Quantum immanants, double Young-Capelli bitableaux

and

Schur shifted symmetric functions

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Abstract

In this paper are introduced two classes of elements in the enveloping algebra $\mathbf{U}(gl(n))$: the *double Young-Capelli bitableaux* [$\boxed{S \mid T}$] and the *central Schur elements* $\mathbf{S}_\lambda(n)$, that act in a remarkable way on the highest weight vectors of irreducible Schur modules.

Any element $\mathbf{S}_\lambda(n)$ is the sum of all double Young-Capelli bitableaux [$\boxed{S \mid S}$], S row (strictly) increasing Young tableaux of shape $\widetilde{\lambda}$. The Schur elements $\mathbf{S}_\lambda(n)$ are proved to be the preimages - with respect to the Harish-Chandra isomorphism - of the *shifted Schur polynomials* $s^*_{\lambda|n} \in \Lambda^*(n)$. Hence, the Schur elements are the same as the Okounkov *quantum immanants*, recently described by the present authors as linear combinations of *Capelli immanants*. This new presentation of Schur elements/quantum immanants doesn't involve the irreducible characters of symmetric groups. The Capelli elements $\mathbf{H}_k(n)$ are column Schur elements and the Nazarov elements $\mathbf{I}_k(n)$ are row Schur elements. The duality in $\boldsymbol{\zeta}(n)$ follows from a combinatorial description of the eigenvalues of the $\mathbf{H}_k(n)$ on irreducible modules that is *dual* (in the sense of shapes/partitions) to the combinatorial description of the eigenvalues of the $\mathbf{I}_k(n)$.

The passage $n \to \infty$ for the algebras $\boldsymbol{\zeta}(n)$ is obtained both as direct and inverse limit in the category of filtered algebras, via the *Olshanski decomposition/projection*.

Keyword: Combinatorial representation theory; shifted symmetric functions; superalgebras; central elements in U(gl(n)); Capelli identities; superstandard Young tableaux; Schur supermodules.

Contents

1 Introduction

The study of the center $\boldsymbol{\zeta}(n)$ of the enveloping algebra $\mathbf{U}(gl(n))$ of the general linear Lie algebra $gl(n,\mathbb{C})$ and the study of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials have noble and rather independent origins and motivations.

The theme of central elements in $\mathbf{U}(gl(n))$ is a standard one in the general theory of Lie algebras, see e.g. [28]. It is an old and actual one, since it is an offspring of the celebrated Capelli identity (see, e.g. [19], [22], [35], [36], [54], [62], [64]), relates to its modern generalizations and applications (see, e.g. [3], [39], [40], [46], [47], [49], [50], [58], [61]) as well as to the theory of *Yangians* (see, e.g. [44], [45], [48]).

The algebra $\Lambda^*(n)$ of shifted symmetric polynomials is a remarkable deformation of the algebra $\Lambda(n)$ of symmetric polynomials and its study fits into the mainstream of generalizations of the classical theory (see, e.g. *factorial symmetric functions*, [6], [7], [23], [32], [33], [42], [43]).

Since the algebras $\boldsymbol{\zeta}(n)$ and $\Lambda^*(n)$ are related by the Harish-Chandra isomorphism χ_n (see, e.g. [51]), their investigation can be essentially regarded as a single topic, and this fact gave rise to a fruitful interplay between representation-theoretic methods (e.g., eigenvalues on irreducible representations) and combinatorial techniques (e.g., generating functions).

We propose a new approach to a systematic study of some of the main features of the theory of the center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$ and of the algebra $\Lambda^*(n)$ of shifted symmetric polynomials that allows the whole theory to be developed, in a transparent and concise way, from a combinatorial representation theoretic point of view, that is entirely in the center $\boldsymbol{\zeta}(n)$.

The paper is organized as follows.

In Section 2, we provide a synthetic presentation of the *superalgebraic method of virtual variables* for $gl(n)$. For details, we refer the reader to [9], [17] and [18]. This method was developed by the present authors for the general linear Lie superalgebras $gl(m|n)$ (see, e.g. [37], [59]) in the series of notes [8], [9], [11], [12], [13], [14], [16]. The technique of virtual variables is an extension of Capelli's method of *variabili ausilarie* (Capelli [22], see also Weyl [64]).

The superalgebraic method of virtual variables allows us to express remarkable classes of elements in $\mathbf{U}(gl(n))$, namely,

- the *Capelli bitableaux* $[S|T] \in \mathbf{U}(gl(n))$

- the *Young-Capelli bitableaux* $[S|\boxed{T}] \in \mathbf{U}(gl(n))$

- the *double Young-Capelli bitableaux* $[\,\boxed{S \mid T}\,] \in \mathbf{U}(gl(n))$

as the images - with respect to the $Ad_{gl(n)}$-adjoint equivariant Capelli *devirtualization epimorphism* - of simple expressions in an enveloping superalgebra $\mathbf{U}(gl(m_0|m_1+n))$.

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Capelli (determinantal) bitableaux are generalizations of the famous *determinantal elements* in $\mathbf{U}(gl(n))$ introduced by Capelli in 1887 [19] (see, e.g. [18]). Young-Capelli bitableaux were introduced by the present authors several years ago [12], [13], [14] and might be regarded as generalizations of the Capelli determinantal elements in $\mathbf{U}(gl(n))$ as well as of the *Young symmetrizers* of the classical representation theory of symmetric groups (see, e.g. [64]). Double Young-Capelli bitableaux are essentially new and play a crucial role in the present paper.

In plain words, the Young-Capelli bitableau $[S \boxed{T}]$ is obtained by adding a *column symmetrization* to the Capelli bitableau $[S|T]$ (Proposition 2.13) and the double Young-Capelli bitableau $[\boxed{S \mid T}]$ is obtained by adding a further *row skew-symmetrization* to the Young-Capelli bitableau $[S\boxed{T}]$ (Proposition 2.14).

In Section 3, we regard the *supersymmetric* superalgebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ as $\mathbf{U}(gl(m_0|m_1+n))$-module and define the *Schur supermodules* $Schur_\lambda(m_0, m_1+n)$ as (irreducible) submodules of $\mathbb{C}[M_{m_0|m_1+n,d}]$. Schur supermodules are isomorphic to the modules constructed by Berele and Regev [4], [5] as tensor modules induced by Young symmetrizers (see, e.g. [64]) when they act by a "signed action" of the symmetric group (see also [38]). The description presented here is simpler than the tensor description, and provides a close connection with the superstraightening theory of Grosshans, Rota and Stein [34]. The classical Schur modules $Schur_\lambda(n)$ - $gl(n)$-irreducible modules with (nonnegative) integral highest weights - are here regarded as $gl(n)$-submodules of suitable Schur supermodules. The crucial and new result is that double Young-Capelli bitableaux act in a remarkable way on the highest weight vectors of Schur modules (subsection 3.4, Theorem 3.9).

In Section 4, we consider a class of central elements $\mathbf{S}_\lambda(n)$, $\tilde{\lambda}_1 \leq n$, which arise in a natural way in the context of the virtual method when dealing with symmetry and skew-symmetry in $\mathbf{U}(gl(n))$.

These elements are expressed as linear combinations

$$\mathbf{S}_\lambda(n) = \frac{1}{H(\tilde{\lambda})} \sum_S [\boxed{S \mid S}] \in \mathbf{U}(gl(n)) \tag{1}$$

of double Young-Capelli bitableaux where the sum is extended to all row (strictly) increasing tableaux S of shape $sh(S) = \tilde{\lambda}$.

We call the elements $\mathbf{S}_\lambda(n)$ the *Schur elements*.

The main results are Theorems 4.2 and 4.3 that follow from Theorem 3.9 and provide notable descriptions of the action of the central Schur elements $\mathbf{S}_\lambda(n)$ on the highest weight vectors of Schur modules. Theorem 4.2 implies that the set of the elements $\mathbf{S}_\lambda(n)$, $\tilde{\lambda}_1 \leq n$ is a basis of the center $\boldsymbol{\zeta}(n)$.

By combining Theorem 4.2 with the *Sahi-Okounkov characterization Theorem* ([57], [51], [49], here quoted as Proposition 4.6), we infer that the Schur elements $\mathbf{S}_\lambda(n)$ are the preimages - with respect to the Harish-Chandra isomorphism - of the *shifted Schur polynomials* $s^*_{\lambda|n} \in \Lambda^*(n)$ [56], [51]. Hence,

the Schur elements are the same as the *quantum immanants*, first presented by Okounkov as traces of *fusion matrices* ([49], [50]) and, recently, described by the present authors as linear combinations (with explicit coefficients) of "diagonal" *Capelli immanants*, see [17]. Presentation (1) of Schur elements/quantum immanants doesn't involve the irreducible characters of symmetric groups. Furthermore, it is better suited to the study of the duality in the algebra $\boldsymbol{\zeta}(n)$ as well as to the study of the limit $n \to \infty$.

We examine two further classes of central elements, namely, the *determinantal Capelli elements* $\mathbf{H}_k(n)$, $k = 1, 2, \ldots, n$ (see, e.g. [19], [20], [21], [9]), and the *permanental Nazarov elements* $\mathbf{I}_k(n)$, $k \in \mathbf{Z}^+$ (see, e.g. [47], [48], [61], [63], see also [9]), which provide two systems of algebra generators of the center $\boldsymbol{\zeta}(n)$.

The Capelli elements $\mathbf{H}_k(n)$ are column Schur elements, specifically,

$$\mathbf{H}_k(n) = \mathbf{S}_{(1^k)}(n).$$

The Nazarov elements $\mathbf{I}_k(n)$ are row Schur elements, specifically,

$$\mathbf{I}_k(n) = \mathbf{S}_{(k)}(n).$$

The duality in $\boldsymbol{\zeta}(n)$ (Theorem 4.27) immediately follows from a combinatorial description of the eigenvalues of the Capelli elements $\mathbf{H}_k(n)$ on irreducible Schur modules (Proposition 4.18) that is *dual* (in the sense of shapes/partitions) to the combinatorial description of the eigenvalues of the Nazarov elements $\mathbf{I}_k(n)$ (Theorem 4.23.1).

The passage to the infinite dimensional case $n \to \infty$ for the algebras $\boldsymbol{\zeta}(n)$ is rather subtle; the "naive" ∞-dimensional analogue of the algebras $\mathbf{U}(gl(n))$, that is the direct limit algebra $\varinjlim \mathbf{U}(gl(n))$ with respect to the *"inclusion" monomorphisms*, has trivial center. In Section 5, the ∞-dimensional analogue $\boldsymbol{\zeta}$ of the algebras $\boldsymbol{\zeta}(n)$ is obtained as the *direct limit algebra* $\varinjlim \boldsymbol{\zeta}(n)$ (in the category of filtered algebras) with respect to the family of monomorphisms $\mathbf{i}_{n+1,n} : \boldsymbol{\zeta}(n) \hookrightarrow \boldsymbol{\zeta}(n+1)$, where

$$\mathbf{i}_{n+1,n}\big(\mathbf{H}_k(n)\big) = \mathbf{H}_k(n+1), \quad k = 1, 2, \ldots, n.$$

An intrinsic/invariant presentation of the monomorphisms $\mathbf{i}_{n+1,n}$ is obtained, in subsection 5.2, via the *Olshanski projections* $\boldsymbol{\mu}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n)$ [52], [53] (see also Molev [44]). The Olshanski projections $\boldsymbol{\mu}_{n,n+1}$ are *left* inverses of the monomorphisms $\mathbf{i}_{n+1,n}$, and they become *two-sided* inverses when restricted to the filtration elements $\boldsymbol{\zeta}(n+1)^{(m)}$ and $\boldsymbol{\zeta}(n)^{(m)}$, for n sufficiently large (Propositions 5.2 and Proposition 5.6). The interplay between the monomorphisms $\mathbf{i}_{n+1,n}$ and the projections $\boldsymbol{\mu}_{n,n+1}$ shows the algebra $\boldsymbol{\zeta}$ admits a double presentation, both as a direct limit and as an inverse limit.

Since the Olshanski projection $\boldsymbol{\mu}_{n,n+1}$ maps $\mathbf{H}_k(n+1)$ to $\mathbf{H}_k(n)$, $\mathbf{I}_k(n+1)$ to $\mathbf{I}_k(n)$ and $\mathbf{S}_\lambda(n+1)$ to $\mathbf{S}_\lambda(n)$, then

$$\mathbf{i}_{n+1,n}\big(\mathbf{H}_k(n)\big) = \mathbf{H}_k(n+1), \qquad \mathbf{i}_{n+1,n}\big(\mathbf{I}_k(n)\big) = \mathbf{I}_k(n+1),$$

and

$$\mathbf{i}_{n+1,n}\big(\mathbf{S}_\lambda(n)\big) = \mathbf{S}_\lambda(n+1),$$

for n sufficiently large.

Hence, the direct limits

$$\mathbf{H}_k \overset{def}{=} \underrightarrow{\lim}\, \mathbf{H}_k(n) \in \boldsymbol{\zeta} \qquad \mathbf{I}_k \overset{def}{=} \underrightarrow{\lim}\, \mathbf{I}_k(n) \in \boldsymbol{\zeta}$$

and

$$\mathbf{S}_\lambda \overset{def}{=} \underrightarrow{\lim}\, \mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}$$

can be consistently written as *formal series* by naturally extending to infinite sums the finite sums (eqs. (3), (30), (33)) that define $\mathbf{H}_k(n)$, $\mathbf{I}_k(n)$ and $\mathbf{S}_\lambda(n)$, respectively.

The algebra $\boldsymbol{\zeta}$ is isomorphic to the algebra Λ^* of *shifted symmetric functions* (Theorem 7.2). The Olshanski projections are the natural counterpart, in the context of the centers $\boldsymbol{\zeta}(n)$, of the Okounkov-Olshanski *stability principle* for the algebras $\Lambda^*(n)$ of shifted symmetric polynomials [51], the isomorphism $\chi : \boldsymbol{\zeta} \to \Lambda^*$ is indeed the "limit" of the Harish-Chandra isomorphisms $\chi_n : \boldsymbol{\zeta}(n) \to \Lambda^*(n)$ and it admits a transparent representation-theoretic interpretation (Sections 6 and 7).

Some parts of this paper are basically reviews of previously published results. We hope it can be beneficial for some readers who are not too familiar with the subject.

2   A glimpse on the superalgebraic method of virtual variables

2.1   The superalgebras $gl(m_0|m_1 + n)$ and $\mathbb{C}[M_{m_0|m_1+n,d}]$

2.1.1   The general linear Lie super algebra $gl(m_0|m_1 + n)$

Given a vector space V_n of dimension n, we will regard it as a subspace of a $\mathbb{Z}_2-$graded vector space $W = W_0 \oplus W_1$, where

$$W_0 = V_{m_0}, \qquad W_1 = V_{m_1} \oplus V_n.$$

The vector spaces V_{m_0} and V_{m_1} (we assume that $dim(V_{m_0}) = m_0$ and $dim(V_{m_1}) = m_1$ are "sufficiently large") are called the *positive virtual (auxiliary) vector space*, the *negative virtual (auxiliary) vector space*, respectively, and V_n is called the *(negative) proper vector space*.

The inclusion $V_n \subset W$ induces a natural embedding of the ordinary general linear Lie algebra $gl(n)$ of V_n into the *auxiliary* general linear Lie *superalgebra* $gl(m_0|m_1 + n)$ of $W = W_0 \oplus W_1$ (see, e.g. [37], [59]).

Let $\mathcal{A}_0 = \{\alpha_1, \ldots, \alpha_{m_0}\}$, $\mathcal{A}_1 = \{\beta_1, \ldots, \beta_{m_1}\}$, $\mathcal{L} = \{1, 2, \ldots, n\}$ denote *fixed bases* of V_{m_0}, V_{m_1} and V_n, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|\beta_t| = |i| = 1 \in \mathbb{Z}_2$.

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Let

$$\{e_{a,b}; a, b \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}\}, \qquad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2$$

be the standard $\mathbb{Z}_2-$homogeneous basis of the Lie superalgebra $gl(m_0|m_1 + n)$ provided by the elementary matrices. The elements $e_{a,b} \in gl(m_0|m_1 + n)$ are $\mathbb{Z}_2-$homogeneous of $\mathbb{Z}_2-$degree $|e_{a,b}| = |a| + |b|$.

The superbracket of the Lie superalgebra $gl(m_0|m_1 + n)$ has the following explicit form:

$$[e_{a,b}, e_{c,d}] = \delta_{bc}\, e_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{ad}\, e_{c,b},$$

$a, b, c, d \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$.

In the following, the elements of the sets $\mathcal{A}_0, \mathcal{A}_1, \mathcal{L}$ will be called *positive virtual symbols*, *negative virtual symbols* and *negative proper symbols*, respectively.

2.1.2 The supersymmetric algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$

Let

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[(i|j)]_{i=1,\dots,n,j=1,\dots,d}$$

be the polynomial algebra in the (commutative) entries $(i|j)$ of the matrix:

$$M_{n,d} = [(i|j)]_{i=1,\dots,n,j=1,\dots,d} = \begin{pmatrix} (1|1) & \dots & (1|d) \\ \vdots & & \vdots \\ (n|1) & \dots & (n|d) \end{pmatrix}.$$

We regard the commutative algebra $\mathbb{C}[M_{n,d}]$ as a subalgebra of the "auxiliary" supersymmetric algebra

$$\mathbb{C}[M_{m_0|m_1+n,d}]$$

generated by the ($\mathbb{Z}_2$-graded) variables

$$(a|j), \quad a \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}, \quad j \in \mathcal{P} = \{j = 1, \dots, d; |j| = 1 \in \mathbb{Z}_2\},$$

with $|(a|j)| = |a| + |j| \in \mathbb{Z}_2$, subject to the commutation relations:

$$(a|h)(b|k) = (-1)^{|(a|h)||(b|k)|}\, (b|k)(a|h).$$

In plain words, $\mathbb{C}[M_{m_0|m_1+n,d}]$ is the free supersymmetric algebra

$$\mathbb{C}\big[(\alpha_s|j), (\beta_t|j), (i|j)\big]$$

generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j), (\beta_t|j), (i|j)$, $j = 1, 2, \dots, d$, where all the variables commute each other, with the exception of pairs of variables $(\alpha_s|j), (\alpha_t|j)$ that skew-commute:

$$(\alpha_s|j)(\alpha_t|j) = -(\alpha_t|j)(\alpha_s|j).$$

In the standard notation of multilinear algebra, we have:

$$\mathbb{C}[M_{m_0|m_1+n,d}] \cong \Lambda\big[W_0 \otimes P_d\big] \otimes \mathrm{Sym}\big[W_1 \otimes P_d\big]$$
$$= \Lambda\big[V_{m_0} \otimes P_d\big] \otimes \mathrm{Sym}\big[(V_{m_1} \oplus V_n) \otimes P_d\big]$$

where $P_d = (P_d)_1$ denotes the trivially $\mathbb{Z}_2-$graded vector space with distinguished basis $\mathcal{P} = \{j = 1, \dots, d; |j| = 1 \in \mathbb{Z}_2\}$.

