Ordinal Relative Satisficing Behavior: Theory and Experiments

Salvador Barberà
Alejandro Neme

This version: June 2016

October 2014

Barcelona GSE Working Paper Series

Working Paper nº 790

Ordinal Relative Satisficing Behavior: Theory and Experiments*

SALVADOR BARBERÀ† and ALEJANDRO NEME‡
06/12/2016

Abstract: We propose a notion of $r-$rationality, a relative version of satisficing behavior based on the idea that, for any set of available alternatives, individuals choose one of their $r-$best according to a single preference. We fully characterize the choice functions satisfying the condition for any r, and provide an algorithm to compute the maximal degree of $r-$rationality associated with any given choice function. The notion is extended to individuals whose r may vary with the set of available alternatives. We provide experimental evidence that the predictive power of our theory, measured by Selten's index, improves upon that of alternative ones.

Journal of Economic Literature Classification Numbers: D01, D60.

Keywords: Choice, Rationality, Satisficing Behavior, Rationalizability, Preferences, Choice Functions .

*We are grateful to Miguel Angel Ballester, Jose Apesteguia, Larry Samuelson, Arunava Sen, William Zwicker, Markus Brill, Ariel Rubinstein for useful comments, and to Adrià Bronchal and Paula Salinas for their help in carrying the experiment. We are especially thankful to Paola Manzini and Marco Mariotti for their careful reading and comments to several drafts of the paper, and to Jordi Brandts and Pedro Rey Biel, who contributed generously their expertise as experimentalists at different stages of the paper.

Salvador Barberà acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centers of Excellence in R&D (SEV-2015-0563) and grant ECO2014-53051-P, and from the Generalitat de Catalunya, through grant SGR2014-515. The work of Alejandro Neme is partially supported by the Universidad Nacional de San Luis, through grant 319502, and by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), through grant PIP 112-200801-00655.

†MOVE, Universitat Autònoma de Barcelona, and Barcelona GSE. Departament d'Economia i d'Història Econòmica, Edifici B, 08193 Bellaterra, Spain. E-mail: barbera.salvador@gmail.com

‡Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Avenida Italia 1554. 5700, San Luis, Argentina. E-mail: neme.alejandroj@gmail.com

1 Introduction

In its most classical expression, an individual's choice behavior is said to be rational if it results (1) from choosing the best available alternative according to (2) a complete, reflexive and transitive preference relation on the set of alternatives. In view of mounting evidence against the observable implications of this simple model of choice, a growing literature has arisen that proposes a variety of departures from it.

Many of these departures, but not ours, assume that agent's choices are guided by considerations that cannot be expressed by means of a single preference. We propose a notion of $r-$rationality based on the idea that individual choices are indeed based on a well-defined preference order, but that agents may be content with selecting one out of their $r-$best alternatives. This provides a purely ordinal and relative version of the classical idea of satisficing behavior. No level of satisfaction is exogenously fixed, agents are not full maximizers, but they follow a clear pattern of behavior whose consequences generate testable implications.

For any r ranging from 1 to the total number n of alternatives, we provide necessary and sufficient conditions for choice functions to be $r-$rationalizable.

Let us informally illustrate the basic intuition for our new conditions by first recalling what we know about classical rational choice in our setting, and then comparing its implications with those of the new notion of $r-$rationality.

Choice functions satisfying classical rationalizability (now called $1-$rationalizability) are usually characterized in our simple setting as the ones satisfying the following necessary and sufficient contraction condition: if an alternative x is chosen for a set B, x must also be chosen from any subset of B that still contains it. This condition provides a "top down" constructive method for the unique rationalization associated to a rationalizable choice function. Just rank in first place the alternative that is chosen when all of them are available, then rank second the one that is chosen after just deleting the first, and so on.

Notice, however, that we could have formulated differently this necessary and sufficient conditions for $1-$rationalizability. Here is another way to describe it, which is the one inspiring the axioms we use in our general case. Take a choice function. Consider the set of all alternatives that are chosen by that function for some subset of alternatives that is not a singleton. If the choice function is $1-$rationalizable, there must be one and only one alternative that is never chosen, and that should be the one ranking in last place in the rationalizing order. If we look at the family of all non-singleton subsets that do not contain this last alternative, and then at all choices for these subsets, again there must be one and only one alternative that does not appear, and it must be ranked second to last.

This "bottom up" construction is the hint to an alternative characterization of classical rationality, through the requirement that choices in a decreasing sequence of sets must exclude one alternative at a time.

Now, the same idea can be tried as a starting point to identify conditions for $r-$rationalizability. For a given choice function, consider those alternatives that are chosen from subsets of size $r+1$ or larger. If all alternatives were in this set, then the choice function could not be rationalized, because the alternative that is last in an eventual rationalizing order can never be chosen. Hence, some alternatives must never be chosen, and they must be at most r. If only one is missing, we can assure that it is last in a rationalizing order, if such exists. If several alternatives are never chosen, we'll show that if a rationalizing order exists, then there is one that places each of these non-chosen alternatives in the last position. Select any one of them to be placed last in the rationalizing order, and look at all subsets that do not contain it but whose cardinality is still above r. Again, some alternative other than the deleted one must never be chosen out of this restricted family of subsets, and so on. Our characterization result is based on a precise formalization of this idea, leading to a natural extension of the axiom that applies to $1-$rationalizable choice functions. Notice, however, that our rationalizations will not be unique for r different than 1.

Here are two examples that anticipate the kind of issues we deal with.

i) A choice function that is not $1-$rationalizable but is $2-$rationalizable. Let $X = \{a_1, a_2, a_3, a_4\}$. Let F such that for any B with $\{a_1, a_4\} \subseteq B$, $F(B) = a_4$ and otherwise $F(B) = a_i$ where i is the minimum value in $\{1, 2, 3, 4\}$ such that $a_i \in B$. This choice function F is not $1-$rationalizable . We can see this because $F(\{a_1, a_2, a_4\}) = a_4$ and $F(\{a_2, a_4\}) = a_2$ a violation of the standard contraction consistency condition. But we can also see that it cannot be $1-$rationalizable because for any $a_i \in X$, there exists $B \subseteq X$ with $\#B \geq 2$ such that $F(B) = a_i$.

Yet, notice that our choice function F is $2-$rationalizable by the preference orders R, R',where $a_4Ra_1Ra_2Ra_3$ and $a_1R'a_4R'a_2R'a_3$.

ii) A choice function that is neither 1 nor $2-$rationalizable, but is $3-$rationalizable. Let again $X = \{a_1, a_2, a_3, a_4\}$ Let F such that $F(X) = a_3$, $F(B) = a_4$ for any B with $\{a_1, a_4\} \subseteq B \subsetneq X$; and otherwise $F(B) = a_i$ where i is the minimum value in $\{1, 2, 3, 4\}$ such that $a_i \in B$. Notice that this is the same function than in the preceding case, except for its value in X. Yet, now the function is no longer $2-$rationalizable, but is $3-$rationalizable by any order that does not rank a_3 in the last position.

Now, any choice function F on a set of size n is obviously $n-$rationalizable, and in fact also $(n-1)-$rationalizable, as shown later. Hence, we can properly speak about the level of rationality exhibited by any choice function F as given by the minimum value $r(F)$ for which F is $r(F)-$rationalizable.[1] We provide an algorithm to compute the rationality level $r(F)$ associated with any given choice function, thus providing that notion with operational content.

We then propose an even more flexible model of choice, where the level of rationality that an agent displays when choosing from any given set B may vary with the set under consideration. An agent's level of rationality at each set can be described by a function α where $\alpha(B)$ is the rank required for satisfaction when choosing from B. Then, a choice function will be $\alpha-$rationalizable if for some order R, the choice from any set B is among the $\alpha(B)-$best ranked alternatives according to R. And, again, we fully characterize the choice functions that are α rationalizable, for any given α.

The notion that agents may decide to stop short of choosing their best alternative has deep roots and multiple expressions. Recent work on demand theory by Gabaix (2014), Aguiar and Serrano (2914), Frick (2016), Halevy, Peisetz and Zrill (2015), and Halevy and Zrill (2016), also assumes non-maximizing behavior. In a different vein, Amartya Sen (see Sen (1993), for example) has described the apparently irrational behavior of agents who consistently choose their second best out of the set of alternatives they are proposed. The consequences of that behavior are discussed in Baigent and Gaertner (2010). And our more direct reference point is the idea of satisficing behavior, first introduced by Herbert Simon (1955). When individuals are guided by utility functions, and the comparisons among utility levels have a meaning, one can think of satisficing behavior as the one where the individual chooses among those alternatives that guarantee her a minimum, satisficing level of utility. Our purpose here is to develop a theory of satisficing behavior that is purely ordinal, and thus cannot appeal to any exogenous level of utility as a reference. Within the ordinal context, one could still think of a formulation where some absolute level of satisfaction, identified with the one provided by some exogenously fixed alternative, could set the frontier between satisfactory choices and those that are not. This is the assumption in recent work by Caplin, Dean, and Martin (2011), Papi (2012), Rubinstein and Salant (2006), and Tyson (2008). Our formulation is in a similar spirit, but our notion of a satisficing choice will be relative: agents will select one of the $r-$best ranked alternatives among those available at any act of choice.

Our work, and that of those authors we just mentioned, does not preclude the assumption that agents are still endowed with a preference ordering. Other papers in the burgeoning literature on behavioral economics do, and propose alternative formulations of the actual decision process followed by individuals, as the likely source of their departures from rationality.

[1] We shall provide a precise statement about non uniqueness in Corollary 1.

Some theories abandon the hypothesis that agents are guided by one order of preferences alone, and consider the possibility that choices might be generated by several preferences, used in some organized manner. These include, for example, Apesteguia and Ballester (2011), De Clippel and Eliaz (2010), Green and Hojman (2008), Houy and Tadenuma (2009), Kalai, Rubinstein and Spiegler (2002), Manzini and Mariotti (2007 and 2012).

Other theories depart from the idea that agents maximize over the set of all feasible alternatives. Observed choices may then be the best elements of some subset of available alternatives, those that have been selected through some screening process. Examples of this approach can be found in Cherepanov, Feddersen and Sandroni (2013), Caplin and Dean (2011), Eliaz, Richter and Rubinstein (2011), Eliaz and Spiegler (2011) Horan (2011 and 2015), Lleras, Masatlioglu, Nakajima and Ozbay (2010), Manzini and Mariotti (2014).

