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ON THE REGULARITY OF THE DISTANCE NEAR THE BOUNDARY OF AN OBSTACLE

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Abstract. We study the regularity of the Euclidean distance function from a given point-wise target of a n-dimensional vector space in the presence of a compact obstacle bounded by a smooth hypersurface. It is known that such a function is semiconcave with fractional modulus one half. We provide a geometrical explanation of the exponent one half. Furthermore, under a natural (weak) assumption on the position of the point-wise target relatively to the obstacle, we show that there exists a point in the boundary of the obstacle so that no better regularity result holds near such a point. As a consequence of this result, we show that the Euclidean metric cannot be extended to a tubular neighborhood of the obstacle, as a Riemannian metric, keeping the property that the associated distances coincide outside the obstacle.

1. Introduction and statement of the results

Let $\mathscr{O} \subset \mathbb{R}^n$ be a closed obstacle bounded by a smooth hypersurface. We study the distance function, d, from a given point k_0 which does not belong to the interior of $\mathscr{O}$. Observe that d can be seen as a point-to-point distance for a particular class of manifolds with boundary. We are interested in the (global) optimal regularity of d near the boundary of $\mathscr{O}$. For this purpose, let us introduce the class of obstacles we deal with, $\mathcal{O}$.

We say that a set $\mathscr{O} \subset \mathbb{R}^n$ belongs to the class $\mathcal{O}$ if there exists open, bounded, and connected set $\Omega \subset \mathbb{R}^n$, with boundary of class C^2, such that $\mathscr{O} = \overline{\Omega}$ is the closure of Ω in $\mathbb{R}^n$. For aesthetic reasons, we also admit the case of $\Omega = \emptyset$ (i.e. $\emptyset \in \mathcal{O}$). Let X be the connected component of $\mathbb{R}^n \setminus \mathscr{O}$ that contains k_0.

Given a curve $\gamma \in C^1([0,1]; X)$, we define its length as

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)|\, dt.$$

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(Here $|\cdot|$ stands for the standard Euclidean norm.) For every $x, y \in X$, the distance between x and y is given by

$$d(x,y) = \inf_{\gamma} L(\gamma), \tag{1.1}$$

where the infimum is taken over all the curves $\gamma \in C^1([0,1]; X)$ so that $\gamma(0) = x$ and $\gamma(1) = y$. Furthermore, for $x \in X$, we denote by

$$d(x) = d(x, k_0)$$

the distance function of x from k_0. We observe that

$$d(x,y) \geq |x-y| \text{ and } d(x) \geq |x-k_0|, \qquad \forall x, y \in X.$$

Remark 1.1. *We notice that the distance function d can be seen as the value function for a constrained minimum time problem with a pointwise target k_0. In particular, d is the viscosity solution of a suitable boundary value problem for the eikonal equation (see, e.g.,* [8, Theorem X.1]*)*

$$|Dd(x)|^2 = 1 \qquad in \qquad X \setminus (\partial\mathscr{O} \cup \{k_0\}). \tag{1.2}$$

We recall that, in order to study the regularity of d, the appropriate class of functions is the one of semiconcave functions with fractional modulus.

Definition 1.1. *Given a set $U \subset \mathbb{R}^n$, we say that $u : U \longrightarrow \mathbb{R}$ is a fractionally semiconcave function on U of exponent $\alpha \in]0,1]$, if*

- *u is locally Lipschitz continuous*[1] *on U,*

and

- *there exists $C \in \mathbb{R}$ such that*

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leq C\lambda(1-\lambda)|x-y|^{1+\alpha}, \tag{1.3}$$

for any $x, y \in U$ such that the line segment $[x,y]$ is contained in U, and for every $\lambda \in [0,1]$.

In the case $\alpha = 1$, u is called semiconcave with linear modulus.
We denote by $SC^\alpha(U)$ the set of all the fractionally semiconcave functions of exponent α in U.

Such a property can obviously be localized, in which case we refer to the space $SC^\alpha_{loc}(U)$. We recall that the following (interior) regularity property which is a consequence of the fact that d is a viscosity solution of equation (1.2) (see e.g. [9] and [1] for more general results).

Theorem 1.1. *Let $\mathscr{O} \subset \mathbb{R}^n$ be a closed set and let d be a (continuous) viscosity solution of Equation* (1.2)*. Then, $d \in SC^1_{loc}(X \setminus (\partial\mathscr{O} \cup \{k_0\}))$.*

[1] We observe that, due to the fact that no assumption is made on the set U, the Lipschitz continuity is not a consequence of (1.3).

We point out that, in the above result, the only assumption on $\mathscr{O}$ is that it is a closed set.

Throughout all this paper we assume

(H) $\mathscr{O} \in \mathcal{O}$ and X is unbounded[2].

