orthogonal axes taken into account. In particular, we consider two half-lines. His or her budget set is accordingly a right triangle belonging to the first quadrant of a two-dimensional Cartesian coordinate system. The vertex of the right angle of the triangle taken into account coincides with the point given by $(0, 0)$. ${}_1X$, ${}_2X$, and ${}_1X\,{}_2X$ are first discrete goods. They are studied as continuous goods when and only when all their coherent previsions are taken into account at the first stage. The budget line is an equation of a linear function expressed in an implicit form. The prices of the prevision bundle[9] denoted by $\mathbf{P}({}_1X\,{}_2X) \equiv (\mathbf{P}({}_1X), \mathbf{P}({}_2X))$ are formally two constants of the straight line expressed in an implicit form such that their ratio gives its slope. The budget constraint of the decision-maker requires that the amount of money spent on the two random goods be no more than the total amount he or she has to spend. The budget constraint derives from

$$c_1\,({}_1X) + c_2\,({}_2X) \leq c. \tag{15}$$

It is written in the form

$$c_1\,\mathbf{P}({}_1X) + c_2\,\mathbf{P}({}_2X) \leq c, \tag{16}$$

where (c_1, c_2) are the objective prices of the two random goods, whereas the objective amount of money the decision-maker has to spend is equal to c. Please note that c_1, c_2, and c are positive real numbers. The slope of the budget line expressed by

$$c_1\,\mathbf{P}({}_1X) + c_2\,\mathbf{P}({}_2X) = c \tag{17}$$

is given by

$$-\frac{c_1}{c_2}. \tag{18}$$

The budget line can always be drawn. It is possible to establish its horizontal and vertical intercepts every time. This means that we pass from m to $m + 1$ possible alternatives for each marginal random good. Structures open to the adjunction of new entities as new circumstances arise are considered by us. They are linear spaces over $\mathbb{R}$ with a different dimension. Structures open are considered because the notion of event is intrinsically subdivisible. The prices of the two random goods taken into account are determined whenever the budget line is drawn. Three convex sets are established. They are two one-dimensional convex sets and one two-dimensional convex set. The first one-dimensional convex set is found between zero expressed by $(0, 0)$ and the horizontal intercept of the budget line given by $\frac{c}{c_1}$. The second one is found between zero, expressed by $(0, 0)$, and the vertical intercept of it given by $\frac{c}{c_2}$. The third two-dimensional convex set is given by all the points that are found inside the plane region bounded by the right triangle. Please note that (17) always passes through the point whose coordinates are given by

$$\big(\sup I({}_1X),\, \sup I({}_2X)\big). \tag{19}$$

If the budget line changes its negative slope, then the budget set of the decision-maker changes. He or she chooses a point belonging to his or her changed budget set. His or her state of information and knowledge changes. It is clear that (16) is analogous to (3). We pass from E^{m+1} to a linear space over $\mathbb{R}$ with its dimension which is equal to 1. There exists a one-to-one correspondence between a one-dimensional linear subspace of E^{m+1} and a one-dimensional straight line, on which an origin, a unit of length, and an orientation are chosen. Two one-dimensional linear subspaces of E^{m+1} are dealt with. These subspaces identify two one-dimensional straight lines, on which an origin, the same unit of length, and an orientation are chosen. They establish the budget set of the decision-maker.

The decision-maker's choice functions for the two marginal random goods under consideration are expressed by

$$\mathbf{P}({}_1X) = \big\{\mathbf{P}({}_1X)[(c_1, c_2, c)]\big\}, \tag{20}$$

and

$$\mathbf{P}({}_2X) = \{\mathbf{P}({}_2X)[(c_1, c_2, c)]\}, \tag{21}$$

where $\mathbf{P}$ is additive and convex as a consequence of its coherence. We note the following:

Remark 2. *The decision-maker estimates both marginal masses associated with ${}_1X$ and ${}_2X$ and the joint ones associated with ${}_1X\,{}_2X$. Marginal masses associated with ${}_1X$ and ${}_2X$ give rise to $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$. Given $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$, a bilinear and disaggregate measure coinciding with $\mathbf{P}({}_1X\,{}_2X)$ is a summarized element of the Fréchet class. Given $(\mathbf{P}({}_1X), \mathbf{P}({}_2X))$, the decision-maker also chooses a summarized element of the Fréchet class such that $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ never change. This element is obtained using the notion of α-product outside the budget set of the decision-maker.*

A remarkable point of this paper is the following. The decision-maker can choose a coherent summary of a joint distribution of mass identifying a summarized element of the Fréchet class such that there is no linear correlation between random good 1 and random good 2, so they are stochastically independent. Given the same marginal masses, ${}_1X$ and ${}_2X$ are stochastically independent if each joint mass in a joint distribution is the product of its corresponding marginal masses. In particular, if ${}_1X$ and ${}_2X$ are two risky assets, then the decision-maker is risk neutral. He or she could also choose a coherent summary of a joint distribution of mass such that there is an inverse or direct linear relationship between ${}_1X$ and ${}_2X$. This means that the decision-maker is, respectively, risk averse or risk loving. In fact, given the same marginal masses, an aggregation of joint masses such that ${}_1X$ tends to increase when ${}_2X$ increases shows a direct linear relationship between ${}_1X$ and ${}_2X$. Conversely, given the same marginal masses, an aggregation of joint masses such that ${}_1X$ tends to decrease when ${}_2X$ increases shows an inverse linear relationship between ${}_1X$ and ${}_2X$. Regarding the Fréchet class, two extreme limit cases together with an intermediate case are dealt with. They are paradigmatic cases.

2.2. Two Jointly Considered Random Goods Depending on the Notion of Ordered Pair and Their α-Product

Two marginal random goods denoted by ${}_1X$ and ${}_2X$ always give rise to a joint random good denoted by ${}_1X\,{}_2X$. All its possible values are obtained by considering the Cartesian product of the possible values for ${}_1X$ and ${}_2X$. The horizontal and vertical intercepts must be added to $I({}_1X)$ and $I({}_2X)$, respectively. We write $I({}_1X) \cup \{\frac{c}{c_1}\}$ and $I({}_2X) \cup \{\frac{c}{c_2}\}$. The values of $I({}_1X) \cup \{\frac{c}{c_1}\}$ and $I({}_2X) \cup \{\frac{c}{c_2}\}$ coincide with the contravariant components of two $(m+1)$-dimensional vectors uniquely expressed as linear combinations of $m+1$ basis vectors of E^{m+1}. We put $I({}_1X) \cup \{\frac{c}{c_1}\} = I^*({}_1X)$ and $I({}_2X) \cup \{\frac{c}{c_2}\} = I^*({}_2X)$.

