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Connecting operation-choice problems by the variation principle: Sixth graders' operational or deeper relational pathways

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ABSTRACT

Many empirical studies documented students' challenges with operation-choice problems, in particular for multiplication and division with rational numbers. The design principle of problem variation was suggested to overcome these challenges by engaging students in making connections between inverse operation-choice problems of multiplication and division, and between problems with natural numbers and fractions/decimals, but so far, this approach was hardly investigated empirically. In this study, we investigate 17 sixth graders' modelling pathways through sets of operation-choice problems that are systematically designed according to the variation principle. In the qualitative analysis, we identify five pathways by which students solve the problems and sometimes connect them. While one pathway uses deep relational connections, others only draw superficial and operational connections and others stay with informal strategies without connecting them to formal operations.

1. Introduction

Multiplication and division word problems have often been shown to be challenging for students as the choice of operations requires modelling decisions (Bell et al., 1984; Fischbein et al., 1985): Students are supposed to discover multiplicative structures in the problem situation, and on this base decide how to mathematize by a division or multiplication. For the same situation (e.g., 24 cookies structured into six boxes of four cookies each), the difficulty of these decisions depends on the *problem structure*, i.e., whether the total is unknown (multiplication problem structure), or the number of groups (quotitive problem structure) or the size of the groups (partitive problem structure; for a review, see Greer, 1992). Students can exploit the inverse relationship between the three problem structures when the problems belong to the same situation (ibid.). For students, the difficulty depends furthermore on the *numbers involved* (Zorrilla et al., 2023): problems are mastered more frequently with natural numbers than with rational numbers (e.g., Bell et al., 1981; De Corte et al., 1988; Greer, 1992) where many students refer to overgeneralizations (such as multiplication makes bigger; e.g., Bell et al., 1984; Fischbein et al., 1985) rather than to the invariance of the problem structures (Levain, 1992; Prediger, 2008; Zorrilla, Fernández et al., 2022).

To overcome these challenges, authors have suggested to engage students in making two kinds of connections: Connecting inverse problems for the same multiplicative situations (i.e., the multiplication problem with its inverse partitive and quotitive division

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problems; Downton, 2009, 2013; Fischbein et al., 1985) and connecting problems with natural numbers to problems with rational numbers (decimals or fractions; e.g., Levain, 1992; Zorrilla, Fernández et al., 2022).

The *variation principle* (*Bianshi* in Chinese) is widely established in China as a design principle to engage students in making both kinds of connections through systematic problem variations, either in Chinese textbooks (Sun, 2011a, 2011b, 2019) or in teachers' own practices (Han et al., 2017). But so far, students' work initiated by the variation principle have not yet been investigated empirically in depth.

In this paper, we contribute to reducing this research gap by pursuing the following empirical research question: *How do sixth graders solve and connect operation-choice problems of multiplication and division when they participate in an instruction based on the variation principle?*

2. Theoretical background

2.1. Typical challenges with operation-choice problems for multiplication and division

Students' challenges of the choice of operations for word problems were identified in many empirical studies in the 1980 s and 1990 s (overview in Greer, 1992). These studies discerned different factors of problem difficulty, among them the structure of the problems, number size and number types (Bell et al., 1981, 1984; Fischbein et al., 1985; Harel et al., 1994), types of context situations, language features of the problem texts (Abedi & Lord, 2001) and others. Among these factors, we focus on problem structure and number types.

The most relevant *problem structures* have already been distinguished by Fischbein et al. (1985): For the same situation being structured in equal groups, multiplication problems ask for the total quantity, partitive division problems ask for the size of the groups (e.g., in situations of fair sharing) and quotitive problems ask for the number of groups (e.g., in measurement problems: How often does the 5 fit into the 15?). Nevertheless, instead of drawing connections of inverse relationships (Sun, 2011a, 2011b), students treat the three problem structures in isolation, presenting difficulties in identifying the multiplicative structure in division problems (Neuman, 1999).

The *number types* have often been reported to form a second important factor for students' choice of operations: Many students who successfully choose the operations for natural numbers do not succeed for decimal numbers (Bell et al., 1981, 1984, 1989; Fischbein et al., 1985) or for fractions (Harel et al., 1994; Prediger, 2008).

In this paper, we will disentangle the underlying cognitive processes (i.e., students' mental actions) while solving operation-choice problems in modelling cycles between the multiple (mental) representations involved (see Fig. 1 adapted from Vom Hofe and Blum, 2016): When students are confronted with a word problem text (in textual representation), they are expected to *structure* the underlying situation in a (mental or graphical) contextual representation (sometimes in episodic form, sometimes graphically), therefrom *mathematize* by choosing an operation (into symbolic representation), *calculate* the result (in numerical representation), *interpret* the operation and the result in the situation structure and *validate* it in the problem situation.

In the step from the situation structure to the operation choice, students' modelling cycles can draw upon mental models for example, division in the quotitive or partitive model (Fischbein et al., 1985; Greer, 1992). Ideally, they draw upon the intended mental model in their processes of mathematizing and interpreting, for example multiplication as counting in groups, multiplication as scaling up, multiplication as Cartesian product, etc. (for further details see Greer, 1992; Prediger, 2008).

In contrast, when students make inaccurate operation choices, different steps in the modelling cycle can be involved. Reading comprehension can notably influence the accuracy of the situation structure to be constructed (Leiss et al., 2019). Indeed, a limited reading comprehension can lead to a deviant mental representation (Dröse & Prediger, 2020; Verschaffel et al., 2000) of the problem situation or *suspended structuring* by immediate direct translations (Hegarty et al., 1995). An inaccurate mental contextual representation due to overgeneralizations derived from natural number properties and their operations can lead to wrong *mathematizing* (Fischbein et al., 1985), and superficial *interpreting* and *validating* processes can hinder students from recognizing their errors (Verschaffel et al., 2000).

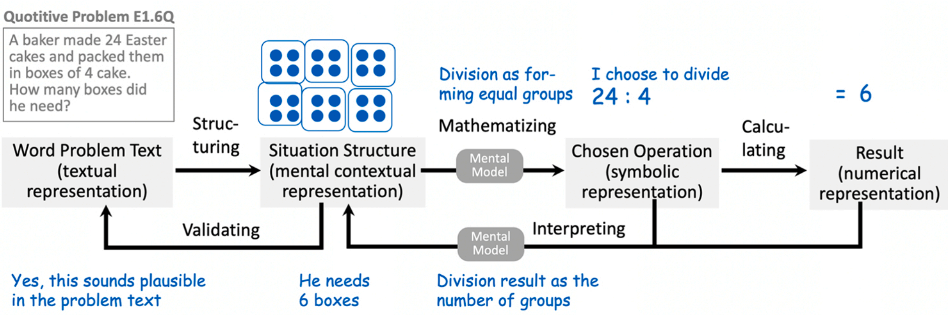


Fig. 1. Modelling cycle with five cognitive processes (structuring, mathematizing, calculating, interpreting, validating) while solving a quotitive division problem.

In particular, when rational numbers below 1 are involved, the suspension of the intended mental models by idiosyncratic implicit models have often been documented (e.g., multiplication makes bigger which is correct for natural factors but not for rational factors below 1, cf. Bell et al., 1984; Fischbein et al., 1985; Levain, 1992). These overgeneralizations are derived from the properties of natural numbers and their operations. The interference of natural numbers knowledge on rational numbers learning has been documented in the literature as the phenomenon of natural number bias (e.g., González-Forte et al., 2020; Van Hoof et al., 2015) or conceptual change (Prediger, 2008). As a reason for students' reluctant use of the invariance of problem structures across number types and suspension of mental models, researchers have identified epistemological obstacles indicating that some of the mental models have to be adapted in the transition from natural to rational numbers (Greer, 1992; Prediger, 2008): Whereas division can still be interpreted in quotitive problem structures (We cut six cakes in halves, how many halves do we get?), the mental model of division as equal sharing (partitive structure) does not work for fractions or decimals below 1 (We share the six cakes with half a person?). Also, multiplication needs to be re-interpreted in the transition from natural to rational numbers, e.g., from counting in groups (3 groups of 2) to taking parts of a whole (2/3 of 3) or of a part (2/3 of 3/4; see Prediger, 2008; Taber, 2007).

2.2. *Instructional approaches suggested for overcoming challenges with operation-choice problems*

Although students' challenges with operation-choice problems have been widely documented for forty years (since Bell et al., 1981), there are still only few studies investigating instructional approaches for overcoming these challenges, as Sun (2019) and Sun et al. (2019) problematized. Although nearly all researchers came to the conclusions that students need to learn to make more connections, this claim has hardly been empirically explored, so far.

For developing students' mental models of partitive and quotitive division for natural numbers, various researchers have shown that students can see and use the inverse relationships between the three problem structures even before the relationship is systematically taught (Correa et al., 1998; Downton, 2009), so this intuitive inverse relationship should be used to develop students' mental models for division. Downton (2013) has provided empirical evidence that when the inverse relationships between division and multiplication is understood by students, it strengthens their ability to correctly choose operations for natural numbers, perceiving the role of each number in the operation.