Remark 1.2. *We observe that we defined the distance function only in the set X: for $x \in \mathbb{R}^n \setminus X$ there is no curve γ, joining x with k_0, such that $\gamma(t) \notin int(\mathscr{O})$, for every t, i.e. in $\mathbb{R}^n \setminus X$ the distance function would be identically infinite. Furthermore, our requirement that k_0 belongs to the unbounded connected component of $\mathbb{R}^n \setminus \mathscr{O}$ forces the fact that the presence of the obstacle is felt by the distance function. On the other hand, one can easily construct examples of obstacles in the class $\mathcal{O}$ such that, taking k_0 in a bounded connected component of $\mathbb{R}^n \setminus \Omega$, we have that $d(x) = |x - k_0|$ on X. For instance, it suffices to consider $k_0 = 0 \in \mathbb{R}^n$ and $\mathscr{O} = \overline{B_2(0) \setminus B_1(0)}$.*

The first result of the present paper is the following

Theorem 1.2. *Under Assumption (H), let $\mathscr{O}$ be nonempty. Then, $d \notin SC^1_{loc}(X \setminus \{k_0\})$.*

Remark 1.3. (1) *As clarified by the example in Remark 1.2, in Theorem 1.2, the assumption "X unbounded" cannot be omitted.*
(2) *Theorems 1.1 and 1.2 show that the lack of regularity happens in $\partial\mathscr{O}$. The failure of the linear semiconcavity is due to the behavior of the length minimizing curves, which will be discussed in the sequel (see Theorem 1.4 below).*

We observe that, in the absence of the obstacle, $d \in C^\infty(\mathbb{R}^n \setminus \{k_0\})$. Hence, Theorem 1.2 yields the following characterization:

Corollary 1.1. *Assume (H). Then, $d \in SC^1_{loc}(X \setminus \{k_0\})$ if and only if $\mathscr{O} = \emptyset$.*

We recall the following regularity result:

Theorem 1.3. *Let $\mathscr{O} \in \mathcal{O}$. Then, we have that $d \in SC^{\frac{1}{2}}_{loc}(X \setminus \{k_0\})$.*

Remark 1.4. *Theorem 1.3 is a special case of* [4, Theorem 1.2], *see also* [11, Formula (A3)]. *We point out that, in the quoted papers, the more general case of a Riemannian distance function is considered. We*

[2]We recall that $\mathscr{O} \in \mathcal{O}$ means that it is the closure of an open connected set with boundary of class C^2, Ω, the case $\Omega = \emptyset$ is admitted. Furthermore, X is the connected component of $\mathbb{R}^n \setminus \Omega$ containing k_0.

also remark that in [4]*, it is assumed that* $k_0 \notin \mathscr{O}$ *and that* $\mathbb{R}^n \setminus \mathscr{O}$ *is a connected set. In fact, the proof of Theorem 1.3 is a verbatim repetition of the one of* [4, Theorem 1.2].

One may wonder if Theorem 1.3 is optimal or if a "better" regularity result may hold, for instance if $d \in SC^{\alpha}_{loc}(X \setminus \{k_0\})$, for some $\alpha > 1/2$. Observe that for $\alpha, \beta \in [0, 1]$,

$$\alpha > \beta \implies SC^{\alpha}_{loc}(U) \subset SC^{\beta}_{loc}(U).$$

In [4, Proposition 1.1], it is shown that, for $\mathscr{O} = \overline{B}_1(0)$ and $k_0 \in \mathbb{R}^n \setminus \mathscr{O}$, $d \notin SC^{\alpha}_{loc}(X \setminus \{k_0\})$ for every $\alpha \in]1/2, 1]$. We recall that the proof of this fact was mainly based on two ingredients: the symmetry of the sphere (which permits a reduction to the dimension $n = 2$) and the explicit knowledge of the geodesics in the circle. The idea behind the proof of [4, Proposition1.1] is the construction of a level curve intersecting the obstacle at a sort of "cuspidal" point, where a blow-up of the curvature of the level curve occur. (Such a blow-up is incompatible with $d \in SC^{\alpha}_{loc}(X \setminus \{k_0\})$, for $\alpha \in]1/2, 1]$.)

One of the motivating question for the present paper is to give a geometrical meaning to the exponent 1/2 and, as a byproduct, we obtain that, if X is unbounded, Theorem 1.3 provides the optimal regularity for $\mathscr{O} \in \mathcal{O}$ nonempty.

We recall that a curve $\gamma : [0, 1] \longrightarrow X$, with $\gamma(0) = x$ and $\gamma(1) = k_0$ is a minimizing curve if $d(x) = L(\gamma)$. For $x \in X \setminus \{k_0\}$, we denote by $\Gamma^*[x]$ the set of all the minimizing curves γ, parametrized by the arc length, with $\gamma(0) = x$, i.e. $d(\gamma(t)) = d(x) - t$, for every $t \in [0, d(x)]$.

Furthermore, we say that $\partial\mathscr{O}$ satisfies Condition (C) near x_0 if there are $r > 0$ and $c_0 > 0$ so that, for each $i = 1, 2, \ldots, n - 1$,

$$\lambda_i(x) \geq c_0, \qquad \forall x \in B_r(x_0) \cap \partial\mathscr{O}. \tag{1.4}$$

(Here λ_i are the principal curvatures of $\partial\mathscr{O}$.)

We observe that Condition (C) is a "nondegenerate" strict convexity assumption near a given point x_0.

Now, we are ready to relate the exponent 1/2 with the behavior of the minimizing curves.

