ON RECURSIVE CONSTRUCTIONS OF $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-LINEAR HADAMARD CODES

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Abstract. The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes are subgroups of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$. A $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code is a Hadamard code, which is the Gray map image of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code. In this paper, we generalize some known results for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes to $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. First, we give a recursive construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ with $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$. It is known that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Unlike $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, in general, this family of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes does not include the family of $\mathbb{Z}_4$-linear or $\mathbb{Z}_8$-linear Hadamard codes. We show that, for example, for length 2^{11}, the constructed nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard, nor to any previously constructed $\mathbb{Z}_{2^s}$-Hadamard code, with $s \geq 2$. Finally, we also present other recursive constructions of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes having the same type, and we show that, after applying the Gray map, the codes obtained are equivalent to the previous ones.

1. Introduction

Let $\mathbb{Z}_{2^s}$ be the ring of integers modulo 2^s with $s \geq 1$. The set of n-tuples over $\mathbb{Z}_{2^s}$ is denoted by $\mathbb{Z}_{2^s}^n$. In this paper, the elements of $\mathbb{Z}_{2^s}^n$ are also called vectors. A code over $\mathbb{Z}_2$ of length n is a nonempty subset of $\mathbb{Z}_2^n$, and it is linear if it is a subspace of $\mathbb{Z}_2^n$. Similarly, a nonempty subset of $\mathbb{Z}_{2^s}^n$ is a $\mathbb{Z}_{2^s}$-additive code if it is a subgroup of $\mathbb{Z}_{2^s}^n$. A $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code is a subgroup of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$. Note that a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code is a linear code over $\mathbb{Z}_2$ when $\alpha_2 = \alpha_3 = 0$, a $\mathbb{Z}_4$-additive or $\mathbb{Z}_8$-additive code when $\alpha_1 = \alpha_3 = 0$ or $\alpha_1 = \alpha_2 = 0$, respectively, and a $\mathbb{Z}_2\mathbb{Z}_4$-additive code when $\alpha_3 = 0$. The order of a vector $u \in \mathbb{Z}_{2^s}^n$, denoted by $o(u)$, is the smallest positive integer m such that $mu = (0, \ldots, 0)$. Also, the order of a vector

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$\mathbf{u} \in \mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$, denoted by $o(\mathbf{u})$, is the smallest positive integer m such that $m\mathbf{u} = (0, \dots, 0 \mid 0, \dots, 0 \mid 0, \dots, 0)$.

The Hamming weight of a vector $u \in \mathbb{Z}_2^n$, denoted by $\mathrm{wt}_H(u)$, is the number of nonzero coordinates of u. The Hamming distance of two vectors $u, v \in \mathbb{Z}_2^n$, denoted by $d_H(u, v)$, is the number of coordinates in which they differ. Note that $d_H(u, v) = \mathrm{wt}_H(u - v)$. The minimum distance of a code C over $\mathbb{Z}_2$ is $d(C) = \min\{d_H(u, v) : u, v \in C, u \neq v\}$.

In [19], a Gray map from $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ is defined as $\phi(0) = (0, 0)$, $\phi(1) = (0, 1)$, $\phi(2) = (1, 1)$ and $\phi(3) = (1, 0)$. There exist different generalizations of this Gray map, which go from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$ [14, 12, 15, 20, 24]. The one given in [20] can be defined in terms of the elements of a Hadamard code [24], and Carlet's Gray map [14] is a particular case of the one given in [24] satisfying $\sum \lambda_i \phi(2^i) = \phi(\sum \lambda_i 2^i)$ [16]. In this paper, we focus on Carlet's Gray map [14], from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$, which is also a particular case of the one given in [33]. Specifically,

$$\phi_s(u) = (u_{s-1}, u_{s-1}, \dots, u_{s-1}) + (u_0, \dots, u_{s-2})Y_{s-1}, \tag{1}$$

where $u \in \mathbb{Z}_{2^s}$; $[u_0, u_1, \dots, u_{s-1}]_2$ is the binary expansion of u, that is, $u = \sum_{i=0}^{s-1} u_i 2^i$ with $u_i \in \{0, 1\}$; and Y is a matrix of size $(s-1) \times 2^{s-1}$ whose columns are all the vectors in $\mathbb{Z}_2^{s-1}$. Without loss of generality, we assume that the columns of Y_{s-1} are ordered in ascending order by considering the elements of $\mathbb{Z}_2^{s-1}$ as the binary expansions of the elements of $\mathbb{Z}_{2^{s-1}}$. Note that ϕ_1 is the identity map, and

$$\begin{array}{rrcl}
\phi_2 : & \mathbb{Z}_4 & \longrightarrow & \mathbb{Z}_2^2 \\
& 0 & \mapsto & (0,0) \\
& 1 & \mapsto & (0,1) \\
& 2 & \mapsto & (1,1) \\
& 3 & \mapsto & (1,0)
\end{array}
\qquad
\begin{array}{rrcl}
\phi_3 : & \mathbb{Z}_8 & \longrightarrow & \mathbb{Z}_2^4 \\
& 0 & \mapsto & (0,0,0,0) \\
& 1 & \mapsto & (0,1,0,1) \\
& 2 & \mapsto & (0,0,1,1) \\
& 3 & \mapsto & (0,1,1,0) \\
& 4 & \mapsto & (1,1,1,1) \\
& 5 & \mapsto & (1,0,1,0) \\
& 6 & \mapsto & (1,1,0,0) \\
& 7 & \mapsto & (1,0,0,1).
\end{array}$$

We define $\Phi_s : \mathbb{Z}_{2^s}^n \to \mathbb{Z}_2^{n2^{s-1}}$ as the component-wise extended map of ϕ_s. We can also define a Gray map Φ from $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ to $\mathbb{Z}_2^n$, where $n = \alpha_1 + 2\alpha_2 + 4\alpha_3$, as follows:

$$\Phi(u_1 \mid u_2 \mid u_3) = (u_1, \Phi_2(u_2), \Phi_3(u_3)),$$

for any $u_i \in \mathbb{Z}_{2^i}^{\alpha_i}$, where $1 \leq i \leq 3$.

Let $\mathcal{C} \subseteq \mathbb{Z}_{2^s}^n$ be a $\mathbb{Z}_{2^s}$-additive code of length n. We say that the Gray map image of $\mathcal{C}$, say $C = \Phi_s(\mathcal{C})$, is a $\mathbb{Z}_{2^s}$-linear code of length $n2^{s-1}$. Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_{2^s}^n$, it is isomorphic to $\mathbb{Z}_{2^s}^{t_1} \times \mathbb{Z}_{2^{s-1}}^{t_2} \times \dots \times \mathbb{Z}_2^{t_s}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi_s(\mathcal{C})$, is of type $(n; t_1, \dots, t_s)$. Note that

$|\mathcal{C}| = 2^{st_1}2^{(s-1)t_2}\cdots 2^{t_s}$. Similarly, if $\mathcal{C} \subseteq \mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code, we say that its Gray map image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of length $\alpha_1 + 2\alpha_2 + 4\alpha_3$. Since $\mathcal{C}$ can be seen as a subgroup of $\mathbb{Z}_8^{\alpha_1+\alpha_2+\alpha_3}$, it is isomorphic to $\mathbb{Z}_8^{t_1} \times \mathbb{Z}_4^{t_2} \times \mathbb{Z}_2^{t_3}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi(\mathcal{C})$, is of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$. We have that a $\mathbb{Z}_2\mathbb{Z}_4$-linear code $\mathcal{C}$ [10, 11] can can be seen as a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of type $(\alpha_1, \alpha_2, 0; 0, t_2, t_3)$. In this case, we also write that the type of $\mathcal{C}$ is directly $(\alpha_1, \alpha_2; t_2, t_3)$. Unlike linear codes over finite fields, linear codes over rings do not have a basis, but a generator matrix exists for these codes with a minimum number of rows. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$, then $|\mathcal{C}| = 8^{t_1}4^{t_2}2^{t_3}$ and there exist a generator matrix with $t_1 + t_2 + t_3$ rows.

Two structural properties of codes over $\mathbb{Z}_2$ are the rank and dimension of the kernel. The rank of a code C over $\mathbb{Z}_2$ is simply the dimension of the linear span of C, say $\langle C\rangle$. The kernel of a code C over $\mathbb{Z}_2$ is defined as $\mathrm{K}(C) = \{\mathbf{x} \in \mathbb{Z}_2^n : \mathbf{x} + C = C\}$ [3]. If the all-zero vector belongs to C, then $\mathrm{K}(C)$ is a linear subcode of C. Note also that if C is linear, then $K(C) = C = \langle C\rangle$. We denote the rank of C as $\mathrm{rank}(C)$ and the dimension of the kernel as $\ker(C)$. These parameters can be used to distinguish between nonequivalent codes since equivalent ones have the same rank and dimension of the kernel.

A binary code with length n, $2n$ codewords, and minimum distance $n/2$ is called a Hadamard code. Hadamard codes can be constructed from Hadamard matrices [1, 26]. Note that linear Hadamard codes are first-order Reed-Muller codes, or equivalently, the dual of extended Hamming codes [26, Ch.13 §3]. The $\mathbb{Z}_{2^s}$-additive codes such that after the Gray map Φ_s give Hadamard codes are called $\mathbb{Z}_{2^s}$-additive Hadamard codes, and the corresponding images are called $\mathbb{Z}_{2^s}$-linear Hadamard codes. Similarly, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes such that after the Gray map Φ give Hadamard codes are called $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes, and the corresponding images are called $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes.

It is well-known that $\mathbb{Z}_4$-linear Hadamard codes (that is, $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 = 0$) and $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ can be classified by using either the rank or the dimension of the kernel [23, 28]. Moreover, in [25], it is shown that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$. Later, in [16, 6, 18, 4], a recursive construction for $\mathbb{Z}_{p^s}$-linear Hadamard codes, with p prime, is described, the linearity is established, and a partial classification by using the dimension of the kernel is obtained, giving the exact amount of nonequivalent such codes for some parameters. In [17], a complete classification of $\mathbb{Z}_8$-linear Hadamard codes by using the rank and dimension of the kernel is provided, giving the exact amount of nonequivalent such codes. For any $t \geq 2$, the full classification of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear Hadamard codes of length p^t, with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $p \geq 3$ prime, is given in [5, 7], by using just the dimension of the kernel.

The paper contributes to the study of codes over rings $\mathbb{Z}_{p^s}$, which were first studied by Blake [9] and Shankar [30] in 1975 and 1979, respectively. These codes have become more significant after the publication of [19]. It is also important to note that Hadamard codes are two weight codes, which have been widely studied in [31, 32]. On the other hand, the classification of nonlinear Hadamard codes is still an open problem. By giving an additive structure, as $\mathbb{Z}_{p^s}$-linear, $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear or $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes, to some of them, and showing whether they are equivalent or not among them, we are providing a partial classification for these codes.

From a more practical point of view, since Hadamard codes are optimal and have a high correction capability, they appear in different aspects related to the transmission of information, such as in digital communication with satellites [21], in CDMA phones to modulate the transmission of information and minimize interference with other transmissions [34] and, in general, in different OCDMA multiple access systems to allow access to multiple users asynchronously and simultaneously [22]. Other applications are found in cryptography [27] or in information hiding (steganography and watermarking) [35]. See [21] for more applications in other fields.

This paper is focused specifically on $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$, generalizing some results given for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ in [28, 29] related to a recursive construction of such codes. These codes are also compared with the $\mathbb{Z}_4$-linear, $\mathbb{Z}_8$-linear, and in general $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 2$ considered in [16]. In general, the construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes allows to construct codes which are not equivalent to $\mathbb{Z}_{2^s}$-linear Hadamard codes, with $s \geq 2$. It is known that each $\mathbb{Z}_4$-linear Hadamard code is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Unlike $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, in general, this family of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadeamard codes does not include the family of $\mathbb{Z}_4$-linear or $\mathbb{Z}_8$-linear Hadamard codes, or $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 4$. In Example 2.3, we show that all the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [16], with $s \geq 2$, of the same length 2^{11}. This paper is organized as follows. In Section 2, we describe a recursive construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. We emphasise that, unlike $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, in general, this family of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadeamard codes does not include the family of $\mathbb{Z}_4$-linear or $\mathbb{Z}_8$-linear Hadamard codes. In Section 3, we present other recursive constructions and show that we obtain $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes equivalent to the previous ones. Finally, in Section 4, we give some conclusions and further research on this topic.

2. Recursive construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes

The description of generator matrices having a minimum number of rows for $\mathbb{Z}_4$-additive Hadamard, some families of $\mathbb{Z}_{2^s}$-additive Hadamard, and in general $\mathbb{Z}_{p^s}$-additive Hadamard codes, with $s \geq 2$ and p prime, are given in [23], [16], and [6], respectively. Similarly, generator matrices having a minimum number of rows for $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive Hadamard codes with $\alpha_1 \neq 0, \alpha_2 \neq 0$ and p prime, as long as a recursive construction of these matrices, are given in [28, 29] when $p = 2$ and in [5] when $p \geq 3$. In this section, we generalize these results for $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. Specifically, we define a recursive construction for the generator matrices of a family of these codes and establish that they generate $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes.

Let $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{7}$ be the vectors having the elements $0, 1, 2, \dots, 7$ repeated in each coordinate, respectively. If A is a generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code, that is, a subgroup of $\mathbb{Z}_2^{\alpha_1} \times \mathbb{Z}_4^{\alpha_2} \times \mathbb{Z}_8^{\alpha_3}$ for some integers $\alpha_1, \alpha_2, \alpha_3 \geq 0$, then we denote by A_1 the submatrix of A with the first α_1 columns over $\mathbb{Z}_2$, A_2 the submatrix with the next α_2 columns over $\mathbb{Z}_4$, and A_3 the submatrix with the last α_3 columns over $\mathbb{Z}_8$. We have that $A = (A_1 \mid A_2 \mid A_3)$, where the number of columns of A_i is α_i for $i \in \{1, 2, 3\}$.

Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Now, we construct recursively matrices A^{t_1,t_2,t_3} having t_1 rows of order 8, t_2 rows of order 4, and t_3 rows of order 2 as follows. First, we consider the following matrix:

$$A^{1,0,1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right). \tag{2}$$

Then, we apply the following constructions. If we have a matrix $A^{\ell-1,0,1} = (A_1 \mid A_2 \mid A_3)$, with $\ell \geq 2$, we may construct the matrix

$$A^{\ell,0,1} = \left(\begin{array}{cc|ccccc|ccccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 & M_2 & A_3 & A_3 & \cdots & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array}\right), \tag{3}$$

where $M_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0, 2\}^{\ell-1}\}$ and $M_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\} \times \{0, 2, 4, 6\}^{\ell-1}\}$. We perform Construction (3) until $\ell = t_1$. If we have a matrix $A^{t_1,\ell-1,1} = (A_1 \mid A_2 \mid A_3)$, with $t_1 \geq 1$ and $\ell \geq 1$, we may construct the matrix

$$A^{t_1,\ell,1} = \left(\begin{array}{cc|ccccc|cccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 & A_3 & A_3 & A_3 & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \end{array}\right), \tag{4}$$

where $M_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0, 2\}^{t_1+\ell-1}\}$. We repeat Construction (4) until $\ell = t_2$. Finally, if we have a matrix $A^{t_1,t_2,\ell-1} = (A_1 \mid A_2 \mid A_3)$, with $t_1 \geq 1$, $t_2 \geq 0$, and $\ell \geq 2$, we may construct the matrix

$$A^{t_1,t_2,\ell} = \left(\begin{array}{cc|cc|cc} A_1 & A_1 & A_2 & A_2 & A_3 & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{4} \end{array}\right). \tag{5}$$

We repeat Construction (5) until $\ell = t_3$. Thus, in this way, we obtain A^{t_1,t_2,t_3}.