For the transition of natural to rational numbers, already Bell et al. (1981) conducted first teaching experiments with 15–16-year-old students to connect the operations with decimals to the operations with natural numbers by varying the problems. Zazkis et al. (2008) also addressed this transition after observing that from secondary school students to pre-service elementary school teachers there were difficulties in solving the following quotitive division problem "A pound of fancy grain cost[*s*] $1.68, how much grain can you buy for $0.50?" (p. 136). Zazkis et al. (2008) turned to *numerical variation* to support similarity recognition to extend the general multiplicative structure. Thus, from the original problem, they modified only 1.68 (the cost per pound of grain) and 0.50 (the total cost) respectively to: 2 and 6, in the second problem; 2 and 20, in the third problem; 2 and 0.50, in the fourth problem… These numerical and other consecutive variations would allow solving the originally proposed problem (Zazkis et al., 2008). Most other researchers only made suggestions for the aforementioned transition without investigating the suggested instructional approach (e.g., Fischbein et al., 1985; Levain, 1992).

An instructional approach for engaging students in making connections draws upon the design principle of problem variation, called variation theory of learning in Sweden (Marton, 2015), and *Bianshi* teaching in the Chinese tradition (Cai & Nie, 2007; Sun, 2019). While variation plays a leading role in both traditions, they differ in their focus: The focus is on differences (variation) in the Swedish tradition and on what remains invariant in the Chinese tradition (Pang et al., 2017).

In the variation theory of learning, task design by the teacher requires attention to the object of learning and the way in which the student intends to learn (*specific keys* and *ways of seeing*; Marton et al., 2004). On the one hand, "discerning the object of learning amounts to discerning its critical aspects" (Marton & Pang, 2006, p. 193). These critical aspects are the *specific keys* necessary for the acquisition of the object of learning (Marton, 2015). Thus, when solving an operation-choice problem, the solver must have the ability to discern the quantities that make it up and the relationships between them (Pang et al., 2017). On the other hand, *ways of seeing* present a fundamental role in the discernment of these specific keys. Marton (2015) identified patterns of (in)variation: contrast, generalization, and fusion. Particularly, the pattern of generalization used in our study intends to separate in an object of learning the necessary aspects from the optional aspects by varying the last ones (Marton, 2015). Returning to the example of solving an operation-choice problem, generalization would take place keeping the relationships between quantities invariant (i.e., problem structure as a necessary aspect) while other aspects would vary (e.g., context as an optional aspect).

In *Bianshi* teaching, variation also plays a relevant role. Although variation is associated with change, as Huang et al. (2006) state in recalling an ancient Chinese saying, variation is about "remaining essentially the same despite all apparent changes" (p. 270). Three widespread activities highlight Chinese mathematics education among teachers (e.g., Cai & Nie, 2007; Sun, 2019): one problem, multiple changes (OPMC); one problem, multiple solution strategies (OPMS); and multiple problems, one solution strategy (MPOS).

Firstly, OPMC consists of presenting an initial problem and, once it is solved, presenting and solving variations of the initial problem. In her insightful analysis of Chinese textbooks, Sun (2011a) (2011b) unpacks possible examples for making connections for multiplication and division, within one number type and between number types, without providing empirical insights. An example within the same number type is a situation with identical quantities (300 g of candies in 3 boxes of 100 g; Sun, 2011b). The multiplication problem is designed "Each box of fruit candies weighs 100 g. How much do the three boxes weigh?" (p. 75) and also its corresponding partitive division problem and quotitive division problem. An example between number types is using the situation 3/10 of a kilogram of candy in 3 boxes of 1/10 of a kilogram, changing the numbers from the previous situation (Sun, 2011b).

Secondly, OPMS consists of presenting a problem and providing the opportunity to solve it using different strategies to promote flexible ways of thinking in choosing/designing a strategy to solve the problem. Sun (2019) illustrates how a multiplication problem can be solved by a counting strategy based on graphical representation, repeated addition, or the multiplication formal operation.

Finally, MPOS consists of using the same strategy to solve a set of problems of identical structure. Sun (2019) illustrates the use of the multiplication formal operation in different multiplication problems in the following activity: "How many liters are there in three barrels if each barrel is 12 L? How many liters are there in 1/2 of a barrel? How many liters are there in 1/4 of a barrel?" (p. 116). Thus, OPMC, OPMS, and MPOS provide the opportunity to perceive structures and relationships by making connections (Sun, 2011a).

The variation principle has also been applied for other typical challenges of word problems, for example, Dröse and Prediger (2020) have used it for raising students' language awareness by engaging them in connecting two word problems which vary only in one subtle language feature (e.g., "Each penguin eats 5 fishes" vs. "Each penguin eats 5 fishes more"). The controlled trial provided quantitative evidence for the overall efficacy of these language variations for enhancing students' success in structuring the situations (Dröse & Prediger, 2021). However, the qualitative in-depth investigation of students' pathways while working with the linguistically varied word problems also identified important shortcomings: Only some students reflected *deeply* about the impact of the linguistic variation to the situation structure in view, whereas others made only *superficial connections* which tuned out to be less productive for their learning (Dröse & Prediger, 2020).

Summing up, we still agree to Sun (2019) that although students' challenges are well-documented and the variation principle is already used for engaging students in making connections in textbooks (Sun, 2011a, 2011b, 2019) and teacher professional development (Han et al., 2017; Runesson, 2005), there is still a research gap in empirical studies providing in-depth insights into students' work.

In the current study, we develop a conceptual framework for an in-depth analysis in order to investigate students' cognitive processes for operation-choice problems, when varying the problem structure and the numbers involved.

3. Conceptual framework of this paper

For the design of the varied problem sets (instruction), we follow the Bianshi approach. To investigate students' cognitive processes while working in the varied problem sets, we outline a conceptual framework that allow us to unpack the intended modelling pathways in the instruction (i.e., the sequence of cognitive processes intended by the design of the tasks).

3.1. Framework for designing the varied problem sets

The first author and other colleagues (Zorrilla, Ivars et al., 2022) have developed an instruction based on the variation principle (Cai & Nie, 2007; Gu et al., 2004) for both kinds of connections in problem-solving processes of operation-choice word problems: (a) The connection between inverse relationships of multiplication, partitive division and quotitive division problems within the same number type, and (b) the connections across natural numbers and proper and improper fractions within the same structure. Considering these two connections, three variation conditions were combined in the instructional design.

In a first variation condition (Table 1), the design followed Chinese textbooks (Sun, 2011a, 2011b, 2019) for focusing students' attention to the inverse relationships: In the variation format of *one problem – multiple changes* (OPMC), each problem set was constructed by three problems with the same situation but different problem structures, varied with respect to the unknown (Greer, 1992) into a multiplication problem, partitive division problem and quotitive division problem. By discerning that the situation structure and the numbers are the same and only the question for the unknown varies, students were expected to discover and use the inverse relationships. Five varied problem sets were designed and the three problems were presented in different orders.

A second variation condition is documented in Table 2: Across the five varied problem sets, also the involved number types were systematically varied, starting with natural numbers in Problem Set 1, one proper fraction and two natural numbers in Problem Set 2, one improper fraction and two natural numbers in Problem Set 3, two proper fractions and a natural number in Problem Set 4 and two improper fractions and a natural number in Problem Set 5. By this variation, we intended to promote students' transition from natural numbers to proper fractions and improper fractions, as in Chinese textbooks (Sun, 2019). The variation format of *one problem – multiple changes* (OPMC) was crucial across problem sets since we keep the structure of the problem (same unknown) and we change the numbers. Furthermore, the variation format *multiple problems – one solution strategy* (MPOS) aimed at inviting students to continue to choose the same operation with fractions (Cai & Nie, 2007; Gu et al., 2004).

A third variation condition aims at supporting students' "disregard" of the context and rather tries to make students switch to generalizing activities independent of the given context: Between the Problem Sets 1–2, 2–3 and after Problem Set 5, additional extra

Table 1
Problem sets varying the problems' unknown.

Multiplication problem	Partitive division problem	Quotitive division problem
We have [number of groups] packages.	We have [number of groups] packages.	We have some packages.
Each package contains [quantity per group] kilos.	All the packages contain the same number of kilos. We have [total quantity] kilos in total.	Each package contains [quantity per group] kilos. We have [total quantity] kilos in total.
How many kilos do we have in total?	How many kilos does each package have?	How many packages do we have?

Table 2
Complete composition of five problem sets designed with three conditions of variations: *variations of the unknown* (vertically within the column, **2.2, 2.1** show order of presented problems, using ":" for partitive division and "÷" for quotitive division), *variations of involved number types* (horizontal change across the columns), and *variations of context situations and non-matching numbers* (in extra problems marked by **E1.4**).

