A COMBINATORIAL INTEGRATION ON THE CANTOR DUST

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ABSTRACT. In this paper, we generalize the Cantor function to 2-dimensional cubes and construct a cyclic 2-cocycle on the Cantor dust. This cocycle is non-trivial on the pullback of the smooth functions on the 2-dimensional torus with the generalized Cantor function while it vanishes on the Lipschitz functions on the Cantor dust. The cocycle is calculated through the integration of 2-forms on the torus by using a combinatorial Fredholm module.

INTRODUCTION

Cyclic cohomology for algebras [2] is a fundamental tool to study non-commutative geometry. One of its application is the study of fractal sets to which powerful tools such as de Rham homology (on smooth manifolds) cannot be applied. K-homology, one of other homotopy invariant theories, of fractal sets is also studied extensively. For some class of fractal sets such as the Cantor set and the Cantor dust, the K^0 homology groups are isomorphic to $\prod_{}^{\infty} \mathbb{Z}$, which is known as the Baer-Specker group [1]. One feature of the Baer-Specker group is that the group does not admit basis. This feature makes the study of K^0 groups for such fractal sets somewhat intractable because local topological features of the spaces cannot induce local algebraic structures. Cyclic cohomology on the one hand is expected to be favorable in these cases because it can be characterized as a "linearization" of K-homology through the Chern character.

There is a study [4] by H. Moriyoshi and T. Natsume which presented a variant of the Riemann-Stieltjes integration on the middle third Cantor set. Let $CS = \bigcap_{n=0}^{\infty} I_n$ be the middle third Cantor set, where $I_0 = [0,1]$, $I_1 = \left[0, \dfrac{1}{3}\right]$, $I_2 = \left[0, \dfrac{1}{3^2}\right] \cup \left[\dfrac{2}{3^2}, \dfrac{3}{3^2}\right] \cup \left[\dfrac{6}{3^2}, \dfrac{7}{3^2}\right] \cup \left[\dfrac{8}{3^2}, 1\right], \dots$. We also denote by (H_n, F_n) the Fredholm module on I_n which is defined by the direct sum of Connes' Fredholm module on intervals. In [4], a new class of algebra is introduced: an algebra $\mathcal{P} = c^* BV(S^1)$ defined by the pull-back of the bounded variation class on a unit circle $BV(S^1)$ with the canonical Cantor function c.

Then they define a functional $\psi(f,g) = \lim_{n\to\infty} \psi_n(f,g) = \lim_{n\to\infty} \mathrm{Tr}(\epsilon f[F_n,g]F_n)$ on $\mathcal{P}$. The existence of the limit is proved by showing that the cocycle is reduced to the Riemann-Stieltjes integration. A key ingredient for the proof is the Cantor function (and its intermediate functions that converge into the Cantor function); the function is used to map the iterated function system of the Cantor set onto 2^n-fold subdivisions of the unit interval on which the Riemann-Stieltjes integration is defined. Because the Cantor function remains surjective onto the unit interval when the function is restricted to the Cantor set, the iterated function system for the Cantor set may be seen as a variant of subdivisions on I.

In the present paper, by following the idea mentioned above, we construct a new cyclic 2-cocycle on the Cantor dust CD and show that the cocycle is not trivial. In order to construct the cocycle, we define a sequence of functionals ϕ_n (see Definition 1), that is a generalization of a functional ψ_n on CS, by using a combinatorial Fredholm module on squares defined by the authors [3] and the product of the Cantor function. The function induces the map $\boldsymbol{c}$ between the Cantor dust and 2-torus $\mathbb{T}^2$:

Theorem (see Theorem 3.3). *Let $C^\infty(\mathbb{T}^2)$ be the smooth functions on the torus $\mathbb{T}^2$. For $f = \boldsymbol{c}^*(\tilde{f}), g = \boldsymbol{c}^*(\tilde{g}), h = \boldsymbol{c}^*(\tilde{h}) \in \boldsymbol{c}^*C^\infty(\mathbb{T}^2)$, we have*

$$\lim_{n\to\infty} \phi_n(f,g,h) = 2 \int_{\mathbb{T}^2} \tilde{f}d\tilde{g} \wedge d\tilde{h}.$$

Moreover, the limit only depends on $f,g,h \in \boldsymbol{c}^*C^\infty(\mathbb{T}^2)$. Therefore, the functional ϕ defined by the limit is a cyclic 2-cocycle on $\boldsymbol{c}^*C^\infty(\mathbb{T}^2)$.