Summarizing, in order to achieve A^{t_1,t_2,t_3} from $A^{1,0,1}$, first we add t_1-1 rows of order 8 by applying Construction (3) t_1-1 times, starting from $A^{1,0,1}$ up to obtain $A^{t_1,0,1}$; then we add t_2 rows of order 4 by applying Construction (4) t_2 times, up to generate $A^{t_1,t_2,1}$; and, finally, we add t_3-1 rows of order 2 by applying Construction (5) t_3-1 times to achieve A^{t_1,t_2,t_3}. Note that the first row always has the row $(\mathbf{1} \mid \mathbf{2} \mid \mathbf{4})$.

Example 2.1. *By using the constructions described in (3), (4), and (5), we obtain the following matrices $A^{2,0,1}$, $A^{1,1,1}$ and $A^{1,1,2}$, respectively, starting from $A^{1,0,1}$ given in (2):*

$$A^{2,0,1} = \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 02 & 1111 & 0246 & 11111111 \\ 00 & 11 & 11 & 0123 & 1111 & 01234567 \end{array}\right), \tag{6}$$

$$A^{1,1,1} = \left(\begin{array}{cc|cc|c} 11 & 11 & 22 & 2222 & 4444 \\ 01 & 01 & 02 & 1111 & 1111 \\ 00 & 11 & 11 & 0123 & 0246 \end{array}\right), \tag{7}$$

$$A^{1,1,2} = \left(\begin{array}{cc|cc|cc} 1111 & 1111 & 222222 & 222222 & 4444 & 4444 \\ 0101 & 0101 & 021111 & 021111 & 1111 & 1111 \\ 0011 & 0011 & 110123 & 110123 & 0246 & 0246 \\ 0000 & 1111 & 000000 & 222222 & 0000 & 4444 \end{array}\right).$$

In order to obtain $A^{2,1,1}$, we start with $A^{1,0,1}$, we apply Construction (3) to obtain $A^{2,0,1} = (A_1 \mid A_2 \mid A_3)$ given in (6), and then we apply (4) to obtain

$$A^{2,1,1} = \left(\begin{array}{cc|ccccc|cccc} A_1 & A_1 & \begin{matrix} 2222 \\ 0022 \\ 0202 \end{matrix} & A_2 & A_2 & A_2 & A_2 & A_3 & A_3 & A_3 & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \end{array}\right).$$

The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code generated by A^{t_1,t_2,t_3} is denoted by $\mathcal{H}^{t_1,t_2,t_3}$, and the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code $\Phi(\mathcal{H}^{t_1,t_2,t_3})$ by H^{t_1,t_2,t_3}.

Lemma 2.1. *Let $t_1 \geq 1$ and $t_2 \geq 0$ be integers. Let $\mathcal{H}^{t_1,t_2,1}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, 1)$ generated by $A^{t_1,t_2,1}$. Then, $2^{t_1+t_2} = \alpha_1$, $4^{t_1+t_2} = \alpha_1 + 2\alpha_2$ and $8^{t_1}4^{t_2} = \alpha_1 + 2\alpha_2 + 4\alpha_3$.*

Proof. First, we prove this lemma for the code $\mathcal{H}^{t_1,0,1}$ by induction on $t_1 \geq 1$. Note that the lemma holds for the code $\mathcal{H}^{1,0,1}$ of type $(2,1,1;1,0,1)$. Assume that the lemma holds for the code $\mathcal{H}^{t_1,0,1}$ of type $(\alpha_1, \alpha_2, \alpha_3; t_1, 0, 1)$, that is,

$$2^{t_1} = \alpha_1, 4^{t_1} = \alpha_1 + 2\alpha_2 \textit{ and } 8^{t_1} = \alpha_1 + 2\alpha_2 + 4\alpha_3. \tag{8}$$

By using Construction (3), the type of $\mathcal{H}^{t_1+1,0,1}$ is $(\alpha'_1, \alpha'_2, \alpha'_3; t_1+1, 0, 1)$, where

$$\alpha'_1 = 2\alpha_1, \alpha'_2 = 2^{t_1} + 4\alpha_2 \textit{ and } \alpha'_3 = 4^{t_1} + 8\alpha_3. \tag{9}$$

Thus, from (8) and (9), $2^{t_1+1} = 2\alpha_1 = \alpha_1'$, $4^{t_1+1} = 4\alpha_1 + 8\alpha_2 = 2\alpha_1 + 2\alpha_1 + 8\alpha_2 = \alpha_1' + 2^{t_1+1} + 8\alpha_2 = \alpha_1' + 2\alpha_2'$ and $8^{t_1+1} = 8\alpha_1 + 16\alpha_2 + 32\alpha_3 = 2\alpha_1 + (2\alpha_1 + 8\alpha_2) + (4\alpha_1 + 8\alpha_2 + 32\alpha_3) = 2\alpha_1 + (2^{t_1+1} + 8\alpha_2) + (4^{t_1+1} + 32\alpha_3) = \alpha_1' + 2\alpha_2' + 4\alpha_3'$. Therefore, the lemma holds for the code $\mathcal{H}^{t_1,0,1}$.

Next, we prove this lemma for the code $\mathcal{H}^{t_1,t_2,1}$ by induction on $t_2 \geq 0$. Assume that the lemma holds for the code $\mathcal{H}^{t_1,t_2,1}$ of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, 1)$, that is,

$$2^{t_1+t_2} = \alpha_1, 4^{t_1+t_2} = \alpha_1 + 2\alpha_2, \textit{ and } 8^{t_1}4^{t_2} = \alpha_1 + 2\alpha_2 + 4\alpha_3. \tag{10}$$

By using Construction (4), the type of $\mathcal{H}^{t_1,t_2+1,1}$ is $(\alpha_1', \alpha_2', \alpha_3'; t_1, t_2 + 1, 1)$, where

$$\alpha_1' = 2\alpha_1, \alpha_2' = 2^{t_1+t_2} + 4\alpha_2 \textit{ and } \alpha_3' = 4\alpha_3. \tag{11}$$

Thus, from (10) and (11), $2^{t_1+(t_2+1)} = 2\alpha_1 = \alpha_1'$, $4^{t_1+(t_2+1)} = 4\alpha_1 + 8\alpha_2 = 2\alpha_1 + 2\alpha_1 + 8\alpha_2 = \alpha_1' + 2^{t_1+t_2+1} + 8\alpha_2 = \alpha_1' + 2\alpha_2'$ and $8^{t_1}4^{t_2+1} = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 = 2\alpha_1 + (2\alpha_1 + 8\alpha_2) + 16\alpha_3 = \alpha_1' + (2^{t_1+t_2+1} + 8\alpha_2) + 4\alpha_3' = \alpha_1' + 2\alpha_2' + 4\alpha_3'$. Therefore, the lemma holds for the code $\mathcal{H}^{t_1,t_2+1,1}$. This completes the proof. □

Proposition 2.1. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let $\mathcal{H}^{t_1,t_2,t_3}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ generated by A^{t_1,t_2,t_3}. Then,*

$$\begin{aligned} \alpha_1 &= 2^{t_1+t_2+t_3-1}, \\ \alpha_1 + 2\alpha_2 &= 4^{t_1+t_2}2^{t_3-1}, \\ \alpha_1 + 2\alpha_2 + 4\alpha_3 &= 8^{t_1}4^{t_2}2^{t_3-1}. \end{aligned} \tag{12}$$

Proof. We prove this result for the code $\mathcal{H}^{t_1,t_2,t_3}$ by induction on $t_3 \geq 1$. By Lemma 2.1, the proposition holds for $t_3 = 1$, that is, for the code $\mathcal{H}^{t_1,t_2,1}$. Assume that it holds for the code $\mathcal{H}^{t_1,t_2,t_3}$ of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$, that is, (12) holds. By using Construction (5), the type of $\mathcal{H}^{t_1,t_2,t_3+1}$ is $(\alpha_1', \alpha_2', \alpha_3'; t_1, t_2, t_3 + 1)$, where

$$\alpha_1' = 2\alpha_1, \alpha_2' = 2\alpha_2, \text{ and } \alpha_3' = 2\alpha_3. \tag{13}$$

Thus, from (12) and (13), $2^{t_1+t_2+t_3} = 2\alpha_1 = \alpha_1'$, $4^{t_1+t_2}2^{t_3} = 2\alpha_1 + 4\alpha_2 = \alpha_1' + 2\alpha_2'$ and $8^{t_1}4^{t_2}2^{t_3} = 2\alpha_1 + 4\alpha_2 + 8\alpha_3 = \alpha_1' + 2\alpha_2' + 4\alpha_3'$. Therefore, the proposition is true for the code $\mathcal{H}^{t_1,t_2,t_3+1}$. This completes the proof. □

Corollary 2.1. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let $\mathcal{H}^{t_1,t_2,t_3}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ generated by A^{t_1,t_2,t_3}. Then,*

$$\begin{aligned} \alpha_1 &= 2^{t_1+t_2+t_3-1}, \\ \alpha_2 &= 4^{t_1+t_2}2^{t_3-2} - 2^{t_1+t_2+t_3-2}, \\ \alpha_3 &= 8^{t_1}4^{t_2-1}2^{t_3-1} - 4^{t_1+t_2-1}2^{t_3-1}. \end{aligned}$$

Remark 2.1. *By Corollary 2.1, we have that the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive codes $\mathcal{H}^{t_1,t_2,t_3}$ of type $(\alpha_1,\alpha_2,\alpha_3;t_1,t_2,t_3)$ generated by A^{t_1,t_2,t_3}, so constructed recursively from (3), (4), and (5), satisfy that $\alpha_1\neq 0$, $\alpha_2\neq 0$, and $\alpha_3\neq 0$.*

Remark 2.2. *The construction of the generator matrices A^{t_1,t_2,t_3} is a generalization of the recursive construction of the generator matrices of the $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes of type $(\alpha_1,\alpha_2;t_2,t_3)$ with $\alpha_1\neq 0$ and $\alpha_2\neq 0$, given in* [29]. *Note that if we do not consider the coordinates over $\mathbb{Z}_8$ in Constructions (3), (4), and (5), we have that (3) and (4) become*

$$A^{\ell,1}=\left(\begin{array}{cc|ccccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2\\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3}\end{array}\right), \tag{14}$$

where $A^{\ell-1,1}=(A_1\mid A_2)$ and $M_1=2A_1=\{\mathbf{z}^T:\mathbf{z}\in\{2\}\times\{0,2\}^{\ell-1}\}$ (up to a column permutation); and Construction (5) become

$$A^{t_2,\ell}=\left(\begin{array}{cc|cc} A_1 & A_1 & A_2 & A_2\\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2}\end{array}\right), \tag{15}$$

where $A^{t_2,\ell-1}=(A_1\mid A_2)$. Then, starting from the following matrix:

$$A^{1,1}=\left(\begin{array}{cc|c} 1 & 1 & 2\\ 0 & 1 & 1\end{array}\right), \tag{16}$$

and applying (14) and (15) in the same way as above, we obtain the generator matrices A^{t_2,t_3} of the known $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes of type $(\alpha_1,\alpha_2;t_2,t_3)$ with $\alpha_1\neq 0$ and $\alpha_2\neq 0$ [28, 29]. *The $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by A^{t_2,t_3} is denoted by $\mathcal{H}^{t_2,t_3}$, and the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code $\Phi(\mathcal{H}^{t_2,t_3})$ by H^{t_2,t_3}.*

When we include all the elements of $\mathbb{Z}_{2^i}$, where $1\leq i\leq 3$, as coordinates of a vector, we place them in increasing order. For a set $S\subseteq\mathbb{Z}_{2^i}$ and $\lambda\in\mathbb{Z}_{2^i}$, where $i\in\{1,2,3\}$, we define $\lambda S=\{\lambda j : j\in S\}$ and $S+\lambda=\{j+\lambda : j\in S\}$. As before, when including all the elements in those sets as coordinates of a vector, we place them in increasing order. For example, $2\mathbb{Z}_8=\{0,4,6,8\}$, $(\mathbb{Z}_4,\mathbb{Z}_4)=(0,1,2,3,0,1,2,3)\in\mathbb{Z}_4^8$ and $(\mathbb{Z}_2\mid\mathbb{Z}_4\mid 2\mathbb{Z}_8,4\mathbb{Z}_8)=(0,1\mid 0,1,2,3\mid 0,2,4,6,0,4)\in\mathbb{Z}_2^2\times\mathbb{Z}_4^4\times\mathbb{Z}_8^6$.

Lemma 2.2. *Let $1\leq i\leq 3$ and $j\in\{0,1,\dots,i-1\}$.*

1. *(1) If $\mu\in 2^j\mathbb{Z}_{2^i}$, then $2^j\mathbb{Z}_{2^i}+\mu=2^j\mathbb{Z}_{2^i}$.*
2. *(2) If $\mu\in 2^j\mathbb{Z}_{2^i}$, then $(2^j\mathbb{Z}_{2^i},\overset{m}{\dots},2^j\mathbb{Z}_{2^i})+\mu\mathbf{1}$, where $m\geq 1$, is a permutation of the vector $(2^j\mathbb{Z}_{2^i},\overset{m}{\dots},2^j\mathbb{Z}_{2^i})$.*
3. *(3) If $\mu\in 2\mathbb{Z}_{2^i}$, then $(\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i})+\mu=\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$.*
4. *(4) If $\mu\in\mathbb{Z}_{2^i}$, then $(\mathbf{0},\dots,\mathbf{2^i-1})+(\mu,\overset{\ell\cdot 2^i}{\dots},\mu)$, where $\ell\geq 1$ and $\mathbf{k}=(k,\overset{\ell}{\dots},k)$ for $k\in\mathbb{Z}_{2^i}$, is a permutation of $(\mathbb{Z}_{2^i},\overset{\ell}{\dots},\mathbb{Z}_{2^i})$.*

Proof. Item 1 follows from the fact that $\mathbb{Z}_{2^i}$ is a ring and $2^j\mathbb{Z}_{2^i}$ is an ideal of $\mathbb{Z}_{2^i}$. Item 2 follows from Item 1.

For Item 3, it $x\in(\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i})+\mu$, then $x-\mu\in\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$. Assume that $x\notin\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$, so $x\in 2\mathbb{Z}_{2^i}$. Since $2\mathbb{Z}_{2^i}$ is an ideal of $\mathbb{Z}_{2^i}$, we have that

$x-\mu \in 2\mathbb{Z}_{2^i}$, which is a contradiction. Thus, $x \in \mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$ and hence $(\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i})+\mu \subseteq \mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$. In the same way, $(\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i})-\mu \subseteq \mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$. Hence, $\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i} \subseteq (\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i})+\mu$ and therefore $(\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i})+\mu = \mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$.

