Ordinary differential equations invariant under two-variable Möbius transformations

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Abstract: We consider two Möbius transformations that map two variables, compute their invariants and describe the ordinary differential equations that are kept invariant under these transformations.

1 Introduction: the Möbius transformation for one variable

It is well know that the Möbius transformation plays an important role in the study of the integrability of certain nonlinear differential equations. This appears, for example, in the Painlevé analysis of differential equations [7] and in the study of symmetry-integrable evolution equations [2]. In this introduction, we sum up the well known cases of the invariance of ordinary differential equations under one-variable Möbius transformations, which sets the stage for the study of the two-variable Möbius transformations.

Consider two variables u and x, where u depends on x. We now apply the Möbius transformation on u, as follows:

$$\mathcal{M}: \begin{cases} u(x) \mapsto v(\bar{x}) = \dfrac{\alpha_1 u(x) + \beta_1}{\alpha_2 u(x) + \beta_2} \\[2ex] x \mapsto \bar{x} = x. \end{cases} \tag{1.1}$$

Here

$$\Phi = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \in SL(2, \mathbb{R}). \tag{1.2}$$

This is an example of a one-variable Möbius transformation. We compute the invariants of $sl(2,\mathbb{R})$ using the basis $\{\frac{\partial}{\partial u}, u\frac{\partial}{\partial u}, u^2\frac{\partial}{\partial u}\}$, in order to find all ordinary differential equations that are kept invariant under (1.1). The two fundamental invariants [5] are $\omega_1 = x$ and the Schwarzian derivative $S = \dfrac{u_{xxx}}{u_x} - \dfrac{3}{2}\dfrac{u_{xx}^2}{u_x^2}$ (see for example [6]), whereby the higher-order invariants are the x-derivatives of S, i.e. $\{S_x, S_{xx}, \ldots\}$. We conclude that all ordinary differential equations that are invariant under (1.1) are of the form

$$\Psi(x, S, S_x, S_{xx}, \ldots, S_{mx}) = 0, \quad S_{mx} := \frac{d^m S}{dx^m}, \tag{1.3}$$

where Ψ is an arbitrary smooth function. The 3rd-order equation $\Psi(x, S) = 0$ is the lowest order equation that is invariant under (1.1). If we assume that $\Psi(x, S) = 0$ can be solved algebraically for S, we obtain the equation $u_{xxx} = \frac{3}{2}\frac{u_{xx}^2}{u_x} + u_x\phi(x)$, where ϕ is an arbitrary smooth function. This has been reported in [3].

Exchanging the roles of u and x in the Möbius transformation (1.1), so that x is mapped instead of u, we obtain the fundamental invariants $\omega_1 = u$ and $\omega_2 = \frac{u_{xxx}}{u_x^3} - \frac{3}{2}\frac{u_{xx}^2}{u_x^4} \equiv \frac{S}{u_x^2}$. The equations that are invariant under this Möbius transformation then take the form [1]

$$\Psi(u, \omega_2, \frac{d\omega_2}{du}, \frac{d^2\omega_2}{du^2}, \dots, \frac{d^m\omega_2}{du^m}) = 0. \tag{1.4}$$

The aim of the current letter is to consider the cases where the Möbius transformation acts on two variables and to obtain the ordinary differential equations that are kept invariant under those transformations. This gives rise to two cases, namely **Case 1** and **Case 2**, as reported in the next section.

2 Möbius transformations for two variables

We discuss two cases, whereby the Möbius transformation acts on two variables.