From a technical point of view, the next result is the core of our proof of the lack of linear semiconcavity.

Theorem 1.4. *Let* $\mathscr{O} \in \mathcal{O}$ *be nonempty, let* $x_1, x_2 \in X \setminus \{k_0\}$*, with* $x_1 \neq x_2$*, and let* $\gamma_i \in \Gamma^*[x_i]$*,* $i = 1, 2$*. Assume that* $\gamma_1(t_*) = \gamma_2(t_*)$*, for a suitable* $t_* \notin \{0, d(x_1), d(x_2)\}$*. Then* $d \notin SC^1_{loc}(X \setminus \{k_0\})$*. Suppose, in addition, that* $\partial\mathscr{O}$ *satisfies Condition (C) near* $\gamma_1(t_*)$ $(= \gamma_2(t_*))$*, then* $d \notin SC^{\alpha}_{loc}(X \setminus \{k_0\})$*, for every* $\alpha \in]1/2, 1]$.

In other words, if two minimizing curves intersect each other at a point of $\partial\mathscr{O}$ (and such a point is not an endpoint for the curves), then the function d fails to be semiconcave with linear modulus. Under a "nondegenerate" convexity assumption on $\partial\mathscr{O}$, 1/2 is the larger exponent which does not exclude such an intersection of the minimizers.

Remark 1.5. *We point out that if $\gamma(t_*)$ is as in Theorem 1.4, due to* (2.12) *below, the function d is differentiable at $\gamma(t_*)$, i.e. the lack of semiconcavity of the distance function and the presence of singularities of d are two unrelated phenomena.*

The following is the main result of this paper.

Theorem 1.5. *Under Assumption* (H)*, let $\mathscr{O}$ be nonempty. Then, for every $\alpha \in]1/2, 1]$, $d \notin SC^{\alpha}_{loc}(X \setminus \{k_0\})$.*

Remark 1.6. *(i) We point out that, in Theorem 1.5, there is no convexity assumption on the obstacle $\mathscr{O}$.*
(ii) We notice that, as a consequence of Theorem 1.5, there is a sort of (weak) regularizing effect:

$$d \in SC^{\alpha}_{loc}(X \setminus \{k_0\}) \text{ for some } \alpha \in]1/2, 1] \implies d \in SC^{1}_{loc}(X \setminus \{k_0\}).$$

(iii) In order to prove Theorems 1.2 and 1.5, in light of Theorem 1.4, it suffices to find two length minimizing curves merging at a boundary point of the obstacle. In the Euclidean case, one can find these length minimizing curves by constructing a convex cone tangent to a strictly convex part of the boundary of the obstacle and using the fact that, outside the obstacle, the length minimizers are straight line segments (see Section 4 below). In the (general) Riemannian setting, this strategy does not work and a different approach seems to be necessary.

We observe that Theorem 1.5 provides a negative answer to the following ("global") extension problem. For the sake of simplicity let us suppose that

- $\mathbb{R}^n \setminus \Omega$ is a connected set ($X = \mathbb{R}^n \setminus \Omega$) and let $\mathscr{O} = \bar{\Omega} \in \mathcal{O}$.

In other words, X is an unbounded manifold (with boundary) of a special form. Consider X equipped with the distance function associated to the Euclidean metric. One can construct examples showing that, if we extend the metric to a tubular neighborhood of the boundary of the obstacle (keeping it Euclidean), then the corresponding distance functions are different in X. It is less clear, if the extension can be done (keeping the equality of the relative distances where both are defined) admitting as an extension a Riemannian (non-constant) metric. More precisely, consider the following question

(Q) let $g(\xi, \xi) = |\xi|^2$ be the Euclidean metric, let $d(\cdot, \cdot)$ the corresponding distance in X given by (1.1), and let $\Omega_1 \subset \Omega$ be an open set, with $\partial\Omega_1 \cap \partial\Omega = \emptyset$. Can we find a Riemannian metric $\tilde{g}$ of class C^2, so that $g = \tilde{g}$ in TX and $d = \tilde{d}$ in X? (Here $\tilde{d}$ is the distance in $X \cup \Omega_1$ corresponding to $\tilde{g}$.)

One may conjecture that suitably penalizing the region $\Omega_1 \setminus X$ with an appropriate extension of the metric, one can force that length minimizing curves are directed toward the exterior of the obstacle. This is not the case. Indeed, fix a point $k_0 \in X$ and consider $d(x) := d(x, k_0)$, $x \in X$. Then, if such a $\tilde{g}$ exists, $\tilde{d}(x) := \tilde{d}(x, k_0)$ would be an extension of d to $X \cup \Omega_1$. But, for every $x \in \partial\mathscr{O}$, there is a neighborhood of V_x such that $\tilde{d} \in SC^1(V_x)$ (this is a consequence of the fact that $\tilde{d}$ is a viscosity solution of the eikonal equation and of the regularity result given in [1]). Then, we would deduce that $d \in SC^1(V_x \cap X)$, for every $x \in \partial\mathscr{O}$, in contrast with Theorem 1.5.

We observe that it is not a matter of the regularity of $\tilde{g}$. Indeed, even if we admit some lower regularity[3] of the extended Riemannian metric, we have that the answer to question (Q) is in a negative sense.

