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On the generalized β-absolute convergence of single and multiple Fourier series

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Abstract. In this paper, we provide sufficient conditions for the generalized β-absolute convergence of multiple Fourier series of a function f of p-$(\Lambda^1, ..., \Lambda^N)$-bounded variation.

1. Introduction

Concerning the absolute convergence of Fourier series, the theorem of Bernstein [2, Vol. II, Theorem 2 of Bernstein, p. 154], the theorem of Szász [2, Vol. II, p. 155], and the theorem of Zygmund [2, Vol. II, p. 160] are classical. Generalizing these classical results of Bernstein, Szász and Zygmund, Gogoladze and Meskhia [4] obtained sufficient conditions for the generalized β-absolute convergence of single Fourier series. In 2007, Móricz and Veres [6] proved the analogues of theorems of Szász and Zygmund for multiple Fourier series. Móricz and Veres [5] have also generalized their results and given a multidimensional analogue of the results of Gogoladze and Meskhia.

In the present paper, we provide sufficient conditions for the generalized β-absolute convergence of multiple Fourier series of a function f of p-$(\Lambda^1, ..., \Lambda^N)$-bounded variation. Our results generalize the earlier results of Gogoladze and Meskhia [4, Corollary 3, p. 32], of Vyas [10, Theorem 3.1 and Corollary 3.2, p. 233–234] and of Vyas and Patadia [12, Theorem 1, for $n_k = k$, for all k] for single Fourier series, and also of Móricz and Veres [5, Theorem 4 and Corollary 4, p. 153; and their extensions Theorem $4^{'}$ and Corollary $4^{'}$, p. 160] and of Vyas and Darji [11, Theorem 3.3, p. 73 and an extension of Theorem 3.3, p. 80] for multiple Fourier series.

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In the sequel, $\mathbb{T} := [-\pi, \pi)$ is the torus, $\mathbb{L}$ is the class of non-decreasing sequences $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$ of positive numbers such that $\sum_k \frac{1}{\lambda_k}$ diverges, and C is a constant whose value may be different at each occurrence.

2. New results for single Fourier series

Given a sequence $\Lambda = \{\lambda_k\}_{k=1}^{\infty} \in \mathbb{L}$ and $p \geq 1$, a complex valued function f defined on $\overline{\mathbb{T}}$ is said to be of p-Λ-bounded variation (that is, $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}})$), if

$$V_{\Lambda_p}(f, \overline{\mathbb{T}}) := \sup_{\mathcal{I}} \left\{ \left(\sum_j \frac{|f(I_j)|^p}{\lambda_j} \right)^{\frac{1}{p}} \right\} < \infty,$$

where $\mathcal{I}$ is a finite collection of non-overlapping subintervals $\{I_j\} = \{[a_j, b_j]\}$ in $\overline{\mathbb{T}}$ and $f(I_j) = f(b_j) - f(a_j)$.

Note that, for $\Lambda = \{1\}$ (that is, $\lambda_k \equiv 1$, for all k) and $p = 1$ one gets the class $BV(\overline{\mathbb{T}})$; for $p = 1$ one gets the class $\Lambda BV(\overline{\mathbb{T}})$; and for $\Lambda = \{1\}$ one gets the class $BV^{(p)}(\overline{\mathbb{T}})$. If $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}})$, then f is bounded on $\overline{\mathbb{T}}$ [9, Lemma 1, p. 771].

For a 2π-periodic complex valued function $f \in L^1(\mathbb{T})$, its Fourier series is defined as

$$f(x) \sim \sum_{m \in \mathbb{Z}} \hat{f}(m)\, e^{imx}, \; x \in \mathbb{T},$$

where the Fourier coefficients $\hat{f}(m)$ are defined by

$$\hat{f}(m) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x)\, e^{-imx}\, dx, \; m \in \mathbb{Z}.$$

A Fourier series of f is said to be β-absolute convergent if

$$\sum_{m \in \mathbb{Z}} |\hat{f}(m)|^{\beta} < \infty.$$

For $\beta = 1$, one gets the absolute convergence of the Fourier series of f.

The modulus of continuity of a function f is defined as

$$\omega(f; \delta) := \sup \left\{ |f(x+h) - f(x)| : \; x \in \mathbb{T}, \; 0 < h \leq \delta \right\}, \; \delta > 0.$$

Following the definition in [4], a sequence $\gamma = \{\gamma_m : \; m \in \mathbb{N}_+\}$ of non-negative numbers is said to belong to the class $\mathfrak{U}_{\alpha}$ for some $\alpha \geq 1$ if the inequality

$$\left(\sum_{m \in \mathcal{D}_{\mu}} \gamma_m^{\alpha} \right)^{1/\alpha} \leq \eta \; 2^{\mu(1-\alpha)/\alpha} \sum_{m \in \mathcal{D}_{\mu-1}} \gamma_m \tag{1}$$

is satisfied for all $\mu \geq 0$, where

$$\mathcal{D}_{-1} := \mathcal{D}_0 = \{1\}, \; \mathcal{D}_{\mu} := \{2^{\mu-1}+1, 2^{\mu-1}+2, ..., 2^{\mu}\} \text{ for } \mu \geq 0 \tag{2}$$

and the constant η does not depend on μ. Without loss of generality, we assume that $\eta \geq 1$. Note that,

$$\mathfrak{U}_{\alpha_2} \subset \mathfrak{U}_{\alpha_1} \text{ if } 1 \leq \alpha_1 < \alpha_2 < \infty. \tag{3}$$

If a sequence γ is such that

$$\max\{\gamma_m : m \in \mathcal{D}_\mu\} \leq \eta \ \min\{\gamma_m : m \in D_{\mu-1}\}, \ \mu \in \mathbb{N}_+,$$

then $\gamma \in \mathfrak{U}_\alpha$ for every $\alpha \geq 1$. This inequality was introduced by Ul'yanov [7]. For convenience in writing, put $\gamma_{-m} := \ \gamma_m, \ m \in \mathbb{N}_+$.

We prove the following result.