2.1.3 Left superderivations and left superpolarizations

A *left superderivation* D^l ($\mathbb{Z}_2-$homogeneous of degree $|D^l|$) (see, e.g. [59], [37]) on $\mathbb{C}[M_{m_0|m_1+n,n}]$ is an element of the superalgebra $End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]]$ that satisfies "Leibniz rule"

$$D^l(\mathbf{p} \cdot \mathbf{q}) = D^l(\mathbf{p}) \cdot \mathbf{q} + (-1)^{|D^l||\mathbf{p}|}\mathbf{p} \cdot D^l(\mathbf{q}),$$

for every $\mathbb{Z}_2-$homogeneous of degree $|\mathbf{p}|$ element $\mathbf{p} \in \mathbb{C}[M_{m_0|m_1+n,d}]$.

Given two symbols $a, b \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$, the *left superpolarization* $D^l_{a,b}$ of b to a is the unique *left* superderivation of $\mathbb{C}[M_{m_0|m_1+n,n}]$ of $\mathbb{Z}_2-$degree $|D^l_{a,b}| = |a| + |b| \in \mathbb{Z}_2$ such that

$$D^l_{a,b}\left((c|j)\right) = \delta_{bc}\ (a|j),\ c \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L},\ j = 1, \ldots, n.$$

Informally, we say that the operator $D^l_{a,b}$ *annihilates* the symbol b and *creates* the symbol a.

2.1.4 The superalgebra $\mathbb{C}[M_{m_0|m_1+n,n}]$ as a $\mathbf{U}(gl(m_0|m_1 + n))$-module

Since

$$D^l_{a,b}D^l_{c,d} - (-1)^{(|a|+|b|)(|c|+|d|)}D^l_{c,d}D^l_{a,b} = \delta_{b,c}D^l_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)}\delta_{a,d}D^l_{c,b},$$

the map

$$e_{a,b} \mapsto D^l_{a,b}, \qquad a, b \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$$

is a Lie superalgebra morphism from $gl(m_0|m_1 + n)$ to $End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]]$ and, hence, it uniquely defines a representation:

$$\varrho : \mathbf{U}(gl(m_0|m_1 + n)) \to End_{\mathbb{C}}[\mathbb{C}[M_{m_0|m_1+n,d}]],$$

where $\mathbf{U}(gl(m_0|m_1 + n))$ is the enveloping superalgebra of $gl(m_0|m_1 + n)$.

In the following, we always regard the superalgebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ as a $\mathbf{U}(gl(m_0|m_1 + n))-$supermodule, with respect to the action induced by the representation ϱ:

$$e_{a,b} \cdot \mathbf{p} = D^l_{a,b}(\mathbf{p}),$$

for every $\mathbf{p} \in \mathbb{C}[M_{m_0|m_1+n,n}]$.

We recall that $\mathbf{U}(gl(m_0|m_1 + n))-$module $\mathbb{C}[M_{m_0|m_1+n,d}]$ is a semisimple module, whose simple submodules are - up to isomorphism - *Schur supermodules* (see, e.g. [11], [12], [8]. For a more traditional presentation, see also [24]).

Clearly, $\mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$ is a subalgebra of $\mathbf{U}(gl(m_0|m_1 + n))$ and the subalgebra $\mathbb{C}[M_{n,d}]$ is a $\mathbf{U}(gl(n))-$submodule of $\mathbb{C}[M_{m_0|m_1+n,d}]$.

2.1.5 The virtual algebra $Virt(m_0 + m_1, n)$ and the virtual presentations of elements in $\mathbf{U}(gl(n))$

We say that a product

$$e_{a_m b_m} \cdots e_{a_1 b_1} \in \mathbf{U}(gl(m_0|m_1 + n)), \quad a_i, b_i \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}, \ i = 1, \ldots, m$$

is an *irregular expression* whenever there exists a right subword

$$e_{a_i, b_i} \cdots e_{a_2, b_2} e_{a_1, b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in \mathcal{A}_0 \cup \mathcal{A}_1$ such that

$$\#\{j; b_j = \gamma, j \leq i\} > \#\{j; a_j = \gamma, j < i\}. \tag{2}$$

The meaning of an irregular expression in terms of the action of $\mathbf{U}(gl(m_0|m_1 + n))$ by left superpolarization on the algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ is that there exists a virtual symbol γ and a right subsequence in which the symbol γ is *annihilated* more times than it was already *created* and, therefore, the action of an irregular expression on the algebra $\mathbb{C}[M_{n,d}]$ is *zero*.

Example 2.1. Let $\gamma \in \mathcal{A}_0 \cup \mathcal{A}_1$ and $x_i, x_j \in \mathcal{L}$. The product

$$e_{\gamma, x_j} e_{x_i, \gamma} e_{x_j, \gamma} e_{\gamma, x_i}$$

is an irregular expression.

$\square$

Let **Irr** be the *left ideal* of $\mathbf{U}(gl(m_0|m_1 + n))$ generated by the set of irregular expressions.

Proposition 2.2. *The superpolarization action of any element of* **Irr** *on the subalgebra* $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m_0|m_1+n,d}]$ *- via the representation ϱ - is identically zero.*

Proposition 2.3. *The sum* $\mathbf{U}(gl(0|n)) + \mathbf{Irr}$ *is a direct sum of vector subspaces of* $\mathbf{U}(gl(m_0|m_1 + n))$.

Proposition 2.4. *The direct sum vector subspace* $\mathbf{U}(gl(0|n)) \oplus \mathbf{Irr}$ *is a* subalgebra *of* $\mathbf{U}(gl(m_0|m_1 + n))$.

The subalgebra

$$Virt(m_0 + m_1, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \subset \mathbf{U}(gl(m_0|m_1 + n)).$$

is called the *virtual algebra*.

Proposition 2.5. *The left ideal* **Irr** *of* $\mathbf{U}(gl(m_0|m_1 + n))$ *is a two sided ideal of* $Virt(m_0 + m_1, n)$.

The *Capelli devirtualization epimorphism* is the surjection

$$\mathfrak{p} : Virt(m_0 + m_1, n) = \mathbf{U}(gl(0|n)) \oplus \mathbf{Irr} \twoheadrightarrow \mathbf{U}(gl(0|n)) = \mathbf{U}(gl(n))$$

with $Ker(\mathfrak{p}) = \mathbf{Irr}$.

Any element in $\mathbf{M} \in Virt(m_0 + m_1, n)$ defines an element in $\mathbf{m} \in \mathbf{U}(gl(n))$
- via the map $\mathfrak{p}$ - and $\mathbf{M}$ is called a *virtual presentation* of $\mathbf{m}$.

Furthermore,

Proposition 2.6. *The subalgebra* $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m_0|m_1+n,d}]$ *is invariant with
respect to the action of the subalgebra* $Virt(m_0 + m_1, n)$.

Proposition 2.7. *For every element* $\mathbf{m} \in \mathbf{U}(gl(n))$, *the action of* $\mathbf{m}$ *on the
subalgebra* $\mathbb{C}[M_{n,d}]$ *is the same of the action of any of its virtual presentation*
$\mathbf{M} \in Virt(m_0 + m_1, n)$. *In symbols,*

$$if \quad \mathfrak{p}(\mathbf{M}) = \mathbf{m} \quad then \quad \mathbf{m} \cdot \mathbf{P} = \mathbf{M} \cdot \mathbf{P}, \quad for \; every \; \mathbf{P} \in \mathbb{C}[M_{n,d}].$$

Since the map $\mathfrak{p}$ a surjection, any element $\mathbf{m} \in \mathbf{U}(gl(n))$ admits several
virtual presentations. In the sequel, we even take virtual presentations as the
definition of special elements in $\mathbf{U}(gl(n))$, and this method will turn out to be
quite effective.

The superalgebra $\mathbf{U}(gl(m_0|m_1 + n))$ is a Lie module with respect to the
adjoint representation $Ad_{gl(m_0|m_1+n)}$. Since $gl(n) = gl(0|n)$ is a Lie subalgebra
of $gl(m_0|m_1 + n)$, then $\mathbf{U}(gl(m_0|m_1 + n))$ is a $gl(n)-$module with respect to
the adjoint action $Ad_{gl(n)}$ of $gl(n)$. We recall a couple of results from [18].

Proposition 2.8. *The virtual algebra* $Virt(m_0 + m_1, n)$ *is a submodule of*
$\mathbf{U}(gl(m_0|m_1 + n))$ *with respect to the adjoint action* $Ad_{gl(n)}$ *of* $gl(n)$.

Proposition 2.9. *The Capelli epimorphism*

$$\mathfrak{p} : Virt(m_0 + m_1, n) \twoheadrightarrow \mathbf{U}(gl(n))$$

is an $Ad_{gl(n)}-$equivariant *map.*

Corollary 2.10. *The isomorphism* $\mathfrak{p}$ *maps any* $Ad_{gl(n)}-invariant$ *element* $\mathbf{m} \in$
$Virt(m_0 + m_1, n)$ *to a* central *element of* $\mathbf{U}(gl(n))$.

Balanced monomials are elements of the algebra $\mathbf{U}(gl(m_0|m_1 + n))$ of the
form:

– $e_{i_1, \gamma_{p_1}} \cdots e_{i_k, \gamma_{p_k}} \cdot e_{\gamma_{p_1}, j_1} \cdots e_{\gamma_{p_k}, j_k}$,

– $e_{i_1, \theta_{q_1}} \cdots e_{i_k, \theta_{q_k}} \cdot e_{\theta_{q_1}, \gamma_{p_1}} \cdots e_{\theta_{q_k}, \gamma_{p_k}} \cdot e_{\gamma_{p_1}, j_1} \cdots e_{\gamma_{p_k}, j_k}$,

– and so on,

where $i_1, \ldots, i_k, j_1, \ldots, j_k \in L$, i.e., the $i_1, \ldots, i_k, j_1, \ldots, j_k$ are k proper (nega-
tive) symbols, and the $\gamma_{p_1}, \ldots, \gamma_{p_k}, \ldots, \theta_{q_1}, \ldots, \theta_{q_k}, \ldots$ are virtual symbols. In
plain words, a balanced monomial is product of two or more factors where the

rightmost one *annihilates* (by superpolarization) the k proper symbols $j_1, \ldots, j_k$ and *creates* (by superpolarization) some virtual symbols; the leftmost one *annihilates* all the virtual symbols and *creates* the k proper symbols $i_1, \ldots, i_k$; between these two factors, there might be further factors that annihilate and create virtual symbols only.

Proposition 2.11. *Every balanced monomial belongs to $Virt(m_0 + m_1, n)$. Hence, the Capelli epimorphism $\mathfrak{p}$ maps balanced monomials to elements of $\mathbf{U}(gl(n))$.*

2.2 Four special classes of elements in $Virt(m_0 + m_1, n)$ and their images in $\mathbf{U}(gl(n))$

We will introduce four classes of remarkable elements of the enveloping algebra $\mathbf{U}(gl(n))$, that we call *bitableaux monomials*, *Capelli bitableaux*, *Young-Capelli bitableaux* and *double Young-Capelli bitableaux*, respectively.

2.2.1 Partitions and Young tableaux

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p) \vdash h$ be a partition of the positive integer $h \in \mathbb{Z}^+$, where $p = l(\lambda)$ is the *length* of λ.

We denote by $\tilde{\lambda} = (\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_q)$ the *conjugate partition* of λ, that is

$$\tilde{\lambda}_i = \#\{j = 1, 2, \ldots, p; \lambda_j \geq i\}, \quad i = 1, 2, \ldots \lambda_1;$$

clearly, $l(\tilde{\lambda}) = \lambda_1$.

Label the boxes of the Ferrers diagram of the partition λ with the numbers $1, 2, \ldots, h$ in the following way:

$$
\begin{array}{ccccc}
1 & 2 & \cdots & \cdots & \lambda_1 \\
\lambda_1 + 1 & \lambda_1 + 2 & \cdots & \lambda_1 + \lambda_2 & \\
\cdots & \cdots & \cdots & & \\
\cdots & \cdots & h & &
\end{array}
.
$$

A *Young tableau* T of shape λ over the alphabet $\mathcal{A} = \{a_1, a_2, \ldots\}$ is a map $T : \underline{h} = \{1, 2, \ldots, h\} \to \mathcal{A}$; the element $T(i)$ is the symbol in the cell i of the tableau T.

The sequences

$$
\begin{aligned}
&T(1)T(2)\cdots T(\lambda_1), \\
&T(\lambda_1 + 1)T(\lambda_1 + 2)\cdots T(\lambda_1 + \lambda_2), \\
&\cdots\cdots
\end{aligned}
$$

are called the *row words* of the Young tableau T.

We will also denote a Young tableau by its sequence of rows words, that is $T = (\omega_1, \omega_2, \ldots, \omega_p)$. Furthermore, the *word of the tableau* T is the concatenation

$$w(T) = \omega_1 \omega_2 \cdots \omega_p. \tag{3}$$

The *content* of a tableau T is the function $c_T : \mathcal{A} \to \mathbb{N}$,

$$c_T(a) = \sharp\{i \in \underline{h};\ T(i) = a\}.$$

Set

$$D_\lambda = \begin{pmatrix} a_1 \ldots\ldots\ldots a_{\lambda_1} \\ a_1 \ldots\ldots a_{\lambda_2} \\ \ldots\ldots \\ a_1 \ldots a_{\lambda_p} \end{pmatrix}. \tag{4}$$

The tableaux of kind (4) are called *Deruyts tableaux* (of shape λ) in honor of Jacques Deruyts $(1862 - 1945)$, who introduced them in his treatment of *semi-invariants/primary covariants* of algebraic forms [26] (see also [31] and [8]).

Set

$$C_\lambda = \begin{pmatrix} a_1 \ldots\ldots\ldots a_1 \\ a_2 \ldots\ldots a_2 \\ \ldots\ldots \\ a_p \ldots a_p \end{pmatrix}. \tag{5}$$

Since C_λ is the conjugate tableau $C_\lambda = \widetilde{D_{\tilde\lambda}}$ of the Deruyts tableau $D_{\tilde\lambda}$ of shape $\tilde\lambda$, we refer to the tableaux of kind (5) as *Coderuyts tableaux* (of shape λ).

Now, assume that the alphabet $\mathcal{A}$ is

$$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$$

as in Section 2.1.1.

Given a shape/partition λ, we assume that $|\mathcal{A}_0| = m_0 \geq \tilde\lambda_1$ and $|\mathcal{A}_1| = m_1 \geq \lambda_1$. Let us denote by $\alpha_1, \ldots, \alpha_p \in \mathcal{A}_0$, $\beta_1, \ldots, \beta_{\lambda_1} \in A_1$ two *arbitrary* families of *distinct positive and negative virtual symbols*, respectively.

Set

$$D_\lambda^* = \begin{pmatrix} \beta_1 \ldots\ldots\ldots \beta_{\lambda_1} \\ \beta_1 \ldots\ldots \beta_{\lambda_2} \\ \ldots\ldots \\ \beta_1 \ldots \beta_{\lambda_p} \end{pmatrix}, \qquad C_\lambda^* = \begin{pmatrix} \alpha_1 \ldots\ldots\ldots \alpha_1 \\ \alpha_2 \ldots\ldots \alpha_2 \\ \ldots\ldots \\ \alpha_p \ldots \alpha_p \end{pmatrix}. \tag{6}$$

The tableaux of kind (6) are called *virtual* Deruyts and Coderuyts tableaux of shape λ, respectively.

2.2.2 Bitableaux monomials in $\mathbf{U}(gl(m_0 + m_1, n))$

Let S and T be two Young tableaux of same shape $\lambda \vdash h$ on the alphabet $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$:

$$S = \begin{pmatrix} z_{i_1} \ldots\ldots\ldots z_{i_{\lambda_1}} \\ z_{j_1} \ldots\ldots z_{j_{\lambda_2}} \\ \ldots\ldots \\ z_{s_1} \ldots z_{s_{\lambda_p}} \end{pmatrix}, \qquad T = \begin{pmatrix} z_{h_1} \ldots\ldots\ldots z_{h_{\lambda_1}} \\ z_{k_1} \ldots\ldots z_{k_{\lambda_2}} \\ \ldots\ldots \\ z_{t_1} \ldots z_{t_{\lambda_p}} \end{pmatrix}. \tag{7}$$

To the pair (S, T), we associate the *bitableau monomial*:

$$e_{S,T} = e_{z_{i_1}, z_{h_1}} \cdots e_{z_{i_{\lambda_1}}, z_{h_{\lambda_1}}} e_{z_{j_1}, z_{k_1}} \cdots e_{z_{j_{\lambda_2}}, z_{k_{\lambda_2}}} \cdots \cdots e_{z_{s_1}, z_{t_1}} \cdots e_{z_{s_{\lambda_p}}, z_{t_{\lambda_p}}} \quad (8)$$

in $\mathbf{U}(gl(m_0|m_1 + n))$.

2.2.3 Capelli bitableaux in $\mathbf{U}(gl(n))$

Given a pair of Young tableaux S, T of the same shape λ on the proper alphabet $\mathcal{L}$, consider the elements

$$e_{S, C_\lambda^*} \, e_{C_\lambda^*, T} \in \mathbf{U}(gl(m_0|m_1 + n)).$$

Since these elements are *balanced monomials* in $\mathbf{U}(gl(m_0|m_1+n))$, then they belong to the *virtual subalgebra* $Virt(m_0 + m_1, n)$.