Finally, let us remark a difference between the Dirichlet boundary condition and the state constrained boundary condition for the eikonal equation. While, in the case of the homogeneous Dirichlet problem for the eikonal equation, one can prescribe a geometrical condition on the boundary of the domain ensuring that the solution of the eikonal equation can be (globally) extended still remaining a solution of the equation (see [2]). Instead, as clarified by the above discussion, in the case of the state constrained boundary condition such a global extension, to a tubular neighborhood of the boundary of the obstacle, cannot be obtained.

2. Preliminaries

2.1. **Preliminaries on semiconcave functions.** We recall a result about the extension of a semiconcave function with fractional modulus (see [3] for the proof of a more precise result).

Theorem 2.6. *Let $A \subset \mathbb{R}^n$ be an open set with nonempty boundary, and let $u \in SC^\alpha_{loc}(\overline{A})$. Then, for every $x \in \partial A$ and there exist $\delta > 0$ and a function $E(u) \in SC^\alpha(B_\delta(x))$ such that*

[3]What we have in mind here is the regularity needed to ensure local linear semiconcavity of the solution of the eikonal equation, see [1].

(1) *$E(u)(y) = u(y)$ for every $y \in B_\delta(x) \cap \overline{A}$;*
(2) *$D^*E(u)(y) = D^*u(y)$ for every $y \in B_\delta(x) \cap \partial A$.*

In particular, for the applications of interest to this paper, one can take $u = d$, $\alpha = 1/2$ and $A = \mathbb{R}^n \setminus (\mathscr{O} \cup \{k_0\})$. For any function $u \in SC^\alpha_{loc}(U)$ and any fixed $x \in U$, we denote by $D^*u(x)$ the nonempty compact set of *reachable gradients* of u at x, i.e.,

$$D^*u(x) = \{p \in \mathbb{R}^n \mid \exists x_h \in \mathrm{int}(U), x_h \to x,\ \exists Du(x_h) \to p\}, \tag{2.5}$$

where $\mathrm{int}(U)$ stands for the interior of U.

We recall that if $u \in SC^\alpha_{loc}(X \setminus \{k_0\})$ for a suitable $\alpha \in]0,1]$, as a consequence of Rademacher's Theorem, we have that $D^*u(x) \neq \emptyset$, for every $x \in \mathbb{R}^n \setminus (\mathcal{O} \cup \{k_0\})$. Furthermore, for every compact set $K \subset X \setminus \{k_0\}$, every $x, y \in K$ such that $[x,y] \subset K$, and every $p \in D^+u(x)$, we have that

$$u(y) \leq u(x) + \langle p, y - x\rangle + C|y-x|^{1+\alpha}, \tag{2.6}$$

for a suitable constant C depending on K. We recall that the superdifferential of u at x is defined as

$$D^+u(x) = \left\{ p \,|\, \limsup_{X\setminus\{k_0\}\ni y \to x} \frac{u(y) - u(x) - \langle p, y-x\rangle}{|y-x|} \leq 0 \right\},$$

for every $x \in X \setminus \{k_0\}$. We point out that (2.6) is a consequence of the assumption that $u \in SC^\alpha_{loc}(X \setminus \{k_0\}))$ for a suitable $\alpha \in]0,1]$. The following "monotonicity" formula is a direct consequence of (2.6). Let $K \subset X \setminus \{k_0\}$ be a compact set and let $x, y \in K$ such that $[x,y] \subset K$. Then

$$\langle p - q, x - y\rangle \leq 2C|x-y|^{1+\alpha}, \tag{2.7}$$

for every $p \in D^+u(x)$ and for every $q \in D^+u(y)$, where C is the constant given in (2.6).

2.2. **Preliminaries on minimizing curves.** We recall a result on the existence and regularity of the minimizing curves (see [11, Lemma A.1], see also [5, 10, 6]).

Theorem 2.7. *For any $x, y \in X$ there exists a curve $\gamma : [0,1] \to X$ of class C^1 such that $\gamma(0) = x$, $\gamma(1) = y$ and $d(x,y) = L(\gamma)$. Furthermore, for any such curve $\dot\gamma$ is uniformly Lipschitz continuous on $[0,1]$.*

In our analysis, the following decomposition of X is important, we set

$$I(k_0) = \{x \in X \,|\, d(x) = |x - k_0|\} \quad \text{and} \quad S(k_0) = X \setminus I(k_0). \tag{2.8}$$

In other words $S(k_0) = \{x \in X \mid d(x) > |x - k_0|\}$ is a relatively open set in X.

Remark 2.7. *Let us also recall that, in the proof of Theorem 2.7 given in* [11]*, some additional properties are given. For instance, it is shown that if $\gamma : [0,1] \to X$ is as in Theorem 2.7, and $\gamma(t) \in \partial\mathscr{O}$, for some $t \in]0,1[$, then*

$$\dot\gamma(t) \text{ is a tangent vector to } \partial\mathscr{O} \text{ at } \gamma(t). \tag{2.9}$$

Furthermore, as a consequence of such a proof, if $x \in \mathbb{R}^n \setminus \mathscr{O}$ belongs to a minimizing curve, γ, then, near x, γ is a straight line segment. Instead, if $\partial\mathscr{O}$ is strictly convex in a neighborhood of $x \in \partial\mathscr{O}$ then, near x, either γ is a minimizing geodesic in $\partial\mathscr{O}$, or γ is a straight line segment, or γ is a concatenation (in a suitable order) of a straight line segment with a minimizing geodesic in $\partial\mathscr{O}$.