Theorem 1. *If $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}})$ $(p \geq 1)$ and $\gamma = \{\gamma_m\} \in \mathfrak{U}_{2/(2-\beta)}$ for some $\beta \in (0,2)$, then*

$$\sum(\gamma; f)_\beta := \sum_{|m|\geq 1} \gamma_m |\hat{f}(m)|^\beta \leq \eta C \sum_{\mu=0}^{\infty} 2^{-\mu\beta/2} \ \Gamma_{\mu-1} \ \left(\frac{\omega^q(f; \frac{\pi}{2^\mu})}{\sum_{j=1}^{2^\mu} \frac{1}{\lambda_j}} \right)^{\frac{\beta}{p+q}},$$

where η is from (1) *corresponding to $\alpha := 2/(2-\beta)$, $q > 0$, $p + q \geq 2$, and*

$$\Gamma_\mu := \sum_{m \in \mathcal{D}_\mu} \gamma_m \ \textit{for} \ \mu \in \mathbb{N}. \tag{4}$$

Proof. For a given $h > 0$, put

$$\Delta f_j\,(x; h) := f\,(x + jh) - f\,(x + (j-1)h)\,.$$

Then, for each $m \in \mathbb{Z}$,

$$\widehat{\Delta f}_j(m) = 2i\hat{f}(m) \ e^{im(j-\frac{1}{2})h} \ \sin\left(\frac{mh}{2}\right).$$

Since $f \in \Lambda BV^{(p)}(\overline{\mathbb{T}})$, f is bounded on $\overline{\mathbb{T}}$ and hence $f \in L^2(\overline{\mathbb{T}})$. Using Parseval formula, we get

$$\sum_{m\in\mathbb{Z}} \left| \hat{f}(m) \ \sin\left(\frac{mh}{2}\right) \right|^2 = O\left(\int_{\overline{\mathbb{T}}} |\Delta f_j(x; h)|^2 dx \right).$$

Putting $h := \frac{\pi}{2^\mu}$, $\mu \in \mathbb{N}$, and taking into account that

$$\frac{\pi}{4} < \frac{|m|\pi}{2^{\mu+1}} \leq \frac{\pi}{2}, \ |m| \in \mathcal{D}_\mu, \tag{5}$$

it follows that

$$S_\mu := \sum_{|m|\in\mathcal{D}_\mu} |\hat{f}(m)|^2 = O\left(\int_{\overline{\mathbb{T}}} \left| \Delta f_j\left(x; \frac{\pi}{2^\mu}\right) \right|^2 \ dx \right),$$

for all $j = 1, ..., 2^\mu$.

Applying Hölder's inequality on the right side of the above inequality, we have

$$S_\mu = O\left(\left(\int_{\overline{\mathbb{T}}} \left|\Delta f_j\left(x; \frac{\pi}{2^\mu}\right)\right|^{p+q} dx\right)^{\frac{2}{p+q}}\right).$$

Since the left hand side of the above inequality is independent of j, multiplying both sides of it by $\frac{1}{\lambda_j}$, summing over j from 1 to 2^μ, and letting $\Lambda_{2^\mu} := \sum_{j=1}^{2^\mu} \frac{1}{\lambda_j}$, we get

$$S_\mu = O\left(\frac{1}{(\Lambda_{2^\mu})^{\frac{2}{p+q}}}\left(\int_{\overline{\mathbb{T}}} \sum_{j=1}^{2^\mu} \frac{\left|\Delta f_j\left(x; \frac{\pi}{2^\mu}\right)\right|^{p+q}}{\lambda_j} dx\right)^{\frac{2}{p+q}}\right).$$

Since $\left|\Delta f_j\left(x; \frac{\pi}{2^\mu}\right)\right| = O\left(\omega\left(f; \frac{\pi}{2^\mu}\right)\right)$, we have

$$S_\mu = O\left(\left(\frac{\omega^q(f; \frac{\pi}{2^\mu})}{\Lambda_{2^\mu}}\right)^{\frac{2}{p+q}}\left(\int_{\overline{\mathbb{T}}} \sum_{j=1}^{2^\mu} \frac{\left|\Delta f_j\left(x; \frac{\pi}{2^\mu}\right)\right|^{p}}{\lambda_j} dx\right)^{\frac{2}{p+q}}\right),$$

where

$$\sum_{j=1}^{2^\mu} \frac{\left|\Delta f_j\left(x; \frac{\pi}{2^\mu}\right)\right|^{p}}{\lambda_j} = O(1) \text{ as } f \in \Lambda BV^{(p)}(\overline{\mathbb{T}}).$$

Hence,

$$S_\mu = O\left(\left(\frac{\omega^q(f; \frac{\pi}{2^\mu})}{\Lambda_{2^\mu}}\right)^{\frac{2}{p+q}}\right).$$

Since $1 = \frac{\beta}{2} + \frac{2-\beta}{2}$, by Hölder's inequality, we have

$$\begin{aligned} R_\mu := \sum_{|m| \in \mathcal{D}_\mu} \gamma_m |\hat{f}(m)|^\beta &\le \left(\sum_{|m| \in \mathcal{D}_\mu} |\hat{f}(m)|^2\right)^{\beta/2} \left(\sum_{|m| \in \mathcal{D}_\mu} \gamma_m^{2/(2-\beta)}\right)^{(2-\beta)/2} \\ &\le \left(\frac{\omega^q(f; \frac{\pi}{2^\mu})}{\Lambda_{2^\mu}}\right)^{\frac{\beta}{p+q}} \left(\sum_{|m| \in \mathcal{D}_\mu} \gamma_m^{2/(2-\beta)}\right)^{(2-\beta)/2}. \end{aligned} \tag{6}$$

In case $\mu \ge 1$, in view of (1) with $\alpha := \frac{2}{2-\beta}$, and (6), we get

$$R_\mu \le \eta C\, 2^{-\mu\beta/2}\, \Gamma_{\mu-1} \left(\frac{\omega^q(f; \frac{\pi}{2^\mu})}{\Lambda_{2^\mu}}\right)^{\frac{\beta}{p+q}}.$$

If $\mu = 0$, then from equation (6) it follows that

$$R_0 := \gamma_1(|\hat{f}(1)|^\beta + |\hat{f}(-1)|^\beta) = O\left(\gamma_1 \left(\frac{\omega^q(f, \pi)}{\frac{1}{\lambda_1}}\right)^{\frac{\beta}{p+q}}\right).$$

Hence, the result follows from

$$\sum_{|m|\geq 1} \gamma_m |\hat{f}(m)|^\beta = \sum_{\mu=0}^{\infty} R_\mu.$$

□

In the case when $p = q = 1$, it follows from Theorem 1 that

$$\sum(\gamma; f)_\beta \leq \eta C \sum_{\mu=0}^{\infty} 2^{-\mu\beta/2}\, \Gamma_{\mu-1} \left(\frac{\omega(f; \frac{\pi}{2^\mu})}{\sum_{j=1}^{2^\mu} \frac{1}{\lambda_j}} \right)^{\frac{\beta}{2}}.$$

This was proved by Vyas [10, Theorem 3.1, p. 233].