For Item 4, note that $(\mathbf{0},\dots,\mathbf{2^i-1})+(\mu,\overset{\ell\cdot 2^i}{\ldots},\mu)$ is a permutation of

$$(\mathbb{Z}_{2^i},\overset{\ell}{\ldots},\mathbb{Z}_{2^i})+(\mu,\overset{\ell\cdot 2^i}{\ldots},\mu). \tag{17}$$

Since $\mathbb{Z}_{2^i}+\mu=\mathbb{Z}_{2^i}$, (17) is a permutation of $(\mathbb{Z}_{2^i},\overset{\ell}{\ldots},\mathbb{Z}_{2^i})$. $\square$

Lemma 2.3. *Let $1\le i\le 3$, $\lambda\in\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$, and $u\in\mathbb{Z}_{2^i}^n$. Then,*

$$(u,\overset{2^i}{\ldots},u)+\lambda(\mathbf{0},\dots,\mathbf{2^i-1})$$

is a permutation of $(\mathbb{Z}_{2^i},\overset{n}{\ldots},\mathbb{Z}_{2^i})$.

Proof. Since $\lambda\in\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i}$, $\lambda(\mathbf{0},\dots,\mathbf{2^i-1})$ is a permutation of $(\mathbf{0},\dots,\mathbf{2^i-1})$ and we may consider $\lambda=1$. Then, $(u,\dots,u)+(\mathbf{0},\dots,\mathbf{2^i-1})$ is a permutation of $(u_1+\mathbb{Z}_{2^i},\dots,u_n+\mathbb{Z}_{2^i})=(\mathbb{Z}_{2^i},\overset{n}{\ldots},\mathbb{Z}_{2^i})$, where $u=(u_1,\dots,u_n)$. $\square$

Lemma 2.4. *Let $u=(\mu,\overset{m}{\ldots},\mu,2\mathbb{Z}_4,\overset{n}{\ldots},2\mathbb{Z}_4,\mathbb{Z}_4\backslash 2\mathbb{Z}_4,\overset{r}{\ldots},\mathbb{Z}_4\backslash 2\mathbb{Z}_4)\in\mathbb{Z}_4^{m+2n+2r}$, where $m,n,r\ge 0$ and $\mu\in\mathbb{Z}_4\backslash 2\mathbb{Z}_4=\{1,3\}$. Then,*

$$(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$$

is a permutation of $(2\mathbb{Z}_4,\overset{4n}{\ldots},2\mathbb{Z}_4,\mathbb{Z}_4\backslash 2\mathbb{Z}_4,\overset{4r+2m}{\ldots},\mathbb{Z}_4\backslash 2\mathbb{Z}_4)$.

Proof. By Items 1 and 3 of Lemma 2.2, $u+\mathbf{2}$ is a permutation of $(\mu+2,\overset{m}{\ldots},\mu+2,2\mathbb{Z}_4,\overset{n}{\ldots},2\mathbb{Z}_4,\mathbb{Z}_4\backslash 2\mathbb{Z}_4,\overset{r}{\ldots},\mathbb{Z}_4\backslash 2\mathbb{Z}_4)$. Let $\mathbf{k}=(\mu,\overset{m}{\ldots},\mu)$. Since $\mu\in\{1,3\}$, we have that $(\mathbf{k},\mathbf{k},\mathbf{k},\mathbf{k})+(\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$ is a permutation of $(\mathbb{Z}_4\backslash 2\mathbb{Z}_4,\overset{2m}{\ldots},\mathbb{Z}_4\backslash 2\mathbb{Z}_4)$. Therefore, $(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$ is a permutation of $(2\mathbb{Z}_4,\overset{4n}{\ldots},2\mathbb{Z}_4,\mathbb{Z}_4\backslash 2\mathbb{Z}_4,\overset{4r+2m}{\ldots},\mathbb{Z}_4\backslash 2\mathbb{Z}_4)$. $\square$

Lemma 2.5. *Let $u=(\mu',\overset{m'}{\ldots},\mu',\mu'',\overset{m'}{\ldots},\mu'',2\mathbb{Z}_8,\overset{n'}{\ldots},2\mathbb{Z}_8,\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{r'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)\in\mathbb{Z}_8^{2m'+4n'+4r'}$, where $m',n',r'\ge 0$ and $\mu,\mu'\in\mathbb{Z}_8\backslash 2\mathbb{Z}_8=\{1,3,5,7\}$. Then,*

1. *(1) $(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(2\mathbb{Z}_8,\overset{4n'}{\ldots},2\mathbb{Z}_8,\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{4r'+2m'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)$;*
2. *(2) $(u,u,u,u)+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ is a permutation of $(\mu',\overset{4m'}{\ldots},\mu',\mu'+4,\overset{4m'}{\ldots},\mu'+4,2\mathbb{Z}_8,\overset{4n'}{\ldots},2\mathbb{Z}_8,\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{4r'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ if $\mu'=\mu''$ or $\mu'=\mu''+4$, or a permutation of $(2\mathbb{Z}_8,\overset{4n'}{\ldots},2\mathbb{Z}_8,\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{4r'+2m'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ otherwise.*

Proof. For Item 1, by Items 1 and 3 of Lemma 2.2, if $j\in\{0,2,4,6\}$, then $u+\mathbf{j}$ is a permutation of $(\mu'+j,\overset{m'}{\ldots},\mu'+j,\mu''+j,\overset{m'}{\ldots},\mu''+j,2\mathbb{Z}_8,\overset{n'}{\ldots},2\mathbb{Z}_8,\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{r'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)$. Let $\mathbf{k}'=(\mu',\overset{m'}{\ldots},\mu',\mu''\ \overset{m'}{\ldots},\mu'')$. Since $\mu',\mu''\in\{1,3,5,7\}$, we have that $(\mathbf{k}',\overset{4}{\ldots},\mathbf{k}')+(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{2m'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ and hence $(u,u,u,u)+(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of $(2\mathbb{Z}_8,\overset{4n'}{\ldots},2\mathbb{Z}_8,\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{4r'+2m'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)$.

For item 2, we have $(\mathbf{k}',\overset{4}{\ldots},\mathbf{k}')+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ is a permutation of $(\mu',\overset{4m'}{\ldots},\mu',\mu'+4,\overset{4m'}{\ldots},\mu'+4)$ if $\mu'=\mu''$ or $\mu'=\mu''+4$, or a permutation of $(\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{2m'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ otherwise. Therefore, $(u,u,u,u)+(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ is a permutation of $(\mu',\overset{4m'}{\ldots},\mu',\mu'+4,\overset{4m'}{\ldots},\mu'+4,2\mathbb{Z}_8,\overset{4n'}{\ldots},2\mathbb{Z}_8,\mathbb{Z}_8\backslash 2\mathbb{Z}_8,\overset{4r'}{\ldots},\mathbb{Z}_8\backslash 2\mathbb{Z}_8)$

if $\mu' = \mu''$ or $\mu' = \mu'' + 4$, or a permutation of $(2\mathbb{Z}_8, \overset{4n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{4r'+2m'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ otherwise. $\square$

Lemma 2.6. *Let* $u = (\mu, \overset{m}{\ldots}, \mu, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8) \in \mathbb{Z}_8^{m+2n+2r}$, *where* $m, n, r \geq 0$ *and* $\mu \in 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8 = \{2, 6\}$. *Then,*

(1) $(u, u, u, u) + (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ *is a permutation of* $(2\mathbb{Z}_8, \overset{2r+2n+m}{\ldots}, 2\mathbb{Z}_8)$;
(2) $(u, u, u, u) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ *is a permutation of* $(4\mathbb{Z}_8, \overset{4n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{4r+2m}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$.

Proof. By Item 1 of Lemma 2.2, if $j \in \{0, 4\}$, then $u + \mathbf{j}$ is a permutation of $(\mu + j, \overset{m}{\ldots}, \mu + j, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$. Similarly, if $j \in \{2, 6\}$, then $u + \mathbf{j}$ is a permutation of $(\mu + j, \overset{m}{\ldots}, \mu + j, 4\mathbb{Z}_8, \overset{r}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{n}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$. Let $\mathbf{k} = (\mu, \overset{m}{\ldots}, \mu)$.

For Item 1, since $\mu \in \{2, 6\}$, we have that $(\mathbf{k}, \overset{4}{\ldots}, \mathbf{k}) + (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{m}{\ldots}, 2\mathbb{Z}_8)$, and hence $(u, u, u, u) + (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{2r+2n+m}{\ldots}, 2\mathbb{Z}_8)$.

For Item 2, we have $(\mathbf{k}, \overset{4}{\ldots}, \mathbf{k}) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ is a permutation of $(2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{2m}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$. Therefore, $(u, u, u, u) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ is a permutation of $(4\mathbb{Z}_8, \overset{4n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{4r+2m}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$. $\square$

Let $t_1 \geq 1, t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let $\mathcal{G}^{t_1,t_2,t_3}$ be the set of all codewords of the code generated by the matrix obtained from A^{t_1,t_2,t_3} after removing the row $(\mathbf{1} \mid \mathbf{2} \mid \mathbf{4})$.

Lemma 2.7. *Let* $t_1 \geq 1$ *be an integer. Let*

$$\mathbf{z} = (u_1, u_1 \mid x_1, u_2, u_2, u_2, u_2 \mid x_2, u_3, \overset{8}{\ldots}, u_3) \in \mathcal{G}^{t_1+1,0,1},$$

where $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,0,1}$ *and* $x_{i-1} \in (2\mathbb{Z}_{2^i})^{2^{(i-1)t_1}}$ *for* $i \in \{2, 3\}$. *Then,*

(1) *if* $o(\mathbf{z}) = 8$, *then* x_{i-1} *is a permutation of* $(2\mathbb{Z}_{2^i}, \overset{2^{(i-1)(t_1-1)}}{\ldots}, 2\mathbb{Z}_{2^i})$ *for* $i \in \{2, 3\}$.
(2) *if* $o(\mathbf{z}) = 4$, *then* $x_1 = \mathbf{0}$ *and* x_2 *is a permutation of* $(4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\ldots}, 4\mathbb{Z}_8)$.
(3) *if* $o(\mathbf{z}) = 2$, *then* $x_1 = \mathbf{0}$ *and* $x_2 = \mathbf{0}$.

Proof. Let $\mathbf{w}_j$, where $j \in \{1, \ldots, t_1 + 2\}$, be the jth row of $A^{t_1+1,0,1}$. Note that $\mathbf{w}_1 = (\mathbf{1} \mid \mathbf{2} \mid \mathbf{4})$, and $\mathbf{w}_2, \ldots, \mathbf{w}_{t_1+2}$ are the rows of order 8, where $\mathbf{w}_{t_1+2} = (\mathbf{0}, \mathbf{1} \mid \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3} \mid \mathbf{1}, \mathbf{0}, \ldots, \mathbf{7})$. Since any element of $\mathcal{G}^{t_1+1,0,1}$ can be written as $\mathbf{z} + \lambda \mathbf{w}_{t_1+2}$, where $\lambda \in \mathbb{Z}_8$, then $\mathbf{z} = \sum_{j=2}^{t_1+1} r_j \mathbf{w}_j = (u_1, u_1 \mid x_1, u_2, u_2, u_2, u_2 \mid x_2, u_3, \overset{8}{\ldots}, u_3)$, where $r_j \in \mathbb{Z}_8$. By construction, x_1 and x_2 are generated by the rows of $M_1' = \{\mathbf{z}^T : \mathbf{z} \in \{0, 2\}^{t_1}\}$ and $M_2' = \{\mathbf{z}^T : \mathbf{z} \in \{0, 2, 4, 6\}^{t_1}\}$, respectively. Thus, $x_1 = \mathbf{0}$ or x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1-1}}{\ldots}, 2\mathbb{Z}_4)$, and $x_2 = \mathbf{0}$ or x_2 is a permutation of $(2\mathbb{Z}_8, \overset{4^{t_1-1}}{\ldots}, 2\mathbb{Z}_8)$ or $(4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\ldots}, 4\mathbb{Z}_8)$.

For Item 1, there exists at least one $j \in \{2, \ldots, t_1 + 1\}$ such that $r_j \in \{1, 3, 5, 7\}$. Therefore, by Item 1 of Lemma 2.2, x_{i-1} is a permutation of $(2\mathbb{Z}_{2^i}, \overset{2^{(i-1)(t_1-1)}}{\ldots}, 2\mathbb{Z}_{2^i})$ for $i \in \{2, 3\}$.

For Item 2, we have that $r_j \in 2\mathbb{Z}_8$ for all $j \in \{2,\dots,t_1+1\}$ and there exist at least one $j \in \{2,\dots,t_1+1\}$ such that $r_j \in \{2,6\}$. Therefore, $x_1 = \mathbf{0}$ and, by Item 1 of Lemma 2.2, x_2 is a permutation of $(4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\dots}, 4\mathbb{Z}_8)$.

For Item 3, we have that $r_j \in 4\mathbb{Z}_8$ for all $j \in \{2,\dots,t_1+1\}$ and there exist at least one $j \in \{2,\dots,t_1+1\}$ such that $r_j = 4$. Therefore, $x_1 = \mathbf{0}$ and $x_2 = \mathbf{0}$. $\square$

Lemma 2.8. *Let $t_1 \geq 1$ and $t_2 \geq 0$ be integers. Let*

$$\mathbf{z} = (u_1, u_1 \mid x_1, u_2, u_2, u_2, u_2 \mid u_3, u_3, u_3, u_3) \in \mathcal{G}^{t_1,t_2+1,1},$$

where $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,t_2,1}$ and $x_1 \in (2\mathbb{Z}_4)^{2^{t_1+t_2}}$. Then,

(1) *if $o(\mathbf{z}) = 8$, then x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\dots}, 2\mathbb{Z}_4)$.*
(2) *if $o(\mathbf{z}) = 4$, then $x_1 = \mathbf{0}$ if $u_1 = \mathbf{0}$, and x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\dots}, 2\mathbb{Z}_4)$ otherwise.*
(3) *if $o(\mathbf{z}) = 2$, then $x_1 = \mathbf{0}$.*

Proof. Let $\mathbf{w}_i$, where $i \in \{1,\dots,t_1+t_2+2\}$, be the ith row of $A^{t_1,t_2+1,1}$. Note that $\mathbf{w}_1 = (\mathbf{1} \mid \mathbf{2} \mid \mathbf{4})$, $\mathbf{w}_2,\dots,\mathbf{w}_{t_1+1}$ are the rows of order 8, and $\mathbf{w}_{t_1+2},\dots,\mathbf{w}_{t_1+t_2+2}$ are the rows of order 4, where $\mathbf{w}_{t_1+t_2+2} = (\mathbf{0},\mathbf{1} \mid \mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3} \mid \mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$. Since any element of $\mathcal{G}^{t_1,t_2+1,1}$ can be written as $\mathbf{z} + \lambda \mathbf{w}_{t_1+t_2+2}$, where $\lambda \in \{0,1,2,3\}$, then $\mathbf{z} = \sum_{i=2}^{t_1+t_2+1} r_i \mathbf{w}_i = (u_1, u_1 \mid x_1, u_2, u_2, u_2, u_2 \mid u_3, u_3, u_3, u_3)$, where $r_i \in \mathbb{Z}_8$ for $i \in \{2,\dots,t_1+1\}$ and $r_i \in \{0,1,2,3\}$ for $i \in \{t_1+2,\dots,t_1+t_2+1\}$. By construction, x_1 is generated by the rows of $M_1' = \{\mathbf{z}^T : \mathbf{z} \in \{0,2\}^{t_1+t_2}\}$. Thus, $x_1 = \mathbf{0}$ or x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\dots}, 2\mathbb{Z}_4)$.

For Item 1, there exists at least one $i \in \{2,\dots,t_1+1\}$ such that $r_i \in \{1,3,5,7\}$. Therefore, since x_1 is of order at most two, $x_1 \neq \mathbf{0}$.