We now proceed to relate the minimizing curve parametrized by the arc length with suitable generalized gradients of d.

In the more general Riemannian setting, the next property is a consequence of the inequalities established in [11, Lemma A.4].

Lemma 2.1. *Let $\mathscr{O} \in \mathcal{O}$. Then, for every $x \in X \setminus \{k_0\}$ and $\gamma \in \Gamma^*[x]$ we have that*

$$\dot\gamma(t) \in D^+ d(\gamma(t)), \qquad \forall t \in [0, d(x)[. \tag{2.10}$$

We need the following refinement[4] of Lemma 2.1 given in [4, Lemma 2.1].

Lemma 2.2. *Let $\mathscr{O} \in \mathcal{O}$. Then, for every $x \in X \setminus \{k_0\}$ and $\gamma \in \Gamma^*[x]$, we have that*

$$-\dot\gamma(t) \in D^* d(\gamma(t)), \qquad \forall t \in [0, d(x)]. \tag{2.11}$$

Furthermore[5]*,*

$$D^* d(\gamma(t)) = \{Dd(\gamma(t))\}, \quad \forall t \in]0, d(x)[. \tag{2.12}$$

Finally, for every $x \in X \setminus \{k_0\}$ and for every $p \in D^* d(x)$ there exists $\gamma \in \Gamma^*[x]$, such that $-\dot\gamma(0) = p$.

(Here and in the sequel we take $\dot\gamma(0) = \lim_{t\to 0^+}[\gamma(t) - \gamma(0)]/t$ and, similarly, $\dot\gamma(d(x)) = \lim_{t\to d(x)^-}[\gamma(d(x)) - \gamma(t)]/[d(x) - t]$.)

We point out that (2.11) and (2.12) imply that for every $x \in X \setminus \{k_0\}$ and for every $\gamma \in \Gamma^*[x]$

$$\dot\gamma(t) = -Dd(\gamma(t)), \qquad t \in]0, d(x)[. \tag{2.13}$$

[4]We recall that $D^+ d(x) = \operatorname{co} D^* d(x)$ for every x in the interior of $X \setminus \{k_0\}$.

[5]Hereafter, even at a point $y \in \partial\mathcal{O}$, we have kept the notation $Dd(y)$ to denote the unique element of $D^* d(y)$ whenever the last set reduces to a singleton.

3. Proof of Theorem 1.4

We begin by showing that if d is locally semiconcave with a linear modulus, then two minimizing curves parameterized by the arc length cannot intersect each other in X (possibly except at the endpoints). The proof is based on the backward uniqueness for the (minus) gradient flow.

Let $x_1, x_2 \in X \setminus \{k_0\}$, with $x_1 \neq x_2$, and let $\gamma_i \in \Gamma^*[x_i]$. We want to show that, if d is locally semiconcave with a linear modulus, then

$$\gamma_1(t) \neq \gamma_2(t), \qquad \text{for every } t \in]0, \min\{d(x_1), d(x_2)\}[. \tag{3.14}$$

Let us suppose, by contradiction, that $\gamma_1(t_0) = \gamma_2(t_0)$ for a suitable $t_0 \in]0, \min\{d(x_1), d(x_2)\}[$. Without loss of generality, we may suppose that

$$t_0 = \inf\{t \geq 0 \mid \gamma_1(t) = \gamma_2(t)\}.$$

Then, for $\varepsilon \in]0, t_0[$, $\gamma_1(t) \neq \gamma_2(t)$, for every $t \in]t_0 - \varepsilon, t_0]$ and, in particular,

$$\tilde{x}_1 := \gamma_1(t_0 - \varepsilon) \neq \gamma_2(t_0 - \varepsilon) =: \tilde{x}_2.$$

As a consequence of the regularity Theorem 1.1 and (2.6), taking $K = \overline{B}_r(\gamma_1(t_0)) \cap X$, for a suitable $r > 0$, one can find a constant $C > 0$ such that

$$\langle p - q, x - y \rangle \leq 2C|x - y|^2, \tag{3.15}$$

for every $x, y \in K$, with $[x, y] \subset K$, and for every $p \in D^+d(x), q \in D^+d(y)$. (We observe that, in order to apply (2.6), we need that the whole segment $[x, y]$ is contained in K, if $\gamma_1(t_0) \notin \mathscr{O}$ it suffices to take r small enough, if $\gamma_1(t_0) \in \partial\mathscr{O}$, we may argue on the extension of d given by Theorem 2.6.)