Still another approach is to reformulate the issue of rationality by expanding the set of observables and assuming that information may be available on more complex objects, like sequences of tentative choices over subsets, eventually leading to a final selection. This route is particularly fruitful to model decision processes that involve search costs and stopping rules, and is taken in papers like, Horan (2010), Masatlioglu, Nakajima, and Ozbay (2013), Papi (2012) and Raymond (2013).

Our notions of $r-$ and $\alpha-$rationability can be connected with some behavioral models, and proven to be incompatible with others. For example, when agents can only observe a limited number of the available alternatives, due to search costs or other limitations, they can still guarantee that their choices on a set B will not be ranked below some threshold $\alpha(B)$. Hence, $\alpha-$rationalizability will be among the necessary conditions to be satisfied by choice functions generated by these models. Similarly, processes based on the initial screening of $r-$best elements, followed by a final choice among them, as in Eliaz, Richter and Rubinstein (2011),[2] will generate $r-$rationalizable choice functions. On the other hand, we can prove that our notion of rationality is not implied, nor implies the properties required by other models of choice, like the ones proposed by Manzini and Mariotti (2007 and 2012), for example.[3]

In order to provide a first test of robustness for our theory that agents may be content to choose among some of their $r-$best alternatives, we have performed a simple experiment, whose results we discuss in Section 6, and that gives support to the idea that even the strongest of our new notions, that of $2-$rationality, may provide a substantial increase in predictive value relative to the full rationality hypothesis and to some of the alternative theories that has been proposed by other authors (Manzini and Mariotti (2009 and 2010)).

[2] see appendix

[3] We will elaborate this point in Section 5.

The paper proceeds as follows. After this introduction, in Section 2 we formalize the idea of $r-$rationalizability and provide a first characterization result. In Section 3 we define the degree of rationality of a choice function and provide an algorithm allowing to compute that value. In Section 4 discuss the notion of $\alpha-$rationalizability, to cover the case where the value of r can depend on the set from which the agent can choose. In section 5 we compare our approach with that of two important papers, just to prove by example that our notion of rationalizability cannot be accommodated within some of the alternative proposals in the behavioral literature. Section 6 describes our experiment and discusses its significant support to our proposal. Section 7 concludes.

2 $r-$Rationalizable Choice Functions

Consider a finite set X of alternatives with $\#X = n \geq 3$. Let $\mathcal{D} = 2^X - \{\emptyset\}$ be the set of all non-empty subset of alternatives. A choice function on X is a function $F : \mathcal{D} \longrightarrow X$ such that $F(A) \in A$, for every $A \in \mathcal{D}$.

Remark 1 More generally, we can define choice functions whose domains are restricted to classes $\Sigma \subseteq \mathcal{D}$ of alternatives. Our general conditions will apply to these cases with essentially no change, and hence we stick, for clarity of exposition, the the simplest case where the domain of the functions is $\mathcal{D}$.

Let R be a preference relation over the set of all alternatives X. Specifically, R is a complete, antisymmetric, and transitive binary relation on X.[4]

Given a preference relation R on X and a subset $A \in \mathcal{D}$, let $h(A, R)$ the maximal alternative of a set A with respect to preference R. Formally

$$h(A, R) = x \Leftrightarrow xRy \text{ for every } y \in A.$$

Because R is complete and antisymmetric, $\#h(A, R) = 1$ for every $A \in \mathcal{D}$.

Denote $h^1(A, R) = h(A, R)$, and define for every t;

$$h^t(A, R) = h(A - \bigcup_{i=1}^{t-1} h^i(A, R))$$

and

$$M^r(A, R) = \bigcup_{i=1}^{r} h^i(A, R)$$

[4] A binary relation R on F is (i) *complete* if for all $x, y \in X$, either xRy or yRx (ii) *transitive* if for all $x, y, z \in X$ such that $xRyRz$; xRz holds, and (iii) *antisymmetric* if, for all $x, y \in X$ such that xRy and yRx, $x = y$ holds.

Hence, $h^t(A, R)$ is the r-th ranked alternative in A according to R, $M^r(A, R)$ is the set of elements in A that R ranks in $r-$th position or better.

To relax the assumption that an agent always chooses her best alternative, we say that a choice function is $r-$rationalizable if there exists a preference relation on the alternatives such that the one chosen for each subset is among its $r-$best ranked elements according to that order. Formally,

Definition 1 A choice function F is $r-$*rationalizable* if there exists a preference relation R over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in \bigcup_{i=1}^{r} h^i(A, R) = M^r(A, R).$$

Remark 2 If a choice function is $r-$rationalizable, it is also $r'-$rationalizable for $r' > r$.

Remark 3 The concept of $r-$rationalizability does not impose any restrictions on the possible choices of an agent for sets of size r or less. In particular, any choice function is $n-$rationalizable when there are n alternatives. In fact, any choice function F on a set of size n is $(n-1)-$rationalizable by any preference relation R such that $h(X, R) = F(X)$, since we then have that for any $A \subseteq X$, $F(A) \in M^{(n-1)}(A, R)$. Yet, not every choice function is $(n-2)-$rationalizable, as already proven by the second example in the introduction.

We now introduce definitions leading to our main characterization result and inspired in the intuitions we provided in the Introduction about our "bottom up" approach.

For each $Y \subseteq X$ and r, a natural number, define the family of sets $\mathcal{D}^r(Y) = \{B \in \mathcal{D} : \#(B \cap Y) > r\}$. It is on such families of subsets that our rationalizability conditions will have bite.

Let

$$\mathcal{C}_F^r(Y) = \{x \in Y : \text{ for all } B \in \mathcal{D}^r(Y),\ F(B) \neq x\}$$

be the set of alternatives that will never be chosen from any set in $\mathcal{D}^r(Y)$[5].

Theorem 1 A choice function F on X is $r-$rationalizable if and only $\mathcal{C}_F^r(Y) \neq \varnothing$ for all $Y \subseteq X$.

Before proving the Theorem, we provide some intuitions and a Lemma that can be seen as an alternative characterization of $r-$rationalizable choice functions. After those intuitions, and having stated and proven the lemma, Theorem 1 will follow easily.

Here is how we can check for rationality and eventually construct the rationalizing order. Let $X = Y_0$, first look at the set $\mathcal{C}_F^r(Y_0)$ of those alternatives that are not chosen

[5] Notice that if $\mathcal{D}_F^r(Y)$ is empty, then $\mathcal{C}_F^r(Y) = Y$ trivially.

for any subset of size at least $(r+1)$. Define

$$L_1 = \mathcal{C}_F^r(Y_0) \text{ and } Y_1 = Y_0 - \mathcal{C}_F^r(Y_0).$$

Clearly, $\#L_1$ must be at least one: otherwise, F is not $r-$rationalizable, because there is no candidate for last position in a rationalizing order. If that first requirement is satisfied, and if the size of Y_1 is equal to r, we are done: F is $r-$rationalizable with elements in Y_1 in the top of the rationalizing order. Otherwise, there will be subsets of Y_1 with size at least $(r+1)$. Look for those alternatives $\mathcal{C}_F^r(Y_1)$ that are not chosen from any subset B of Y_0 with size of $(B \cap Y_1)$ larger than r. Define

$$L_2 = \mathcal{C}_F^r(Y_1) \text{ and } Y_2 = Y_1 - \mathcal{C}_F^r(Y_1),$$

this set must be strictly smaller than Y_1. Otherwise, there is no candidate to be the worse alternative before those in Y_1 in the rationalizing order, and F is not $r-$rationalizable. If that second test is still passed, and if the size of Y_2 is equal to r, we are done: F is $r-$rationalizable with elements in Y_2 in the top of the rationalizing order. Otherwise, there will be subsets of Y_2 with size at least $(r+1)$. Compute the set $\mathcal{C}_F^r(Y_2)$, define

$$L_3 = \mathcal{C}_F^r(Y_2) \text{ and } Y_3 = Y_2 - \mathcal{C}_F^r(Y_2).$$

Again, the set Y_3 is smaller than Y_2, and so on. Since the necessary conditions in that sequence implies the nestedness of the sets Y_i, and Y_0 is finite, either they stop holding at some point, with $\#Y_{\overline{k}} > r$ ($\mathcal{C}_F^r(Y_{\overline{k}}) = \varnothing$), and define

$$L_{\overline{k}} = Y_{\overline{k}}$$

in which case F is not $r-$rationalizable, or else they lead to a set $Y_{\overline{k}}$ of size equal to r, defining

$$L_{\overline{k}} = Y_{\overline{k}} = \mathcal{C}_F^r(Y_{\overline{k}})$$

and rationalizability holds. Sufficiency is easily derived by ranking different subsets in such a way that those that are still chosen in a certain iteration are ranked above those who have disappeared from the Y_i's in preceding steps.

Consider the partition $\mathcal{P} = \{L_1, .., L_{\overline{k}}\}$ on X, observe that for every $k < \overline{k}$;

$$\#L_k \leq r.$$

Now, we can look at a preference relation R_F over X satisfying:

$$yR_Fx \text{ for every } x, y \text{ such that } y \in L_j \text{ and } x \in L_j \text{ with } i < j$$

Lemma 1 A choice function F is $r-$rationalizable if and only if $\#L_{\overline{k}} \leq r$.

Proof $\Rightarrow$) Let F be an $r-$rationalizable choice function. Assume that R is a preference relation over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in M^r(A, R)$$

Let $\mathcal{P} = \{L_1, .., L_{\overline{k}}\}$ be the above defined partition on X.

Assume otherwise, that is $L_{\overline{k}} = Y_{\overline{k}}$,

$$\#L_{\overline{k}} > r$$

and $\mathcal{C}_F^r(Y_{\overline{k}}) = \varnothing$. Let $y \in Y_{\overline{k}}$ be such that

$$xRy$$

for every $x \in Y_{\overline{k}}$. Because $\mathcal{C}_F^r(Y_{\overline{k}}) = \varnothing$, there exists $B \in \mathcal{D}^r(X_{\overline{k}})$ such that $F(B) = y$. Since $\#(B \cap Y_{\overline{k}}) > r$, we have that $y \notin M^r(B, R)$, contradicting that $F(B) \in M^r(B, R)$.

