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GAME POSITIONS OF MULTIPLE HOOK REMOVING GAME

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Abstract. Multiple Hook Removing Game (MHRG for short) introduced in [1] is an impartial game played in terms of Young diagrams. In this paper, we give a characterization of the set of all game positions in MHRG. As an application, we prove that for $t \in \mathbb{Z}_{\geq 0}$ and $m, n \in \mathbb{N}$ such that $t \leq m \leq n$, and a Young diagram Y contained in the rectangular Young diagram $Y_{t,n}$ of size $t \times n$, Y is a game position in MHRG with $Y_{m,n}$ the starting position if and only if Y is a game position in MHRG with $Y_{t,n-m+t}$ the starting position, and also that the Grundy value of Y in the former MHRG is equal to that in the latter MHRG.

1. Introduction.

The Sato-Welter game is an impartial game studied by Welter [8] and Sato [5], independently. This game is played in terms of Young diagrams. The rule is given as follows:

(i) The starting position is a Young diagram Y.
(ii) Assume that a Young diagram Y' appears as a game position. A player chooses a box $(i, j) \in Y'$, and moves game position from Y' to $Y'\langle i, j\rangle$, where $Y'\langle i, j\rangle$ is the Young diagram which is obtained by removing the hook at (i, j) from Y' and filling the gap between two diagrams (see Figure 2 below).
(iii) The (unique) ending position is the empty Young diagram $\emptyset$. The winner is the player who makes $\emptyset$ after his/her operation (ii).

Kawanaka [2] introduced the notion of a plain game, as a generalization of the Sato-Welter game. A plain game is played in terms of d-complete posets which was introduced and classified by Proctor [3, 4], and can be thought of as a generalization of Young diagrams. It is known that d-complete posets are closely related to not only the combinatorial game theory, but also the representation theory and the algebraic geometry associated with simply-laced finite-dimensional simple Lie algebras. For example, the weight system of a minuscule representation (which is identical to the Weyl group orbit of a minuscule fundamental weight) for a simply-laced finite-dimensional simple Lie algebra can be described in terms of a d-complete poset. Applying

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the “folding” technique to this fact for the simply-laced case, Tada [7] described the Weyl group orbits of some weights (see [7, Section 7, Table 4]) for multiply-laced finite-dimensional simple Lie algebras in terms of d-complete posets with “colorings” ([7, Theorem 7.2]); note that the weights are not minuscule weights, except for a special case in type C_n.

$$
\begin{tikzcd}
\text{Sato-Welter game} \arrow[r, leftrightarrow, "{[8,\,5]}"] & \text{Young diagram} \arrow[r, leftrightarrow] \arrow[d, "\text{generalization}"] & \text{type A} \\
\text{plain game} \arrow[r, leftrightarrow, "{[2]}"] & d\text{-complete poset} \arrow[r, leftrightarrow] \arrow[d, "\text{``folding''}"] & \text{simply-laced} \arrow[d, "\text{folding}"] \\
 & \begin{array}{c} d\text{-complete poset} \\ \text{with a ``coloring''} \end{array} \arrow[r, leftrightarrow, "{[7]}"] \arrow[d, "\text{special case}"] & \text{multiply-laced} \\
\text{MHRG} \arrow[r, leftrightarrow, "{[1]}"] & \begin{array}{c} \text{Young diagram} \\ \text{with the unimodal numbering} \end{array} \arrow[r, leftrightarrow] & \text{types B and C}
\end{tikzcd}
$$

FIGURE 1.

Based on [7], Abuku and Tada [1] introduced a new impartial game, named Multiple Hook Removing Game (MHRG for short). MHRG is played in terms of Young diagrams with the unimodal numbering; for the definition of unimodal numbering, see Section 3. Let us explain the rule of MHRG. We fix positive integers $m, n \in \mathbb{N}$ such that $m \leq n$. Let $Y_{m,n} := \{(i,j) \in \mathbb{N}^2 \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ be the rectangular Young diagram of size $m \times n$. We denote by $\mathcal{F}(Y_{m,n})$ the set of all Young diagrams contained in the rectangular Young diagram $Y_{m,n}$. For a game position G of an impartial game, we denote by $\mathcal{O}(G)$ the set of all options of G. The rule of MHRG is given as follows:

(1) All game positions are some Young diagrams contained in $\mathcal{F}(Y_{m,n})$ with the unimodal numbering. The starting position is the rectangular Young diagram $Y_{m,n}$.
(2) Assume that $Y \in \mathcal{F}(Y_{m,n})$ appears as a game position. If $Y \neq \emptyset$ (the empty Young diagram), then a player chooses a box $(i,j) \in Y$, and remove the hook at (i,j) in Y. We denote by $Y\langle i,j\rangle$ the resulting Young diagram. Then we know from [1, Lemma 9] (see also Lemma 4.4 below) that $f := \#\{(i',j') \in Y\langle i,j\rangle \mid \mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j) \text{ (as multisets)}\} \leq 1$,where $\mathcal{H}_Y(i,j)$ (resp., $\mathcal{H}_{Y\langle i,j\rangle}(i',j')$) is the numbering multiset for the

hook at $(i, j) \in Y$ (resp., $(i', j') \in Y\langle i, j\rangle$); see Section 3. If $f = 0$, then a player moves Y to $Y\langle i, j\rangle \in \mathcal{O}(Y)$. If $f = 1$, then a player moves Y to $(Y\langle i, j\rangle)\langle i', j'\rangle \in \mathcal{O}(Y)$, where $(i', j') \in Y\langle i, j\rangle$ is the unique element such that $\mathcal{H}_{Y\langle i,j\rangle}(i', j') = \mathcal{H}_Y(i, j)$.

(3) The (unique) ending position is the empty Young diagram $\emptyset$. The winner is the player who makes $\emptyset$ after his/her operation (2).

In general, not all Young diagrams in $\mathcal{F}(Y_{m,n})$ appear as game positions of MHRG (see Example 4.3). The purpose of this paper is to give a characterization of the set of all game positions in MHRG. Let us explain our results more precisely. Let $\binom{[1,m+n]}{m}$ denote the set of all subsets of $[1, m + n] := \{x \in \mathbb{N} \mid 1 \le x \le m + n\}$ having m elements. Then there exists a bijection I from $\mathcal{F}(Y_{m,n})$ onto $\binom{[1,m+n]}{m}$ (see Subsection 2.1). Let Y^D denote the dual Young diagram of Y in $Y_{m,n}$ (see Subsection 2.1). We set $c := (m + n - 1 + \chi) / 2$, where $\chi = 0$ (resp., $\chi = 1$) if $m + n$ is odd (resp., even). For $Y \in \mathcal{F}(Y_{m,n})$, we set $I_R(Y) := I(Y) \cap [c + 1 - \chi, m + n]$. We denote by $\mathcal{S}(Y_{m,n})$ the set of all those Young diagrams in $\mathcal{F}(Y_{m,n})$ which appear as game positions of MHRG.

