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HOMFLY POLYNOMIALS FROM THE HILBERT SCHEMES OF A PLANAR CURVE,
[after D. Maulik, A. Oblomkov, V. Shende...]

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INTRODUCTION

Among the most interesting invariants one can associate with an oriented link $\mathcal{L} \subset S^3$ is its HOMFLY-PT polynomial $\mathbf{P}(\mathcal{L}, v, s) \in \mathbb{Z}[v^{\pm 1}, (s - s^{-1})^{\pm 1}]$ ([13, 33]). In 2010 A. Oblomkov and V. Shende ([32]) conjectured that this polynomial can be expressed in algebraic geometric terms when $\mathcal{L}$ is an *algebraic link,* that is, it is obtained as the intersection of a plane curve singularity $(C, p) \subset \mathbb{C}^2$ with a small sphere centered at p. More precisely, let $C_p^{[n]}$ be the punctual Hilbert scheme of C at p, parameterizing the length n subschemes of C supported at p. If $m : C_p^{[n]} \to \mathbb{Z}$ is the function associating with the subscheme $Z \in C_p^{[n]}$ the minimal number $m(I)$ of generators of its defining ideal I in the local ring $\mathcal{O}_{C,p}$, they conjecture that the generating function

$$Z(C, v, s) = \sum_{n \geq 0} s^{2n} \int_{C_p^{[n]}} (1 - v^2)^{m(I)} d\chi_{\mathrm{top}}$$

coincides with $\mathbf{P}(\mathcal{L}, v, s)$, up to the monomial term $\left(\frac{v}{s}\right)^{\mu(f)-1}$, where $\mu(f)$ is the Milnor number. In the formula the integral is done with respect to the Euler characteristic measure $d\chi_{\mathrm{top}}$. Shortly afterwards, this surprising identity was generalized in two different directions:

1. In [31], Oblomkov, Rasmussen and Shende propose a “homological version”: while the equality of Oblomkov and Shende is at the level of Euler characteristics, they conjecture a relation between the HOMFLY homology of Khovanov and Rozansky and the virtual Poincaré polynomial of the Hilbert schemes of (C, p). This conjecture, still open, will be shortly discussed in Section 7.
2. In [8], E. Diaconescu, Z. Hua and Y. Soibelman conjectured an equality in case the data of C and $\mathcal{L}$ are “colored” by choosing an array $\overrightarrow{\mu}$ of partitions, one for every branch of the curve, or equivalently for every component of the link. This choice allows one to define two enhancements of the original objects:
   - On the algebraic geometric side a thickening $C_{\overrightarrow{\mu}}$ of C, using the correspondence between partitions and monomial ideals in the plane (see Definition 5.8).

– On the link side a "satellite link" $\mathcal{L}_{\overrightarrow{\mu}} = \mathcal{L} * (Q_{\mu_1}, \cdots, Q_{\mu_N})$, by associating special braids with the partitions, closing them to links Q_μ and finally wrapping them around $\mathcal{L}$ (see Eq. (61)).

In this setting, the Hilbert schemes are replaced by the moduli spaces $\mathcal{P}(Y, C, \mu, r, n)$ of stable pairs framed on $C_{\overrightarrow{\mu}}$, whereas $\mathbf{P}(\mathcal{L}, v, s)$ is replaced by the colored HOMFLY-PT polynomial $W(\mathcal{L}, \overrightarrow{\mu}; v, s)$. Diaconescu, Hua and Soibelman conjectured that a generating function arising from the topological Euler characteristic of the spaces $\mathcal{P}(Y, C, \mu, r, n)$ should coincide with $W(\mathcal{L}, {}^t\overrightarrow{\mu}; v, s)$, where ${}^t\overrightarrow{\mu}$ is the vector of transposed partitions, see Theorem 5.41 for the precise statement.

It is shown in [31] that choosing all partitions to be trivial yields the conjecture of Oblomkov and Shende as a special case. The conjecture of Diaconescu, Hua and Soibelman was proved by D. Maulik in 2012 in the striking paper [24]. The proof proceeds by showing that the two sides of the identity have the same behaviour when the singular point is blown up, thus reducing to the case when the singularity is a single node, where a direct verification is possible. It is worth noticing that, even starting in the original uncolored setting of Oblomkov and Shende, the blow-up procedure leads to colored links and curves. Therefore, even though the set-up in the colored version is much more technical than the one required to explain the original conjecture, we need to discuss this level of generality, besides its intrinsic interest and beauty. As the details of the proof of Theorem 5.41 are quite involved, but well presented in the original paper [24], this seminar will only give a sketch of the main ideas used in the proof, and focus instead on presenting the definitions and foundations needed, along with some examples, so as to provide the necessary background for the reading of [24].

1. ALGEBRAIC LINKS

We summarize a few classical facts on singular points of a plane curve and their links (see [28] for a historical account and references to the original papers). Let (C, p) be a germ of a reduced plane curve singularity, defined as the zero set of a local equation $f = 0$, where $f \in \mathbb{C}[X, Y]$, with $f(0, 0) = 0$. We denote by $\mathfrak{m}$ the maximal ideal of functions vanishing at the point $p = (0, 0)$. We denote also by f and $\mathfrak{m}$ their images in $\mathbb{C}[[X, Y]]$. The point p is singular if $\partial_x f, \partial_y f \in \mathfrak{m}$. Under these hypotheses, the ideal $(\partial_x f, \partial_y f)$ is $\mathfrak{m}$-primary, and the quotient algebra $\mathbb{C}[[X, Y]]/(\partial_x f, \partial_y f)$ is a finite-dimensional vector space, whose dimension $\mu(f)$ is the *Milnor number* of the singular point. If $f = \sum_{k \in \mathbb{N}} f_k$, with f_k homogeneous of degree k, let f_d be the first nonzero homogeneous component. Then $d =: \mathrm{mult}_p(C)$ is called the *multiplicity* of C at p, and the scheme defined by $f_d(X, Y) = 0$ is the *tangent cone*. It is a union of lines, possibly with multiplicities. Let $Bl_p : \widetilde{\mathbb{A}^2(\mathbb{C})} \to \mathbb{A}^2(\mathbb{C})$ be the blow up at p. The points in the intersection of the proper transform $\widetilde{C}$ of C with the exceptional divisor correspond to the lines in the tangent cone. By the theorem on embedded resolution of singularities, [43, Theorem 3.4.4], there exists a sequence of blow-ups so that the (reduced) total transform of C is a normal crossing curve.

Let $f = \prod_i f_i$, with $f_i \in \mathfrak{m}$, be the factorization in irreducibles of f in $\mathbb{C}[[X,Y]]$: since C is reduced, no multiple factors appear. The curves C_i defined by the equations $f_i = 0$ are called the *branches* of the germ (C,p). Let $S^3_\epsilon \subset \mathbb{A}^2(\mathbb{C})$ be a sphere of radius ϵ centered at p. For small enough ϵ the sphere and C intersect transversally, therefore $\mathcal{L} := S^3_\epsilon \cap C$ is a nonsingular oriented one-dimensional submanifold of $S^3_\epsilon \simeq S^3$, whose isotopy class is independent of ϵ, *the link of the singularity.*

If f is irreducible in $\mathbb{C}[[X,Y]]$, then its link is connected, so actually a *knot.* More generally, the connected components of $\mathcal{L}$ correspond to the branches of (C,p).