Corollary 1. *Under the hypothesis of Theorem 1, we have*

$$\sum(\gamma; f)_\beta \leq \eta C \sum_{m=1}^{\infty} m^{-\beta/2}\, \gamma_m \left(\frac{\omega^q(f; \frac{\pi}{m})}{\sum_{j=1}^{m} \frac{1}{\lambda_j}} \right)^{\frac{\beta}{p+q}}.$$

In the case when $q = 2 - p$ and $\{\lambda_j\} = \{1\}$, it follows from Corollary 1 that

$$\sum(\gamma; f)_\beta \leq \eta C \sum_{m=1}^{\infty} m^{-\beta}\, \gamma_m\, \omega^{\frac{(2-p)\beta}{2}} \left(f; \frac{\pi}{m}\right).$$

This was proved by Gogoladze and Meskhia [4, Corollary 3, p. 32].

Similarly, Corollary 1 reduces to the result concerning the generalized β-absolute convergence of single Fourier series of Vyas [10, Corollary 3.2, p. 234] in the case when $p = q = 1$; and also reduces to the result proved in [3, Corollary 3.6, p. 366] in the case $p = q$. Further, Corollary 1 was proved by Vyas and Patadia [12, Theorem 1, with $n_k = k$, for all k, p. 131] in the case when $\{\gamma_m\} = \{1\}$ and $p = q = 1$.

3. New results for double Fourier series

Consider function f on $\mathbb{R}^k$. For $k = 1$ and $I = [a, b]$, define $f(I) := f(b) - f(a)$. For $k = 2$, $I = [a, b]$ and $J = [c, d]$, define

$$f(I \times J) := f(I, d) - f(I, c) = f(b, d) - f(a, d) - f(b, c) + f(a, c).$$

Given (Λ^1, Λ^2), where $\Lambda^r = \{\lambda_n^r\}_{n=1}^{\infty} \in \mathbb{L}$, for $r = 1, 2$, and $p \geq 1$, a complex valued measurable function f defined on $\overline{\mathbb{T}}^2$ is said to be of p-(Λ^1, Λ^2)-bounded variation (that is, $f \in (\Lambda^1, \Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2)$), if

$$V_{(\Lambda^1,\Lambda^2)_p}(f, \overline{\mathbb{T}}^2) := \sup_{I^1,\, I^2} \left\{ \left(\sum_j \sum_k \frac{|f(I_j^1 \times I_k^2)|^p}{\lambda_j^1 \lambda_k^2} \right)^{\frac{1}{p}} \right\} < \infty,$$

where I^1 and I^2 are finite collections of non-overlapping subintervals $\{I^1_j\}$ and $\{I^2_k\}$ in $\overline{\mathbb{T}}$, respectively.

Consider a function $f : \overline{\mathbb{T}}^2 \to \mathbb{R}$ defined by $f(x, y) = g(x) + h(y)$, where g and h are any two arbitrary functions from $\overline{\mathbb{T}}$ into $\mathbb{R}$ which need not be bounded (or need not be measurable). Then $V_{(\Lambda^1,\Lambda^2)_p}(f, \overline{\mathbb{T}}^2) = 0$. Thus, a function f with $V_{(\Lambda^1,\Lambda^2)_p}(f, \overline{\mathbb{T}}^2) < \infty$ need not be bounded (or need not be measurable).

If $f \in (\Lambda^1, \Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2)$ is such that the marginal functions $f(0, \cdot) \in \Lambda^2 BV^{(p)}(\overline{\mathbb{T}})$ and $f(\cdot, 0) \in \Lambda^1 BV^{(p)}(\overline{\mathbb{T}})$, then f is said to be of p-$(\Lambda^1, \Lambda^2)^*$-bounded variation (that is, $f \in (\Lambda^1, \Lambda^2)^* BV^{(p)}(\overline{\mathbb{T}}^2)$).

If $f \in (\Lambda^1, \Lambda^2)^* BV^{(p)}(\overline{\mathbb{T}}^2)$, then f is bounded on $\overline{\mathbb{T}}^2$ [8, Lemma 5.1, with $p(n) = p$, for all n].

Note that, for $p = 1$ and $\Lambda^1 = \Lambda^2 = \{1\}$, the classes $(\Lambda^1, \Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2)$ and $(\Lambda^1, \Lambda^2)^* BV^{(p)}(\overline{\mathbb{T}}^2)$ reduce to the classes $BV_V(\overline{\mathbb{T}}^2)$, the class of functions of bounded variation in the sense of Vitali (refer to [6, p. 279] for the definition of $BV_V(\overline{\mathbb{T}}^2)$) and $BV_H(\overline{\mathbb{T}}^2)$, the class of functions of bounded variation in the sense of Hardy (refer to [6, p. 280] for the definition of $BV_H(\overline{\mathbb{T}}^2)$), respectively; for $p = 1$, the classes $(\Lambda^1, \Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2)$ and $(\Lambda^1, \Lambda^2)^* BV^{(p)}(\overline{\mathbb{T}}^2)$ reduce to the classes $(\Lambda^1, \Lambda^2)BV(\overline{\mathbb{T}}^2)$ [1, Definition 2] and $(\Lambda^1, \Lambda^2)^* BV(\overline{\mathbb{T}}^2)$, respectively; and for $\Lambda^1 = \Lambda^2 = \{1\}$, the classes $(\Lambda^1, \Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2)$ and $(\Lambda^1, \Lambda^2)^* BV^{(p)}(\overline{\mathbb{T}}^2)$ reduce to the classes $BV_V^{(p)}(\overline{\mathbb{T}}^2)$ (refer to [5, p. 153]) and $BV_H^{(p)}(\overline{\mathbb{T}}^2)$, respectively.