Theorem 1.1 (= Theorem 5.1)**.** Let $Y \in \mathcal{F}(Y_{m,n})$, and $\lambda = (\lambda_1, \dots, \lambda_m)$ the partition corresponding to Y. The following (I), (II), (III), and (IV) are equivalent.
(I) $Y \in \mathcal{S}(Y_{m,n})$. (II) $Y^D \in \mathcal{S}(Y_{m,n})$. (III) $I_R(Y) \cap I_R(Y^D) = \emptyset$.
(IV) $\lambda_i + \lambda_j \neq n - m + i + j - 1$ for all $1 \le i \le j \le m$.

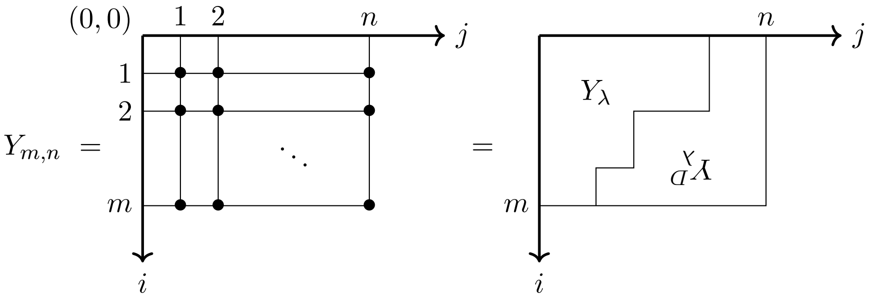
Theorem 1.2 (= Theorem 6.1)**.** Let $t \in \mathbb{Z}_{\ge 0}$ and $m, n \in \mathbb{N}$ such that $t \le m \le n$. For a Young diagram Y having at most t rows, $Y \in \mathcal{S}(Y_{m,n})$ if and only if $Y \in \mathcal{S}(Y_{t,n-m+t})$. Moreover, the Grundy value of Y as an element of $\mathcal{S}(Y_{m,n})$ is equal to the Grundy value of Y as an element of $\mathcal{S}(Y_{t,n-m+t})$.

Tada proved that there exists a natural bijection between the set of all game positions of MHRG and a set of Young diagrams with the unimodal numbering, which corresponds to the Weyl group orbit of a certain weight in types B and C (recall the last line of Figure 1). Based on our proof of Theorem 1.1, Tada also gave a description of the Weyl group orbit of the weight in types B and C; see [7, Theorem 9.4].

This paper is organized as follows. In Section 2, we fix our notation for Young diagrams, and recall some basic facts on the combinatorial game theory. In Section 3, we recall the definition of the unimodal numbering and the diagonal expression for Young diagrams. In Section 4, we recall the rule of MHRG, and a basic property (Lemma 4.4). In Sections 5 and 6, we prove Theorems 1.1 and 1.2 above, respectively.

2. Preliminaries.

2.1. **Young diagrams.** Let $\mathbb{N}$ denote the set of positive intgers. For $a, b \in \mathbb{Z}$, we set $[a, b] \coloneqq \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Throughout this paper, we fix $m, n \in \mathbb{N}$ such that $m \leq n$. For a positive integer $x \in \mathbb{N}$, we set $\overline{x} \coloneqq m + n + 1 - x$. Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition of length at most m such that $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$. We can identify $\lambda = (\lambda_1, \dots, \lambda_m)$ with the *Young diagram* $Y_\lambda \coloneqq \{(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$ of shape λ; if $\lambda = (0, 0, \dots, 0)$, then we denote Y_λ by $\emptyset$, and call it the *empty Young diagram*. We identify $(i, j) \in Y_\lambda$ with the square in $\mathbb{R}^2$ whose vertices are $(i-1, j-1), (i-1, j), (i, j-1)$, and (i, j); elements in Y_λ are called *boxes* in Y_λ. Let $Y_{m,n} \coloneqq \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ be the rectangular Young diagram of size $m \times n$, which corresponds to $(n, n, \dots, n)$. Set $\mathcal{F}(Y_{m,n}) \coloneqq \{Y_\lambda \mid n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0\}$; notice that $\mathcal{F}(Y_{m,n})$ is identical to the set of all Young diagrams contained in the rectangular Young diagram $Y_{m,n}$. We set $\lambda^D \coloneqq (n - \lambda_m, \dots, n - \lambda_1)$. The Young diagram $Y_\lambda^D \coloneqq Y_{\lambda^D}$ is called the *dual Young diagram* of Y_λ (in $Y_{m,n}$).



Let $\binom{[1,m+n]}{m}$ denote the set of all subsets of $[1, m+n]$ having m elements. For $\lambda = (\lambda_1, \dots, \lambda_m)$, we set $i'_t \coloneqq \lambda_{m-t+1} + t$ for $1 \leq t \leq m$; observe that $I_\lambda \coloneqq \{i'_1 < \cdots < i'_m\} \in \binom{[1,m+n]}{m}$. It is well-known that the map $Y_\lambda \mapsto I_\lambda$ is a bijection from $\mathcal{F}(Y_{m,n})$ onto $\binom{[1,m+n]}{m}$; write it as I. Let $Y \in \mathcal{F}(Y_{m,n})$. For $(i, j) \in Y$, we set $H_Y(i, j) \coloneqq \{(i, j)\} \cup \{(i, j') \in Y \mid j < j'\} \cup \{(i', j) \in Y \mid i < i'\}$, and call it the *hook at* (i, j) in Y. Also, for $(i, j) \in Y$, we set

$$\begin{aligned} Y\langle i, j\rangle \coloneqq \{(i', j') \mid (i', j') \in Y, \text{ and } i' < i \text{ or } j' < j\} \\ \cup \{(i'-1, j'-1) \mid (i', j') \in Y,\ i' > i \text{ and } j' > j\}. \end{aligned}$$