	Problem Set 1	Problem Set 2	Problem Set 3	Problem Set 4	Problem Set 5
Same situation, other unknowns	3 packages of 4 kg are 12 kg	8 packages of $\frac{1}{2}$ kg are 4 kg	$5\frac{1}{2}$ packages of 2 kg are 11 kg	8 packages of $\frac{1}{10}$ kg are $\frac{4}{5}$ kg	$2\frac{1}{4}$ packages of 2 kg are $4\frac{1}{2}$ kg
Multiplication problem	**1.1M** $3 \times 4 = ?$	**2.3M** $8 \times \frac{1}{2} = ?$	**3.2M** $5\frac{1}{2} \times 2 = ?$	**4.1M** $8 \times \frac{1}{10} = ?$	**5.1M** $2\frac{1}{4} \times 2 = ?$
Partitive division problem	**1.2P** $12 : 3 = ?$	**2.1P** $4 : 8 = ?$	**3.3P** $11 : 5\frac{1}{2} = ?$	**4.3P** $\frac{4}{5} : 8 = ?$	**5.2P** $4\frac{1}{2} : 2\frac{1}{4} = ?$
Quotitive division problem	**1.3Q** $12 \div 4 = ?$	**2.2Q** $4 \div \frac{1}{2} = ?$	**3.1Q** $11 \div 2 = ?$	**4.2Q** $\frac{4}{5} \div \frac{1}{10} = ?$	**5.3Q** $4\frac{1}{2} \div 2 = ?$
Extra problems (other numbers and situations)	**E1.4M** $8 \times 2 = ?$ **E1.5P** $20 : 5 = ?$ **E1.6Q** $24 \div 4 = ?$	**E2.4P** $2 : 20 = ?$ **E2.5Q** $2 \div \frac{1}{4} = ?$ **E2.6M** $10 \times \frac{2}{5} = ?$			**E5.4M** $5\frac{1}{4} \times 3 = ?$ **E5.5P** $8\frac{4}{5} : 2\frac{1}{5} = ?$ **E5.6Q** $7\frac{1}{2} \div 2 = ?$

problems (E1.4-E1.6, E5.4-E5.6) were included that used other context situations and not the same numbers in the three problems. This variation of context situations aimed at encouraging students to extrapolate the discovered structures to other contexts and, thus, generalize the structure (Marton, 2015).

Moreover, students work on these problems in a format of *one problem – multiple solution strategies* (OPMS). Students worked in small groups on each problem set without any specific intervention from the teacher. Then, the teacher shared and discussed different students' strategies in the whole class. During these group discussions, references to the strategies used in previous problems were also made, promoting MPOS and, by connecting the structures of different problems, OPMC. Since Problem Set 2 was students' first encounter with multiplication and division with fractions, the teachers encouraged the students in the whole class discussion to work with informal solution strategies such as counting in units to connect these strategies with the formalized operations.

3.2. Framework for investigating students' modelling pathways

For empirically investigating students' modelling pathways, modelling cycles were developed as conceptual frameworks by which the intermediate steps (problem text, situation structure, chosen operation, result) and the cognitive processes of modelling (structuring, mathematizing, calculating, interpreting, validating) can be articulated.

We adapted the modelling cycle from Fig. 1 (an intended modelling pathway in our study) to capture also the cognitive processes involved in the two types of connections for the transition of meanings for multiplication and division from natural numbers to fractions and decimals (connections across number types, connections between inverse relationships, similar to Zazkis et al., 2008). To

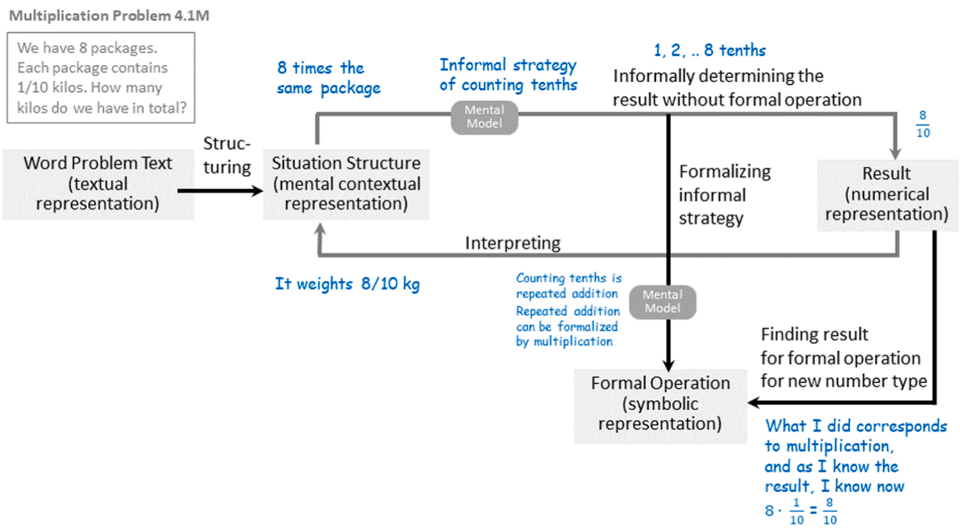


Fig. 2. Intended emergent modelling pathway with cognitive processes while extending their mental model of multiplication as counting in units from natural numbers to the multiplication of a natural number with a fraction supported by the variation format *one problem – multiple solution strategies* (OPMS).

distinguish these meaning-making processes from modelling with already learned meanings, we call these cognitive processes *emerging modelling processes*, borrowing the term from the tradition of Realistic Mathematics Education (Gravemeijer, 2007). While there are distinct differences, our variation approach for emergent modelling shares with Realistic Mathematics Education the key idea that (here, systematically varied) imaginable context problems can support students' construction or extension of meanings and draws upon similar processes as modelling. In our instruction, students are encouraged to draw upon informal strategies and connect them to the newly emerging meanings of operations for fractions and decimals (see also Empson & Levi, 2011). In this way, we can articulate the commonalities and differences between (intended and enacted) *modelling pathways* with operations for which meanings are already learned and *emergent modellings* through which the meanings are to be learned.

Fig. 2 shows an example of an intended emergent modelling pathway for students who do not yet know meanings of the multiplication of a natural number with a fraction. When solving Multiplication Problem 4.1M, they might draw upon informal strategies of counting tenths, and only after that, they realize that this informal strategy corresponds to counting in units of tenths (which they know) and this corresponds to a multiplication, when they (perhaps with the help of the teacher) connect their informal strategies to the established mental model of multiplication as counting in units. In this emergent modelling step, they construct the meaning of multiplication of a natural number multiplied by a fraction by formalizing their informal strategy.

More complex is the articulation of the intended emergent modelling pathways when students are invited to draw upon the inverse relationship between division and multiplication problems (the other connection in our instruction), as this requires to extent the conceptual framework to two problems at the same time. Fig. 3 shows an example of the intended emergent modelling pathway when students solve the Multiplication Problem 2.3M by connecting it to the Partitive Division Problem 2.1P. Students can start with modelling the Partitive Division Problem 2.1P by structuring the situation in the idea to share 4 kg among 8 packages. With the mental model of division as equal sharing, they can choose the division 4 : 8 and solve it by drawing upon their knowledge of fractions as parts of wholes, with the numerical result 1/2. This modelling process can then be used for solving Multiplication Problem 2.3M. For connecting both problems, students might realize that both problem texts refer to the same situation (8 times the same package of 1/2 kg, so 4 kg in total), yet with another unknown. By discovering this commonality and the inverse relationship in the problem texts, they can mathematize the second problem (not by the mental model of multiplication as counting equal groups, but) by the mental model of multiplication as the inverse relationship to division. By this, they can arrive at the intended operation $8 \times 1/2$, perhaps extending multiplication to fractions for their first time. For the new operation just invented, they determine the result not by standardized calculation rules, but by deriving it from the structured situation.

Fig. 2 and Fig. 3 exemplify how the modelling cycle was adapted to provide a conceptual framework for describing some of the intended emergent modelling pathways in students' work in our instruction considering the two connections developed. These intended (emergent) modelling pathways allow us to have a conceptual framework to empirically identify the enacted pathways followed by students when working on the varied problem sets. Thereby, we refine our research question with respect to the conceptual framework:

What modelling pathways do sixth graders enact and articulate while solving operation-choice problems when these are designed in systematically varied problem sets focused on the connection between inverse relationships of multiplication, partitive division and quotitive division problems within the same number type, and the connections across natural numbers and proper and improper fractions within the same structure?

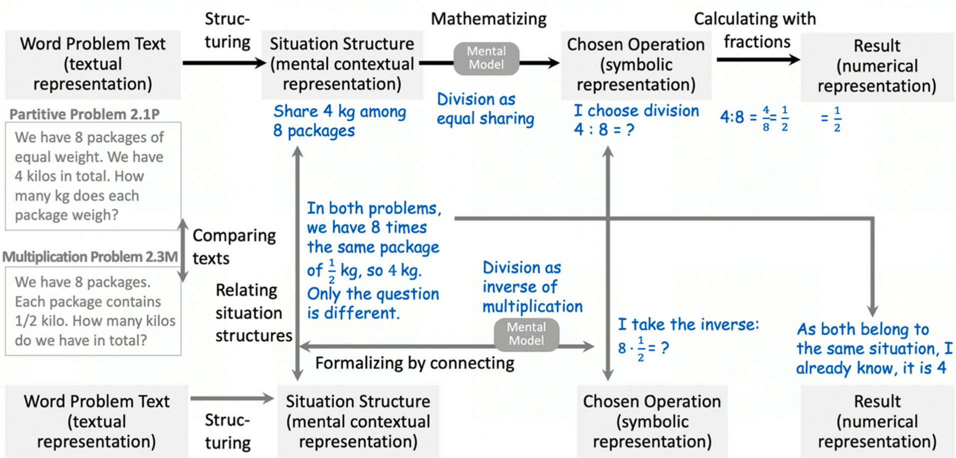


Fig. 3. Intended emergent modelling pathway by connecting varied problems for extending their mental model of multiplication as counting in units from natural numbers to multiplications of a natural number with a fraction, supported by problem variation in format *one problem – multiple changes* (OPMC). (Black arrows document the intended modelling pathway of the first problem 2.1P, grey arrow belong to the intended emergent modelling pathway while solving problem 2.3M).