Another remarkable point of this paper is that the notion of ordinal utility is a metric measure (Maturo and Angelini 2023). Prevision (probability) and utility are formally the two sides of the same coin, so it is possible to present the following:

Theorem 1. *Let ${}_1X$ and ${}_2X$ be two logically independent random goods. They are jointly considered inside the budget set of the decision-maker. Their possible values are expressed by $I({}_1X) \cup \{\frac{c}{c_1}\}$ and $I({}_2X) \cup \{\frac{c}{c_2}\}$. If each coherent prevision of ${}_1X\,{}_2X$ denoted by $\mathbf{P}({}_1X\,{}_2X)$ is decomposed into two linear previsions, then its properties coincide with the ones of well-behaved preferences.*

We prove this theorem in another paper of ours. Since indifference curves cannot cross, given any two prevision bundles belonging to two different indifference curves, this theorem tells us that the decision-maker can rank them as to their distance from $(0,0)$ measured along the 45-degree line. One of the prevision bundles is strictly better than the other if and only if its distance from $(0,0)$ measured along the 45-degree line is greater than the other. A numerical example of this can easily be shown using the Pythagorean theorem. It is possible to write

$$^{2}d(O,\mathbf{P}) = \sqrt{\sum_{i=1}^{2} \mathbf{P}({}_{i}X)^{2}} \tag{22}$$

to denote the distance of $\mathbf{P}$ from $O = (0,0)$, where $\mathbf{P}$ stands for $(\mathbf{P}({}_{1}X),\, \mathbf{P}({}_{2}X))$. We write

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}({}_{1}X) \\ \mathbf{P}({}_{2}X) \end{pmatrix}. \tag{23}$$

The bundles for which the decision-maker is indifferent to $(\mathbf{P}({}_{1}X),\, \mathbf{P}({}_{2}X))$ form the indifference curve. Its slope is negative. It can be imagined by identifying preferences for perfect substitutes without loss of generality. It intersects the 45-degree line in a point only. All other indifference curves intersect the 45-degree line. Each of them intersects the 45-degree line in a point only. Preferences are not directly observable. In our approach, the notion of utility has then an independent meaning other than its being what a given decision-maker maximizes.

We establish the following:

Definition 4. *All the events associated with an ordered pair of random goods are obtained by considering the Cartesian product of the possible values for two logically independent random goods denoted by ${}_{1}X$ and ${}_{2}X$. Such random goods give rise to a joint random good denoted by ${}_{1}X\,{}_{2}X$. The latter is a function written in the form ${}_{1}X\,{}_{2}X\colon I^{*}({}_{1}X) \times I^{*}({}_{2}X) \to \mathbb{R}$, where we have ${}_{1}X\,{}_{2}X({}_{(1)}x^{i}, {}_{(2)}x^{j}) = {}_{(1)}x^{i}\,{}_{(2)}x^{j}$, with $i, j = 1, \ldots, m+1$.*

We are faced with

$${}_{1}X\,{}_{2}X = {}_{(1)}x^{1}\,{}_{(2)}x^{1}|{}_{(1)}E_{1}||{}_{(2)}E_{1}| + \ldots + {}_{(1)}x^{m+1}\,{}_{(2)}x^{m+1}|{}_{(1)}E_{m+1}||{}_{(2)}E_{m+1}|, \tag{24}$$

where it is possible to write

$$|{}_{(1)}E_{i}||{}_{(2)}E_{j}| = \begin{cases} 1, & \text{if } {}_{(1)}E_{i} \text{ and } {}_{(2)}E_{j} \text{ are both true} \\ 0, & \text{otherwise} \end{cases} \tag{25}$$

for every $i, j = 1, \ldots, m+1$.

Since ${}_{1}X$ and ${}_{2}X$ are two random goods, where each of them has $m+1$ possible values, two random goods giving rise to ${}_{1}X\,{}_{2}X$ are logically independent whenever there exist $[(m+1)\cdot(m+1)]$ possible values for ${}_{1}X\,{}_{2}X$ (the notion of measure associated with possible values for a random entity is dealt with by Nunke and Savage (1952)). Given $({}_{1}X,\, {}_{2}X)$, we are faced with two different partitions. Each of them is characterized by $m+1$ incompatible and exhaustive events[10]. The covariant components of an affine tensor of order 2 represent the joint masses of the nonparametric joint distribution of ${}_{1}X$ and ${}_{2}X$. Their number is overall equal to $(m+1)^{2}$ (coherent probabilities associated with possible values for random entities are handled by Regazzini (1985)). We say that an ordered pair of random goods denoted by $({}_{1}X,\, {}_{2}X)$ is represented by an ordered triple of geometric entities denoted by

$$\left({}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x}, p_{ij}\right), \tag{26}$$

with $(i,\, j) \in I_{m+1} \times I_{m+1}$, where we write $I_{m+1} = \{1, 2, \ldots, m+1\}$. We consider the notion of α-product between ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$. It is possible to establish a quadratic metric on E^{m+1} in this way. This notion is a scalar or inner product obtained using the joint masses denoted by p_{ij} of the nonparametric joint distribution of ${}_{1}X$ and ${}_{2}X$ together with the contravariant components of ${}_{(1)}\mathbf{x}$ and ${}_{(2)}\mathbf{x}$. We then write

$$\langle {}_{(1)}\mathbf{x},\, {}_{(2)}\mathbf{x} \rangle_{\alpha} = {}_{(1)}x^{i}\,{}_{(2)}x^{j} p_{ij} = {}_{(1)}x^{i}\,{}_{(2)}x_{i} = \mathbf{P}({}_{1}X\,{}_{2}X), \tag{27}$$

where

$$ {}_{(2)}x^j p_{ij} = {}_{(2)}x_i \tag{28} $$

is a vector homography by means of which we pass from ${}_{(2)}x^j$ to ${}_{(2)}x_i$ using p_{ij}. All covariant components of an $(m+1)$-dimensional vector are obtained by means of vector homographies involving p_{ij}. For instance, from the following Table.

Random Good 1 \ **Random Good 2**	**0**	**4**	**5**	**Sum**
0	0	0	0	0
2	0	0.1	0.2	0.3
3	0	0.5	0.2	0.7
Sum	0	0.6	0.4	1

It follows that we have $\mathbf{P}({}_1X\,{}_2X) = 11.8$. Given the contravariant components of ${}_{(2)}\mathbf{x}$ identifying the following column vector

$$ \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}, $$

its covariant components are expressed by

$$ 0 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 = 0, $$

$$ 0 \cdot 0 + 4 \cdot 0.1 + 5 \cdot 0.2 = 1.4, $$

and

$$ 0 \cdot 0 + 4 \cdot 0.5 + 5 \cdot 0.2 = 3, $$

so it is possible to write the following result

$$ \left\langle \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.4 \\ 3 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha = \mathbf{P}({}_1X\,{}_2X) = 11.8. $$

On the other hand, after calculating the covariant components of ${}_{(1)}\mathbf{x}$ in a similar way, we write

$$ \left\langle \begin{pmatrix} 0 \\ 1.7 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \langle {}_{(1)}\mathbf{x}, {}_{(2)}\mathbf{x} \rangle_\alpha = \mathbf{P}({}_1X\,{}_2X) = 11.8. $$

After transferring the possible values for ${}_1X$ and ${}_2X$ on two one-dimensional straight lines, $\mathbf{P}({}_1X\,{}_2X)$ lives inside a subset of a two-dimensional linear space over $\mathbb{R}$. $\mathbf{P}({}_1X\,{}_2X)$ is a measure of a metric nature living inside a subset of a linear space over $\mathbb{R}$ denoted by $\mathbb{R} \times \mathbb{R}$. Please note that $\mathbf{P}({}_1X\,{}_2X)$ is identified with a two-dimensional point. We write

$$ (\mathbf{P}({}_1X), \mathbf{P}({}_2X)) \tag{29} $$

to identify $\mathbf{P}({}_1X\,{}_2X)$ inside the budget set of the decision-maker. This means that $\mathbf{P}({}_1X\,{}_2X)$ is always decomposed into two linear measures. Each of them is identified with a one-dimensional point inside the budget set of the decision-maker. The notion of α-norm is a particular case of the one of α-product. From the following Table.

Random Good 1 \ Random Good 1	**0**	**2**	**3**	**Sum**
0	0	0	0	0
2	0	0.3	0	0.3
3	0	0	0.7	0.7
Sum	0	0.3	0.7	1

It follows that we write $\|_{(1)}\mathbf{x}\|^2_{\alpha} = \mathbf{P}({}_1X\,{}_1X) = 7.5$, whereas from the following Table.