For Item 2, we have that $r_i \in 2\mathbb{Z}_8$ for all $i \in \{2,\dots,t_1+1\}$ and $r_i \in \{0,1,2,3\}$ for all $i \in \{t_1+2,\dots,t_1+t_2+1\}$. Note that, since x_1 and u_1 are of order at most two, $x_1 \neq \mathbf{0}$ if and only if there exists at least one i for $i \in \{t_1+2,\dots,t_1+t_2+1\}$ such that $r_i \in \{1,3\}$, or equivalently, if and only if $u_1 \neq \mathbf{0}$.

For Item 2, we have that $r_i \in 4\mathbb{Z}_8 = \{0,4\}$ for all $i \in \{2,\dots,t_1+1\}$ and $r_i \in \{0,2\}$ for all $i \in \{t_1+2,\dots,t_1+t_2+1\}$. Therefore, since x_1 is of order at most two, $x_1 = \mathbf{0}$. $\square$

Lemma 2.9. *Let $t_1 \geq 1$ be an integer. Let $\mathcal{H}^{t_1,0,1}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1,\alpha_2,\ \alpha_3;\ t_1,0,1)$ generated by $A^{t_1,0,1}$. Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,0,1}$. Then,*

(1) *if $o(\mathbf{u}) = 8$, then u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of $(\mu, \overset{m}{\dots}, \mu, 2\mathbb{Z}_4, \overset{n}{\dots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\dots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ for some integers $m,n,r \geq 0$ and $\mu \in \{1,3\}$, and u_3 is a permutation of $(\mu', \overset{m'}{\dots}, \mu', \mu'', \overset{m'}{\dots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\dots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\dots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ for some integers $m',n',r' \geq 0$ and $\mu,\mu' \in \{1,3,5,7\}$.*
(2) *if $o(\mathbf{u}) = 4$, then $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\frac{1}{2}(\frac{\alpha_1}{2} + \alpha_2) = 4^{t_1-1}$ times and $\frac{\alpha_2}{2} - \frac{\alpha_1}{4} = 4^{t_1-1} - 2^{t_1-1}$*

times the element 0*, and* u_3 *is a permutation of* $(\mu,\overset{m}{\ldots},\mu,4\mathbb{Z}_8,\overset{n}{\ldots},4\mathbb{Z}_8,2\mathbb{Z}_8\backslash 4\mathbb{Z}_8,\overset{r}{\ldots},2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$ *for some integers* $m,n,r\geq 0$ *and* $\mu\in\{2,6\}$*.*

(3) *if* $o(\mathbf{u})=2$*, then* $u_1=\mathbf{0}$*,* $u_2=\mathbf{0}$*, and* u_3 *contains the element in* $4\mathbb{Z}_8\backslash\{0\}=\{4\}$ *exactly* $\frac{1}{4}(\frac{\alpha_1}{2}+\alpha_2+2\alpha_3)=8^{t_1-1}$ *times and* $\frac{\alpha_3}{2}-\frac{1}{4}(\frac{\alpha_1}{2}+\alpha_2)=8^{t_1-1}-4^{t_1-1}$ *times the element* 0*.*

Proof. We perform a proof by induction on $t_1\geq 1$. If $t_1=1$, then by Lemma 2.1, $\alpha_1=2$, $\alpha_2=1$, $\alpha_3=1$, and $\mathcal{G}^{1,0,1}=\langle(0,1\mid 1\mid 1)\rangle$. Let $\mathbf{u}=(u_1\mid u_2\mid u_3)\in\mathcal{G}^{1,0,1}$. Then, $\mathbf{u}=\lambda(0,1\mid 1\mid 1)$, where $\lambda\in\mathbb{Z}_8$. Thus, we have that $u_1=\lambda(0,1)$, $u_2=(\lambda)$, and $u_3=(\lambda)$. If $o(\mathbf{u})=8$, then $\lambda\in\mathbb{Z}_8\backslash 2\mathbb{Z}_8$. Therefore, $\mathbf{u}$ satisfies Property 1. If $o(\mathbf{u})=4$, then $\lambda\in\{2,6\}$. In this case, $u_1=(0,0)$, $u_2=(2)$ contains the element in $2\mathbb{Z}_4\backslash\{0\}=\{2\}$ exactly $1=\frac{1}{2}(\frac{\alpha_1}{2}+\alpha_2)$ time and $0=\frac{\alpha_2}{2}-\frac{\alpha_1}{4}$ times the element 0, and $u_3=(\lambda)$. Thus, $\mathbf{u}$ satisfies Property 2. If $o(\mathbf{u})=2$, then $\lambda=4$. In this case, $u_1=(0,0)$, $u_2=(0)$, and $u_3=(4)$ contains the element in $4\mathbb{Z}_8\backslash\{0\}=\{4\}$ exactly $1=\frac{1}{4}(\frac{\alpha_1}{2}+\alpha_2+2\alpha_3)$ time and $0=\frac{\alpha_3}{2}-\frac{1}{4}(\frac{\alpha_1}{2}+\alpha_2)$ times the element 0. Thus, $\mathbf{u}$ satisfies Property 3. Therefore, the lemma holds for $t_1=1$.

Assume now that the lemma holds for the code $\mathcal{H}^{t_1,0,1}$ of type $(\alpha_1,\alpha_2,\alpha_3;t_1,0,1)$ with $t_1\geq 1$. By Lemma 2.1, we have that

$$2^{t_1}=\alpha_1, 4^{t_1}=\alpha_1+2\alpha_2, \text{ and } 8^{t_1}=\alpha_1+2\alpha_2+4\alpha_3. \tag{18}$$

We must show that the lemma is also true for the code $\mathcal{H}^{t_1+1,0,1}$.

Let $\mathbf{v}=(v_1\mid v_2\mid v_3)\in\mathcal{G}^{t_1+1,0,1}$. We can write

$$\mathbf{v}=\mathbf{z}+\lambda\mathbf{w},$$

where $\mathbf{z}=(u_1,u_1\mid x_1,u_2,u_2,u_2,u_2\mid x_2,u_3,\overset{8}{\ldots},u_3)$, $\mathbf{w}=(\mathbf{0},\mathbf{1}\mid\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\mid\mathbf{1},\mathbf{0},\ldots,\mathbf{7})$, $\mathbf{u}=(u_1\mid u_2\mid u_3)\in\mathcal{G}^{t_1,0,1}$, $\lambda\in\mathbb{Z}_8$, $x_1\in(2\mathbb{Z}_4)^{2^{t_1}}$ such that either $x_1=\mathbf{0}$ or x_1 is a permutation of $(2\mathbb{Z}_4,\overset{2^{t_1-1}}{\ldots},2\mathbb{Z}_4)$, and $x_2\in(2\mathbb{Z}_8)^{4^{t_1}}$ such that either $x_2=\mathbf{0}$ or x_2 is a permutation of $(2\mathbb{Z}_8,\overset{4^{t_1-1}}{\ldots},2\mathbb{Z}_8)$ or $(4\mathbb{Z}_8,\overset{2\cdot 4^{t_1-1}}{\ldots},4\mathbb{Z}_8)$. Then, $v_1=(u_1,u_1)+\lambda(\mathbf{0},\mathbf{1})$ and, for $i\in\{2,3\}$,

$$v_i=(x_{i-1},u_i,\overset{2^i}{\ldots},u_i)+\lambda(\mathbf{1},\mathbf{0},\ldots,\mathbf{2^i}-\mathbf{1}). \tag{19}$$

If $\mathbf{z}=\mathbf{0}$, then $\mathbf{v}=\lambda\mathbf{w}$ and it is easy to see that $\mathbf{v}$ satisfies Property 1 if $\lambda\in\mathbb{Z}_8\backslash 2\mathbb{Z}_8=\{1,3,5,7\}$, Property 2 if $\lambda\in\{2,6\}$, and Property 3 if $\lambda=4$. Therefore, we focus on the case when $\mathbf{z}\neq\mathbf{0}$.

Case 1: Assume that $o(\mathbf{v})=8$. We have two subcases: when $o(\mathbf{z})$ is arbitrary and $\lambda\in\mathbb{Z}_8\backslash 2\mathbb{Z}_8$, and when $o(\mathbf{z})=8$ and $\lambda\in 2\mathbb{Z}_8$. In both subcases, note that v_1 contains every element of $\mathbb{Z}_2$ the same number of times. For the first subcase, we have that $(u_i,\overset{2^i}{\ldots},u_i)+\lambda(\mathbf{0},\ldots,\mathbf{2^i}-\mathbf{1})$, for $i\in\{2,3\}$, is a permutation of $(\mathbb{Z}_{2^i},\overset{\alpha_i}{\ldots},\mathbb{Z}_{2^i})$ by Lemma 2.3. Thus, from (19), v_i is a permutation of $(x_{i-1}+\lambda\mathbf{1},\mathbb{Z}_{2^i},\overset{\alpha_i}{\ldots},\mathbb{Z}_{2^i})$. Since either $x_{i-1}+\lambda\mathbf{1}=\lambda\mathbf{1}$, or $x_{i-1}+\lambda\mathbf{1}$ is a permutation of $(\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i},\overset{2^{(i-1)(t_1-1)}}{\ldots},\mathbb{Z}_{2^i}\backslash 2\mathbb{Z}_{2^i})$, $\mathbf{v}$ satisfies Property 1.

For the second subcase when $o(\mathbf{v}) = 8$, that is, when $o(\mathbf{z}) = 8$ and $\lambda \in 2\mathbb{Z}_8$, we have that $o(\mathbf{u}) = 8$ and, by Item 1 of Lemma 2.7, x_{i-1} is a permutation of $(2\mathbb{Z}_{2^i}, \overset{2^{(i-1)(t_1-1)}}{\ldots}, 2\mathbb{Z}_{2^i})$ for $i \in \{2,3\}$. By induction hypothesis, $\mathbf{u}$ satisfies Property 1 and then u_2 is a permutation of

$$(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4),$$

where $m,n,r \geq 0$ and $\mu \in \{1,3\}$, and u_3 is a permutation of

$$(\mu', \overset{m'}{\ldots}, \mu', \mu'', \overset{m'}{\ldots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8),$$

where $m', n', r' \geq 0$ and $\mu', \mu'' \in \{1,3,5,7\}$. From (19), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3})$. If $\lambda \in \{0,4\}$, then $v_2 = (x_1, u_2, u_2, u_2, u_2)$ in $\mathbf{v}$ satisfies the same property as u_2 in $\mathbf{u}$; that is, Property 1. If $\lambda \in \{2,6\}$, then $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$. By Item 1 of Lemma 2.2, we have that $x_1 + \mathbf{2}$ is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1-1}}{\ldots}, 2\mathbb{Z}_4)$. Thus, by Lemma 2.4, v_2 is a permutation of

$$(2\mathbb{Z}_4, \overset{4n+2^{t_1-1}}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{4r+2m}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4).$$

Therefore, for $\lambda \in 2\mathbb{Z}_8$, v_2 satisfies Property 1. Now, we consider the coordinates in v_3. From (19), $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + \lambda(\mathbf{1},\mathbf{0},\ldots,\mathbf{7})$. By Item 1 of Lemma 2.2, we have that, for $\lambda \in 2\mathbb{Z}_8$, $x_2 + \lambda\mathbf{1}$ is a permutation of $(2\mathbb{Z}_8, \overset{4^{t_1-1}}{\ldots}, 2\mathbb{Z}_8)$. If $\lambda = 0$, it is easy to see that v_3 satisfies Property 1. Note that $\lambda(\mathbf{0},\ldots,\mathbf{7})$ is a permutation of $(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6},\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ if $\lambda \in \{2,6\}$, and a permutation of $(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ if $\lambda = 4$. Thus, by Lemma 2.5, v_3 satisfies Property 1. Therefore, if $o(\mathbf{v}) = 8$, then $\mathbf{v}$ satisfies Property 1.

Case 2: Assume that $o(\mathbf{v}) = 4$. We have two subcases: when $o(\mathbf{z}) = 4$ and $\lambda \in 2\mathbb{Z}_8$, and when $o(\mathbf{z}) = 2$ and $\lambda \in \{2,6\}$. For the first subcase, since $o(\mathbf{z}) = 4$, we have that $o(\mathbf{u}) = 4$. Moreover, $x_1 = \mathbf{0}$ and x_2 is a permutation of $(4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\ldots}, 4\mathbb{Z}_8)$ by Item 2 of Lemma 2.7. By induction hypothesis, $\mathbf{u}$ satisfies Property 2. Then, $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly 4^{t_1-1} times and $4^{t_1-1} - 2^{t_1-1}$ times the element 0, and u_3 is a permutation of

$$(\mu, \overset{m}{\ldots}, \mu, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$$

for some integers $m,n,r \geq 0$ and $\mu \in \{2,6\}$. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0},\mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in 2\mathbb{Z}_8$, we have that $v_1 = \mathbf{0}$. From (19), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3})$. If $\lambda \in \{0,4\}$, then $v_2 = (x_1, u_2, u_2, u_2, u_2)$. Since $x_1 = \mathbf{0}$ is of length 2^{t_1}, it is easy to see that v_2 in $\mathbf{v}$ satisfies the same property as u_2 in $\mathbf{u}$; that is, Property 2. If $\lambda \in \{2,6\}$, then $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$, where $x_1 = \mathbf{0}$ is of length 2^{t_1}. Note that $u_2 + \mathbf{2}$ contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ as many times as u_2 contains the element 0, and the element 0 as many times as u_2 contains the element 2. Thus, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $2^{t_1} + 2(4^{t_1-1}) + 2(4^{t_1-1} - 2^{t_1-1}) = 4^{t_1}$ times and $2(4^{t_1-1}) + 2(4^{t_1-1} - 2^{t_1-1}) = 4^{t_1} - 2^{t_1}$ times the element 0. Therefore, for $\lambda \in 2\mathbb{Z}_8$, v_2 satisfies Property 2. Now, we consider the coordinates in v_3. From (19), $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + \lambda(\mathbf{1},\mathbf{0},\ldots,\mathbf{7})$.

If $\lambda = 0$, it is easy to see that v_3 satisfies Property 2. For $\lambda = 4$, $x_2 + \lambda\mathbf{1}$ is a permutation of $(4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\ldots}, 4\mathbb{Z}_8)$, and for $\lambda \in \{2,6\}$, it is a permutation of

$$(2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{2\cdot 4^{t_1-1}}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8).$$

Note that $\lambda(\mathbf{0},\ldots,\mathbf{7})$ is a permutation of $(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6},\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ if $\lambda \in \{2,6\}$, and a permutation of $(\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$ if $\lambda = 4$. Hence, by Lemma 2.6, v_3 also satisfies Property 2, and so does $\mathbf{v}$.