$\Leftarrow$) Let $\mathcal{P} = \{L_1, .., L_{\overline{k}}\}$ be a partition on X such that $L_k = \mathcal{C}_F^r(Y_k)$ with $k = 1, .., \overline{k}$ and $\#\mathcal{C}_F^r(Y_{\overline{k}}) \leq r$. We will show that F is $r-$rationalizable by R_F. That is, for every $A \in \mathcal{D}$;

$$F(A) \in M^r(A, R_F).$$

Assume otherwise, that there existed $A \in \mathcal{D}$ such that

$$F(A) = z \notin M^r(A, R_F)$$

Notice that

$$xR_F z. \tag{1}$$

for every $x \in M^r(A, R_F)$. Let i' be such that $M^r(A, R_F) \cup \{z\} \subseteq Y_{i'}$ and $M^r(A, R_F) \cup \{z\} \nsubseteq Y_{i'+1}$. Thus, there exists $\overline{x} \in \mathcal{C}_F^r(Y_{i'})$ and $z \notin \mathcal{C}_F^r(Y_{i'})$. This implies that there exists j' with $i' < j'$ and $z \in \mathcal{C}_F^r(Y_{j'})$, and consequently $zR_F\overline{x}$, a contradiction to (1).

This concludes the proof. ■

Now, we prove Theorem 1.

Proof of Theorem 1 $\Rightarrow$) Let F be an $r-$rationalizable choice function. Assume that R is a preference relation over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in M^r(A, R).$$

Let Y be any subset of alternatives and $\overline{y} \in Y$ such that

$$yR\overline{y} \text{ for every } y \in Y.$$

Notice that

$$\overline{y} \notin M^r(B,R)$$

for any B such that $\#(B \cap Y) > r$. Thus, $F(B) \neq \overline{y}$. This implies that $\overline{y} \in \mathcal{C}_F^r(Y)$.

$\Leftarrow$) Assume that $\mathcal{C}_F^r(Y) \neq \varnothing$ for every $Y \subseteq X$. Let $\mathcal{P} = \{L_1, .., L_{\overline{k}}\}$ be a partition on X such that $L_k = \mathcal{C}_F^r(Y_k)$ with $k = 1, .., \overline{k}$. Thus, $\#\mathcal{C}_F^r(Y_{\overline{k}}) = r$.

Lemma 1 concludes the proof. ∎

Let us show how we check for rationalizability and at the same time eventually construct a rationalizable order, by examining two examples.

Example 2 Let $X = \{a_1, ..., a_5\}$. The choice function $F : \mathcal{D} \longrightarrow X$ is defined as follows;

- If $\#B = 2$, then
  - i) If $B = \{a_1, a_5\}$, then $F(B) = a_5$.
  - ii) Otherwise, let $B = \{a_i, a_j\}$ and $i^* = \min\{i, j\}$; then $F(B) = a_{i^*}$.
- If $\#B = 3$, then
  - i) If $a_3 \in B$, then $F(B) = a_3$.
  - ii) If $a_4 \in B$ and $a_3 \notin B$, then $F(B) = a_4$.
  - iii) $F(\{a_1, a_2, a_5\}) = a_2$.
- If $\#B = 4$, then

$F(\{a_1, a_2, a_3, a_4\}) = a_4$	$F(\{a_1, a_2, a_3, a_5\}) = a_3$	$F(\{a_2, a_3, a_4, a_5\}) = a_3$
$F(\{a_1, a_2, a_4, a_5\}) = a_4$	$F(\{a_1, a_3, a_4, a_5\}) = a_4$	

- If $\#B = 5$, then $F(\{a_1, a_2, a_3, a_4, a_5\}) = a_3$.

Let us check whether F is $2-$rationalizable[6].

According to definitions,

$$\mathcal{C}_F^2(X) = \left\{x : \text{for all } B \in \mathcal{D}_F^2(X),\ F(B) \neq x\right\} = \{a_5\} = L_1$$

Then $Y_1 = X - \{a_5\}$

[6]Notice that F is not $1-$ rationalizable, since for any i there exists B such that $F(B) = a_i$.

Next, consider

$$\mathcal{C}_F^2(Y_1) = \{x : \text{for all } B \in \mathcal{D}_F^2(Y_1),\ F(B) \neq x\} = \{a_1, a_2\}$$

Define $Y_2 = Y_1 - \{a_1, a_2\}$.

Since $\#Y_2 = 2$, therefore $\mathcal{C}_F^2(Y_2) = Y_2 = L_3$.

Lemma 1 implies that F is $2-$rationalizable. $\square$

Example 2 illustrates Theorem 1, with a function F is $2-$rationalizable. Our next example involves a function that is not $2-$rationalizable.

Example 3 Let $X = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of alternatives and $F : \mathcal{D} \longrightarrow X$ the choice function defined as follows,

- If $\#B = 2$, then
  - i) If $B = \{a_1, a_5\}$, then $F(B) = a_1$.
  - ii) Otherwise, let $B = \{a_i, a_j\}$ and $i^* = \min\{i, j\}$; then $F(B) = a_{i^*}$.
- If $\#B = 3$, then
  - i) If $a_3 \in B$, then $F(B) = a_3$.
  - ii) If $a_4 \in B$ and $a_3 \notin B$, then $F(B) = a_4$.
  - iii) $F(\{a_1, a_2, a_5\}) = a_2$.
- If $\#B = 4$, then

$F(\{a_1, a_2, a_3, a_4\}) = a_1$	$F(\{a_1, a_2, a_3, a_5\}) = a_3$	$F(\{a_1, a_2, a_4, a_5\}) = a_4$
$F(\{a_1, a_3, a_4, a_5\}) = a_3$	$F(\{a_2, a_3, a_4, a_5\}) = a_3$	

- If $\#B = 5$, then $F(\{a_1, a_2, a_3, a_4, a_5\}) = a_2$.

Let us check whether F is $2-$rationalizable[7].

Now clearly,

$$\mathcal{C}_F^2(X) = \{a_5\}.$$

[7] Notice that F is not $1-$ rationalizable, since for any i there exists B such that $F(B) = a_i$.

Which implies that $Y_1 = X - \{a_5\}$

Notice that

$$\mathcal{C}_F^2(Y_1) = \{a_1, a_2, a_3, a_4\}.$$

But this means that, $\mathcal{C}_F^2(Y_1) = \varnothing$. Theorem 1 implies that F is not $2-$rationalizable. $\square$

Our conditions for $r-$rationalizability allow us to discuss the following issues. What is the number of possible $r-$rationalizations for a given choice function?. How much can we learn about the actual ranking of any given alternative in a given preference order, by observing the choice function of an $r-$rational agent who has that order?. The next Corollary and Remark give answers to these questions for the case where choice data are available for all subsets.[8]

Given $r-$rationalizable choice function F, let $\mathcal{P} = \{L_1, .., L_{\overline{k}}\}$ be a partition on X such that $L_k = \mathcal{C}_F^r(Y_k)$ with $k = 1, .., \overline{k}$ and $\#\mathcal{C}_F^r(Y_{\overline{k}}) \leq r$.

This allows us to provide an exact count of the number of rationalization that the choice function F will admit.

Corollary 1 Consider an $r-$rationalizable choice function F. This function is rationalizable by exactly $t(F)$ different orders, where

$$t(F) = \prod_{k=1}^{\overline{k}} [(\#L_k)!]$$

The bounds for that number are

$$(s_1! + s_2!) \leq t(F) \leq q(r!) + s!$$

where $n = qr + s$, with $0 \leq s < r$ and $q \geq 0$, $s_1 + s_2 = r + 1$, $s_i \in \mathbb{N}$ and $0 \leq s_2 - s_1 \leq 1$, corresponding to the case where the cardinality of L_k is minimal and maximal for every k, respectively.

Uniqueness only arises for the classical case where $r = 1$.

Remark 4 Notice that the rank of any alternative in different rationalizations of the same choice function will move between bounds that can be computed from the values of the sets $\overline{Y}_j$ in our iterative constructive process. These bounds may be very tight or very loose. For some choice functions the rank of some alternatives in any rationalization may be completely determined, while in some others it may be completely undetermined.

[8] If the domain Σ of the choice function did not contain all subsets, the multiplicity would be more pervasive.

3 The degree of rationality of a choice function

In this section we define a natural measure of the degree of rationality that is exhibited by a choice function F, and we then provide an algorithm that shows how we can actually compute that degree of rationality in an effective manner.

Definition 2 A choice function F exhibits a degree of rationality $r(F)$ iff F is $r(F)-$rationalizable, and it is not $r'-$rationalizable for any $r' < r(F)$.[9]

We may naturally associate this degree of rationalizability with the search for a "best approximation" to a fully rational preference, in a similar spirit than Afriat (1973), Houtman and Maks (1985), Varian (1990) or, more recently, Apesteguia and Ballester (2014), Halevy, Peisetz and Zrill (2015), and Halevy and Zrill (2016).

In our case, for any given choice function F, and any given linear order P, compute the vector indicating, for each subset B of alternatives, the rank of $F(B)$ according to P. Find a $\overline{P}$ that minimizes the maximal component of these vectors across all possible preferences, and let $\overline{r}$ be the value of the maximal component of the vector associated to $\overline{P}$. Then, $\overline{r}$ will correspond to F's degree of rationality, and any such $\overline{P}$ is an $\overline{r}-$rationalization for F.

We turn to our proposed algorithm. We do not claim it to be particularly efficient, but it is certainly simple enough to prove that it is possible to associate a degree of rationality to every choice function.

The algorithm follows the basic steps suggested by Remark 3 and Theorem 2. We start by identifying, iteratively, the smallest set size such that, when choosing from all sets of at least that size, the set of alternatives that would be eventually chosen is smaller than the initial set of alternatives. This gives us a first bound on the rationality level. That bound is definitely chosen if no sets of its size or more are left when removing the unchosen alternatives from X.

Otherwise, the algorithm continues in a similar manner, but considering only the choices from classes of sets that are nested, and eventually increasing, if necessary, the rationality bound.

Algorithm:
Input: A finite set of alternatives X, with $\#X = n \geq 3$ and F a choice function on X.

[9] This notion is in a similar spirit than the exercise in Salant and Rubinstein (2006), where the minimum number of different lists necessary to rationalize a given choice function is also calculated. But of course, we refer to diferent concepts of rationality.

Step 1: For $r = 0, .., n-1$; compute $\mathcal{C}_F^r(X)$;

$$\mathcal{C}_F^r(X) = \{x \in Y : \text{ for all } B \in \mathcal{D}^r(Y),\ F(B) \neq x\}$$

Step 2: Let r_0 be such that $\mathcal{C}_F^{r_0}(X) = \varnothing$ and $\mathcal{C}_F^{r_0+1}(X) \neq \varnothing$. Define $r = r_0 + 1$.