Hence, we can consider their images in $\mathbf{U}(gl(n))$ with respect to the Capelli epimorphism $\mathfrak{p}$.

We set

$$\mathfrak{p}\Big(e_{S, C_\lambda^*} \, e_{C_\lambda^*, T}\Big) = [S|T] \in \mathbf{U}(gl(n)), \quad (9)$$

and call the element $[S|T]$ a *Capelli bitableau*.

The elements defined in (9) do not depend on the choice of the virtual Coderuyts tableau C_λ^*.

From [18], we recall that the Capelli bitableaux $[S|T] \in \mathbf{U}(gl(n))$ are the preimages of the *determinantal bitableaux* $(S|T)$ in the polynomial algebra $\mathbb{C}[M_{n,n}]$ (see section 2.3 below) with respect to the *Koszul equivariant isomorhism*

$$\mathcal{K} : \mathbf{U}(gl(n)) \to \mathbb{C}[M_{n,n}] \cong \mathbf{Sym}(gl(n)).$$

Hence, Capelli bitableaux $[S|T]$ admit explicit Laplace expansions: let S and T be the Young tableaux

$$S = \begin{pmatrix} i_{p_1} \ldots \ldots \ldots i_{p_{\lambda_1}} \\ i_{q_1} \ldots \ldots i_{q_{\lambda_2}} \\ \ldots \ldots \\ i_{r_1} \ldots i_{r_{\lambda_m}} \end{pmatrix}, \quad T = \begin{pmatrix} j_{s_1} \ldots \ldots \ldots j_{s_{\lambda_1}} \\ j_{t_1} \ldots \ldots j_{t_{\lambda_2}} \\ \ldots \ldots \\ j_{v_1} \ldots j_{v_{\lambda_m}} \end{pmatrix}.$$

Proposition 2.12. ([18], Corollary 8.3) *We have*

$$[S|T] = \sum_{\sigma_1, \ldots, \sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \begin{bmatrix} i_{p_{\sigma_1(1)}} & j_{s_1} \\ . & . \\ i_{p_{\sigma_1(\lambda_1)}} & j_{s_{\lambda_1}} \\ \vdots & \vdots \\ i_{r_{\sigma_m(1)}} & j_{v_1} \\ . & . \\ i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}} \end{bmatrix}$$

$$
= \sum_{\sigma_1,\ldots,\sigma_m} (-1)^{\sum_{k=1}^{m} |\sigma_k|}
\begin{bmatrix}
i_{p_1} & j_{s_{\sigma_1(1)}} \\
\cdot & \cdot \\
i_{p_{\lambda_1}} & j_{s_{\sigma_1(\lambda_1)}} \\
\vdots & \vdots \\
i_{r_1} & j_{v_{\sigma_m(1)}} \\
\cdot & \cdot \\
i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}}
\end{bmatrix},
$$

where

$$
\begin{bmatrix}
i_{p_{\sigma_1(1)}} & j_{s_1} \\
\cdot & \cdot \\
i_{p_{\sigma_1(\lambda_1)}} & j_{s_{\lambda_1}} \\
\vdots & \vdots \\
i_{r_{\sigma_m(1)}} & j_{v_1} \\
\cdot & \cdot \\
i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}}
\end{bmatrix},
\qquad
\begin{bmatrix}
i_{p_1} & j_{s_{\sigma_1(1)}} \\
\cdot & \cdot \\
i_{p_{\lambda_1}} & j_{s_{\sigma_1(\lambda_1)}} \\
\vdots & \vdots \\
i_{r_1} & j_{v_{\sigma_m(1)}} \\
\cdot & \cdot \\
i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}}
\end{bmatrix}
$$

are column Capelli bitableaux *in* $\mathbf{U}(gl(n))$ (see e.g. [18], [17]).

2.2.4 Young-Capelli bitableaux in $\mathbf{U}(gl(n))$

Given a pair of Young tableaux S, T of the same shape λ on the proper alphabet $\mathcal{L}$, consider the elements

$$
e_{S,C_\lambda^*}\, e_{C_\lambda^*,D_\lambda^*}\, e_{D_\lambda^*,T} \in \mathbf{U}(gl(m_0|m_1+n)).
$$

Since these elements are *balanced monomials* in $\mathbf{U}(gl(m_0|m_1+n))$, then they belong to the *virtual subalgebra* $Virt(m_0 + m_1, n)$.

Hence, we can consider their images in $\mathbf{U}(gl(n))$ with respect to the Capelli epimorphism $\mathfrak{p}$.

We set

$$
\mathfrak{p}\Big(e_{S,C_\lambda^*}\, e_{C_\lambda^*,D_\lambda^*}\, e_{D_\lambda^*,T}\Big) = [S\|\boxed{T}\,] \in \mathbf{U}(gl(n)). \tag{10}
$$

and call the element $[S\|\boxed{T}\,]$ a *Young-Capelli bitableau.*

The elements defined in (10) do not depend on the choice of the virtual Deruyts and Coderuyts tableaux D_λ^* and C_λ^*.

In plain words, the Young-Capelli bitableau $[S\|\boxed{T}\,]$ is obtained from the Capelli bitableau $[S|T]$ by adding a *column symmetrization* on the right Young tableau T. Indeed, we have

Proposition 2.13. *Any Young-Capelli bitableau equals the sum of Capelli bitableaux:*

$$
[S\|\boxed{T}\,] = \sum [S|\overline{T}],
$$

where the sum is extended to all Young tableaux $\overline{T}$ obtained from T by permutations of the elements of each column.

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2.2.5   Double Young-Capelli bitableaux in $\mathbf{U}(gl(n))$

Given a pair of Young tableaux S, T of the same shape λ on the proper alphabet $\mathcal{L}$ consider the elements

$$e_{S,C_\lambda^*} \cdot e_{C_\lambda^*,D_\lambda^*} \cdot e_{D_\lambda^*,C_\lambda^*} \cdot e_{C_\lambda^*,T} \in \mathbf{U}(gl(m_0|m_1+n)).$$

Since these elements are *balanced monomials* in $\mathbf{U}(gl(m_0|m_1+n))$, then they belong to the *virtual subalgebra* $Virt(m_0 + m_1, n)$.

Hence, we can consider their images in $\mathbf{U}(gl(n))$ with respect to the Capelli epimorphism $\mathfrak{p}$.

We set

$$\mathfrak{p}\Big(e_{S,C_\lambda^*} \cdot e_{C_\lambda^*,D_\lambda^*} \cdot e_{D_\lambda^*,C_\lambda^*} \cdot e_{C_\lambda^*,T}\Big) = [\, \boxed{S \mid T} \,] \in \mathbf{U}(gl(n)). \qquad (11)$$

and call the element $[\, \boxed{S \mid T} \,]$ a *double Young-Capelli bitableau*.

The elements defined in (11) do not depend on the choice of the virtual Deruyts and Coderuyts tableaux D_λ^* and C_λ^*.

In plain words, the double Young-Capelli bitableau $[\, \boxed{S \mid T} \,]$ is obtained from the Young-Capelli bitableau $[S\boxed{\boxed{T}}$ by adding a further *row skew symmetrization*. Indeed, we have

Proposition 2.14. *Any double Young-Capelli bitableau equals the sum of Young-Capelli bitableaux:*

$$[\, \boxed{S \mid T} \,] = (-1)^{\binom{h}{2}} \sum_\sigma (-1)^{|\sigma|} \, [S\boxed{\boxed{T^\sigma}},$$

where the sum is extended to all Young tableaux T^σ obtained from T by permutations of the elements of each row, and $(-1)^{|\sigma|}$ is the product of the signatures of row permutations.

2.3   Bitableaux in $\mathbb{C}[M_{m_0|m_1+n,d}]$ and the standard monomial theory

2.3.1   Biproducts in $\mathbb{C}[M_{m_0|m_1+n,d}]$

Embed the algebra

$$\mathbb{C}[M_{m_0|m_1+n,d}] = \mathbb{C}[(\alpha_s|j), (\beta_t|j), (i|j)]$$

into the (supersymmetric) algebra $\mathbb{C}[(\alpha_s|j), (\beta_t|j), (i|j), (\gamma|j)]$ generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j), (\beta_t|j), (i|j), (\gamma|j)$, $j = 1, 2, \ldots, d$, where

$$|(\gamma|j)| = 1 \in \mathbb{Z}_2 \quad for\ every\ j = 1, 2, \ldots, d,$$

and denote by $D_{z_i,\gamma}^l$ the superpolarization of γ to z_i.

Let $\omega = z_1 z_2 \cdots z_p$ be a word on $\mathcal{A}_0 \cup \mathcal{A}_c up \mathcal{L}$, and $\varpi = j_{t_1} j_{t_2} \cdots j_{t_q}$ a word on the alphabet $P = \{1, 2, \ldots, d\}$. The *biproduct*

$$(\omega | \varpi) = (z_1 z_2 \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$$

is the element

$$D^l_{z_1, \gamma} D_{z_2, \gamma} \cdots D^l_{z_p, \gamma} \Big((\gamma | j_{t_1})(\gamma | j_{t_2}) \cdots (\gamma | j_{t_q}) \Big) \in \mathbb{C}[M_{m_0 | m_1 + n, d}]$$

if $p = q$ and is set to be zero otherwise.

Claim 2.15. *The biproduct $(\omega | \varpi) = (z_1 z_2 \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$ is supersymmetric in the z's and skew-symmetric in the j's. In symbols*

1. $(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q}) =$
$$(-1)^{|z_i||z_{i+1}|}(z_1 z_2 \cdots z_{i+1} z_i \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_q})$$

2. $(z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} j_{t_2} \cdots j_{t_i} j_{t_{i+1}} \cdots j_{t_q}) =$
$$- (z_1 z_2 \cdots z_i z_{i+1} \cdots z_p | j_{t_1} \cdots j_{t_{i+1}} j_{t_i} \cdots j_{t_q}).$$

Proposition 2.16. *(Laplace expansions)* We have

1. $(\omega_1 \omega_2 | \varpi) = \Sigma_{(\varpi)} \ (-1)^{|\varpi_{(1)}||\omega_2|} \ (\omega_1 | \varpi_{(1)})(\omega_2 | \varpi_{(2)}).$

2. $(\omega | \varpi_1 \varpi_2) = \Sigma_{(\omega)} \ (-1)^{|\varpi_1||\omega_{(2)}|} \ (\omega_{(1)} | \varpi_1)(\omega_{(2)} | \varpi_2.)$

where

$$\triangle(\varpi) = \Sigma_{(\varpi)} \ \varpi_{(1)} \otimes \varpi_{(2)}, \quad \triangle(\omega) = \Sigma_{(\omega)} \ \omega_{(1)} \otimes \omega_{(2)}$$

denote the coproducts in the Sweedler notation (see, e.g. [1]) of the elements ϖ and ω in the supersymmetric Hopf algebra of W (see, e.g. [8]) and in the free exterior Hopf algebra generated by $j = 1, 2, \ldots, d$, respectively.

Example 2.17. Let $\omega = \alpha_1 \alpha_2 3$, $\varpi = 123$, where $|(\alpha_1 | j)| = |(\alpha_2 | j)| = 1$, $j = 1, 2, 3$ and $|(3 | j)| = 0$, $j = 1, 2, 3$. Then

$$(\omega | \varpi) = D^l_{\alpha_1, \gamma} D^l_{\alpha_2, \gamma} D^l_{3, \gamma} \big((\gamma | 1)(\gamma | 2)(\gamma | 3) \big)$$

$$= D_{\alpha_1, \gamma} D_{\alpha_2, \gamma} \Big((3 | 1)(\gamma | 2)(\gamma | 3) - (\gamma | 1)(3 | 2)(\gamma | 3) + (\gamma | 1)(\gamma | 2)(3 | 3) \Big)$$

$$= D_{\alpha_1, \gamma} \Big((3 | 1)(\alpha_2 | 2)(\gamma | 3) + (3 | 1)(\gamma | 2)(\alpha_2 | 3) - (\alpha_2 | 1)(3 | 2)(\gamma | 3)$$
$$- (\gamma | 1)(3 | 2)(\alpha_2 | 3) + (\alpha_2 | 1)(\gamma | 2)(3 | 3) + (\gamma | 1)(\alpha_2 | 2)(3 | 3) \Big)$$

$$= (3 | 1)(\alpha_2 | 2)(\alpha_1 | 3) + (3 | 1)(\alpha_1 | 2)(\alpha_2 | 3) - (\alpha_2 | 1)(3 | 2)(\alpha_1 | 3)$$
$$- (\alpha_1 | 1)(3 | 2)(\alpha_2 | 3) + (\alpha_2 | 1)(\alpha_1 | 2)(3 | 3) + (\alpha_1 | 1)(\alpha_2 | 2)(3 | 3).$$

From Proposition 2.16.1, by setting $\varpi_1 = 12$, $\varpi_2 = 3$, it follows

$$(\omega | \varpi) = (\alpha_1 \alpha_2 | 12)(3 | 3) + (\alpha_1 3 | 12)(\alpha_2 | 3) + (\alpha_2 3 | 12)(\alpha_1 | 3).$$

From Proposition 2.16.2, by setting $\omega_1 = \alpha_1 \alpha_2$, $\omega_2 = 3$, it follows

$$(\omega | \varpi) = (\alpha_1 \alpha_2 | 12)(3 | 3) - (\alpha_1 \alpha_2 | 13)(3 | 2) + (\alpha_1 \alpha_2 | 23)(3 | 1).$$

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2.3.2 Biproducts in $\mathbb{C}[M_{n,d}]$

Let $\omega = i_1 i_2 \cdots i_p$, $\varpi = j_1 j_1 \cdots j_p$ be words on the negative alphabet $\mathcal{L} = \{1, 2, \ldots, n\}$ and on the negative alphabet $\mathcal{P} = \{1, 2, \ldots, d\}$.

From Proposition 2.16, we infer

Corollary 2.18. *The biproduct of the two words ω and ϖ*

$$(\omega|\varpi) = (i_1 i_2 \cdots i_p | j_1 j_2 \cdots j_p) \tag{12}$$

is the signed minor*:*

$$(\omega|\varpi) = (-1)^{\binom{p}{2}} \, det\Big(\, (i_r|j_s) \, \Big)_{r,s=1,2,\ldots,p} \in \mathbb{C}[M_{n,d}].$$

2.3.3 Biproducts and polarization operators

Following the notation introduced in the previous sections, let

$$Super[W] = Sym[W_0] \otimes \Lambda[W_1]$$

denote the *(super)symmetric* algebra of the space

$$W = W_0 \oplus W_1$$

(see, e.g. [59]).

By multilinearity, the algebra $Super[W]$ is the same as the superalgebra $Super[\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}]$ generated by the "variables"

$$\alpha_1, \ldots, \alpha_{m_0} \in \mathcal{A}_0, \quad \beta_1, \ldots, \beta_{m_1} \in A_1, \quad x_1, \ldots, x_n \in L,$$

modulo the congruences

$$zz' = (-1)^{|z||z'|} z'z, \quad z, z' \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}.$$

Let $d^l_{z,z'}$ denote the (left)polarization operator of z' to z on

$$Super[W] = Super[\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}],$$

that is the unique superderivation of $\mathbb{Z}_2$-degree

$$|z| + |z'| \in \mathbb{Z}_2$$

such that

$$d^l_{z,z'}(z'') = \delta_{z',z''} \cdot z,$$

for every $z, z', z'' \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$.

Clearly, the map

$$e_{z,z'} \to d^l_{z,z'}$$

is a Lie superalgebra map and, therefore, induces a structure of

$$gl(m_0|m_1 + n) - module$$

on $Super[\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}] = Super[W]$.

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Proposition 2.19. *Let $\varpi = j_{t_1} j_{t_2} \cdots j_{t_q}$ be a word on $P = \{1, 2, \ldots, d\}$. The map*

$$\Phi_\varpi : \omega \mapsto (\omega | \varpi),$$

ω any word on $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$, uniquely defines $gl(m_0|m_1 + n)-$equivariant linear operator

$$\Phi_\varpi : Super[\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}] \to \mathbb{C}[M_{m_0|m_1+n,d}],$$

that is

$$\Phi_\varpi \big(e_{z,z'} \cdot \omega\big) = \Phi_\varpi \big(d^l_{z,z'}(\omega)\big) = D^l_{z,z'} \big((\omega | \varpi)\big) = e_{z,z'} \cdot (\omega | \varpi), \qquad (13)$$

for every $z, z' \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$.

With a slight abuse of notation, we will write (13) in the form

$$D^l_{z,z'} \big((\omega | \varpi)\big) = (d^l_{z,z'}(\omega) | \varpi). \qquad (14)$$

2.3.4   Bitableaux in $\mathbb{C}[M_{m_0|m_1+n,d}]$

Let $S = (\omega_1, \omega_2, \ldots, \omega_p$ and $T = (\varpi_1, \varpi_2, \ldots, \varpi_p)$ be Young tableaux on $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$ and $P = \{1, 2, \ldots, d\}$ of shapes λ and μ, respectively.

If $\lambda = \mu$, the *Young bitableau $(S|T)$* is the element of $\mathbb{C}[M_{m_0|m_1+n,d}]$ defined as follows:

$$(S|T) = \begin{pmatrix} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{pmatrix} = \pm (\omega_1)|\varpi_1)(\omega_2)|\varpi_2) \cdots (\omega_p)|\varpi_p),$$

where

$$\pm = (-1)^{|\omega_2||\varpi_1| + |\omega_3|(|\varpi_1| + |\varpi_2|) + \cdots + |\omega_p|(|\varpi_1| + |\varpi_2| + \cdots + |\varpi_{p-1}|)}.$$

If $\lambda \neq \mu$, the *Young bitableau $(S|T)$* is set to be zero.