Now, we consider the second subcase, when $o(\mathbf{z}) = 2$ and $\lambda \in \{2,6\}$. Since $o(\mathbf{z}) = 2$, we have that $o(\mathbf{u}) = 2$. Then, by Item 3 of Lemma 2.7, $x_1 = \mathbf{0}$ and $x_2 = \mathbf{0}$. By induction hypothesis, $\mathbf{u}$ satisfies Property 3, so $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $m = 8^{t_1-1}$ times and $m' = 8^{t_1-1} - 4^{t_1-1}$ times the element 0. Since $v_1 = (u_1,u_1) + \lambda(\mathbf{0},\mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in \{2,6\}$, we have that $v_1 = \mathbf{0}$. From (19), $v_2 = (x_1,u_2,u_2,u_2,u_2) + (\mathbf{2},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$. Since $x_1 = \mathbf{0}$ and $u_2 = \mathbf{0}$, of length α_1 and α_2, respectively, we have that $v_2 = (\mathbf{2},\mathbf{0},\mathbf{2},\mathbf{0},\mathbf{2})$. Therefore, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\alpha_1 + 2\alpha_2 = 4^{t_1}$ times and $2\alpha_2 = 4^{t_1} - 2^{t_1}$ times the element 0, by (18). Therefore, v_2 satisfies Property 2. Now, we consider the coordinates in v_3. From (19), $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + \lambda(\mathbf{1},\mathbf{0},\ldots,\mathbf{7})$. Since $x_2 = \mathbf{0}$, $x_2 + \lambda\mathbf{1} = (\lambda, \overset{4^{t_1}}{\ldots}, \lambda)$. Note that u_3 is a permutation of

$$(4, \overset{m-m'}{\ldots}, 4, 4\mathbb{Z}_8, \overset{m'}{\ldots}, 4\mathbb{Z}_8).$$

Moreover, since $\lambda \in \{2,6\}$, $\lambda(\mathbf{0},\ldots,\mathbf{7})$ is a permutation of $(\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6},\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$. Thus, by Item 1 of Lemma 2.2, $(u_3, \overset{8}{\ldots}, u_3) + (\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6},\mathbf{0},\mathbf{2},\mathbf{4},\mathbf{6})$ is a permutation of

$$(2\mathbb{Z}_8, \overset{2(m-m')+4m'}{\ldots}, 2\mathbb{Z}_8).$$

Thus, v_3 is a permutation of $(\lambda, \overset{4^{t_1}}{\ldots}, \lambda, 2\mathbb{Z}_8, \overset{2(m-m')+4m'}{\ldots}, 2\mathbb{Z}_8)$ with $\lambda \in \{2,6\}$, and hence v_3 also satisfies Property 2 and so does $\mathbf{v}$. Therefore, if $o(\mathbf{v}) = 4$, then $\mathbf{v}$ satisfies Property 2.

Case 3: Assume that $o(\mathbf{v}) = 2$. Then, $o(\mathbf{z}) = 2$ and $\lambda \in \{0,4\}$. Since $o(\mathbf{z}) = 2$, then $o(\mathbf{u}) = 2$. Moreover, $x_1 = \mathbf{0}$ and $x_2 = \mathbf{0}$ by Item 3 of Lemma 2.7. By induction hypothesis, $\mathbf{u}$ satisfies Property 3, and then $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly 8^{t_1-1} times and $8^{t_1-1} - 4^{t_1-1}$ times the element 0. Since $v_1 = (u_1,u_1) + \lambda(\mathbf{0},\mathbf{1})$, $v_1 = \mathbf{0}$. From (19), $v_2 = (x_1,u_2,u_2,u_2,u_2) + \lambda(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3})$, where $x_1 = \mathbf{0}$ and $u_2 = \mathbf{0}$, so $v_2 = \mathbf{0}$. From (19), $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + \lambda(\mathbf{1},\mathbf{0},\ldots,\mathbf{7})$, where $x_2 = \mathbf{0}$ is of length 4^{t_1}. If $\lambda = 0$, it is easy to see that v_3 satisfies Property 3. If $\lambda = 4$, $v_3 = (x_2, u_3, \overset{8}{\ldots}, u_3) + (\mathbf{4},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4},\mathbf{0},\mathbf{4})$. Note that $u_3 + \mathbf{4}$ contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ as many times as u_3 contains the element 0, and the element 0 as many times as u_3 contains the element 4. Then, v_3 contains the element 4 exactly $4^{t_1} + 4(8^{t_1-1}) + 4(8^{t_1-1} - 4^{t_1-1}) = 8^{t_1}$ times and $4(8^{t_1-1}) + 4(8^{t_1-1} - 4^{t_1-1}) = 8^{t_1} - 4^{t_1}$ the element 0. Therefore, $\mathbf{v}$ satisfies Property 3. This completes the proof. $\square$

Lemma 2.10. *Let $t_1 \geq 1$ and $t_2 \geq 0$ be integers. Let $\mathcal{H}^{t_1,t_2,1}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, 1)$ generated by $A^{t_1,t_2,1}$. Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,t_2,1}$.*

(1) If $o(\mathbf{u}) = 8$, then $\mathbf{u}$ has the following property:
   (a) u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of $(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ for some integers $m, n, r \geq 0$ and $\mu \in \{1, 3\}$, and u_3 is a permutation of $(\mu', \overset{m'}{\ldots}, \mu', \mu'', \overset{m'}{\ldots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8)$ for some integers $m', n', r' \geq 0$ and $\mu, \mu' \in \{1, 3, 5, 7\}$.

(2) If $o(\mathbf{u}) = 4$, then $\mathbf{u}$ has one of the following properties:
   (a) $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\frac{1}{2}(\frac{\alpha_1}{2} + \alpha_2) = 4^{t_1+t_2-1}$ times and $\frac{\alpha_2}{2} - \frac{\alpha_1}{4} = 4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of $(\mu, \overset{m}{\ldots}, \mu, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$ for some integers $m, n, r \geq 0$ and $\mu \in \{2, 6\}$.
   (b) u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of $(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ for some integers $m, n, r \geq 0$ and $\mu \in \{1, 3\}$, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{t}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{t'}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$ for some integers $t, t' \geq 0$.

(3) If $o(\mathbf{u}) = 2$, then $\mathbf{u}$ has one of the following properties:
   (a) $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $\frac{1}{4}(\frac{\alpha_1}{2} + \alpha_2 + 2\alpha_3) = 8^{t_1-1}4^{t_2}$ times and $\frac{\alpha_3}{2} - \frac{1}{4}(\frac{\alpha_1}{2} + \alpha_2) = 8^{t_1-1}4^{t_2} - 4^{t_1+t_2-1}$ times the element 0.
   (b) $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\frac{1}{2}(\frac{\alpha_1}{2} + \alpha_2) = 4^{t_1+t_2-1}$ times and $\frac{\alpha_2}{2} - \frac{\alpha_1}{4} = 4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{m}{\ldots}, 4\mathbb{Z}_8)$ for some $m \geq 0$.

Proof. We prove this lemma by induction on $t_2 \geq 0$. The lemma holds for the code $\mathcal{H}^{t_1,0,1}$ by Lemma 2.9. Assume that the lemma holds for the code $\mathcal{H}^{t_1,t_2,1}$ of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, 1)$ with $t_1 \geq 1$ and $t_2 \geq 0$. By Lemma 2.1, we have that

$$2^{t_1+t_2} = \alpha_1, 4^{t_1+t_2} = \alpha_1 + 2\alpha_2, \text{ and } 8^{t_1}4^{t_2} = \alpha_1 + 2\alpha_2 + 4\alpha_3. \tag{20}$$

We must show that the lemma is also true for the code $\mathcal{H}^{t_1,t_2+1,1}$.

Let $\mathbf{v} = (v_1 \mid v_2 \mid v_3) \in \mathcal{G}^{t_1,t_2+1,1}$. We can write

$$\mathbf{v} = \mathbf{z} + \lambda\mathbf{w},$$

where $\mathbf{z} = (u_1, u_1 \mid x_1, u_2, u_2, u_2, u_2 \mid u_3, u_3, u_3, u_3)$, $\mathbf{w} = (\mathbf{0}, \mathbf{1} \mid \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3} \mid \mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$, $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{G}^{t_1,t_2,1}$, $\lambda \in \{0, 1, 2, 3\}$, and $x_1 \in (2\mathbb{Z}_4)^{2^{t_1+t_2}}$

such that either $x_1 = \mathbf{0}$ or a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$. Then,

$$\begin{aligned} v_1 &= (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1}), \\ v_2 &= (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}), \\ v_3 &= (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}). \end{aligned} \tag{21}$$

If $\mathbf{z} = \mathbf{0}$, then $\mathbf{v} = \lambda\mathbf{w}$. It is easy to see that $\mathbf{v}$ satisfies Property 2b if $\lambda \in \{1, 3\}$ and Property 3b if $\lambda = 2$. Therefore, we focus on the case when $\mathbf{z} \neq \mathbf{0}$.

Case 1: Assume that $o(\mathbf{v}) = 8$. Then, $o(\mathbf{z}) = 8$ and $\lambda \in \{0, 1, 2, 3\}$. We have that $o(\mathbf{u}) = 8$ and, by Item 1 of Lemma 2.8, x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$. By induction hypothesis, $\mathbf{u}$ satisfies Property 1a. Then, u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of

$$(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4),$$

where $m, n, r \geq 0$ and $\mu \in \{1, 3\}$, and u_3 is a permutation of

$$(\mu', \overset{m'}{\ldots}, \mu', \mu'', \overset{m'}{\ldots}, \mu'', 2\mathbb{Z}_8, \overset{n'}{\ldots}, 2\mathbb{Z}_8, \mathbb{Z}_8\backslash 2\mathbb{Z}_8, \overset{r'}{\ldots}, \mathbb{Z}_8\backslash 2\mathbb{Z}_8),$$

where $m', n', r' \geq 0$ and $\mu', \mu'' \in \{1, 3, 5, 7\}$. First, since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, v_1 contains every element of $\mathbb{Z}_2$ the same number of times, for any $\lambda \in \{0, 1, 2, 3\}$. Second, from (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$. If $\lambda = 0$, then v_2 clearly satisfies 1a. If $\lambda \in \{1, 3\}$, then we have that $(u_2, u_2, u_2, u_2) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$ is a permutation of $(\mathbb{Z}_4, \overset{\alpha_2}{\ldots}, \mathbb{Z}_4)$ by Lemma 2.3. For $\lambda \in \{1, 3\}$, since $x_1 + \lambda\mathbf{1}$ is a permutation of $(\mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ by Item 3 of Lemma 2.2, we have that v_2 satisfies Property 1a. If $\lambda = 2$, $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$. By Item 1 of Lemma 2.2, we have that $x_1 + \mathbf{2}$ is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$. Therefore, by Lemma 2.4, v_2 is a permutation of $(2\mathbb{Z}_4, \overset{4n+2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{4r+2m}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ and then v_2 satisfies Property 1a. Finally, we consider the coordinates in v_3. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. If $\lambda = 0$, then v_3 clearly satisfies 1a. Note that $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}) = (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ if $\lambda = 2$ and $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ if $\lambda \in \{1, 3\}$. Therefore, by Lemma 2.5, v_3 satisfies Property 1a, and so does $\mathbf{v}$.

Case 2: Assume that $o(\mathbf{v}) = 4$. We have two subcases: when $o(\mathbf{z}) = 4$ and $\lambda \in \{0, 1, 2, 3\}$, and when $o(\mathbf{z}) = 2$ and $\lambda \in \{1, 3\}$. For the first subcase, since $o(\mathbf{z}) = 4$, $o(\mathbf{u}) = 4$. By induction hypothesis, $\mathbf{u}$ satisfies Property 2a or 2b. Assume that $\mathbf{u}$ satisfies Property 2a. Then, $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $4^{t_1+t_2-1}$ times and $4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of

$$(\mu, \overset{m}{\ldots}, \mu, 4\mathbb{Z}_8, \overset{n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{r}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$$

for some integers $m, n, r \geq 0$ and $\mu \in \{2, 6\}$. Note that, in this case, $x_1 = \mathbf{0}$ by Item 2 of Lemma 2.8. If $\lambda = 0$, then it is easy to see that $\mathbf{v}$ satisfies Property 2a. If $\lambda = 2$, we show that $\mathbf{v}$ satisfies Property 2a. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda = 2$, we have that $v_1 = \mathbf{0}$.

From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. Note that $u_2 + \mathbf{2}$ contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ as many times as u_2 contains the element 0, and the element 0 as many times as u_2 contains the element 2. Thus, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $2^{t_1+t_2} + 2(4^{t_1+t_2-1}) + 2(4^{t_1+t_2-1} - 2^{t_1+t_2-1}) = 4^{t_1+t_2}$ times and $2(4^{t_1+t_2-1}) + 2(4^{t_1+t_2-1} - 2^{t_1+t_2-1}) = 4^{t_1+t_2} - 2^{t_1+t_2}$ times the element 0, so v_2 satisfies Property 2a. From (21), $v_3 = (u_3, u_3, u_3, u_3) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$. By Item 2 of Lemma 2.6, v_3 is a permutation of

$$(4\mathbb{Z}_8, \overset{4n}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{4r+2m}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8).$$

Therefore, for $\lambda = 2$, $\mathbf{v}$ satisfies Property 2a. Finally, if $\lambda \in \{1, 3\}$, we show that $\mathbf{v}$ satisfies Property 2b. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in \{1, 3\}$, we have that v_1 contains every element of $\mathbb{Z}_2$ the same number of times. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. Since $\lambda \in \{1, 3\}$, by Lemma 2.3, we have that v_2 is a permutation of $(\lambda, \overset{2^{t_1+t_2}}{\ldots}, \lambda, \mathbb{Z}_4, \overset{\alpha_2}{\ldots}, \mathbb{Z}_4)$. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Note that, for $\lambda \in \{1, 3\}$, $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Thus, by Item 1 of Lemma 2.6, v_3 satisfies Property 2b, and so does $\mathbf{v}$. Therefore, if $o(\mathbf{u}) = 4$ and $\mathbf{u}$ satisfies Property 2a, we have that $\mathbf{v}$ satisfies either Property 2a or 2b.

We continue with the first subcase, when $o(\mathbf{z}) = 4$ and $\lambda \in \{0, 1, 2, 3\}$. Again, we have that $o(\mathbf{u}) = 4$. Now, we assume that $\mathbf{u}$ satisfies Property 2b. Then, u_1 contains every element of $\mathbb{Z}_2$ the same number of times, u_2 is a permutation of

$$(\mu, \overset{m}{\ldots}, \mu, 2\mathbb{Z}_4, \overset{n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{r}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$$

for some integers $m, n, r \geq 0$ and $\mu \in \{1, 3\}$, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{t}{\ldots}, 4\mathbb{Z}_8, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8, \overset{t'}{\ldots}, 2\mathbb{Z}_8\backslash 4\mathbb{Z}_8)$ for some integers $t, t' \geq 0$. Note that, in this case, x_1 is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$ by Item 2 of Lemma 2.8. Now, we show that $\mathbf{v}$ satisfies Property 2b. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$ and u_1 contains every element of $\mathbb{Z}_2$ the same number of times, we have that v_1 contains every element of $\mathbb{Z}_2$ the same number of times, for any $\lambda \in \{0, 1, 2, 3\}$. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$. If $\lambda = 0$, it is clear that v_2 satisfies Property 2b. Note that $x_1 + \lambda\mathbf{1}$ is a permutation of $(2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, 2\mathbb{Z}_4)$ if $\lambda = 2$, and a permutation of $(\mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{2^{t_1+t_2-1}}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4)$ if $\lambda \in \{1, 3\}$. If $\lambda = 2$, then by Lemma 2.4, $(u_2, u_2, u_2, u_2) + (\mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$ is a permutation of

$$(2\mathbb{Z}_4, \overset{4n}{\ldots}, 2\mathbb{Z}_4, \mathbb{Z}_4\backslash 2\mathbb{Z}_4, \overset{4r+2m}{\ldots}, \mathbb{Z}_4\backslash 2\mathbb{Z}_4).$$

If $\lambda \in \{1, 3\}$, then by Lemma 2.3, $(u_2, u_2, u_2, u_2) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$ is a permutation of $(\mathbb{Z}_4, \overset{\alpha_2}{\ldots}, \mathbb{Z}_4)$. Therefore, v_2 satisfies Property 2b. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. If $\lambda = 0$, it is clear that v_3 satisfies Property 2b. Note that $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ if $\lambda \in \{1, 3\}$, and $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}) = (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$ if $\lambda = 2$. Therefore, by Lemma 2.6, v_3 satisfies Property 2b, and so does $\mathbf{v}$.