Step 3: Define $Y_1^r = X - \mathcal{C}_F^r(X)$. Set $j := 1$.

Step 4: If $\#Y_j^r \leq r$, then r^* and go to step 8.

Step 5: Compute $\mathcal{C}_F^r(Y_j^r)$. Notice that $r < \#Y_j^r$.

Step 6: If $\mathcal{C}_F^r(Y_j^r) = \varnothing$, set $r := r + 1$ and go to step 3.

Step 7: If $\mathcal{C}_F^r(Y_j^r) \neq \varnothing$, define $Y_{j+1}^r = Y_j^r - \mathcal{C}_F^r(X_j^r)$. Set $j := j + 1$ and go to step 4.

Step 8: The choice function F is r^*-rationalizable. Define $r^* = r(F)$.

END.

Theorem 2 The natural number $r(F)$ is the minimum such that the function F is $r(F)-$rationalizable.

Proof Let $r^* = r(F)$ be obtained in step 7.

First, we will prove that F is r^*-rationalizable. It follows clearly from step 4 and Lemma 1, because $L_j = \mathcal{C}_F^r(Y_j^r)$ with $\#L_j \leq r$.

Now, we have to prove that F is not $(r^* - 1)-$rationalizable. Assume otherwise, that there exists R such that for any $Y \subset X$;

$$F(Y) \in M^{(r^*-1)}(Y, R). \tag{2}$$

The algorithm stops for $r = r^*$. Then for $r = r^* - 1$ the algorithm did not lead to Step 8, but proceeded to steps 5 and 6. But then, the only chance for the algorithm to finally stop after having re-visited step 4 is that at some point it reverted to step 3, and this implies that there exists j such that $\mathcal{C}_F^r(Y_j^r) = \varnothing$ and $r < \#Y_j^r$. Notice that, for $Y = Y_j^r$ $(\#Y > r)$, and $\mathcal{C}_F^r(Y) = \varnothing$. But then, Theorem 1 implies that F is not $(r^* - 1)-$rationalizable.

This concludes the proof. ■

We illustrate how the algorithm works with the following example.

Example 4 Let $X = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of alternatives and $F : \mathcal{D} \longrightarrow X$ the choice function defined as follows for each subset of size 2 or larger:

- $F(\{a_i, a_j\}) = a_j$, with $j > i$
- If $\#B = 3$, then

- i) If $a_3 \in B$, then $F(B) = a_3$.
- ii) If $a_4 \in B$ and $a_3 \notin B$, then $F(B) = a_4$.
- iii) $F(\{a_1, a_2, a_5\}) = a_2$.

- If $\#B = 4$, then

$F(\{a_1, a_2, a_3, a_4\}) = a_1$	$F(\{a_1, a_2, a_3, a_5\}) = a_3$	$F(\{a_1, a_2, a_4, a_5\}) = a_4$
$F(\{a_1, a_3, a_4, a_5\}) = a_3$	$F(\{a_2, a_3, a_4, a_5\}) = a_3$	

- If $\#B = 5$, then $F(\{a_1, a_2, a_3, a_4, a_5\}) = a_2$.

The algorithm

Step 1: For $r = 0, .., 4$; compute $\mathcal{C}_F^0(X) = \mathcal{C}_F^1(X) = \varnothing$; $\mathcal{C}_F^2(X) = \{a_5\}$; $\mathcal{C}_F^3(X) = \{a_5\}$; $\mathcal{C}_F^4(X) = \{a_5\}$.

Step 2: Set $\mathcal{C}_F^0(X) = \mathcal{C}_F^1(X) = \varnothing$, and $\mathcal{C}_F^2(X) \neq \varnothing$, $r_0 = 1$ and $r = 2$.

Step 3: Define $Y_1^2 = X - \mathcal{C}_F^2(X) = \{a_1, a_2, a_3, a_4\}$.

Step 4: Because $\#(Y_1^2) > 2$, then go to step 5.

Step 5: Compute $\mathcal{C}_F^2(X_1^2)$;

$$\mathcal{C}_F^2(X_1^2) = \varnothing.$$

Step 6: Because $\mathcal{C}_F^2(Y_1^2) = \varnothing$, then define $r = 3$ and go to step 3. Notice F is not 2$-$rationalizable.

Step 3: Define $Y_1^3 = X - \mathcal{C}_F^3(X) = \{a_1, a_2, a_3, a_4\}$. Set $j := 1$, then go to step 4.

Step 4: Because $\#(Y_j^3) > 3$, then go to step 5.

Step 5: Compute $\mathcal{C}_F^2(Y_j^3)$;

$$\mathcal{C}_F^2(Y_j^3) = \{a_3, a_4\}$$

Because $\mathcal{C}_F^2(Y_1^3) \neq \varnothing$, go to step 7.

Step 7: Define $Y_{j+1}^3 = Y_j^3 - \mathcal{C}_F^2(Y_j^3) = \{a_1, a_2\}$. Set $j := 2$ and go to step 4.

Step 4: Because $\#Y_j^3 = 2 \leq 3 = r$, then $\mathcal{C}_F^2(Y_j^3) = Y_j^3$ and go to step 8.

Step 8: The choice function F is 3$-$rationalizable.

END.

4 A further extension of the rationalizability concept

In this section we consider the general case where the same agent may be content, or not, with getting her $r-$ranked alternative, depending on the context where this choice occurs. For example, a larger r may be required when choosing from a set of similar alternatives, while a smaller level of r may apply when the states involved when making a potential mistake are larger. Our definitions and results are similar to those already presented, and we shall thus be a bit more expedient in the presentation. The proofs and examples illustrating the basic aspects of the proposed extension are presented in Appendix A.

Consider a finite set X of alternatives with $\#X \geq 3$, and a function $\alpha : \mathcal{D} \longrightarrow \{1, .., n\}$ that determines a level of relative ordinal satisficing behavior for each subset B of alternatives. We say that a choice function is $\alpha-$rationalizable if there exists a preference relation on the alternatives such that the one chosen for each subset A of alternatives is among the $\alpha(A)-$best ranked elements of the order among those in the sets. Formally:

Definition 3 A choice function F is *α-rationalizable* if there exists a preference relation R over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in \bigcup_{i=1}^{\alpha(A)} h^i(A, R) = M^{\alpha(A)}(A, R).$$

Let r be a natural number, we say that F is $r-$rationalizable if it is $\alpha-$rationalizable with $\alpha(B) = r$.

We now introduce definitions leading to our main characterization result and inspired in the intuitions we provided in the Introduction about our "bottom up" approach.

For each $Y \subseteq X$ and a function $\alpha : \mathcal{D} \longrightarrow \{1, .., n\}$, define the family of sets $\mathcal{D}^{\alpha}(Y) = \{B \in \mathcal{D} : \#(B \cap Y) > \alpha(B)\}$. It is on such families of subsets that our rationalizability conditions will have bite.

Let $\mathcal{C}_F^{\alpha}(Y) = \{x \in Y : \text{for all } B \in \mathcal{D}^{\alpha}(Y),\ F(B) \neq x, \}$. This is the set of alternatives that will never be chosen from any set in $\mathcal{D}^{\alpha}(Y)$.

Theorem 3 A choice function F on X is $\alpha-$rationalizable if and only $\mathcal{C}_F^{\alpha}(Y) \neq \varnothing$ for all $Y \subseteq X$.

Proof See appendix. ■

Again, we can reformulate the characterization in a form that is parallel to Lemma 1. Define $Y_0 = X$, and for $k = 1, .., \overline{k} - 1$;

$$L_k = \mathcal{C}_F^{\alpha}(Y_{k-1}) \text{ and } Y_k = Y_{k-1} - \mathcal{C}_F^{\alpha}(Y_{k-1})$$

and $L_{\overline{k}} = Y_{\overline{k}} = \mathcal{C}_F^{\alpha}(Y_{\overline{k}-1})$, where $\overline{k}$ is either equal to $\min\limits_{\mathcal{C}_F^{\alpha}(Y_k)=\varnothing} k$ or $\min\limits_{\mathcal{C}_F^{\alpha}(Y_k)=Y_k} k$.

Lemma 2 A choice function F is $\alpha-$rationalizable if and only if $Y_{\overline{k}} = \mathcal{C}_F^{\alpha}(Y_{\overline{k}})$.

Proof See appendix. ■

Notice that given a function $\alpha(B) = \#B$, every choice function F is $\alpha-$rationalizable. Moreover, let α, α' be such that $\alpha(B) \leq \alpha'(B)$ for every B. If the choice function F is $\alpha-$rationalizable, then F is $\alpha'-$rationalizable.

5 A comparison of $r-$rationalizability with alternative restrictions on choice functions

As an illustration that our notion of rationalizability characterizes behavior that is independent from the one predicated by other models, we compare its implications with those of two proposal by Manzini and Mariotti (2007) and (2012), which they call the Rational Shortlist Method and the Categorize-Then-Choose, respectively.

Definition 4 A choice function F is a Rational Shortlist Method (RSM) whenever there exists an ordered pair (P_1, P_2) of asymmetric relations, with $P_i \subseteq X \times X$ such that

$$F(A) = \max(\max(A; P_1); P_2).$$

Manzini and Mariotti show that choice functions of this form do not need to be $1-$rational. Yet, they identify two properties that fully characterize them.

Expansion: For all $S, T \subseteq X$, if $x = F(S) = F(T)$, then $x = F(S \cup T)$.

WARP* : If $T \subseteq R \subseteq S$, $F(S) = F(T) = x$ and $y \in T$, then $y \neq F(R)$.[10]

Theorem (Manzini-Mariotti (2007)) The choice function F is RSM if and only if it satisfies $WARP^*$ and Expansion.

A second, more general proposal of Manzini and Mariotti (2012), involved boundedly rational agents who categorize alternatives before choosing.

A **shading relation** on X is an asymmetric relation (possibly incomplete) $\succ$ on $\mathcal{D}$.

Given a shading relation $\succ$ on X and a set $A \in \mathcal{D}$, define the $\succ-$maximal set on A as follows:

$$\max(A, \succ) = \{x \in A : \forall B \in D \text{ with } x \in B, \ \nexists\, B' \text{ such that } B' \succ B\}$$

[10] Manzini-Mariotti (2007) show that the axiom we write here is equivalent to one where T is limited to be a pair, whenever all pairs are in the domain of the choice function. We formulate the axiom in this alternative manner to take into account the possibility that not all pairs be in that domain.