Example 1.1. — *Let $f = y^r - x^s$, with $r \leq s$. If $r < s$, the tangent cone is the line $y = 0$ with multiplicity r, while if $r = s$ it consists of the r distinct lines $y - \xi^i x = 0$, with ξ a primitive r-th root of unity. If r and s are coprime there is a unique branch, whose link is the toral (r,s) knot $\mathcal{L}_{r,s} \subset S^1 \times S^1$, parameterized by*

$$x = \exp(\sqrt{-1}rt),\ \ y = \exp(\sqrt{-1}st)\ \textit{with}\ t \in [0, 2\pi].$$

Otherwise, let $r = da, s = db$, with a and b coprime, where d is the greatest common divisor of r and s. Letting ξ be a primitive d-th root of unity, the factorization

$$y^r - x^s = (y^a)^d - (x^b)^d = \prod_{\ell=0}^{d-1} (y^a - \xi^\ell x^b) \tag{1}$$

shows that $\mathcal{L}_{r,s}$ has d connected components, each isomorphic to the (a,b) toral knot. Notice that for $r = s = 2$ we obtain the Hopf link. As every link (Alexander's Theorem), $\mathcal{L}_{r,s}$ can be obtained as the closure of a braid: it is isomorphic to the closure of $(\beta_r)^s$, where β_r is the braid with r strands in which the first strand passes under all the other ones (if the strands are oriented from top to bottom, see Section 3.1 for the sign convention), that is, the product of the standard generators σ_i of the braid group $\mathscr{B}_r$.

Definition 1.2. — *Given two germs of curves C (resp D) through p, of equations $f = 0$ (resp $g = 0$), with no common factor, their* intersection number *at p is*

$$C \bullet D = \dim \mathbb{C}[[X,Y]]/(f,g). \tag{2}$$

The corresponding notion on the link side is that of linking number ([19, Chapter I]):

Definition 1.3. — *Given two disjoint oriented knots $K_1, K_2 \subset S^3$, let U_1 be a tubular neighborhood, homeomorphic to $S^1 \times D^2$, of K_1, disjoint from K_2. The homology group $H_1(S^3 \setminus U_1)$ is canonically isomorphic to $\mathbb{Z}$, and generated by a meridian of U_1, i.e. a circle bounding a disk in U_1 and meeting K_1 positively in only one point. Then the* linking number *of K_1 and K_2 is defined as the homology class $L(K_1, K_2) \in \mathbb{Z}$ of $K_2 \subset S^3 \setminus U_1$.*

It is easy to see that $L(K_1, K_2) = L(K_2, K_1)$ (see [19, Chapter I]). The relation between the two notions just defined is:

PROPOSITION 1.4. — *If K_1 is the link of C and K_2 is the link of D, then $L(K_1, K_2) = C \bullet D$. In particular the linking numbers of components of algebraic knots are strictly positive.*

The links arising from curve singularities via this construction are called *algebraic links*, and, among their several distinctive features, probably the most important is the description of their single components as *iterated torus knots* (also called cable knots), which is the topological counterpart of the Newton-Puiseux theorem: assume that f is irreducible in $\mathbb{C}[[x, y]]$ and $f(x, y) \neq x$. Up to a change of coordinates we can assume that it is a monic polynomial in y with coefficients in $\mathbb{C}[[X]]$. Then one can "solve in y as a function of x" and the Newton-Puiseux theorem states that y can be expressed as a power series in fractional powers of x. It will be useful to write this series as

$$y(x) = x^{\frac{q_0}{p_0}}(a_0 + x^{\frac{q_1}{p_0 p_1}}(a_1 + x^{\frac{q_2}{p_0 p_1 p_2}}(a_2 + \dots))) \tag{3}$$

where $a_i \neq 0$, each Newton pair (p_i, q_i) consists of relatively prime positive integers, and, eventually, $p_k = 1$. This leads to an inductive description of the knot as an iterated toral knot: We consider $y(x) = x^{\frac{q_0}{p_0}}$ as the first approximation (a toral knot K_0 of type (p_0, q_0)). Then $y(x) = x^{\frac{q_0}{p_0}}(a_0 + a_1 x^{\frac{q_1}{p_0 p_1}})$ gives the second approximation, describing a toral knot K_1 wrapped around K_0 and so on. In order to state this iterative description precisely, one needs at each step to have a *framing* of the knot: this notion will be discussed in a more general framework later (Section 5.2.1) and for the time being we shall limit ourselves to a "carousel" description of a specific example.

EXAMPLE 1.5. — [10] *Let $f(x, y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$. There is a unique branch which admits the parameterization*

$$x = t^4,\ y = t^6 + t^7,\ \text{or equivalently the Puiseux series } y = x^{\frac{3}{2}} + x^{\frac{7}{4}}. \tag{4}$$

Up to a rescaling, the link $\mathcal{L}$ is described by

$$x = \exp(4\sqrt{-1}t),\ y = \exp(6\sqrt{-1}t) + \rho \exp(7\sqrt{-1}t),$$

with $\rho << 1$. Since ρ is small, $\mathcal{L}$ is contained in a tubular neighborhood of the "leading knot" L of equations $x = \exp(4\sqrt{-1}t)$, $y = \exp(6\sqrt{-1}t)$ (a $(2, 3)$ knot) of which $\mathcal{L}$ is a satellite: for any point of L there are two points orbiting around. In a proper parametrization of the tubular neighborhood they can be seen describing a torus knot of type $(2, 13)$ (with respect to the natural *framing, see Example 5.15).*

In general the Puiseux parameterization may contain infinitely many terms, but only a finite number of them will be relevant for the topology of the knot, which will be then described as an iteration of the construction of Example 1.5, in which the types of the toric knots can be determined by the series of the Puiseux exponents [43, 10].

REMARK 1.6. — *Another important distinctive property of algebraic links is that their topology is uniquely determined by the topology of their components and their pairwise linking numbers (*[28, Theorem 1.1]*).*

2. PUNCTUAL HILBERT SCHEMES AND NESTED HILBERT SCHEMES

Given a plane curve $C \subset \mathbb{A}^2(\mathbb{C})$ and a point $p \in C$, its punctual Hilbert scheme of length n, denoted $C_p^{[n]}$, parameterizes 0-dimensional subschemes $Z \subset C$, such that $\dim \Gamma(Z, \mathcal{O}_Z) = n$ and $Z_{red} = p$. Let $\mathcal{O}_{C,p}$ be the local ring of C at p and denote by $\mathfrak{m}_p$ its maximal ideal. The points $Z \in C_p^{[n]}$ will be identified with their defining ideals $I \subset \mathcal{O}_{C,p}$. The condition that $Z_{red} = p$ translates into $\sqrt{I} = \mathfrak{m}_p$, and there is a natural constructible function $m : C_p^{[n]} \to \mathbb{Z}_{>0}$, defined as

$$m(I) = \text{minimal number of generators of } I = \dim_{\mathbb{C}} I/\mathfrak{m}_p I, \tag{5}$$

the last equality stemming from Nakayama's lemma.