Now, we consider the second subcase when $o(\mathbf{v}) = 4$, that is, when $o(\mathbf{z}) = 2$ and $\lambda \in \{1, 3\}$. Since $o(\mathbf{z}) = 2$, $o(\mathbf{u}) = 2$. By induction hypothesis, $\mathbf{u}$ satisfies Property 3a or 3b. Assume that $\mathbf{u}$ satisfies Property 3a. Then, $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $m = 8^{t_1-1}4^{t_2}$ times and $m' = 8^{t_1-1}4^{t_2} - 4^{t_1+t_2-1}$ times the element 0. By Item 3 of Lemma 2.8, we have that $x_1 = \mathbf{0}$. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in \{1, 3\}$, we have that v_1 contains every element of $\mathbb{Z}_2$ the same number of times. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. By Lemma 2.3, we have that v_2 is a permutation of

$$(\lambda, \overset{2^{t_1+t_2}}{\ldots}, \lambda, \mathbb{Z}_4, \overset{\alpha_2}{\ldots}, \mathbb{Z}_4),$$

where $\lambda \in \{1, 3\}$. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Note that u_3 is a permutation of $(4, \overset{m-m'}{\ldots}, 4, 4\mathbb{Z}_8, \overset{m'}{\ldots}, 4\mathbb{Z}_8)$ and, since $\lambda \in \{1, 3\}$, $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Thus, by Item 1 of Lemma 2.2, $v_3 = (u_3, u_3, u_3, u_3) + (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{m+m'}{\ldots}, 2\mathbb{Z}_8)$, so v_3 satisfies Property 2b, and so does $\mathbf{v}$. Therefore, if $o(\mathbf{u}) = 2$ and $\mathbf{u}$ satisfies Property 3a, we have that $\mathbf{v}$ satisfies Property 2b.

We continue with the second subcase, when $o(\mathbf{z}) = 2$ and $\lambda \in \{1, 3\}$. Again, we have that $o(\mathbf{u}) = 2$. Now, we assume that $\mathbf{u}$ satisfies Property 3b. Then, $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $4^{t_1+t_2-1}$ times and $4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{m}{\ldots}, 4\mathbb{Z}_8)$ for some $m \geq 0$. By Item 3 of Lemma 2.8, we have that $x_1 = \mathbf{0}$. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda \in \{1, 3\}$, we have that v_1 contains every element of $\mathbb{Z}_2$ the same number of times. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + \lambda(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. By Lemma 2.3, we have that v_2 is a permutation of

$$(\lambda, \overset{2^{t_1+t_2}}{\ldots}, \lambda, \mathbb{Z}_4, \overset{\alpha_2}{\ldots}, \mathbb{Z}_4),$$

where $\lambda \in \{1, 3\}$. From (21), $v_3 = (u_3, u_3, u_3, u_3) + \lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Since $\lambda \in \{1, 3\}$, $\lambda(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$. Thus, by Item 1 of Lemma 2.2, $v_3 = (u_3, u_3, u_3, u_3) + (\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6})$ is a permutation of $(2\mathbb{Z}_8, \overset{2m}{\ldots}, 2\mathbb{Z}_8)$. Therefore, v_3 satisfies Property 2b, and so does $\mathbf{v}$.

Case 3: Assume that $o(\mathbf{v}) = 2$. Then, $o(\mathbf{z}) = 2$ and $\lambda \in \{0, 2\}$. Since $o(\mathbf{z}) = 2$, we have that $o(\mathbf{u}) = 2$ and, by Item 3 of Lemma 2.8, $x_1 = \mathbf{0}$. By induction hypothesis, $\mathbf{u}$ satisfies Property 3a or 3b. Assume that $\mathbf{u}$ satisfies Property 3a. Then, $u_1 = \mathbf{0}$, $u_2 = \mathbf{0}$, and u_3 contains the element in $4\mathbb{Z}_8\backslash\{0\} = \{4\}$ exactly $m = 8^{t_1-1}4^{t_2}$ times and $m' = 8^{t_1-1}4^{t_2} - 4^{t_1+t_2-1}$ times the element 0. If $\lambda = 0$, then $\mathbf{v} = (\mathbf{0} \mid \mathbf{0} \mid v_3)$ satisfies Property 3a, since v_3 contains $4m$ times the element 4 and $4m'$ the element 0. Now, we assume that $\lambda = 2$. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda = 2$, we have that $v_1 = \mathbf{0}$. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$ and $u_2 = \mathbf{0}$. Therefore, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $\alpha_1 + 2\alpha_2 = 4^{t_1+t_2}$ times and $2\alpha_2 = 4^{t_1+t_2} - 2^{t_1+t_2}$ times the element 0, by (20). From (21), $v_3 = (u_3, u_3, u_3, u_3) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$.

Note that u_3 is a permutation of

$$(4, \overset{m-m'}{\ldots}, 4, 4\mathbb{Z}_8, \overset{m'}{\ldots}, 4\mathbb{Z}_8).$$

Thus, by Item 1 of Lemma 2.2, v_3 is a permutation of $(4\mathbb{Z}_8, \overset{2m+2m'}{\ldots}, 4\mathbb{Z}_8)$, so v_3 satisfies Property 3b, and so does $\mathbf{v}$. Therefore, if $o(\mathbf{u}) = 2$ and $\mathbf{u}$ satisfies Property 3a, we have that $\mathbf{v}$ satisfies Property 3b.

We continue with the case when $o(\mathbf{z}) = 2$ and $\lambda \in \{0, 2\}$. Again, we have that $o(\mathbf{u}) = 2$ and $x_1 = \mathbf{0}$. Now, we assume that $\mathbf{u}$ satisfies Property 3b. Then, $u_1 = \mathbf{0}$, u_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $4^{t_1+t_2-1}$ times and $4^{t_1+t_2-1} - 2^{t_1+t_2-1}$ times the element 0, and u_3 is a permutation of $(4\mathbb{Z}_8, \overset{m}{\ldots}, 4\mathbb{Z}_8)$ for some $m \geq 0$. If $\lambda = 0$, then it is easy to see that $\mathbf{v}$ satisfies Property 3b. Now, we assume that $\lambda = 2$. Since $v_1 = (u_1, u_1) + \lambda(\mathbf{0}, \mathbf{1})$, $u_1 = \mathbf{0}$, and $\lambda = 2$, we have that $v_1 = \mathbf{0}$. From (21), $v_2 = (x_1, u_2, u_2, u_2, u_2) + (\mathbf{2}, \mathbf{0}, \mathbf{2}, \mathbf{0}, \mathbf{2})$, where $x_1 = \mathbf{0}$ is of length $2^{t_1+t_2}$. Note that $u_2 + \mathbf{2}$ contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ as many times as u_2 contains the element 0, and the element 0 as many times as u_2 contains the element 2. Therefore, v_2 contains the element in $2\mathbb{Z}_4\backslash\{0\} = \{2\}$ exactly $2^{t_1+t_2} + 2(4^{t_1+t_2-1}) + 2(4^{t_1+t_2-1} - 2^{t_1+t_2-1}) = 4^{t_1+t_2}$ times and $2(4^{t_1+t_2-1}) + 2(4^{t_1+t_2-1} - 2^{t_1+t_2-1}) = 4^{t_1+t_2} - 2^{t_1+t_2}$ times the element 0. From (21), $v_3 = (u_3, u_3, u_3, u_3) + (\mathbf{0}, \mathbf{4}, \mathbf{0}, \mathbf{4})$. By Item 1 of Lemma 2.2, v_3 is a permutation of $(4\mathbb{Z}_8, \overset{4m}{\ldots}, 4\mathbb{Z}_8)$. Therefore, v_3 satisfies Property 3b, and so does $\mathbf{v}$. This completes the proof. $\square$

From [6], related to the generalized Gray map (1) considered in this paper, we have the following results:

Lemma 2.11. [6] *Let $\lambda, \mu \in \mathbb{Z}_2$. Then, $\phi_s(\lambda\mu 2^{s-1}) = \lambda\phi_s(\mu 2^{s-1}) = \lambda\mu\phi_s(2^{s-1})$.*

Lemma 2.12. [6] *Let $u, v \in \mathbb{Z}_{2^s}$. Then, $\phi_s(2^{s-1}u + v) = \phi_s(2^{s-1}u) + \phi_s(v)$.*

Proposition 2.2. [14, 6] *Let $u, v \in \mathbb{Z}_{2^s}$. Then,*

$$d_H(\phi_s(u), \phi_s(v)) = \mathrm{wt}_H(\phi_s(u - v)).$$

By Proposition 2.2, the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes obtained from the Gray map Φ are distance invariant, that is, the Hamming weight distribution is invariant under translation by a codeword. Therefore, their minimum distance coincides with the minimum weight.

Proposition 2.3. *Let $t_1 \geq 1$ and $t_2 \geq 0$ be integers. The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{H}^{t_1,t_2,1}$, generated by $A^{t_1,t_2,1}$, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code.*

Proof. Let $\mathcal{H}^{t_1,t_2,1}$ be the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, 1)$ and $H^{t_1,t_2,1}$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of length N. We have that $N = \alpha_1 + 2\alpha_2 + 4\alpha_3$. The cardinality of $H^{t_1,t_2,1}$ is $8^{t_1} \cdot 4^{t_2} \cdot 2 = 2(\alpha_1 + 2\alpha_2 + 4\alpha_3) = 2N$ by Lemma 2.1. By Proposition 2.2, the minimum distance of $H^{t_1,t_2,1}$ is equal to the minimum weight of $H^{t_1,t_2,1}$. Therefore, we need to prove that the minimum weight of $H^{t_1,t_2,1}$ is $N/2$.

We can write that $\mathcal{H}^{t_1,t_2,1} = \mathcal{G}^{t_1,t_2,1} \cup (\mathcal{G}^{t_1,t_2,1} + (\mathbf{1} \mid \mathbf{2} \mid \mathbf{4}))$. By Lemma 2.12, $H^{t_1,t_2,1} = \Phi(\mathcal{G}^{t_1,t_2,1}) \cup (\Phi(\mathcal{G}^{t_1,t_2,1}) + \mathbf{1})$. Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in \mathcal{H}^{t_1,t_2,1} \backslash \{\mathbf{0}, (\mathbf{1} \mid \mathbf{2} \mid \mathbf{4})\}$. We show that $\mathrm{wt}_H(\Phi(\mathbf{u})) = N/2$. First, consider $\mathbf{u} \in \mathcal{G}^{t_1,t_2,1} \backslash \{\mathbf{0}\}$. If $o(\mathbf{u}) = 8$, then by Lemma 2.10, u_1 contains every element of $\mathbb{Z}_2$ the same number of times, and for $i \in \{2,3\}$, u_i contains every element of $2^{i-1}\mathbb{Z}_{2^i}$ exactly s_i times, $s_i \geq 0$, and the remaining $\alpha_i - 2s_i$ coordinates of u_i are from $\mathbb{Z}_{2^i} \backslash 2^{i-1}\mathbb{Z}_{2^i}$. Thus, from the definition of Φ, we have that $\mathrm{wt}_H(\Phi(\mathbf{u})) = \alpha_1/2 + 2s_2 + (\alpha_2 - 2s_2)\cdot 1 + 4s_3 + (\alpha_3 - 2s_3)\cdot 2 = \alpha_1/2 + \alpha_2 + 2\alpha_3 = N/2$. If $o(\mathbf{u}) = 4$, then $\mathbf{u}$ satisfies Property 2a or 2b given in Lemma 2.10. If $\mathbf{u}$ satisfies Property 2a, then u_3 contains every element of $4\mathbb{Z}_8$ exactly m times, $m \geq 0$, and the remaining coordinates of u_3 are from $\mathbb{Z}_8 \backslash 4\mathbb{Z}_8$. Thus, $\mathrm{wt}_H(\Phi(\mathbf{u})) = \alpha_1/2 + \alpha_2 + 4m + (\alpha_3 - 2m)\cdot 2 = \alpha_1/2 + \alpha_2 + 2\alpha_3 = N/2$. Otherwise, if $\mathbf{u}$ satisfies Property 2b, then $\mathrm{wt}_H(\Phi(\mathbf{u})) = \alpha_1/2 + 2n + (\alpha_2 - 2n)\cdot 1 + 4t + (\alpha_3 - 2t)\cdot 2 = N/2$. If $o(\mathbf{u}) = 2$, then $\mathbf{u}$ satisfies Property 3a or 3b given in Lemma 2.10. If $\mathbf{u}$ satisfies Property 3a, then $\mathrm{wt}_H(\Phi(\mathbf{u})) = \frac{1}{4}(\alpha_1/2 + \alpha_2 + 2\alpha_3)\cdot 4 = N/2$. Otherwise, if $\mathbf{u}$ satisfies Property 3b, then $\backslash = 2 \cdot \frac{1}{2}(\alpha_1/2 + \alpha_2) + 4m + (\alpha_3 - 2m)\cdot 2 = N/2$.

Finally, note that $\mathrm{wt}_H(\Phi(\mathbf{u}) + \mathbf{1}) = N/2$. Therefore, we have that the weight of every element of $H^{t_1,t_2,1} \backslash \{\mathbf{0}, \mathbf{1}\}$ is $N/2$, that is, the minimum weight of $H^{t_1,t_2,1}$ is $N/2$. □

Proposition 2.4. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. If $\mathcal{H}^{t_1,t_2,t_3}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$, then, by applying Construction (5), $\mathcal{H}^{t_1,t_2,t_3+1}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(2\alpha_1, 2\alpha_2, 2\alpha_3; t_1, t_2, t_3+1)$.*

Proof. By Construction (5), $\mathcal{H}^{t_1,t_2,t_3+1}$ is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code of type $(\alpha_1', \alpha_2', \alpha_3'; t_1, t_2, t_3+1)$, where $\alpha_1' = 2\alpha_1$, $\alpha_2' = 2\alpha_2$, and $\alpha_3' = 2\alpha_3$.