The procedure which obtains $Y\langle i, j\rangle$ from Y is called *removing the hook* at (i, j) from Y (see Figure 2 below). Recall that the hook at (i, j) in Y determines a unique element (l, r) in $([1, m+n] \setminus I(Y)) \times I(Y)$ which satisfies $I(Y\langle i, j\rangle) = (I(Y) \setminus \{r\}) \cup \{l\}$, since the map

$$(i, j) \mapsto (j + \#\{k \in \{1, 2, \dots, m\} \mid \lambda_k < j\}, \lambda_i + m - i + 1)$$

is an injection from Y into $([1, m+n] \setminus I(Y)) \times I(Y)$ such that $I(Y\langle i,j\rangle) = (I(Y) \setminus \{\lambda_i + m - i + 1\}) \cup \{j + \#\{k \in \{1, 2, \ldots, m\} \mid \lambda_k < j\}\}$. Let us write $Y \xrightarrow{l,r} Y\langle i,j\rangle$ if the hook at (i,j) in Y corresponds to $(l,r) \in ([1, m+n] \setminus I(Y)) \times I(Y)$ under the injection above.

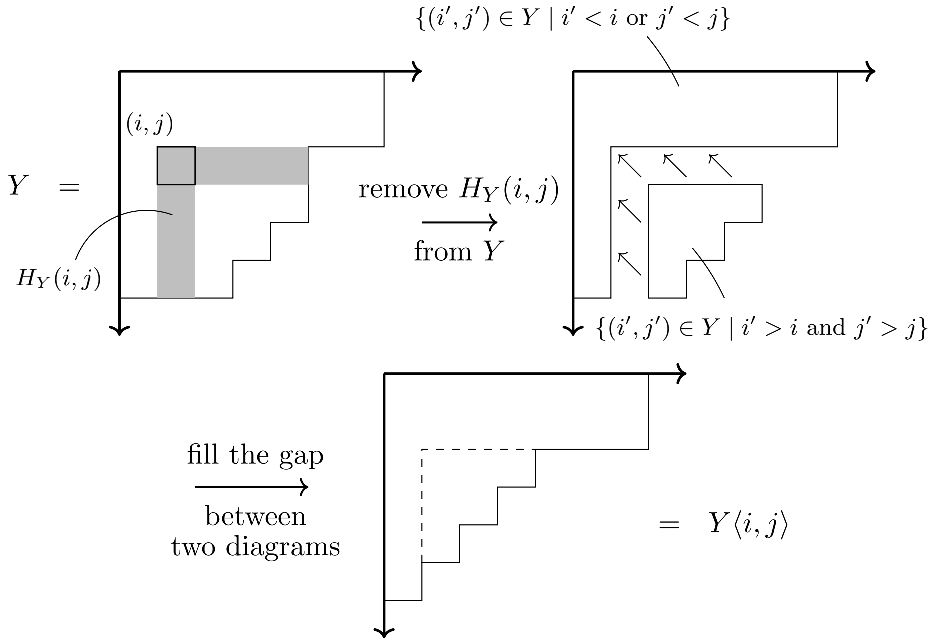


FIGURE 2. Removing the hook at (i,j) from Y.

2.2. **Combinatrial game theory.** For the general theory of combinatorial games, we refer the reader to [6, Chapters 1 and 2]. In this subsection, we fix an impartial game in normal play whose game positions are all short (in the sense of [6, pages 4 and 9]).

Definition 2.1. A game position of an impartial game is called an $\mathcal{N}$-*position* (resp., a $\mathcal{P}$-*position*) if the next player (resp., the previous player) has a winning strategy.

Definition 2.2. For a (proper) subset X of $\mathbb{Z}_{\geq 0}$, we set $\operatorname{mex} X := \min(\mathbb{Z}_{\geq 0} \setminus X)$.

For a game position G of an impartial game, we denote by $\mathcal{O}(G)$ the set of all options of G.

Definition 2.3. Let G be a game position. The *Grundy value* $\mathcal{G}(G)$ of G is defined by $\mathcal{G}(G) := \operatorname{mex}\{\mathcal{G}(P) \mid P \in \mathcal{O}(G)\}$.

Recall from [6, page 6] that each game position of an impartial game is either an $\mathcal{N}$-position or a $\mathcal{P}$-position. The following result is well-known in the combinatorial game theory.

Theorem 2.4 ([6, Theorem 2.1])**.** A game position G is a $\mathcal{P}$-position if and only if $\mathcal{G}(G) = 0$.

3. Unimodal numbering on Young diagrams.

Let $Y \in \mathcal{F}(Y_{m,n})$. For each box $(i, j) \in Y$, we write

$$c\,(i, j) := \min\,(j - i + m, i - j + n)$$

on it; we call this numbering on Y the *unimodal numbering* on Y.

Example 3.1. Assume that $m = 3$ and $n = 5$. The Young diagram $Y = Y_{(4,4,2)} \in \mathcal{F}(Y_{3,5})$ with the unimodal numbering is as follows:

$$\begin{array}{l} \xrightarrow{\hspace{6em}} j \\ \begin{array}{|c|c|c|c|}\hline 3&4&3&2\\ \hline 2&3&4&3\\ \hline 1&2 \\ \cline{1-2}\end{array} \\ \downarrow \\ i \end{array}$$

It can be easily checked that $c := (m + n - 1 + \chi)\,/\,2$ is the maximum number appearing in the unimodal numbering, where

$$\chi := \begin{cases} 1 & \text{if } m + n \in 2\mathbb{N}, \\ 0 & \text{if } m + n \in 2\mathbb{N} + 1. \end{cases}$$

For a subset S of $Y \in \mathcal{F}(Y_{m,n})$, we define $\mathcal{H}_Y(S)$ to be the multiset consisting of $c\,(i, j)$ for $(i, j) \in S$. The multiset $\mathcal{H}_Y(S)$ is called the *numbering multiset* for S. In particular, if $S = H_Y(i, j)$ for some $(i, j) \in Y$, then we denote $\mathcal{H}_Y(S)$ by $\mathcal{H}_Y(i, j)$. We deduce that $\mathcal{H}_Y(Y) = \mathcal{H}_Y(Y\langle i, j\rangle) \cup \mathcal{H}_Y(i, j)$ (the union of multisets).

4. Multiple Hook Removing Game.

Abuku and Tada [1] introduced an impartial game, named Multiple Hook Removing Game (MHRG for short), whose rule is given as follows; recall that m and n are fixed positive integers such that $m \leq n$:

(1) All game positions are some Young diagrams contained in $\mathcal{F}(Y_{m,n})$ with the unimodal numbering. The starting position is the rectangular Young diagram $Y_{m,n}$.

