Aeroelastic Dynamic Feedback Control of a Volterra Airfoil

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1 Introduction

Dynamic behavior analysis of the nonlinear aeroelastic system is one of the interesting topics among researchers that have been studied in recent years. Nonlinear airfoil instability and behavior analysis in subsonic flow are one of the main parts in this field. Aeroelastic systems are characterized by complex nonlinear phenomena due to structural oscillations coupled with the fluid dynamic. The coexistence of phenomena, such as limit cycle oscillation and chaotic vibrations induced by the fluid, can lead the dynamic systems to instability such as flutter which can decrease the system performance, as well as the damage of the structure itself.

Historically, the main approach to analyze the dynamic instability of nonlinear aeroelastic systems has been developed by Theodorsen [1] in the frequency domain. His theory aimed to model the aerodynamic loads on an airfoil when the wake releasing is considered as a memory effect on the global fluid-structure interaction dynamic. Wagner proposed a time-domain analysis where the memory effects are represented by convolution Volterra integrals [2]. Both these traditional models are linear, but, in many cases, the dynamic equations of an airfoil became nonlinear due to the presence of nonlinear elements (dampers, stiffness) or for the instabilities generated from the fluid-structure interaction. The nonlinear aerodynamic model in the time domain can be solved by numerical techniques or analytical methods. In the first case, the solutions such as the finite difference method, Runge–Kutta,

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and cyclic method were employed to solve differential equations of a nonlinear system. Moreover, analytical and semi-analytical solutions including describing function technique were used to analyze the instability of control surfaces of a wing with nonlinear stiffness [3]. In the last decade, different methods have been used to investigate nonlinear dynamic systems [4, 5], some, based on perturbation method and stochastic approach [6], have been developed to analyze nonlinear aeroelastic systems such as perturbation incremental or transformation point method and homotropy method [7, 8].

In the literature, several authors have provided extensive reviews about nonlinear control methods for the minimization of oscillations of elastic wings and aircraft; moreover, in recent years, many control strategies for flutter avoidance have been developed. Partial feedback linearization methodology was also applied to the design of nonlinear controllers for the nonlinear aeroelastic system [9]. The state-dependent Riccati equation (SDRE) method was developed for nonlinear control problems and used to design suboptimal control laws of nonlinear aeroelastic systems considering both quasi-steady [10] and unsteady aerodynamics. Recently, an output feedback and an adaptive decoupled fuzzy sliding-mode control laws have been implemented for suppressing flutter and reducing the vibrational level in the subcritical flight speed range [11]. Moreover, based on the tensor-product model transformation and the parallel distributed compensation, a control law for the prototypical aeroelastic wing section was designed and presented in [12].

This work aims to develop a novel variational optimal control strategy to control the aerodynamic behavior of an aerofoil which presents a memory effect from wake production. Integral memory terms, representing, in this case, the release of wake, are normally not included in variational control algorithms. From the control point of view, the Volterra models are solved through direct methods, discretizing the equations, and then the optimal problem is solved through nonlinear programming [11]. The proposed optimal control, called Proportional-Nth-order-Integral control, PI(N), fills this gap and it belongs to the category of Variational Feedback Controls (VFCs) [13–18]. The solution of the optimal problem is provided through a particular solution of Riccati's equation including the memory terms generated by the past system evolution [19, 20]. The structure of the control law shows how the optimal solution is related to the kernel function order, that is, of the Volterra integral typology.

Finally, the analytical solution PI(N) is tested on a prototypical wing and the numerical results show how is possible to reach the best performance of the proposed controller in comparison with the classical Linear Quadratic Regulator (LQR) method.

2 Mathematical Model of Wagner's Controlled Wing

Theodorsen's theory is widely used to achieve the mathematical model of the aerodynamic problem [1]. It provides the generalized unsteady aerodynamic forces

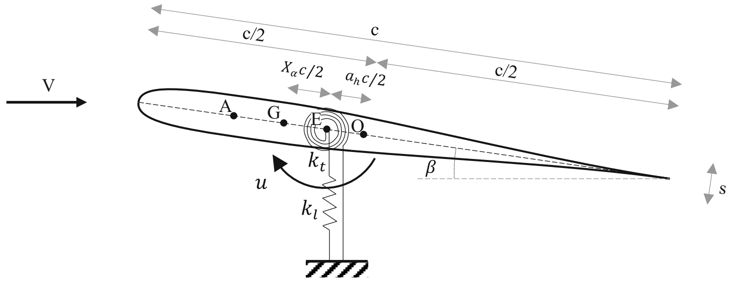

Fig. 1 Theodorsen's airfoil section geometry

due to an arbitrary motion of the airfoil, which generates a vortex wake. The time-domain counterpart of the Theodorsen's theory, formulated by Wagner [2], is considered in this work to formulate an optimal control law which includes Volterra's memory effects.