Since H^{t_1,t_2,t_3} is a Hadamard code of length $N = \alpha_1 + 2\alpha_2 + 4\alpha_3$, then its minimum distance is $N/2$ and $|H^{t_1,t_2,t_3}| = 2N$. Note that H^{t_1,t_2,t_3+1} is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code of length $N' = \alpha_1' + 2\alpha_2' + 4\alpha_3' = 2N$ and $|H^{t_1,t_2,t_3+1}| = 8^{t_1}4^{t_2}2^{t_3+1} = 2|H^{t_1,t_2,t_3}| = 2\cdot 2N = 2N'$. By Proposition 2.2, the minimum distance of H^{t_1,t_2,t_3+1} is equal to the minimum weight of H^{t_1,t_2,t_3+1}. We only have to prove that the minimum weight of H^{t_1,t_2,t_3+1} is $N'/2$. Let $\mathcal{H}^{t_1,t_2,t_3} = (\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)$. Note that

$$\mathcal{H}^{t_1,t_2,t_3+1} = \bigcup_{\lambda \in \{0,1\}} ((\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1} \mid \mathbf{0}, \mathbf{2} \mid \mathbf{0}, \mathbf{4})).$$

By Lemmas 2.11 and 2.12,

$$\begin{aligned} H^{t_1,t_2,t_3+1} &= \bigcup_{\lambda \in \{0,1\}} (\Phi(\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})) \\ &= A_0 \cup A_1, \end{aligned} \tag{22}$$

where $A_\lambda = \Phi(\mathcal{H}_1, \mathcal{H}_1 \mid \mathcal{H}_2, \mathcal{H}_2 \mid \mathcal{H}_3, \mathcal{H}_3) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})$, $\lambda \in \{0,1\}$. Next, we show that the minimum weight of A_λ is $N'/2$. Any element in A_λ is of the form $\Phi(u_1, u_1 \mid u_2, u_2 \mid u_3, u_3) + \lambda(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})$, for $\mathbf{u} = (u_1 \mid u_2 \mid$

$u_3) \in (\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)$. Let $\mathbf{u} = (u_1 \mid u_2 \mid u_3) \in (\mathcal{H}_1 \mid \mathcal{H}_2 \mid \mathcal{H}_3)\backslash\{\mathbf{0}\}$. When $\lambda = 0$, we have that $\mathrm{wt}_H(\Phi(u_1, u_1 \mid u_2, u_2 \mid u_3, u_3)) = 2\mathrm{wt}_H(\Phi(\mathbf{u}))$. Thus, the minimum weight of A_0 is $2 \cdot N/2 = N'/2$. Otherwise, when $\lambda = 1$, we have that $\mathrm{wt}_H(\Phi(u_1, u_1 \mid u_2, u_2 \mid u_3, u_3) + (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})) = \mathrm{wt}_H(\Phi(\mathbf{u})) + \alpha_1 - \mathrm{wt}_H(u_1) + 2\alpha_2 - \mathrm{wt}_H(\Phi_2(u_2)) + 4\alpha_3 - \mathrm{wt}_H(\Phi_3(u_3)) = \mathrm{wt}_H(\Phi(\mathbf{u})) + \alpha_1 + 2\alpha_2 + 4\alpha_3 - \mathrm{wt}_H(\Phi(\mathbf{u})) = N = N'/2$. Thus, the minimum weight of A_1 is $N'/2$. Therefore, from (22), the minimum weight of H^{t_1,t_2,t_3+1} is $N'/2$. $\square$

Next, from Proposition 2.3 and Proposition 2.4, one can derive the result below.

Theorem 2.1. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{H}^{t_1,t_2,t_3}$, generated by A^{t_1,t_2,t_3}, is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code.*

To illustrate, we present an example.

Example 2.2. *The $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive code $\mathcal{H}^{1,0,1}$ generated by $A^{1,0,1}$, given in (2), is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard code of type $(2,1,1;1,0,1)$. We can write $\mathcal{H}^{1,0,1} = \bigcup_{\alpha\in\mathbb{Z}_2}(\mathcal{A} + \alpha\mathbf{1})$, where $\mathcal{A} = \{\lambda(0,1 \mid 1 \mid 1) : \lambda \in \mathbb{Z}_8\}$. Thus, $H^{1,0,1} = \Phi(\mathcal{H}^{1,0,1}) = \bigcup_{\alpha\in\mathbb{Z}_2}(\Phi(\mathcal{A}) + \alpha\mathbf{1})$, where $\Phi(\mathcal{A})$ consists of all the rows of the Hadamard matrix*

$$H(2,4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $\Phi(\mathcal{A})$ is linear and the minimum distance of $\Phi(\mathcal{A})$ is 4, so $H^{1,0,1}$ is a binary linear Hadamard code of length 8.

Proposition 2.5. *Let $t_1 \geq 1$, $t_2 \geq 0$, and $t_3 \geq 1$ be integers. Let H^{t_1,t_2,t_3} be a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code of length 2^t. Then, $t+1 = 3t_1 + 2t_2 + t_3$.*

Proof. Since H^{t_1,t_2,t_3} is a binary Hadamard code of length 2^t, we have that $|H^{t_1,t_2,t_3}| = 2 \cdot 2^t = 2^{t+1}$. Note that $|H^{t_1,t_2,t_3}| = 2^{3t_1+2t_2+t_3}$, and hence $t + 1 = 3t_1 + 2t_2 + t_3$. $\square$

Now, we recall Theorem 2.2 in order to compare the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} (with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$) with the $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes (with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$) of the same length. Also recall that the type of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code can be given as $(\alpha_1, \alpha_2, 0; 0, t_2, t_3)$ if we see the code as a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear code with $\alpha_3 = 0$, or directly $(\alpha_1, \alpha_2; t_2, t_3)$. Note that there are no $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes neither with only $\alpha_1 = 0$ nor with only $\alpha_2 = 0$ [24, 33].

Theorem 2.2. [28] *Let $t \geq 3$ and $t_2 \in \{0, \ldots, \lfloor t/2 \rfloor\}$. Let H^{t_2,t_3} be the nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^t and type $(\alpha_1, \alpha_2; t_2, t_3)$, where $\alpha_1 = 2^{t-t_2}$, $\alpha_2 = 2^{t-1} - 2^{t-t_2-1}$, and $t_3 = t + 1 - 2t_2$. Then,*

$$\operatorname{rank}(H^{t_2,t_3}) = t_3 + 2t_2 + \binom{t_2}{2} \quad \textit{and} \quad \ker(H^{t_2,t_3}) = t_2 + t_3.$$

We also recall the construction of the $\mathbb{Z}_{2^s}$-linear Hadamard codes with $s \geq 2$ studied in [16], and Theorem 2.3 given in [18], in order to compare these codes with the $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} (with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$) of the same length. Let $T_i = \{j \cdot 2^{i-1} \,:\, j \in \{0, 1, \ldots, 2^{s-i+1} - 1\}\}$ for all $i \in \{1, \ldots, s\}$. Note that $T_1 = \{0, \ldots, 2^s - 1\}$. Let t_1, t_2,…,t_s be non-negative integers with $t_1 \geq 1$. Consider the matrix $\bar{A}^{t_1,\ldots,t_s}$ whose columns are exactly all the vectors of the form $\mathbf{z}^T$, $\mathbf{z} \in \{1\} \times T_1^{t_1-1} \times T_2^{t_2} \times \cdots \times T_s^{t_s}$. Let $\bar{\mathcal{H}}^{t_1,\ldots,t_s}$ be the $\mathbb{Z}_{2^s}$-additive code of type $(n; t_1, \ldots, t_s)$ generated by $\bar{A}^{t_1,\ldots,t_s}$. Let $\bar{H}^{t_1,\ldots,t_s} = \Phi_s(\bar{\mathcal{H}}^{t_1,\ldots,t_s})$ be the corresponding $\mathbb{Z}_{2^s}$-linear Hadamard code.

Theorem 2.3. [18] *Let $\bar{H}^{t_1,\ldots,t_s}$ be the $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 2$ and $t_s \geq 1$. Then, for all $\ell \in \{1, \ldots, t_s\}$, $\bar{H}^{t_1,\ldots,t_s}$ is permutation equivalent to the $\mathbb{Z}_{2^{s+\ell}}$-linear Hadamard code $\bar{H}^{1,\mathbf{0}^{\ell-1},t_1-1,t_2,\ldots,t_{s-1},t_s-\ell}$.*

For $5 \leq t \leq 11$, Tables 1 and 3 given in [16] show all possible values of $(t_1, \ldots, t_s)$ corresponding to nonlinear $\mathbb{Z}_{2^s}$-linear Hadamard codes, with $s \geq 2$, of length 2^t. For each of them, the values (r, k) are shown, where r is the rank, and k is the dimension of the kernel. Note that if two codes have different values (r, k), they are not equivalent. The following example shows that all the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [16], with $s \geq 2$, of the same length 2^{11}.

Example 2.3. *Consider $t = 11$. By solving equation $t + 1 = 3t_1 + 2t_2 + t_3$ given in Proposition 2.5, all $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are the ones in*

$$T = \{H^{1,0,9}, H^{1,1,7}, H^{1,2,5}, H^{1,3,3}, H^{1,4,1}, H^{2,0,6}, H^{2,1,4}, H^{2,2,2}, H^{3,0,3}, H^{3,1,1}\}.$$

By using the computer algebra system Magma [13], *their corresponding values of (r, k), where r is the rank and k is the dimension of the kernel, are* $(12, 12)$*,* $(14, 9)$*,* $(17, 8)$*,* $(21, 7)$*,* $(26, 6)$*,* $(17, 8)$*,* $(22, 7)$*,* $(28, 6)$*,* $(28, 6)$*, and* $(37, 5)$*, respectively. The code $H^{1,0,9}$ is the only linear code in T since $r = k = 12$. Using* Magma*, we can check that the following codes in each pair are nonequivalent to each other: $(H^{1,2,5}, H^{2,0,6})$, $(H^{2,2,2}, H^{3,0,3})$. Therefore, the codes in T are not equivalent to each other.*

Let $\bar{T} = T \setminus \{H^{1,0,9}\}$. Similarly, by solving equation $t + 1 = 2t_2 + t_3$ given in Theorem 2.2, all nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of length 2^{11} are $H^{2,8}$, $H^{3,6}$, $H^{4,4}$ and $H^{5,2}$, and by Theorem 2.2, their corresponding values of (r, k) are $(13, 10)$*,* $(15, 9)$*,* $(18, 8)$*, and* $(22, 7)$*, respectively. Using* Magma*, we can check that $H^{2,1,4}$ and $H^{5,2}$ are nonequivalent. Therefore,* *all the codes in $\bar{T}$ are nonequivalent to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code of length 2^{11}.*

Finally, note that all the codes in $\bar{T}$, except $H^{1,1,7}$ and $H^{2,1,4}$, are not equivalent to any $\mathbb{Z}_{2^s}$-linear Hadamard code, with $s \geq 2$, of length 2^{11}, since they have different values of (r,k). The $\mathbb{Z}_{2^s}$-linear Hadamard codes of length 2^{11}, having the same values $(r,k) = (14,9)$ as $H^{1,1,7}$, are $\bar{H}^{2,0,6}$, $\bar{H}^{1,1,0,5}$, $\bar{H}^{1,0,1,0,4}$, $\bar{H}^{1,0,0,0,1,0,2}$, and $\bar{H}^{1,0,0,0,0,0,1,0,0}$, which are equivalent to each other by Theorem 2.3. The $\mathbb{Z}_4$-linear Hadamard code $\bar{H}^{6,0}$ is the only $\mathbb{Z}_{2^s}$-linear Hadamard code of length 2^{11}, having the same values $(r,k) = (22,7)$ as $H^{2,1,4}$. However, using Magma*, we can check that the following codes in each pair are nonequivalent to each other: $(H^{1,1,7}, \bar{H}^{2,0,6})$, $(H^{2,1,4}, \bar{H}^{6,0})$.*

Therefore, all the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [16]*, with $s \geq 2$, of the same length 2^{11}.*

Finally, the following example shows that other $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes can not be constructed by Construction (3). However, in the next section, we also show that other constructions of these codes do generate equivalent codes.

Example 2.4. *Consider the matrix*

$$B = \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 02 & 1111 & 0646 & 11113333 \\ 00 & 11 & 31 & 0123 & 1771 & 01234725 \end{array}\right).$$

Using Magma*, we can check that the code generated by B is a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard code of type $(4,6,12;2,0,1)$, and it is nonequivalent to the code $H^{2,0,1}$ generated by $A^{2,0,1}$ given in (6).*

3. Same type equivalent $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes

In this section, we see that if we consider other specific starting matrices, instead of the matrix $A^{1,0,1}$ given in (2), and apply the same recursive Construction (3), (4) and (5), or new constructions more general than (3) and (4), and the same Construction (5), we also obtain $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-additive Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Indeed, the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes, after applying the Gray map Φ, are equivalent to the codes $\Phi(\mathcal{H}^{t_1,t_2,t_3})$ of the same type constructed in Section 2.

Let $\mathbb{Z}^*_{2^i}$ be the group of units of $\mathbb{Z}_{2^i}$ for $i \in \{2,3\}$. Then, $\mathbb{Z}^*_4 = \{1,3\}$ and $\mathbb{Z}^*_8 = \{1,3,5,7\}$.

Proposition 3.1. *Let $\bar{a}_1 = (a_1)$ and $\bar{b}_1 = (b_1)$, where $a_1 \in \mathbb{Z}^*_4$ and $b_1 \in \mathbb{Z}^*_8$. Then, the code generated by*

$$\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & a_1 & b_1 \end{array}\right), \tag{23}$$

denoted by $\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$*, is a* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-additive Hadamard code of type* $(2,1,1;1,0,1)$*. Moreover, the corresponding* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-linear Hadamard code* $\Phi(\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1})$ *is permutation equivalent to* $\Phi(\mathcal{H}^{1,0,1})$*.*

Proof. Let $\overline{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ be the matrix obtained from $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ by applying the row operation described in Table 1, depending on the values of $a_1 \in \mathbb{Z}_4^*$ and $b_1 \in \mathbb{Z}_8^*$. Note that $\overline{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ is also a generator matrix of $\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$. After permuting the first two columns of $\overline{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$, if necessary, we obtain $A^{1,0,1}$. Thus, $\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ and $\mathcal{H}^{1,0,1}$ are permutation equivalent, and so are the codes $\Phi(\hat{\mathcal{H}}^{1,0,1}_{\bar{a}_1,\bar{b}_1})$ and $\Phi(\mathcal{H}^{1,0,1})$. $\square$

Theorem 3.1. *Let* $\ell \geq 1$*. Let* $\bar{a}_\ell = (a_1,\ldots,a_\ell) \in (\mathbb{Z}_4^*)^\ell$*,* $\bar{b}_\ell = (b_1,\ldots,b_\ell) \in (\mathbb{Z}_8^*)^\ell$*,* $\mathbf{a}_i = (a_i, \overset{2^{i-1}}{\ldots}, a_i)$*, and* $\mathbf{b}_i = (b_i, \overset{4^{i-1}}{\ldots}, b_i)$*,* $1 \leq i \leq \ell$*. Let* $\hat{A}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$*, with* $t_1 \geq 1$*, be the matrix obtained by using the following construction (instead of Construction (3)). We start with* $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ *given in (23). If we have* $\hat{A}^{\ell-1,0,1}_{\bar{a}_{\ell-1},\bar{b}_{\ell-1}} = (\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$*, with* $\ell \geq 2$*, we may construct*

$$\hat{A}^{\ell,0,1}_{\bar{a}_\ell,\bar{b}_\ell} = \left(\begin{array}{cc|ccccc|ccccc} \hat{A}_1 & \hat{A}_1 & \hat{M}_1 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{M}_2 & \hat{A}_3 & \hat{A}_3 & \cdots & \hat{A}_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{a}_\ell & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{b}_\ell & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array}\right), \tag{24}$$

where $\hat{M}_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\}\times\{0,2\}^{\ell-1}\}$*,* $\hat{M}_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\}\times\{0,2,4,6\}^{\ell-1}\}$*. We repeat Construction (24) until* $\ell = t_1$*. Then, the code generated by* $\hat{A}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$*, denoted by* $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$*, is a* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-additive Hadamard code of type* $(\alpha_1,\alpha_2,\alpha_3;t_1,0,1)$ *with* $\alpha_1 \neq 0$*,* $\alpha_2 \neq 0$ *and* $\alpha_3 \neq 0$*. Moreover, the corresponding* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-linear Hadamard code* $\Phi(\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}})$ *is permutation equivalent to* $\Phi(\mathcal{H}^{t_1,0,1})$*.*

Proof. It is enough to show that $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,0,1}$ are permutation equivalent. We prove this by induction on $t_1 \geq 1$. By Proposition 3.1, this is true for $t_1 = 1$. Assume that $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,0,1}$ are permutation equivalent. Let $\hat{A}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}} = (\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$ and $A^{t_1,0,1} = (A_1 \mid A_2 \mid A_3)$. By Construction (24), we have

$$\hat{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}} = \left(\begin{array}{cc|ccccc|ccccc} \hat{A}_1 & \hat{A}_1 & \hat{M}_1 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{M}_2 & \hat{A}_3 & \hat{A}_3 & \cdots & \hat{A}_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{a}_{t_1+1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{b}_{t_1+1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array}\right),$$

where $\hat{M}_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\}\times\{0,2\}^{t_1}\}$, $\hat{M}_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\}\times\{0,2,4,6\}^{t_1}\}$, $\mathbf{a}_{t_1+1} = (a_{t_1+1}, \overset{2^{t_1}}{\ldots}, a_{t_1+1})$, and $\mathbf{b}_{t_1+1} = (b_{t_1+1}, \overset{4^{t_1}}{\ldots}, b_{t_1+1})$. By Construction (3), we have

$$A^{t_1+1,0,1} = \left(\begin{array}{cc|ccccc|ccccc} A_1 & A_1 & M_1 & A_2 & A_2 & A_2 & A_2 & M_2 & A_3 & A_3 & \cdots & A_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{array}\right),$$

where $M_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{t_1}\}$, $M_2 = \{\mathbf{z}^T : \mathbf{z} \in \{4\} \times \{0,2,4,6\}^{t_1}\}$. Let $\hat{\mathcal{H}}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ and $\mathcal{H}^{t_1+1,0,1}$ be the codes generated by $\hat{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ and $A^{t_1+1,0,1}$, respectively.