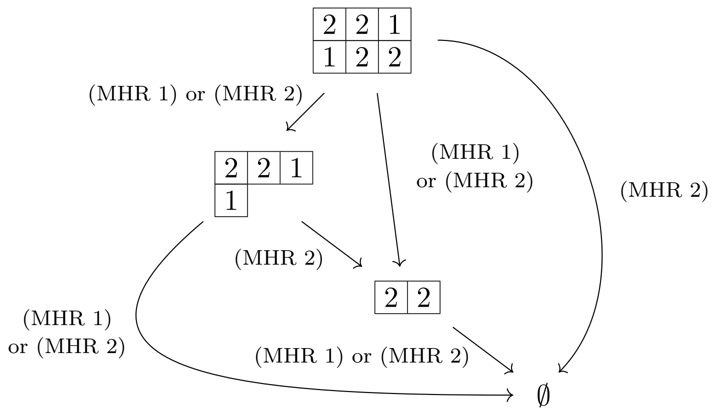
(2) Assume that $Y \in \mathcal{F}(Y_{m,n})$ appears as a game position. If $Y \neq \emptyset$ (the empty Young diagram), then a player chooses a box $(i,j) \in Y$, and remove the hook at (i,j) in Y; recall from Subsection 2.1 that the resulting Young diagram is $Y\langle i,j\rangle$. Then we know from [1, Lemma 9] (see also Lemma 4.4 below) that $f := \#\{(i',j') \in Y\langle i,j\rangle \mid \mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$ (as multisets)$\} \leq 1$. If $f = 0$, then a player moves Y to $Y\langle i,j\rangle \in \mathcal{O}(Y)$; we call this case and this operation (MHR 1). If $f = 1$, then a player moves Y to $(Y\langle i,j\rangle)\langle i',j'\rangle \in \mathcal{O}(Y)$, where $(i',j') \in Y\langle i,j\rangle$ is the unique element such that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$; we call this case and this operation (MHR 2).

(3) The (unique) ending position is the empty Young diagram $\emptyset$. The winner is the player who makes $\emptyset$ after his/her operation (2).

Definition 4.1. We denote by $\mathcal{S}(Y_{m,n})$ the set of all those Young diagrams in $\mathcal{F}(Y_{m,n})$ which appear as game positions of MHRG (with $Y_{m,n}$ the starting position); in general, $\mathcal{S}(Y_{m,n}) \subsetneq \mathcal{F}(Y_{m,n})$ as Example 4.3 below shows.

Definition 4.2. Let $Y \in \mathcal{S}(Y_{m,n})$, and $Y' \in \mathcal{O}(Y)$. If a player moves Y to Y' by operation (MHR 1) (resp., (MHR 2)), then we write $Y \xrightarrow{\text{(MHR 1)}} Y'$ (resp., $Y \xrightarrow{\text{(MHR 2)}} Y'$).

Example 4.3. Assume that $m = 2$ and $n = 3$. The elements of $\mathcal{S}(Y_{2,3})$ are



The following elements of $\mathcal{F}(Y_{2,3})$ are not contained in $\mathcal{S}(Y_{2,3})$:

$$\begin{array}{|c|c|c|}\hline 2&2&1\\\hline 1&2\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|}\hline 2&2\\\hline 1&2\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 2&2\\\hline 1\\\cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 2&2&1\\\hline\end{array}\,,\quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\,,\quad \begin{array}{|c|}\hline 2\\\hline\end{array}\,.$$

Lemma 4.4 ([1, Lemma 9])**.** Let $Y \in \mathcal{F}(Y_{m,n})$, and $(i,j) \in Y$. Assume that there exists a box $(i',j') \in Y\langle i,j\rangle$ such that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$ (as multisets). If $Y \xrightarrow{l,r} Y\langle i,j\rangle$, then $Y\langle i,j\rangle \xrightarrow{\overline{r},\overline{l}} (Y\langle i,j\rangle)\langle i',j'\rangle$. In particular, $\#\{(i',j') \in Y\langle i,j\rangle \mid \mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$ (as multisets)$\} \leq 1$.

Remark 4.5. In fact, the following holds (see [1, Lemma 9]), although we do not use these facts in this paper.

(1) Keep the notation and setting in Lemma 4.4. There does not exist $(i'', j'') \in (Y\langle i,j\rangle)\langle i',j'\rangle$ such that $\mathcal{H}_{(Y\langle i,j\rangle)\langle i',j'\rangle}(i'', j'') = \mathcal{H}_Y(i,j)$.
(2) Let $(i,j), (k,l) \in Y$. Assume that $\mathcal{H}_Y(i,j) = \mathcal{H}_Y(k,l)$. If there exists a box $(i',j') \in Y\langle i,j\rangle$ such that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$, then there exists a (unique) box $(k',l') \in Y\langle k,l\rangle$ such that $\mathcal{H}_{Y\langle k,l\rangle}(k',l') = \mathcal{H}_Y(i,j)$. Moreover, in this case, we have $(Y\langle i,j\rangle)\langle i',j'\rangle = (Y\langle k,l\rangle)\langle k',l'\rangle$.

5. Description of $\mathcal{S}(Y_{m,n})$.

Recall that $m, n \in \mathbb{N}$ are such that $m \leq n$, and that $c = \max\{c(i,j) \mid (i,j) \in Y_{m,n}\}$ is equal to $(m+n-1+\chi)/2$, where $\chi = 0$ (resp., $\chi = 1$) if $m+n$ is odd (resp., even). Also, we have a canonical bijection $I \colon \mathcal{F}(Y_{m,n}) \to \binom{[1,m+n]}{m}$ (see Subsection 2.1).

Let $Y \in \mathcal{F}(Y_{m,n})$. We set $I_R(Y) \coloneqq I(Y) \cap [c+1-\chi, m+n]$; note that $\overline{c+1-\chi} = m+n+1-(c+1-\chi) = c+1 \geq c+1-\chi$.

Theorem 5.1. Let $Y \in \mathcal{F}(Y_{m,n})$, and $\lambda = (\lambda_1, \ldots, \lambda_m)$ the partition corresponding to Y, that is, $Y = Y_\lambda$. The following (I), (II), (III), and (IV) are equivalent.
(I) $Y \in \mathcal{S}(Y_{m,n})$. (II) $Y^D \in \mathcal{S}(Y_{m,n})$. (III) $I_R(Y) \cap I_R(Y^D) = \emptyset$.
(IV) $\lambda_i + \lambda_j \neq n - m + i + j - 1$ for all $1 \leq i, j \leq m$.