A variant of this construction, which is relevant for our purposes, is the nested Hilbert scheme: given $\ell, n \in \mathbb{N}$,

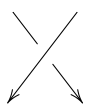
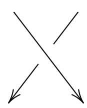
$$C_p^{[\ell,\ell+n]} = \{\mathfrak{m}_p J \subset I \subset J,\, I \in C_p^{[\ell+n]}, J \in C_p^{[\ell]}\} \subset C_p^{[\ell+n]} \times C_p^{[\ell]}. \tag{6}$$

Remark 2.1. — *The projection $C_p^{[\ell,\ell+n]} \to C_p^{[\ell]}$, sending (I, J) to J is, when restricted to a level set of m, a fibration, with fibre the Grassmannian* $\mathrm{Gr}(n, m(I))$*. In fact, by Nakayama's lemma we have* $\dim J/\mathfrak{m}_p J = m(J)$*, and, given J, every n-codimensional subspace $W \subset J/\mathfrak{m}_p J$ defines the colength $\ell + n$ ideal $W + \mathfrak{m}_p J \subset \mathcal{O}_{C,p}$.*

3. KNOTS AND LINKS AND THE HOMFLY-PT POLYNOMIAL

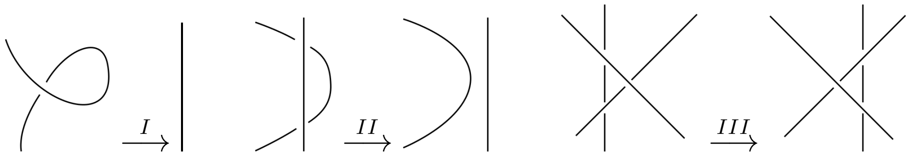
3.1. Diagrams of links

An oriented link is represented by a planar diagram, a collection of oriented closed curves, which we will assume differentiable, with at most simple crossings and the indication of which arc lies over the other. In other words, a neighborhood of a crossing is oriented diffeomorphic to one of the following:

(7) $\mathcal{L}_+ :$  $\epsilon = 1$ $\qquad \mathcal{L}_- :$  $\epsilon = -1$.

As indicated in (7), we associate a sign ϵ with a crossing. We will often use, without explicit mention, two basic theorems on knots:

- The theorem of Reidemeister, stating that two different diagrams represent the same link if and only if they are related by a sequence of the three Reidemeister moves and an oriented diffeomorphism of the plane (see [19, Chapter I]).



The three Reidemeister moves

– The theorem of Alexander, stating that every link can be realized as the closure of an appropriate braid (see [14, §2.3]).

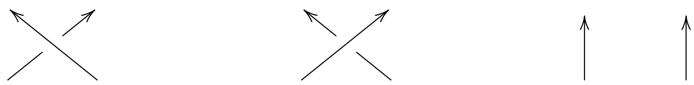
3.2. The HOMFLY-PT polynomial

The HOMFLY-PT polynomial of an oriented link $\mathcal{L}$ is defined as the unique element in $\mathbf{P}(\mathcal{L}, v, s) \in \mathbb{Z}[v^{\pm 1}, (s - s^{-1})^{\pm}]$ normalized by

$$\mathbf{P}(\text{unknot}, v, s) = \frac{v - v^{-1}}{s - s^{-1}}, \tag{8}$$

and satisfying the following *skein* relation (for which we follow the convention of [24]): Assume the diagrams of three links $\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0$ coincide except in the neighborhood of a point, where they look like:

(9)



$\mathcal{L}_-$ $\mathcal{L}_+$ $\mathcal{L}_0$

Then

$$v\mathbf{P}(\mathcal{L}_-) - v^{-1}\mathbf{P}(\mathcal{L}_+) = (s - s^{-1})\mathbf{P}(\mathcal{L}_0). \tag{10}$$

These two conditions determine a well-defined invariant on oriented links in S^3.

Example 3.1. — **Two unlinked unknots :** *Applying the skein relation we get* $v\mathbf{P}(\text{unknot}) - v^{-1}\mathbf{P}(\text{unknot}) = (s - s^{-1})\mathbf{P}(\text{two unlinked unknots})$, *hence*

$$\mathbf{P}(\text{two unlinked unknots}) = \left(\frac{v - v^{-1}}{s - s^{-1}}\right)^2 = \mathbf{P}(\text{unknot})^2. \tag{11}$$

The Hopf link : *The skein relation gives:*

$$v\mathbf{P}(\text{two unlinked unknots}) - v^{-1}\mathbf{P}(\text{Hopf}) = (s - s^{-1})\mathbf{P}(\text{unknot}).$$

Hence

$$\mathbf{P}(\text{Hopf}) = \left(\frac{v - v^{-1}}{s - s^{-1}}\right)\left(\frac{v^3 - v}{s - s^{-1}} - v(s - s^{-1})\right). \tag{12}$$

The $(2,3)$ **toral knot (trefoil) :** *Again applying the skein relation to a positive crossing of the diagram,*

$$v\mathbf{P}(\text{unknot}) - v^{-1}\mathbf{P}(\text{trefoil}) = (s - s^{-1})\mathbf{P}(\text{Hopf}).$$

Hence

$$\mathbf{P}(\text{trefoil}) = \left(\frac{v - v^{-1}}{s - s^{-1}}\right)\left(v^2(s^2 + s^{-2}) - v^4\right). \tag{13}$$

We notice that the HOMFLY-PT polynomial does not change if all the orientations of the components of the link are changed, but it does if only some of them are. This will not be relevant for us since algebraic links are given a canonical orientation coming from the orientation of C and that of S^3_ϵ.

4. THE CONJECTURE OF OBLOMKOV-SHENDE

In [32] A. Oblomkov and V. Shende [(1)] conjecture the following surprising equality, relating algebraic geometric data on (C, p) with topological invariants on $\mathcal{L}$:

Theorem 4.1 (D. Maulik, [24]). — *Let (C, p) be the germ of a singular plane curve, with Milnor number $\mu(f)$, and $\mathcal{L}$ its associated oriented link. Let $C_p^{[\ell,\ell+n]}$ denote the punctual nested Hilbert scheme, defined in Eq.* (6)*. Let* $\mathbf{P}$ *denote the HOMFLY-PT polynomial of $\mathcal{L}$. Then*

$$\left(\frac{v}{s}\right)^{\mu(f)-1} \sum_{\ell,n\geq 0} s^{2\ell}(-v^2)^n \chi_{\text{top}}(C_p^{[\ell,\ell+n]}) = \mathbf{P}(\mathcal{L}, v, s), \tag{14}$$

where χ_{top} denotes the topological Euler characteristic.

A remarkable completely unexpected consequence of Eq. (14) is that the left hand side depends only on the topology of the link, and does not detect the analytic moduli of the singularities. This can be tested explicitly in the case of the singularities $xy(x - y)(x - \alpha y) = 0$ which are not analytically equivalent for different values of α, but have equivalent links (four circles, each simply linked with every other one). Another completely non obvious feature is that the series on the left hand side represents a rational function in v and s.