Definition 5 A choice function F on X is Categorize-Then-Choose (CTC) whenever there exists a shading relation $\succ$ and a preference P such that for all $A \in \mathcal{D}$,

$$F(A) \in \max(A, \succ) \text{ and } F(A)Py, \text{ for all } y \in \max(A, \succ).$$

Manzini and Mariotti show that choice functions of this form do not need to be 1−rational. Yet, they show that $WARP^*$ fully characterizes them.

Theorem (Manzini-Mariotti (2010)) The choice function F is CTC if and only if it satisfies $WARP^*$.

We'll show that choice function derived from their two models may not satisfy our notion of rationality, and conversely, that our functions need not be of their forms.

Example 5 The example shows a 2-rationalizable choice function F that is not CTC (and then not RSM). Let $X = \{x_1, x_2, x_3, x_4\}$, P an strict order on X; $x_1Px_2Px_3Px_4$, and F be a choice function defined as follows:

$$F(A) = \begin{cases} \max(A; P) & if \quad A \neq \{x_1, x_2, x_3\} \\ x_2 & if \quad A = \{x_1, x_2, x_3\} \end{cases}$$

Observe that F is 2−rationalizable but is not CTC. This is because, $\{x_1, x_2\} \subseteq \{x_1, x_2, x_3\} \subseteq X$, $F(\{x_1, x_2\}) = x_1 = F(X)$, but $F(\{x_1, x_2, x_3\}) = x_2$, and therefore F does not satisfy $WARP^*$.

Example 6 The following example shows that there exists a RSM, and consequently CTC choice function F, that is not 2-rationalizable. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of alternatives. Let R_1 be the following partial order;

$$x_4R_1x_1 \text{ and } x_5R_1x_2.$$

Let R_2 be a strict order on X;

$$x_1R_2x_2R_2x_3R_2x_4R_2x_5$$

Observe that,

$$\max(\{x_1, x_2, x_3\}; R_1) = \{x_1, x_2, x_3\},$$

$$\max(\{x_1, x_2, x_3, x_4\}; R_1) = \{x_2, x_3, x_4\},$$

and

$$\max(\{x_1, x_2, x_3, x_4, x_5\}; R_1) = \{x_3, x_4, x_5\}.$$

Observe that for every $S \subseteq X$;

$$F(S) = \max\left(\max(S; R_1), R_2\right).$$

Then,

$$F(\{x_1, x_2, x_3\}) = x_1,$$

$$F(\{x_1, x_2, x_3, x_4\}) = x_2,$$

and

$$F(\{x_1, x_2, x_3, x_4, x_5\}) = x_3.$$

Let $Y = \{x_1, x_2, x_3\}$, observe that for each $x_i \in Y$ there B such that $\#(B \cap Y) \geq 3$ and $F(B) = x_i$. Theorem 2, implies that F is not $2-$rationalizable.

The above examples show that our model and theirs lead to independent restrictions on the set of possible choice functions they generate.

6 A class experiment

As a first test of robustness of our proposed theory, we have performed a simple experiment. Its main purpose was to check the consistency of our data with the more demanding of our new possible levels of rationality ($2-$rationality), and to compare the predictive power of this hypothesis with that of full rationality and of some alternative views of the choice processes.

6.1 Experimental Design

Our experiment consists in eliciting the (partial) choice function of each subject over a set of alternatives. The task is straightforward: pick the preferred one that among a set of alternatives, for different subsets of a grand set.

The grand set of alternatives consisted of the following five remuneration plans. All of them add to the same amount of 120 cents, and propose different partial payments staggered over three fixed dates, just three, six and nine weeks after the experiment.

Three period sequences					
A	**B**	**C**	**D**	**E**	
80	240	160	80	160	in three weeks
160	160	160	80	240	in six weeks
240	80	160	320	80	in nine weeks

The choice of these alternatives was prompted by two different considerations. A substantial one is that choosing among money payment schedules rather than combinations of goods reduces potential emotive connotations. A strategic reason was to shape the experiment in a way that would allow comparisons with alternative theories, following the lines of Manzini and Mariotti (2010). These authors generated full choice functions for a grand set of four payment schedules. We enlarged the grand set by adding an additional schedule to the same ones that they proposed. Then we elicited the partial choice function over all subsets of size larger than two. We did not ask questions regarding pairs, since our theory does not impose any restrictions on such choices. As a result the number of questions and the challenge for the subjects, or their eventual fatigue, were kept very close to those of the experiments we shall compare ours with.

The experiment was carried out at the Universitat Autònoma de Barcelona. We ran four sessions with second-year students in two groups of the microeconomics course. The experiments with both groups were simultaneous. The members of each group were asked to respond to the experiment in two separate dates, with a five-week lag between the two. The payment dates for the different sessions did not overlap. We shall separately treat the aggregate results for each of the different dates as an experiment on its own, but also discuss the relevance of having performed the exercise twice with the same subjects.

Questionnaires offered the same questions in many different orders, in a stapled stack of 16 pages, starting by instructions and followed by one page per choice problem, and each subject had to make the choices in order, without turning pages back. Subjects did not communicate with each other. Experiments lasted about 15 minutes, with 10 minutes of effective play, preceded by a period where an experimenter read aloud the instruction that were in the front page of the stack, including the method of payment to participants[11].

All agents were paid 120 cents participation fee at the date of the experiment, and the staggered payments were made according to individual responses, randomly chosen in public after thee experiment was completed.

Table 1 and 2 display sample sets of plans from which subjects were asked to choose one.

Plan A		Plan B		Plan C	
how much	when	how much	when	how much	when
80 cts	in 3 weeks	240 cts	in 3 weeks	160 cts	in 3 weeks
160 cts	in 6 weeks	160 cts	in 6 weeks	160 cts	in 6 weeks
240 cts	in 9 weeks	80 cts	in 9 weeks	160 cts	in 9 weeks

Table 1

and

[11] See the instructions in Appendix B.

Plan A		Plan B	
how much	when	how much	when
160 cts	in 3 weeks	80 cts	in 3 weeks
160 cts	in 6 weeks	240 cts	in 6 weeks
160 cts	in 9 weeks	160 cts	in 9 weeks
Plan C		**Plan D**	
how much	when	how much	when
240 cts	in 3 weeks	80 cts	in 3 weeks
80 cts	in 6 weeks	160 cts	in 6 weeks
160 cts	in 9 weeks	240 cts	in 9 weeks

Table 2

The order in which the questions were posed was randomized.

6.2 Experimental results

Let us first remark that individuals did certainly not choose at random. With a uniform probability distribution on each choice set, the probability of observing even only two subjects with the same choice is effectively zero for all practical purposes. In fact, as there are a possible $3^{10} \cdot 4^5 \cdot 5 \approx 3 \cdot 10^8$ choice function on the universal set, that probability is $(3.10^8)^{-1}$. But since there are 20 students with the same modal choice in the first session, and 55 for the second, this clearly the hypothesis of randomness.

A total of 117 subjects participated in the first session, and 113 in the second. Here are the results regarding how many of their choice functions could be fully rationalized, versus those that would admit a $2-$rationalization.

	First session		Second session	
	#	%	#	%
full rationality	43	37	70	60
2-rationality	106	90	105	93

Table 3: Overall satisfaction of axioms

In order to compare the ability of different theories to fit the observed data, it is not enough to just check the proportion of favorable observations, since these will definitely increase as the theories become less demanding. Hence, we use Selten's Measure of Predictive Success (Selten (1991)) to compare the relative success of $1-$rationality vs $2-$rationality, and later on we shall also use this measure to compare the success of $2-$rationality relative to an alternative theory discussed in the literature. Selten's measure was specifically designed to evaluate area theories like the ones considered in this paper, namely theories that exclude deterministically a subset of the possible outcomes as not being consistent with them. The measure takes into account not only the descriptive power of the model (measured by the proportion of "hits", the observed outcomes consistent with the model), but also its parsimony. The lower the proportion of theoretically possible outcomes consistent with the model, the more parsimonious the model. Precisely, the measure, denoted s, is expressed as

$$s = r - a$$

where r is the descriptive power (number of actually observed outcomes compatible with the model divided by the number of possible outcomes) and a is the relative area of the model, namely the number of outcomes in principle compatible with the model divided by the number of all possible outcomes.

We find that Selten's index for $1-$rationality and $2-$rationality in our first and second experiment, respectively:

	$1-$rationality	$2-$rationality
first experiment	$s_1^{FR} \approx 0,367$	$s_1^{2-R} \approx 0,786$
second experiment	$s_2^{FR} \approx 0,619$	$s_2^{2-R} \approx 0,818$

The exact calculations we shall now summarize can be found in Appendix 2. Hence, it is not only the proportion of favorable cases that has increased dramatically, but also the predictive success of our theory for $r = 2$, rather than $r = 1$.

6.3 Comparing theories

We have concentrated in $2-$rationalizability because it is the most restrictive of our nested concepts of $r-$rationality, after standard $1-$rationality. But of course our theory is in competition with other possible models, and we shall provide here a comparison that is in accordance with the available data.

As an example, we compare our theory with that of Manzini and Mariotti , since these authors already established that their theory was clearly superior to a variety of others. Since even our strongest concept of $2-$rationality does not impose any restrictions on choices over pairs, it would be unfair to judge others in terms of eventual failures directly or indirectly related to pairwise choices. As already explained in Section 5, Manzini and Mariotti's SRM theory imposes two restrictions on choice functions, Expansion and $WARP^*$, and their CTC theory only imposes $WARP^*$.

Because in our experiment we did not obtain the choice function data on sets of cardinality 2, our calculations for RSM and CTC are an upper bound.

	First session		Second session	
	#	%	#	%
full rationality	43	37	70	60
2-rationality	106	90	105	93
SRM	44	38	73	62
CTC	74	63	84	75

Table 4: Satisfaction of the axioms by participants in two sessions

Selten's index for the SMR and CTC theories, according to our experimental data, is not only lower than the corresponding index for $2-$rationality, but even than that of full rationality.

	first experiment	second experiment
$1-$rationality	$s_1^{FR} \approx 0,367$	$s_2^{FR} \approx 0,619$
$2-$rationality	$s_1^{2-R} > 0,786$	$s_2^{2-R} > 0,818$
SRM	$s_1^{SRM} << 0,366$	$s_2^{SRM} << 0,646$
CTC	$s_1^{CTC} << 0,463$	$s_2^{CTC} << 0,743$

7 Conclusions

We conclude by acknowledging some of the limitations of our present approach and by suggesting some further lines of work.