The rest of this section is devoted to a proof of Theorem 5.1. We can easily show the following lemma.

Lemma 5.2. It holds that $I(Y^D) = \{\bar{i} = m+n+1-i \mid i \in I(Y)\} = \overline{I(Y)}$ for $Y \in \mathcal{F}(Y_{m,n})$.

Remark 5.3. Let $Y \in \mathcal{F}(Y_{m,n})$, and $(i,j) \in Y$. Let $1 \leq l < r \leq m+n$ be such that $Y \xrightarrow{l,r} Y\langle i,j\rangle$. By Lemma 4.4, it follows that $\bar{r} \notin I(Y\langle i,j\rangle)$ and $\bar{l} \in I(Y\langle i,j\rangle)$ if and only if there exists a (unique) box $(i',j') \in Y\langle i,j\rangle$ such that $Y\langle i,j\rangle \xrightarrow{\bar{r},\bar{l}} (Y\langle i,j\rangle)\langle i',j'\rangle$; in particular, in this case, it holds that $\mathcal{H}_{Y\langle i,j\rangle}(i',j') = \mathcal{H}_Y(i,j)$ (as multisets).

We first show (I) $\Rightarrow$ (III). Since $Y \in \mathcal{S}(Y_{m,n})$ by (I), there exists a sequence of game positions of the form

$$Y_{m,n} = Y_0 \xrightarrow{t_1} Y_1 \xrightarrow{t_2} Y_2 \xrightarrow{t_3} \cdots \xrightarrow{t_p} Y_p = Y,$$

where t_i is either (MHR 1) or (MHR 2) for each $1 \leq i \leq p$. For $1 \leq i \leq p$ such that t_i is (MHR 2), we see from Lemma 4.4 that $Y_{i-1} \xrightarrow{l_i,r_i} Y_i' \xrightarrow{\overline{r_i},\overline{l_i}} Y_i$ for some $1 \leq l_i < r_i \leq m+n$ with $l_i \notin I(Y_{i-1})$, $r_i \in I(Y_{i-1})$, and

$Y_i' \in \mathcal{F}(Y_{m,n})$. Similarly, for $1 \le i \le p$ such that t_i is (MHR 1), there exists $1 \le l_i < r_i \le m+n$ with $l_i \notin I(Y_{i-1})$ and $r_i \in I(Y_{i-1})$ such that $Y_{i-1} \xrightarrow{l_i, r_i} Y_i$; we set $Y_i' := Y_i$ by convention. We show by induction on p that $I_R(Y_p) \cap I_R(Y_p^D) = \emptyset$. If $p = 0$, then it is obvious that $I_R(Y_{m,n}) \cap I_R(Y_{m,n}^D) = \emptyset$, since $I_R(Y_{m,n}) = \{n+1, n+2, \dots, m+n\}$ and

$$I_R(Y_{m,n}^D) = I_R(\emptyset) = \begin{cases} \emptyset & \text{if } m < n, \\ \{m\} & \text{if } m = n. \end{cases}$$

Assume that $p > 0$; by the induction hypothesis,

$$I_R(Y_{p-1}) \cap I_R(Y_{p-1}^D) = \emptyset. \tag{5.1}$$

Also, we have

$$I_R(Y_p') \setminus \{l_p\} = I_R(Y_{p-1}) \setminus \{r_p\}, \tag{5.2}$$

$$I_R(Y_p'^D) \setminus \{\overline{l_p}\} = I_R(Y_{p-1}^D) \setminus \{\overline{r_p}\}. \tag{5.3}$$

Lemma 5.4. It holds that $I_R(Y_p') \cap I_R(Y_p'^D) \neq \emptyset$ if and only if $\overline{l_p} \in I(Y_{p-1}) \setminus \{r_p\}$ or $l_p = \overline{l_p}$; notice that $l_p = \overline{l_p}$ if and only if $\chi = 0$ and $l_p = c+1$.

Proof. Assume first that $l_p < c+1-\chi$; recall that $\overline{l_p} > \overline{c+1-\chi} = c+1 \ge c+1-\chi$. It follows from (5.2) and (5.3) that

$$I_R(Y_p') = I_R(Y_{p-1}) \setminus \{r_p\}, \quad I_R(Y_p'^D) = (I_R(Y_{p-1}^D) \setminus \{\overline{r_p}\}) \cup \{\overline{l_p}\}.$$

Because $I_R(Y_{p-1}) \cap I_R(Y_{p-1}^D) = \emptyset$ by the induction hypothesis, we see that $I_R(Y_p') \cap I_R(Y_p'^D) \neq \emptyset$ if and only if $\overline{l_p} \in I_R(Y_{p-1}) \setminus \{r_p\}$. Assume next that $l_p \ge c+1-\chi$. It follows from (5.2) and (5.3) that

$$I_R(Y_p') = (I_R(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p\},$$

$$I_R(Y_p'^D) = \begin{cases} I_R(Y_{p-1}^D) \setminus \{\overline{r_p}\} & \text{if } \overline{l_p} < c+1-\chi, \\ (I_R(Y_{p-1}^D) \setminus \{\overline{r_p}\}) \cup \{\overline{l_p}\} & \text{if } \overline{l_p} \ge c+1-\chi. \end{cases}$$

Here we note that $\overline{l_p} \in I(Y_{p-1}) \setminus \{r_p\}$ if and only if $l_p \in I(Y_{p-1}^D) \setminus \{\overline{r_p}\}$ by Lemma 5.2. If $\overline{l_p} < c+1-\chi$ (resp., $\overline{l_p} \ge c+1-\chi$), then it holds that $I_R(Y_p') \cap I_R(Y_p'^D) \neq \emptyset$ if and only if $\overline{l_p} \in I(Y_{p-1}) \setminus \{r_p\}$ (resp., $\overline{l_p} \in I_R(Y_{p-1}) \setminus \{r_p\}$ or $l_p = \overline{l_p}$). Thus we have proved the lemma. □

Proposition 5.5. (1) The operation t_p is (MHR 1) if and only if either of the following (a) or (b) holds.

(a) $\overline{l_p} \notin I(Y_{p-1})$ and $l_p \neq \overline{l_p}$.

(b) $l_p = \overline{r_p}$ (notice that $l_p \neq \overline{l_p}$ also in this case since $l_p \neq r_p = \overline{l_p}$).