For a complex valued function $f \in L^1(\mathbb{T}^2)$, where f is 2π-periodic in each variable, its double Fourier series is given by

$$f(x, y) \sim \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \hat{f}(m, n)\ e^{i(mx+ny)},\ (x, y) \in \mathbb{T}^2,$$

where the Fourier coefficients $\hat{f}(m, n)$ are defined by

$$\hat{f}(m, n) := \frac{1}{4\pi^2} \int\int_{\mathbb{T}^2} f(x, y)\ e^{-i(mx+ny)}\ dx\ dy,\ (m, n) \in \mathbb{Z}^2.$$

A double Fourier series of f is said to be β-absolute convergent if

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\hat{f}(m, n)|^\beta < \infty,$$

where

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\hat{f}(m, n)|^\beta \quad = \quad \sum_{|m|\geq 1} \sum_{|n|\geq 1} |\hat{f}(m, n)|^\beta + \sum_{m\in\mathbb{Z}} |\hat{f}(m, 0)|^\beta$$

$$+\sum_{n\in\mathbb{Z}}|\hat{f}(0,n)|^\beta-|\hat{f}(0,0)|^\beta. \tag{7}$$

In the special cases, when $m=0$ or $n=0$, we write

$$\hat{f}(m,0)=\hat{f}_1(m),\ \text{where } f_1(x):=\frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)\ dy,\ x\in\mathbb{T} \tag{8}$$

and

$$\hat{f}(0,n)=\hat{f}_2(n),\ \text{where } f_2(y):=\frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)\ dx,\ y\in\mathbb{T}. \tag{9}$$

We may write

$$\sum_{m\in\mathbb{Z}}|\hat{f}_1(m)|^\beta=\sum_{m\in\mathbb{Z}}|\hat{f}(m,0)|^\beta \text{ and } \sum_{n\in\mathbb{Z}}|\hat{f}_2(n)|^\beta=\sum_{n\in\mathbb{Z}}|\hat{f}(0,n)|^\beta.$$

Combining this with (7) gives

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}|\hat{f}(m,n)|^\beta=\sum_{|m|\ge 1}\sum_{|n|\ge 1}|\hat{f}(m,n)|^\beta+\sum_{m\in\mathbb{Z}}|\hat{f}_1(m)|^\beta+\sum_{n\in\mathbb{Z}}|\hat{f}_2(n)|^\beta-|\hat{f}(0,0)|^\beta.$$

Thus, the Fourier series of f is β-absolute convergent if

$$\sum_{|m|\ge 1}\sum_{|n|\ge 1}|\hat{f}(m,n)|^\beta<\infty,\ \sum_{m\in\mathbb{Z}}|\hat{f}_1(m)|^\beta<\infty \text{ and } \sum_{n\in\mathbb{Z}}|\hat{f}_2(n)|^\beta<\infty.$$

For $\beta=1$, one gets the absolute convergence of the double Fourier series of f. The modulus of continuity of a function f is defined as

$$\omega(f;\delta_1,\delta_2):=\sup\left\{|f([x,x+h_1]\times[y,y+h_2])|:\ 0<h_1\le\delta_1,\ 0<h_2\le\delta_2\right\}.$$

Following the definition in [5], a double sequence $\gamma=\{\gamma_{mn}:(m,n)\in\mathbb{N}_+^2\}$ of nonnegative numbers belongs to the class $\mathfrak{U}_\alpha$ for some $\alpha\ge 1$ if the inequality

$$\left(\sum_{m\in\mathcal{D}_\mu}\sum_{n\in\mathcal{D}_\nu}\gamma_{mn}^\alpha\right)^{1/\alpha}\le\eta 2^{(\mu+\nu)(1-\alpha)/\alpha}\sum_{m\in\mathcal{D}_{\mu-1}}\sum_{n\in\mathcal{D}_{\nu-1}}\gamma_{mn} \tag{10}$$

is satisfied for all $\mu,\nu\ge 0$, where $\mathcal{D}_\mu$ is as defined in (2) for $\mu\ge 0$. For instance, if $\mu\ge 1$ and $\nu=0$, then inequality (10) is of the form

$$\left(\sum_{m\in\mathcal{D}_\mu}\gamma_{m1}^\alpha\right)^{1/\alpha}\le\eta 2^{\mu(1-\alpha)/\alpha}\sum_{m\in\mathcal{D}_{\mu-1}}\gamma_{m1}.$$

It is easy to check that the inclusion (3) remains valid; and if a double sequence $\gamma=\{\gamma_{mn}\ge 0\}$ is such that

$$\max\{\gamma_{mn}:\ m\in\mathcal{D}_\mu,\ n\in\mathcal{D}_\nu\}$$
$$\le\eta\ \min\{\gamma_{mn}:\ m\in\mathcal{D}_{\mu-1},\ n\in\mathcal{D}_{\nu-1}\},\ (\mu,\nu)\in\mathbb{N}^2,$$

where η is a constant, then $\gamma \in \mathfrak{U}_\alpha$ for every $\alpha \geq 1$. For convenience in writing, put

$$\gamma_{-m,n} = \gamma_{m,-n} = \gamma_{-m,-n} := \gamma_{m,n}, \ (m,n) \in \mathbb{N}_+^2.$$

We prove the following result.