Case 1: Consider the following Möbius transformation that acts on the two dependent variables $u_1(x)$ and $u_2(x)$, as follows:

$$\mathcal{M} : \begin{cases} u_1(x) \mapsto v_1(\bar{x}) = \dfrac{a_{11}u_1(x) + a_{12}u_2(x) + b_{11}}{c_{11}u_1(x) + c_{12}u_2(x) + \beta} \\ \\ u_2(x) \mapsto v_2(\bar{x}) = \dfrac{a_{21}u_1(x) + a_{22}u_2(x) + b_{21}}{c_{11}u_1(x) + c_{12}u_2(x) + \beta} \\ \\ x \mapsto \bar{x} = x. \end{cases} \tag{2.1}$$

Here

$$\Phi = \begin{pmatrix} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{21} \\ c_{11} & c_{12} & \beta \end{pmatrix} \in SL(3, \mathbb{R}). \tag{2.2}$$

We construct the Lie generators of the corresponding infinitesimal transformation of (2.1) given by SL$(3, \mathbb{R})$. A basis for the 8-dimensional matrix Lie algebra $sl(3, \mathbb{R})$ in given by eight 3×3 matrices $\{X_j\}$. Recall that $\Phi = \exp(\epsilon X)$, where $\Phi \in SL(3, \mathbb{R})$ and $X \in sl(3, \Re)$. Since $\det\Phi = 1$ and $\det\Phi = \exp(\epsilon \operatorname{Tr} X)$, it follows that $\operatorname{Tr}(X_j) = 0$ for all $j = 1, 2, \dots, 8$. Applying the above matrices, we obtain the following Lie generators for the basis of the 8-dimensional Lie algebra $sl(3, \mathbb{R})$:

$$\{Z_1 = \frac{\partial}{\partial u_1},\ Z_2 = \frac{\partial}{\partial u_2},\ Z_3 = u_1\frac{\partial}{\partial u_1},\ Z_4 = u_2\frac{\partial}{\partial u_2},\ Z_5 = u_1\frac{\partial}{\partial u_2},\ Z_6 = u_2\frac{\partial}{\partial u_1},$$

$$Z_7 = u_1^2\frac{\partial}{\partial u_1} + u_1u_2\frac{\partial}{\partial u_2},\ Z_8 = u_1u_2\frac{\partial}{\partial u_1} + u_2^2\frac{\partial}{\partial u_2}\}. \tag{2.3}$$

We refer the reader to the books [5] and [4] for details. We now compute the invariants for the Möbius transformation (2.1) up to order four. That is, we find the invariants I of $sl(3,\mathbb{R})$ with generators (2.3). The conditions are $Z_j^{(3)}I = 0,\ j = 1, 2, \ldots, 8$, where $Z^{(3)}$ denotes the 3rd prolongation of Z_j, I is assumed to depend on x, u_1, u_2 as well as all derivatives with respect to x up to order four. This leads to the following general solution for this condition: $I = F(\omega_0, \omega_1, \omega_2)$, where, besides the obvious invariant $\omega_0 = x$, we obtain two more invariants ω_1 and ω_2, namely

$$\begin{aligned}\omega_1 = {} & \frac{1}{(u_{1,x}u_{2,xx} - u_{1,xx}u_{2,x})^2}\Big(3u_{1,x}^2u_{2,xx}u_{2,xxxx} - 4u_{1,x}^2u_{2,xxx}^2 - 6u_{1,x}u_{1,xxx}u_{2,xx}^2 \\ & -3u_{1,x}u_{1,xxxx}u_{2,x}u_{2,xx} + 6u_{1,x}u_{1,xx}u_{2,xx}u_{2,xxx} - 3u_{1,x}u_{1,xx}u_{2,x}u_{2,xxxx} \\ & +8u_{1,x}u_{1,xxx}u_{2,x}u_{2,xxx} + 6u_{1,xx}u_{1,xxx}u_{2,x}u_{2,xx} - 6u_{1,xx}^2u_{2,x}u_{2,xxx} \\ & - 4u_{1,xxx}^2u_{2,x}^2 + 3u_{1,xx}u_{1,xxxx}u_{2,x}^2\Big)\end{aligned} \tag{2.4a}$$