We can take the value $\phi_n(f,g,h)$ for Lipschitz functions $f,g,h \in C^{\mathrm{Lip}}(CD)$ by the definition of ϕ_n. However, we have $\lim_{n\to\infty} \phi_n(f,g,h) = 0$ for any $f,g,h \in C^{\mathrm{Lip}}(CD)$ (see Proposition 3.2). Thus the algebra $\boldsymbol{c}^*C^\infty(\mathbb{T}^2)$ is far different from $C^{\mathrm{Lip}}(CD)$.

All the main results to be shown also hold for the Sierpinski carpet. $\mathbb{T}^2$ can be also replaced with a 2-dimensional sphere. In order to generalize our main theorem to general dimension, we need a general method to prove the existence of a cyclic cocycle. We will leave the study for future work.

1. A COMBINATORIAL FREDHOLM MODULE ON SQUARES

In this section, we review a combinatorial Fredholm module on squares constructed by the authors [3]. Let $\gamma \subset \mathbb{R}^2$ be a square of dimension 2 and $V = \{v_0,\ v_1,\ v_2,\ v_3\}$ the set of vertices of γ; see the following figure for the numbering of the vertices.

Set $V_0 = \{v_0,\ v_2\}$ and $V_1 = \{v_1,\ v_3\}$, so we have $V = V_0 \cup V_1$. Set

$$\mathcal{H}^+ = \ell^2(V_0) = \ell^2(v_0) \oplus \ell^2(v_2), \qquad \mathcal{H}^- = \ell^2(V_1) = \ell^2(v_1) \oplus \ell^2(v_3)$$

$$
\begin{array}{ccc}
v_3 & \text{------} & v_2 \\
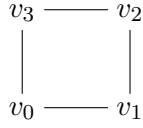| & & | \\
v_0 & \text{------} & v_1
\end{array}
$$

FIGURE 1. Numbering of the vertices

and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. The vector space $\mathcal{H}(\cong \mathbb{C}^4)$ is a Hilbert space of dimension 4 with an inner product

$$\langle f, g \rangle = \sum_{i=0}^{3} f(v_i) \overline{g(v_i)}.$$

We assume that $\mathcal{H}$ is $\mathbb{Z}_2$-graded with the grading $\epsilon = \pm 1$ on $\mathcal{H}^{\pm}$, respectively. The C^*-algebra $C(V)$ of continuous functions on V acts on $\mathcal{H}$ by multiplication:

$$\rho(f) = (f(v_0) \oplus f(v_2)) \oplus (f(v_1) \oplus f(v_3)).$$

Set $U = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} & U^* \\ U & \end{bmatrix} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} & & 1 & 1 \\ & & -1 & 1 \\ 1 & -1 & & \\ 1 & 1 & & \end{bmatrix}$. Then

F is a bounded operator on $\mathcal{H}$ and we have $F\epsilon + \epsilon F = O$. So $(\mathcal{H}, F)$ is the Fredholm module on $C(V)$.

Set $I = [0,1] \times [0,1]$ and let $f_s : I \to I$ $(s = 1, \ldots, N)$ be similitudes. Denote by

$$r_s = \frac{\|f_s(x) - f_s(y)\|_{\mathbb{R}^n}}{\|x - y\|_{\mathbb{R}^n}} \quad (< 1) \quad (x \neq y)$$

the similarity ratio of f_s. An iterated function system (IFS) $(I, S = \{1, \ldots, N\}, \{f_s\}_{s \in S})$ defines the unique non-empty compact set $K = K(\gamma_n, S = \{1, \ldots, N\}, \{f_s\}_{s \in S})$ called the self-similar set such that $K = \bigcup_{s=1}^{N} f_s(K)$. Denote by $\dim_S(K)$ the similarity dimension of K, that is, the number s that satisfies

$$\sum_{s=1}^{N} r_s^s = 1.$$

If an IFS $(I, S, \{f_s\}_{s \in S})$ satisfies the open set condition, we have $\dim_H(K) = \dim_S(K)$, where we denote by $\dim_H(K)$ the Hausdorff dimension of K.