2.3.5   Bitableaux and polarization operators

By naturally extending the slight abuse of notation (14), the action of any polarization on bitableaux can be explicitly described:

Proposition 2.20. *Let $z, z' \in \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$, and let $S = (\omega_1, \ldots, \omega_p)$, $T =$*

$(\varpi_1, \ldots, \varpi_p)$. *We have the following identity:*

$$e_{z,z'} \cdot (S \,|\, T) \;=\; D^l_{z,z'} \left(\begin{pmatrix} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{pmatrix} \right)$$

$$= \sum_{s=1}^{p} (-1)^{(|z|+|z'|)\epsilon_s} \begin{pmatrix} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ d^l_{z,z'}(\omega_s) & \vdots \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{pmatrix},$$

where

$$\epsilon_1 = 1, \quad \epsilon_s = |\omega_1| + \cdots + |\omega_{s-1}|, \quad s = 2, \ldots, p.$$

Example 2.21. Let $\alpha_i \in \mathcal{A}_0$, $1,2,3,4 \in L$, $|D_{\alpha_i,2}| = 1$. Then

$$e_{\alpha_i,2} \cdot \begin{pmatrix} 1\,3\,2 & 1\,2\,3 \\ 2\,3 & 2\,3 \\ 4\,2 & 3\,1 \end{pmatrix} = D^l_{\alpha_i,2} \left(\begin{pmatrix} 1\,3\,2 & 1\,2\,3 \\ 2\,3 & 2\,3 \\ 4\,2 & 3\,1 \end{pmatrix} \right) =$$

$$= \begin{pmatrix} 1\,3\,\alpha_i & 1\,2\,3 \\ 2\,3 & 2\,3 \\ 4\,2 & 3\,1 \end{pmatrix} - \begin{pmatrix} 1\,3\,2 & 1\,2\,3 \\ \alpha_i\,3 & 2\,3 \\ 4\,2 & 3\,1 \end{pmatrix} + \begin{pmatrix} 1\,3\,2 & 1\,2\,3 \\ 2\,3 & 2\,3 \\ 4\,\alpha_i & 3\,1 \end{pmatrix}.$$

2.3.6 The straightening algorithm and the standard basis theorem for $\mathbb{C}[M_{m_0|m_1+n,d}]$

Consider the set of all bitableaux $(S|T) \in \mathbb{C}[M_{m_0|m_1+n,d}]$, where $sh(S) = sh(T) \vdash h$, h a given positive integer. In the following, let denote by $\leq$ the partial order on this set defined by the following two steps:

1. $(S|T) < (S'|T')$ whenever $sh(S) <_l sh(S')$,

2. $(S|T) < (S'|T')$ whenever $sh(S) = sh(S')$, $w(S) >_l w(S')$, $w(T) >_l w(T')$,

where the shapes and the row-words are compared in the lexicographic order.

The next results are superalgebraic versions of classical, well-known results for the symmetric algebra $\mathbb{C}[M_{n,d}]$ ([30], [27], [25], for the general theory of standard monomials see, e.g. [54], Chapt. 13) and of their skew-symmetric analogues ([29], [2]).

Theorem 2.22. *(The straightening algorithm)* [34]

Let $(P|Q) \in \mathbb{C}[M_{m_0|m_1+n,d}]$. Then $(P|Q)$ can be written as a linear combination, with rational coefficients,

$$(P|Q) = \sum_{S,T} c_{S,T} \ (S|T), \tag{15}$$

of standard bitableaux $(S|T)$, where $(S|T) \geq (P|Q)$ and $c_S = c_P$, $c_T = c_Q$.

Since standard bitableaux are linearly independent in $\mathbb{C}[M_{m_0|m_1+n,d}]$, the expansion (15) is unique.

Following [4], [5], [11], a partition λ satisfies the $(m_0, m_1+n)-hook\ condition$ (in symbols, $\lambda \in H(m_0, m_1 + n)$) if and only if $\lambda_{m_0+1} \leq m_1 + n$. We have:

Lemma 2.23. *There exists a standard tableau on $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$ of shape λ if and only if $\lambda \in H(m_0, m_1 + n)$.*

Given a positive integer $h \in \mathbb{Z}^+$, let $\mathbb{C}_h[M_{m_0|m_1+n,d}]$ denote the $h-$th homogeneous component of $\mathbb{C}[M_{m_0|m_1+n,d}]$.

From Theorem 2.22, it follows

Corollary 2.24. *(The Standard basis theorem for $\mathbb{C}_h[M_{m_0|m_1+n,d}]$, [34])*
The following set is a basis of $\mathbb{C}_h[M_{m_0|m_1+n,d}]$:

$$\{(S|T)\ standard;\ sh(S) = sh(T) = \lambda \vdash h, \lambda \in H(m_0, m_1 + n), \lambda_1 \leq d\ \}.$$

3 The Schur (covariant) modules and supermodules

3.1 The Schur $\mathbf{U}(gl(m_0|m_1 + n))$-supermodules as submodules of $\mathbb{C}[M_{m_0|m_1+n,d}]$

Given $\lambda \in H(m_0, m_1 + n)$, the *Schur supermodule $Schur_\lambda(m_0, m_1 + n)$* is the subspace of $\mathbb{C}[M_{m_0|m_1+n,d}]$, $d \geq \lambda_1$, spanned by the set of all bitableaux $(S|D_\lambda^P)$ of shape λ, where S is a Young tableau on the alphabet $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{L}$, and D_λ^P is the *Deruyts* tableau on $P = \{1, 2, \ldots, d\}$

$$D_\lambda^P = \begin{pmatrix} 1\ 2\ 3 \ldots \ldots & \lambda_1 \\ 1\ 2\ 3 \ldots \lambda_2 & \\ \ldots \ldots & \\ 1\ 2\ 3 \ldots \lambda_p & \end{pmatrix}, \quad p = l(\lambda).$$

From Corollary 2.24 and the *straightening algorithm* (see, e.g. [8]), it follows

Proposition 3.1. *The set*

$$\left\{ (S|D_\lambda^P);\ S\ superstandard \right\}$$

is a $\mathbb{C}$-linear basis of $Schur_\lambda(m_0, m_1 + n)$.

Furthermore, we recall

Proposition 3.2. ([11], [8]) *The submodule $Schur_\lambda(m_0, m_1 + n)$ is an irreducible $\mathbf{U}(gl(m_0|m_1 + n))$-submodule of $\mathbb{C}[M_{m_0|m_1+n,d}]$, with highest weight*

$$(\lambda_1, \ldots, \lambda_{m_0}; \ \widetilde{\lambda}_1 - m_0, \widetilde{\lambda}_2 - m_0, \ldots).$$

3.2 The classical Schur $\mathbf{U}(gl(n))$-modules

Given λ such that $\lambda_1 \leq n$, the *Schur module $Schur_\lambda(n)$* is the subspace of $\mathbb{C}[M_{n,d}]$, $d \geq \lambda_1$, spanned by the set of all bitableaux $(X|D_\lambda^P)$ of shape λ and X is a Young tableau on the alphabet L.

Proposition 3.3. *The set*

$$\left\{ (X|D_\lambda^P); X \ standard \right\}$$

is a $\mathbb{C}$-linear basis of $Schur_\lambda(n)$.

Furthermore, $Schur_\lambda(n)$ is an irreducible $\mathbf{U}(gl(n))$-submodule of $\mathbb{C}[M_{n,d}]$, with highest weight $\widetilde{\lambda}$.

Let

$$D_\lambda = \begin{pmatrix} 1\ 2\ 3 \ldots \ldots \quad \lambda_1 \\ \\ 1\ 2\ 3 \ldots \lambda_2 \\ \ldots \ldots \\ 1\ 2\ 3 \ldots \lambda_p \end{pmatrix}$$

denote the (proper) Deruyts tableau on the alphabet $\mathcal{L} = \{1, 2, \ldots, n\}$. The bitableau

$$v_{\widetilde{\lambda}} = (D_\lambda|D_\lambda^P)$$

is the "canonical" highest weight vector of the irreducible $gl(n)$-module $Schur_\lambda(n)$ with highest weight $\widetilde{\lambda}$.

3.3 The classical Schur modules as $gl(n)$-submodules of Schur supermodules

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition such that $\lambda_1 \leq n$.

Consider a Schur supermodule

$$Schur_\lambda(m_0, m_1 + n)$$

(clearly, $\lambda \in H(m_0, m_1 + n)$, for every m_0, m_1).

The Schur module $Schur_\lambda(n)$ can be regarded as a $\mathbf{U}(gl(n))$-submodule of the $\mathbf{U}(gl(m_0|m_1 + n))$-supermodule $Schur_\lambda(m_0, m_1 + n)$.

Let $\mathfrak{p}$ be the Capelli epimorphism

$$\mathfrak{p} : Virt(m_0 + m_1, n) \twoheadrightarrow \mathbf{U}(gl(n)), \qquad Ker(\mathfrak{p}) = \mathbf{Irr}.$$

Proposition 3.4. *The Schur module $Schur_\lambda(n)$ is invariant (as a subspace of $Schur_\lambda(m_0, m_1 + n)$) with respect to the action of the subalgebra*

$$Virt(m_0 + m_1, n) \subset \mathbf{U}(gl(m_0|m_1 + n)).$$

From Proposition 2.7, we infer

Proposition 3.5. *For every element $\mathbf{M} \in Virt(m_0 + m_1, n)$, the action of $\mathbf{M}$ on the Schur module $Schur_\lambda(n)$ is the same of the action of its image $\mathfrak{p}(\mathbf{M}) = \mathbf{m} \in \mathbf{U}(gl(n))$.*

3.4 The action of double Young-Capelli bitableaux on highest weight vectors of Schur modules

Let's start with some lemmas.

In the following, given partitions λ, μ and their conjugates $\widetilde{\lambda}$ and $\widetilde{\mu}$, we assume that

$$m_0 \geq \lambda_1, \ \mu_1, \qquad m_1, d \geq \widetilde{\lambda}_1, \ \widetilde{\mu}_1.$$

.

Let $v_\mu = (D_{\widetilde{\mu}}|D_{\widetilde{\mu}}^P)$ be the "canonical" highest weight vector of weight μ of the irreducible $gl(n)$-module $Schur_{\widetilde{\mu}}$.

Lemma 3.6. *We have*

$$\text{If } |\widetilde{\mu}| < |\widetilde{\lambda}|, \ \text{then} \qquad\qquad e_{C_{\widetilde{\lambda}}^*, S} \cdot (D_{\widetilde{\mu}}|D_{\widetilde{\mu}}^P) = 0, \quad \forall S \qquad (16)$$

$$\text{If } |\widetilde{\mu}| = |\widetilde{\lambda}|, \ \widetilde{\mu} \neq \widetilde{\lambda}, \ \text{then} \qquad e_{D_{\widetilde{\lambda}}^*, C_{\widetilde{\lambda}}^*} e_{C_{\widetilde{\lambda}}^*, S} \cdot (D_{\widetilde{\mu}}|D_{\widetilde{\mu}}^P) = 0, \quad \forall S. \qquad (17)$$

The assertions of eqs. (16), (17) are special cases of standard elementary facts of the method of virtual variables (see, e.g. [8]).

Lemma 3.7. *If $\widetilde{\lambda} \not\subseteq \widetilde{\mu}$, then*

$$e_{D_{\widetilde{\lambda}}^*, C_{\widetilde{\lambda}}^*} e_{C_{\widetilde{\lambda}}^*, S} \cdot (D_{\widetilde{\mu}}|D_{\widetilde{\mu}}^P) = 0, \quad \forall S.$$

Proof. Assume that $|\widetilde{\mu}| \geq |\widetilde{\lambda}|$ to avoid trivial cases (by eq. (16)). The action $e_{C_{\widetilde{\lambda}}^*, S} \cdot (D_{\widetilde{\mu}}|D_{\widetilde{\mu}}^P)$ produces a linear combination of bitableaux $(T|D_{\widetilde{\mu}}^P) \in Schur_{\widetilde{\mu}}(m_0, m_1 + n)$, where each tableau T contains exactly $\widetilde{\lambda}_i$ occurrences of the positive virtual symbols $\alpha_i \in \mathcal{A}_0$. By *straightening* each of them *(see, e.g. [8])*, the element $e_{C_{\widetilde{\lambda}}^*, S} \cdot (D_{\widetilde{\mu}}|D_{\widetilde{\mu}}^P)$ is uniquely expressed as a linear combination of (super)standard tableaux

$$e_{C_{\widetilde{\lambda}}^*, S} \cdot (D_{\widetilde{\mu}}|D_{\widetilde{\mu}}^P) = \sum_i (S_i|D_{\widetilde{\mu}}^P) \in Schur_{\widetilde{\mu}}(m_0, m_1 + n), \qquad (18)$$

where in each S_i the positive virtual symbols $\alpha_i \in \mathcal{A}_0$ occupies a subshape $\widetilde{\lambda}' \subseteq \widetilde{\mu}$ such that $\widetilde{\lambda}' \trianglerighteq \widetilde{\lambda}$. If $\widetilde{\lambda} \not\subseteq \widetilde{\mu}$, any element $(S_i|D_{\widetilde{\lambda}}^P)$ in the canonical form (18) is such that $\widetilde{\lambda}' \trianglerighteq \widetilde{\lambda}$, $\widetilde{\lambda}' \neq \widetilde{\lambda}$. Then $e_{D_{\widetilde{\lambda}}^*, C_{\widetilde{\lambda}}^*} \cdot (S_i|D_{\widetilde{\mu}}^P) = 0$, by skew-symmetry, and the assertion follows. $\square$

We recall that, given a shape/partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, the *hook length* $H(x)$ of a box x in the Ferrers diagram F_λ of the shape λ is the number of boxes that are in the same row to the right of it plus those boxes in the same column below it, plus one (for the box itself). The *hook number* the shape λ is the product $H(\lambda) = \prod_{x \in F_\lambda} H(x)$. Furthermore, we write $\lambda!$ for the product $\lambda_1! \lambda_2! \cdots \lambda_p!$.

Lemma 3.8. *(Regonati's Hook Lemma, [55]) Let $H(\lambda) = H(\widetilde{\lambda})$ denotes the hook number of the shape/partition λ. We have*

$$e_{C^*_{\widetilde{\lambda}}, D_{\widetilde{\lambda}}} \cdot v_\lambda = e_{C^*_{\widetilde{\lambda}}, D_{\widetilde{\lambda}}} \cdot (D_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}}) \tag{19}$$

$$= (-1)^{\binom{k}{2}} \frac{H(\widetilde{\lambda})}{\widetilde{\lambda}!} \ (C^*_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}}) \tag{20}$$

and

$$e_{C^*_{\widetilde{\lambda}}, D^*_{\widetilde{\lambda}}} \cdot (D^*_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}}) = (-1)^{\binom{k}{2}} \frac{H(\widetilde{\lambda})}{\widetilde{\lambda}!} \ (C^*_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}}). \tag{21}$$

Furthermore

$$e_{D^*_{\widetilde{\lambda}}, C^*_{\widetilde{\lambda}}} \cdot (C^*_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}}) = \ \widetilde{\lambda}! \ (D^*_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}}). \tag{22}$$

Theorem 3.9. *We have:*

1. *If $sh(S) = sh(S') = \widetilde{\lambda}$, $|\widetilde{\mu}| < |\widetilde{\lambda}|$, then*

$$[\ \boxed{S' \mid S}\ ] \cdot v_\mu = 0,$$

2. *If $sh(S) = sh(S') = \widetilde{\lambda}$, $|\widetilde{\mu}| = |\widetilde{\lambda}|$, $\widetilde{\mu} \neq \widetilde{\lambda}$, then*

$$[\ \boxed{S' \mid S}\ ] \cdot v_\mu = 0,$$

3. *If $\widetilde{\mu} = \widetilde{\lambda}$, then*

$$[\ \boxed{D_{\widetilde{\lambda}} \mid D_{\widetilde{\lambda}}}\ ] \cdot v_\lambda = H(\widetilde{\lambda})^2 \ v_\lambda,$$

4. *If $sh(S) = sh(S') = \widetilde{\lambda}$, $\widetilde{\lambda} \nsubseteq \widetilde{\mu}$, then*

$$[\ \boxed{S' \mid S}\ ] \cdot v_\mu = 0.$$

Proof. Since

$$[\ \boxed{S' \mid S}\ ] = \mathfrak{p}\Big(e_{S', C^*_{\widetilde{\lambda}}} \cdot e_{C^*_{\widetilde{\lambda}}, D^*_{\widetilde{\lambda}}} \cdot e_{D^*_{\widetilde{\lambda}}, C^*_{\widetilde{\lambda}}} \cdot e_{C^*_{\widetilde{\lambda}}, S}\Big) \in \mathbf{U}(gl(n)),$$

then

$$[\ \boxed{S' \mid S}\ ] \cdot v_\mu = e_{S' C^*_{\widetilde{\lambda}}} \cdot e_{C^*_{\widetilde{\lambda}}, D^*_{\widetilde{\lambda}}} \cdot e_{D^*_{\widetilde{\lambda}}, C^*_{\widetilde{\lambda}}} \cdot e_{C^*_{\widetilde{\lambda}}, S} \cdot v_\mu,$$

from Proposition 2.7.

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Hence, item 1) follows from eq. (16), item 2) follows from eq. (17) and item 4) follows from Lemma 3.6.