$$\begin{aligned}\omega_2 = {} & \frac{1}{(u_{1,x}u_{2,xx} - u_{1,xx}u_{2,x})^3}\Big(9u_{1,x}^2u_{1,xxxx}u_{2,xx}^3 + 8u_{1,xxx}^3u_{2,x}^3 - 8u_{1,x}^3u_{2,xxx}^3 \\ & +24u_{1,x}^2u_{1,xxx}u_{2,x}u_{2,xxx}^2 - 24u_{1,x}u_{1,xx}^2u_{2,x}u_{2,xxx}^2 - 24u_{1,x}u_{1,xxx}^2u_{2,x}^2u_{2,xxx} \\ & -24u_{1,x}^2u_{1,xxx}u_{2,xx}^2u_{2,xxx} + 24u_{1,x}u_{1,xxx}^2u_{2,x}u_{2,xx}^2 + 9u_{1,xx}^2u_{1,xxxx}u_{2,x}^2u_{2,xx} \\ & -24u_{1,xx}u_{1,xxx}^2u_{2,x}^2u_{2,xx} - 6u_{1,xx}u_{1,xxx}u_{1,xxxx}u_{2,x}^3 + 24u_{1,xx}^2u_{1,xxx}u_{2,x}^2u_{2,xxx} \\ & +24u_{1,x}^2u_{1,xx}u_{2,xx}u_{2,xxx}^2 + 6u_{1,x}^3u_{2,xx}u_{2,xxx}u_{2,xxxx} - 9u_{1,x}^2u_{1,xx}u_{2,xx}^2u_{2,xxxx} \\ & -6u_{1,x}^2u_{1,xxx}u_{2,x}u_{2,xx}u_{2,xxxx} + 6u_{1,x}u_{1,xx}u_{1,xxx}u_{2,x}^2u_{2,xxxx} \\ & +18u_{1,x}u_{1,xx}^2u_{2,x}u_{2,xx}u_{2,xxxx} - 6u_{1,x}^2u_{1,xx}u_{2,x}u_{2,xxx}u_{2,xxxx} \\ & -9u_{1,xx}^3u_{2,x}^2u_{2,xxxx} - 6u_{1,x}^2u_{1,xxxx}u_{2,x}u_{2,xx}u_{2,xxx} \\ & +6u_{1,x}u_{1,xx}u_{1,xxxx}u_{2,x}^2u_{2,xxx} - 18u_{1,x}u_{1,xx}u_{1,xxxx}u_{2,x}u_{2,xx}^2 \\ & + 6u_{1,x}u_{1,xxx}u_{1,xxxx}u_{2,x}^2u_{2,xx}\Big).\end{aligned} \tag{2.4b}$$

Higher-order invariants are given by the x-derivatives of ω_1 and ω_2. From the invariants ω_0, ω_1 and ω_2, we conclude that the 4th-order semilinear system of two ordinary differential equations that are invariant under the Möbius transformation (2.1) has the following form:

$$u_{1,xxxx} = -\frac{1}{6(u_{1,x}u_{2,xx} - u_{1,xx}u_{2,x})}\Big[8u_{1,xxx}^2u_{2,x} - 8u_{1,xxx}u_{1,x}u_{2,xxx}$$

$$-12u_{1,xxx}u_{1,xx}u_{2,xx}+12u_{1,xx}^2u_{2,xxx}+k_1(x)(6u_{1,x}^2u_{2,xxx}-6u_{1,x}u_{2,x}u_{1,xxx}$$

$$\left.-9u_{1,x}u_{1,xx}u_{2,xx}+9u_{1,xx}^2u_{2,x})\right]+k_2(x)u_{1,x}, \tag{2.5a}$$

$$u_{2,xxxx}=-\frac{1}{6(u_{1,x}u_{2,xx}-u_{1,xx}u_{2,x})}\Big[-8u_{1,x}u_{2,xxx}^2+8u_{1,xxx}u_{2,x}u_{2,xxx}$$

$$+12u_{2,xxx}u_{1,xx}u_{2,xx}-12u_{2,xx}^2u_{1,xxx}+k_1(x)(-6u_{2,x}^2u_{1,xxx}+6u_{1,x}u_{2,x}u_{2,xxx}$$