Set $f_{\boldsymbol{s}} = f_{s_1} \circ \cdots \circ f_{s_j}$ for $\boldsymbol{s} = (s_1, \ldots, s_j) \in S^{\infty} = \bigcup_{j=0}^{\infty} S^{\times j}$ and $f_{\emptyset} = \text{id}$. For the simplicity, we will denote by i the vertex $f_{\boldsymbol{s}}(v_i)$ of a square $f_{\boldsymbol{s}}(I)$. We also denote by $V_{\boldsymbol{s}}$ the vertices of a square $f_{\boldsymbol{s}}(I)$. Denote by $e_{\boldsymbol{s}}$ the length of edge of $f_{\boldsymbol{s}}(I)$, which equals $\prod_{k=1}^{j} r_{s_k}$. As introduced above, we set

the Hilbert space $\mathcal{H}_{\boldsymbol{S}} = \ell^2(V_{\boldsymbol{S}})$ on $f_{\boldsymbol{S}}(I)$ of the length $e_{\boldsymbol{S}}$, which splits the positive part $\mathcal{H}_{\boldsymbol{S}}^+$ and the negative part $\mathcal{H}_{\boldsymbol{S}}^-$. Taking direct sum on squares, we define

$$\mathcal{H}_n = \bigoplus_{j=1}^{n} \bigoplus_{\boldsymbol{S} \in S^{\times j}} \mathcal{H}_{\boldsymbol{S}} \text{ and } F_n = \bigoplus_{j=1}^{n} \bigoplus_{\boldsymbol{S} \in S^{\times j}} F.$$

The C^*-algebra $C(K)$ acts on $\mathcal{H}_n$ by multiplication:

$$\rho_n(f) = \bigoplus_{j=1}^{n} \bigoplus_{\boldsymbol{S} \in S^{\times j}} \rho(f).$$

2. Review on the Cantor dust

We introduce an IFS of the Cantor dust and 2-dimensional analogue of the Cantor function defined on a 1-dimensional interval. The IFS of the Cantor dust is our main interest in the paper. The analogue of the Cantor function plays a crucial role in construction of cyclic cocycle in Section 3. Let $I = [0,1] \times [0,1]$. The IFS of the Cantor dust is defined as a set of functions $\{f_s : I \to \mathbb{R}^2\}_{s=1,2,3,4}$ described as follows:

$$f_1(x) = \frac{1}{3}x, \ \ f_2(x) = \frac{1}{3}x + \frac{1}{3}\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ \ f_3(x) = \frac{1}{3}x + \frac{1}{3}\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \ \ f_4(x) = \frac{1}{3}x + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

This IFS induces a unique non-empty compact set CD in I such that $CD = \bigcup_{i=1}^{4} f_s(CD)$ holds; CD is called the Cantor dust.

Let $c_0 = x : [0,1] \to \mathbb{R}$. We define in an inductive manner a sequence of $\mathbb{R}$-valued continuous functions defined on the unit interval $\{c_n\}_{n\in\mathbb{N}}$ as follows:

$$c_{n+1}(x) = \begin{cases} \frac{1}{2}c_n(3x), & 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{2}, & \frac{1}{3} \leq x \leq \frac{2}{3}, \\ \frac{1}{2}c_n(3x-2) + \frac{1}{2}, & \frac{2}{3} \leq x \leq 1. \end{cases}$$

The limit of $\{c_n\}_{n\in\mathbb{N}}$ exists and is called the Cantor function. We then generalize the construction to 2-dimensional case by taking the pair of the two identical sequences:

$$\boldsymbol{c}_n = (c_n, c_n) : I \to \mathbb{R}^2.$$

The limit of the sequence $\{\boldsymbol{c}_n\}$ also exists and equals the product of the Cantor functions. We call the limit $\lim_{n\to\infty} \boldsymbol{c}_n$ *2-dimensional Cantor dust function*. We note that the construction of the 2-dimensional Cantor dust function can be generalized to higher dimensional case.