Then, we prove item 3). We have

$$[\ \boxed{D_{\widetilde{\lambda}} \mid D_{\widetilde{\lambda}}}\ ] \cdot v_\lambda = e_{D_{\widetilde{\lambda}}, C^*_{\widetilde{\lambda}}}\ e_{C^*_{\widetilde{\lambda}}, D^*_{\widetilde{\lambda}}}\ e_{D^*_{\widetilde{\lambda}}, C^*_{\widetilde{\lambda}}}\ e_{C^*_{\widetilde{\lambda}}, D_{\widetilde{\lambda}}} \cdot (D_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}}).$$

From eq. (20), this equals

$$\frac{1}{\widetilde{\lambda}!}(-1)^{\binom{k}{2}} H(\lambda)\ e_{D_\lambda, C^*_{\widetilde{\lambda}}}\ e_{C^*_{\widetilde{\lambda}}, D^*_{\widetilde{\lambda}}}\ e_{D^*_{\widetilde{\lambda}}, C^*_{\widetilde{\lambda}}} \cdot (C^*_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}});$$

from eq. (22) this equals

$$(-1)^{\binom{k}{2}} H(\widetilde{\lambda})\ e_{D_{\widetilde{\lambda}}, C^*_{\widetilde{\lambda}}}\ e_{C^*_{\widetilde{\lambda}}, D^*_{\widetilde{\lambda}}} \cdot (D^*_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}});$$

from eq. (21) this equals

$$\begin{aligned}
&= (-1)^{\binom{k}{2}} H(\widetilde{\lambda})\frac{1}{\widetilde{\lambda}!}(-1)^{\binom{k}{2}}\ e_{D_{\widetilde{\lambda}}, C^*_{\widetilde{\lambda}}} \cdot (C^*_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}})\\
&= H(\widetilde{\lambda})^2\ (D_{\widetilde{\lambda}} | D^P_{\widetilde{\lambda}}) = H(\widetilde{\lambda})^2\ v_\lambda.
\end{aligned}$$

$\square$

4 The center $\boldsymbol{\zeta}(n)$ of $\mathbf{U}(gl(n))$

4.1 The Schur elements $\mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}(n)$

Let μ be a partition, let $\tilde{\mu}$ be its conjugate partition. Assume $\mu_1 \leq n$, and $m_1 \geq \mu_1$, $m_0 \geq \widetilde{\mu}_1$; hence, the virtual Deruyts tableau D^*_μ and the virtual Coderuyts tableau C^*_μ can be constructed. Let S_1, S_2 be tableaux on the proper alphabet $\mathcal{L} = \{1, 2, \ldots, n\}$ of shape μ. We notice that any element

$$e_{S_1, C^*_\mu} \cdot e_{C^*_\mu, D^*_\mu} \cdot e_{D^*_\mu, C^*_\mu} \cdot e_{C^*_\mu, S_2} \in Virt(m_0 + m_1, n),$$

is *skew-symmetric* in the rows of S_1 and S_2, respectively.

Given a partition λ, assume $\tilde{\lambda}_1 \leq n$, $m_1 \geq \tilde{\lambda}_1$, $m_0 \geq \lambda_1$. We set

$$\mathbf{S}_\lambda(n) = \frac{1}{H(\tilde{\lambda})}\ \sum_S\ \mathfrak{p}\Big(e_{S, C^*_{\tilde{\lambda}^*}}\, e_{C^*_{\tilde{\lambda}}, D^*_{\tilde{\lambda}}} \cdot e_{D^*_{\tilde{\lambda}}, C^*_{\tilde{\lambda}}} \cdot e_{C^*_{\tilde{\lambda}}, S}\Big) \qquad (23)$$

$$= \frac{1}{H(\tilde{\lambda})}\ \sum_S\ [\ \boxed{S \mid S}\ ] \in \mathbf{U}(gl(n)), \qquad (24)$$

where the sum is extended to all row (strictly) increasing tableaux S of shape $\tilde{\lambda}$ on the proper alphabet $\mathcal{L} = \{1, 2, \ldots, n\}$. Notice that $H(\tilde{\lambda}) = H(\lambda)$.

By convention, if λ is the empty partition, we set $\mathbf{S}_\emptyset(n) = \mathbf{1} \in \boldsymbol{\zeta}(n)$.

The element $\mathbf{S}_\lambda(n) \in \mathbf{U}(gl(n))$ is called the *Schur element* of *weight* λ (and shape $\tilde{\lambda}$) in dimension n.

Theorem 4.1. *The Schur elements* $\mathbf{S}_\lambda(n)$ *are central in* $\mathbf{U}(gl(n))$.

Proof. Consider the element

$$\sum_S e_{S,C^*_{\widetilde{\lambda}}} \cdot e_{C^*_{\widetilde{\lambda}},D^*_{\widetilde{\lambda}}} \cdot e_{D^*_{\widetilde{\lambda}},C^*_{\widetilde{\lambda}}} \cdot e_{C^*_{\widetilde{\lambda}},S} \in Virt(m_0 + m_1, n),$$

where the sum is extended to all row (strictly) increasing tableaux S on the proper alphabet $\mathcal{L} = \{1, 2, \ldots, n\}$.

Since the adjoint representation acts by derivation, we have

$$ad(e_{ij})\big(\sum_S e_{S,C^*_{\widetilde{\lambda}}} \cdot e_{C^*_{\widetilde{\lambda}},D^*_{\widetilde{\lambda}}} \cdot e_{D^*_{\widetilde{\lambda}},C^*_{\widetilde{\lambda}}} \cdot e_{C^*_{\widetilde{\lambda}},S}\big) = 0,$$

for every $e_{ij} \in gl(n)$. Hence, the assertion follows from Corollary 2.10. $\qquad\square$

Let $\boldsymbol{\zeta}(n)^{(m)}$ denote the m-th filtration element of $\boldsymbol{\zeta}(n)$ with respect to the filtration induced by the standard filtration of $\mathbf{U}(gl(n))$. Clearly,

$$\mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}(n)^{(m)}, \tag{25}$$

for every $m \geq |\lambda|$.

Theorem 4.2. *(Triangularity/orthogonality of the actions on highest weight vectors) We have:*

$$\text{If } |\mu| < |\lambda|, \text{ then} \qquad \mathbf{S}_\lambda(n) \cdot v_\mu = 0, \tag{26}$$

$$\text{If } |\mu| = |\lambda|, \text{ then} \qquad \mathbf{S}_\lambda(n) \cdot v_\mu = \delta_{\lambda,\mu}\, H(\lambda)\, v_\lambda. \tag{27}$$

Proof. The first assertion is an immediate consequence of Theorem 3.9, item 1). The fact that, if $|\mu| = |\lambda|$, $\mu \neq \lambda$, then $\mathbf{S}_\lambda(n) \cdot v_\mu = 0$, is an immediate consequence of Theorem 3.9, item 2).

We examine the case $\lambda = \mu$. The value

$$\mathbf{S}_\lambda(n) \cdot v_\lambda \overset{def}{=} \frac{1}{H(\widetilde{\lambda})} \sum_S \mathfrak{p}\big(e_{S,C^*_{\widetilde{\lambda}}}\, e_{C^*_{\widetilde{\lambda}},D^*_{\widetilde{\lambda}}}\, e_{D^*_{\widetilde{\lambda}},C^*_{\widetilde{\lambda}}} e_{C^*_{\widetilde{\lambda}},S}\big) \cdot (D_{\widetilde{\lambda}}|D_{\widetilde{\lambda}}^P)$$

equals

$$\frac{1}{H(\widetilde{\lambda})} \sum_S e_{S,C^*_{\widetilde{\lambda}}}\, e_{C^*_{\widetilde{\lambda}},D^*_{\widetilde{\lambda}}}\, e_{D^*_{\widetilde{\lambda}},C^*_{\widetilde{\lambda}}} e_{C^*_{\widetilde{\lambda}},S} \cdot (D_{\widetilde{\lambda}}|D_{\widetilde{\lambda}}^P),$$

by Proposition 2.7. Clearly, this reduces to

$$\frac{1}{H(\widetilde{\lambda})}\, e_{D_{\widetilde{\lambda}},C^*_{\widetilde{\lambda}}}\, e_{C^*_{\widetilde{\lambda}},D^*_{\widetilde{\lambda}}}\, e_{D^*_{\widetilde{\lambda}},C^*_{\widetilde{\lambda}}}\, e_{C^*_{\widetilde{\lambda}},D_{\widetilde{\lambda}}} \cdot (D_{\widetilde{\lambda}}|D_{\widetilde{\lambda}}^P).$$

This value equals

$$\frac{1}{H(\widetilde{\lambda})}\, [\,\boxed{D_{\widetilde{\lambda}} \mid D_{\widetilde{\lambda}}}\,]\cdot v_\lambda = H(\widetilde{\lambda})\, v_\lambda,$$

by item 3) of Theorem 3.9. $\qquad\square$

Theorem 4.3. *(Vanishing theorem) If $\lambda \nsubseteq \mu$, then* $\mathbf{S}_\lambda(n) \cdot v_\mu = 0$.

Proof. It is an immediate consequence of item 4) of Theorem 3.9. $\qquad\square$

Theorem 4.4. *For every $m \in \mathbb{Z}^+$, the set*

$$\left\{ \mathbf{S}_\lambda(n);\ \tilde{\lambda}_1 \leq n,\ |\lambda| \leq m \right\}$$

is a linear basis of $\boldsymbol{\zeta}(n)^{(m)}$.
  The set

$$\left\{ \mathbf{S}_\lambda(n);\ \tilde{\lambda}_1 \leq n \right\}$$

is a linear basis of the center $\boldsymbol{\zeta}(n)$.

4.2 The Sahi-Okounkov Characterization Theorem

We reword Theorem 4.2 in terms of the *Harish-Chandra isomorphism*

$$\chi_n : \boldsymbol{\zeta}(n) \longrightarrow \Lambda^*(n),$$

where $\Lambda^*(n)$ denotes the algebra of *shifted symmetric polynomials* in n variables (see Section 6 below).

Proposition 4.5. *Given λ, $\tilde{\lambda}_1 \leq n$ and μ, $\tilde{\mu}_1 \leq n$, we have:*

$$\begin{aligned}
\text{If } |\mu| < |\lambda|, \quad &\text{then} \quad & \chi_n\left(\mathbf{S}_\lambda(n)\right)(\mu) &= 0, \\
\text{If } |\mu| = |\lambda|, \quad &\text{then} \quad & \chi_n\left(\mathbf{S}_\lambda(n)\right)(\mu) &= \delta_{\lambda,\mu}\ H(\lambda).
\end{aligned}$$

We recall the *Sahi-Okounkov Characterization Theorem* for the *Schur shifted symmetric polynomials*

$$s^*_{\lambda|n}, \qquad \lambda_1 \leq n$$

(Theorem 1 of [57] and Theorem 3.3 of [51], see also [49]).

Proposition 4.6. *Given λ, $\widetilde{\lambda}_1 \leq n$, the polynomial $s^*_{\lambda|n}$ is the unique element of $\Lambda^*(n)$ such that $\deg s^*_{\lambda|n} \leq |\lambda|$ and*

$$s^*_{\lambda|n}(\mu) = \delta_{\lambda\mu}\ H(\lambda),$$

for all partitions μ such that $|\mu| \leq |\lambda|$ and $\mu_1 \leq n$.

From Propositions 4.5 and 4.6, we obtain

Corollary 4.7. *Given λ, $\tilde{\lambda}_1 \leq n$, we have*

$$\chi_n\left(\mathbf{S}_\lambda(n)\right) = s^*_{\lambda|n}.$$

It follows that the Schur elements $\mathbf{S}_\lambda(n)$ are the same as the *Okounkov quantum immanant* associated to λ ([49], see also [50] and [48]).

We recall that the Schur element/Okounkov quantum immanant $\mathbf{S}_\lambda(n)$ also admits a representation as linear combination of *Capelli immanants* (see [17], section 5)

$$
Cimm_\mu[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h] = \sum_{\sigma \in \mathbf{S}_h} \chi^\mu(\sigma) \begin{bmatrix} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{bmatrix}
$$

$$
= \sum_{\sigma \in \mathbf{S}_h} \chi^\mu(\sigma) \begin{bmatrix} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{bmatrix},
$$

where χ^μ denotes the *irreducible character* associated to the irreducible representation of shape μ of the symmetric group $\mathbf{S}_h$ and

$$
\begin{bmatrix} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{bmatrix}, \quad \begin{bmatrix} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{bmatrix}
$$

are *column Capelli bitableaux* (see, e.g. [17], [18]):

Proposition 4.8. ([17], Theorem 6.2) *Given λ, $\lambda_1 \leq n$, we have*

$$
\mathbf{S}_\lambda(n) = (-1)^{\binom{h}{2}} \sum_{h_1 + \cdots + h_n = h} \frac{1}{h_1! \cdots h_n!} \, Cimm_{\tilde\lambda}[1^{h_1} \ldots n^{h_n}; 1^{h_1} \ldots n^{h_n}].
$$

Hence, we have the remarkable identity:

Corollary 4.9.

$$
\mathbf{S}_\lambda(n) = \frac{1}{H(\tilde\lambda)} \sum_S [\, \boxed{S \mid S}\,] \tag{28}
$$

$$
= (-1)^{\binom{h}{2}} \sum_{h_1 + \cdots + h_n = h} \frac{1}{h_1! \cdots h_n!} \, Cimm_{\tilde\lambda}[1^{h_1} \ldots n^{h_n}; 1^{h_1} \ldots n^{h_n}], \tag{29}
$$

where the sum of double Young-Capelli bitableaux in eq. (28) is extended to all row (strictly) increasing tableaux S of shape $\tilde\lambda$ on the proper alphabet $\mathcal{L} = \{1, 2, \ldots, n\}$ and eq. (29) is a sum of diagonal Capelli immanants of shape $\tilde\lambda$.

Corollary 4.9 was announced, without proof, in our recent paper [17].

4.3 The determinantal Capelli generators $\mathbf{H}_k(n)$

Let (k) be the row shape of length k, $\alpha \in \mathcal{A}_0$ a positive virtual symbol. The element

$$[i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k] = \mathfrak{p}\big(e_{i_1,\alpha} e_{i_2,\alpha} \cdots e_{i_k,\alpha} e_{\alpha,j_1} e_{\alpha,j_2} \cdots e_{\alpha,j_k}\big)$$

is the Capelli bitableau

$$\mathfrak{p}\big(e_{S,C^*_{(k)}} \, e_{C^*_{(k)},T}\big) \in \mathbf{U}(gl(n)),$$

where $S = (i_1 i_2 \cdots i_k)$ and $T = (j_1 j_2 \cdots j_k)$. Clearly, the elements $[i_1 i_2 \cdots i_k | j_1 j_2 \cdots j_k]$ are skew-symmetric both in the left and the right sequences. In particular,

$$[i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k] = (-1)^{\binom{k}{2}} [i_1 i_2 \cdots i_k | i_1 i_2 \cdots i_k].$$

In the enveloping algebra $\mathbf{U}(gl(n))$, given any integer $k = 1, 2, \ldots, n$, consider the *Capelli elements*

$$\mathbf{H}_k(n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]. \tag{30}$$

We recall that the Capelli elements admit a classical presentation as a column determinant [1] [19].

Proposition 4.10. *For every $k = 1, 2, \ldots, n$, we have:*

$$\mathbf{H}_k(n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{cdet} \begin{pmatrix} e_{i_1,i_1} + (k-1) & e_{i_1,i_2} & \cdots & e_{i_1,i_k} \\ e_{i_2,i_1} & e_{i_2,i_2} + (k-2) & \cdots & e_{i_2,i_k} \\ \vdots & \vdots & \vdots & \vdots \\ e_{i_k,i_1} & e_{i_k,i_2} & \cdots & e_{i_k,i_k} \end{pmatrix}.$$

Proof. See [18], Proposition 8.6 (see also [41]). $\qquad \square$

Proposition 4.11. *Let $(1)^k$ be the column shape of depth k. Then,*

$$\mathbf{H}_k(n) = \mathbf{S}_{(1^k)}(n) \in \boldsymbol{\zeta}(n).$$

Proof. We have

$$\mathbf{S}_{(1^k)}(n) = \frac{1}{H((k))} \sum_S \mathfrak{p}\big(e_{S,C^*_{(k)}} \, e_{C^*_{(k)},D^*_{(k)}} \, e_{D^*_{(k)},C^*_{(k)}} \, e_{C^*_{(k)},S}\big)$$

,

---

[1] The *column determinant* of a matrix $A = [a_{ij}]$ with noncommutative entries is, by definition, $\mathbf{cdet}(A) = \sum_\sigma (-1)^{|\sigma|} \, a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.

where the sum is extended to all strictly increasing row tableaux S of shape (k) and $H((k)) = k!$.

Notice that

$$\mathfrak{p}\big(e_{S,C^*_{(k)}} \cdot e_{C^*_{(k)},D^*_{(k)}} \cdot e_{D^*_{(k)},C^*_{(k)}} \cdot e_{C^*_{(k)},S}\big)$$

equals

$$(-1)^{\binom{k}{2}} \, \mathfrak{p}\big(e_{S,C^*_{(k)}} \cdot e_{C^*_{(k)},C^*_{(k)}} \cdot e_{C^*_{(k)},S}\big),$$

that, in turn, equals

$$(-1)^{\binom{k}{2}} \, k! \, \mathfrak{p}\big(e_{S,C^*_{(k)}} \cdot e_{C^*_{(k)},S}\big) = (-1)^{\binom{k}{2}} \, k! \, \mathfrak{p}\big(e_{i_1,\alpha} e_{i_2,\alpha} \cdots e_{i_k,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big).$$

Hence,

$$\mathbf{S}_{(1^k)}(n) = (-1)^{\binom{k}{2}} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathfrak{p}\big(e_{i_1,\alpha} e_{i_2,\alpha} \cdots e_{i_k,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big)$$

$$= \sum_{1 \le i_1 < \cdots < i_k \le n} \mathfrak{p}\big(e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big)$$

$$= \sum_{1 \le i_1 < \cdots < i_k \le n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k] = \mathbf{H}_k(n).$$

$\square$

Corollary 4.12. *The Capelli elements* $\mathbf{H}_k(n)$ *are central. Furthermore,*

$$\mathbf{H}_k(n) \in \boldsymbol{\zeta}(n)^{(m)},$$

for every $m \ge k$.

We recall the following fundamental result, indeed proved by Capelli in two papers ([20], [21]) with deceiving titles.

Theorem 4.13. *(Capelli, 1893)*
The set

$$\mathbf{H}_1(n), \mathbf{H}_2(n), \ldots, \mathbf{H}_n(n)$$

is a set of algebraically independent generators of the center $\boldsymbol{\zeta}(n)$ *of* $\mathbf{U}(gl(n))$.

As usual in the theory of symmetric functions, given a shape

$$\lambda = (\lambda_1 \ge \cdots \ge \lambda_p), \ \lambda_1 \le n,$$

we set

$$\mathbf{H}_\lambda(n) = \mathbf{H}_{\lambda_1}(n)\mathbf{H}_{\lambda_2}(n) \cdots \mathbf{H}_{\lambda_p}(n).$$

By convention, if λ is the empty partition, we set $\mathbf{H}_\emptyset(n) = \mathbf{1} \in \boldsymbol{\zeta}(n)$.