Theorem 2. *If a measurable $f \in (\Lambda^1, \Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2)$ $(p \geq 1)$, f is bounded, and $\gamma = \{\gamma_{mn}\} \in \mathfrak{U}_{2/(2-\beta)}$ for some $\beta \in (0,2)$, then*

$$\sum(\gamma; f)_\beta := \sum_{|m|\geq 1} \sum_{|n|\geq 1} \gamma_{mn} |\hat{f}(m,n)|^\beta$$

$$\leq \eta C \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} 2^{-(\mu+\nu)\beta/2} \ \Gamma_{\mu-1,\nu-1} \ \left(\frac{\omega^q(f; \frac{\pi}{2^\mu}, \frac{\pi}{2^\nu})}{\sum_{j=1}^{2^\mu} \sum_{k=1}^{2^\nu} \frac{1}{\lambda_j^1 \lambda_k^2}} \right)^{\frac{\beta}{p+q}}, \tag{11}$$

where η is from (10) *corresponding to $\alpha := 2/(2-\beta)$, $q > 0$, $p + q \geq 2$ and*

$$\Gamma_{\mu\nu} := \sum_{m \in \mathcal{D}_\mu} \sum_{n \in \mathcal{D}_\nu} \gamma_{mn} \ for \ \mu, \nu \in \mathbb{N}. \tag{12}$$

Proof. For given $h_1, h_2 > 0$, put

$$\Delta f_{jk}(x, y; h_1, h_2) := f\left([x + (j-1)h_1, x + jh_1] \times [y + (k-1)h_2, y + kh_2]\right).$$

Then, for each $m, n \in \mathbb{Z}$,

$$\widehat{\Delta f}_{jk}(m,n) = -4\hat{f}(m,n) \ e^{im(j-\frac{1}{2})h_1} \ e^{in(k-\frac{1}{2})h_2} \ \sin\left(\frac{mh_1}{2}\right) \sin\left(\frac{nh_2}{2}\right).$$

Since f is bounded, $f \in L^2(\overline{\mathbb{T}}^2)$. Therefore the Parseval formula gives

$$\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \left| \hat{f}(m,n) \sin\left(\frac{mh_1}{2}\right) \sin\left(\frac{nh_2}{2}\right) \right|^2 = O\left(\int\int_{\overline{\mathbb{T}}^2} |\Delta f_{jk}(x, y; h_1, h_2)|^2 dxdy \right).$$

Putting $h_1 := \frac{\pi}{2^\mu}$, $h_2 := \frac{\pi}{2^\nu}$, $\mu, \nu \in \mathbb{N}$, taking into account the inequality (5) and using that an analogous inequality holds for $|n| \in \mathcal{D}_\nu$, we have

$$S_{\mu\nu} := \sum_{|m|\in\mathcal{D}_\mu} \sum_{|n|\in\mathcal{D}_\nu} |\hat{f}(m,n)|^2 = O\left(\int\int_{\overline{\mathbb{T}}^2} \left| \Delta f_{jk}\left(x, y; \frac{\pi}{2^\mu}, \frac{\pi}{2^\nu}\right) \right|^2 \ dx \ dy \right),$$

for all $j = 1, ..., 2^\mu$ and for all $k = 1, ..., 2^\nu$.

Applying Hölder's inequality on the right side of the above inequality, we have

$$S_{\mu\nu} = O\left(\left(\int\int_{\overline{\mathbb{T}}^2} \left| \Delta f_{jk}\left(x, y; \frac{\pi}{2^\mu}, \frac{\pi}{2^\nu}\right) \right|^{p+q} \right)^{\frac{2}{p+q}} \right).$$

Since the left hand side of the above inequality is independent of j and k, multiplying both sides of it by $\frac{1}{\lambda_j^1\lambda_k^2}$, summing over j from 1 to 2^μ and k from 1 to 2^ν, and letting $\Lambda_{2^\mu,2^\nu} := \sum_{j=1}^{2^\mu}\sum_{k=1}^{2^\nu}\frac{1}{\lambda_j^1\lambda_k^2}$, we get

$$S_{\mu\nu} = O\left(\frac{1}{(\Lambda_{2^\mu,2^\nu})^{\frac{2}{p+q}}}\left(\int\int_{\overline{\mathbb{T}}^2}\sum_{j=1}^{2^\mu}\sum_{k=1}^{2^\nu}\frac{\left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|^{p+q}}{\lambda_j^1\lambda_k^2}\right)^{\frac{2}{p+q}}\right).$$

Since $\left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right| = O\left(\omega\left(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right)$, we have $S_{\mu\nu} =$

$$O\left(\left(\frac{\omega^q\left(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)}{\Lambda_{2^\mu,2^\nu}}\right)^{\frac{2}{p+q}}\left(\int\int_{\overline{\mathbb{T}}^2}\sum_{j=1}^{2^\mu}\sum_{k=1}^{2^\nu}\frac{\left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|^{p}}{\lambda_j^1\lambda_k^2}\,dx\,dy\right)^{\frac{2}{p+q}}\right),$$

where

$$\sum_{j=1}^{2^\mu}\sum_{k=1}^{2^\nu}\frac{\left|\Delta f_{jk}\left(x,y;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu}\right)\right|^{p}}{\lambda_j^1\lambda_k^2} = O(1)\ \text{as} f\in(\Lambda^1,\Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2).$$

Hence,

$$S_{\mu\nu} = O\left(\left(\frac{\omega^q(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu})}{\Lambda_{2^\mu,2^\nu}}\right)^{\frac{2}{p+q}}\right).$$

Since $1 = \frac{\beta}{2} + \frac{2-\beta}{2}$, by Hölder's inequality, we have

$$\begin{aligned}
R_{\mu\nu} &:= \sum_{|m|\in\mathcal{D}_\mu}\sum_{|n|\in\mathcal{D}_\nu}\gamma_{mn}|\hat{f}(m,n)|^\beta \\
&\le \left(\sum_{|m|\in\mathcal{D}_\mu}\sum_{|n|\in\mathcal{D}_\nu}|\hat{f}(m,n)|^2\right)^{\beta/2}\left(\sum_{|m|\in\mathcal{D}_\mu}\sum_{|n|\in\mathcal{D}_\nu}\gamma_{mn}^{2/(2-\beta)}\right)^{(2-\beta)/2} \\
&\le \left(\frac{\omega^q(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu})}{\Lambda_{2^\mu,2^\nu}}\right)^{\frac{\beta}{p+q}}\left(\sum_{|m|\in\mathcal{D}_\mu}\sum_{|n|\in\mathcal{D}_\nu}\gamma_{mn}^{2/(2-\beta)}\right)^{(2-\beta)/2}. \qquad (13)
\end{aligned}$$