4. Methodological framework

4.1. Methods of data gathering and focus sample

The instruction was conducted in three regular Grade 6 classes of Spanish Public Primary schools, with in total 61 students (11–12 years old). Prior to the instruction, the students had been introduced to the part-whole meaning of fractions and to the addition, subtraction and multiplication algorithms for fractions and decimal numbers, and division algorithm for decimal numbers. In six sessions (of approximately 45 min each), small groups of 2–4 students worked collaboratively on the worksheets with problem sets (listed in Table 2), followed by whole-class discussions to explore and compare their solution strategies, encourage children to consider other strategies and connect informal strategies with the formal operations. The first author was in charge of the implementation of the sessions.

For the in-depth analysis of students' individual modelling pathways in view of this paper, we selected a focus sample of n = 17 students who worked in six small groups. To ensure that all students had the same order of problem sets and the same learning opportunities in whole-class discussions, the six groups were selected from the same class. We selected the most participatory and talkative small groups to gain insights into their thinking. Other groups were succeeding equally well, but with less explicit cooperation so that their pathways can be only partially traced.

In total, the focus data corpus for the current paper comprises students' worksheets of all 24 problems listed in Table 2 and the audio-recordings of the six sessions 45 min) from six groups (1620 min of audio-recorded small group work on these problems). Although students worked in groups, the analysis, in this paper, focused on what each student articulated in the small group communication.

4.2. Method of data analysis

In order to infer the students' enacted modelling pathways (i.e., the sequence of cognitive processes really initiated in students' work with the tasks), audio-recordings of the six groups were transcribed and analyzed in four steps:

Step 1: Deductive coding of the addressed representations. Each of the students' utterances was coded with respect to the implicitly or explicitly addressed representation: the *textual representation* of the problem text, the *mental contextual or graphical representation* of the situation structure, the *symbolic representation* of the chosen or constructed operation and the *numerical representation* of the result. Additionally, we coded when students made implicit or explicit *reference to other problems* (of the same problem set or of earlier problem sets). Utterances addressing more than one representation were double-coded or divided into parts of the utterance with unique codes. After coding, the turns of the utterances were located in a navigation space (see Fig. 4), a table that is organized in columns according to the addressed representation and in rows according to the chronology from top to bottom.

Step 2: Deductive - Inductive coding of cognitive (emergent) subprocesses. To characterize the students' subprocesses when changing or connecting representations, their utterances were segmented into sense-making units referring to the same ideas (while omitting interjections not carrying to the students' work process). For these sense-making units, we started by deductively coding the cognitive processes (see Figs. 1–3) and inductively identifying the subprocesses informed by the processes (Mayring, 2015). Examples for the identified cognitive subprocesses are listed in Table 3.

Step 3: Identifying pathways as chains of cognitive subprocesses across the navigation space. For the 17 students and 24 problems, Step 1 and Step 2 resulted in 408 chains of cognitive subprocesses in particular cells across the navigation spaces that we call navigations. These navigations were then systematically compared with each other and classified into re-occurring patterns. In this way, we inductively identified re-occurring patterns of navigation that we conceptualize as enacted modelling pathways. This analysis revealed five enacted modelling pathways across the 408 analyzed problem-solving navigations.

Step 4: Categorizing 408 navigations according to the students' enacted modelling pathway. Having identified the five enacted modelling pathways, we were able to classify the complete data corpus of the focus sample. In the last table, we present the

Problem text (textual representation)	Reference to other problems	Situation structure (Contextual or graphical representation)	Chosen operation (Symbolic representation)	Result (Numerical representation)
E1.6Q $24 \div 4 = ?$			Before Turn 239 (individual writing): *Choosing an operation*	Before Turn 239 (individual writing): *Stating result*
				Turn 239-240 *Stating result*
			Turn 241-243 *Choosing an operation*	
				Turn 244-245 *Stating result*

Fig. 4. Analysis of the students' navigation space in Transcript E1.6Q-Group C (Tíscar, Lola, & Jose).

Table 3
Examples of codes for students' subprocesses.

Cognitive processes	Code for some cognitive subprocesses	Explanation
Structuring	Contextualizing the data of the problem	Assigning meaning to the problem number(s) within the contextual representation
	Reporting the mental contextual representation and/or graphical representation	Structuring the problem text in a contextual representation and/or graphical representation
Relating situation structures	Relating the problems	Naming a different problem without making clear what the relation to the problem is based on (context, numbers or mathematical structure)
	Referring to the context of the problems	Naming a different problem referring to a relation based on the context
	Comparing the numbers of the problems	Naming a different problem comparing the occurring numbers of the problems
	Connecting the mathematical structures of the problems	Naming a different problem connecting their mathematical structures
Mathematizing	Symbolizing the context	Representing relationships between quantities in a symbolic way
	Choosing an operation	Naming the chosen operation to solve the problem
Calculating	Converting the symbolic representation	Performing transformations within numerical representations: e.g., from mixed number to decimal number
	Stating result	Naming the numerical result
Interpreting	Contextualizing the operation	Interpreting the numbers and the operation according to the contextual representation (word problem text)
	Contextualizing the result	Interpreting the result according to the contextual representation (word problem text)
Validating	Checking the result according to the number obtained from another problem	Validating the result by comparing the numbers or the contextualized numbers with other word problem text

modelling pathways summarized across the problems. It allows us to study the variety of pathways for each task.

For a subset of the data (about 50 % of the data) the analytic steps were intensively discussed by the first and second author until agreement was achieved. Critical cases were discussed also with the other authors until 100 % of agreement was achieved. Main point here was to understand the discrepancies and come to a consensus, if necessary, by sharpening the categories (Mayring, 2015).

5. Empirical findings

5.1. Enacted modelling pathways identified in the data corpus

We identified five modelling pathways: Pathway of Direct Translation, Pathway of Contextualizing the Chosen Operation, Pathway of Informal Strategies, Pathway of Operational Connection and Pathway of Relational Connection. In this section, we illustrate each of these five modelling pathways by the analysis of one transcript.

Table 4
Coded Transcript E1.6Q-Group C (Tíscar, Lola, & Jose).

E1.6Q	Quotitive Division Problem	"A baker made 24 *Easter cakes* and packed them in boxes of 4 cakes, how many boxes did he need?"			
- 239	Students' individual writing	Tíscar 24 |4 0, 6 6 cajas (6 boxes)	Jose 24|4 0 6 6 cajas de monas (6 boxes of Easter cakes)	Lola 24 ÷ 4 = 6 cajas (6 boxes)	*Choosing an operation* *Stating result*
239	Tíscar:	In Exercise 3 [*Problem E1.6Q from* Table 2], did you put 6?			*Stating result*
240	Jose:	Yes.			
241	Tíscar:	24 divided by 4, right?			*Choosing an operation*
242	Lola:	It's just that it's super easy, 24 divided by 4.			
243	Tíscar:	That's it.			
244	Jose:	6.			*Stating result*
245	Tíscar:	Yes, it's 6.			

5.1.1. *Pathway of direct translation: jump from text to symbolic-numerical representations*

Whereas the intended modelling pathway in Fig. 1 is assumed to go through the stage of a mental contextual representation of the situation structure, many students seem to jump immediately from the problem text to the symbolic and numerical representations.

We exemplify this pathway by a transcript from Session 1 in which Group C compares their individual solutions for the Quotitive Division Problem E1.6Q. In their worksheets, all the three have individually written the division and minimally recontextualized the result 6 by "6 cajas" (Spanish for "six boxes"). Table 4 shows their individual writings, their conversation, and the coding of the cognitive subprocesses. In Fig. 4, the analysis of the students' navigations in the navigation space is shown.

The analysis reveals that the students' conversation concentrates exclusively on the comparison of individual results and chosen operations in the numerical and symbolic representation. First, they compare the numbers they obtained as a result in their individual writing (Turns 239–240) and then they justify their result by naming the corresponding operation (Turns 241–243) that was chosen to obtain the result (Turns 244–245).

A characteristic for this *pathway* is the predominant location of students' cognitive subprocesses in the columns of the symbolic and numerical representation for finding and justifying the result, without addressing the other three columns in the navigation space (see Fig. 4).

5.1.2. *Pathway of contextualizing the chosen operation*

Closer to the intended modelling pathway from Fig. 1 is a second class of navigation that we summarized as the *Pathway of Contextualizing the Chosen Operation.* In this pathway, students (i) directly articulate the chosen operation, but they provide evidence of constructing a mental contextual representation from the problem text that allows them to contextualize the operation or (ii) construct a mental contextual representation from the problem text and then, choose the operation.

We exemplify the case (i) by a transcript from Session 4 in which Pere (from Group E) works on the Multiplication Problem 4.1M (Table 5 and Fig. 5).