From Theorem 4.13, one infers

Corollary 4.14. *The set*

$$\left\{\mathbf{H}_\lambda(n); \lambda_1 \le n, \ |\lambda| \le m \right\}$$

is a linear basis of $\boldsymbol{\zeta}(n)^{(m)}$.

We recall that $v_{\widetilde{\mu}} = (D_\mu | D_\mu^P)$ denotes the "canonical" highest weight vector of the Schur module $Schur_\mu(n)$, $\mu_1 \le n$, which is indeed of weight $\widetilde{\mu}$ (Subsection 3.2).

Furthermore, we will write $\mathbf{H}_k(n) \cdot v_{\widetilde{\mu}}$ to mean the action of the central element $\mathbf{H}_k(n)$ on $v_{\widetilde{\mu}}$.

For every $k = 1, 2, \ldots, n$, let

$$e_k^*(\widetilde{\mu}) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (\widetilde{\mu}_{i_1} + k - 1)(\widetilde{\mu}_{i_2} + k - 2) \cdots (\widetilde{\mu}_{i_k}). \qquad (31)$$

Remark 4.15. *In formula (31), the sum can be regarded as extended over all Ferrers subdiagrams obtained from the Ferrers diagram of the partition μ by selecting k columns $i_1 < i_2 < \cdots < i_k$, and each summand*

$$(\widetilde{\mu}_{i_1} + k - 1)(\widetilde{\mu}_{i_2} + k - 2) \cdots (\widetilde{\mu}_{i_k})$$

is the product of the hook length $H(x)$ of the boxes of the first row of each Ferrers subdiagram.

$\square$

The classical determinatal presentation (Proposition 4.10) of the $\mathbf{H}_k(n)$'s implies the following result.

Proposition 4.16. *We have*

$$\mathbf{H}_k(n) \cdot v_{\widetilde{\mu}} = e_k^*(\widetilde{\mu}) \cdot v_{\widetilde{\mu}}, \quad e_k^*(\widetilde{\mu}) \in \mathbb{Z}.$$

Corollary 4.17. *If $\mu_1 < k$, then*

$$\mathbf{H}_k(n) \cdot v_{\widetilde{\mu}} = 0,$$

The "*virtual definition*" (30) of the $\mathbf{H}_k(n)$'s leads to a further combinatorial description of the integer eigenvalues $e_k^*(\widetilde{\mu})$, which will turn out to be crucial in the section on *duality*.

Proposition 4.18. *We have*

$$e_k^*(\widetilde{\mu}) = \sum hstrip_\mu(k)!,$$

where the sum is extended to all "horizontal strips" [2] *of length k in the Ferrers diagram of the partition μ, and the symbol $hstrip_\mu(k)!$ denotes the products of the factorials of the cardinality of each "horizontal component"* [3] *of the horizontal strip.*

---

[2] In this work, we use the expression *horizontal strip* in a generalized sense. To wit, a horizontal strip in a Ferrers diagram is a subset of cells such that no two cells in the subset appear in the same column.

[3] For each each generalized horizontal strip, a *horizontal component* is the set all cells on the same row.

Proof. Let

$$\mathbf{H}_k(n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k]$$

$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathfrak{p}\big(e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big).$$

Let $v_{\widetilde{\mu}} = (D_\mu | D_\mu^P)$ be the canonical highest weight vector of the irreducible $gl(n)$-module $Schur_\mu(n)$ (of weight $\widetilde{\mu}$.)

Recall that

$$\mathfrak{p}\big(e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}\big) \cdot (D_\mu | D_\mu^P) =$$

$$= e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha} e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k} \cdot (D_\mu | D_\mu^P),$$

by Proposition 2.7.

The action of each summand of the "virtualizing part"

$$e_{\alpha,i_1} e_{\alpha,i_2} \cdots e_{\alpha,i_k}$$

distributes the k occurrences of α in all horizontal strips of length k (with column positions $i_1, i_2, \ldots, i_k$) in the Ferrers diagram of the partition μ, with signs - according to Remark 2.20 - since $|e_{\alpha,i_h}| = 1$. By applying the "devirtualizing part"

$$e_{i_k,\alpha} \cdots e_{i_2,\alpha} e_{i_1,\alpha}$$

it is easy to see that, for each horizontal strip, we obtain a sum of tableaux that:

– to be non zero, have the occurrences of α - in any horizontal component of the strip - replaced by a permutation of the elements that have been previously polarized into α,

– have a sign that is easily seen to be the product of the signs of the permutations of the elements in each horizontal component.

By reordering each horizontal component, all the signs cancel out. Therefore, we get the "canonical" highest weight vector $v_{\widetilde{\mu}} = (D_\mu | D_\mu^P)$ with a positive integer coefficient that is the product of the factorials of the lengths of the horizontal components. $\square$

4.4 The permanental Nazarov generators $\mathbf{I}_k(n)$

In this section we provide the virtual form of the set of the preimages in $\boldsymbol{\zeta}(n)$ - with respect to the Harish-Chandra isomorphism - of the sequence of *shifted complete symmetric polynomials* $\mathbf{h}_k^*(x_1, x_2, \ldots, x_n)$, $k \in \mathbb{Z}^+$ (see [51] and [44], Theorem 4.9).

The central elements $\mathbf{I}_k(n)$, $k \in \mathbb{Z}^+$, coincide (see [9]) with the "permanental generators" of $\boldsymbol{\zeta}(n)$ originally discovered and studied - through the machinery of *Yangians* - by Nazarov [47] and later described by Umeda [63] as sums of

column permanents[4] in $\mathbf{U}(gl(n))$ (see e.g. Example 4.20 below, see also [46], [48], and Turnbull [60]).

The element

$$[n^{h_n} \cdots 2^{h_2} 1^{h_1} | 1^{h_1} 2^{h_2} \cdots n^{h_n}]^* = \mathfrak{p}\big(e_{n,\beta}^{h_n} \cdots e_{2,\beta}^{h_2} e_{1,\beta}^{h_1} e_{\beta,1}^{h_1} e_{\beta,2}^{h_2} \cdots e_{\beta,n}^{h_n}\big),$$

where $\beta \in A_1$ denotes *any* negative virtual symbol, is symmetric both in the left and the right sequences. In particular,

$$[n^{h_n} \cdots 2^{h_2} 1^{h_1} | 1^{h_1} 2^{h_2} \cdots n^{h_n}]^* = [1^{h_1} 2^{h_2} \cdots n^{h_n} | 1^{h_1} 2^{h_2} \cdots n^{h_n}]^*.$$

Remark 4.19. *Let $k = h_1 + h_2 + \cdots + h_n$, and let $(1)^k$ be the column shape of depth k. Since*

$$e_{n,\beta}^{h_n} \cdots e_{2,\beta}^{h_2} e_{1,\beta}^{h_1} e_{\beta,1}^{h_1} e_{\beta,2}^{h_2} \cdots e_{\beta,n}^{h_n} = e_{1,\beta}^{h_1} e_{2,\beta}^{h_2} \cdots e_{n,\beta}^{h_n} e_{\beta,1}^{h_1} e_{\beta,2}^{h_2} \cdots e_{\beta,n}^{h_n} \qquad (32)$$

the element (32) *equals the* bitableau monomial *(see formula* (8))

$$e_{T,D^*_{(1)^k}} e_{D^*_{(1)^k},T} \in \mathbf{U}(gl(m_0|m_1 + n)),$$

where T is the column tableau of shape $(1)^k$ with $\widetilde{T} = (1^{h_1} 2^{h_2} \cdots n^{h_n})$, $sh(\widetilde{T}) = (k)$.

In the enveloping algebra $\mathbf{U}(gl(n))$, given any positive integer $k \in \mathbb{Z}^+$, consider the *Nazarov elements*

$$\mathbf{I}_k(n) = \sum_{(h_1, h_2, \ldots, h_n)} (h_1! h_2! \cdots h_n!)^{-1} \, [n^{h_n} \cdots 2^{h_2} 1^{h_1} | 1^{h_1} 2^{h_2} \cdots n^{h_n}]^*, \qquad (33)$$

where the sum is extended to all n-tuples $(h_1, h_2, \ldots, h_n)$ such that $h_1 + h_2 + \cdots + h_n = k$. Clearly, formula (33) can be rewritten as

$$\sum_{\underline{i} = (1 \leq i_1 \leq \cdots \leq i_k \leq n)} \big(h_1(\underline{i})! \cdots h_n(\underline{i})!\big)^{-1} \quad \mathfrak{p}\big(e_{i_k,\beta} \cdots e_{i_1,\beta} e_{\beta,i_1} \cdots e_{\beta,i_k}\big), \qquad (34)$$

where, given a non decreasing k-tuple $\underline{i} = (1 \leq i_1 \leq \cdots \leq i_k \leq n)$, we set

$$h_j(\underline{i}) = \sharp\{i_s = j; s = 1, \ldots, k\}, \quad j = 1, 2, \ldots, n.$$

In "nonvirtual form", the summands

$$[n^{h_n} \cdots 2^{h_2} 1^{h_1} | 1^{h_1} 2^{h_2} \cdots n^{h_n}]^*$$

can be written as column permanent in the algebra $\mathbf{U}(gl(n))$ (see, e.g. [63]).

---

[4]The *column permanent* of a matrix $A = [a_{ij}]$ with noncommutative entries is, by definition, $\mathbf{cper}(A) = \sum_\sigma a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$.

Example 4.20.

$$\mathbf{I}_3(3) = \frac{1}{3!}[111|111]^* + \frac{1}{2!}[211|112]^* + \frac{1}{2!}[311|113]^* + \frac{1}{2!}[221|122]^* + [321|123]^* +$$

$$+ \frac{1}{2!}[331|133]^* + \frac{1}{3!}[222|222]^* + \frac{1}{2!}[322|223]^* + \frac{1}{2!}[332|233]^* + \frac{1}{3!}[333|333]^* =$$

$$= \frac{1}{3!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,1}-1 & e_{1,1} \\ e_{1,1}-2 & e_{1,1}-1 & e_{1,1} \\ e_{1,1}-2 & e_{1,1}-1 & e_{1,1} \end{pmatrix} + \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,1}-1 & e_{1,2} \\ e_{1,1}-2 & e_{1,1}-1 & e_{1,2} \\ e_{2,1} & e_{2,1} & e_{2,2} \end{pmatrix} +$$

$$+ \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,1}-1 & e_{1,3} \\ e_{1,1}-2 & e_{1,1}-1 & e_{1,3} \\ e_{3,1} & e_{3,1} & e_{3,3} \end{pmatrix} + \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,2} & e_{1,2} \\ e_{2,1} & e_{2,2}-1 & e_{2,2} \\ e_{2,1} & e_{2,2}-1 & e_{2,2} \end{pmatrix} +$$

$$+ \mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2}-1 & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix} + \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{1,1}-2 & e_{1,3} & e_{1,3} \\ e_{3,1} & e_{3,3}-1 & e_{3,3} \\ e_{3,1} & e_{3,3}-1 & e_{3,3} \end{pmatrix} +$$

$$+ \frac{1}{3!}\mathbf{cper}\begin{pmatrix} e_{2,2}-2 & e_{2,2}-1 & e_{2,2} \\ e_{2,2}-2 & e_{2,2}-1 & e_{2,2} \\ e_{2,2}-2 & e_{2,2}-1 & e_{2,2} \end{pmatrix} + \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{2,2}-2 & e_{2,2}-1 & e_{2,3} \\ e_{2,2}-2 & e_{2,2}-1 & e_{2,3} \\ e_{3,2} & e_{3,2} & e_{3,3} \end{pmatrix} +$$

$$+ \frac{1}{2!}\mathbf{cper}\begin{pmatrix} e_{2,2}-2 & e_{2,3} & e_{2,3} \\ e_{3,2} & e_{3,3}-1 & e_{3,3} \\ e_{3,2} & e_{3,3}-1 & e_{3,3} \end{pmatrix} + \frac{1}{3!}\mathbf{cper}\begin{pmatrix} e_{3,3}-2 & e_{3,3}-1 & e_{3,3} \\ e_{3,3}-2 & e_{3,3}-1 & e_{3,3} \\ e_{3,3}-2 & e_{3,3}-1 & e_{3,3} \end{pmatrix}.$$

$\square$

Proposition 4.21. *Let (k) be the row shape of length k. Then,*

$$\mathbf{I}_k(n) = \mathbf{S}_{(k)}(n).$$

Proof. By formula (33), we have

$$\mathbf{S}_{(k)}(n) = \frac{1}{H((1)^k)} \sum_S \mathfrak{p}\left(e_{S,C^*_{(1^k)}} \; e_{C^*_{(1^k)},D^*_{(1^k)}} \; e_{D^*_{(1^k)},C^*_{(1^k)}} \; e_{C^*_{(1^k)},S}\right),$$

where the sum is extended to all column tableaux S of shape (1^k) and $H((1)^k) = k!$.

Since S is a column tableaux of shape (1^k) and the column tableau $C^*_{(1^k)}$ is

$$C^*_{(1^k)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix},$$

where the α_i's are *distinct positive* virtual symbols, then each summand

$$\mathfrak{p}\left(e_{S,C^*_{(1^k)}} \cdot e_{C^*_{(1^k)},D^*_{(1^k)}} \cdot e_{D^*_{(1^k)},C^*_{(1^k)}} \cdot e_{C^*_{(1^k)},S}\right)$$

33

equals

$$\mathfrak{p}\big(e_{S,D^*_{(1^k)}} \cdot e_{D^*_{(1^k)},S}\big).$$

Hence,

$$\mathbf{S}_{(k)}(n) = \frac{1}{k!} \sum_{S} \mathfrak{p}\big(e_{S,D^*_{(1^k)}} e_{D^*_{(1^k)},S}\big)$$

$$= \frac{1}{k!} \sum_{(h_1,\ldots,h_n)} \sum_{T} \mathfrak{p}\big(e_{T,D^*_{(1^k)}} e_{D^*_{(1^k)},T}\big),$$

where the outer sum is extended over all indexes $h_1 + \cdots + h_n = k$ and inner sum is extended over all column tableaux T with h_1 occurrences of 1, h_2 occurrences of 2, ..., h_n occurrences of n. Moreover, since each element $e_{T,D^*_{(1^k)}}$ and $e_{D^*_{(1^k)},T}$ are commutative, then the inner sum

$$\sum_{T} e_{T,D^*_{(1^k)}} e_{D^*_{(1^k)},T}$$

equals

$$\binom{k}{h_1,h_2,\ldots,h_n} e^{h_1}_{1,\beta_1} \cdots e^{h_n}_{n,\beta_1} e^{h_1}_{\beta_1,1} \cdots e^{h_n}_{\beta_n,1},$$

where there are h_1 occurrences of 1, h_2 occurrences of 2, ..., h_n occurrences of n. Hence, from Remark 4.19, we infer

$$\mathbf{S}_{(k)}(n) = \frac{1}{k!} \sum_{(h_1,\ldots,h_n)} \sum_{T} \mathfrak{p}\big(e_{T,D^*_{(1^k)}} e_{D^*_{(1^k)},T}\big)$$

$$= \frac{1}{k!} \sum_{(h_1,\ldots,h_n)} \binom{k}{h_1,h_2,\ldots,h_n} \mathfrak{p}\big(e^{h_1}_{1,\beta_1} \cdots e^{h_n}_{n,\beta_1} e^{h_1}_{\beta_1,1} \cdots e^{h_n}_{\beta_n,1}\big)$$

$$= \sum_{(h_1,h_2,\ldots,h_n)} \frac{1}{h_1!h_2!\cdots h_n!} [n^{h_n} \cdots 2^{h_2} 1^{h_1} | 1^{h_1} 2^{h_2} \cdots n^{h_n}]^* = \mathbf{I}_k(n).$$

$$\square$$

Corollary 4.22. *The Nazarov elements* $\mathbf{I}_k(n)$ *are central. Furthermore,*

$$\mathbf{I}_k(n) \in \boldsymbol{\zeta}(n)^{(m)},$$

for every $m \geq k$.

The next characterization of the eigenvalues $h_k^*(\widetilde{\mu})$ of the elements $\mathbf{I}_k(n)$, in combination with characterization of the eigenvalues $e_k^*(\widetilde{\mu})$ of the elements $\mathbf{H}_k(n)$ (see Proposition 4.18), will play a crucial role in our treatment of *duality* in the center $\boldsymbol{\zeta}(n)$ (see Section 4.5 below).