Our analysis is limited to finite sets of alternatives. Extending the notion of second best or of $r-$best alternatives to sets with a continuum of alternatives is non-trivial. We have already mentioned the important and recent literature on demand theory that also considers non-maximizing agents (Gabaix X. 2014, Aguiar V. and Serrano R., 2014). Obviously, it starts from an opposite end, where a continuous set of alternatives is the natural assumption. Even if these two ends do not meet, we feel that our very simple formulation of the basic choice problem is also a natural starting point. In particular, Aguiar and Serrano's definition of the "size" of bounded rationality is in a similar spirit than our calculations of the rationality level of a choice function, in Section 3.

We also limit attention to choice functions, and one may want to see similar results for the case of correspondences. For example, in the case of $2-$rationalizability, we may want to characterize the behavior of agents may who choose several alternatives belonging to their best and second best indifference classes within a set. That would be consistent with the assumption that agents' preferences may be weak orders. There are several ways to make this idea more precise, and we discuss the issue in Appendix D.

On the positive side, the assumption that we have information on the choices over all possible sets is not a limitative one. We can still discuss the rationalizability of choice functions defined on any family of subsets, by just assuming that our α function assigns to those sets on which we have no information a value equal to its cardinality.

Finally, if one was convinced that the present proposal is a reasonable alternative to full rationality, it would be worth investigating the consequences on the theory of games that would derive from assuming that agents behave accordingly.

References

[1] Afriat, S. N. (1973), "On a System of Inequalities in Demand Analysis: An Extension of the Classical Method", International Econimic Review, 14, 469-472.

[2] Aguiar V. and Serrano R. (2016), "Slutsky Matrix Norms and the Size of Bounded Rationality". Working paper.

[3] Apesteguia, J., Ballester, M.A. (2011), "The Computational Complexity of Rationalizing Behavior", Journal of Mathematical Economics 46, 356–363.

[4] Apesteguia, J., Ballester, M.A. (2015), "A Measure of Rationality and Welfare", Forthcoming to Journal of Political Economy.

[5] Baigent, N., and Gaertner, W. (1996), "Never Choose the Uniquely Largest: a Characterization". Economic Theory 8, 239-249.

[6] Bajraj G. and Ülkü L. (2015), "Choosing two Finalists and the Winner". Forthcoming to Social Choice Welfare.

[7] Caplin, A., and Dean, M. (2011), "Search, Choice, and Revealed Preference". Theoretical Economics 6, 19–48.

[8] Caplin, A., Dean, M., and Martin, D. (2011), "Search and Satisficing". American Economic Review 101 (7), 2899–2922.

[9] Cherepanov V., Feddersen T., and Sandroni A. (2013), "Rationalization", Theoretical Economics. Vol. 8, Issue 3, pages 775–800.

[10] De Clippel, G, and Eliaz, K. (2012), "Reason-Based Choice: A Bargaining Rationale for the Attraction and Compromise Effects". Theoretical Economics. Vol. 7, Issue 1, pages 125–162.

[11] Eliaz, K., Richter, M., and Rubinstein, A. (2011), "Choosing the Two Finalists". Economic Theory 46, 211–219.

[12] Eliaz, K., and Spiegler, R. (2011), "Consideration Sets and Competitive Marketing". Review of Economic Studies 78, 235–262.

[13] Frick M., (2016), "Monotone Threshold Representations", Theoretical Economics (2016)

[14] Gabaix X. (2014), "A Sparsity-Based Model of Bounded Rationality". Quarterly Journal of Economics, vol. 129(4), p.1661-1710.

[15] Green, J., and Hojman, D. (2009), "Choice, Rationality and Welfare Measurement", Harvard University, Working Paper.

[16] Halevy, Y, D. Peisetz and L. Zrill, (2015) “Parametric Recoverability of Preferences”. Working paper 2015.

[17] Halevy,Y, and L. Zrill, (2016). “Parametric Recovery Methods: A Comparative Experimental Study”. Working paper 2016.

[18] Houtman, M. and J. A. Maks (1985), "Determining all Maximal Data Subsets Consistent with Revealed Preference." Kwantitatieve Methoden, 19:89-104.

[19] Horan, S. (2011), "Sequential Search and Choice From Lists". Université du Québec à Montréal.

[20] Horan, S. (2015), "A Simple Model of Two-Stage Choice, ". Université du Québec à Montréal. Working Paper 2015.

[21] Houy N., and Tadenuma, K. (2009), "Lexicographic Compositions of Multiple Criteria for Decision Making", Journal of Economic Theory 144 1770–1782.

[22] Kalai G., Rubinstein A., Spiegler R. (2002), "Rationalizing Choice Functions by Multiple Rationales", Econometrica 70, 2481–2488.

[23] Lleras, J. S., Masatlioglu Y., Nakajima D., and Ozbay E. (2015), "When More is Less: Limited Consideration", Working Paper.

[24] Manzini, P, and Mariotti, M. (2007), "Sequentially Rationalizable Choice". American Economic Review 97 (5), 1824–1839.

[25] Manzini, P, and Mariotti, M. (2009), "Categorize Then Choose: Boundedly Rational Choice and Welfare- Technical and date appedix". Working Paper.

[26] Manzini, P, and Mariotti, M. (2010), Revealed Preferences and Boundedly Rational Choice Procedures: an Experiment". Working Paper.

[27] Manzini, P., and Mariotti, M. (2012), "Categorize Then Choose: Boundedly Rational Choice and Welfare". Journal of the European Economic Association. 10, 5, p. 1141-1165.

[28] Manzini, P, and Mariotti, M. (2014), "Stochastic Choice and Consideration Sets", Econometrica, 82 (3): 1153-1176, 2014.

[29] Masatlioglu, Y., Nakajima, D., and Ozbay, E.Y. (2012), "Revealed Attention". American Economic Review, 102(5), 2183-2205.

[30] McGarvey, D. C. (1953), "A Theorem on the Construction of Voting Paradoxes". Econometrica 21, 608-10.

[31] Moulin H. (1988), " Axioms of Cooperative Decision Making". Econometric Society Monographs.

[32] Papi, M. (2012), "Satisficing Choice Procedures", Journal of Economic Behavior & Organization 84 ,451– 462.

[33] Raymond C. (2013), "Revealed Search Theory", University of Michigan. Working paper.

[34] Rubinstein, A., and Salant, Y. (2006), "A Model of Choice From Lists". Theoretical Economics 1 (1), 3–17.

[35] Selten, R. (1991), "Properties of a measure of predictive success", Mathematical Social Sciences, 21: 153-167.

[36] Sen, A. (1993), "Maximization and the Act of Choice". Econometrica, 65, 745-779.

[37] Simon, Herbert A. (1955), "A Behavioral Model of Rational Choice". Quarterly Journal of Economics, 69 (1): 99–118.

[38] Stearns R. (1959), "The Voting Problem". American Mathematical Monthly, 66, 761-3.

[39] Tyson, C. J. (2008), "Cognitive Constraints, Contraction Consistency and the Satisficing Criterion", Journal of Economic Theory 138, 51-70.

[40] Varian, H. R. (1990), "Goodness-of-Fit in Optimizing Models", Journal of Econometrics, 46, 125-140.

8 Appendix A: More on Lemma 2 and Theorem 3

Lemma 2 A choice function F is $\alpha-$rationalizable if and only if $Y_{\overline{k}} = \mathcal{C}_F^{\alpha}(Y_{\overline{k}})$.

Proof $\Rightarrow$) Let F be an $\alpha-$rationalizable choice function. Assume that R is a preference relation over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in M^{\alpha}(A, R)$$

Let $\mathcal{P} = \{L_1, .., L_{\overline{k}}\}$ be the partition on X.

Assume otherwise, that is $\mathcal{D}^{\alpha}(Y_{\overline{k}}) = \{B \in \mathcal{D} : \#(B \cap Y_{\overline{k}}) > \alpha(B)\} \neq \varnothing$ and $\mathcal{C}_F^{\alpha}(Y_{\overline{k}}) = \varnothing$. Let $y \in Y_{\overline{k}}$ be such that

$$xRy$$

for every $x \in Y_{\overline{k}}$. Because $\mathcal{C}_F^{\alpha}(Y_{\overline{k}}) = \varnothing$, there exists $B \in \mathcal{D}^{\alpha}(Y_{\overline{k}})$ such that $F(B) = y$. Since $\#(B \cap Y_{\overline{k}}) > \alpha(B)$, we have that $y \notin M^{\alpha}(B, R)$. Contradicting that $F(B) \in M^{\alpha}(B, R)$.

$\Leftarrow$) Let $\mathcal{P} = \{L_1, .., L_{\overline{k}}\}$ be a partition on X such that $L_k = Y_k - \mathcal{C}_F^{\alpha}(Y_{k-1})$ with $k = 1, .., \overline{k}$ and $\mathcal{D}^{\alpha}(Y_{\overline{k}}) = \varnothing$. We will show that F is $\alpha-$rationalizable by R_F. That is, for every $A \in \mathcal{D}$;

$$F(A) \in M^{\alpha(A)}(A, R_F).$$

Assume otherwise, that there existed $A \in \mathcal{D}$ such that

$$F(A) = z \notin M^{\alpha(A)}(A, R_F)$$

Notice that

$$xR_F z. \tag{3}$$

for every $x \in M^{\alpha(A)}(A, R_F)$. Since $\mathcal{D}^{\alpha}(Y_{\overline{k}}) = \varnothing$; we have that $\mathcal{C}_F^{\alpha}(Y_{\overline{k}}) = Y_{\overline{k}}$. This implies that there exists i' such that $M^{\alpha}(A, R_F) \cup \{z\} \subseteq Y_{i'}$and $M^{\alpha}(A, R_F) \cup \{z\} \nsubseteq Y_{i'+1}$. Thus, there exists $\overline{x} \in \mathcal{C}_F^{\alpha}(Y_{i'})$ and $z \notin \mathcal{C}_F^{\alpha}(Y_{i'})$. This implies that there exists j' with $i' < j'$ and $z \in \mathcal{C}_F^{\alpha}(Y_{j'})$. Consequently $zR_F\overline{x}$, contradicting (3).

This concludes the proof. ■

Theorem 3 A choice function F on X is $\alpha-$rationalizable if and only $\mathcal{C}_F^{\alpha}(Y) \neq \varnothing$ for all $Y \subseteq X$.