The engineering control application of a two-degree-of-freedom airfoil is here investigated, where the pitch degree of freedom is actively controlled.

The sketch in Fig. 1 shows the geometry of the typical Theodorsen airfoil section where the x-axis is chord-wise axis (positive towards the trailing edge) and E is the center of rotation. The airfoil is a simple two degree-of-freedom system, elastically constrained by a pair of springs, k_l and k_t, oscillating in plunge and pitch. The actuation force, u, is meant here as dimensionless torque applied on the pitch rotation degree-of-freedom β. By using standard notations, the nondimensional plunge deflection at the elastic center is denoted by w,while β represents the pitch motion. The elastic center, E, is located at a distance $OE = a_h c/2$ from the mid-chord (a_h is the dimensionless distance, considered with respect to the half-cord length, $c/2$, between the center of the foil O, and the elastic axis), while the mass center, G, is located at a distance $EG = x_\alpha c/2$. With these assumptions, the aeroelastic equations of the typical wing are:

$$\begin{cases} \ddot{w} + x_\alpha \ddot{\beta} + \Omega^2 w = -p\,(w, \beta) \\ \frac{x_\alpha}{r_\alpha^2} \ddot{w} + \ddot{\beta} + \beta = r\,(w, \beta) + u \end{cases} \tag{1}$$

where the overdot denotes differentiation with respect to a dimensionless time $\tau = \omega_\alpha t$; $\Omega = \omega_w / \omega_\alpha$ is the heave stiffness, being ω_w and ω_α the uncoupled natural frequencies of heave and pitch modes; $r_\alpha = \sqrt{4 I_\beta / m c^2}$is the dimensionless radius of gyration about the elastic axis and m, I_β, c are the mass, the moment of inertia per unit length with respect to the elastic center, and the wing chord. For an incompressible two-dimensional flow, Wagner defines the aerodynamic loads as the sum of two contributions [2]: (i) a linear composition of degree-of-freedom which represents the added mass, damping, and stiffness due to the fluid-structure interaction and (ii) a convolution term including the memory effects defined as follows:

$$
\begin{aligned}
p(w,\beta) &= \frac{1}{\mu}\left(\ddot{w} - a_h\ddot{\beta} + U\dot{\beta}\right) + \frac{2U}{\mu}\int_{-\infty}^{t} K_W(t-\tau)\,\dot{\tilde{w}}_{3/4}(\tau)\,d\tau \\
r(w,\beta) &= \frac{1}{\mu r_\alpha^2}\left[a_h\left(\ddot{w} - a_h\ddot{\beta}\right) - \frac{1}{2}U(1-a_h)\dot{\beta} - \frac{1}{8}\ddot{\beta}\right] \\
&\quad + \frac{U(1+2a_h)}{\mu r_\alpha^2}\int_{-\infty}^{t} K_W(t-\tau)\,\dot{\tilde{w}}_{3/4}(\tau)\,d\tau
\end{aligned}
\tag{2}
$$

where p and r are the lift and pitching moment, respectively, $\mu = \pi\rho c^2/4m$ is the mass ratio, and $U = 2V/c\omega_\alpha$ is the dimensionless inflow velocity and V is the inflow velocity, oriented along the x-axis. The time-dependent known function $\tilde{w}_{3/4}(\tau) = \dot{w}(\tau) - \left(\frac{1}{2} - a_h\right)\dot{\beta}(\tau) + U\beta(\tau)$ and $K_W(t-\tau) = \sum_{k=1}^{N}\alpha_k e^{-\beta_k(t-\tau)}$ are the downwash and the Wagner function, respectively. Using a standard notation in control theory, by defining a new state vector $\boldsymbol{x} = \left[w, \beta, \dot{w}, \dot{\beta}\right]^T$, the system can be arranged in its matrix notation:

$$
\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \mathbf{K}(t) * \boldsymbol{x} + \boldsymbol{B}u \tag{3}
$$

where the following definitions are used:

$$
\begin{gathered}
\boldsymbol{A} = \begin{bmatrix} 0_{2\times2} & \boldsymbol{I}_{2\times2} \\ -\boldsymbol{M}^{-1}\boldsymbol{\Omega} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix};\ \boldsymbol{M} = \begin{bmatrix} 1+\frac{1}{\mu} & x_\alpha - \frac{a_h}{\mu} \\ \frac{x_\alpha}{r_\alpha^2} - \frac{a_h}{\mu r_\alpha^2} & 1 + \frac{a_h^2}{\mu r_\alpha^2} + \frac{1}{8\mu r_\alpha^2} \end{bmatrix};\ \boldsymbol{C} = \begin{bmatrix} 0 & \frac{U}{\mu} \\ 0 & \frac{U(1-a_h)}{2\mu r_\alpha^2} \end{bmatrix} \\
\mathbf{K}(t) = \begin{bmatrix} 0_{2\times2} & 0_{2\times2} \\ -\boldsymbol{M}^{-1}\boldsymbol{\Phi}(t) & 0_{2\times2} \end{bmatrix};\quad \boldsymbol{B} = \begin{bmatrix} 0_{2\times1} \\ 8\mu r_\alpha^2(a_h - \mu x_\alpha)/\epsilon \\ 8\mu r_\alpha^2(\mu+1)/\epsilon \end{bmatrix} \\
\boldsymbol{\Omega} = \begin{bmatrix} \Omega^2 & 0 \\ 0 & 1 \end{bmatrix};\ \boldsymbol{\Phi}(t) = \begin{bmatrix} \frac{2U}{\mu}\ddot{K}_w & \frac{2U}{\mu}\left[\left(a_h - \frac{1}{2}\right)\ddot{K}_w + U\dot{K}_w\right] \\ -\frac{(1+2a_h)U}{\mu r_\alpha^2}\ddot{K}_w & -\frac{(1+2a_h)U}{\mu r_\alpha^2}\left[\left(a_h - \frac{1}{2}\right)\ddot{K}_w + U\dot{K}_w\right] \end{bmatrix} \\
\epsilon = 8a_h^2\mu + 16a_h\mu x_\alpha + 8r_\alpha^2\mu(\mu+1) - 8\mu^2x_\alpha^2 + \mu + 1
\end{gathered}
\tag{4}
$$

and $\boldsymbol{I}$ is indicating the identity matrix. The term expressed by the operator $*$ indicates the convolution integral or Volterra integral between the element i-row j-column of $\mathbf{K}$ matrix and the state $\boldsymbol{x}$.

$$
K_{ij} * x_i = \int_0^{\mathrm{t}} K_{ij}(t-\tau)\,x_i(\tau)\,d\tau \tag{5}
$$

This work aims to find an optimal control that can minimize a given objective function J under the constraint hypotheses of the differential system (3). Usually, the Pontryagin problem is not easily solved except through numerical approaches based on the discretization of the equations making use of direct control methods such as single-multiple shooting or collocation methods. In this case, the open-loop control solution is founded by the solution of a nonlinear programming system, which requires high computational efforts and therefore cannot be used for real-time applications. Here, the authors propose an indirect and analytical solution of the Pontryagin problem for Volterra equations, which can be used in feedback, making the algorithm suitable for real-time applications. The objective function is defined as the classical quadratic form in terms of both the state $\boldsymbol{x}$ and the control u, and the constraint expressed by the integral differential equation of first species Volterra:

$$\min J = \min \left\{ \int_0^T \tfrac{1}{2}\boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} + \tfrac{1}{2}Ru^2 + \boldsymbol{\lambda}^T \left(\dot{\boldsymbol{x}} - \boldsymbol{A}\boldsymbol{x} - \boldsymbol{K} * \boldsymbol{x} - \boldsymbol{B}u\right) dt \right\} \quad i.c.\ \ \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{6}$$

Equation (6) represents a typical optimal problem in which the integral differential constraint is considered using the Lagrange multiplier $\boldsymbol{\lambda}$. The matrix $\boldsymbol{Q}$ and the coefficient R are generally gains of the objective function or also called penalty parameters suitably selected to best tune the optimal control solution.