In case $\max\{\mu,\nu\}\ge 1$, in view of (10), with $\alpha := \frac{2}{2-\beta}$, and (13), we get

$$R_{\mu\nu} \le \eta C\ 2^{-(\mu+\nu)\beta/2}\ \Gamma_{\mu-1,\nu-1}\left(\frac{\omega^q(f;\frac{\pi}{2^\mu},\frac{\pi}{2^\nu})}{\Lambda_{2^\mu,2^\nu}}\right)^{\frac{\beta}{p+q}}.$$

If $\mu=\nu=0$, then from equation (13) it follows that

$$\begin{aligned}
R_{00} &:= \gamma_{11}(|\hat{f}(1,1)|^\beta + |\hat{f}(-1,1)|^\beta + |\hat{f}(1,-1)|^\beta + |\hat{f}(-1,-1)|^\beta) \\
&= O\left(\gamma_{11}\left(\frac{\omega^q(f,\pi,\pi)}{\frac{1}{\lambda_1^1\lambda_1^2}}\right)^{\frac{\beta}{p+q}}\right).
\end{aligned}$$

Hence, the result follows from

$$\sum_{|m|\geq 1}\sum_{|n|\geq 1}\gamma_{mn}|\hat{f}(m,n)|^{\beta}=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}R_{\mu\nu}.$$

□

In the case $q=2-p$ and $\{\lambda_j^1\}=\{\lambda_k^2\}=\{1\}$, it follows from Theorem 2 that

$$\sum(\gamma;f)_{\beta}\leq \eta C\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}2^{-(\mu+\nu)\beta}\ \Gamma_{\mu-1,\nu-1}\ \omega^{(2-p)\frac{\beta}{2}}\left(f;\frac{\pi}{2^{\mu}},\frac{\pi}{2^{\nu}}\right).$$

This was proved by Móricz and Veres [5, Theorem 4, p. 153].

Corollary 2. *If a measurable* $f\in(\Lambda^1,\Lambda^2)^*BV^{(p)}(\overline{\mathbb{T}}^2)$, *then* (11) *holds true, where* $p,q,\gamma,\ \beta,\ \eta,\ \alpha$ *and* Γ *are as in Theorem* 2.

Proof. Since $f\in(\Lambda^1,\Lambda^2)^*BV^{(p)}(\overline{\mathbb{T}}^2)$ is bounded and $(\Lambda^1,\Lambda^2)^*BV^{(p)}(\overline{\mathbb{T}}^2)\subset(\Lambda^1,\Lambda^2)BV^{(p)}(\overline{\mathbb{T}}^2)$, the corollary follows from Theorem 2. □

Corollary 3. *Under the hypothesis of Theorem* 2, *we have*

$$\sum(\gamma;f)_{\beta}\leq \eta C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\beta/2}\ \gamma_{mn}\ \left(\frac{\omega^{q}(f;\frac{\pi}{m},\frac{\pi}{n})}{\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{\lambda_j^1\lambda_k^2}}\right)^{\frac{\beta}{p+q}}. \tag{14}$$

Proof. In the case $\mu,\nu\geq 1$ from (2) and (12) it follows that

$$\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}2^{-(\mu+\nu)\beta/2}\ \Gamma_{\mu-1,\nu-1}\ \left(\frac{\omega^{q}(f;\frac{\pi}{2^{\mu}},\frac{\pi}{2^{\nu}})}{\sum_{j=1}^{2^{\mu}}\sum_{k=1}^{2^{\nu}}\frac{1}{\lambda_j^1\lambda_k^2}}\right)^{\frac{\beta}{p+q}}$$

$$\leq\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\beta/2}\gamma_{mn}\left(\frac{\omega^{q}(f;\frac{\pi}{m},\frac{\pi}{n})}{\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{\lambda_j^1\lambda_k^2}}\right)^{\frac{\beta}{p+q}}.$$

In case $\mu\geq 1$ and $\nu=0$, it follows that

$$\sum_{\mu=0}^{\infty}2^{-\mu\beta/2}\,\Gamma_{\mu-1,-1}\ \left(\frac{\omega^{q}(f;\frac{\pi}{2^{\mu}},\pi)}{\sum_{j=1}^{2^{\mu}}\frac{1}{\lambda_j^1\lambda_1^2}}\right)^{\frac{\beta}{p+q}}\leq\sum_{m=1}^{\infty}m^{-\beta/2}\gamma_{m1}\left(\frac{\omega^{q}(f;\frac{\pi}{m},\pi)}{\sum_{j=1}^{m}\frac{1}{\lambda_j^1\lambda_1^2}}\right)^{\frac{\beta}{p+q}}.$$

In case $\mu=0$ and $\nu\geq 1$, an analogous inequality holds; while in case $\mu=0$ and $\nu=0$, we have

$$\Gamma_{-1,-1}\ \left(\frac{\omega^{q}(f;\pi,\pi)}{\frac{1}{\lambda_1^1\lambda_1^2}}\right)^{\frac{\beta}{p+q}}\ \leq\ \gamma_{11}\ \left(\frac{\omega^{q}(f;\pi,\pi)}{\frac{1}{\lambda_1^1\lambda_1^2}}\right)^{\frac{\beta}{p+q}}.$$

Hence, the corollary follows from Theorem 2. □

Corollary 3 was proved by Móricz and Veres [5, Corollary 4, p. 153] in the case when $q = 2 - p$ and $\{\lambda_j^1\} = \{\lambda_k^2\} = \{1\}$, and also proved by Vyas and Darji [11, Theorem 3.3, p. 73] in the case when $\{\gamma_{mn}\} = \{1\}$ and $p = q = 1$.

Corollary 4. *Under the hypothesis of Corollary* 2*, the inequality* (14) *holds true.*

Proof of Corollary 4 is similar to that of Corollary 2.

Combining Corollary 1 and Corollary 3, we can easily find sufficient conditions imposed on f, f_1 and f_2 for the convergence of the double series

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \gamma_{mn}|\hat{f}(m,n)|^\beta.$$

For $\{\gamma_{mn}\} = \{\gamma_m\} = \{\gamma_n\} = \{1\}$, combining Corollary 1 and Corollary 3, we obtain the following corollary.