3. A COMBINATORIAL INTEGRATION

3.1. **Approximating combinatorial integration.** We apply the construction of the combinatorial Fredholm module to the IFS of the Cantor dust $(I, S = \{1, 2, 3, 4\}, \{f_s\}_{s \in S})$ defined in Section 2. In order to construct our cyclic cocycle, we need the multiplication operator $\rho(f)$ and the commutator $[F, f]$ on $\mathcal{H}_{\boldsymbol{S}}$ for $f \in C(CD)$. Since $\rho(f)$ is even, we can express $\rho(f) = \begin{bmatrix} f^+ & \\ & f^- \end{bmatrix}$. We can write down the commutator $[F, f]$ as follows:

$$[F, f] = \frac{1}{\sqrt{2}} \begin{bmatrix} & & -(f(0) - f(1)) & -(f(0) - f(3)) \\ & & f(2) - f(1) & -(f(2) - f(3)) \\ f(0) - f(1) & -(f(2) - f(1)) & & \\ f(0) - f(3) & f(2) - f(3) & & \end{bmatrix}.$$

Here, for the simplicity, we denote $i = v_i$ the vertices on the squares $V_{\boldsymbol{S}}$. We denote the upper right 2×2 block of $[F, f]$ by $d^- f$ and the lower left by $d^+ f$.

We now construct a sequence of operators on $\bigoplus_{\boldsymbol{s} \in S^{\times n}} \mathcal{H}_{\boldsymbol{S}}$ that will give rise to a cyclic cocycle. For $f, g, h \in C(K)$, we have

$$f[F, g][F, h] = \begin{bmatrix} f^+ & \\ & f^- \end{bmatrix} \begin{bmatrix} & d^- g \\ d^+ g & \end{bmatrix} \begin{bmatrix} & d^- h \\ d^+ h & \end{bmatrix}$$

$$= \begin{bmatrix} & f^+ d^- g \\ f^- d^+ g & \end{bmatrix} \begin{bmatrix} & d^- h \\ d^+ h & \end{bmatrix} = \begin{bmatrix} f^+ d^- g d^+ h & \\ & f^- d^+ g d^- h \end{bmatrix}.$$

By setting $f_{i,j} = f(j) - f(i)$, the two diagonal 2×2 blocks of $f[F, g][F, h]$ can be expressed as

$$f^+ d^- g d^+ h = \frac{1}{2} \begin{bmatrix} f(0) & \\ & f(2) \end{bmatrix} \begin{bmatrix} g_{0,1} & g_{0,3} \\ -g_{2,1} & g_{2,3} \end{bmatrix} \begin{bmatrix} -h_{0,1} & h_{2,1} \\ -h_{0,3} & -h_{2,3} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} f(0) g_{0,1} & f(0) g_{0,3} \\ -f(2) g_{2,1} & f(2) g_{2,3} \end{bmatrix} \begin{bmatrix} -h_{0,1} & h_{2,1} \\ -h_{0,3} & -h_{2,3} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -f(0)(g_{0,1} h_{0,1} + g_{0,3} h_{0,3}) & f(0)(g_{0,1} h_{2,1} - g_{0,3} h_{2,3}) \\ f(2)(g_{2,1} h_{0,1} - g_{2,3} h_{0,3}) & -f(2)(g_{2,1} h_{2,1} + g_{2,3} h_{2,3}) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} f(0)(g_{0,1} h_{1,0} + g_{0,3} h_{3,0}) & -f(0)(g_{0,1} h_{1,2} - g_{0,3} h_{3,2}) \\ f(2)(g_{2,3} h_{3,0} - g_{2,1} h_{1,0}) & f(2)(g_{2,1} h_{1,2} + g_{2,3} h_{3,2}) \end{bmatrix}$$