(2) The operation t_p is (MHR 2) if and only if $\overline{l_p} \in I(Y_{p-1}) \setminus \{r_p\}$ or $l_p = \overline{l_p}$.

Proof. It suffices to show only part (2). We first show the "only if" part of (2). Assume that t_p is (MHR 2); recall that $Y_{p-1} \xrightarrow{l_p, r_p} Y'_p \xrightarrow{\overline{r_p}, \overline{l_p}} Y_p$. It follows that $\overline{l_p} \in I(Y'_p) = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p\}$. Thus we have $\overline{l_p} \in I(Y_{p-1}) \setminus \{r_p\}$ or $\overline{l_p} = l_p$. We next show the "if" part of (2); by Remark 5.3 and Lemma 4.4, it suffices to show that $\overline{r_p} \notin I(Y'_p)$ and $\overline{l_p} \in I(Y'_p)$. Because $I(Y'_p) = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p\}$, it is obvious from the assumption that $\overline{l_p} \in I(Y'_p)$. Let us show that $\overline{r_p} \notin I(Y'_p)$. Suppose, for a contradiction, that $\overline{r_p} \in I(Y'_p)$. Since $I(Y'_p) = (I(Y_{p-1}) \setminus \{r_p\}) \cup \{l_p\}$, and since $\overline{r_p} \neq l_p$, we have $\overline{r_p} \in I(Y_{p-1}) \setminus \{r_p\} \subset I(Y_{p-1})$, and hence $r_p \in I(Y^D_{p-1})$ by Lemma 5.2. If $c + 1 - \chi \leq r_p$, then $r_p \in I_R(Y^D_{p-1})$. Since $r_p \in I_R(Y_{p-1})$, we get $r_p \in I_R(Y_{p-1}) \cap I_R(Y^D_{p-1})$, which contradicts the induction hypothesis (5.1). If $c + 1 - \chi > r_p$, then $c + 1 - \chi \leq c + 1 = \overline{c + 1 - \chi} < \overline{r_p}$, which implies that $\overline{r_p} \in I_R(Y_{p-1})$. Since $r_p \in I(Y_{p-1})$, we have $\overline{r_p} \in I_R(Y^D_{p-1})$ by Lemma 5.2. Hence we get $\overline{r_p} \in I_R(Y_{p-1}) \cap I_R(Y^D_{p-1})$, which contradicts the induction hypothesis (5.1). Therefore we obtain $\overline{r_p} \notin I(Y'_p)$, as desired. Thus we have proved the proposition. □

If t_p is (MHR 1) (recall that $Y'_p = Y_p$ and $Y^{D\prime}_p = Y^D_p$ in this case), then we see by Lemma 5.4 and Proposition 5.5 (1) that $I_R(Y_p) \cap I_R(Y^D_p) = \emptyset$. Assume that t_p is (MHR 2), or equivalently, $\overline{l_p} \in I(Y_{p-1}) \setminus \{r_p\}$ or $l_p = \overline{l_p}$ by Proposition 5.5 (2). Because $Y_{p-1} \xrightarrow{l_p, r_p} Y'_p \xrightarrow{\overline{r_p}, \overline{l_p}} Y_p$ in this case, it follows that

$$I_R(Y_p) \setminus \{\overline{r_p}, l_p\} = I_R(Y_{p-1}) \setminus \{r_p, \overline{l_p}\}, \tag{5.4}$$

$$I_R(Y^D_p) \setminus \{r_p, \overline{l_p}\} = I_R(Y^D_{p-1}) \setminus \{\overline{r_p}, l_p\}. \tag{5.5}$$

Hence, by (5.4) and (5.5), together with the induction hypothesis (5.1), we obtain $I_R(Y_p) \cap I_R(Y^D_p) = \emptyset$. Thus we have proved (I) ⇒ (III) in Theorem 5.1.

Conversely, we prove (III) ⇒ (I), that is, $Y \in \mathcal{S}(Y_{m,n})$ if $I_R(Y) \cap I_R(Y^D) = \emptyset$. We show by (descending) induction on $\langle I(Y) \rangle := \sum_{i \in I(Y)} i$. It is obvious that $Y_{m,n} \in \mathcal{S}(Y_{m,n})$. Assume that $\langle I(Y) \rangle < \langle I(Y_{m,n}) \rangle$. Since $I(Y_{m,n}) = [n+1, m+n]$, and $I(Y) \neq I(Y_{m,n})$ with $\# I(Y) = m$, there exists $r \notin I(Y)$ such that $n + 1 \leq r$. Also, there exists $l < r$ such that $l \in I(Y)$. Here we show that $\bar{l} \notin I(Y)$. Suppose, for a contradiction, that $\bar{l} \in I(Y)$. If $c + 1 - \chi \geq l$, then $c + 1 - \chi \leq c + 1 = \overline{c + 1 - \chi} \leq \bar{l}$, and hence $\bar{l} \in I_R(Y)$. By Lemma 5.2 applied to $l \in I(Y)$, it follows that $\bar{l} \in I_R(Y^D)$. Thus we obtain $\bar{l} \in I_R(Y) \cap I_R(Y^D)$, which contradicts the assumption that $I_R(Y) \cap I_R(Y^D) = \emptyset$. If $c + 1 - \chi < l$, then $l \in I_R(Y^D)$ because $\bar{l} \in I(Y)$. Since $l \in I_R(Y)$, we get $l \in I_R(Y) \cap I_R(Y^D)$, which contradicts the assumption that $I_R(Y) \cap I_R(Y^D) = \emptyset$. Therefore we obtain $\bar{l} \notin I(Y)$.

Proposition 5.6. Keep the setting above.

(1) If $\bar{r} \notin I(Y)$ or $\bar{r} = l$, then there exists a (unique) Young diagram Y' such that $I(Y') = (I(Y) \setminus \{l\}) \cup \{r\}$ and $I(Y'^D) = (I(Y^D) \setminus \{\bar{l}\}) \cup \{\bar{r}\}$. Furthermore, $Y' \in \mathcal{S}(Y_{m,n})$, and $Y' \xrightarrow{\text{(MHR 1)}} Y$.

(2) If $\bar{r} \in I(Y)$ and $\bar{r} \neq l$, then there exists a (unique) Young diagram Y'' such that $I(Y'') = (I(Y) \setminus \{\bar{r}, l\}) \cup \{r, \bar{l}\}$ and $I(Y''^D) = (I(Y^D) \setminus \{r, \bar{l}\}) \cup \{\bar{r}, l\}$. Furthermore, $Y'' \in \mathcal{S}(Y_{m,n})$, and $Y'' \xrightarrow{\text{(MHR 2)}} Y$.