Proof: $\Rightarrow$) Let F be an $\alpha-$rationalizable choice function. Assume that R is a preference relation over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$F(A) \in M^{\alpha(A)}(A, R).$$

Let Y be any subset of alternatives and $\overline{y} \in Y$ such that

$$yR\overline{y} \text{ for every } y \in Y.$$

Notice that for every B such that $\#(B \cap Y) > \alpha(B)$, we have that

$$\overline{y} \notin M^{\alpha(B)}(B, R).$$

Thus, $F(B) \neq \overline{y}$. This implies that $\overline{y} \in \mathcal{C}_F^{\alpha}(Y)$.

$\Leftarrow$) Assume that $\mathcal{C}_F^{\alpha}(Y) \neq \varnothing$ for every $Y \subseteq X$. Let $\mathcal{P} = \{L_1, .., L_{\overline{k}}\}$ be a partition on X such that $L_k = \mathcal{C}_F^{\alpha}(Y_k)$ with $k = 1, .., \overline{k}$. This implies that $L_{\overline{k}} = \mathcal{C}_F^{\alpha}(Y_{\overline{k}}) = Y_{\overline{k}}$. Thus, $\mathcal{D}^{\alpha}(Y_{\overline{k}}) = \varnothing$. Lemma 1, concludes the proof. ■

The following example illustrates the family of subsets of alternatives that we are constructing.

Example 7: $X = \{a_1, ..., a_5\}$ the set of alternatives. The choice function $F : \mathcal{D} \longrightarrow X$ is defined as follows; for each subset of size 3 or larger:

- If $\#B = 3$, then
  - i) If $a_3 \in B$, then $F(B) = a_3$.
  - ii) If $a_4 \in B$ and $a_3 \notin B$, then $F(B) = a_4$.
  - iii) $F(\{a_1, a_2, a_5\}) = a_2$.
- If $\#B = 4$, then

$F(\{a_1, a_2, a_3, a_4\}) = a_4$	$F(\{a_1, a_2, a_3, a_5\}) = a_3$	$F(\{a_1, a_2, a_4, a_5\}) = a_4$
$F(\{a_1, a_3, a_4, a_5\}) = a_1$	$F(\{a_2, a_3, a_4, a_5\}) = a_3$	

- If $\#B = 5$, then $F(\{a_1, a_2, a_3, a_4, a_5\}) = a_3$.

Let $\alpha : \{1, .., 5\} \to \{1, .., 5\}$ be defined by

$\alpha(1) = 1$	$\alpha(2) = 2$	$\alpha(3) = 2$	$\alpha(4) = 3$	$\alpha(5) = 1$

That is, it has the same value for two subsets of equal cardinality.

Now clearly,

$$\mathcal{D}^{\alpha}(X) = \{a_1, a_2, a_3, a_4\}$$

This implies that $\mathcal{C}_F^{\alpha}(X) = \{a_5\}$. Define $Y_1 = X - \{a_5\} = \{a_1, a_2, a_3, a_4\}$.

Notice that

$$\mathcal{D}^{\alpha}(Y_1) = \{a_3, a_4\}.$$

Because $\mathcal{D}^{\alpha}(Y_1) = \{a_1, a_2\}$, define $Y_2 = Y_1 - \{a_1, a_2\} = \{a_3, a_4\}$.

Consider

$$\mathcal{D}^{\alpha}(Y_2) = \{a_3\}.$$

Since $\mathcal{C}_F^{\alpha}(Y_2) = \{a_4\}$, define $Y_3 = Y_2 - \{a_4\} = \{a_3\}$.

Therefore,

$$\mathcal{D}^{\alpha}(Y_3) = \varnothing$$

Lemma 2 implies that the function F is $\alpha-$rationalizable. □

9 Appendix B: Instructions for the Experiment

You should not communicate with other participants during the experiment. Instructions are the same for all participants. You are participating in an experiment to study intertemporal preferences, financed by research projects ECO 2014-53052 and SGR2014-515.

You will be paid 1,20 euros (120cts) today for having participated in the experiment. The money is an envelope, which also contains a receipt. Take the money, sign your receipt with your name and id number, and leave it on the table.

You will receive additional payments, as explained below.

The task.

You have in front of you a stack of 16 pages, each one displaying several payment plans. Each plan implies payments to be made three, six and nine weeks from today. The payments always add up to 4,8 euros, and they are expressed in cents. In each page you will find several possible plans, and you must choose one of them. After you've made the choice, turn to the next page and proceed to select one of the plans that are proposed in it. Continue until the task is completed. Do not return to previous pages.

Payments.

Today, after the experiment is over, one of the pages you have been presented with will be chosen at random. You will be paid according to the payment plan that you chose in that page, in three, six, and nine weeks, after the class periods of (March10, April 20 and May 11). Payments will be made as today, and each time you must sign a receipt.

An example

In order to become familiar with the way in which the plans you have to choose from will be presented to you, here is an example for a hypothetical case involving payments of 4 euros (400 cents) over three periods.

Plan A		**Plan B**		**Plan C**	
How much:	When:	How much:	When:	How much:	When:
60 cts	on march 30	100 cts	on march 30	50 cts	on march 30
200 cts	on april 20	100 cts	on april 20	250 cts	on april 20
140 cts	on may 11	200 cts	on may 11	100 cts	on may 11

Your answer must indicate the plan according to which you would prefer to be paid, by ticking one of the boxes.

i) Plan **A** ☐ ii) Plan **B** ☐ iii) Plan C ☐.

Thank you for participating.

10 Appendix C: Calculations for Selten's index

In order to compare different models we calculate Selten's index for the first and second sessions of the experiments.

Full Rationality.

In order to compute a, the "relative area" for a model we have to compute the proportion of choice functions compatible with the set of axioms characterizing that model. The number of all possible outcomes $5 \cdot 4^5 \cdot 3^{10}$. For each universal set of alternatives, the number of choice functions satisfying $WARP$ is $5 \cdot 4 \cdot 3$. Then the areas are:

$$a^{FR} = \frac{1}{4^4 \cdot 3^9}.$$

In order to compute r, we have to compute the number of actually observed outcomes compatible with the model divided by the number of possible outcomes. The value of r

for the first session,

$$r_1^{FR} = \frac{43}{117}$$

and for the second session,

$$r_2^{FR} = \frac{70}{113}.$$

Then, Selten's measure for the first session will be,

$$s_1^{FR} = \frac{43}{117} - \frac{1}{4^4 \cdot 3^9} \approx 0,367$$

and for the second

$$s_2^{FR} = \frac{70}{113} - \frac{1}{4^4 \cdot 3^9} \approx 0,619$$

2−rationality.

The number of all possible choice function are $5 \cdot 4^5 \cdot 3^{10}$. To avoid tedious computations, we set a lower bounded of s, using the following definition,

$$T = \# \{f \in \mathcal{F} : f \text{ not violating the 2-rationality axioms}\}$$

observe that,

$$T << \sum_{t \in X} \# \{f \in \mathcal{F} : f(\mathcal{D}) = X - \{t\}\}.$$

Since $\# \{f \in \mathcal{F} : f(\mathcal{D}) = X - \{t\}\} = 2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2$, then

$$T << 2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2 \cdot 5$$

that is,

$$a^{2-R} << \frac{2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2 \cdot 5}{3^{10} \cdot 4^5 \cdot 5} = \frac{1}{9}$$

Value of r for the first session,

$$r_1^{2-R} = \frac{105}{117}$$

and for the second session,

$$r_2^{2-R} = \frac{105}{113}.$$

Then, Selten's measure will be,

$$s_1^{2-R} >> \frac{106}{117} - \frac{1}{9} \approx 0,786$$

and

$$s_2^{2-R} >> \frac{105}{113} - \frac{1}{9} \approx 0,818$$

Observe that

$$s_2^{2-R} > s_1^{2-R} > s_2^{FR} >> s_1^{FR}.$$

<u>**Calculating data on SRM and CTC**</u>

Since full rationality implies SRM and CTC, and the number of all possible choice functions satisfying $WARP$ are smaller that satisfying $WARP^*$ or $WARP^*$ and Expansion, then the areas are

$$a^{SRM} > a^{FR} \quad \text{and} \quad a^{CTC} > a^{FR}$$

Because in our experiment we did not obtain the choice function data of the on sets of cardinality 2, then we calculate an upper bound value of r for the SRM and CTC models. for the first session,

$$r_1^{SRM} \leq \frac{44}{117} \quad \text{and} \quad r_1^{CTC} \leq \frac{74}{117}$$

and for the second session, it is

$$r_2^{SRM} \leq \frac{73}{113} \quad \text{and} \quad r_2^{CTC} \leq \frac{84}{113}$$

Then, Selten's measure for the first session will be,

$$s_1^{SRM} < \frac{44}{117} - \frac{1}{4^4 \cdot 3^9} \approx 0,366 \quad \text{and} \quad s_1^{CTC} < \frac{74}{117} - \frac{1}{4^4 \cdot 3^9} \approx 0,632$$

and for the second session, it is

$$s_2^{SRM} < \frac{73}{113} - \frac{1}{4^4 \cdot 3^9} \approx 0,646 \quad \text{and} \quad s_2^{CTC} < \frac{84}{113} - \frac{1}{4^4 \cdot 3^9} \approx 0,743$$

Then for the first session,

$$s_1^{SRM} < s_1^{FR} < s_1^{CTC} < s_1^{2-R}$$

and for the second session

$$s_2^{SRM} < s_2^{FR} < s_2^{CTC} < s_2^{2-R}.$$

11 Appendix D: Choice Correspondences

In different contexts, including much of demand theory, it is sometimes natural to describe choice behavior in terms of correspondences, rather than functions. Our suggested departures from full rationality may be extended to formulations where the basic data involve the choice of sets, rather than that of single alternatives. In this section we provide an elementary extension of our model to choice correspondences, we comment on related work that is also based on that formulation, and we use our basic characterization result to further discuss some of our experimental data.

11.1 Extending our work to consider choice correspondences

Let X be a finite set of alternatives, with $\#X \geq 3$. Let $\mathcal{D} = 2^X - \{\emptyset\}$ be the set of all non-empty subset of alternatives. A choice correspondence on $\mathcal{D}$ is a correspondence $H : \mathcal{D} \longrightarrow \mathcal{D}$ such that $H(A) \subseteq A$, for every $A \in \mathcal{D}$.

Let R be a preference relation over the set of all alternatives X. Specifically, R is a complete, reflexive, antisymmetric, and transitive binary relation on X.