and

$$f^- d^+ g d^- h = \frac{1}{2} \begin{bmatrix} f(1) & \\ & f(3) \end{bmatrix} \begin{bmatrix} -g_{0,1} & g_{2,1} \\ -g_{0,3} & -g_{2,3} \end{bmatrix} \begin{bmatrix} h_{0,1} & h_{0,3} \\ -h_{2,1} & h_{2,3} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -f(1) g_{0,1} & f(1) g_{2,1} \\ -f(3) g_{0,3} & -f(3) g_{2,3} \end{bmatrix} \begin{bmatrix} h_{0,1} & h_{0,3} \\ -h_{2,1} & h_{2,3} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} f(1)(-g_{0,1}h_{0,1} - g_{2,1}h_{2,1}) & f(1)(-g_{0,1}h_{0,3} + g_{2,1}h_{2,3}) \\ f(3)(-g_{0,3}h_{0,1} + g_{2,3}h_{2,1}) & f(3)(-g_{0,3}h_{0,3} - g_{2,3}h_{2,3}) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} f(1)(g_{1,0}h_{0,1} + g_{1,2}h_{2,1}) & f(1)(g_{1,0}h_{0,3} - g_{1,2}h_{2,3}) \\ -f(3)(g_{3,2}h_{2,1} - g_{3,0}h_{0,1}) & f(3)(g_{3,0}h_{0,3} + g_{3,2}h_{2,3}) \end{bmatrix}.$$

Here, we define

$$M = -\frac{2}{(e_{\boldsymbol{s}})^2}[F,x][F,y] = \begin{bmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}.$$

M can be expressed as $N \oplus N$ by denoting the off-diagonal 2×2 matrices by N. The following lemma indicates that the trace of an operator $f[F,g][F,h]M$ gives rise to a discretized version of integration on a square.

Lemma 3.1. *For any $n \in \mathbb{N}$ and $\boldsymbol{s} \in S^{\times n}$, we have*

$$2 \cdot \mathrm{Tr}(f[F,g][F,h]M) = f(0)(g_{0,1}h_{1,2} - g_{0,3}h_{3,2}) + f(2)(g_{2,3}h_{3,0} - g_{2,1}h_{1,0})$$
$$- f(1)(g_{1,0}h_{0,3} - g_{1,2}h_{2,3}) - f(3)(g_{3,2}h_{2,1} - g_{3,0}h_{0,1}).$$

Proof. The proof follows straightforward calculation. By multiplying M with $f[F,g][F,h]$, we have

$$f[F,g][F,h]M = \begin{bmatrix} f^+ d^- g d^+ h & \\ & -f^- d^+ g d^- h \end{bmatrix} \begin{bmatrix} N & \\ & N \end{bmatrix}$$
$$= \begin{bmatrix} f^+ d^- g d^+ h N & \\ & f^- d^+ g d^- h \, N \end{bmatrix}.$$

Then we have

$$f^+ d^- g d^+ h N = \frac{1}{2} \begin{bmatrix} f(0)(g_{0,1}h_{1,0} + g_{0,3}h_{3,0}) & -f(0)(g_{0,1}h_{1,2} - g_{0,3}h_{3,2}) \\ f(2)(g_{2,3}h_{3,0} - g_{2,1}h_{1,0}) & f(2)(g_{2,1}h_{1,2} + g_{2,3}h_{3,2}) \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} f(0)(g_{0,1}h_{1,2} - g_{0,3}h_{3,2}) & f(0)(g_{0,1}h_{1,0} + g_{0,3}h_{3,0}) \\ -f(2)(g_{2,1}h_{1,2} + g_{2,3}h_{3,2}) & f(2)(g_{2,3}h_{3,0} - g_{2,1}h_{1,0}) \end{bmatrix}$$

and

$$f^- d^+ g d^- h \, N = \frac{1}{2} \begin{bmatrix} f(1)(g_{1,0}h_{0,1} + g_{1,2}h_{2,1}) & f(1)(g_{1,0}h_{0,3} - g_{1,2}h_{2,3}) \\ -f(3)(g_{3,2}h_{2,1} - g_{3,0}h_{0,1}) & f(3)(g_{3,0}h_{0,3} + g_{3,2}h_{2,3}) \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} -f(1)(g_{1,0}h_{0,3} - g_{1,2}h_{2,3}) & f(1)(g_{1,0}h_{0,1} + g_{1,2}h_{2,1}) \\ -f(3)(g_{3,0}h_{0,3} + g_{3,2}h_{2,3}) & -f(3)(g_{3,2}h_{2,1} - g_{3,0}h_{0,1}) \end{bmatrix}.$$

Therefore, the trace of $f[F,g][F,h]M$ is given by the lemma. $\square$

When $g = x$ and $h = y$ are the coordinate functions of $\mathbb{R}^2$, respectively, each term of the right hand side of Lemma 3.1 is nothing but the summand of the Riemannian sum of f. So the sum of the trace in the left hand side can be regard as an analogue of the Riemannian sum on the Cantor dust.