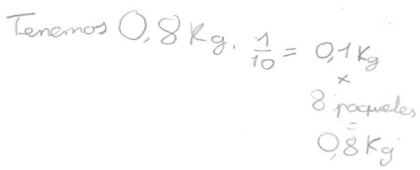
Being confronted with the unknown problem asking for multiplying with a fraction, Pere converts the fraction into a decimal number (Turn 34), the number type in which he knows how to multiply. Moreover, he puts the obtained decimal number in the corresponding context (Turn 34) and stays in this context for applying appropriately the chosen operation (multiplication). Pere states the correct result (Turn 36) and puts it in the context (Turn 38).

A characteristic for this *pathway* is the contextualization that takes place simultaneously or in other cases in short switches between the contextual-graphical representation and the symbolic representation (see Fig. 5). In contrast to the *Pathway of Direct Translation* (Fig. 4), this *pathway* is strongly rooted in the contextual representation. In contrast to the *Pathway of Informal Strategies* (Figs. 6 and 7 below), this pathway is strongly connected with the underlying operations like multiplication or division.

5.1.3. *Pathway of informal strategies*

In our data, we identified a deviation from the intended emergent modelling pathway from Fig. 2 that we termed *Pathway of Informal Strategies.* It deviates from the intended emergent modelling pathway in that it does not connect to multiplications and divisions. Instead, students mainly work (i) in a contextual or graphical representation (Table 6 and Fig. 6) or (ii) start to mathematize (using symbolic representation; Table 7 and Fig. 7) for obtaining or justifying the result without reaching the formal operation.

Table 5
Coded Transcript 4.1M-Group E (Salva, Pere, & Melania).

4.1M	Multiplication problem	"We have 8 packages. Each package contains 1/10 kilos. How many kilos do we have in total?"	
33	Salva:	1 over 10…	
34	Pere:	1 over 10 is equal to 0.1 kg. We multiply this by 8 packages [*Writes down his solution steps*] …  (We have 0.8 kg; 0.1 kg × 8 packages = 0.8 kg)	*Converting the symbolic representation: Converting fraction to decimal number* *Choosing an operation* *Contextualizing the operation*
35	Salva:	And it will give us the result…	
36	Pere:	0.8.	*Stating result*
[…]			
38	Pere:	0.8 kg.	*Contextualizing the result*

Problem text (textual representation)	Reference to other problems	Situation structure (Contextual or graphical representation)	Chosen operation (Symbolic representation)	Result (Numerical representation)
4.1M 8 × 1/10 = ?		Turn 34 *Contextualizing the operation*	Turn 34 *Converting the symbolic representation: Converting fraction to decimal number* *Choosing an operation*	
				Turn 36 *Stating result*
		Turn 38 *Contextualizing the result*		Turn 38 *Contextualizing the result*

Fig. 5. Analysis of the Pere's navigation space in Transcript 4.1M-Group E (Salva, Pere, & Melania).

Problem text (textual representation)	Reference to other problems	Situation structure (Contextual or graphical representation)	Chosen operation (Symbolic representation)	Result (Numerical representation)
3.3P 11 : 5 ½ = ?		Before Turn 14 (Alba's writing) *Reporting the mental contextual representation and/or graphical representation (Informal strategy: Direct modelling)*		Before Turn 14 (Alba's writing) *Stating result*
		Turn 15 *Reporting the mental contextual representation and/or graphical representation* *(Informal strategy: Direct modelling)*		
		Turn 19 *Contextualizing the data of the problem*		
		Turn 21 *Contextualizing the result*		Turn 21 *Stating result*
		Turns 23-25 *Reporting the mental contextual representation and/or graphical representation*		

Fig. 6. Analysis of Alba's navigation space in Transcript 3.3P-Group B (Alba, Bianca, & Merlín).

Focusing on the first case (i), Table 6 shows a transcript from Session 4 in which Alba (from Group B) works on the Partitive Division Problem 3.3P (Table 6 and Fig. 6); while Bianca works on the *Pathway of Operational Connection* (see below in Fig. 8).

Alba draws a model of the 5 ½ packages for which the weight adds to the total amount of 11 kg (before Turn 14). From the drawings and the audio, we assume that she did not take the 2 kg from earlier problems, but determined them by trial-and-error in graphically adding up. In Turn 14, when Bianca asks where the 2 kg come from, she seems to assume that Alba knows the result 2 kg from the earlier problems. In her question, she erroneously uses the word total (in the Spanish original "total", Turn 14) for the result. Alba, who is focused on the situation structure ($2 + 2 + 2 + 2 + 2 + 1$) rather than on the numbers, reacts to the word total, not to where the 2 kg came from. In Turn 15, Alba corrects that the total is 11 kg, but Bianca is totally focused on the numbers. After Bianca strengthens her point of view by reading out the question in the problem text (Turn 20), Alba clarifies the different perspectives by making explicit that the 2 kg is the weight of each package (Turn 21). We assume Alba points to her drawing (not traceable in the audio-recording), because in Turn 22, Bianca explicitly refers to it by "seeing" the total 11. The episode continues in that Alba explains her informal strategy of counting up in a graphical representation for justifying her result 2 kg. However, they do not connect this informal strategy to a formal operation.

In the second case (ii), Table 7 shows a transcript from Session 4 in which Jose (from Group C) works on the Quotitive Division Problem 4.2Q. The difference of this second case considering the previous one is that students start to mathematize from the contextual representation establishing relationships between quantities perhaps even using mathematical symbols. However, still not using the formal operation of multiplication or division.

Problem text (textual representation)	**Reference to other problems**	**Situation structure (Contextual or graphical representation)** Deviation: Mathematizing the situation with the symbolic representation	**Chosen operation (Symbolic representation)**	**Result (Numerical representation)**
4.2Q $4/5 \div 1/10 = ?$		Before Turn 110 (Jose's writing): *Symbolizing the context (Informal strategy: Multiplicative strategy)*		Before Turn 110-Turn 110 *Stating result*
		Turn 114 *Contextualizing the data of the problem*		
		Turns 116-122 *Reporting the mental contextual representation and/or graphical representation*		
		Turn 124 *Contextualizing the result*		Turn 124 *Stating result*

Fig. 7. Analysis of Jose's navigation space in Transcript 4.2Q-Group C (Tíscar, Lola, & Jose).

Jose looks for an equivalent fraction of 4/5 (before Turn 110). Later, Jose explains that he was aware that the denominator had to be "10", as it is the "how much" (Turn 118). We could understand that the expression "how much" would refer to the number of kilos in each package (1/10 is the amount that must be repeated). After the prompts of the teacher, by pointing at the numerator (also in Turn 118), he indicates that this number (8) would be the number of packages. Later, he specifies that 8/10 would represent the total number of kilos (Turn 120). Finally, he specifies that the result is "8 packages", quantifying the number of tenths ($8 \times 1/10 = 8/10$).

The episode shows how Jose from a contextual representation can symbolize his thinking by using equivalent fractions. Nevertheless, this process does not imply the connection of this strategy with the formal operation of division.

Figs. 6 and 7 show the characteristics of the *Pathway of Informal Strategies*: the use of contextual or graphical representations for determining the result or for validating and justifying it. However, without making the connection to the formal multiplications and divisions (column of symbolic representation). This characteristic distinguishes this pathway from the *Pathway of Contextualizing the Chosen Operation* that is strongly connected with the chosen operation (formal multiplication or division).

5.1.4. Pathway of operational connection: using reference to other problems for finding or validating result

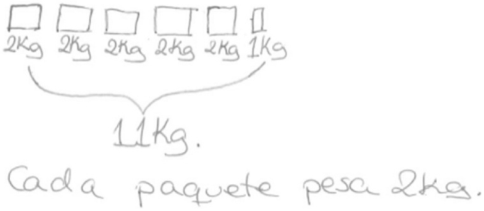
The intended emergent modelling pathway exemplified in Fig. 3 invites students in connecting the problems in order to make use of the inverse relationships between multiplication and division. However, we found a navigation pattern that we classified as a *Pathway of Operational Connection*. In these navigations, students make connections to other problems, but not for relating mathematical problem structures, but for operationally finding or validating the result.

We exemplify the *Pathway of Operational Connection* by a transcript from Session 2 (in Table 8) in which Group B works on the Quotitive Division Problem 2.2Q (n packages $\times$ 1/2 kg = 4 kg) and immediately relate it to the other Problems 2.1P (8 packages $\times$ n kg = 4 kg) and 2.3M (asking for 8 packages $\times$ 1/2 kg = n kg) of the same varied problem set. They use the identified connection to immediately derive the result of Problem 2.2Q in one of the two other problems. After finding the result, they start searching for the operation, yet without any relation to the structure of the situation (Table 8 and Fig. 8).