Random Good 2 \ Random Good 2	**0**	**4**	**5**	**Sum**
0	0	0	0	0
4	0	0.6	0	0.6
5	0	0	0.4	0.4
Sum	0	0.6	0.4	1

It follows that we have $\|_{(2)}\mathbf{x}\|^2_{\alpha} = \mathbf{P}({}_2X\,{}_2X) = 19.6$.

2.3. Two Jointly Considered Random Goods That Are Independent of the Notion of Ordered Pair

Let ${}_1X$ and ${}_2X$ be two random goods, where each of them is characterized by $m+1$ possible values. We note the following:

Remark 3. *Given an orthonormal basis of E^{m+1}, the possible values for two separately considered random goods are represented by the contravariant components of two vectors of E^{m+1}. If we are not interested in fusing ${}_1X$ and ${}_2X$, then the possible values for two logically independent random goods which are jointly considered could coincide with the contravariant components of an affine tensor of order* 2*. If we are conversely interested in fusing ${}_1X$ and ${}_2X$, then the possible values for a stand-alone and double random good denoted by X_{12} are represented by the contravariant components of an antisymmetric tensor of order* 2*.*

Since we want to pass from an ordered pair of marginal random goods to two marginal random goods which are jointly considered regardless of the notion of ordered pair, we define a double random good denoted by

$$X_{12} = \{{}_1X, {}_2X\}. \tag{30}$$

It is a multiple random good of order 2. The possible values for X_{12} coincide with the contravariant components of an antisymmetric tensor of order 2. After choosing $(m+1)^2$ joint masses connected with ${}_1X\,{}_2X$, where we write

$${}_1X\,{}_2X\colon I^*({}_1X) \times I^*({}_2X) \to \mathbb{R}, \tag{31}$$

it is necessary to consider four nonparametric joint distributions characterizing ${}_1X\,{}_1X$, ${}_1X\,{}_2X$, ${}_2X\,{}_1X$, and ${}_2X\,{}_2X$, with

$${}_1X\,{}_1X\colon I^*({}_1X) \times I^*({}_1X) \to \mathbb{R}, \tag{32}$$

$${}_2X\,{}_2X\colon I^*({}_2X) \times I^*({}_2X) \to \mathbb{R}, \tag{33}$$

and

$$ {}_2X\,{}_1X\colon I^*({}_2X) \times I^*({}_1X) \to \mathbb{R}, \tag{34}$$

to release X_{12} from the notion of ordered pair. Please note that ${}_1X\,{}_1X$ and ${}_2X\,{}_2X$ give rise to joint distributions such that all off-diagonal joint masses of a two-way table, where the number of rows is equal to the one of columns, coincide with zero. After choosing $(m+1)^2$ joint masses connected with ${}_1X\,{}_2X$, the distributions associated with ${}_1X\,{}_1X$, ${}_1X\,{}_2X$, ${}_2X\,{}_1X$, and ${}_2X\,{}_2X$ are automatically determined.

The mathematical expectation of ${}_iX\,{}_jX$, with $i, j = 1, 2$, is of a bilinear nature. This means that it is separately linear in each marginal random good (the notion of prevision of a random entity is studied by Berti et al. (2001)).

Thus, we present the following:

Theorem 2. *The mathematical expectation of $X_{12} = \{{}_1X,\, {}_2X\}$ denoted by $\mathbf{P}(X_{12})$ coincides with the determinant of a square matrix of order 2. Each element of such a determinant is a real number coinciding with the mean value of ${}_iX\,{}_jX$ denoted by $\mathbf{P}({}_iX\,{}_jX)$, where we have $i, j = 1, 2$.*

This theorem is proved by us in another paper of ours.

What is actually demanded for X_{12} by the decision-maker coincides with $\mathbf{P}(X_{12})$. It is a multiple choice associated with a multiple good. It is an aggregate measure that is obtained after observing what the decision-maker actually chooses inside his or her budget set. He or she chooses $(\mathbf{P}({}_1X),\, \mathbf{P}({}_2X))$ whenever the prices and income are, respectively, c_1, c_2, and c. He or she also chooses $\mathbf{P}({}_1X\,{}_2X)$, so he or she chooses those joint masses such that an element of the Fréchet class is summarized. A remarkable point of this paper is the following. Since a given decision-maker is indifferent to the exchange of ${}_1X$ for $\mathbf{P}({}_1X)$ and of ${}_2X$ for $\mathbf{P}({}_2X)$, he or she is also indifferent to the exchange of X_{12} for $\mathbf{P}(X_{12})$, where we write

$$\mathbf{P}(X_{12}) = \begin{vmatrix} \mathbf{P}({}_1X\,{}_1X) & \mathbf{P}({}_1X\,{}_2X) \\ \mathbf{P}({}_2X\,{}_1X) & \mathbf{P}({}_2X\,{}_2X) \end{vmatrix}. \tag{35}$$

$\mathbf{P}(X_{12})$ extends the notion of barycenter of masses. In particular, the property of the barycenter known as stable equilibrium is extended. Given ${}_1X$ and ${}_2X$ and their average quantities, we consider all deviations from $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ of the possible values for ${}_1X$ and ${}_2X$ (Rockafellar et al. 2006).

We then present the following:

Theorem 3. *The variance of $X_{12} = \{{}_1X,\, {}_2X\}$ denoted by $\mathrm{Var}(X_{12})$ coincides with the determinant of a square matrix of order 2. Each element of such a determinant is a real number coinciding with the variance of ${}_1X$ and ${}_2X$, and with their covariance.*

This theorem is proved by us in another paper of ours.

We note the following:

Remark 4. *The origin of the variability of X_{12} depends on the variable state of information and knowledge associated with a given decision-maker. This is because all deviations from $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ of the possible values for ${}_1X$ and ${}_2X$ depend on his or her variable state of information and knowledge.*

A nonlinear (multilinear) metric is the expression given by

$$\|{}_{12}d\|^2_\alpha = \begin{vmatrix} \|{}_{(1)}\mathbf{d}\|^2_\alpha & \langle {}_{(1)}\mathbf{d},\, {}_{(2)}\mathbf{d}\rangle_\alpha \\ \langle {}_{(2)}\mathbf{d},\, {}_{(1)}\mathbf{d}\rangle_\alpha & \|{}_{(2)}\mathbf{d}\|^2_\alpha \end{vmatrix} = \|{}_{(1)}\mathbf{d}\|^2_\alpha \|{}_{(2)}\mathbf{d}\|^2_\alpha - \left(\langle {}_{(1)}\mathbf{d},\, {}_{(2)}\mathbf{d}\rangle_\alpha\right)^2. \tag{36}$$

It is the area of a 2-parallelepiped. Its edges are two marginal random goods with their possible values that are subjected to two changes of origin. The strict components of ${}_{12}d$ are the coordinates of such edges denoted by ${}_{(1)}\mathbf{d}$ and ${}_{(2)}\mathbf{d}$. We also write

$$\|{}_{12}d\|_{\alpha}^{2} = \mathrm{Var}(X_{12}) = \begin{vmatrix} \mathrm{Var}({}_1X) & \mathrm{Cov}({}_1X, {}_2X) \\ \mathrm{Cov}({}_2X, {}_1X) & \mathrm{Var}({}_2X) \end{vmatrix}, \tag{37}$$

so the property of the barycenter known as the minimum of the moment of inertia is extended.