Since γ_1 and γ_2 are solutions of Equation (2.13), due to (3.15), we find that

$$\frac{d}{dt}|\gamma_2(t) - \gamma_1(t)|^2 \geq -4C|\gamma_2(t) - \gamma_1(t)|^2,$$

for every $t \in [t_0 - \varepsilon, t_0[$. Hence,

$$|\gamma_2(t) - \gamma_1(t)| \geq e^{-2C(t - t_0 + \varepsilon)}|\tilde{x}_2 - \tilde{x}_1|, \qquad \text{for every } t \in [t_0 - \varepsilon, t_0[,$$

and we find the contradiction $0 = |\gamma_2(t_0) - \gamma_1(t_0)| > 0$, so (3.14) follows. This completes the proof of the first part of Theorem 1.4. We observe that, due to the (local) linear semiconcavity of d in $X \setminus (\partial\mathscr{O} \cup \{k_0\})$, if two length minimizing curves intersect each other at a point x_* (which is not an endpoint for the curves), then x_* belongs to $\partial\mathscr{O}$.

Now, we want to show that if there exist $x_1, x_2 \in X \setminus \{k_0\}$, with $x_1 \neq x_2$, $\gamma_i \in \Gamma^*[x_i]$ and $t_0 \in]0, \min\{d(x_1), d(x_2)\}[$ so that $\gamma_1(t_0) = \gamma_2(t_0)$ and $\partial\mathscr{O}$ satisfies Condition (C) near such a point, then $d \notin SC^{\alpha}_{loc}(X \setminus \{k_0\})$

for every $\alpha \in]1/2, 1]$. We argue, once more, by contradiction assuming that $d \in SC^{\alpha}_{loc}(X \setminus \{k_0\})$, for some $\alpha \in]1/2, 1[$, that there exist $x_1, x_2 \in X \setminus \{k_0\}$, with $x_1 \neq x_2$, $\gamma_i \in \Gamma^*[x_i]$ and $t_0 \in]0, \min\{d(x_1), d(x_2)\}[$ such that $\gamma_1(t_0) = \gamma_2(t_0)$. We observe that, as already observed, we have that $\gamma_1(t_0) = \gamma_2(t_0) \in \partial\mathscr{O}$. Without loss of generality, we may assume once more that

$$t_0 = \inf\{t \geq 0 \mid \gamma_1(t) = \gamma_2(t)\}. \tag{3.16}$$

Then, we can find $r > 0$ such that $k_0 \notin \overline{B}_r(\gamma_1(t_0))$ and $\gamma_i(t) \in \overline{B}_r(\gamma_1(t_0))$, for every $t \in [t_0 - \varepsilon, t_0]$, $i = 1, 2$ (for $\varepsilon > 0$ small enough). Furthermore, we point out that from the structure of the minimizing curves (see Remark 2.7), we may assume that

$$\gamma_1(t) = \gamma_1(t_0) + \dot{\gamma}_1(t_0)(t - t_0), \qquad t \in [t_0 - \varepsilon, t_0]$$

and $\gamma_2(t)$ is a geodesic on $\partial\mathscr{O}$ (for $[t_0 - \varepsilon, t_0]$). (The other eventualities are excluded by the local uniqueness for the Cauchy problem for the geodesics on $\partial\mathscr{O}$.)

Now, (2.6) and (2.13) yield that

$$\frac{d}{dt}|\gamma_2(t) - \gamma_1(t)|^2 \geq -4C|\gamma_2(t) - \gamma_1(t)|^{1+\alpha},$$

for every $t \in [t_0 - \varepsilon, t_0[$. Then, we find that

$$|\gamma_2(t) - \gamma_1(t)|^{1-\alpha} \geq |\gamma_2(t_0 - \varepsilon) - \gamma_1(t_0 - \varepsilon)|^{1-\alpha} - 2(1-\alpha)C(t - t_0 + \varepsilon),$$

for every $t \in [t_0 - \varepsilon, t_0[$. It follows that if $|\gamma_2(t_0) - \gamma_1(t_0)| = 0$ then

$$|\gamma_2(t_0 - \varepsilon) - \gamma_1(t_0 - \varepsilon)|^{1-\alpha} \leq 2(1-\alpha)C\varepsilon. \tag{3.17}$$

Now, we have that

$$\gamma_2(t_0 - \varepsilon) = \gamma_2(t_0) - \dot{\gamma}_2(t_0)\varepsilon + \varepsilon \int_0^1 [\dot{\gamma}_2(t_0) - \dot{\gamma}_2(t_0 - s\varepsilon)]\, ds,$$

and, since γ_2 is a geodesic on $\partial\mathscr{O}$, we have that $\gamma_2 \in C^2([t_0 - \varepsilon, t_0], \partial\mathscr{O})$ and

$$\lim_{\varepsilon \to 0^+} \frac{\gamma_2(t_0 - \varepsilon) - \gamma_2(t_0) + \dot{\gamma}_2(t_0)\varepsilon}{\varepsilon^2} = \ddot{\gamma}_2(t_0). \tag{3.18}$$

(Here and in the sequel $\ddot{\gamma}_2(t_0)$ stands for the left second derivative.) We observe that, since $\gamma_2(t) \in \partial\mathscr{O}$ for $t \in [t_0 - \varepsilon, t_0]$, then find that

$$\langle \nu(\gamma_2(t)), \dot{\gamma}_2(t) \rangle = 0, \qquad \forall t \in [t_0 - \varepsilon, t_0].$$

Then, taking the derivative with respect to t of both the sides of the above identity, we deduce that

$$\langle \nu(\gamma_2(t)), \ddot{\gamma}_2(t) \rangle + \langle D\nu(\gamma_2(t))\dot{\gamma}_2(t), \dot{\gamma}_2(t) \rangle = 0,$$

for $t \in [t_0 - \varepsilon, t_0[$.