To write the left hand side of the equality in a more elegant form, we first recall the notion of integration of a constructible function against the Euler characteristic measure: since the Euler characteristic of a compact space stratified by odd-dimensional manifolds vanishes ([40]), the Euler characteristic of a complex analytic space, possibly singular and noncompact, coincides with the "compactly supported Euler characteristic" $\chi_c(Z) := \sum(-1)^k \dim \mathrm{H}^k_c(Z)$, thus it satisfies the additivity property

1. According to private communication from the authors of [32], a course of R. Pandharipande at the University of Princeton, in the Fall term of 2008, played a catalyzing role for the elaboration of the conjecture.

$\chi_{\text{top}}(X) = \chi_{\text{top}}(Y) + \chi_{\text{top}}(X \setminus Y)$ if $Y \subset X$ is a closed subset. Given a complex analytic variety Y and a commutative ring R, let $f : Y \to R$ be an R-valued *constructible function* namely a finite sum $f = \sum f_\alpha \mathbf{1}_{Y_\alpha}$, where $f_\alpha \in R$ and $\mathbf{1}_{Y_\alpha}$ are the characteristic functions of locally closed complex subvarieties $Y_\alpha \subset Y$. We define the integral:

$$\int_Y f d\chi_{\text{top}} := \sum_\alpha f_\alpha \chi_{\text{top}}(Y_\alpha). \tag{15}$$

The additivity property ensures well-definedness, i.e. the "integral" does not change if we subdivide the Y_α's. Furthermore, for a locally trivial fibration $f : X \to Y$, with fibre F, one has $\chi_{\text{top}}(X) = \chi_{\text{top}}(Y)\chi_{\text{top}}(F)$, so that "Fubini" theorem holds. The conjecture of Oblomkov and Shende conjecture can be stated as follows:

Theorem 4.2. — *Let (C, p) be the germ of a singular plane curve, with Milnor number $\mu(f)$, and $\mathcal{L}$ its associated link. Let $C_p^{[n]}$ be the punctual Hilbert scheme and let $\mathcal{P}$ denote the HOMFLY-PT polynomial. Then:*

$$\left(\frac{v}{s}\right)^{\mu(f)-1} \sum_{\ell \geq 0} s^{2\ell} \int_{C_p^{[\ell]}} (1 - v^2)^{m(I)} d\chi_{\text{top}}(I) = \mathbf{P}(\mathcal{L}, v, s). \tag{16}$$

The integral on the left hand side of the equality requires some word of explanation: as we noticed in Remark 2.1, the forgetful map $C_p^{[\ell,\ell+n]} \to C_p^{[\ell]}$ is, when restricted to the level sets $m^{-1}(r)$, a fibration with fibre the Grassmannian $\mathrm{Gr}(n, m(I))$. It is easily seen that

$$\chi_{\text{top}}\left(\mathrm{Gr}(n, m(I))\right) = \binom{m(I)}{n}, \tag{17}$$

hence, from Fubini theorem and Eq. (15), we have

$$\sum_{\ell, n} s^{2\ell}(-v^2)^n \chi_{\text{top}}(C_p^{[\ell,\ell+n]}) = \sum_{\ell \geq 0} s^{2\ell} \int_{C_p^{[\ell]}} (1 - v^2)^{m(I)} d\chi_{\text{top}}(I). \tag{18}$$

Example 4.3. — *We give some examples of direct verification of this equality, taken from* [32]*:*

- **nonsingular point :** *In case the point is nonsingular, the Milnor number vanishes and the link is the unknot. The ideals of $\mathbb{C}[[T]]$ are all of the form (T^k), hence the left hand side of Eq.* (16) *is*

(19)
$$\left(\frac{v}{s}\right)^{\mu(f)-1} \sum_{\ell \geq 0} s^{2\ell} \int_{C_p^{[\ell]}} (1 - v^2)^{m(I)} d\chi_{\text{top}}(I) = \frac{s}{v} \sum_{\ell \geq 0} s^{2\ell}(1 - v^2) = \frac{s(1 - v^2)}{v(1 - s^2)} = \frac{v - v^{-1}}{s - s^{-1}}.$$

  which equals the HOMFLY-PT polynomial of the unknot, Eq. (8)*.*
- **Node :** *In case we have a node, of equation $y^2 - x^2$, the Milnor number is one. The two nonsingular branches have linking number one, therefore we have the Hopf link. The finite length ideals of $\mathbb{C}[[X, Y]]/(Y^2 - X^2) \simeq \mathbb{C}[[T_1, T_2]]/T_1T_2$ are, besides the trivial* (1) *of length* 0*, with $m((1)) = 1$, either principal of the form $(T_1^k + \alpha T_2^{i-k})$, with $\alpha \neq 0$ for $1 \leq k < i$, or generated by two elements and*

of the form (T_1^k, T_2^{i-k+1}), *for* $1 \leq k \leq i$ *(of length* i*). The principal ideals are parameterized by* $\mathbb{C}^\times$ *so that the Euler characteristic of the corresponding locus vanishes. The only contribution to the integral comes from the monomial ideals with two generators, and there are* i *of them with length* $i > 0$*. The left hand side of Eq.* (16) *is therefore:*

(20)
$$\left(\frac{v}{s}\right)^{\mu(f)-1} \sum_{\ell \geq 0} s^{2\ell} \int_{C_p^{[\ell]}} (1-v^2)^{m(I)} d\chi_{\mathrm{top}}(I) = (1-v^2) + \sum_{\ell > 0} \ell s^{2\ell} (1-v^2)^2 = (1-v^2)\left(1 + \frac{1-v^2}{(s-s^{-1})^2}\right)$$

which equals the HOMFLY-PT polynomial of the Hopf link, Eq. (12).

Cusp : *For the cusp, of equation* $y^2 - x^3$*, the Milnor number is two, while its link is the trefoil knot. The only ideals of length* i *in* $\mathbb{C}[[X,Y]]/(Y^2 - X^3) \simeq \mathbb{C}[[T^2, T^3]]$ *contributing to the Euler characteristic are the monomial ones, namely the principal ones* (T^i) *for* $i \geq 2$*, and those with two generators* (T^{i+1}, T^{i+2})*, for* $i \geq 1$*. Thus*

(21)
$$\left(\frac{v}{s}\right)^{\mu(f)-1} \sum_{\ell \geq 0} s^{2\ell} \int_{C_p^{[\ell]}} (1-v^2)^{m(I)} d\chi_{\mathrm{top}}(I) = \frac{v - v^3}{s}\left(1 + \sum_{\ell \geq 2} s^{2\ell} + (1-v^2)\sum_{\ell \geq 1} s^{2\ell}\right),$$

which equals the HOMFLY-PT polynomial of the trefoil, Eq. (13).

In [32, §5] *the authors compute the left hand side of Eq.* (14) *for the general singularity of Example 1.1, using the following remark: if an algebraic variety* Z *admits the action of a torus* $T := (\mathbb{C}^\times)^N$ *with a finite number of fixed points, then* $\chi_{\mathrm{top}}(Z) = \sharp Z^T$*. This fact allows them to reduce the evaluation of the integral to counting the monomial ideals.*

Remark 4.4. — 1. *It is evident from Eqs.* (8) *and* (10) *and the skein definition of the Alexander polynomial* $\nabla_{\mathcal{L}}(s)$ *that* $\lim_{v \to -1} \frac{\mathbf{P}(\mathcal{L}, v, s)}{\mathbf{P}(unknot)} = \nabla_{\mathcal{L}}(s)$*. It is not hard to see that in the limit* $v \to -1$ *of the left hand side of Eq.* (16)*, divided by* $\frac{v - v^{-1}}{s - s^{-1}}$*, only* principal *ideals (*$m(I) = 1$*) give a nonzero contribution. In* [32, §3]*, Oblomkov and Shende show that this special case follows from the main result of* [7].