$$\left.+9u_{2,x}u_{1,xx}u_{2,xx}-9u_{2,xx}^2u_{1,x})\right]+k_2(x)u_{2,x}, \tag{2.5b}$$

where k_1 and k_2 are arbitrary functions of x. We remark that the same system of 4th-order equations (2.5a)-(2.5b) follows by applying the Lie symmetry invariance condition with the Lie symmetry basis generators (2.3) for a general system of two ordinary differential equations up to order four. There exists no system of two 1st-order, two 2nd-order or two 3rd-order ordinary differential equations that are invariant under the Möbius transformation (2.1). Of course higher order systems that are invariant under (2.1) can be constructed by considering the x-derivatives of ω_1 and ω_2. Those systems are of the form

$$\Psi_1(x,\omega_1,\omega_2,\omega_{1,x},\omega_{2,x},\ldots)=0,\quad \Psi_2(x,\omega_1,\omega_2,\omega_{1,x},\omega_{2,x},\ldots)=0.$$

Since the transformation (2.1) maps solutions to solutions for those equations that are kept invariant, we can use the transformation to map special solutions of the equations to new solutions that will contain the arbitrary parameters in the transformation (2.1). For example, for system (2.5a)-(2.5b) with $k_1=k_2=0$, (2.1) maps the special solution $u_1=x$, $u_2=x^2$ to the solution $u_1=\dfrac{1}{x^2+s_1x+1}$ and $u_2=\dfrac{x+s_2}{x^2+s_1x+1}$, where s_1 and s_2 are arbitrary constants (we have set some of the constants in (2.1) to zero and some to one). We then map the latter solution again by (2.1) to obtain the following general solution of system (2.5a)-(2.5b) with $k_1=k_2=0$:

$$u_1(x)=\frac{b_{11}x^2+q_1x+q_2}{x^2+q_3x+q_4},\quad u_2(x)=\frac{b_{21}x^2+q_5x+q_6}{x^2+q_3x+q_4}. \tag{2.6a}$$

Here $q_1=a_{12}+s_1b_{11}$, $q_2=a_{11}+b_{11}+s_2a_{12}$, $q_3=s_1+c_{12}$, $q_4=c_{11}+s_2c_{12}+1$, $q_5=a_{22}+s_1b_{21}$ and $q_6=a_{21}+b_{21}+s_2a_{22}$. Note that the solution (2.6a) contains eight arbitrary constants, taking condition (2.2) into account.

Case 2: Another possibility for a two-variable Möbius transformation that is associated with $SL(3,\mathbb{R})$ is to consider one dependent variable $u(x)$ and then apply the transformation on both u and x, as follows:

$$\mathcal{M}:\begin{cases} x\mapsto \bar{x}=\dfrac{a_{11}x+a_{12}u(x)+b_{11}}{c_{11}x+c_{12}u(x)+\beta}\\[2ex] u(x)\mapsto v(\bar{x})=\dfrac{a_{21}x+a_{22}u(x)+b_{21}}{c_{11}x+c_{12}u(x)+\beta},\end{cases} \tag{2.7}$$

taking into account condition (2.2). In this case we obtain the following basis for the corresponding 8-dimensional Lie algebra $sl(3,\mathbb{R})$:

$$\{Z_1=\frac{\partial}{\partial x},\ Z_2=\frac{\partial}{\partial u},\ Z_3=x\frac{\partial}{\partial x},\ Z_4=u\frac{\partial}{\partial u},\ Z_5=x\frac{\partial}{\partial u},\ Z_6=u\frac{\partial}{\partial x},$$
$$Z_7=x^2\frac{\partial}{\partial x}+xu\frac{\partial}{\partial u},\ Z_8=xu\frac{\partial}{\partial x}+u^2\frac{\partial}{\partial u}\}. \tag{2.8}$$