Proof. (1) Recall that $l \in I(Y)$ and $r \notin I(Y)$, which implies that

$$(I(Y) \setminus \{l\}) \cup \{r\} \in \binom{[1, m+n]}{m}.$$

Since $I \colon \mathcal{F}(Y_{m,n}) \to \binom{[1,m+n]}{m}$ is a bijection, there exists unique $Y' \in \mathcal{F}(Y_{m,n})$ such that $I(Y') = (I(Y) \setminus \{l\}) \cup \{r\}$; note that $I(Y'^D) = (I(Y^D) \setminus \{\bar{l}\}) \cup \{\bar{r}\}$ by Lemma 5.2. Then it follows that $Y' \xrightarrow{l,r} Y$. Because $\bar{r} \notin I(Y)$ or $\bar{r} = l$ by the assumption of (1), and $I_R(Y) \cap I_R(Y^D) = \emptyset$ by assumption, it can be easily verified that $I_R(Y') \cap I_R(Y'^D) = \emptyset$. Since $l < r$, we have $\langle I(Y') \rangle > \langle I(Y) \rangle$, and hence $Y' \in \mathcal{S}(Y_{m,n})$ by the induction hypothesis. Because $\bar{l} \notin I(Y)$, we see from Remark 5.3 that there does not exist a box $(i, j) \in Y$ such that $Y \xrightarrow{\bar{r},\bar{l}} Y\langle i, j \rangle$. Thus we obtain $Y' \xrightarrow{\text{(MHR 1)}} Y$, as desired.

(2) Let Y' be as in the proof of part (1). Since $\bar{r} \in I(Y)$ and $\bar{r} \neq l$ by the assumption of (2), and $\bar{l} \notin I(Y)$ as seen above,

$$(I(Y') \setminus \{\bar{r}\}) \cup \{\bar{l}\} = (I(Y) \setminus \{\bar{r}, l\}) \cup \{r, \bar{l}\} \in \binom{[1, m+n]}{m}.$$

Thus there exists $Y'' \in \mathcal{F}(Y_{m,n})$ such that $I(Y'') = (I(Y) \setminus \{\bar{r}, l\}) \cup \{r, \bar{l}\}$; note that $I(Y''^D) = (I(Y^D) \setminus \{r, \bar{l}\}) \cup \{\bar{r}, l\}$ by Lemma 5.2. It follows that $Y'' \xrightarrow{\bar{r},\bar{l}} Y' \xrightarrow{l,r} Y$. Because $\bar{r} \in I(Y)$ and $\bar{r} \neq l$ by the assumption of (2), and $I_R(Y) \cap I_R(Y^D) = \emptyset$ by assumption, it can be easily verified that $I_R(Y'') \cap I_R(Y''^D) = \emptyset$. Since $l < r$ and $\bar{l} > \bar{r}$, we have $\langle I(Y'') \rangle > \langle I(Y) \rangle$, and hence $Y'' \in \mathcal{S}(Y_{m,n})$ by the induction hypothesis. We see from Lemma 4.4 that $Y'' \xrightarrow{\text{(MHR 2)}} Y$, as desired. □

By Proposition 5.6, we obtain $Y \in \mathcal{S}(Y_{m,n})$. This completes the proof of (III) ⇒ (I), and hence (I) ⇔ (III). The equivalence (II) ⇔ (III) follows from the equivalence (I) ⇔ (III) since $I_R(Y^D) \cap I_R((Y^D)^D) = I_R(Y) \cap I_R(Y^D)$.

Finally, let us show the equivalence (III) $\Leftrightarrow$ (IV). Let $Y \in \mathcal{F}(Y_{m,n})$, and $\lambda = (\lambda_1, \dots, \lambda_m)$ be such that $Y = Y_\lambda$. We first show (IV) $\Rightarrow$ (III). Obviously, if $I_R(Y) \cap I_R(Y^D) \neq \emptyset$, then $I(Y) \cap I(Y^D) \neq \emptyset$. It follows from Subsection 2.1 that

$$\begin{aligned} I(Y) &= \{\lambda_p + m - p + 1 \mid 1 \le p \le m\}, \\ I(Y^D) &= \{n - \lambda_q + q \mid 1 \le q \le m\}. \end{aligned}$$

Hence, $I(Y) \cap I(Y^D) \neq \emptyset$ if and only if $\lambda_i + m - i + 1 = n - \lambda_j + j$ (or equivalently, $\lambda_i + \lambda_j = n - m + i + j - 1$) for some $1 \le i, j \le m$. Thus we have shown (IV) $\Rightarrow$ (III).

We next show (III) $\Rightarrow$ (IV). Assume that $\lambda_i + \lambda_j = n - m + i + j - 1$ for some $1 \le i, j \le m$; we may assume that $i \le j$. As seen above, we have $\lambda_i + m - i + 1 \in I(Y) \cap I(Y^D)$. Hence it suffices to show that if $\lambda_i + \lambda_j = n - m + i + j - 1$, then $\lambda_i + m - i + 1 \in [c + 1 - \chi, m + n]$. Indeed, suppose, for a contradiction, that $\lambda_i + m - i + 1 \notin [c + 1 - \chi, m + n]$. Then, $\lambda_i + m - i + 1 < c + 1 - \chi$ or $m + n < \lambda_i + m - i + 1$. Because $\lambda_i + m - i + 1 \le n + m - i + 1 \le n + m$, we get $\lambda_i + m - i + 1 < c + 1 - \chi$. Since $i \le j$ (and hence $\lambda_i \ge \lambda_j$) and $\lambda_i < c - m - \chi + i$, we have $\lambda_i + \lambda_j \le 2\lambda_i < (m + n - 1 + \chi) - 2m - 2\chi + 2i = n - m - \chi + 2i - 1 \le n - m + i + j - 1 = \lambda_i + \lambda_j$, which is a contradiction. Therefore, we conclude that $\lambda_i + m - i + 1 \in [c + 1 - \chi, m + n]$. Thus we have shown (III) $\Rightarrow$ (IV), thereby completing the proof of (III) $\Leftrightarrow$ (IV).