The students' problem-solving process for the given Problem 2.2Q starts by seeing connections to the Problem 2.1P or 2.3M (Turns 55–68). They explicitly state that they have "the same results" (Turn 56), meaning that the given quantities and the respecting numbers in the different problems are the same. Based on the discovered operational connection, they identify "8 packages" that is a given quantity in another problem as the unknown solution of this problem (Turn 68). After determining the result 8, they start looking for a suitable operation (Turn 71) by which that results in 8. In this regard, Merlín is joking about an arbitrary operation (Turn 77). Merlín's joke is an indicator of his awareness that he should not write an operation with arbitrary numbers. Instead, they search for an operation that contains 4 and 1/2. It is remarkable that they split 1/2 into the two digits 1 and 2 and start a trial-and-error procedure for finding an operation that results in 8 (Turns 81–82 and Turns 85–87). They result not in $4 \div 1/2$, but in an arbitrary combination of digits (which procedurally corresponds to the right operation; see Turns 88–90).

A characteristic for this pathway is the starting point: Students know the result from one of the inverse problems (column of reference to other problems in Fig. 8). Then, they try to find a suitable operation that leads to the already known result by trial-and-error. As shown in the example above this process of finding an operation can happen in a rather arbitrary and not conceptually based way. In these specific navigations, the search for the operation is not inspired by the reflection of the same situation structure.

Table 6
Coded Transcript 3.3P-Group B (Alba, Bianca, & Merlín).

3.3P	Partitive Division Problem	"We have 5 ½ packages. All the packages contain the same number of kilos. We have 11 kilos in total. How many kilos does each package have?"	
-14	Alba's individual drawing	 (Each package weights 2 kg.)	*Reporting the mental contextual representation and/or graphical representation (Informal strategy: Direct modelling)* *Stating result*
14	Bianca:	[…] What are you doing? You know that the total has to be 2, don't you?	
15	Alba:	11 [*corrects Bianca's "total" of 2 kg into the total of 11*]!	*Reporting the mental contextual representation and/or graphical representation (Informal strategy: Direct modelling)*
[…]			
18	Bianca:	Let's see, it has to give 2 kilos, you know?	
19	Alba:	It has to give 11 in total.	*Contextualizing the data of the problem*
20	Bianca:	No, because the total has to be… "How many kilos does each package have?" [*reads out*]. It has to be 2 kilos.	
21	Alba:	2 kilos each package [*presumably referring to her drawing*].	*Stating result* *Contextualizing the result*
22	Bianca:	Yeah, and I see 11 there [*referring to Alba's drawing*].	
23	Alba:	11 in total. Can't you see? 2, 2, 2, 2, 2 and 1.	*Reporting the mental contextual representation and/or graphical representation*
24	Bianca:	That's not 2. That's half.	
25	Alba:	That's it. That's why it's 1.	
26	Bianca:	Ah, you've put a 1…	
[…]			
44	Bianca:	So, 2… Right, that's why… And the statement says [*reads out*], "How many kilos does each package have?" So, 2 kilos.	

5.1.5. Pathway of relational connection: using reference to other problems for finding similar structures

The intended emergent modelling pathway exemplified in Fig. 3 invites students in connecting the problems in order to make use of the inverse relationships between multiplication and division. We found a navigation pattern that we classified as a *Pathway of Relational Connection* in which students use indeed the reference to other problems for finding similar structures of problems.

We exemplify the *Pathway of Relational Connection* by a transcript from Session 1 (in Table 9) in which Tíscar and Jose (from Group C) work on the Partitive Division Problem 1.2P and relate $12 : 3 = ?$ to $3 \times 4 = 12$ from the Problem 1.1M of the same problem set (Table 9 and Fig. 9).

At the beginning of this transcript, Jose realizes that between Problem 1.2P and Problem 1.1M, there exists a connection. He articulates a first idea of this connection by referring to the mathematical structure of inverse operations, in that case, a division as the inverse of a multiplication (Turn 13). Also, Tíscar notices a connection between the problems in a more contextual mental representation (Turn 23). Then, Jose focuses on the result of the given problem (Turn 24). Lola justifies the result by referring to Problem 1.1M (Turn 25). With the prompt of the teacher (Turn 38), Jose repeats his idea (from Turn 13) by "it's all backwards" (Turn 39) and refers that they have all the data of the situation ("you've got it all"). Tíscar brings in again a focus on the structural connections (inverse relationship) repeating both, the division $12 : 3$ and the multiplication 4×3 (Turn 40).

Although this pathway looks similar to the *Pathway of Operational Connection* at a first glance, as it also contains references to the other problems, it is characterized by one crucial difference: the reference to other problems is not restricted to the result, but on the multiplicative structure (inverse relationship between multiplication and division), approaching the intended emergent modelling pathway exemplified in Fig. 3.

5.2. Prevalence of the enacted modelling pathways identified

In total, we have identified five modelling pathways. Most of the analyzed 408 navigations could be subsumed under one of these pathways. In this subsection, we summarize the prevalence of the enacted modelling pathways identified.

In total, 158 out of the 408 navigations were categorized as *No communication*, when students did not communicate enough to classify their navigations in any of the pathways. The remaining 250 navigations were categorized under one or more of the pathways. Table 10 shows the prevalence of pathways within problems and across problems. The sum of occurrences was 276, since 24 navigations were categorized by two pathways and one by three pathways.

The most frequent pathway is the *Pathway of Direct Translation* which was identified 126 times, followed by the *Pathway of Contextualizing the Chosen Operation*. The least frequent is the pathway for one of the two connections for which the design was originally developed: *Pathway of Relational Connection*. In contrast, students made operational connections (*Pathway of Operational Connection*) in 47 navigations.

Table 10 reveals also insights into the variation of the prevalence of the pathways along the problem sets. In Problem Set 1 (1.1M–1.3Q and E1.4M–E1.6Q), as long as the problems only contained natural numbers, the students often moved directly to the symbolic representation and the result, at least in their articulation (*Direct Translation*).

References to other problems to validate or justify the result of the problem (*Operational Connection*) become more relevant with the introduction of fractions in Problem Set 2 (2.1P–2.3M with one proper fraction). This pathway is less frequent in the other problem sets, in which *Contextualizing the Chosen Operation* is more frequent, reflecting a greater need to contextualize the symbolic representation for several fractions and improper fractions.

The *Pathway of Informal Strategies* appears more frequently in the extra problems. This shows the need of students to work in a contextual or graphical representation or start to mathematize using the symbolic representation for obtaining or justifying the results when they have to generalize their knowledge to other context situations.

6. Discussion

6.1. Intended (emergent) modelling pathways versus enacted modelling pathways

Five students modelling pathways were identified that provide information related to student's cognitive processes used in the varied problem sets and thereby allow to compare them with the intended modelling pathways for which the design was developed: (i) *Direct Translation pathway*: students jump immediately from the problem text to the symbolic and numerical representations; (ii) *Contextualizing the Chosen Operation pathway*: students directly articulate the chosen operation, but they construct a mental contextual representation from the problem text that allows them to contextualize the operation; (iii) *Informal Strategies pathway*: students work in a contextual or graphical representation or start to mathematize using the symbolic representation for obtaining or justifying the results without achieving the formal operations; (iv) *Operational Connection pathway:* students reference to other problems for finding or validate the result without understanding the situation structure; and (v) *Relational Connection pathway*: students reference to other problems for identifying similar structures. We need to emphasize that these are the communicational pathways, not necessarily the students' mental pathways. The thinking might have been richer than what was articulated in the communication.

Fig. 10 reveals the comparison between the intended (emergent) modelling pathways in the design and the modelling pathways empirically identified when students work in the problem sets. Whereas the *Relational Connection pathway* corresponds to an intended emergent modelling pathway, the other pathways deviate from the intended pathways in varying degrees and with varying consequences for the learning, so disentangling these backgrounds in this analytic summary allows us to make explicit in how far the findings not only reveal descriptive, but also explanative empirical outcomes.

Within the originally intended modelling pathway, students go from the textual representation of the contextual problem to the chosen operation by structuring first a mathematical model of the situation (Vom Hofe & Blum, 2016). The pathways of *Direct Translation* and *Contextualizing the Chosen Operation* are deviations from this intended modelling pathway. In *Direct Translation*, students do not draw upon the cognitive processes of structuring and interpreting. In *Contextualizing the Chosen Operation*, students might also first jump to the chosen operation, but then construct a mental contextual representation from the problem text that allows them to contextualize their choice, that means, structuring and interpreting occur, but later than expected.

The pathway of *Direct Translation* has been problematized in the literature as underlying many inaccurate modelling decisions due to a superficial focus on numbers and key terms that "define" relationships (Hegarty et al., 1995). These inaccurate modelling decisions may also be influenced by intuitive tendencies (e.g., multiplication always makes bigger; Fischbein et al., 1985) that unconsciously interfere with correct reasoning. In our analysis, some students' navigations were classified as *Direct Translation* although their written documents give evidence of successful problem-solving activities in this pathway with a deep mental contextual representation, but they seem to have felt no need to verbalize their contextual representation. These students' written products give also evidence for their mental capacity to control the intuitive tendencies, reflecting and monitoring the different mental processes (i.e., *inhibitory control*, Van Dooren & Inglis, 2015), even if the communication does not reflect this capacity.

Additionally, also the pathway of *Contextualizing the Chosen Operation* is an extension of existing findings on *Direct Translation*, as many students seem to structure the situation before or after choosing the operation. It can be seen as a step in a productive direction as students engage in reflections about the meanings in the situation structure (Borromeo Ferri, 2006).