3. Random Goods Whose Possible Values Are of a Monetary Nature: Risky Assets

3.1. Risky Assets Studied inside the Budget Set of the Decision-Maker

Let ${}_1X$ and ${}_2X$ be two risky assets. In this subsection, we study them inside the budget set of the decision-maker. In the first stage, all coherent expected returns on the portfolio denoted by $X_{12} = \{{}_1X, {}_2X\}$ consisting of two risky assets are expressed by

$$\frac{c_1}{c_1 + c_2}\,\mathbf{P}({}_1X) + \frac{c_2}{c_1 + c_2}\,\mathbf{P}({}_2X) \leq \frac{c}{c_1 + c_2}. \tag{38}$$

Given ${}_1X$ and ${}_2X$, where ${}_1X$ and ${}_2X$ are the components of X_{12}, whenever we use the principle characterizing a linear and quadratic metric to establish the expected return on a two-risky asset portfolio, we focus on the components of X_{12} only. We focus on ${}_1X$ and ${}_2X$ only. The left-hand side of (38) is a weighted average of the two expected returns on the two risky assets taken into account (Markowitz 1952). The two expected returns on the two risky assets taken into account are themselves two weighted averages. A coherent expected return on a joint risky asset denoted by ${}_1X\,{}_2X$ is given by $\mathbf{P}({}_1X\,{}_2X)$. A nonparametric joint distribution of mass is summarized by means of $\mathbf{P}({}_1X\,{}_2X)$. The latter is decomposed into $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ inside the budget set of the decision-maker. In the first stage, all coherent expected returns on a joint risky asset give rise to a two-dimensional convex set. The decision-maker divides his or her relative monetary wealth given by

$$\frac{c_1}{c_1 + c_2} \tag{39}$$

and

$$\frac{c_2}{c_1 + c_2} \tag{40}$$

between the two risky assets taken into account, where we observe

$$\frac{c_1}{c_1 + c_2} + \frac{c_2}{c_1 + c_2} = 1. \tag{41}$$

The budget set of the decision-maker established by the budget constraint given by (16) does not change whenever we multiply all objective prices and income by a positive number. The best rational choice being made by him or her from his or her budget set does not change either. His or her best rational choice depends on his or her subjective preferences (Angelini and Maturo 2022a). His or her best rational choice depends on further hypotheses of an empirical nature. Please note that (39) and (40) can be viewed as the prices associated with average quantities chosen by a given decision-maker, whereas $\frac{c}{c_1+c_2}$ is the amount of money he or she has to spend. Formally, the two prices are constants expressed by real numbers. Their ratio identifies the slope of a hyperplane embedded in a two-dimensional linear space over $\mathbb{R}$. We note the following:

Remark 5. *It is possible to study real data given by time series connected with annual returns referred to marginal risky assets. It is possible to make a coherent prevision about the return associated with each marginal risky asset based on observed data in different stock markets. Each time series is associated with a stock market. Real data given by time series are possible alternatives* *that are summarized. Their nature is intrinsically objective. From the slope of the budget line which can be drawn, it is possible to observe the prices of the two risky assets viewed to be as two marginal random goods. It is also possible to wonder if the decision-maker taken into account maximizes, or does not maximize, his or her subjective utility connected with weighted averages. This is because the notion of ordinal utility is itself a metric measure.*

3.2. Risky Assets Studied outside the Budget Set of the Decision-Maker

Given ${}_1X$ and ${}_2X$ and their expected returns, we consider all deviations from $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ of the possible values for ${}_1X$ and ${}_2X$. We denote them by ${}_{(1)}\mathbf{d}$ and ${}_{(2)}\mathbf{d}$, respectively. Please note that $\mathbf{P}({}_1X)$ and $\mathbf{P}({}_2X)$ are chosen by the decision-maker inside his or her budget set. We are now found outside it. Given

$$\mathbf{y} = \lambda_1 \, {}_{(1)}\mathbf{d} + \lambda_2 \, {}_{(2)}\mathbf{d}, \tag{42}$$

with $\lambda_1 = \frac{c_1}{c_1+c_2}$, $\lambda_2 = \frac{c_2}{c_1+c_2} \in \mathbb{R}$, it is possible to obtain

$$\|\mathbf{y}\|_\alpha^2 = (\lambda_1)^2 \, \|{}_{(1)}\mathbf{d}\|_\alpha^2 + 2\lambda_1 \lambda_2 \langle {}_{(1)}\mathbf{d}, {}_{(2)}\mathbf{d}\rangle_\alpha + (\lambda_2)^2 \, \|{}_{(2)}\mathbf{d}\|_\alpha^2, \tag{43}$$

with

$$\|{}_{(1)}\mathbf{d}\|_\alpha^2 = \mathrm{Var}({}_1X), \tag{44}$$

$$\|{}_{(2)}\mathbf{d}\|_\alpha^2 = \mathrm{Var}({}_2X), \tag{45}$$

and

$$\langle {}_{(1)}\mathbf{d}, {}_{(2)}\mathbf{d}\rangle_\alpha = \mathrm{Cov}({}_1X, {}_2X). \tag{46}$$

Whenever we use a linear and quadratic metric, we focus on the riskiness of ${}_1X$ and ${}_2X$ only. In fact, we consider $\mathrm{Var}({}_1X)$, $\mathrm{Var}({}_2X)$, and $\mathrm{Cov}({}_1X, {}_2X)$. A linear metric is the α-norm of $\mathbf{y}$ given by (43). In particular, it is possible to write

$$\|{}_{(1)}\mathbf{d} - {}_{(2)}\mathbf{d}\|_\alpha^2 = \|{}_{(1)}\mathbf{d}\|_\alpha^2 + \|{}_{(2)}\mathbf{d}\|_\alpha^2 - 2\langle {}_{(1)}\mathbf{d}, {}_{(2)}\mathbf{d}\rangle_\alpha. \tag{47}$$

Such an expression shows the notion of α-distance between two marginal risky assets. Their possible values are subjected to two changes of origin.

4. Conditions Allowing the Studying of a Marginal Risky Asset as a Double Risky Asset

Given a marginal risky asset denoted by ${}_1X$, we want to study it as a double risky asset denoted by X_{12}, where X_{12} intrinsically consists of four joint risky assets. We must study four joint distributions of mass. They must be all summarized[11]. We note that two conditions must be satisfied to represent ${}_1X$ as X_{12}. First, we write

$$\mathbf{P}({}_1X) = \sum_{i_1=1}^{m+1} {}_{(1)}x^{i_1} \, p_{i_1} \tag{48}$$

to denote the expected return on ${}_1X$. We say that ${}_1X$ is the component of a double risky asset, where

$$p = p_{i_1 i_2} \tag{49}$$

is an affine tensor of order 2 whose covariant components express all joint masses taken into account (the conditions of coherence are studied by Berti and Rigo (2002)). Such an affine tensor must satisfy the following relationship given by

$$\sum_{i_1=1}^{m+1} {}_{(1)}x^{i_1} \, p_{i_1} = \sum_{i_1, i_2=1}^{m+1} {}_{(1)}x^{i_1} \, p_{i_1 i_2}. \tag{50}$$

We then say that the two sides of (50) are equal if and only if we have

$$\sum_{i_1=1}^{m+1} p_{i_1} = \sum_{i_1,\, i_2=1}^{m+1} p_{i_1 i_2}. \tag{51}$$

Since we write

$$\sum_{i_1=1}^{m+1} p_{i_1} = \sum_{i_1,\, i_2=1}^{m+1} p_{i_1 i_2} = 1, \tag{52}$$

it follows that the first condition tells us that ${}_1X$ and ${}_1X\,{}_2X$ are two finite partitions of events such that the sum of their associated masses is equal to 1.