We observe that, as usual, the gradient of ν, $D\nu$, should be appropriately defined. For instance, taking r suitably small, we have that

$$\mathscr{O} \cap B_r(\gamma_1(t_0)) = \{x \in B_r(\gamma_1(t_0)) \mid \Phi(x) \le 0\}$$

where Φ is a function of class C^2 simply vanishing on $\partial\mathscr{O} \cap B_r(\gamma_1(t_0))$. Then, for $x \in \partial\mathscr{O} \cap B_r(\gamma_1(t_0))$, $\nu(x) = D\Phi(x)/|D\Phi(x)|$ and

$$D\nu(x) = \left(D^2\Phi(x) - \frac{D\Phi(x)}{|D\Phi(x)|} \otimes \frac{D\Phi(x)}{|D\Phi(x)|} \right).$$

Then, (1.4) reads as

$$D\nu(x)\xi \cdot \xi \ge c_0|\xi|^2, \qquad \forall \xi \in T_x\partial\mathscr{O}, \tag{3.19}$$

for each $x \in \partial\mathscr{O} \cap B_r(\gamma_1(t_0))$.

Since, γ_2 is a geodesic on $\partial\mathscr{O}$, we have that

$$\ddot\gamma_2(t) = \langle \ddot\gamma_2(t), \nu(\gamma(t))\rangle \nu(\gamma(t))$$

for $t \in [t_0 - \varepsilon, t_0[$ (i.e. the tangential component of the acceleration vanishes). Due to (3.19), we conclude that

$$|\ddot\gamma_2(t_0)| = \langle D\nu(\gamma_2(t_0))\dot\gamma_2(t_0), \dot\gamma_2(t_0)\rangle \ge c_0 > 0. \tag{3.20}$$

We observe that this point is the only part of the proof where we used the fact that $\partial\mathscr{O}$ satisfies Condition (C) near $\gamma_1(t_0)$.

Now, Lemma 2.2 implies that $\dot\gamma_2(t_0) = \dot\gamma_1(t_0)$. Hence, due to (3.17), we find that

$$\left| \frac{\gamma_2(t_0 - \varepsilon) - \gamma_2(t_0) + \dot\gamma_2(t_0)\varepsilon}{\varepsilon^2} \right|^{1-\alpha} \varepsilon^{2(1-\alpha)} \le 2(1-\alpha)C\varepsilon,$$

i.e.

$$\left| \frac{\gamma_2(t_0 - \varepsilon) - \gamma_2(t_0) + \dot\gamma_2(t_0)\varepsilon}{\varepsilon^2} \right|^{1-\alpha} \varepsilon^{1-2\alpha} \le 2(1-\alpha)C.$$

Since $\varepsilon > 0$ can be taken arbitrarily small, in light of (3.18) and (3.20), we conclude that for $1 - 2\alpha < 0$ the inequality above cannot be true. This completes the proof of Theorem 1.4.

4. Proofs of Theorems 1.2 and 1.5

The proofs of Theorems 1.2 and 1.5 are based on Theorem 1.4. To proof both the results[6], it suffices to find two minimizing curves (parametrized by the arc length) γ_1 and γ_2 so that $\gamma_1(t_0) \ne \gamma_2(t_0)$ and $\gamma_1(t_1) = \gamma_2(t_1)$, for suitable $t_0 < t_1$.

[6]Really, the proof of Theorem 1.5 requires some more care since we need to verify that $\partial\mathscr{O}$ satisfies Condition (C) near $\gamma_1(t_1)$.

For this purpose, we consider the maximum of the function

$$\mathscr{O} \ni x \mapsto |x - k_0|.$$

Since $\mathscr{O}$ is compact and the function $x \mapsto |x - k_0|$ has no critical points then the maximum is attained at some $x_0 \in \partial\mathscr{O}$. We claim that $x_0 \in S(k_0)$. In order to prove such a claim, we argue by contradiction assuming that $x_0 \in I(k_0)$. In particular, we have that

$$Dd(x_0) = \frac{x_0 - k_0}{|x_0 - k_0|} \tag{4.21}$$

We observe that, due to the definition of x_0,

$$\mathscr{O} \subset \overline{B}_R(k_0), \text{ and } x_0 \in \partial\mathscr{O} \cap \partial\overline{B}_R(k_0), \tag{4.22}$$

where $R := |x_0 - k_0|$. Now, in light of the regularity assumptions on $\mathscr{O}$, we can find $\delta > 0$ such that

$$\mathscr{O} \cap B_\delta(x_0) = \{x \in B_\delta(x_0) \mid \Phi(x) \le 0\}, \tag{4.23}$$

for a suitable Φ of class C^2 without critical points inside $B_\delta(x_0)$, and

$$\partial\mathscr{O} \cap B_\delta(x_0) = \{x \in B_\delta(x_0) \mid \Phi(x) = 0\}. \tag{4.24}$$