The pathway of *Informal Strategies* is an important *part* of the originally intended emergent modelling pathway: It was intended to use invariance of number types across problem sets to encourage students to draw upon informal strategies (Empson & Levi, 2011). But

Table 7
Coded Transcript 4.2Q-Group C (Tíscar, Lola, & Jose).

4.2Q	Quotitive Division Problem	"We have some packages. Each package contains 1/10 kilos. We have 4/5 kilos in total. How many packages do we have?"	
-110	Jose's individual writing	$\underbrace{\overbrace{\frac{4}{5} = \frac{8}{10}}^{\times 2}}_{\times 2}$ Hay 8 paquetes (There are 8 packages.)	*Symbolizing the context* *(Informal strategy: Multiplicative strategy)* *Stating result*
110	Jose:	4.5 [*incorrectly indicating "4/5"*] into 8.	*Stating result*
111	Lola:	Can I see?	
112	Jose:	Yes, I've done equivalent fractions.	
113	Teacher:	What have you done here?	
114	Jose:	I've been looking for equivalent fractions. 4 over 5 is how many kilos there are.	*Contextualizing the data of the problem*
115	Teacher:	Yes.	
116	Jose:	So, if I multiply it by 2, that gives 10, which is this [*presumably pointing at "1/10"*].	*Reporting the mental contextual representation and/or graphical representation*
117	Teacher:	Yes.	
118	Jose:	This is. So to speak. "How much", so to speak. And this [*presumably pointing at the numerator*] is the packages.	
119	Teacher:	I mean, what?	
120	Jose:	This [*presumably pointing at "8/10"*] is what all the packages weigh. A fraction. I mean.	
121	Teacher:	A fraction, yes.	
122	Jose:	So, 10 over 1.	
123	Teacher:	A tenth.	
124	Jose:	A tenth. 8 tenths would be 8 packages.	*Stating result* *Contextualizing the result*

the design intended to elicit informal strategies for the purpose of connecting them to the newly emerging mental models of multiplication and division when students move from natural numbers to fractions (see Fig. 10). In contrast to these intentions, students worked in a contextual or graphical representation or started to mathematize using the symbolic representation for obtaining or justifying the results without connecting them with the new formal operation model (multiplication or division). According to Gravemeijer (2007), *emergent modelling* points to an activity based on the construction of new mathematical ideas, in our case, the formal models of multiplication and division with fractions. In this activity, modelling and symbolizing are interrelated. Our results document how students model their own informal strategies. However, symbolization is not necessarily linked to a formal model, so students can mathematize their informal strategies by using symbolic representations. Although students do not connect their approach to the formal operation of multiplication or division in this pathway, they seem to start structuring the situation with graphical representations and mathematizing the relations between quantities by symbols. This can be interpreted as indication that they have found other productive ways of solving the problems when they are working with fractions. From this interpretation, we argue that this pathway is an important first step toward the connection of mathematizing-interpreting processes that is key to the intended emergent modelling pathway (Fig. 2).

One of the intended emergent modelling pathway is based upon inverse relationships and focused on relating situation structures by discovering the commonality between structures and using the mental model of multiplication as the inverse relationship to division (Downton, 2013). In contrast, the identified pathway of *Operational Connection* deviates from this intended emergent modelling pathway in that students do not relate situation structures, but use the result known from one of the inverse problems and, in many cases, they try to find a suitable operation that leads to the already known result by trial-and-error. This pathway can be interpreted as

Problem text (textual representation)	Reference to other problems	Situation structure	Chosen operation (Symbolic representation)	Result (Numerical representation)
2.2Q 4 ÷ 1/2 = ?	Reference to **2.3M** (8 × 1/2 = ?) **2.1P** (4 ÷ 8 = ?) Turns 55-56 *Comparing the numbers of the problems*			
	Turn 68 *Comparing the numbers of the problems* *Referring to the context of the problems*			Turn 68 *Stating result* *Contextualizing the result*
			Turns 81-85 *Choosing an operation*	
	Turns 86-87 *Checking the result according to the number obtained from another problem*			Turns 86-87 *Checking the result according to the number obtained from another problem*
			Turns 88-90 *Choosing an operation*	

Fig. 8. Analysis of the group's navigation space in Transcript 2.2Q-Group B (Bianca, Alba, & Merlín).

Table 8
Coded Transcript 2.2Q-Group B (Bianca, Alba, & Merlín).

2.2Q	Quotitive Division Problem	"We have some packages. Each package contains 1/2 kilos. We have 4 kilos in total. How many packages do we have?"	
55	Bianca:	It says it here.	*Comparing the numbers of the problems*
56	Alba:	It's the same. It's the same results, but…	
[…]			
68	Bianca:	[…] "We have 4 kilos in total… How many packages…?" Well… Yes! It says it here: "We have 8 packages!"	*Comparing the numbers of the problems* *Referring to the context of the problems* *Stating result* *Contextualizing the result*
[…]			
71	Alba:	So what? It's just that we have to do the operation.	
[…]			
77 […]	Merlín:	Yeah, and what do we do? 5 plus 4 minus 1 [*joking*]?	
81	Bianca:	It's 4 multiplied by 1 divided by 2.	*Choosing an operation*
82	Merlín:	What did you do? What do you mean, 4 multiplied by 1…?	
[…]			
85	Bianca:	It's just that to know the result you have to make 4 multiplied by one divided by two.	
86	Alba:	What? That doesn't give 8…	*Checking the result according to the number obtained from another problem*
87	Bianca:	4 divided by 2? Ah, no. It doesn't give 8.	
88	Alba:	No, it's not like that… It's 4 multiplied by 2 divided by 1.	*Choosing an operation*
89	Bianca:	Oh, yeah?	
90	Alba:	Yes… 4 multiplied by 2 divided by 1.	

Table 9
Coded Transcript 1.2 P-Group C (Tíscar, Lola, & Jose).

1.2P	Partitive Division Problem	“We have 3 packages. All the packages contain the same number of kilos. We have 12 kilos in total. How many kilos does each package have?”	
13a/b	Jose:	But, well, if this is the other way around… The next problem is to do the inverse. 12 divided by 3.	*Connecting the mathematical structures of the problems* *Choosing an operation*
[…]			
23	Tíscar:	Well, it’s true, it’s the same thing. If he’s already telling you here. “How many kilos does each package have?” [*reading out the text of Problem 1.2P*].	*Comparing the numbers of the problems* *Referring to the context of the problems*
24	Jose:	In the second one [*referring to Problem 1.1M*] it’s 4 packages, just so you know.	*Stating result*
25	Lola:	Yes, because it says so in the first one [*referring to Problem 1.1M*].	
26	Tíscar:	It says it in the first one.	
27	Jose:	It’s the other way around.	*Connecting the mathematical structures of the problems*
[…]			
38	Teacher:	Did you talk about it?	*Relating the problems*
39	Jose:	It’s just that it’s all backwards, I mean, it’s… With the first exercise you’ve got it all.	
40	Tíscar:	12 divided by 3… 4 times 3… 12 divided by…	*Connecting the mathematical structures of the problems* *Choosing an operation*
[…]			
50	Tíscar:	Lola, I think it’s the other way round. Because if you do 3 divided by 12, it’s different.	*Choosing an operation*
51	Lola:	So, 12 divided by 3?	
52	Tíscar:	Yes, it’s because it would give you commas [*Spanish expression for decimal points*]…	
53	Lola:	Yeah, yeah…	

an indication of only superficial recognition of the connection between the situation structure and the operation needed.

Whereas all students started first steps in the intended emergent modelling pathway based upon invariance of number types across problem sets (with pathways of *Informal Strategies* or *Operational Connection*), only few of them completed the intended emergent modelling pathway based upon inverse relationships (n = 9, *Relational Connection Pathway*) with an explicit focus on situation structures to be formalized. This rarely observed pathway substantiates the original design intentions that the problem sets designed according to the principle of variation could built a “space of conceptual relations” (Sun, 2011b, p. 77). In this space, the comparison between problems is the primary condition to identify structures and relations (Sun, 2011a). Nevertheless, the low frequencies obtained indicate that the learning of new mental models for operations requires deeper thinking and perhaps more initial support by collective reflections. In this way, the analysis leads to identifying affordances and critical conditions of success for the implementation

Problem text (textual representation)	**Reference to other problems**	**Situation structure (Contextual or graphical representation)**	**Chosen operation (Symbolic representation)**	**Result (Numerical representation)**
1.2P 12 : 3 = ?	Reference to **1.1M** 3 × 4 = ? Turn 13 *Connecting the mathematical structures of the problems*		Turn 13 *Choosing an operation*	
	Turns 23-26 *Comparing the numbers of the problems* *Referring to the context of the problems*	Turns 23-26 *Referring to the context of the problems*		Turn 24 *Stating result*
	Turn 27 *Connecting the mathematical structures of the problems*			
	Turns 38-39 *Relating the problems*			
	Turn 40 *Connecting the mathematical structures of the problems*		Turn 40 *Choosing an operation*	
			Turns 50-53 *Choosing an operation*	

Fig. 9. Analysis of Tíscar’s and Jose’s navigation space in Transcript 1.2P-Group C (Tíscar, Lola, & Jose).

Table 10
Frequency of identified pathways in 250 navigations, listed by problem.