The second condition tells us that ${}_1X$ and ${}_1X\,{}_2X$ must have the same summarized measure which is obtained using $\mathbf{P}$. This means that ${}_1X$ and ${}_1X\,{}_2X$ must have the same expected return. We therefore write

$$\sum_{i_1,\, i_2=1}^{m+1} {}_{(1)}x^{i_1}\, {}_{(2)}x^{i_2}\, p_{i_1 i_2} = \sum_{i_1,\, i_2=1}^{m+1} {}_{(1)}x^{i_1}\, p_{i_1 i_2}. \tag{53}$$

It follows that the two sides of (53) are equal if and only if we have

$${}_{(2)}x^{i_2} = 1, \quad \forall\, i_2 \in I_{m+1}. \tag{54}$$

Hence, we note the following:

Remark 6. *Let $\mathcal{B}_{m+1} = \{\mathbf{e}_i\}$, $i = 1, \ldots, m+1$, be an orthonormal basis of E^{m+1}. The possible values for the other risky asset such that ${}_1X$ is studied as X_{12} are the contravariant components, all of them coinciding with 1, of a vector of E^{m+1}. They form the set denoted by*

$$\{1^i\}. \tag{55}$$

Its number of elements is equal to $m+1$. Such components are not vectorially intrinsic because they depend on the basis of E^{m+1} being chosen. If we pass from $\mathcal{B}_{m+1}$ to $\mathcal{B}'_{m+1} = \{\mathbf{e}_{i'}\}$, $i' = 1, \ldots, m+1$, then the contravariant components of such a vector transform like the ones of any other vector of E^{m+1}. We therefore write

$$1^{i'} = a^{i'}_i\, 1^i = \sum_{i=1}^{m+1} a^{i'}_i, \tag{56}$$

where $A = (a^{i'}_i)$ is an $(m+1) \times (m+1)$ matrix expressing a change of basis.

Remark 7. *The vector of E^{m+1} whose contravariant components form the set expressed by*

$$\{\phi^1 = 1,\, \phi^2 = 1,\, \ldots,\, \phi^{m+1} = 1\} \tag{57}$$

is denoted by $\boldsymbol{\phi}$.

It is evident that $\boldsymbol{\phi}$ identifies a degenerate risky asset. It has 1 as its unique possible value.

From a Marginal Distribution of Mass to Four Joint Distributions: A Numerical Example

A nonparametric marginal distribution of mass of ${}_1X$ can be interpreted as a joint distribution of ${}_1X$ and ${}_2X = \boldsymbol{\phi}$. For instance, from the following Table.

${}_2X = \boldsymbol{\phi}$ / ${}_1X$	**1**	**1**	**1**	**Sum**
0	0	0	0	0
2	0	0.3	0	0.3
3	0	0	0.7	0.7
Sum	0	0.3	0.7	1

It follows that we have $\mathbf{P}({}_1X) = \mathbf{P}({}_1X\,{}_2X) = \mathbf{P}({}_2X\,{}_1X) = 2.7$. Since we observe $\mathbf{P}({}_1X\,{}_1X) = 7.5$ and $\mathbf{P}({}_2X\,{}_2X) = 1$, the riskiness of ${}_1X$ can be expressed by

$$\sigma^2_{{}_1X} = \begin{vmatrix} \mathbf{P}({}_1X\,{}_1X) = 7.5 & \mathbf{P}({}_1X\,{}_2X) = 2.7 \\ \mathbf{P}({}_2X\,{}_1X) = 2.7 & \mathbf{P}({}_2X\,{}_2X) = 1 \end{vmatrix} = 0.21.$$

The riskiness of ${}_1X$ is expressed through a known index. It is shown in a more general fashion. In fact, the riskiness of ${}_1X$ is determined as if ${}_1X$ coincides with $X_{12} = \{{}_1X, \boldsymbol{\phi}\}$ (other specific risk measures are handled by Herdegen and Khan (2022)).

5. A Marginal Risky Asset Identified with a Variability Tensor

The possible values for a double risky asset denoted by X_{12} coincide with the strict contravariant components of an antisymmetric tensor of order 2. In general, let ${}_{12}f$ be an antisymmetric tensor of order 2. We write

$$_{12}f^{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}x^{i_1} & {}_{(1)}x^{i_2} \\ {}_{(2)}x^{i_1} & {}_{(2)}x^{i_2} \end{vmatrix} = {}_{(1)}x^{i_1}\,{}_{(2)}x^{i_2} - {}_{(1)}x^{i_2}\,{}_{(2)}x^{i_1} \tag{58}$$

to identify the strict contravariant components of it. If ${}_1X$ is viewed as a double risky asset, then the strict contravariant components of an antisymmetric tensor of order 2 identifying ${}_1X$ are given by

$${}_{(1)}f^{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}x^{i_1} & {}_{(1)}x^{i_2} \\ \phi^{i_1} = 1 & \phi^{i_2} = 1 \end{vmatrix}. \tag{59}$$

We prove the following:

Theorem 4. *A nonparametric distribution of mass characterizing a marginal risky asset denoted by ${}_1X$ is summarized using the notion of α-norm of an antisymmetric tensor of order 2 denoted by ${}_{(1)}f$. A measure of riskiness of ${}_1X$ is obtained by calculating the α-norm of ${}_{(1)}f$ denoted by $\|{}_{(1)}f\|^2_\alpha$.*

Proof. Since it is possible to write

$$\phi^{i_1} p_{i_1 i_2} = \phi_{i_2} = p_{i_2}, \tag{60}$$

the covariant components of $\boldsymbol{\phi}$ represent the masses associated with the possible values for ${}_1X$ by a given decision-maker (Angelini and Maturo 2020). It follows that we observe

$$\phi^{i_1}\phi_{i_1} = 1, \tag{61}$$

where (61) can also be written in the form expressed by

$$\|\boldsymbol{\phi}\|^2_\alpha = 1. \tag{62}$$

The expected return on ${}_1X$ is vectorially expressed by

$${}_{(1)}x^{i_1}\phi_{i_1} = {}_{(1)}x_{i_1}\phi^{i_1} = {}_{(1)}\bar{\mathbf{x}}, \tag{63}$$

where we have

$$ {}_{(1)}\bar{\mathbf{x}} = \begin{pmatrix} {}_{(1)}\bar{x}^1 = \mathbf{P}({}_1X) \\ {}_{(1)}\bar{x}^2 = \mathbf{P}({}_1X) \\ \vdots \\ {}_{(1)}\bar{x}^{m+1} = \mathbf{P}({}_1X) \end{pmatrix}. \tag{64} $$

The strict covariant components of ${}_{(1)}f$ are given by

$$ {}_{(1)}f_{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}x_{i_1} & {}_{(1)}x_{i_2} \\ \phi_{i_1} & \phi_{i_2} \end{vmatrix}. \tag{65} $$

They are obtained by considering all feasible decompositions of two expected returns on the two elements of X_{12} (a geometric approach connected with more general random entities is shown by Pompilj (1957)). We consider different vector homographies to obtain all covariant components of the two vectors denoted by ${}_{(1)}\mathbf{x}$ and $\boldsymbol{\phi}$ identifying the two elements of X_{12}. We compute the mean quadratic difference of ${}_1X$ by taking two different requirements into account (variability measures put forward by Corrado Gini are handled by Berkhouch et al. (2018)). First, the α-norm of an antisymmetric tensor of order 2 is always calculated by considering its strict components. Second, the notion of mean quadratic difference of ${}_1X$ requires that all possible differences be considered (Furman et al. 2017). This means that the non-strict components of ${}_{(1)}f$ are even taken into account. We then write

$$ {}^2\Delta^2({}_1X) = \|{}_{(1)}f\|_\alpha^2 = {}_{(1)}f^{(i_1 i_2)}\,{}_{(1)}f_{(i_1 i_2)} = \frac{1}{2}\,{}_{(1)}f^{i_1 i_2}\,{}_{(1)}f_{i_1 i_2}, \tag{66} $$

where we have

$$ \frac{1}{2} = \frac{1}{2!}. \tag{67} $$

Please note that (67) appears whenever we do not consider the strict components of an antisymmetric tensor of order 2. By taking (59) and (65) into account, we obtain

$$ \frac{1}{2}\,{}_{(1)}f^{i_1 i_2}\,{}_{(1)}f_{i_1 i_2} = \frac{1}{2}\begin{vmatrix} {}_{(1)}x^{i_1} & {}_{(1)}x^{i_2} \\ \phi^{i_1} & \phi^{i_2} \end{vmatrix}\begin{vmatrix} {}_{(1)}x_{i_1} & {}_{(1)}x_{i_2} \\ \phi_{i_1} & \phi_{i_2} \end{vmatrix}. \tag{68} $$