2. *The symmetry* $s \mapsto -s^{-1}$ *of the HOMFLY-PT polynomial, which is not immediately evident in the left hand side of Eq.* (16)*, is shown to follow from Serre duality in* [32, §4].
3. *The paper* [9] *gives a physical interpretation of Eq.* (16) *in terms of large* N *duality for conifold transitions, based in the conjectural equivalence between Gromov-Witten and Donaldson-Thomas theory.*

5. A COLORED REFINEMENT: THE CONJECTURE OF DIACONESCU, HUA AND SOIBELMAN

The proof of the conjecture of Oblomkov and Shende given by Maulik in [24] descends from a “colored” refinement, first proposed in [8].

Assumption 5.1. — *From now on we assume, without loss of generality, that every irreducible component of C gives a unique branch at p, so let $C = \bigcup_{i=1}^{N} C_i$ be the decomposition of C into irreducible components, and let $f_i = 0$ be the equation of C_i, with $f_i \in \mathbb{C}[X, Y]$.*

A *coloring* of C is the choice, for every irreducible component C_i of C, of a partition μ_i of a positive integer $|\mu_i|$. We set $\overrightarrow{\mu} := (\mu_1, \cdots, \mu_N)$. These partitions are used:

- on the algebraic geometric side to define a "thickening" $C_{\overrightarrow{\mu}}$ of the curve (C, p), namely a nonreduced structure, see Section 5.1.3.
- on the link side they are used to construct satellites $\mathcal{L}_{\overrightarrow{\mu}}$ of the link $\mathcal{L}$, by associating special braids with the partitions, see Section 5.2.6.

The correspondence between the invariants of these two enhancements of the original objects, which will be stated in Theorem 5.41, is quite remarkable.

While the objects appearing on the two sides of Eq. (16) can be easily described, their colored variants are more involved and we will devote a good part of this exposition to define them and put them in proper context.

5.1. Framed stable pairs

We start with the algebraic geometric side of the subject, dealing with the colored variant of the Hilbert scheme, which turns out to be the moduli space of stable pairs framed on a thickening of C associated with $\overrightarrow{\mu}$.

5.1.1. *Moduli spaces of framed stable pairs.* — Let us first recall some basic definitions:

Definition 5.2. — *Let $\mathscr{F}$ be a coherent sheaf on X.*

- **Associated point :** *A (not necessarily closed) point $x \in X$ is an* associated point *of $\mathscr{F}$ if the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the annihilator of some element of $\mathscr{F}_x$.*
- **Embedded point :** *An associated point of $\mathscr{F}$ is said to be* embedded *if it is contained in the closure of another associated point of $\mathscr{F}$.*
- **Schematic support :** *The schematic support* $\mathrm{Supp}(\mathscr{F})$ *of $\mathscr{F}$ is the subscheme of X defined by the annihilator ideal sheaf* $\mathrm{Ann}(\mathscr{F}) := \ker \mathcal{O}_X \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathscr{F})$.
- **Pure sheaf :** *A coherent sheaf $\mathscr{F}$ is* pure of dimension d *if* $\dim \mathrm{Supp}(\mathscr{G}) = d$ *for every subsheaf $\mathscr{G} \subset \mathscr{F}$, or, equivalently, if all the associated points of $\mathscr{F}$ have dimension d.*
- **Schematic closure :** *If $i : Z \to X$ is a locally closed embedding, its* schematic closure *is the smallest closed subscheme $Z' \subseteq X$ such that i factors through it. Equivalently, the sheaf of ideals of Z' is given by the regular functions on X which vanish when pulled back to Z via i.*

Recall that a one-dimensional scheme is Cohen-Macaulay if and only if it has no embedded points The schematic closure of a one-dimensional Cohen-Macaulay scheme is therefore Cohen-Macaulay.

We consider the following set-up

– X is a nonsingular quasi-projective threefold,
– $E \subset X$ a closed *projective* subvariety
– $Z \subset X$ is a one-dimensional Cohen-Macaulay subscheme which coincides with the schematic closure of $Z \cup (X \setminus E)$. In particular Z has no embedded points.

Definition 5.3. — *A* Z-framed stable pair *on* X *is a pair* $(\mathscr{F}, \sigma)$, *where*

Purity : $\mathscr{F}$ *is a pure coherent sheaf of dimension* 1.
Support : $\sigma : \mathcal{O}_X \to \mathscr{F}$ *is a section with zero-dimensional cokernel,*
Framing : *There is an isomorphism* $\mathscr{F}_{|X\setminus E} \xrightarrow{\simeq} \mathcal{O}_{Z|X\setminus E}$ *making the diagram*

$$\begin{tikzcd} \mathcal{O}_{X|X\setminus E} \arrow[r, "\sigma_{|X\setminus E}"] \arrow[d, "Id"] & \mathscr{F}_{|X\setminus E} \arrow[d, "\simeq"] \\ \mathcal{O}_{X|X\setminus E} \arrow[r] & \mathcal{O}_{Z|X\setminus E} \end{tikzcd}$$

commutative.

Remark 5.4. — *The appearance of stable pairs is probably made less misterious by noticing that in case* Z *is* Gorenstein, *for instance if it is a planar curve, the datum of a stable pair supported on* Z *is equivalent to that of a zero-dimensional subscheme of* Z, *i.e. a point in the Hilbert scheme of* Z *(*[35, Prop. B5]*). In fact, under the above mentioned hypothesis, the sequence*

$$0 \to \mathcal{H}om_{\mathcal{O}_Z}(\mathscr{F}, \mathcal{O}_Z) \to \mathcal{O}_Z \to \mathcal{E}xt^1_{\mathcal{O}_Z}(Q, \mathcal{O}_Z) \to 0,$$

obtained applying the functor $\mathcal{H}om_{\mathcal{O}_Z}(-, \mathcal{O}_Z)$ *to the stable pair sequence*

$$0 \to \mathcal{O}_Z \xrightarrow{\sigma} \mathscr{F} \to Q = \operatorname{Coker} \sigma \to 0$$

is exact, thus defining a subscheme of Z *with associated sheaf of ideals* $\mathcal{H}om_{\mathcal{O}_Z}(\mathscr{F}, \mathcal{O}_Z)$.

Remark 5.5. —
– *As explained in* [34, Lemma 1.3]*, where stable pairs were introduced (see* [36] *for some motivation coming from enumerative geometry), the requirements that* $\mathscr{F}$ *be pure and that the cokernel of* σ *be zero-dimensional should be interpreted as a stability condition, whence the name.*
– *The support* $W := \operatorname{Supp}(\mathscr{F})$ *is, by the purity condition, a Cohen-Macaulay subscheme of dimension one. The condition on the cokernel of* σ *implies that* $\mathscr{F}$ *coincides with* $\mathcal{O}_W$ *outside a finite set of points.*
– *It follows from the framing condition that* Z *and* W *differ only on* E. *By the universal property of the schematic closure,* Z *is a subscheme of* W, *whose ideal sheaf we denote by* $I_{Z,W}$.