		Direct Translation	Contextualizing the Chosen Operation	Informal Strategies	Operational Connection	Relational Connection
Problems	1.1M	10	2			
	1.2P	5	2		5	2
	1.3Q	10	1		3	2
	E1.4M	7	2	2		
	E1.5P	3	1	1		1
	E1.6Q	10	1	1		
	2.1P	6	6		5	
	2.2Q	2	6		7	
	2.3M	5	4	1	4	3
	E2.4P	2		9		
	E2.5Q	3	5	3		
	E2.6M	2	7	1		
	3.1Q	10	2	1	1	
	3.2M	4	4		1	
	3.3P	3	4	1	4	1
	4.1M	4	4		2	
	4.2Q	3	2	1	2	
	4.3P	1	1		5	
	5.1M	6	4	2		
	5.2P	4			4	
	5.3Q	8		1	3	
	E5.4M	7	4			
	E5.5P	6	4	1	1	
	E5.6Q	5	1	2		
Total		126	67	27	47	9

of the design principle as discussed in the following section.

6.2. Implications for the design: varied problem sets to connect inverse problems for the same multiplicative situations and problems with natural numbers to problems with rational numbers

Our results have highlighted that as expected, problems containing only natural numbers were easy to solve for students, so they often took *Direct Translation pathways* when more elaborate modelling was not necessary for them (Borromeo Ferri, 2006). In contrast, confronted with new problems with fractions, students followed other pathways such as the *Operational Connection pathway* or the *Relational Connection pathway*. The occurrence of these two pathways can be interpreted as a first indicator for the functioning of our design principle of variation: As intended, the problem structure of *one problem, multiple changes* (OPMC; Sun, 2019) provided students with opportunities to connect operations, sometimes for example for validating the result.

However, the prevalence of the *Operational Connection pathway* also indicates a critical condition of success, as it draws only upon superficial relationships (trial-and-error), where the major potential of OPMC unfolds with the *Relational Connection pathway*. Furthermore, many students started their work with fractions with pathways of *Contextualizing the Chosen Operation* or *Informal Strategies*. In both pathways, they worked in contextual or graphical representations as the mathematization into symbolic representation was not straightforward. The high prevalence of these two pathways can be interpreted as indicators for the functioning of *one problem, multiple solution strategies* (OPMS) that can invite the use of different informal strategies. The same applies for the variation format *multiple problems, one solution strategy* (MPOS) that seem to have invited students to continue to choose the same strategy with fractions (independently of the numbers given). In this way, the analysis helped us to generate not only descriptive but also explanative findings, allowing to disentangle students' thinking and possible backgrounds in depth of their transitioning between representations and number domains.

The identified modelling pathways can inform the instructional designers and teachers how to support the development of multiplicative structures in the transition from natural numbers to fractions.

The two kinds of connections used in the instruction focused on the inverse relationships of multiplication, partitive division and quotitive division problems within the same number type, and the connection across natural numbers and proper and improper fractions within the same structure allow us to identify key features of modelling processes. The features of the pathway of *Informal Strategies* and *Contextualizing the Chosen Operation* can be considered as starting point in the development of relational thinking in the intended pathway of *Relational Connection*, and their explicit identification can support teachers' future work in leveraging students' ideas to these pathways (Empson & Levi, 2011).

This is important, as the analysis also revealed that working with varied problem sets can simplify students' modelling processes. In the *Direct Translation Pathway,* students chose the operation without verbalizing the mental contextual situation, but it does not mean that they did not have it in their minds. Teacher who are aware of these findings can deliberately elicit these implicit ideas.

Moreover, the *Operational Connection Pathway* shows how students do not reflect on the contextual situation or on the mathematical structure of the problem. Many students on this pathway use a trial-and-error strategy to find the formal operation that has as a result the number they know from the other inverse problems; thus, they evidenced a superficial recognition of the connection between the

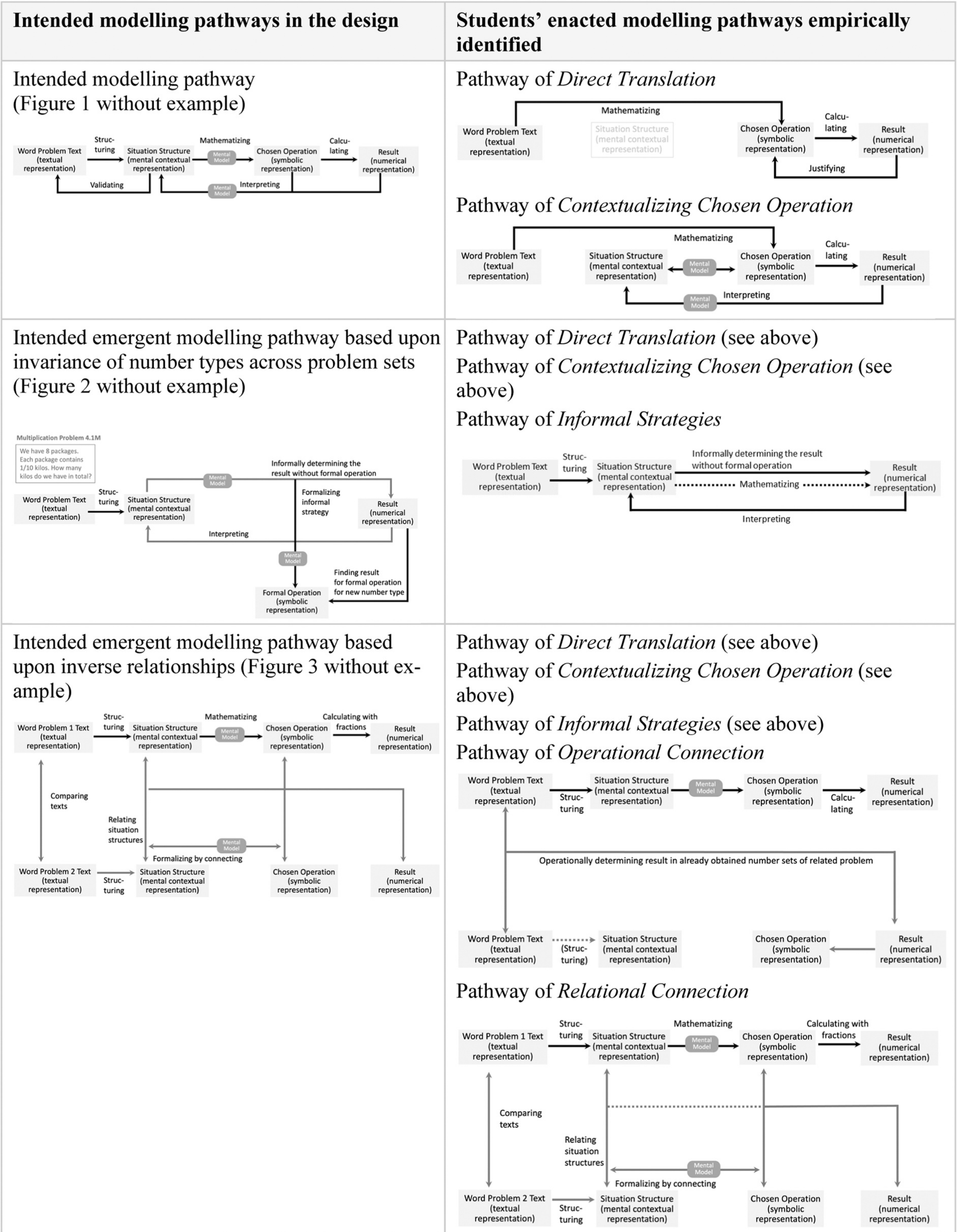


Fig. 10. Overview of intended modelling pathways and students' enacted modelling pathways empirically identified.

situation structure and the operation needed. Designers and teachers aware of this pathway can be alert to support students by collective reflections to deepen their processes and to successively learn to draw *Relational Connections*.

6.3. Methodological limitations and outlook to future research

The findings are strongly tied to the context condition of teaching and learning, e.g., the particular chosen problems, and the particular selected focus students. The selection of the more talkative small groups of students can be seen as a limitation of the study. Nevertheless, we might suspect that the less talkative groups would have produced less deep or untraceable pathways, so we believe to have identified the upper limit of effects.

Moreover, this paper only reports on an instruction with six sessions. Future instructions can build on our findings for a refinement of the design. For instance, first, the frequency of superficial pathways such as the *Operational Connection* or *Direct Translation pathway* reveals the need to extend the sessions of the instruction, since five sets of problems do not seem sufficient to introduce connections on inverse relationships in multiplicative structure problems and connections about these problems with natural and rational numbers. Furthermore, the design could be enriched by further in class discussions about the different representations and their connections. Second, the teacher should guide students with some prompts to promote reflection for drawing connections on inverse relationships between problem structures.

For future work, it would be interesting to analyze how the prompts of the teacher and other students of the group influence the pathways used by the students. We hypothesize that if the pathway followed by the students allows them to obtain the result of the problem, they would probably not modify the pathway used to a more meaningful one. For that, students need reasons to modify their pathway (Hegarty et al., 1995). Furthermore, the fact that the students go through different pathways when solving the different sets of problems reveals the need to examine the students' modelling pathways in each task to identify the changes in the students' pathways along the instruction.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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