The right-hand side of (68) contains all contravariant and covariant components of ${}_{(1)}f$ at the same time. After reminding (61)–(63), we finally write

$$ {}^2\Delta^2({}_1X) = \frac{1}{2}\begin{vmatrix} 2\,\|{}_{(1)}\mathbf{x}\|_\alpha^2 & 2\,{}_{(1)}\bar{\mathbf{x}} \\ 2\,{}_{(1)}\bar{\mathbf{x}} & 2 \end{vmatrix}. \tag{69} $$

We always associate ${}_{(1)}x^{i_1}$ with ${}_{(1)}x_{i_1}$, ${}_{(1)}x^{i_2}$ with ϕ_{i_2}, ϕ^{i_1} with ${}_{(1)}x_{i_1}$, and ϕ^{i_2} with ϕ_{i_2}. Nevertheless, there are two variable indices separately appearing twice in each single term (monomial). After computing the determinant appearing on the right-hand side of (69), it is then possible to obtain

$$ {}^2\Delta^2({}_1X) = \frac{4}{2}\left(\|{}_{(1)}\mathbf{x}\|_\alpha^2 - {}_{(1)}\bar{\mathbf{x}}^2\right) = 2\,\sigma^2_{{}_1X}. \tag{70} $$

We wrote the square of the relationship between the mean quadratic difference of ${}_1X$ denoted by ${}^2\Delta({}_1X)$ and its standard deviation (Gerstenberger and Vogel 2015). □

The relationship between the mean quadratic difference of ${}_1X$ denoted by ${}^2\Delta({}_1X)$ and its standard deviation has been established by Corrado Gini (Ji et al. 2017). We consider the square of it (Li et al. 2016). In this paper, a tensorial approach to the mean quadratic difference is dealt with. More generally, in our opinion, a tensorial approach to the theory of decision-making is well-grounded because of various reasons. First, the object of decision-maker choice naturally embraces various elements made clear in this research work and it is closely connected with the notion of ordinal utility from an operational point of view. Second, the space where a given decision-maker chooses has a precise mathematical structure. Its technical characteristics must be taken into account to try to find out new results. Third, the conditions of certainty are an extreme simplification. In our opinion, they may produce a sterilization of the connection of choice problems with their applications to reality. Fourth, axiomatic constructions generally lead to accepting for certain the alternative based on which a given decision-maker decides to act. Such constructions link choice problems to reality and to applications by replacing a well-founded probability issue with an impossible translation of it into the logic of certainty. In our opinion, this replacement must not take place.

6. A Variability Tensor Based on Deviations from a Mean Value

Let ${}_{(1)}\mathbf{d}$ be the deviation vector corresponding to the vector denoted by ${}_{(1)}\mathbf{x}$ identifying ${}_1X$. By taking (59) into account, we write

$$ {}_{(1)}\psi^{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}d^{i_1} & {}_{(1)}d^{i_2} \\ \phi^{i_1} & \phi^{i_2} \end{vmatrix}, \tag{71} $$

where only the first row of (71) is different from the one of (59). The second row of (71) is the same as the one of (59). Hence, we prove the following:

Theorem 5. *Given ${}_{(1)}\psi$, its α-norm denoted by $\|{}_{(1)}\psi\|^2_\alpha$ represents the mean quadratic difference of ${}_1X$.*

Proof. The strict covariant components of $\|{}_{(1)}\psi\|^2_\alpha$ are given by

$$ {}_{(1)}\psi_{(i_1 i_2)} = \begin{vmatrix} {}_{(1)}d_{i_1} & {}_{(1)}d_{i_2} \\ \phi_{i_1} & \phi_{i_2} \end{vmatrix}, \tag{72} $$

so we can compute the α-norm of ${}_{(1)}\psi$ denoted by $\|{}_{(1)}\psi\|^2_\alpha$. We consequently write

$$ \|{}_{(1)}\psi\|^2_\alpha = {}_{(1)}\psi^{(i_1 i_2)}\, {}_{(1)}\psi_{(i_1 i_2)} = \frac{1}{2}\, {}_{(1)}\psi^{i_1 i_2}\, {}_{(1)}\psi_{i_1 i_2}, \tag{73} $$

where we have

$$ \frac{1}{2}\, {}_{(1)}\psi^{i_1 i_2}\, {}_{(1)}\psi_{i_1 i_2} = \frac{1}{2} \begin{vmatrix} {}_{(1)}d^{i_1} & {}_{(1)}d^{i_2} \\ \phi^{i_1} & \phi^{i_2} \end{vmatrix} \begin{vmatrix} {}_{(1)}d_{i_1} & {}_{(1)}d_{i_2} \\ \phi_{i_1} & \phi_{i_2} \end{vmatrix}. \tag{74} $$

The right-hand side of (74) contains all contravariant and covariant components of ${}_{(1)}\psi$ at the same time. We note that ${}_{(1)}\mathbf{d}$ and $\boldsymbol{\phi}$ are α-orthogonal. We therefore write

$$ d^{i_1}\, \phi_{i_1} = 0. \tag{75} $$

It follows that we obtain

$$\|_{(1)}\psi\|_\alpha^2 = \frac{1}{2}\begin{vmatrix} 2\,\|_{(1)}\mathbf{d}\|_\alpha^2 & 0 \\ 0 & 2 \end{vmatrix} = \frac{1}{2}\cdot 2\Big(2\,\|_{(1)}\mathbf{d}\|_\alpha^2\Big), \tag{76}$$

where the expression enclosed in parentheses represents twice the α-norm of $_{(1)}\mathbf{d}$. We always associate $_{(1)}d^{i_1}$ with $_{(1)}d_{i_1}$, $_{(1)}d^{i_2}$ with ϕ_{i_2}, ϕ^{i_1} with $_{(1)}d_{i_1}$, and ϕ^{i_2} with ϕ_{i_2}. There are two variable indices separately appearing twice in each single term. Thus, we write

$$\|_{(1)}\psi\|_\alpha^2 = 2\,\|_{(1)}\mathbf{d}\|_\alpha^2, \tag{77}$$

so we observe

$$^2\Delta^2({}_1X) = \|_{(1)}\psi\|_\alpha^2 = 2\,\|_{(1)}\mathbf{d}\|_\alpha^2 = 2\,\sigma^2_{{}_1X}. \tag{78}$$

The mean quadratic difference of $_1X$ denoted by $^2\Delta({}_1X)$ is evidently the same (Shalit and Yitzhaki 2005). We can use both $_{(1)}f$ and $_{(1)}\psi$ to obtain it. They are both of them variability tensors identifying the riskiness of $_1X$. □

The mean quadratic difference of $_1X$ measures the spread of the nonparametric distribution of mass taken into account (Jasso 1979). It is a measure of how far the possible values for $_1X$ are from $\mathbf{P}({}_1X)$ (La Haye and Zizler 2019).