Definition 1. For any $n \in \mathbb{N}$ and $f \otimes g \otimes h \in C(CD) \otimes C(CD) \otimes C(CD)$, define a map

$$\phi_n(f, g, h) = \sum_{\boldsymbol{s} \in S^{\times n}} \operatorname{Tr}(f[F, g][F, h]M).$$

We call the sequence of the maps $\{\phi_n\}$ the approximating combinatorial integration.

Proposition 3.2. *For any functions $f \in C(CD)$ and $g, h \in C^{\operatorname{Lip}}(CD)$, the sequence $\{\phi_n(f, g, h)\}$ converges to 0.*

Proof. By Lemma 3.1, we have

$$\left|\phi_n(f, g, h)\right| \leq \sum_{\boldsymbol{s} \in S^{\times n}} \Big(\left|f(0)(g_{0,1}h_{1,2} - g_{0,3}h_{3,2})\right| + \left|f(2)(g_{2,3}h_{3,0} - g_{2,1}h_{1,0})\right|$$

$$+ \left|f(1)(g_{1,0}h_{0,3} - g_{1,2}h_{2,3})\right| + \left|f(3)(g_{3,2}h_{2,1} - g_{3,0}h_{0,1})\right| \Big)$$

$$\leq 2\operatorname{Lip}(g)\operatorname{Lip}(h)\frac{1}{9^n} \sum_{\boldsymbol{s} \in S^{\times n}} \left(|f(0)| + |f(1)| + |f(2)| + |f(3)|\right)$$

$$\leq 8\|f\|\operatorname{Lip}(g)\operatorname{Lip}(h)\frac{4^n}{9^n}$$

for any $n \in \mathbb{N}$, $f \in C(CD)$, and $g, h \in C^{\operatorname{Lip}}(CD)$. Here, $\|f\|$ means the supnorm of f and $\operatorname{Lip}(g)$ means the Lipschitz constant of $g \in C^{\operatorname{Lip}}(CD)$. Therefore, we have $\phi_n(f, g, h) \to 0$ $(n \to \infty)$. $\qquad\square$

By Proposition 3.2, our approximating combinatorial integration does not restore rich structure on the Lipschitz functions. So we need another class of functions on the Cantor dust.

3.2. **Non-triviality of a combinatorial integration.** Let $CD_n = \bigcup_{\boldsymbol{s} \in S^{\times n}} f_{\boldsymbol{s}}(I)$ be a space obtained by applying the n-times composition of $\bigcup_{s=1}^{4} f_s$ to I. CD_n has 4^n distinct squares $\{\gamma_i^n\}_{1 \leq i \leq 4^n}$ with length of edge $\dfrac{1}{3^n}$. For every $n \in \mathbb{N}$, the restriction of $\boldsymbol{c}_n$ to CD_n is still surjective. The image of $\{\gamma_i^n\}_{1 \leq i \leq 4^n}$ by $\boldsymbol{c}_n$ is then a subdivision of I each of whose cell is a square with length $\dfrac{1}{2^n}$. Therefore, the maximum length across all the cells tends to 0 as $n \to \infty$.

We introduce an equivalence relation $\sim$ on ∂I: $(a, 0) \sim (a, 1)$ and $(0, b) \sim (1, b)$. The quotient space turns out to be the torus $\mathbb{T}^2 = I/\sim$. The map $\boldsymbol{c}_n$ induces a surjective map $CD_n \to \mathbb{T}^2$. We denote the map by the same letter $\boldsymbol{c}_n$. Similarly, we can also construct a continuous function $\boldsymbol{c} : CD \to \mathbb{T}^2$ by the restriction of the 2-dimensional Cantor dust function to CD.