Then, in view of (4.22), $x_0 - k_0$ is normal to $\partial\mathscr{O}$ at x_0, and from (4.21), (4.23) and (4.24) we deduce that

$$D\Phi(x_0) = |D\Phi(x_0)|\, Dd(x_0).$$

From the fact that $x_0 \in I(k_0)$, we have that

$$\Gamma^*[x_0] = \{\gamma(t) = x_0 - tDd(x_0), \quad t \in [0, d(x_0)]\}.$$

Furthermore, we have that

$$\frac{d}{dt}\Phi(\gamma(t))\,|_{t=0} = \langle D\Phi(x_0), \dot\gamma(0)\rangle = -|D\Phi(x_0)| < 0,$$

i.e. $\gamma(t) \in \Omega = \mathrm{int}(\mathscr{O})$, for t near 0, in contrast with $x_0 \in I(k_0)$. Then, our claim holds, i.e. $x_0 \in S(k_0)$.

We point out that, due to (4.22) and the C^2 regularity of $\mathscr{O}$, we have that the obstacle satisfies Condition (C) near x_0, i.e. there exists $r > 0$ such that $\partial\mathscr{O} \cap B_r(x_0)$ is a strictly convex hypersurface of class C^2 (with nondegenerate curvatures).

For $\varepsilon > 0$, let V_ε be the minimal cone (with respect to the inclusion), with vertex at $x_0 + \varepsilon\nu(x_0)$, containing $\mathscr{O}$.

We claim that there exists $\varepsilon > 0$ such that

$$\partial V_\varepsilon \cap \partial\mathscr{O} \subset B_r(x_0). \tag{4.25}$$

Indeed, assume by contradiction that there exist a sequence of numbers $\varepsilon_j > 0$ which converges to 0 and a sequence $y_j \in \partial V_{\varepsilon_j} \cap \partial\mathscr{O} \setminus B_r(x_0)$.

Since $\partial V_{\varepsilon_j} \cap \partial\mathscr{O}$ is a compact set we may assume that, possibly taking a subsequence, $y_j \to y \in \partial\mathscr{O} \setminus B_r(x_0)$.

Then there is a unit vector v_j such that

$$\begin{cases} r_j^+(\varepsilon) := \{x_0 + \varepsilon_j \nu(x_0) + t v_j \mid t \geq 0\} \subset \partial V_{\varepsilon_j}, \\ \\ y_j = x_0 + \varepsilon_j \nu(x_0) + t_j v_j \in \partial V_{\varepsilon_j} \cap \partial\mathscr{O} \setminus B_r(x_0), \end{cases}$$

for a suitable $t_j > 0$. Hence, we would find that the straight half line $\{x_0 + tv \mid t \geq 0\}$ is tangent to $\partial\mathscr{O}$ at y in contrast with (4.22), and (4.25) follows.

Let $\varepsilon > 0$ be such that (4.25) holds. We observe that, in view of the strict convexity of $B_r(x_0) \cap \partial\mathscr{O}$ and (4.25), V_ε is a convex cone. Then, $\partial V_\varepsilon \cap \partial\mathscr{O}$ is the boundary of a connected subset of $\partial\mathscr{O}$ containing x_0. Let $\gamma \in \Gamma^*[x_0]$. Then, there exists $t_* \in]0, d(x_0)[$ such that $\gamma(t_*) \in \partial V_\varepsilon \cap \partial\mathscr{O}$. We observe that, for $t \in [0, t_*]$, γ is a length minimizing geodesic in $\partial\mathscr{O}$. In particular, we have that $\gamma(t) \notin \partial V_\varepsilon$, for every $t \in [0, t_*[$. Take $\delta \in]0, t_*[$ small enough so that

$$\gamma(t_*) - \delta\dot{\gamma}(t_*) \in \partial V_\varepsilon \setminus \partial\mathscr{O},$$

and set

$$x_1 = \gamma(t_*) - \delta\dot{\gamma}(t_*), \qquad x_2 = \gamma(t_* - \delta) \ (\in \partial\mathscr{O}).$$

(We observe that $x_1 \neq x_2$.) Then, taking

$$\gamma_1(t) = \begin{cases} x_1 + t\frac{\gamma(t_*) - x_1}{|\gamma(t_*) - x_1|}, \ t \in [0, \delta], \\ \\ \gamma(t - \delta + t_*), \ t \in]\delta, d(x_0) - t_* + \delta], \end{cases}$$

and

$$\gamma_2(t) = \gamma(t + t_* - \delta), \ t \in [0, d(x_0) - t_* + \delta],$$

we have two different minimizing curves starting at x_1 and x_2 and merging at $\gamma(t_*)$. This completes the proof of Theorem 1.2.

Finally, as already remarked, we have that $\partial\mathscr{O}$ satisfies Condition (C) near $\gamma(t_*)$. Then, applying once more Theorem 1.4, Theorem 1.5 follows.

Declarations

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