7. The Sharpe Ratio Obtained Using Multilinear Measures

It is possible to write

$$\mathbf{P}(X_{12\ldots n}) = \begin{vmatrix} \mathbf{P}({}_1X\,{}_1X) & \mathbf{P}({}_1X\,{}_2X) & \ldots & \mathbf{P}({}_1X\,{}_nX) \\ \mathbf{P}({}_2X\,{}_1X) & \mathbf{P}({}_2X\,{}_2X) & \ldots & \mathbf{P}({}_2X\,{}_nX) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{P}({}_nX\,{}_1X) & \mathbf{P}({}_nX\,{}_2X) & \ldots & \mathbf{P}({}_nX\,{}_nX) \end{vmatrix}, \tag{79}$$

and

$$\mathrm{Var}(X_{12\ldots n}) = \begin{vmatrix} \frac{1}{2}\,{}^2\Delta^2({}_1X) & \mathrm{Cov}({}_1X,\,{}_2X) & \ldots & \mathrm{Cov}({}_1X,\,{}_nX) \\ \mathrm{Cov}({}_2X,\,{}_1X) & \frac{1}{2}\,{}^2\Delta^2({}_2X) & \ldots & \mathrm{Cov}({}_2X,\,{}_nX) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}({}_nX,\,{}_1X) & \mathrm{Cov}({}_nX,\,{}_2X) & \ldots & \frac{1}{2}\,{}^2\Delta^2({}_nX) \end{vmatrix}, \tag{80}$$

where $\mathbf{P}(X_{12\ldots n})$ denotes the expected return on an n-risky asset portfolio, whereas $\mathrm{Var}(X_{12\ldots n})$ denotes its riskiness. We do not observe $2^2 = 4$ pairs of risky assets anymore, but we deal with n^2 pairs of them. Regarding the budget sets of a given decision-maker, there exist n^2 budget lines. In particular, the slope of the budget line is always equal to -1 whenever the two risky assets taken into account are the same. In these cases, the budget sets of a given decision-maker always consist of points whose number is infinite. Nevertheless, the joint masses of $_1X\,{}_1X$, $_2X\,{}_2X, \ldots, {}_nX\,{}_nX$ must be estimated in such a way that all off-diagonal joint masses of each two-way table with the same number of rows and columns coincide with zero. An interesting study for bear markets is made by Scholz (2007). In this section, the Sharpe ratio is obtained using a multilinear approach (other return-risk ratios are dealt with by Cheridito and Kromer (2013)). Let r_f be the risk-free asset paying a fixed rate of return. The Sharpe ratio is accordingly given by

$$\mathrm{SR} = \frac{\mathbf{P}(X_{12\,\dots\,n}) - r_f}{\sqrt{\mathrm{Var}(X_{12\,\dots\,n})}}, \tag{81}$$

where $\mathbf{P}(X_{12\,\dots\,n})$ and $\sqrt{\mathrm{Var}(X_{12\,\dots\,n})}$ are two determinants of two square matrices of order n connected with two tensors of the same order (Angelini and Maturo 2022b). It measures how risk and return can be traded off in making portfolio choices. Such choices are studied inside the budget set of the decision-maker (Dowd 2000). The marginal rate of substitution between risk and return is given by (81). The slope of the budget line measuring the cost of achieving a larger expected return on $X_{12\,\dots\,n}$ in terms of the increased standard deviation of the return is given by (81), where we assume $\mathbf{P}(X_{12\,\dots\,n}) > r_f$ (a specific model about uncertainty is studied by Pham et al. (2022)). Please note that $\mathrm{Var}(X_{12\dots n})$ is obtained through the notion of mean quadratic difference. In this section, an extension of the mean-variance model is computationally shown. Moreover, since the beta of a given stock i can statistically be defined by considering the covariance of the return on the stock with the market return divided by the variance of the market return, and specifically it is then possible to write

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_m)}{\mathrm{Var}(r_m)}, \tag{82}$$

what is said in this section can operationally be associated with the Capital Asset Pricing Model, which has many uses in the study of financial markets. The expected market return r_m can accordingly be expressed using a measure with the same structure as (79).

8. Conclusions, Discussion, and Future Perspectives

This paper answers different questions. Two of them are essential. First, the number of points of the budget set of the decision-maker is infinite because all admissible (rational) choices at the first stage derive from masses that are subjectively established. In the second stage, the object of decision-maker choice depends on further hypotheses of an empirical nature, but the distribution of masses identifying this object of decision-maker choice is always characterized by subjective and objective elements. Each point of the budget set of the decision-maker is a metric measure. Every measure is obtained after summarizing a nonparametric joint distribution of mass. Different distributions of mass are different measures. Nevertheless, when talking in terms of measure one does not make of it something fixed, with a special status. A given decision-maker accordingly focuses on masses because there is always the physical perception of being able to move them in whatever way he or she likes. In our approach, a mechanical transposition of all the notions, procedures, and results of measure theory into the calculus of probability does not happen. Every measure is not directly visible inside the budget set of the decision-maker because it is a real number. It appears as a two-dimensional point. Second, the role played by objective alternatives is fundamental. Structures open to the adjunction of new entities as new circumstances arise are studied. They are linear spaces over $\mathbb{R}$. Their dimensions are different. We can know $\mathbf{P}(X_{12})$ and $\mathrm{Var}(X_{12})$ using a multilinear and quadratic metric, where X_{12} is a two-risky asset portfolio. We can also know $\mathbf{P}(X_{12\dots n})$ and $\mathrm{Var}(X_{12\dots n})$, where $X_{12\dots n}$ is an n-risky asset portfolio. Since we use a quadratic metric, we always consider two random goods at a time. We never consider more than two goods at a time. The notion of ordinal utility is a metric measure as well. In this paper, a more general approach to the riskiness of random goods is proposed. We use the notion of mean quadratic difference put forward by Corrado Gini. We develop it using a tensorial approach. If the decision-maker uses mean quadratic differences, then he or she expresses, from time to time, the knowledge hypothesis underlying the variability of his or her choices. It is possible to understand that the notion of mean quadratic difference is also connected with the Bravais–Pearson correlation coefficient. Regarding random goods, this coefficient is intrinsically referred to a double random good denoted by X_{12}. If ${}_{(1)}\mathbf{d}$ and ${}_{(2)}\mathbf{d}$ are α-orthogonal vectors, then we obtain

$$\|_{12}\hat{d}\|_{\alpha}^{2} = \begin{vmatrix} \|_{(1)}\mathbf{d}\|_{\alpha}^{2} & 0 \\ 0 & \|_{(2)}\mathbf{d}\|_{\alpha}^{2} \end{vmatrix} = \|_{(1)}\mathbf{d}\|_{\alpha}^{2}\,\|_{(2)}\mathbf{d}\|_{\alpha}^{2}. \tag{83}$$

Since it is possible to write

$$-1 \leq \left(1 - \frac{\|_{12}d\|_{\alpha}^{2}}{\|_{12}\hat{d}\|_{\alpha}^{2}}\right)^{1/2} \leq +1, \tag{84}$$

the above expression within the parentheses coincides with the Bravais–Pearson correlation coefficient referred to X_{12}, where $\|_{12}d\|_{\alpha}^{2}$ and $\|_{12}\hat{d}\|_{\alpha}^{2}$ are two aggregate measures obtained using a multilinear and quadratic metric. In this paper, the origin of the variability of a nonparametric distribution of mass depends on the variable state of information and knowledge associated with a given decision-maker. It is susceptible to being continuously enriched by the flow of new pieces of information. It can also be enriched by the results that are gradually learned or observed in relation to more or less analogous situations and cases. For this reason, the riskiness of a two-risky asset portfolio is studied using the notion of α-norm of an antisymmetric tensor of order 2.