Theorem 4.23. *We have:*

$$\mathbf{I}_k(n) \cdot v_{\widetilde{\mu}} = h_k^*(\widetilde{\mu}) \cdot v_{\widetilde{\mu}}, \quad h_k^*(\widetilde{\mu}) \in \mathbb{N}$$

with

$$h_k^*(\widetilde{\mu}) = \sum\ vstrip_\mu(k)!,$$

where the sum is extended to all "vertical strips" [5] *of length k in the Ferrers diagram of the partition μ, and the symbol $vstrip_\mu(k)!$ denotes the product of the factorials of the cardinality of each vertical component of the vertical strip.*

Proof. The action of the "virtualizing part"

$$e_{\beta,1}^{h_1} e_{\beta,2}^{h_2} \cdots e_{\beta,n}^{h_n}$$

of each summand in expression (33) distributes k occurrences of the virtual variable β in the Ferrers diagram of the shape μ, with h_1 occurrences in column 1, h_2 occurrences in column 2, and so on. Since $|\beta| = 1$, in order to get a non zero result, these β's must appear in different rows - by skew-symmetry - and, therefore, they form a vertical strip. Clearly, this configuration is created $h_1!h_2!\cdots h_n!$ times. Again by skew-symmetry, the action of the "devirtualizing part"

$$e_{n,\beta}^{h_n} \cdots e_{2,\beta}^{h_2} e_{1,\beta}^{h_1}$$

gives a non zero result if and only if the β's in column 1 are replaced by 1, the β's in column 2 are replaced by 2, and so on. Therefore we obtain again the highest weight vector $v_{\widetilde{\mu}} = (D_\mu | D_\mu^P)$ with multiplicity $h_1!h_2!\cdots h_n!$. Note that, since $|e_{\beta,p}| = |e_{p,\beta}| = 0$, for every $p = 1, 2, \ldots, n$, no signs are involved in the proof. $\qquad\square$

Corollary 4.24. *If $\widetilde{\mu}_1 < k$, then*

$$\mathbf{I}_k(n) \cdot v_{\widetilde{\mu}} = 0.$$

The eigenvalue $h_k^*(\widetilde{\mu})$ admits a further description that relates it to *complete homogeneous shifted symmetric polynomials.*

Theorem 4.25. *We have:*

$$h_k^*(\widetilde{\mu}) = \sum_{1 \leq i_1 \leq i_2 < \cdots \leq i_k \leq n} (\widetilde{\mu}_{i_1} - k + 1)(\widetilde{\mu}_{i_2} - k + 2) \cdots (\widetilde{\mu}_{i_{k-1}} - 1)\widetilde{\mu}_{i_k} \qquad (35)$$

Proof. The action of the "virtualizing part"

$$e_{\beta,i_1} e_{\beta,i_2} \cdots e_{\beta,i_k}, \quad \underline{i} = (i_1 \leq i_2 \leq \cdots \leq i_k),$$

---

[5] In this work, we use the expression *vertical strip* in a generalized sense. To wit, a vertical strip in a Ferrers diagram is a subset of cells such that no two cells in the subset appear in the same row.

of each summand in expression (34) of $\mathbf{I}_k(n)$, distributes *one* occurrence of the virtual variable β in the Ferrers diagram of the shape μ, in column i_k, ..., i_2, i_1. Since $|\beta| = 1$, in order to get a non zero result, these β's must be distributed into different rows, by skew-symmetry. Clearly, this procedure can be done into

$$(\widetilde{\mu}_{i_1} - k + 1)(\widetilde{\mu}_{i_2} - k + 2) \cdots (\widetilde{\mu}_{i_{k-1}} - 1)\widetilde{\mu}_{i_k}$$

ways. Again by by skew-symmetry, the action of the "devirtualizing part"

$$e_{i_k,\beta} e_{i_{k-1},\beta} \cdots e_{i_1,\beta}$$

gives a non zero result of and only if the β in column i_1 is replaced by i_1, the β in column i_2 is replaced by i_2, and so on. Therefore we obtain again the highest weight vector $v_{\widetilde{\mu}} = (D_\mu | D_\mu^P)$ with multiplicity $h_1(\underline{i})! \cdots h_n(\underline{i})!$. □

4.5 Duality in $\boldsymbol{\zeta}(n)$

Let

$$\mathcal{W}_n : \boldsymbol{\zeta}(n) \to \boldsymbol{\zeta}(n)$$

be the algebra automorphism defined by setting

$$\mathcal{W}_n\Big(\mathbf{H}_k(n)\Big) = \mathbf{I}_k(n), \quad k = 1, 2, \ldots, n.$$

Clearly, Proposition 4.18 and Theorem 4.23.2 imply the following result.

Proposition 4.26. *If $\mu_1, \widetilde{\mu}_1 \leq n$, then*

$$e_k^*(\widetilde{\mu}) = h_k^*(\mu), \tag{36}$$

that is, the eigenvalue of $\mathbf{H}_k(n)$ on the Schur module of shape μ equals the eigenvalue of $\mathbf{I}_k(n)$ on the Schur module of shape $\widetilde{\mu}$.

Notice that the following *Duality Theorem* is an immediate consequence of the preceding Proposition.

Theorem 4.27. *Let μ be such that $\mu_1, \widetilde{\mu}_1 \leq n$. For every $\varrho \in \boldsymbol{\zeta}(n)$ the eigenvalue of ϱ on the $gl(n)$-irreducible module $Schur_{\widetilde{\mu}}(n)$ (with highest weight μ) equals eigenvalue of $\mathcal{W}_n\Big(\varrho\Big)$ on the $gl(n)$-irreducible module $Schur_\mu(n)$ (with highest weight $\widetilde{\mu}$).*

The preceding result, in combination with the characterization results of subsection 4.2, implies

Corollary 4.28. *Let $\widetilde{\lambda}_1, \lambda_1 \leq n$. Then*

$$\mathcal{W}_n\Big(\mathbf{S}_\lambda(n)\Big) = \mathbf{S}_{\widetilde{\lambda}}(n).$$

Proof. We have:

$$\text{If } |\mu| < |\lambda|, \text{ then} \qquad \mathbf{S}_\lambda(n) \cdot v_\mu = 0,$$
$$\text{If } |\mu| = |\lambda|, \text{ then} \qquad \mathbf{S}_\lambda(n) \cdot v_\mu = \delta_{\lambda,\mu} \text{Ħ}(\lambda) \, v_\mu.$$

On the other hand, from Theorem 4.27, it follows:

$$\text{If } |\widetilde{\mu}| < |\widetilde{\lambda}|, \text{ then} \qquad \mathcal{W}_n\Big(\mathbf{S}_\lambda(n)\Big) \cdot v_{\widetilde{\mu}} = 0,$$
$$\text{If } |\widetilde{\mu}| = |\widetilde{\lambda}|, \text{ then} \qquad \mathcal{W}_n\Big(\mathbf{S}_\lambda(n)\Big) \cdot v_{\widetilde{\mu}} = \delta_{\lambda,\mu} \text{Ħ}(\lambda) \cdot v_{\widetilde{\mu}}.$$

Since $\delta_{\lambda,\mu} \text{Ħ}(\lambda) = \delta_{\widetilde{\lambda},\widetilde{\mu}} \text{Ħ}(\widetilde{\lambda})$, the assertion follows from Propositions 4.5 and 4.6. $\qquad\qquad\square$

Since $\mathbf{I}_k(n) = \mathbf{S}_{(k)}(n)$ and $\mathbf{H}_k(n) = \mathbf{S}_{(1^k)}(n)$, then

$$\mathcal{W}_n\Big(\mathbf{I}_k(n)\Big) = \mathbf{H}_k(n), \quad k = 1, 2, \ldots, n.$$

by Corollary 4.28.

Corollary 4.29. *The algebra automorphism $\mathcal{W}_n$ is an involution.*

5 The limit $n \to \infty$ for $\boldsymbol{\zeta}(n)$: the algebra $\boldsymbol{\zeta}$

5.1 The monomorphisms $\mathbf{i}_{n+1,n}$ and the epimorphisms $\boldsymbol{\pi}_{n,n+1}$

Given $n \in \mathbb{Z}^+$, let

$$\mathbf{H}_k(n), \quad k = 1, \ldots, n$$

be the Capelli free generators of the center $\boldsymbol{\zeta}(n)$ of the enveloping algebra $\mathbf{U}(gl(n))$, for every $n \in \mathbb{Z}^+$.

For every $n \in \mathbb{Z}^+$, let

$$\mathbf{i}_{n+1,n} : \boldsymbol{\zeta}(n) \hookrightarrow \zeta(n+1)$$

be the algebra monomorphism

$$\mathbf{i}_{n+1,n} : \mathbf{H}_k(n) \to \mathbf{H}_k(n+1), \quad k = 1, 2, \ldots, n.$$

Given $m \in \mathbb{Z}^+$, let $\boldsymbol{\zeta}(n)^{(m)}$ denote the m-th filtration element of $\boldsymbol{\zeta}(n)$ (with respect to the filtration induced by the standard filtration of $\mathbf{U}(n)$). Clearly, the monomorphisms $\mathbf{i}_{n+1,n}$ are *morphisms in the category of filtered algebras,* that is

$$\mathbf{i}_{n+1,n}\Big[\boldsymbol{\zeta}(n)^{(m)}\Big] \subseteq \boldsymbol{\zeta}(n+1)^{(m)}$$

We consider the *direct limit* (in the category of filtered algebras):

$$\varinjlim \boldsymbol{\zeta}(n) = \boldsymbol{\zeta}. \tag{37}$$

The algebra $\boldsymbol{\zeta}$ inherits a structure of filtered algebra, where

$$\boldsymbol{\zeta}^{(m)} = \varinjlim \boldsymbol{\zeta}(n)^{(m)}.$$

On the other hand, given $n \in \mathbb{Z}^+$, we may consider the algebra epimorphism

$$\boldsymbol{\pi}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n),$$

such that

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{H}_k(n+1)) = \mathbf{H}_k(n) \quad k = 1, 2, \ldots, n,$$

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{H}_{n+1}(n+1)) = 0.$$

The following Propositions are fairly obvious from the definitions.

Proposition 5.1. *We have*

1. $Ker(\boldsymbol{\pi}_{n,n+1}) = \Big(\mathbf{H}_{n+1}(n+1)\Big)$, *the bilateral ideal of* $\boldsymbol{\zeta}(n+1)$ *generated by the element* $\mathbf{H}_{n+1}(n+1)$.

2. *The epimorphism* $\boldsymbol{\pi}_{n,n+1}$ *is the (filtered) left inverse of the monomorphism* $\mathbf{i}_{n+1,n}$. *In symbols,*
$$\boldsymbol{\pi}_{n,n+1} \circ \mathbf{i}_{n+1,n} = Id_{\boldsymbol{\zeta}(n)}.$$

Proposition 5.2. *If* $n \geq m$, *then the restriction* $\boldsymbol{\pi}_{n,n+1}^{(m)}$ *of* $\boldsymbol{\pi}_{n,n+1}$ *to* $\boldsymbol{\zeta}(n+1)^{(m)}$ *and the restriction* $\mathbf{i}_{n+1,n}^{(m)}$ *of* $\mathbf{i}_{n+1,n}$ *to* $\boldsymbol{\zeta}(n)^{(m)}$ *are the inverse of each other.*

The crucial point is that the projections $\boldsymbol{\pi}_{n,n+1}$ admit an *intrinsic/invariant* presentation that is founded upon the *Olshanski decomposition*.

5.2 The Olshanski decomposition/projection

We recall a special case of an essential costruction due to Olshanski [52], [53]. For the sake of simplicity, we follow Molev ([44], pp. 928 ff.).

Let $\mathbf{U}(gl(n+1))^0$ be the centralizer in $\mathbf{U}(gl(n+1))$ of the element $e_{n+1,n+1}$ of the standard basis of $gl(n+1)$, regarded as an element of $\mathbf{U}(gl(n+1))$.

Let $\mathcal{I}(n+1)$ be the *left ideal* of $\mathbf{U}(gl(n+1))$ generated by the elements

$$e_{i,n+1}, \quad i = 1, 2, \ldots, n+1.$$

Let $\mathcal{I}(n+1)^0$ be the intersection

$$\mathcal{I}(n+1)^0 = \mathcal{I}(n+1) \cap \mathbf{U}(gl(n+1))^0. \tag{38}$$

We recall that $\mathcal{I}(n+1)^0$ is a *bilateral ideal* of $\mathbf{U}(gl(n+1)^0$, and the following *direct sum decomposition* hold

$$\mathbf{U}((gl(n+1))^0 = \mathbf{U}(gl(n)) \oplus \mathcal{I}(n+1)^0. \tag{39}$$

Therefore, the *Olshanski map*

$$\mathcal{M}_{n+1} : \mathbf{U}((gl(n+1))^0 \twoheadrightarrow \mathbf{U}(gl(n))$$

that maps any element in the direct summand $\mathbf{U}(gl(n))$ to itself and any element in the direct summand $\mathcal{I}(n+1)^0$ to zero is a well-defined algebra epimorphism.

Since $\boldsymbol{\zeta}(n+1)$ is a subalgebra of $\mathbf{U}((gl(n+1))^0$, the direct sum decomposition (39) induces a direct sum decomposition of any element in $\boldsymbol{\zeta}(n+1)$ and the $\mathcal{M}_{n+1}$ map defines, by restriction, an algebra epimorphism

$$\boldsymbol{\mu}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n).$$

In plain words, any element $\boldsymbol{\varrho} \in \boldsymbol{\zeta}(n+1)$ admits a *unique* decomposition

$$\boldsymbol{\varrho} = \boldsymbol{\varrho}' \dotplus \boldsymbol{\varrho}^0, \quad \boldsymbol{\varrho}' \in \boldsymbol{\zeta}(n), \ \boldsymbol{\varrho}^0 \in \mathcal{I}(n+1)^0. \tag{40}$$

We call the decomposition (40) the *Olshanski decomposition* of the element $\boldsymbol{\varrho} \in \boldsymbol{\zeta}(n+1)$.

In this notation, the projection

$$\boldsymbol{\mu}_{n,n+1} : \boldsymbol{\zeta}(n+1) \twoheadrightarrow \boldsymbol{\zeta}(n),$$

$$\boldsymbol{\mu}_{n+1,n}(\boldsymbol{\varrho}) = \boldsymbol{\varrho}', \quad \boldsymbol{\varrho} \in \boldsymbol{\zeta}(n+1)$$

is defined.

Proposition 5.3. *We have*

  1. *if $k \leq n$, then*
  $$\mathbf{H}_k(n+1) = \mathbf{H}_k(n) \dotplus \mathbf{H}_k(n+1)^0,$$

  where
  $$\mathbf{H}_k(n+1)^0 = \mathbf{H}_k(n+1) - \mathbf{H}_k(n) \in \mathcal{I}(n+1)^0,$$

  and
  $$\mathbf{H}_k(n) \in \boldsymbol{\zeta}(n);$$

  2. $\mathbf{H}_{n+1}(n+1) = \mathbf{H}_{n+1}(n+1)^0.$

Example 5.4. We have:

$$\mathbf{H}_2(4) = [21|12] + [31|13] + [41|14] + [32|23] + [42|24] + [43|34]$$
$$= \mathbf{H}_2(3) \dotplus \mathbf{H}_2(4)^0,$$

where
$$\mathbf{H}_2(3) = [21|12] + [31|13] + [32|23] \in \boldsymbol{\zeta}(3),$$

and
$$\mathbf{H}_2(4)^0 = [41|14] + [42|24] + [43|34] \in \mathcal{I}(4)^0.$$

$\square$

Corollary 5.5. *We have*

1. *if $k \leq n$, then $\boldsymbol{\mu}_{n,n+1}\big(\mathbf{H}_k(n+1)\big) = \mathbf{H}_k(n)$,*

2. $\boldsymbol{\mu}_{n,n+1}\big(\mathbf{H}_{n+1}(n+1)\big) = 0.$

Proposition 5.6. *The map $\boldsymbol{\mu}_{n,n+1}$ is the same as the map $\boldsymbol{\pi}_{n,n+1}$.*

Proof. The family

$$\big\{\mathbf{H}_1(n+1), \mathbf{H}_2(n+1), \ldots, \mathbf{H}_n(n+1), \mathbf{H}_{n+1}(n+1)\big\}$$

is a system of algebraically independent generators of the algebra $\boldsymbol{\zeta}(n+1)$. From Proposition 5.5, we obtain:

– if $k \leq n$, then

$$\boldsymbol{\mu}_{n,n+1}(\mathbf{H}_k(n+1)) = \mathbf{H}_k(n) = \boldsymbol{\pi}_{n,n+1}(\mathbf{H}_k(n+1));$$

– $\boldsymbol{\mu}_{n,n+1}(\mathbf{H}_{n+1}(n+1)) = 0 = \boldsymbol{\pi}_{n,n+1}(\mathbf{H}_{n+1}(n+1)).$

$\square$

In the following, we refer to the projections

$$\boldsymbol{\mu}_{n,n+1} = \boldsymbol{\pi}_{n,n+1}$$

as the *Capelli-Olshanski projections*.

From Proposition 5.2, the algebra $\boldsymbol{\zeta}$ (direct limit) is the same as the *inverse limit in the category of filtered algebras*

Proposition 5.7. *We have*

$$\boldsymbol{\zeta} = \underleftarrow{\lim}\, \boldsymbol{\zeta}(n)$$

with respect to the system of Capelli-Olshanski projections.

5.3   Main results

From Theorem 4.13 and Proposition 5.5, we infer

Proposition 5.8. *We have*

1. $\mathbf{I}_k(n+1) = \mathbf{I}_k(n) \dotplus \mathbf{I}_k(n+1)^0$, *where*

$$\mathbf{I}_k(n+1)^0 = \mathbf{I}_k(n+1) - \mathbf{I}_k(n) \in \mathcal{I}_k(n+1)^0,$$

and

$$\mathbf{I}_k(n) \in \boldsymbol{\zeta}(n).$$

Then

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{I}_k(n+1)) = \mathbf{I}_k(n). \tag{41}$$

2. $\mathbf{S}_\lambda(n+1) = \mathbf{S}_\lambda(n) \dotplus \mathbf{S}_\lambda(n+1)^0$, *where*

$$\mathbf{S}_\lambda(n+1)^0 = \mathbf{S}_\lambda(n+1) - \mathbf{S}_\lambda(n) \in \mathcal{I}_k(n+1)^0,$$

and

$$\mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}(n).$$

Then

$$\boldsymbol{\pi}_{n,n+1}(\mathbf{S}_\lambda(n+1)) = \mathbf{S}_\lambda(n). \tag{42}$$

By combining the preceding Proposition with Proposition 5.2, we get

Theorem 5.9. *We have:*

1. *Given a positive integer k, if $n \geq k$ then*

$$\mathbf{i}_{n+1,n}(\mathbf{I}_k(n)) = \mathbf{I}_k(n+1);$$

2. *Given a partition λ, if $n \geq |\lambda|$ then*

$$\mathbf{i}_{n+1,n}(\mathbf{S}_\lambda(n)) = \mathbf{S}_\lambda(n+1).$$

Passing to the direct limit $\varinjlim \boldsymbol{\zeta}(n) = \boldsymbol{\zeta}$, we set:

1. $\mathbf{H}_k \overset{def}{=} \varinjlim \mathbf{H}_k(n) \in \boldsymbol{\zeta}$.

2. $\mathbf{I}_k \overset{def}{=} \varinjlim \mathbf{I}_k(n) \in \boldsymbol{\zeta}$.

3. $\mathbf{S}_\lambda \overset{def}{=} \varinjlim \mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}$.