3 Theory of the Proportional Integrative N-Order PI(N) Control

The proposed optimal control theory of the Volterra Eq. (3) is here presented in a scalar formulation without loss of generality for more clarification. Therefore, starting with the minimizing of the following cost function:

$$\min J = \min \left\{ \int_0^T \frac{1}{2}qx + \frac{1}{2}ru^2 + \lambda \left(\dot{x} - ax - k * x - bu\right) dt \right\} \tag{7}$$

and doing the δ variations and imposing $\delta J = 0$, a set of differential equations in x and λ is found (see Ref. [16] for more details):

$$\begin{cases} \dot{x} = ax + bu + \int_0^\infty k\left(t - \tau\right) x\left(\tau\right) d\tau \\ \dot{\lambda} = qx - a\lambda - \int_0^\infty k\left(\tau - t\right) \lambda\left(\tau\right) d\tau \\ \lambda(T) = 0, \quad u = \frac{b}{r}\lambda \end{cases} \tag{8}$$

The first integral of (8) is the convolutional term $k * x$; instead, the second integral is associated to the δ-variation of $k * x$ in the x variable that is a nontrivial problem. Both can be derived under the kernel causality proprieties $k(t - \tau) = 0$ for $t < 0$ and $\tau > t$ (see Ref. [16] for more details). The transversality condition of Eq. (8), $\lambda(T) = 0$, makes the problem difficult to solve, and only an open loop control solution is naturally stated, precluding the chance of a direct feedback control. As it happens for physical systems described by differential equations, the infinite time-horizon presents the chance of a direct feedback control passing through some assumptions. The proposed method is based on the use of a specific exponential kernel function, which may well represent most convolutional memory phenomena or hysteresis models:

$$k(t) = \sum_{k=1}^{N} \alpha_k e^{-\beta_k t} \tag{9}$$

where α_k and β_k are general coefficients of the exponential series. Now, eliminating u through the third of Eq. (8) and considering the Eq. (9), one can easily obtain

$$\begin{cases} \dot{x} = ax + \lambda \frac{b^2}{r} + \int_0^\infty \sum_{k=1}^{N} \alpha_k e^{-\beta_k t} x(\tau)\, d\tau \\ \dot{\lambda} = qx - a\lambda - \int_0^\infty \sum_{k=1}^{N} \alpha_k e^{-\beta_k t} \lambda(\tau)\, d\tau \end{cases} \tag{10}$$

Thanks to the special form of k, the two integral terms of (10) can be easily Laplace-transformed $\mathcal{L}\{\}$ with variable s:

$$\begin{aligned} (sX(s) - x_0) = {} & (a - d_{N-1})\, X(s) \\ & + \sum_{j=1}^{N} \Big(ad_{N-j} X(s) + p_{N-j} X(s) + \frac{b^2}{r} d_{N-j} \Lambda(s) + x_0 d_{N-j} \\ & - d_{N-1-j} X(s) \Big) s^{-j} + \frac{b^2}{r} \Lambda(s) \\ (s\Lambda(s) - \lambda_0) = {} & -\left(a + \tilde{d}_{N-1}\right) \Lambda(s) \\ & + \sum_{j=1}^{N} \Big(q\tilde{d}_{N-j} X(s) + \tilde{p}_{N-j} \Lambda(s) - a\tilde{d}_{N-j} \Lambda(s) + \lambda_0 \tilde{d}_{N-j} \\ & - \tilde{d}_{N-1-j} \Lambda(s) \Big) s^{-j} + qX(s) \end{aligned} \tag{11}$$

where d_j, $\tilde{d}_j$, p_j, $\tilde{p}_j$ are general coefficients. Now Laplace antitransforming $\mathcal{L}^{-1}\{\}$ and reducing the integral-differential equation to a set of first order, the Eq. (11) can be reduced to an LTI system with the state vector $\boldsymbol{q} = [\boldsymbol{\xi}, \boldsymbol{\eta}]^T$, $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_{N+1}]^T$, $\boldsymbol{\eta} = [\eta_1, \ldots, \eta_{N+1}]^T$ where $\xi_1 = \mathcal{L}^{-1}\{X(s)\}$ and $\eta_1 = \mathcal{L}^{-1}\{\Lambda(s)\}$; the following variables ξ_k, η_k present integrals of state and lambda up to the k-order and matrix $\boldsymbol{H} = [\boldsymbol{H}_{\xi\xi} \boldsymbol{H}_{\xi\eta}; \boldsymbol{H}_{\eta\xi} \boldsymbol{H}_{\eta\eta}] \in \mathbb{R}^{(2N+2) \times (2N+2)}$:

$$\dot{\boldsymbol{q}} = \boldsymbol{H}\boldsymbol{q} \tag{12}$$

Its solution can be expressed in function of its $2N + 2$ eigenvectors $\boldsymbol{\psi}_k$ $\boldsymbol{\theta}_k$ and eigenvalues $\boldsymbol{p} = [p_1, \ldots p_k, \ldots, p_{2N+2}]$. Arranging the eigenvalues p_k with $\mathcal{R}$-positive and Y-negative real part, Eq. (12) can be written as follows:

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix} = \sum_{k=1}^{Y} c_k^{\{-\}} \begin{bmatrix} \boldsymbol{\psi}_k^{\{-\}} \\ \boldsymbol{\theta}_k^{\{-\}} \end{bmatrix} e^{p_k^{\{-\}}t} + \sum_{k=1}^{\mathcal{R}} c_k^{\{+\}} \begin{bmatrix} \boldsymbol{\psi}_k^{\{+\}} \\ \boldsymbol{\theta}_k^{\{+\}} \end{bmatrix} e^{p_k^{\{+\}}t} \tag{13}$$

where $Y + \mathcal{R} = 2N + 2$ and the superscript $\{+\}$ identify the set of values $\boldsymbol{\psi}_k$, $\boldsymbol{\theta}_k$, c_k referred to the positive real parts $\mathcal{R}e\{p_k\} \geq 0$ and vice versa the superscript $\{-\}$ is referred to the set of $\mathcal{R}e\{p_k\} < 0$. The c_k are the unknown coefficients found by imposing the boundary conditions. The system (13) can be solved assuming $\mathcal{R} \geq Y$ or the number of eigenvalues p_k with negative real part are more than the others. The solution can be found by imposing$c_k^{\{+\}} = 0$ to satisfy the boundary condition $\lim_{T\to\infty} \lambda(T) = 0$ and its derivatives, that is, $\lim_{T\to\infty \boldsymbol{\eta}} (T) = 0$. By selecting $N + 1$ equations with $p_k^{\{-\}}$, from Eq. (13), one obtains:

$$\boldsymbol{\eta} = \left[\boldsymbol{\theta}_1^{\{-\}}, \ldots, \boldsymbol{\theta}_{N+1}^{\{-\}}\right] \left[\boldsymbol{\psi}_1^{\{-\}}, \ldots, \boldsymbol{\psi}_{N+1}^{\{-\}}\right]^{-1} \boldsymbol{\xi} = \boldsymbol{P}\boldsymbol{\xi} \tag{14}$$

Finally, from Eq. (8), since $u = \frac{b}{r}\lambda$, extracting from $\boldsymbol{\eta}$ the last subvector $\eta_{N+1} = \lambda$**,** the optimal feedback control variable u can be found as follows:

$$\begin{aligned} u = & - P_{1,1}x + \sum\nolimits_{i=1}^{N} P_{1,i+1} \underbrace{\int \cdots \int}_{i} \left(\hat{d}_i x + \hat{p}_i u\right) \underbrace{dt \ldots dt}_{i} \\ & + \sum\nolimits_{i=1}^{N} x_0 d_{N-i} \frac{t^{i-1}}{(j-1)!} \end{aligned} \tag{15}$$

with $\hat{d}_i$, $\hat{p}_i$, $P_{1,j}$ are general coefficients. The structure of Eq. (15) shows the explicit optimal control solution of Volterra's differential equations here called PI(N). The form of u is mainly related to the structure of the kernel $k(t)$, because it presents a combination of state integral of order equal to the number of the k exponential terms. The explicit control solution has been obtained not only by satisfying the transversality condition stated in (8), but also imposing the same condition for its higher derivatives $\lim_{T\to\infty} \frac{d^i \boldsymbol{\lambda}(T)}{dt^i} = 0$. Moreover, the control solution is also strictly related to the state initial condition x_0.

4 Results and Discussion

In this section, the numerical results are discussed. The proposed PI(N) controller is here developed to control the airfoil motion of Eq. (3) by minimizing the release of vortex wake. The novel algorithm is compared with the benchmarking LQR method, in terms of cost function 1/2($\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + Ru^2$). The benchmarking LQR method has been developed for the model in analysis by disregarding convolutive memory effects that cannot be included in the method $\mathbf{K}(t) * \boldsymbol{x} = \mathbf{0}$. Moreover, both controllers, LQR and PI(N), have been applied to the real Volterra wing model. Table 1 below shows the geometric and dynamic adimensional parameters considered as input for the numerical simulations.