What is said in this paper can be extended. This is because $m+1$ possible values for a risky asset have an objective nature in the same way as $m+1$ sampling units that are observed regarding a specific population. Multilinear relationships between variables with parametric probability distributions such as normal distributions can be dealt with using measures of a multilinear nature. A multilinear regression model based on this multilinear approach has been made by us. The paper containing this model is currently under review by an international journal.

Given $m+1$ possible values for a risky asset, they identify a vector belonging to E^{m+1}. Two linearly independent vectors of E^{m+1} generate a linear subspace of E^{m+1}. Its dimension is equal to 2. The Grassmann coordinates of this linear subspace over $\mathbb{R}$ are the components of a tensor of order 2. Two linearly independent vectors of E^{m+1} are transferred on two mutually orthogonal one-dimensional straight lines, on which an origin, the same unit of length, and an orientation are established. It is possible to show that at least mean quadratic differences, the correlation coefficient, Jensen's inequality, revealed preference theory viewed to be as a branch of the theory of decision-making, the least-squares model, and principal component analysis can be based on intrinsic conditions of uncertainty characterized by objective and subjective elements that are studied inside subsets of linear spaces over $\mathbb{R}$ provided with a specific dimension.

It is possible to overcome the limits of the current research by focusing one's attention on a stochastic view of bound choices. Such a view can be based on subjective opinions or attitudes of a given person. The subjective opinion, meant as something known by the decision-maker taken into account, is something objective in the sense that can be a reasonable object of a rigorous study. Even when one point of a specific convex set is chosen, there is no reason that would lead a given person to consider correct from a philosophical point of view this one, or that one, among the infinitely many possible opinions about the evaluations of probability. Thus, whenever a given decision-maker is indifferent to the exchange of ${}_1X$ for $\mathbf{P}({}_1X)$, a finite number of deviations or errors which are normally distributed can be determined. Whenever he or she is indifferent to the exchange of ${}_2X$ for $\mathbf{P}({}_2X)$, a finite number of deviations or errors which are normally distributed can be determined. Finally, since he or she is also indifferent to the exchange of X_{12} for $\mathbf{P}(X_{12})$, a finite number of deviations or errors can be dealt with in an aggregate fashion.

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Notes

1 In economics, normal and ordinary goods are nonrandom goods. What is demanded for them does not depend on a usual random process. Only a degenerate random process implicitly appears. Only a degenerate probability distribution is implicitly handled. We do not deal with a prevision, but we deal with a prediction. In other words, given a finite number of possible alternatives, a prediction always reduces to the choice of a point in the set of possible alternatives, and not the barycenter of masses distributed over this set. To choose the barycenter of masses distributed over this set is that which characterizes a prevision. In our opinion, it is necessary to make explicit the latter process with respect to choices being made under claimed conditions of certainty.

2 Reductions of dimension are considered in this paper. Hence, we pass from m to 1. Accordingly, we pass from m^2 to 2. Regarding reductions of dimension, a theorem has elsewhere been proved by us. The paper containing this theorem is currently under review by an international journal.

3 Given $(x_1,\, x_2)$, we first handle a closed neighborhood of x_1 denoted by $[x_1 - \epsilon\,;\, x_1 + \epsilon']$ on the horizontal axis, as well as a closed neighborhood of x_2 denoted by $[x_2 - \epsilon\,;\, x_2 + \epsilon']$ on the vertical one, where both ϵ and ϵ' are two small positive quantities. Since the state of information and knowledge associated with a given decision-maker is assumed to be incomplete at the time of choice, m possible quantities which can be demanded for good 1 belong to $[x_1 - \epsilon\,;\, x_1 + \epsilon']$ and m possible quantities which can be demanded for good 2 belong to $[x_2 - \epsilon\,;\, x_2 + \epsilon']$. These quantities belong to two one-dimensional convex sets. One of m possible alternatives does not need to coincide with x_1. The same is true regarding x_2. It follows that m^2 possible quantities which can be demanded for good 1 and good 2 are handled. After determining $\{x_1^1, \ldots, x_1^m\}$, $\{x_2^1, \ldots, x_2^m\}$, and $\{x_1^1, \ldots, x_1^m\} \times \{x_2^1, \ldots, x_2^m\}$, two nonparametric marginal distributions of mass together with a nonparametric joint distribution of mass are estimated in such a way that $(x_1,\, x_2)$ is their chosen summary. m possible quantities which can be demanded for good 1 are found between zero and the horizontal intercept of the budget line, whereas m possible quantities which can be demanded for good 2 are found between zero and the vertical intercept of it.

4 This element is not directly visible because it is a real number. It appears as a two-dimensional point belonging to the two-dimensional convex set. The latter is the budget set of the decision-maker. The budget set of the decision-maker is, therefore, a right triangle belonging to the first quadrant of a two-dimensional Cartesian coordinate system, where the vertex of the right angle of the triangle taken into account coincides with the point given by $(0, 0)$.

5 We do not use the term "random variable", but we use the term "random quantity" because to say random variable might suggest that we are thinking of the statistical interpretation of repeated events, where many trials in which the random quantity under consideration can vary are involved. The random quantity taken into account could assume different values from trial to trial according to the statistical interpretation of repeated events, but this interpretation is contrary to our way of understanding the problem. We do not use the word event in a generic sense. In this paper, an event is always a single event. The sense of it is not generic, but it is specific. A nonparametric distribution of probability characterizing a random quantity can vary from individual to individual. It can also vary with the state of information and knowledge associated with a given individual.

6 Since a larger space containing points that are already known to be impossible is always considered by us within this context, if a set is empty, then it is empty of possible points.

7 A unique symbol $\mathbf{P}$ denotes both probability and prevision, thus avoiding duplication. This is because we use the indicator of an event E expressed by $|E|$. The indicator of E is a random quantity I_E taking values 1 or 0 whenever uncertainty ceases. The mathematical expectation or prevision of the indicator of an event E is denoted by $\mathbf{M}(I_E)$. Since the mathematical expectation of the indicator of an event E is equal to the probability of the same event, we write $\mathbf{M}(I_E) = \mathbf{P}(E)$. If we write $\mathbf{M}(I_E) = \mathbf{P}(E)$, then we must observe $\mathbf{P}(E) = \mathbf{P}(E)$. It follows that a unique symbol $\mathbf{P}$ can be used.

8 If $\mathbf{x}$ is a vector belonging to E^m, then all collinear vectors regarding $\mathbf{x}$ are expressed by $\lambda\,\mathbf{x}$, $\forall \lambda \in \mathbb{R}$.

9 The prevision bundle $(\mathbf{P}({}_1X),\, \mathbf{P}({}_2X))$ is nothing but the object of decision-maker choice under conditions of uncertainty and riskiness.

10 In our approach, to consider larger spaces containing, in addition, impossible points in the light of more recent information and knowledge is never wrong. With respect to $[(m+1)\cdot(m+1)]$ points dealt with by the function denoted by ${}_1X\,{}_2X$, only $m^2 + 2$ points of them are really uncertain. Thus, there are points in which the evaluation of the probability is predetermined, rather than permitting the subjective choice of any value in the interval from 0 to 1, endpoints included.

11 Given the masses of all possible values which are finite in number, their barycenter is a function of them. With regard to a double risky asset, we are not interested in establishing its exact distribution, but we are interested in knowing its barycenter. Whenever

an aggregate choice is studied, the notion of the barycenter of masses is extended together with its properties which are stable equilibrium and minimum of the moment of inertia. The same is true regarding a multiple risky asset of order greater than 2.

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