From the definition of the monomorphisms $\mathbf{i}_{n+1,n}$ and Theorem 5.9, the elements $\mathbf{H}_k, \mathbf{I}_k, \mathbf{S}_{\bar{\lambda}} \in \boldsymbol{\zeta}$ can be consistently written as *formal infinite sums*.

Proposition 5.10. *We have*

1.

$$\mathbf{H}_k = \sum_{i_1 < \cdots < i_k} [i_k \cdots i_2 i_1 | i_1 i_2 \cdots i_k],$$

where the sum is extended to all increasing k-tuples $i_1 < i_2 < \cdots < i_k$ in $\mathbb{Z}^+$.

2.

$$\mathbf{I}_k = \sum_{j_1 < j_2 < \cdots < j_p} (i_{j_1}! \, i_{j_2}! \cdots i_{j_p}!)^{-1} \, [j_p^{i_{j_p}} \cdots j_2^{i_{j_2}} j_1^{i_{j_1}} | j_1^{i_{j_1}} j_2^{i_{j_2}} \cdots j_p^{i_{j_p}}]^*,$$

where the sum is extended to all p-tuples $j_1 < j_2 < \cdots < j_p$ in $\mathbb{Z}^+$, and to all the p-tuples of exponents $(i_{j_1}, i_{j_2}, \cdots, i_{j_p})$ such that

$$i_{j_1} + i_{j_2} + \cdots + i_{j_p} = k.$$

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3.
$$\mathbf{S}_\lambda(n) = \frac{1}{H(\tilde{\lambda})} \ \sum_S [\ \boxed{S \mid S}\ ],$$

where the sum is extended to all row-increasing tableaux S of shape $\tilde{\lambda}$ on the alphabet $\mathbb{Z}^+$.

From Proposition 5.7, it follows

Corollary 5.11. *We have:*

1. *$\varprojlim \mathbf{H}_k(n) = \mathbf{H}_k \in \boldsymbol{\zeta}$,*

2. *$\varprojlim \mathbf{I}_k(n) = \mathbf{I}_k \in \boldsymbol{\zeta}$,*

3. *$\varprojlim \mathbf{S}_\lambda(n) = \mathbf{S}_\lambda \in \boldsymbol{\zeta}$.*

Due the fact that the algebra $\boldsymbol{\zeta}$ is defined as a direct limit, we infer:

Theorem 5.12.

1. *The set*
$$\left\{ \mathbf{H}_k; \ k \in \mathbb{Z}^+ \right\}$$
   is a system of free algebraic generators of $\boldsymbol{\zeta}$.

2. *The set*
$$\left\{ \mathbf{I}_k; \ k \in \mathbb{Z}^+ \right\}$$
   is a system of free algebraic generators of $\boldsymbol{\zeta}$.

3. *The set*
$$\left\{ \mathbf{S}_\lambda; \ \lambda \ any \ partition \ \right\}$$
   is a linear basis of $\boldsymbol{\zeta}$.

5.4 Duality in $\boldsymbol{\zeta}$

Let
$$\mathcal{W} : \boldsymbol{\zeta} \to \boldsymbol{\zeta}$$
denote the automorphism such that
$$\mathcal{W}\left(\mathbf{H}_k\right) = \mathbf{I}_k, \quad for \ every \ k \in \mathbb{Z}^+.$$

Since $\varinjlim \boldsymbol{\zeta}(n) = \boldsymbol{\zeta}$, Corollary 4.28 implies

Theorem 5.13.

1. *For every partition λ,*
$$\mathcal{W}\left(\mathbf{S}_\lambda\right) = \mathbf{S}_{\tilde{\lambda}}.$$

2. *In particular,*
$$\mathcal{W}\left(\mathbf{I}_k\right) = \mathbf{H}_k, \quad for \ every \ k \in \mathbb{Z}^+;$$
   then, the automorphisms $\mathcal{W}$ is an involution.

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6 The algebra $\Lambda^*(n)$ of shifted symmetric polynomials and the Harish-Chandra Isomorphism

In this subsection we follow Okounkov and Olshanski [51].

The algebra $\Lambda^*(n)$ of *shifted symmetric polynomials* is the algebra of polynomials $p(x_1, x_2, \ldots, x_n)$ that satisfy the *shifted symmetry* condition:

$$p(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = p(x_1, \ldots, x_{i+1} - 1, x_i + 1, \ldots, x_n),$$

for $i = 1, 2, \ldots, n - 1$.

The *Harish-Chandra isomorphism* χ_n is the algebra isomorphism

$$\chi_n : \boldsymbol{\zeta}(n) \longrightarrow \Lambda^*(n), \qquad A \mapsto \chi_n(A),$$

where $\chi_n(A)$ is the shifted symmetric polynomial such that, for every highest weight module V_μ, the evaluation $\chi_n(A)(\mu_1, \mu_2, \ldots, \mu_n)$ equals the eigenvalue of $A \in \boldsymbol{\zeta}(n)$ in V_μ (see, e.g. [51]).

From Corollary 4.16.1, it follows

Proposition 6.1.

$$\chi_n(\mathbf{H}_k(n)) = \mathbf{e}_k^*(x_1, x_2, \ldots, x_n)$$
$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \cdots (x_{i_k})$$

for every $k = 1, 2, \ldots, n$.

Clearly, $\chi_n(\mathbf{H}_0(n)) = \mathbf{e}_0^*(x_1, x_2, \ldots, x_n) = \mathbf{1}$.

The polynomials $\mathbf{e}_k^*(x_1, x_2, \ldots, x_n) \in \Lambda^*(n)$ are the *elementary* shifted symmetric polynomials.

From Theorem 4.23.2, it follows

Proposition 6.2.

$$\chi_n(\mathbf{I}_k(n)) = \mathbf{h}_k^*(x_1, x_2, \ldots, x_n)$$
$$= \sum_{1 \leq i_1 \leq i_2 < \cdots \leq i_k \leq n} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots (x_{i_k}),$$

for every $k = 1, 2, \ldots, n$.

Clearly, $\chi_n(\mathbf{I}_0(n)) = \mathbf{h}_0^*(x_1, x_2, \ldots, x_n) = \mathbf{1}$.

The polynomials $\mathbf{h}_k^*(x_1, x_2, \ldots, x_n) \in \Lambda^*(n)$ are the *complete* shifted symmetric polynomials.

Recall that, given a variable z and a natural integer p, the symbol $(z)_p$ denotes the *falling factorial polynomial*:

$$(z)_p = z(z - 1) \cdots (z - p + 1), \quad p \geq 1, \qquad (z)_0 = 1.$$

Let μ be a partition, $\widetilde{\mu}_1 \leq n$.

Following [51], consider the polynomial

$$\mathbf{s}_\lambda^*(x_1,\ldots,x_n) = \frac{det\Big[(x_i + n - i)_{\lambda_i+n-j}\Big]}{det\Big[(x_i + n - i)_{n-j}\Big]} \tag{43}$$

$$= \sum_{T \in RSSYT(\lambda)} (x_{T(s)} - c(s)), \tag{44}$$

where $RSSYT(\mu)$ denotes the set of all *reverse semistandard* [6] Young tableaux T of shape λ over the set $\{1, 2, \ldots, n\}$, $T(s)$ denotes the symbol of in the cell s of the Ferrers diagram of μ and $c(s) = j - i$ is the content of the cell s in position (i, j).

The polynomials $\mathbf{s}_\mu^*(x_1, \ldots, x_n) \in \Lambda^*(n)$ are the *shifted Schur* polynomials.

From the Characterization Theorem for the Schur elements $\mathbf{S}_\lambda(n) \in \boldsymbol{\zeta}(n)$ (see subsection 4.2) and the Characterization Theorem for the shifted Schur polynomials [51], we have:

Theorem 6.3. *For every λ, $\widetilde{\lambda}_1 \leq n$,*

$$\chi_n(\mathbf{S}_\lambda(n)) = \mathbf{s}_\lambda^*(x_1, \ldots, x_n).$$

From Theorem 4.13 and Proposition 6.1, it follows

Proposition 6.4.

1. *The set*
$$\left\{\mathbf{e}_k^*(x_1, x_2, \ldots, x_n); \ k = 1, 2, \ldots, n\right\}$$
   is a set of free algebra generators of the polynomial algebra $\Lambda^*(n)$.

2. *The set*
$$\left\{\mathbf{h}_k^*(x_1, x_2, \ldots, x_n); \ k = 1, 2, \ldots, n\right\}$$
   is a set of free algebra generators of the polynomial algebra $\Lambda^*(n)$.

3. *The set*
$$\left\{\mathbf{s}_\lambda^*(x_1, \ldots, x_n); \ \widetilde{\lambda}_1 \leq n\right\}$$
   is a linear basis of the polynomial algebra $\Lambda^*(n)$.

7 The algebra Λ^* of shifted symmetric functions

Let

$$\mathbf{i}_{n+1,n}^* : \Lambda^*(n) \hookrightarrow \Lambda^*(n+1)$$

---

[6] A Young tableau whose entries belong to $\{1, ..., n\}$ and weakly decrease from left to right along each row and strictly decrease down each column.

be the algebra monomorphism such that

$$\mathbf{i}^*_{n+1,n}\big(\mathbf{e}^*_k(x_1, x_2, \ldots, x_n)\big) = \mathbf{e}^*_k(x_1, x_2, \ldots, x_n, x_{n+1}),$$

for $k = 1, 2, \ldots, n$.

Given $m \in \mathbb{Z}^+$, let $\Lambda^*(n)^{(m)}$ denote the m-th filtration element of $\Lambda^*(n)$ (with respect to the filtration induced by the standard filtration of the algebra of polynomials in the variables $x_1, x_2, \ldots, x_n$).

Clearly, the monomorphisms $\mathbf{i}^*_{n+1,n}$ are *morphisms in the category of filtered algebras*, that is

$$\mathbf{i}^*_{n+1,n}\Big[\Lambda^*(n)^{(m)}\Big] \subseteq \Lambda^*(n+1)^{(m)}.$$

The algebra of *shifted symmetric functions* Λ^* is the *direct limit* (in the category of filtered algebras):

$$\Lambda^* = \varinjlim \Lambda^*(n). \tag{45}$$

The algebra Λ^* inherits a structure of filtered algebra, where

$$\Lambda^{*(m)} = \varinjlim \Lambda^*(n)^{(m)}.$$

Let

$$\boldsymbol{\pi}^*_{n,n+1} : \Lambda^*(n+1) \twoheadrightarrow \Lambda^*(n)$$

be the algebra epimorphism such that

$$\boldsymbol{\pi}^*_{n,n+1}\big(\mathbf{e}^*_k(x_1, x_2, \ldots, x_n, x_{n+1})\big) = \mathbf{e}^*_k(x_1, x_2, \ldots, x_n),$$

for $k = 1, 2, \ldots, n$, and

$$\boldsymbol{\pi}^*_{n,n+1}\big(\mathbf{e}^*_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1})\big) = 0.$$

Clearly

$$\boldsymbol{\pi}^*_{n,n+1}\big(\mathbf{f}^*(x_1, x_2, \ldots, x_n, x_{n+1})\big) = \mathbf{f}^*(x_1, x_2, \ldots, x_n, 0),$$

for every $\mathbf{f}^*(x_1, x_2, \ldots, x_n, x_{n+1}) \in \Lambda^*(n+1)$.

As for the centers $\boldsymbol{\zeta}(n+1)$ and $\boldsymbol{\zeta}(n)$, the following Remarks and Proposition on $\Lambda^*(n+1)$ and $\Lambda^*(n)$ are obvious from the definitions.

Proposition 7.1. *We have:*

1. $Ker\big(\boldsymbol{\pi}^*_{n,n+1}\big)$ *is the bilateral ideal*

$$\Big(\mathbf{e}^*_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1})\Big)$$

   of $\Lambda^*(n+1)$ *generated by the element* $\mathbf{e}^*_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1})$.

2. *The projection* $\boldsymbol{\pi}^*_{n,n+1}$ *is the left inverse of the monomorphism* $\mathbf{i}^*_{n+1,n}$. *In symbols,*

$$\boldsymbol{\pi}^*_{n,n+1} \circ \mathbf{i}^*_{n+1,n} = Id_{\Lambda^*(n)}.$$

3. *If $m \leq n$, then the restriction $\boldsymbol{\pi}^{*}_{n,n+1}{}^{(m)}$ of $\boldsymbol{\pi}^{*}_{n,n+1}$ to $\Lambda^{*}(n+1)^{(m)}$ and the restriction $\mathbf{i}^{*}_{n+1,n}$ of $\mathbf{i}^{*}_{n+1,n}$ to $\Lambda^{*}(n)^{(m)}$ are the inverse of each other.*

From Proposition 7.1, the algebra Λ^{*} (direct limit) is the same as the inverse limit in the category of filtered algebras

$$\Lambda^{*} = \varprojlim \Lambda^{*}(n)$$

with respect to the system of the projections $\boldsymbol{\pi}^{*}_{n,n+1}$, and therefore, the algebra Λ^{*} is the algebra of shifted symmetric functions of [51].

Consider the commutative diagram:

$$
\begin{array}{ccc}
\boldsymbol{\zeta}^{(m)}(n) & \xrightleftharpoons[\boldsymbol{i}_{n+1,n}]{\boldsymbol{\pi}_{n,n+1}} & \boldsymbol{\zeta}^{(m)}(n+1) \\
\Big\downarrow{\chi_n} & & \Big\downarrow{\chi_{n+1}} \\
\Lambda^{*(m)}(n) & \xrightleftharpoons[\boldsymbol{i}^{*}_{n+1,n}]{\boldsymbol{\pi}^{*}_{n,n+1}} & \Lambda^{*(m)}(n+1)
\end{array}
\tag{46}
$$

Theorem 7.2. *If $m \leq n$, the pairs of horizontal arrows in the commutative diagram (46) denote mutually inverse isomorphisms.*

Passing to the direct limit, we get the isomorphism of filtered algebras:

$$\chi : \boldsymbol{\zeta} \to \Lambda^{*}.$$

Given $\boldsymbol{\varrho} = \varinjlim \boldsymbol{\varrho}(n) \in \boldsymbol{\zeta}^{*(m)}$ and every partion μ, if

$$n \geq max\{m,\ l(\widetilde{\mu}) = \mu_1\},$$

then

$$\chi_n(\boldsymbol{\varrho}(n))(\widetilde{\mu}) = \chi_{n+1}\big(\mathbf{i}^{*}_{n+1,n}{}^{(m)}(\boldsymbol{\varrho}(n))\big)(\widetilde{\mu}) = \chi_{n+1}(\boldsymbol{\varrho}(n+1))(\widetilde{\mu}).$$

Therefore, the sequence

$$\Big(\chi_n\big((\boldsymbol{\varrho}(n))\big)(\widetilde{\mu}) \Big)_{n\in\mathbb{N}^+}$$

is *definitively constant* and the eigenvalue

$$\chi(\boldsymbol{\varrho})(\widetilde{\mu}) = \chi_n(\boldsymbol{\varrho}(n))(\widetilde{\mu}),
\tag{47}$$

n sufficiently large, is well-defined.

Corollary 7.3.

1. For every $k \in \mathbb{Z}^+$,
$$\chi(\mathbf{H}_k) = \mathbf{e}_k^* \in \Lambda^*,$$

    where
$$\mathbf{e}_k^* = \sum_{i_1 < i_2 < \cdots < i_k} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \cdots (x_{i_k}), \quad i_s \in \mathbb{Z}^+,$$

    $\mathbf{e}_k^*$ the k-th elementary shifted symmetric function;

2. For every $k \in \mathbb{Z}^+$,
$$\chi(\mathbf{I}_k) = \mathbf{h}_k^* \in \Lambda^*,$$

    where
$$\mathbf{h}_k^* = \sum_{i_1 \leq i_2 < \cdots \leq i_k} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots (x_{i_k}), \quad i_s \in \mathbb{Z}^+,$$

    $\mathbf{h}_k^*$ the k-th complete shifted symmetric function.

Since $\chi_n(\mathbf{S}_\lambda(n)) = s_\lambda^*(n)$, from Proposition 5.8, item 2) and Theorem 5.9, item 2) we have

Corollary 7.4.

1. (stability property [51]) $\boldsymbol{\pi}_{n,n+1}^*(s_\lambda^*(n+1)) = s_\lambda^*(n), \quad n \in \mathbb{Z}^+$.

2. If $n \geq |\lambda|$, then
$$\boldsymbol{i}_{n+1,n}^*(s_\lambda^*(n)) = s_\lambda^*(n+1).$$

The shifted symmetric Schur *function* s_λ^* is the (direct/inverse) limit
$$s_\lambda^* = \varinjlim(s_\lambda^*(n)) = \varprojlim(s_\lambda^*(n)).$$

Then

Corollary 7.5. *For every λ, we have*

1.
$$\chi(\mathbf{S}_\lambda) = s_\lambda^*.$$

2.
$$\mathbf{s}_\lambda^* = \sum_{T \in RSSYT(\lambda)} (x_{T(s)} - c(s)),$$

    where $RSSYT(\lambda)$ is the set of all reverse semistandard Young tableaux T of shape λ over the set $\mathbb{Z}^+$.

47

Let
$$\mathcal{W} : \boldsymbol{\zeta} \to \boldsymbol{\zeta}$$
denote the automorphism such that
$$\mathcal{W}\left(\mathbf{H}_k\right) = \mathbf{I}_k, \quad for\ every\ k \in \mathbb{Z}^+,$$

and let
$$w : \Lambda^* \to \Lambda^*$$
denote the automorphism such that
$$w\left(\mathbf{e}_k^*\right) = \mathbf{h}_k^*, \quad for\ every\ k \in \mathbb{Z}^+.$$

Clearly,
$$\chi \circ \mathcal{W} = w \circ \chi.$$

Corollary 7.6.

1. *For every partition λ,*
$$w\left(s_\lambda^*\right) = \mathbf{s}^*{}_{\widetilde{\lambda}}.$$

2. *In particular,*
$$w\left(\mathbf{h}_k^*\right) = \mathbf{e}_k^*, \quad for\ every\ k \in \mathbb{Z}^+;$$

   then, the automorphism w is an involution.

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