The homology class of the difference $[W] - [Z]$ is denoted by $\beta(\mathscr{F}, \sigma) \in \mathrm{H}_2(E)$: it is the sum of the classes of the components of W contained in E with their generic multiplicities, i.e. the lengths of their local rings at the generic points. We also set $\chi(\mathscr{F}, \sigma) := \chi(\operatorname{Coker} \sigma) - \chi(I_{Z,W})$.

Framed stable pairs with fixed discrete invariants $(\beta, \chi) \in \mathrm{H}_2(E) \times \mathbb{Z}$ are parameterized by a *projective* variety: the set-valued functor $\mathcal{P}(X, E, Z)_{\beta,\chi}$ associating with a reduced scheme T the set of families of $(Z \times T)$-framed stable pairs on $X \times T$ with given invariants is represented by a projective variety, still denoted $\mathcal{P}(X, E, Z)_{\beta,\chi}$ ([24, Lemma 2.1]). This existence theorem relies on the existence of a moduli space for stable (i.e. not framed) pairs on a projective variety, due to R. Pandharipande and R. Thomas [34], and based on previous work of J. Le Potier [17, 18]. If X is projective one just has to prove that the subset corresponding to framed stable pairs is a closed subset of the moduli space of stable pairs. The case when X is quasi-projective is reduced to the projective case by choosing a projective compactification $\overline{X}$ and proving that the restriction map between the functors

$$\mathcal{P}(\overline{X}, E, \overline{Z})_{\beta,\chi} \longrightarrow \mathcal{P}(X, E, Z)_{\beta,\chi} \tag{22}$$

is an equivalence, where $\overline{Z}$ denotes the closure if Z in $\overline{X}$. In particular, the choice of the compactification is irrelevant.

Remark 5.6. — *More precisely, it can be proved that $\mathcal{P}(X, E, Z)_{\beta,\chi}$ only depends on the completion $\widehat{X}$ of X along $E \cup Z$.*

Remark 5.7. — *In the case we are interested in, X is a threefold and E is a nonsingular curve, isomorphic to $\mathbb{P}^1(\mathbb{C})$, as explained in Section 5.1.2. Since $\mathrm{H}_2(E) = \mathbb{Z}$, the class β may be identified with a nonnegative integer r, which is just the generic multiplicity of* $\mathrm{Supp}(\mathscr{F})$ *along E.*

5.1.2. *Set-up: the flop.* — The key technical tool used in [24] is the study of wall-crossing in the derived category of coherent sheaves on CY 3-folds, the culmination of ideas of many people such as Kontsevich-Soibelman, Joyce, Toda and Bridgeland. One of the fundamental insights contained in [8], based on the physical background of [9], is seeing how to give a Calabi-Yau threefold interpretation to what looks like a question about curves on a surface. The local model on which the construction is based goes back to Atiyah [3]. It is a map known as "flop" in birational geometry, and as "conifold transition" in the physics literature.

Let Y be the total space of the vector bundle

$$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

over $\mathbb{P}^1(\mathbb{C})$. An atlas for Y is given by two open subsets $U_1, U_2 \simeq \mathbb{A}^3(\mathbb{C})$, with coordinates (z, ξ_1, ξ_2) and (w, η_1, η_2). The coordinate change in the intersection $U_1 \cap U_2$ is $w = z^{-1}, \eta_1 = z\xi_1, \eta_2 = z\xi_2$. We denote the zero section by $E \simeq \mathbb{P}^1(\mathbb{C})$.

We have the vector bundle projection

$$p : Y \to \mathbb{P}^1(\mathbb{C}), \text{ given by } p(z, \xi_1, \xi_2) = z, \tag{23}$$

and we identify $\mathbb{A}^2(\mathbb{C})$ with $p^{-1}(0)$, so that C sits inside Y:

$$C \subset \mathbb{A}^2(\mathbb{C}) = p^{-1}(0) \subset Y = \text{total space of } \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \tag{24}$$

as the complete intersection

$$C = \{f(\xi_1, \xi_2) = 0,\ z = 0\}. \tag{25}$$

The zero section E can be blown down to a point, thus obtaining

$$\pi : Y \longrightarrow Q := \{(x_1, \cdots, x_4) \in \mathbb{A}^4(\mathbb{C}),\ \text{such that } x_1x_4 - x_2x_3 = 0\}. \tag{26}$$

A coordinate description of the map π in terms of the previous atlas is:
$\pi(z, \xi_1, \xi_2) = (\xi_1, \xi_2, z\xi_1, z\xi_2) \in Q$ on U_1 and $\pi(w, \eta_1, \eta_2) = (w\eta_1, w\eta_2, \eta_1, \eta_2) \in Q$ on U_2.

The singular quadric threefold Q admits two "small" resolutions, related by a birational isomorphism ϕ,

$$\begin{tikzcd} Y \arrow[rr, dotted, "\phi"] \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\ & Q & \end{tikzcd} , \tag{27}$$

where Y' is also isomorphic to the total space of the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and $E' := \pi'^{-1}(0) \simeq \mathbb{P}^1(\mathbb{C})$.

There is a nice description of the maps in the diagram (27) in terms of two natural resolutions of singularities of a Schubert variety: Let $\mathrm{Gr}(2, 4)$ be the Grassmanian of two-dimensional vector spaces in $\mathbb{C}^4$ with canonical basis $\{e_1, \cdots, e_4\}$. Fix the point corresponding to the plane $V_0 = \mathrm{Span}\{e_1, e_2\}$, and let $(x_1, \cdots, x_4)$ be the coordinates on the corresponding $\mathbb{A}^4(\mathbb{C})$-chart, namely $(x_1, \cdots, x_4)$ corresponds to $\mathrm{Span}\{e_1 + x_1e_3 + x_2e_4, e_2 + x_3e_3 + x_4e_4\}$. The closed subset

$$\mathscr{S} := \{V \in \mathrm{Gr}(2, 4) \text{ such that } \dim(V \cap V_0) \geq 1\} \cap \mathbb{A}^4(\mathbb{C}), \tag{28}$$

easily seen to be isomorphic to Q, admits two natural desingularizations: set

$$Y = \{(U, V) \in \mathrm{Fl}(1, 2, \mathbb{C}^4) \text{ such that } V \in \mathscr{S} \text{ and } U \subseteq V \cap V_0\} \overset{\pi(U,V)=V}{\longrightarrow} \mathscr{S}, \tag{29}$$

$$Y' = \{(V, W) \in \mathrm{Fl}(2, 3, \mathbb{C}^4) \text{ such that } V \in \mathscr{S} \text{ and } V + V_0 \subseteq W\} \overset{\pi'(V,W)=V}{\longrightarrow} \mathscr{S}, \tag{30}$$

where $\mathrm{Fl}(d_1, d_2, \mathbb{C}^4)$ denotes the flag variety of nested pairs of linear subspaces in $\mathbb{C}^4$ of dimensions d_1 and d_2 respectively.