Given a preference relation R on X and a subset $A \in \mathcal{D}$, let $h(A, R)$ be maximal alternative of a set A with respect to preference R. Formally

$$h(A, R) = x \Leftrightarrow xRy \text{ for every } y \in A.$$

Because R is complete and antisymmetric, $\#h(A, R) = 1$ for every $A \in \mathcal{D}$.

Denote $h^1(A, R) = h(A, R)$, and define for every t;

$$h^t(A, R) = h(A - \bigcup_{i=1}^{t-1} h^i(A, R))$$

and

$$M^r(A, R) = \bigcup_{i=1}^{r} h^i(A, R)$$

We will relax the assumption that an agent always chooses exactly the best alternatives according to a preference order, by assuming that it will be content with a subset of those alternatives.

Given a natural number $r \in \mathbf{N}$, we say that a choice correspondence is $r-$rationalizable if there exists a preference relation on the alternatives such that the subset chosen is a subset of its $r-$best ranked elements according to that order. Formally,

Definition D1 A choice correspondence H is *r-rationalizable* if there exists a preference relation R over the set of all alternatives X such that for every $A \in \mathcal{D}$;

$$H(A) \subseteq \bigcup_{i=1}^{r} h^i(A, R) = M^r(A, R).$$

We now introduce definitions leading to our characterization result, and inspired in the intuitions we provided in the Introduction about our "bottom up" approach.

Given a natural number $r \in \mathbf{N}$, and $Y \subseteq X$, define the subset of alternatives that impose some restrictions on possible choice of an agent,

$$\mathcal{D}^r(Y) = \{B \in \mathcal{D} : \#(B \cap Y) > r\}.$$

Let

$$\mathcal{C}_H^r(Y) = \{x \in Y : \text{ for all } B \in \mathcal{D}^r(Y),\, x \notin H(B)\}$$

be the set of alternatives that will never be chosen from any set in $\mathcal{D}^r(Y)$.

Theorem D1 A choice function F on X is $r-$rationalizable if and only $\mathcal{C}_H^r(Y) \neq \varnothing$ for all $Y \subseteq X$.

Remark D1 When $\mathcal{D}^r(Y) = \varnothing$, we have $\mathcal{C}_H^r(Y) = Y$. Hence, the restriction that $\mathcal{C}_H^r(Y) \neq \varnothing$ only has bite when the size of Y is at least $r + 1$.

11.2 Some related results

Recent work by Eliaz, Richter and Rubinstein (2011), and Frick (2016) considers agents whose behavior is expressed in terms of choice correspondences, who do not fully maximize (like ours) , and whose choices may be contingent to the menu from which they must choose (as in our extension to α rationality in Section 4). Although other features of the models and results are clearly different, we wanted to point at the clear analogy of purpose.

The paper by Eliaz, Richter and Rubinstein on "Choosing the two Finalists (2011) is the closest to our work, although much less general. These authors characterize "top two" correspondences that select the best two outcomes from an order, given each subset of alternatives. Notice that this is a special case of our $2-$ratinalizable choice functions, because we did also allow the choice of subsets of cardinality less than two. We can see that our set of $2-$rationalizable choice functions would consist of all selections from some of their "Top Two" choice correspondence which are obviously $2-$rationalizable in our setting. In a natural manner, $r-$rationalizable choice correspondences could also be defined and characterized, and selections from them would coincide with our $r-$rationalizable choice functions.

Hence, our notion of $r-$rationalizability is a necessary condition for all those processes that may be described as choosing r finalists in a first stage, and then using some additional criterion to narrow down the choice from any set to a singleton. Of course, additional restrictions could be imposed on choice functions when being specific about the criterion for final selection. Just to illustrate this point, here are some suggestions on how to complement the choice of two finalists.

Assume, for example, that the final choice between two finalists was to be made by a committee that uses majority rule with tie breaking. Given any preference profile on the set of alternatives X, the committee's majority rule will be a tournament, and for each pair of finalists the tournament will be used as the selection device. Then, clearly the choice function that we shall obtain will be $2-$rationalizable, but not any $2-$rationalizable choice function could be derived from that process. This is because the committee will choose the same alternative out of each pair x, y regardless of the set B from which these two were picked as finalists. Thus, a choice function selected by majority from a two-top correspondence will satisfy the following additional condition.

Condition D1 Given $\{a, b\} \subsetneq B$ and $F(\{a, b\}) = a \neq F(B) = b$, there must exist $c \in B - \{a, b\}$ such that $a \neq F(C)$, for every C with $\{a, b, c\} \subseteq C$.

In fact, rationalizability plus condition 1 fully characterize the set of choice functions that can be obtained from the two-stage process we describe, provided the number of committee members is large enough. We express this fact as follows.

Remark D2 A choice function F can be generated by choosing the best two candidates from an order and then selecting the one that has a majority in a committee of size larger than the total number of alternatives if and only if it is $2-$rationalizable and satisfies condition 1.

We leave the detailed proof of Remark 6 to the reader, but note that it relies on the fact that any given tournament on a set of alternatives X can be generated as the majority relation for some committee whose size is larger than that of X (McGarvey (1953), Stearns (1959), see also Moulin (1988)).

A further narrowing down of the preceding class would obtain if we required the selection to be made by choosing the maximal element of a transitive relation, not necessarily the same as the one used to choose the finalists. This particular case, where further restrictions should be imposed on the resulting choice function, has been studied by Bajraj and Ülkü (2014).

Our main point is made through this analysis of the case for two finalists. These screening processes, coupled with a criterion to choose from pairs, generate $2-$rationalizable choice functions, which may be more or less restricted in scope depending on the second stage selection criterion.

Similarly, the choice of $r-$finalists, along with a choice function on subsets of size r, would give rise to $r-$rationalizable choice functions, whose additional properties will be determined by the choice function in question.

11.3 A further look at our experimental results

Here we discuss some consequences of having performed twice a similar experiment with the same set of subjects[12]. Let us start by remarking that although the payment schedules had the same structure, the payment dates were not the same: for example, the last payment

[12] See the experiment description in Section 6.

in the second experiment was made after the course had already come to an end, during the exams period. Hence, alternatives, interpreted as payment schedules in specific dates, were not identical.

The fact is that the behavior of a large majority of the agents was $2-$rationalizable in both cases, and that prompted us to ask whether their behavior would be compatible with the existence of a common preference on payment patterns (rather than preferences for fully specified alternatives, which would make the two cases non-comparable).

Notice then that one could have the same agent to be $1-$rational in each of the two experiments, and yet not even $2-$rationalizable. And similar reasons could prevent the $2-$rationalizability of joint actions of subjects that are either $1-$or $2-$rationalizable under each separate experiment. In fact, we have calculated this joint compatibility and it provides an interesting, even if not overwhelming coincidence, as shown in the following table (94 participants in the two sections):

	First session	Second session	Both sessions
1-rational	37	66	12
2-rational	87	91	45
3-rational	93	94	90

Table 5: Satisfaction of the axioms of participants in two sections

Had the results contained more cases of joint rationalizability we might have argued for some deep stability of preferences that our present results do not strongly support. But several caveats apply. We have evidence that agents changed preferences for patterns of payment between both experiments, but this does not necessarily mean that they changed preferences, since the alternatives also include the dates of payment. Moreover, the subjects did talk to each other in between experiments, and probably became aware that what they had learned in the consumer theory part of the course could have a bearing on their answer.

In order to compare the models we calculate the Selten's index for data of both sessions of the experiments.

Full Rationality.

In order to compute a the "relative area" for a model we have to compute the proportion of choice functions compatible with the set of axioms characterizing that model. The number of all possible outcomes $(5 \cdot 4^5 \cdot 3^{10})^2$. For each universal set of alternatives, the number of choice functions satisfying $WARP$ are $5 \cdot 4 \cdot 3$. Then the area is:

$$a^{FR} = \frac{1}{(4^4 \cdot 3^9)^2}.$$

In order to compute where r, we have to compute the number of actually observed outcomes compatible with the model divided by the number of possible outcomes. The values of r for the first session,

$$r_b^{FR} = \frac{12}{94}$$

and Selten's measure will be,

$$s_b^{FR} = \frac{12}{94} - \frac{1}{\left(4^4 \cdot 3^9\right)^2} \approx 0,127.$$

<u>2−**rationality.**</u>

The number of all possible choice function are $\left(5 \cdot 4^5 \cdot 3^{10}\right)^2$. To avoid tedious computations, we set a lower bounded of s, using the following definition,

$$T = \# \left\{ f \in \mathcal{F} : f \text{ not violating the 2-rationality axioms} \right\}$$

observe that,

$$T << \sum_{t \in X} \# \left\{ f \in \mathcal{F} : f(\mathcal{D}) = X - \{t\} \right\}.$$

Since $\# \left\{ f \in \mathcal{F} : f(\mathcal{D}) = X - \{t\} \right\} = \left(2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2\right)^2$, then

$$T << \left(2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2\right)^2 \cdot 5$$

that is,

$$a^{2-R} << \frac{\left(2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2\right)^2 \cdot 5}{\left(3^{10} \cdot 4^5 \cdot 5\right)^2} = \frac{1}{405}$$

Value of r for both session,

$$r_b^{2-R} = \frac{45}{94}$$

and Selten's measure will be,

$$s_b^{2-R} >> \frac{45}{94} - \frac{1}{405} \approx 0,476.$$

<u>3−**rationality.**</u>

The number of all possible choice correspondence are $\left(5 \cdot 4^5 \cdot 3^{10}\right)^2$. To avoid tedious computations, we set a lower bounded of s, using the following definition,

$$T = \# \left\{ f \in \mathcal{F} : f \text{ not violating the 3-rationality axioms} \right\}$$

observe that,

$$T << \sum_{t \in X} \# \left\{ f \in \mathcal{F} : f(\mathcal{D}) = X - \{t\} \right\}.$$

Since $\#\{f \in \mathcal{F} : f(\mathcal{D}) = X - \{t\}\} = (2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2)^2$, then

$$T << \left(2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2\right)^2 \cdot 5$$

that is,

$$a^{3-R} << \frac{(2^6 \cdot 3^4 \cdot 3^4 \cdot 4^2)^2 \cdot 5}{(3^{10} \cdot 4^5 \cdot 5)^2} = \frac{1}{405}$$

Value of r for both sessions,

$$r_b^{3-R} = \frac{90}{94}$$

and Selten's measure will be,

$$s_b^{3-R} >> \frac{90}{94} - \frac{1}{405} \approx 0,955.$$