Results in Fig. 2 show the behavior of heave (a) and pitch wing (b) motion, when the PIN and LQR controllers are, respectively, acting on the system. For both degree-of-freedom the novel PIN algorithm presents a faster attitude to reach the rest condition, that is, the minimization of memory wake effect, compared to the LQR, which is around 20% of the maximum reached value of the pitch degree-of-freedom. Particularly, the LQR solution presents discontinuity in the heave evolution possibly caused by the presence of memory effects generated by the kernel function. Indeed these effects, which take place during the first seconds of the simulation, are not taken into account in the LQR algorithm.

Also, the control in Fig. 3a underlying a lower effort for the PI(N) controller and in Fig. 3b the cost functions for both methods are compared. The PIN solution presents a lower value of the cost function J than the LQR, confirming a better behavior of the proposed control both in terms of minimization of memory wake effects and cost function itself. This result is due to the fact that the LQR method is not taking into account the memory effects differently from PIN whose behavior is favored by this feature.

Table 1 Dimensionless parameters (see [21] for more details)

Description	Parameters	Value
Dimensionless distance OE	a_h	0.2
Dimensionless mass	μ	26.18
Dimensionless radius of gyration	r_α	4.8
Dimensionless distance GE	x_α	0.2
Frequency	Ω	0.55
Dimensionless inflow velocity	U	2
Kernel function	K_W	$1 - 0.165e^{-0.0009Ut} - 0.335e^{-0.006Ut}$
Control gain	**Q**; R	*diag*(5*e*4; 250; 500; 40); 0.2
Initial displacement	$w(0), \beta(0), \dot{w}(0), \dot{\beta}(0)$	[10, 0, 0, 0]

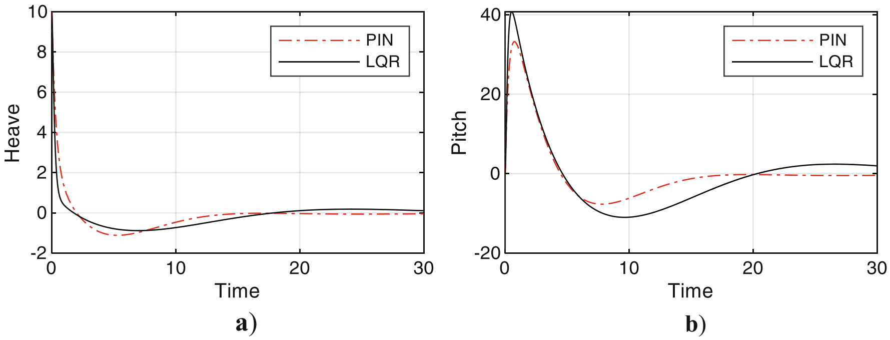

Fig. 2 Heave (**a**) and pitch (**b**) motion: PIN vs LQR

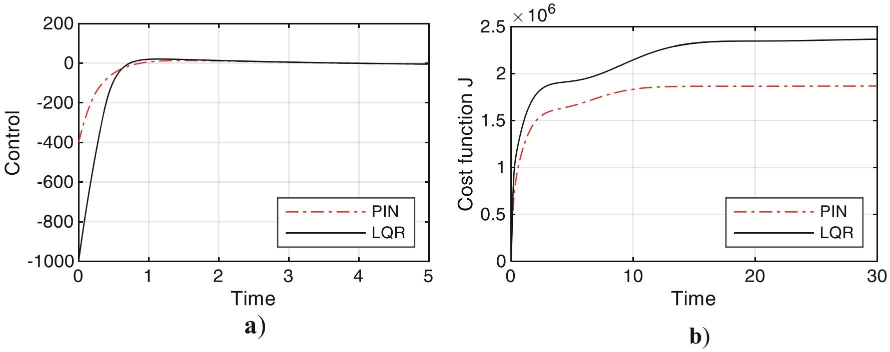

Fig. 3 Control law (**a**) and cost function (**b**) PI(N) vs LQR

5 Conclusions

This work aims to develop a novel optimal control strategy for Volterra models, based on the variational optimal control theory, which normally is applied only to differential equations. In this paper, the proposed algorithm is applied to the control of the aerodynamic behavior of an airfoil which presents a memory effect from wake production. The optimal control, called Proportional-Nth-order-Integral control, PI(N), is here proposed as the indirect solution of the Pontryagin theory applied to the Volterra equation of motions. The analytical control solution has an integral form of order equal to the order of the kernel function series expansion for modelling the wake vortex. Numerical results show the better performances of the proposed PI(N) controller compared with the classic LQR method in terms of reaching rest conditions and minimizing the cost function value.

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