Computing the invariants up to order seven, we find only one, namely the following 7th-order invariant:

$$\begin{aligned}\omega_1 &= \frac{u_{xx}^4 u_{7x}}{(45u_{xx}u_{xxx}u_{4x}-9u_{xx}^2u_{5x}-40u_{xxx}^3)^{5/3}}\\ &+\frac{7}{18(45u_{xx}u_{xxx}u_{4x}-9u_{xx}^2u_{5x}-40u_{xxx}^3)^{8/3}}\Big[675u_{xx}^4u_{4x}^4+1125u_{4x}^3u_{xxx}^2u_{xx}^3\\ &-1890u_{5x}u_{4x}^2u_{xxx}u_{xx}^4-270u_{6x}u_{4x}^2u_{xx}^5-4500u_{4x}^2u_{xxx}^4u_{xx}^2+405u_{5x}^2u_{4x}u_{xx}^5\\ &+1800u_{5x}u_{4x}u_{xxx}^3u_{xx}^3+450u_{6x}u_{4x}u_{xxx}^2u_{xx}^4+4800u_{4x}u_{xxx}^6u_{xx}+108u_{5x}^2u_{xxx}^2u_{xx}^4\\ &-162u_{6x}u_{xxx}u_{xx}^5u_{5x}-960u_{5x}u_{xxx}^5u_{xx}^2+27u_{6x}^2u_{xx}^6-1600u_{xxx}^8\Big].\end{aligned} \tag{2.9}$$

Searching for all ordinary differential equations that admit the Lie symmetries (2.8) up to order seven, we obtain two equations, namely the 5th-order equation

$$u_{5x}=\frac{5u_{xxx}}{9u_{xx}^2}\left(9u_{4x}u_{xx}-8u_{xxx}^2\right), \tag{2.10}$$

as well as the 7th-order equation

$$\omega_1=k, \tag{2.11}$$

where ω_1 is given by (2.9). Obviously equation (2.10) is not related to an invariant of the Möbius transformation (2.7) but this equation does admit the Lie symmetry algebra $sl(3,\mathbb{R})$ with basis (2.8). Therefore, the Möbius transformation (2.7) does map solutions to new solutions for both (2.10) and (2.11). We remark that (2.10) can be solved in general. One way to obtain the general solution is to use the substitution $r(x)=u_{xx}$, by which (2.10) reduces to a 3rd-order equation which admits seven Lie point symmetries, where the 3rd-order equation is linearizable by a point transformation. An alternate way is to make us to use the Möbius transformation (2.7) as follows. Consider (2.10) in terms of the variable $v(\bar{x})$ with the special solution $v(\bar{x})=\bar{x}^2$. Applying the transformation (2.7) we have

$$\bar{x}^2=\frac{a_{21}x+a_{22}u(x)+b_2}{c_{11}x+c_{12}u(x)+\beta}\quad\text{where}\quad \bar{x}=\frac{a_{11}x+a_{12}u(x)+b_{11}}{c_{11}x+c_{12}u(x)+\beta}. \tag{2.12}$$

By solving $u(x)$ from (2.12) and taking into account condition (2.2), we easily obtain the general solution of (2.10).

Special solutions of the 7th-order equation (2.11) can of course also be mapped into new multi-parameter solutions by the transformation (2.7). A further analysis of equation (2.11), including a detailed Lie symmetry analysis, will be published elsewhere.

Concluding Remarks: The two reported cases of two-variable Möbius transformations reveal that the lowest order ordinary differential equations that are invariant under these transformations are rather complicated and of high order, namely a 4th-order system of two equations in Case 1 and a 7th-order scalar equation in Case 2. In Case 1, for the Möbius transformation (2.1), we show that the equations that result from the invariants of the Lie algebra with basis (2.3) coincided with the Lie symmetry invariance classification of the equations for this Lie algebra. However, in Case 2, for the Möbius transformation (2.7), we find that the Lie symmetry algebra with basis (2.8) results in both the 5th-order equation (2.10) and the 7th-order equation (2.11), whereby the 5th-order equation (2.10) is not related to an invariant of this transformation.

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