The following theorem shows that the limit of approximating combinatorial integration with nontrivial value exists for some class of function on CD.

Theorem 3.3. *Let $C^\infty(\mathbb{T}^2)$ be the smooth functions on the torus $\mathbb{T}^2$. For $f = \boldsymbol{c}^*(\tilde{f}), g = \boldsymbol{c}^*(\tilde{g}), h = \boldsymbol{c}^*(\tilde{h}) \in \boldsymbol{c}^* C^\infty(\mathbb{T}^2)$, we have*

$$\lim_{n \to \infty} \phi_n(f, g, h) = 2 \int_{\mathbb{T}^2} \tilde{f} d\tilde{g} \wedge d\tilde{h}.$$

Moreover, the limit only depends on $f, g, h \in \boldsymbol{c}^* C^\infty(\mathbb{T}^2)$. Therefore, the functional $\phi(f, g, h) = \lim\limits_{n \to \infty} \phi_n(f, g, h)$ is a cyclic 2-cocycle on $\boldsymbol{c}^* C^\infty(\mathbb{T}^2)$.

Proof. Let $\boxplus_n$ be the set of equilateral cells that consist of 2^n-fold subdivision of I. By the first-order approximation, for $\tilde{g}, \tilde{h} \in C^\infty(\mathbb{T}^2)$ we have

$$\tilde{g}_{0,1} = \tilde{g}(1) - \tilde{g}(0) = 2^{-n} \tilde{g}_x(0) + o(2^{-n}) \quad (n \to \infty),$$
$$\tilde{h}_{1,2} = \tilde{h}(2) - \tilde{h}(1) = 2^{-n} \tilde{h}_y(0) + o(2^{-n}) \quad (n \to \infty),$$
$$\tilde{g}_{0,3} = \tilde{g}(3) - \tilde{g}(0) = 2^{-n} \tilde{g}_y(0) + o(2^{-n}) \quad (n \to \infty),$$
$$\tilde{h}_{3,2} = \tilde{h}(2) - \tilde{h}(3) = 2^{-n} \tilde{h}_x(0) + o(2^{-n}) \quad (n \to \infty).$$

Apply the above equations to $\tilde{f}(0)(\tilde{g}_{0,1}\tilde{h}_{1,2} - \tilde{g}_{0,3}\tilde{h}_{3,2})$ in Lemma 3.1, and we get

$$\tilde{f}(0)(\tilde{g}_{0,1}\tilde{h}_{1,2} - \tilde{g}_{0,3}\tilde{h}_{3,2}) = 4^{-n}\tilde{f}(0)(\tilde{g}_x(0)\tilde{h}_y(0) - \tilde{g}_y(0)\tilde{h}_x(0)) + o(4^{-n}) \quad (n \to \infty).$$

Therefore, we have

$$\lim_{n \to \infty} \sum_{\square \in \boxplus_n} \tilde{f}(0)(\tilde{g}_{0,1}\tilde{h}_{1,2} - \tilde{g}_{0,3}\tilde{h}_{3,2})$$

$$= \lim_{n \to \infty} 4^{-n} \sum_{\square \in \boxplus_n} \tilde{f}(0)(\tilde{g}_x(0)\tilde{h}_y(0) - \tilde{g}_y(0)\tilde{h}_x(0)) + 4^n o(4^{-n})$$

$$= \int_{\mathbb{T}^2} \tilde{f}(\tilde{g}_x\tilde{h}_y - \tilde{g}_y\tilde{h}_x) dx \wedge dy$$

$$= \int_{\mathbb{T}^2} \tilde{f} d\tilde{g} \wedge d\tilde{h}.$$

Similary, we obtain

$$\lim_{n \to \infty} \sum_{\square \in \boxplus_n} \tilde{f}(2)(\tilde{g}_{2,3}\tilde{h}_{3,0} - \tilde{g}_{2,1}\tilde{h}_{1,0}) = -\lim_{n \to \infty} \sum_{\square \in \boxplus_n} \tilde{f}(1)(\tilde{g}_{1,0}\tilde{h}_{0,3} - \tilde{g}_{1,2}\tilde{h}_{2,3})$$

$$= -\lim_{n \to \infty} \sum_{\square \in \boxplus_n} \tilde{f}(3)(\tilde{g}_{3,2}\tilde{h}_{2,1} - \tilde{g}_{3,0}\tilde{h}_{0,1}) = \int_{\mathbb{T}^2} \tilde{f} d\tilde{g} \wedge d\tilde{h}.$$