Corollary 5. *If a measurable $f \in (\Lambda^1, \Lambda^2)^* BV^{(p)}(\overline{\mathbb{T}}^2)$, $p \geq 1$, $q > 0$, $p + q \geq 2$, $\beta \in (0, 2)$,*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\beta/2}\left(\frac{\omega^q(f;\frac{\pi}{m},\frac{\pi}{n})}{\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{1}{\lambda_j^1\lambda_k^2}}\right)^{\frac{\beta}{p+q}} < \infty,$$

$$\sum_{m=1}^{\infty} m^{-\beta/2}\left(\frac{\omega^q(f_1;\frac{\pi}{m})}{\sum_{j=1}^{m}\frac{1}{\lambda_j^1}}\right)^{\frac{\beta}{p+q}} < \infty,$$

and

$$\sum_{n=1}^{\infty} n^{-\beta/2}\left(\frac{\omega^q(f_2;\frac{\pi}{n})}{\sum_{k=1}^{n}\frac{1}{\lambda_k^2}}\right)^{\frac{\beta}{p+q}} < \infty,$$

where f_1 and f_2 are as defined in (8) *and* (9)*, respectively, then the double Fourier series of f is β-absolute convergent.*

4. Extension to multiple Fourier series

Let $I^k = [a_k, b_k] \subset \mathbb{R}$, for $k = 1, 2, \cdots, N$. In Section 3, we defined $f(I^1)$ for a function f of one variable and $f(I^1 \times I^2)$ for a function f of two variables. Similarly, for a function f on $\mathbb{R}^N$, by induction, defining the expression $f(I^1 \times \cdots \times I^{N-1})$ for a function of $N-1$ variables, one gets

$$f(I^1 \times \cdots \times I^N) = f(I^1 \times \cdots \times I^{N-1}, b_N) - f(I^1 \times \cdots \times I^{N-1}, a_N).$$

Given $(\Lambda^1, ..., \Lambda^N)$, where $\Lambda^r = \{\lambda^r_k\}_{k=1}^{\infty} \in \mathbb{L}$, for $r = 1, ..., N$, and $p \geq 1$, a complex valued measurable function f defined on $\overline{\mathbb{T}}^N$ is said to be of p-$(\Lambda^1, ..., \Lambda^N)$-bounded variation (that is, $f \in (\Lambda^1, ..., \Lambda^N)BV^{(p)}(\overline{\mathbb{T}}^N)$), if

$$V_{(\Lambda^1,...,\Lambda^2)_p}(f, \overline{\mathbb{T}}^N) := \sup_{J^1,\cdots,J^N} \left\{ \left(\sum_{k_1} \cdots \sum_{k_N} \frac{|f(I^1_{k_1} \times \cdots \times I^N_{k_N})|^p}{\lambda^1_{k_1} ... \lambda^N_{k_N}} \right)^{\frac{1}{p}} \right\} < \infty,$$

where $J^1, ..., J^{N-1}$ and J^N are finite collections of non-overlapping subintervals $\{I^1_{k_1}\}, ..., \{I^{N-1}_{k_{N-1}}\}$ and $\{I^N_{k_N}\}$ in $\overline{\mathbb{T}}$, respectively.

Moreover, a function $f \in (\Lambda^1, ..., \Lambda^N)BV^{(p)}(\overline{\mathbb{T}}^N)$ is said to be of p-$(\Lambda^1, ..., \Lambda^N)^*$-bounded variation (that is, $f \in (\Lambda^1, ..., \Lambda^N)^* BV^{(p)}(\overline{\mathbb{T}}^N)$) if for each of its marginal functions

$$f(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_N) \in (\Lambda^1, ..., \Lambda^{i-1}, \Lambda^{i+1}, ..., \Lambda^N)^* BV^{(p)}(\overline{\mathbb{T}}^{N-1}),$$

for all $i = 1, 2, ..., N$. If $f \in (\Lambda^1, ..., \Lambda^N)^* BV^{(p)}(\overline{\mathbb{T}}^N)$ then f is bounded on $\overline{\mathbb{T}}^N$ [8, Lemmma 6.3, with $p(n) = n$, for all n].

Note that the classes $(\Lambda^1, ..., \Lambda^N)BV^{(p)}(\overline{\mathbb{T}}^N)$ and $(\Lambda^1, ..., \Lambda^N)^* BV^{(p)}(\overline{\mathbb{T}}^N)$, for $p = 1$ and $\Lambda^1 = ... = \Lambda^N = \{1\}$, reduce to the classes $BV_V(\overline{\mathbb{T}}^N)$ (the class of functions of bounded variation in the sense of Vitali) and $BV_H(\overline{\mathbb{T}}^N)$ (the class of functions of bounded variation in the sense of Hardy), respectively; for $p = 1$, the classes $(\Lambda^1, ..., \Lambda^N)BV^{(p)}(\overline{\mathbb{T}}^N)$ and $(\Lambda^1, ..., \Lambda^N)^* BV^{(p)}(\overline{\mathbb{T}}^N)$ reduce to the classes $(\Lambda^1, ..., \Lambda^N)BV(\overline{\mathbb{T}}^N)$ and $(\Lambda^1, ..., \Lambda^N)^* BV(\overline{\mathbb{T}}^N)$, respectively; and for $\Lambda^1 = ... = \Lambda^N = \{1\}$, the classes $(\Lambda^1, ..., \Lambda^N)BV^{(p)}(\overline{\mathbb{T}}^N)$ and $(\Lambda^1, ..., \Lambda^N)^* BV^{(p)}(\overline{\mathbb{T}}^N)$ reduce to the classes $BV_V^{(p)}(\overline{\mathbb{T}}^N)$ and $BV_H^{(p)}(\overline{\mathbb{T}}^N)$, respectively.