Since $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,0,1}$ are permutation equivalent, there exist some row operations and column permutations so that after applying these operations on $(\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$, we obtain $(A_1 \mid A_2 \mid A_3)$. First, we apply the same row operations to $\hat{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ and the corresponding column permutations to each submatrix

$$\begin{pmatrix} \hat{A}_i \\ \mathbf{k_i} \end{pmatrix},$$

for $i \in \{1,2,3\}$, $k_i \in \mathbb{Z}_{2^i}$. Thus, for $i \in \{1,2,3\}$, $\hat{A}_i$ becomes A_i. Then, we change the last row by applying the row operation described in Table 1, depending on the values of $a_{t_1+1} \in \mathbb{Z}_4^*$ and $b_{t_1+1} \in \mathbb{Z}_8^*$. After that, we permute the blocks of the form

$$\begin{pmatrix} A_1 \\ \mathbf{k_1} \end{pmatrix}, \begin{pmatrix} A_2 \\ \mathbf{k_2} \end{pmatrix} \text{ and } \begin{pmatrix} A_3 \\ \mathbf{k_3} \end{pmatrix},$$

for $k_i \in \mathbb{Z}_{2^i}$, so that we obtain the submatrices

$$\begin{pmatrix} A_1 & A_1 \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} A_2 & A_2 & A_2 & A_2 \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix} \text{ and } \begin{pmatrix} A_3 & A_3 & \cdots & A_3 \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{7} \end{pmatrix},$$

respectively. Let $\overline{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ be the matrix obtain from $\hat{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ after applying all these operations. Let M_1' and M_2' be the matrices $\hat{M}_1$ and $\hat{M}_2$, respectively, after all these operations. Finally, after a suitable permutation of the columns corresponding to the blocks of the form

$$\begin{pmatrix} M_1' \\ \mathbf{1} \end{pmatrix} \text{ and } \begin{pmatrix} M_2' \\ \mathbf{1} \end{pmatrix}$$

in $\overline{A}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$, we obtain $A^{t_1+1,0,1}$. Thus, the codes $\hat{\mathcal{H}}^{t_1+1,0,1}_{\bar{a}_{t_1+1},\bar{b}_{t_1+1}}$ and $\mathcal{H}^{t_1+1,0,1}$ are permutation equivalent, and so are the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes. This completes the proof. $\square$

Example 3.1. *Let $\bar{a}_2 = (1,3)$ and $\bar{b}_2 = (3,5)$. Then,*

$$\hat{A}^{2,0,1}_{\bar{a}_2,\bar{b}_2} = \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 02 & \mathbf{a}_1 & 0246 & \mathbf{b}_1 \\ 00 & 11 & \mathbf{a}_2 & 0123 & \mathbf{b}_2 & 01234567 \end{array}\right)$$
$$= \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 02 & 1111 & 0246 & 33333333 \\ 00 & 11 & 33 & 0123 & 5555 & 01234567 \end{array}\right),$$

which is obtained by using Construction (24), starting with the matrix $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ given in (23).

a_i	b_i	*row operation*
1	1	$r_{i+1} \leftarrow r_{i+1}$
1	3	$r_{i+1} \leftarrow r_1 - r_{i+1}$
1	7	$r_{i+1} \leftarrow 5r_1 - 5r_{i+1}$
1	5	$r_{i+1} \leftarrow 5r_{i+1}$
3	1	$r_{i+1} \leftarrow r_1 - 3r_{i+1}$
3	3	$r_{i+1} \leftarrow 3r_{i+1}$
3	5	$r_{i+1} \leftarrow r_1 + r_{i+1}$
3	7	$r_{i+1} \leftarrow 7r_{i+1}$

TABLE 1. Row operations depending on the values of $a_i \in \mathbb{Z}_4^*$ and $b_i \in \mathbb{Z}_8^*$.

First, note that we have

$$\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 3 \end{array}\right) \text{ and } A^{1,0,1} = \left(\begin{array}{cc|c|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right).$$

Therefore, by using Table 1, $A^{1,0,1} = (A_1 \mid A_2 \mid A_3)$ can be obtained from $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1} = (\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$ by applying the row operation $r_2 \leftarrow r_1 - r_2$ and the column permutation $(1,2)$. No column permutation is performed on the submatrices $\hat{A}_2$ and $\hat{A}_3$. Then, we apply the same row operation $r_2 \leftarrow r_1 - r_2$ to $\hat{A}^{2,0,1}_{\bar{a}_2,\bar{b}_2}$ and the column permutation $(1,2)$ to each submatrix $\begin{pmatrix} \hat{A}_1 \\ \mathbf{k_1} \end{pmatrix}$, for $k_1 \in \mathbb{Z}_2$. Thus, for $i \in \{1,2,3\}$, $\hat{A}_i$ becomes A_i. Then, we apply $r_3 \leftarrow r_1 + r_3$, described in Table 1, so we obtain

$$\left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 01 & 01 & 20 & 1111 & 4206 & 11111111 \\ 11 & 00 & 11 & 2301 & 1111 & 45670123 \end{array}\right).$$

After that, we permute the blocks of the form $\begin{pmatrix} A_i \\ \mathbf{k_i} \end{pmatrix}$, for $k_i \in \mathbb{Z}_{2^i}$ and $i \in \{1,2,3\}$, so that we obtain

$$\overline{A}^{2,0,1}_{\bar{a}_2,\bar{b}_2} = \left(\begin{array}{cc|cc|cc} 11 & 11 & 22 & 2222 & 4444 & 44444444 \\ 10 & 10 & 20 & 1111 & 4206 & 11111111 \\ 00 & 11 & 11 & 0123 & 1111 & 01234567 \end{array}\right).$$

Finally, after applying a suitable column permutation to the submatrices,

$$\begin{pmatrix} 22 \\ 20 \\ 11 \end{pmatrix} \text{ and } \begin{pmatrix} 4444 \\ 4206 \\ 1111 \end{pmatrix}$$

in $\overline{A}^{2,0,1}_{\bar{a}_2,\bar{b}_2}$, we can obtain $A^{2,0,1}$. Thus, the codes $\hat{\mathcal{H}}^{2,0,1}_{\bar{a}_2,\bar{b}_2}$ and $\mathcal{H}^{2,0,1}$ are permutation equivalent, which is equivalent to say that the codes $\Phi(\hat{\mathcal{H}}^{2,0,1}_{\bar{a}_2,\bar{b}_2})$ and $\Phi(\mathcal{H}^{2,0,1})$ are permutation equivalent.

Theorem 3.2. *Let* $t_1 \geq 1$, $t_2 \geq 0$ *and* $\ell \geq 0$. *Let* $\bar{a}_{t_1+\ell} = (a_1, \ldots, a_{t_1+\ell}) \in (\mathbb{Z}_4^*)^{t_1+\ell}$, $\bar{b}_{t_1} = (b_1, \ldots, b_{t_1}) \in (\mathbb{Z}_8^*)^{t_1}$, *and* $\mathbf{a}_{t_1+i} = (a_{t_1+i}, \overset{2^{t_1+i-1}}{\ldots}, a_{t_1+i})$ *for* $0 \leq i \leq \ell$. *Let* $\hat{A}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ *be the matrix obtained by using the following construction (instead of Construction (4)), starting with* $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ *given in (23). If we have* $\hat{A}^{t_1,\ell-1,1}_{\bar{a}_{t_1+\ell-1},\bar{b}_{t_1}} = (\hat{A}_1 \mid \hat{A}_2 \mid \hat{A}_3)$, *with* $\ell \geq 1$, *we may construct*

$$\hat{A}^{t_1,\ell,1}_{\bar{a}_{t_1+\ell},\bar{b}_{t_1}} = \left(\begin{array}{cc|ccccc|cccc} \hat{A}_1 & \hat{A}_1 & \hat{M}_1 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{A}_2 & \hat{A}_3 & \hat{A}_3 & \hat{A}_3 & \hat{A}_3 \\ \mathbf{0} & \mathbf{1} & \mathbf{a}_{t_1+\ell} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} & \mathbf{2} & \mathbf{4} & \mathbf{6} \end{array}\right), \tag{25}$$

where $\hat{M}_1 = \{\mathbf{z}^T : \mathbf{z} \in \{2\} \times \{0,2\}^{t_1+\ell-1}\}$. *We repeat Construction (25) until* $\ell = t_2$. *Then, the code generated by* $\hat{A}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, *denoted by* $\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, *is a* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-additive Hadamard code of type* $(\alpha_1, \alpha_2, \alpha_3;\ t_1, t_2, 1)$ *with* $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ *and* $\alpha_3 \neq 0$. *Moreover, the corresponding* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-linear Hadamard code* $\Phi(\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}})$ *is permutation equivalent to* $\Phi(\mathcal{H}^{t_1,t_2,1})$.

Proof. It is enough to show that $\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2,1}$ are permutation equivalent. We prove this by induction on $t_2 \geq 0$. By Theorem 3.1, $\hat{\mathcal{H}}^{t_1,0,1}_{\bar{a}_{t_1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,0,1}$ are permutation equivalent. Assume that $\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2,1}$, generated by $\hat{A}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $A^{t_1,t_2,1}$, respectively, are permutation equivalent. We have that $a_{t_1+t_2+1} \in \mathbb{Z}_4^* = \{1,3\}$. Let $\hat{\mathcal{H}}^{t_1,t_2+1,1}_{\bar{a}_{t_1+t_2+1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2+1,1}$ be the codes generated by $\hat{A}^{t_1,t_2+1,1}_{\bar{a}_{t_1+t_2+1},\bar{b}_{t_1}}$ using Construction (25) and $A^{t_1,t_2+1,1}$ using Construction (4), respectively. Then, by the same arguments as in the proof of Theorem 3.1 and applying the row operation $r_{t_1+t_2+1} \leftarrow -r_{t_1+t_2+1}$ if $a_{t_1+t_2+1} = 3$, the codes $\hat{\mathcal{H}}^{t_1,t_2+1,1}_{\bar{a}_{t_1+t_2+1},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2+1,1}$ are permutation equivalent, and so are the corresponding $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear codes. This completes the proof. $\square$

Theorem 3.3. $t_1 \geq 1$, $t_2 \geq 0$ *and* $t_3 \geq 1$. *Let* $\bar{a}_{t_1+t_2} = (a_1, \ldots, a_{t_1+t_2}) \in (\mathbb{Z}_4^*)^{t_1+t_2}$, $\bar{b}_{t_1} = (b_1, \ldots, b_{t_1}) \in (\mathbb{Z}_8^*)^{t_1}$. *Let* $\hat{A}^{t_1,t_2,t_3}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ *be the matrix obtained by using Construction (5) in the following way, starting with* $\hat{A}^{1,0,1}_{\bar{a}_1,\bar{b}_1}$ *given in (23). If we have* $\hat{A}^{t_1,t_2,\ell-1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, $\ell \geq 2$, *we may construct* $\hat{A}^{t_1,t_2,\ell}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ *by Construction (5). We repeat Construction (5) until* $\ell = t_3$. *Then, the code generated by* $\hat{A}^{t_1,t_2,t_3}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, *denoted by* $\hat{\mathcal{H}}^{t_1,t_2,t_3}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$, *is a* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-additive Hadamard code of type* $(\alpha_1, \alpha_2, \alpha_3; t_1, t_2, t_3)$ *with* $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ *and* $\alpha_3 \neq 0$. *Moreover, the corresponding* $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$*-linear Hadamard code* $\Phi(\hat{\mathcal{H}}^{t_1,t_2,t_3}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}})$ *is permutation equivalent to* $\Phi(\mathcal{H}^{t_1,t_2,t_3})$.

Proof. We have that that $\hat{\mathcal{H}}^{t_1,t_2,1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2,1}$ are permutation equivalent by Theorem 3.2. Then, by the same arguments as in the proof of Theorem

3.1, the codes $\hat{\mathcal{H}}^{t_1,t_2,t_3+1}_{\bar{a}_{t_1+t_2},\bar{b}_{t_1}}$ and $\mathcal{H}^{t_1,t_2,t_3+1}$ are permutation equivalent and the result follows. □

4. Conclusions and further research

In this paper, we give several recursive constructions of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. We show that they all allow us to construct the same family of codes since they generate permutation equivalent codes. Moreover, from Example 2.3, we see that all the nonlinear $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes H^{t_1,t_2,t_3} of length 2^{11} are not equivalent to each other, nor to any $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code, nor to any $\mathbb{Z}_{2^s}$-linear Hadamard code [16], with $s \geq 2$, of the same length 2^{11}. Therefore, we have that some nonlinear Hadamard codes, without any known structure, now can be seen as the Gray map image of a $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes with $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. As further research, it would be interesting to generalize this result, given only for 2^{11}, to any length 2^t.

Another further research could be to generalize the given construction of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes to $\mathbb{Z}_2\mathbb{Z}_4 \dots \mathbb{Z}_{2^s}$-linear Hadamard codes with $\alpha_1, \dots, \alpha_s$ different to zero, or even to $\mathbb{Z}_p\mathbb{Z}_{p^2} \dots \mathbb{Z}_{p^s}$-linear generalized Hadamard codes with p prime. The study of $\mathbb{Z}_2\mathbb{Z}_4\mathbb{Z}_8$-linear Hadamard codes may represent an important step to study the general case, and other papers [2, 36] have also focused on this particular case. However, the generalizations are not feasible using the same techniques employed in this paper.

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