Hence, by Lemma 3.1, we get

$$\lim_{n\to\infty} \phi_n(f,g,h) = \frac{1}{2} \lim_{n\to\infty} \sum_{\boldsymbol{s}\in S^{\times n}} \mathrm{Tr}\left(\tilde{f}\circ \boldsymbol{c}\Big[F,\tilde{g}\circ\boldsymbol{c}\Big]\Big[F,\tilde{h}\circ\boldsymbol{c}\Big]M\right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{\square\in\boxplus_n} \mathrm{Tr}\left(\tilde{f}[F,\tilde{g}][F,\tilde{h}]M\right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{\square\in\boxplus_n} \Big(\tilde{f}(0)(\tilde{g}_{0,1}\tilde{h}_{1,2} - \tilde{g}_{0,3}\tilde{h}_{3,2})$$

$$+ \tilde{f}(2)(\tilde{g}_{2,3}\tilde{h}_{3,0} - \tilde{g}_{2,1}\tilde{h}_{1,0})$$

$$- \tilde{f}(1)(\tilde{g}_{1,0}\tilde{h}_{0,3} - \tilde{g}_{1,2}\tilde{h}_{2,3})$$

$$- \tilde{f}(3)(\tilde{g}_{3,2}\tilde{h}_{2,1} - \tilde{g}_{3,0}\tilde{h}_{0,1})\Big)$$

$$= 2\int_{\mathbb{T}^2} \tilde{f}d\tilde{g}\wedge d\tilde{h}.$$

Assume that $f = \boldsymbol{c}^*(\tilde{f}) = \boldsymbol{c}^*(\tilde{f}')$, $g = \boldsymbol{c}^*(\tilde{g}) = \boldsymbol{c}^*(\tilde{g}')$ and $h = \boldsymbol{c}^*(\tilde{h}) = \boldsymbol{c}^*(\tilde{h}')$. In general, if $\boldsymbol{c}^*(\tilde{p}) = \boldsymbol{c}^*(\tilde{q})$ for $\tilde{p},\tilde{q}\in C(\mathbb{T}^2)$, then $\boldsymbol{c}_n^*(\tilde{p}) = \boldsymbol{c}_n^*(\tilde{q})$ on the boundary of CD_n for any $n\in\mathbb{N}$. Therefore, for every $n\in\mathbb{N}$, we have

$$\sum_{\boldsymbol{s}\in S^{\times n}} \mathrm{Tr}\left(\tilde{f}\circ\boldsymbol{c}\Big[F,\tilde{g}\circ\boldsymbol{c}\Big]\Big[F,\tilde{h}\circ\boldsymbol{c}\Big]M\right) = \sum_{\boldsymbol{s}\in S^{\times n}} \mathrm{Tr}\left(\tilde{f}'\circ\boldsymbol{c}\Big[F,\tilde{g}'\circ\boldsymbol{c}\Big]\Big[F,\tilde{h}'\circ\boldsymbol{c}\Big]M\right)$$

Thus as $n\to\infty$, the value $\phi(f,g,h)$ depends only on $f,g,h\in\boldsymbol{c}^*C^\infty(\mathbb{T}^2)$. □

Corollary 3.4. *Given $[p]\in K_0(\boldsymbol{c}^*(C^\infty(\mathbb{T}^2)))$ written as $p = e\circ\boldsymbol{c} : CD \to M_N(\mathbb{C})$, the Connes' pairing of the cyclic 2-cocycle $[\phi]$ with $[p]$ is expressed as follows:*

$$\langle[\phi],[p]\rangle = \frac{1}{\pi i}\int_{\mathbb{T}^2} \mathrm{Tr}(e(de)^2).$$

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