The fibrations $p : Y \to \mathbb{P}(V_0) = \mathbb{P}^1(\mathbb{C})$ and $p' : Y' \to \mathbb{P}(\mathbb{C}^4/V_0) = \mathbb{P}^1(\mathbb{C})$, defined as $p(U, V) = U$ and $p'(V, W) = W$ respectively, show that Y and Y' are nonsingular.

If $V_0 \neq V \in \mathscr{S}$, then $\dim V \cap V_0 = 1$: in this case $(V \cap V_0, V)$ is the unique point in Y over V and $(V, V + V_0)$ is the unique point in Y' over V, while if $V = V_0$ we have $E := \pi^{-1}(V_0) = \mathbb{P}(V_0) = \mathbb{P}^1(\mathbb{C})$ in Y and $E' := \pi'^{-1}(V_0) = \mathbb{P}(\mathbb{C}^4/V_0) = \mathbb{P}^1(\mathbb{C})$ in Y'. Thus, there is an isomorphism $\phi : Y \setminus E \overset{\simeq}{\longrightarrow} Y' \setminus E'$ over $Q \setminus 0$, defining a rational map $\phi : Y \dashrightarrow Y'$.

The map ϕ is called a *flop*. The proper transform by ϕ of $\mathbb{A}^2(\mathbb{C})$ is its blow-up $\widetilde{\mathbb{A}^2}(\mathbb{C}) \subset Y'$ at the origin, with exceptional divisor E'. Similarly, the proper transform by ϕ of C is the blow-up $C' \subset \widetilde{\mathbb{A}^2}(\mathbb{C}) \subset Y'$ of C at p. One of the main steps in [24] is to relate the generating functions of framed stable pairs on Y to the generating functions of framed stable pairs on Y', see Theorem 6.3.

5.1.3. *Coloring C.* — To motivate the definition of the "coloring" of C, first recall that, given a partition $\mu = (\mu^{(1)}, \cdots, \mu^{(\ell)})$, with $\mu^{(1)} \geq \cdots \geq \mu^{(\ell)} > \mu^{(\ell+1)} = 0$, there is a corresponding monomial $\mathfrak{m}$-primary ideal of $\mathbb{C}[X, Y]$ (see [27, §7.2]), namely the one generated by the monomials

$$X^{j-1}Y^{\mu^{(j)}}, \text{ for } j = 1, \cdots, \ell + 1. \tag{31}$$

We denote by $Z_\mu \subset \mathbb{A}^2(\mathbb{C})$ the corresponding subscheme. Recall that C is embedded in Y, the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, as a complete intersection with equations $f(\xi_1, \xi_2) = 0$ and $z = 0$. We will thicken C to a one-dimensional scheme $C_{\overrightarrow{\mu}}$, whose intersection with a two-dimensional slice through a smooth point of C_i is the monomial scheme Z_{μ_i}.

Definition 5.8. — *Let $\overrightarrow{\mu} := (\mu_1, \cdots, \mu_N)$ denote the partitions attached to the components, where $\mu_i = (\mu_i^{(1)}, \cdots, \mu_i^{(\ell_i)})$ is the partition associated with the component $C_i = \{f_i = 0\}$. Let C_{i,μ_i} be the subscheme defined by the ideal generated by*

$$z^{j-1} f_i(\xi_1, \xi_2)^{\mu_i^{(j)}}, \text{ for } j = 1, \cdots, \ell_i + 1, \tag{32}$$

where z is the local coordinate on $\mathbb{P}^1$ vanishing at 0 introduced above. The (nonreduced) one-dimensional subscheme $C_{\overrightarrow{\mu}} \subset Y$ is defined as the schematic closure of

$$\bigcup_1^N C_{i,\mu_i} \cap (Y \setminus \textit{zero section}).$$

Since $\bigcup_1^N C_{i,\mu_i} \cap (Y \setminus \text{zero section})$ is Cohen-Macaulay, and the schematic closure does not create new associated points, it follows that $C_{\overrightarrow{\mu}}$ is Cohen-Macaulay. Notice that on the generic (nonsingular) point of the branch C_i, the scheme $C_{\overrightarrow{\mu}}$ is analytically isomorphic to the product of Z_{μ_i} with the germ of a nonsingular curve.

Remark 5.9. — *When all the partitions are (1) we write $\overrightarrow{\mu} = (1)$. In this case $C_{(1)} = C$.*

We now consider $C_{\overrightarrow{\mu}}$-framed stable pairs $(\mathscr{F}, \sigma)$, as in Section 5.1.1, asking the restriction of σ to $Y \setminus E$ to coincide with the canonical surjection $\mathcal{O}_Y \longrightarrow \mathcal{O}_{C_{\overrightarrow{\mu}}}$. As discussed in Section 5.1.1, there are projective moduli spaces $\mathcal{P}(Y, E, C_{\overrightarrow{\mu}}, Z)_{\beta,\chi}$. In this case the class β is the generic multiplicity r along E of the support of $\mathscr{F}$ (see Remark 5.7). We denote

$$\mathcal{P}(Y, C, \overrightarrow{\mu}, r, n) := \mathcal{P}(Y, E, C_{\overrightarrow{\mu}}, Z)_{r[E],n} \tag{33}$$

and we define the generating function of the Euler characteristics of the moduli spaces of pairs

$$Z'(Y, C, \overrightarrow{\mu}; q, Q) = \frac{\sum_{r,n} q^n Q^r \chi_{\text{top}}\left(\mathcal{P}(Y, C, \overrightarrow{\mu}, r, n)\right)}{\prod_k (1 + q^k Q)^k} \in \mathbb{C}[[q, Q]]. \tag{34}$$

Notice that, when $\overrightarrow{\mu} = (1)$, we recover the C-framed stable pairs, and that setting $Q = 0$ amounts to consider only the moduli spaces of pairs $(\mathscr{F}, \sigma)$ with $\mathrm{Supp}(\mathscr{F}) = C_{\overrightarrow{\mu}}$.

Remark 5.10. — *Even in the uncolored case, the blow-up identity relating $Z'(Y, C, \overrightarrow{\mu}; q, Q)$ to its analogue for the total transform of C by a blow-up, which lies at the heart of Maulik's proof (see Theorem 6.6 and Proposition 6.4 in Section 6.1), requires arbitrary partition labels on the total transform of C.*

Remark 5.11. — *In* [8, Thm.1.1]*, Diaconescu, Hua and Soibelman prove that, in the set-up of Section 5.1.2, $Z'(Y, C, (1); s^2, -v^2)$ and $\sum_{\ell \geq 0} s^{2\ell} \int_{C_p^{[\ell]}} (1 - v^2)^{m(I)} d\chi_{\mathrm{top}}(I)$ coincide after multiplying by a power of s, depending on the normalization chosen here for the invariant χ of $(\mathscr{F}, \sigma)$.*

5.2. Colored HOMFLY-PT polynomials

The link invariants we are going to discuss first arose in connection with the quantum groups $U_q(\mathfrak{sl}(N))$ in the seminal works [38, 39]. We will avoid this approach, though, and, following [24], adopt a more down to earth point of view, ultimately relying on the classical construction of a satellite knot (see [19]). In order to have a well-posed definition one needs to consider *framed* knots, which we now discuss. Good references for this section are the introductory parts of [2, 21].

5.2.1. *Framing.* — Recall that we associated a sign with every crossing in the diagram $\Delta_{\mathcal{L}}$ of a link $\mathcal{L}$ (see (7)).