For a complex valued function $f \in L^1(\mathbb{T}^N)$, where f is 2π-periodic in each variable, its multiple Fourier series is given by

$$f(x_1, ..., x_N) \sim \sum_{m_1 \in \mathbb{Z}} ... \sum_{m_N \in \mathbb{Z}} \hat{f}(m_1, ..., m_N)\ e^{i(m_1 x_1 + ... + m_N x_N)},$$

where the Fourier coefficients $\hat{f}(m_1, ..., m_N)$ are defined by

$$\hat{f}(m_1, ..., m_N) := \frac{1}{(2\pi)^N} \int ... \int_{\overline{\mathbb{T}}^N} f(x_1, ..., x_N)\ e^{-i(m_1 x_1 + ... + m_N x_N)} dx_1 ... dx_N.$$

The multiple Fourier series of f is said to be β-absolute convergent if

$$\sum_{m_1 \in \mathbb{Z}} ... \sum_{m_N \in \mathbb{Z}} |\hat{f}(m_1, ..., m_N)|^{\beta} < \infty.$$

The modulus of continuity of a function f is defined as $\omega(f;\delta_1,...,\delta_N) := \sup\{|f([x_1,x_1+h_1]\times...\times[x_N,x_N+h_N])| : 0<h_j\le\delta_j,\ j=1,...,N\}$.

Analogously to (1) and (10), an $N-$multiple sequence $\gamma=\{\{\gamma_{m_1,...,m_N}\} : (m_1,...,m_N)\in\mathbb{N}_+^N\}$ of nonnegative numbers is said to belong to the class $\mathfrak{U}_\alpha$ for some $\alpha\ge 1$ if the inequality

$$\left(\sum\nolimits_{m_1\in\mathcal{D}_{\mu_1}}\cdots\sum\nolimits_{m_N\in\mathcal{D}_{\mu_N}}\gamma^\alpha_{m_1,...,m_N}\right)^{1/\alpha} \le \eta\ 2^{(\mu_1+...+\mu_N)(1-\alpha)/\alpha}\sum\nolimits_{m_1\in\mathcal{D}_{\mu_1-1}}\cdots\sum\nolimits_{m_N\in\mathcal{D}_{\mu_N-1}}\gamma_{m_1,...,m_N} \tag{15}$$

is satisfied for all $\mu_1,...,\mu_N\ge 0$, where $\mathcal{D}_\mu$ is as defined in (2) for $\mu\ge 0$.

The following statements are the extensions of the results of Section 3.

Theorem 3. *If a measurable* $f\in(\Lambda^1,...,\Lambda^N)BV^{(p)}(\overline{\mathbb{T}}^N)$ $(p\ge 1)$, *f is bounded, and* $\gamma=\{\gamma_{m_1,...,m_N}\}\in\mathfrak{U}_{2/(2-\beta)}$ *for some* $\beta\in(0,2)$, *then*

$$\sum(\gamma;f)_\beta := \sum_{|m_1|\ge1}\cdots\sum_{|m_N|\ge1}\gamma_{m_1,...,m_N}|\hat f(m_1,...,m_N)|^\beta$$

$$\le \eta C\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_N=0}^{\infty}2^{-(\mu_1+...+\mu_N)\beta/2}\,\Gamma_{\mu_1-1,...,\mu_N-1}\left(\frac{\omega^q(f;\frac{\pi}{2^{\mu_1}},...,\frac{\pi}{2^{\mu_N}})}{\sum_{k_1=1}^{2^{\mu_1}}\cdots\sum_{k_N=1}^{2^{\mu_N}}\frac{1}{\lambda^1_{k_1}...\lambda^N_{k_N}}}\right)^{\frac{\beta}{p+q}}, \tag{16}$$

where η is from (15) *corresponding to* $\alpha:=2/(2-\beta)$, $q>0$, $p+q\ge 2$,

$$\Gamma_{\mu_1,...,\mu_N} := \sum_{m_1\in\mathcal{D}_{\mu_1}}\cdots\sum_{m_N\in\mathcal{D}_{\mu_N}}\gamma_{m_1,...,m_N}\ \ for\ \mu_1,...,\mu_N\in\mathbb{N}.$$

In the case when $q=2-p$ and $\{\lambda^1_{k_1}\}=\cdots=\{\lambda^N_{k_N}\}=\{1\}$, it follows from Theorem 3 that $\sum(\gamma;f)_\beta$

$$\le \eta C\sum_{\mu_1=0}^{\infty}\cdots\sum_{\mu_N=0}^{\infty}2^{-(\mu_1+\cdots+\mu_N)\beta}\,\Gamma_{\mu_1-1,...,\mu_N-1}\,\omega^{(2-p)\frac{\beta}{2}}\left(f;\frac{\pi}{2^{\mu_1}},...,\frac{\pi}{2^{\mu_N}}\right).$$

This was proved by Móricz and Veres [5, Theorem 4$'$, p. 160].

Corollary 6. *If a measurable* $f\in(\Lambda^1,...,\Lambda^N)^*BV^{(p)}(\overline{\mathbb{T}}^N)$, *then* (16) *holds true, where* $p,q,\gamma,\ \beta,\ \eta,\ \alpha$ *and* Γ *are as in Theorem* 3.

Corollary 7. *Under the hypothesis of Theorem* 3, *we have* $\sum(\gamma;f)_\beta\le$

$$\eta C\sum_{m_1=1}^{\infty}\cdots\sum_{m_N=1}^{\infty}(m_1\cdots m_N)^{-\beta/2}\gamma_{m_1,...,m_N}\left(\frac{\omega^q(f;\frac{\pi}{m_1},...,\frac{\pi}{m_N})}{\sum_{k_1=1}^{m_1}\cdots\sum_{k_N=1}^{m_N}\frac{1}{\lambda^1_{k_1}...\lambda^N_{k_N}}}\right)^{\frac{\beta}{p+q}}. \tag{17}$$

Corollary 7 was proved by Móricz and Veres [5, Corollary 4′, p. 160] in the case when $q = 2 - p$ and $\{\lambda^1_{k_1}\} = \cdots = \{\lambda^N_{k_N}\} = \{1\}$; and also proved by Vyas and Darji [11, Theorem 5.3, p. 80] in the case when $\{\gamma_{m_1,\dots,m_N}\} = \{1\}$ and $p = q = 1$.

Corollary 8. *Under the hypothesis of Corollary* 6, *the inequality* (17) *holds true.*

Extended results of this section can be proved in the same way as we proved the results in Section 3.

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