6. Application.

Let $t \in \mathbb{Z}_{\ge 0}$ and $m, n \in \mathbb{N}$ such that $t \le m \le n$. For a partition $(\lambda_1, \dots, \lambda_t)$, we set

$$[\![\lambda_1, \dots, \lambda_t]\!] := (\lambda_1, \dots, \lambda_t, \lambda_{t+1}, \dots, \lambda_m),$$

with $\lambda_k := 0$ for $t + 1 \le k \le m$.

Theorem 6.1. Under the notation and setting above, $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ if and only if $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$. Moreover, the Grundy value of $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ is equal to the Grundy value of $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$.

Proof. Since $\lambda_k = 0$ for $t + 1 \le k \le m$, it follows from Theorem 5.1 that $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ if and only if $\lambda_i + \lambda_j \neq n - m + i + j - 1$ for all $1 \le i \le j \le t$ and

$$\lambda_s \neq n - m + s + k - 1 \text{ for all } 1 \le s \le t \text{ and } t + 1 \le k \le m; \tag{6.1}$$

note that $0 \neq n - m + k + l - 1$ for all $t + 1 \le k$, $l \le m$ since $m \le n$. Also, notice that (6.1) is equivalent to $\lambda_1 \le n - m + t$. Therefore, we deduce that $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$ if and only if $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$.

Next, we show the assertion on the Grundy values. We can easily check that there exists a natural correspondence between the unimodal numbering on $Y_{(\lambda_1,\dots,\lambda_t)} \in \mathcal{S}(Y_{t,n-m+t})$ and that on $Y_{[\![\lambda_1,\dots,\lambda_t]\!]} \in \mathcal{S}(Y_{m,n})$. Indeed, let (i,j) be a box of $Y_{(\lambda_1,\dots,\lambda_t)}$, or equivalently, a box of $Y_{[\![\lambda_1,\dots,\lambda_t]\!]}$. If $d\,(i,j)$ (resp., $c\,(i,j)$) is the number in the box (i,j) in $Y_{(\lambda_1,\dots,\lambda_t)}$ (resp., $Y_{[\![\lambda_1,\dots,\lambda_t]\!]}$) with respect to the unimodal numbering for $\mathcal{S}(Y_{t,n-m+t})$ (resp., $\mathcal{S}(Y_{m,n})$), then $d\,(i,j) = c\,(i,j) - m + t$. By this observation, we deduce that if some hooks are removed from $Y_{(\lambda_1,\dots,\lambda_t)}$ by the rule of MHRG starting from $Y_{m,n}$, then the same hooks are removed from $Y_{[\![\lambda_1,\dots,\lambda_t]\!]}$ by the rule of MHRG starting from $Y_{t,n-m+t}$, and vice versa. Hence there exists a canonical bijection between the sets $\mathcal{O}(Y_{(\lambda_1,\dots,\lambda_t)}) \subset \mathcal{S}(Y_{t,n-m+t})$ and $\mathcal{O}(Y_{[\![\lambda_1,\dots,\lambda_t]\!]}) \subset \mathcal{S}(Y_{m,n})$ of options. Therefore, the inductive argument shows that the Grundy value of $Y_{(\lambda_1,\dots,\lambda_t)}$ is equal to that of $Y_{[\![\lambda_1,\dots,\lambda_t]\!]}$. This completes the proof of Theorem 6.1. $\square$

Assume that $m = 2$. Set $c_i(q) := c + i + 4q$ for $i \in \mathbb{Z}$ and $q \geq 0$. We know from [1, Theorem 3] that a Young diagram $Y_\lambda \in \mathcal{S}(Y_{2,n})$ with $\lambda = (\lambda_1, \lambda_2)$ is a $\mathcal{P}$-position if and only if

(6.2)
$$\lambda \in \begin{cases} \mathcal{C} \cup \{(c_1(q), c_0(q)), (c_2(q), c_1(q)) \mid 0 \leq q \leq (p-1)\,/\,2\} & \text{if } n-2 = 4p, \\ \mathcal{C} \cup \{(c_2(q), c_1(q)), (c_3(q), c_2(q)) \mid 0 \leq q \leq (p-1)\,/\,2\} & \text{if } n-2 = 4p+1, \\ \mathcal{C} \cup \{(c_0(q), c_{-1}(q)), (c_1(q), c_0(q)) \mid 0 \leq q \leq p\,/\,2\} & \text{if } n-2 = 4p+2, \\ \mathcal{C} \cup \{(2p+4, 2p+2), (2p+5, 2p+4)\} & \\ \qquad \cup \{(c_1(q), c_0(q)), (c_2(q), c_1(q)) \mid 1 \leq q \leq p\,/\,2\} & \text{if } n-2 = 4p+3, \end{cases}$$

where $p \in \mathbb{Z}_{\geq 0}$, and $\mathcal{C} = \mathcal{C}(p) := \{(2q, 2q) \mid 0 \leq q \leq p\}$.

The following is an immediate consequence of Theorem 6.1 and (6.2).

Corollary 6.2. We set $d_i(q) := c - m + 2 + i + 4q$ for $i \in \mathbb{Z}$ and $q \geq 0$. A Young diagram $Y_\lambda \in \mathcal{S}(Y_{m,n})$ having at most two rows is a $\mathcal{P}$-position if and only if

$$\lambda \in \begin{cases} \mathcal{D} \cup \{[\![d_1(q), d_0(q)]\!], [\![d_2(q), d_1(q)]\!] \mid 0 \leq q \leq (p-1)\,/\,2\} & \text{if } n-m = 4p, \\ \mathcal{D} \cup \{[\![d_2(q), d_1(q)]\!], [\![d_3(q), d_2(q)]\!] \mid 0 \leq q \leq (p-1)\,/\,2\} & \text{if } n-m = 4p+1, \\ \mathcal{D} \cup \{[\![d_0(q), d_{-1}(q)]\!], [\![d_1(q), d_0(q)]\!] \mid 0 \leq q \leq p\,/\,2\} & \text{if } n-m = 4p+2, \\ \mathcal{D} \cup \{[\![2p+4, 2p+2]\!], [\![2p+5, 2p+4]\!]\} & \\ \qquad \cup \{[\![d_1(q), d_0(q)]\!], [\![d_2(q), d_1(q)]\!] \mid 1 \leq q \leq p\,/\,2\} & \text{if } n-m = 4p+3, \end{cases}$$

where $p \in \mathbb{Z}_{\geq 0}$, and $\mathcal{D} = \mathcal{D}(p) := \{[\![2q, 2q]\!] \mid 0 \leq q \leq p\}$.

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