Definition 5.12. — *The writhe $w(\Delta_{\mathcal{L}})$ is the sum, over all crossings, of their signs.*

Remark 5.13. —
1. *The second and third Reidemeister moves preserve the writhe, whereas the first changes it: adding a positive curl increases the writhe by one.*
2. *If the link is represented by a diagram in the plane, the linking number $L(K_1, K_2)$ can be computed as the sum*

$$L(K_1, K_2) = \sum_{i \in \frac{K_1}{K_2}} \epsilon_i, \tag{35}$$

where $\frac{K_1}{K_2}$ is the set of crossings in which K_1 passes over K_2, and $\epsilon_i = \pm 1$ is the sign of the crossing.

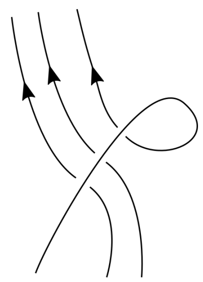
Definition 5.14. — *Given a link $\mathcal{L} = \bigcup \mathcal{L}_i$, a* framing *is the choice of a normal, never vanishing, vector field on each component.*

A framing defines a *parallel curve*, obtained by a little movement along the vector field. Intuitively, the choice of a framing replaces every component of the link with a "ribbon" (homeomorphic with an annulus, since the boundary consists of two connected components). The *self-linking number* of a framed knot is defined to be the linking number of the link with its parallel. This number fixes the framing up to isotopy.

Example 5.15. — 1. *If a link lies on a two-dimensional torus, as in Example 1.1, a natural framing is given by choosing at each point $x \in \mathcal{L}$ a normal vector completing the tangent vector in x to a positively oriented basis.*

2. *The* unlinked, *or* natural, *framing is given by choosing for each component $\mathcal{L}_i$ of the link the unique, up to isotopy, nearby knot which has zero linking number with $\mathcal{L}_i$.*
3. *The choice of a diagram representing the link selects the* blackboard *framing, in which each curve of the link diagram is thought of as a "ribbon" lying on the plane containing the diagram. In this case the self-linking number equals its writhe (Definition 5.12).*
4. *Since the blackboard framing is not invariant under the first Reidemeister move, every framing can be realized as a blackboard framing of a diagram just adding a few curls.*

Remark 5.16. — *The framing of a toral knot (Example 1.1) inherited by its embedding in the torus differs from the blackboard one associated with the presentation as the closure of the braid $(\beta_r)^s$ defined in Example 1.1. In order to fix this discrepancy it is enough to add a positive curl to the diagram of β_r. One may get an intuition of this fact imagining a braid that is wrapping around the outside of the torus and smashing it onto the plane, thus obtaining the curl. We denote by $\beta_{r,\looparrowright}$ the diagram thus obtained.*



$\beta_{3,\looparrowright}$

5.2.2. *Skein theory.* — In this section we collect some facts of skein theory, an efficient way to organize the colored HOMFLY-PT polynomials of a link. In particular, the skein algebra of a rectangle, with n inputs and n outputs, and that of an annulus, play a major role. They turn out to be isomorphic respectively to the Hecke algebra of type A_n and to a commutative algebra, see Theorem 5.26.

Let F be a surface, possibly with boundary and with two sets of marked points P, Q on the boundary. For our purpose F will be one of the following surfaces

1. $F = \mathbb{R}^2$, the euclidean plane, with $P = Q = \emptyset$,
2. the annulus $A = \{(x, y) \in \mathbb{R}^2 \text{ such that} 1 < x^2 + y^2 < 4\}$ with $P = Q = \emptyset$,
3. the square $S_n = [0, 1] \times [0, 1]$ with n marked points $Q = \{q_1, \cdots, q_n\}$ on the top side and n marked points $P = \{p_1, \cdots, p_n\}$ on the bottom side.

It will be useful to think of the square S_n as embedded in the annulus A as an angular sector $1 \leq \rho \leq 2$, $\theta \in [0, \pi/2]$.

Definition 5.17. — *A* diagram Δ *in* F *is a series of oriented closed curves and oriented arcs joining the points in* P *to those in* Q*, with the condition that every point of* P *is the starting point of a unique arc and every point of* Q *is the end point of a unique arc. As in the diagram of a generic planar projection of a link, these arcs and curves are allowed to have only simple crossings. We identify diagrams obtained by an ambient isotopy (fixing the boundary) or obtained one from the other by a sequence of Reidemeister moves II and III.*

Remark 5.18. — *Notice that an element of the braid group* $\mathscr{B}_n$ *defines a diagram in the square* S_n*.*

Remark 5.19. — *Given a diagram* Δ *in* S_n*, thought of as a subset of* A*, this can be closed to a diagram* $\widehat{\Delta}$ *in* A *by joining, for every* $i = 1, \cdots n$*, the point* q_i *to the point* p_i *with the circular arc* $\theta \in [\pi/2, 2\pi]$*. For diagrams given by braids this is just the standard operation of closure of a braid.*

Remark 5.20. — *The embedding of* A *into* $\mathbb{R}^2$ *sends diagrams in* A *to diagrams in* $\mathbb{R}^2$*.*

Let Λ be the ring

$$\Lambda = \mathbb{Z}[v^{\pm 1}, s^{\pm 1}, (s^r - s^{-r})^{-1}] \text{ for all } r \geq 1. \tag{36}$$

Definition 5.21. — *The* framed HOMFLY skein *of* F*, denoted* $\mathcal{S}[F]$*, is the* Λ*-module generated by diagrams in* F *(up to isotopy and II and III Reidemeister moves), modulo the skein relations:*

1. *If* $\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0$ *are as in the diagram (9), then*

$$\mathcal{L}_+ - \mathcal{L}_- = (s - s^{-1})\mathcal{L}_0. \tag{37}$$

2. *If* Δ *is a diagram in* F*, then*

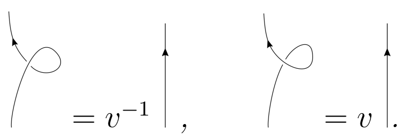
$$\Delta \coprod \text{unknot} = \left(\frac{v^{-1} - v}{s^{-1} - s}\right)\Delta, \tag{38}$$

where $\coprod$ *is meant to denote that* Δ *and the unknot are unlinked. In particular, if* $F = \mathbb{R}^2$ *or* $F = A$*, we have*

$$\text{unknot} = \frac{v^{-1} - v}{s^{-1} - s}[\emptyset] = \frac{v^{-1} - v}{s^{-1} - s} \in \Lambda, \tag{39}$$

by setting the empty diagram to equal 1.

3. *Deleting a curl with positive crossing amounts to multiplying by* v^{-1}*, deleting a curl with negative crossing amounts to multiplying by* v*:*



In this framework diagrams should be thought of as endowed with the "blackboard framing".

Remark 5.22. — *Given the local nature of the relations in Definition 5.21, the embeddings $S_n \hookrightarrow A$ and $A \hookrightarrow \mathbb{R}^2$, and the closure operation $\Delta \to \widehat{\Delta}$, discussed in Remarks 5.19 and 5.20, define Λ-module morphisms*

$$\mathcal{S}[S_n] \xrightarrow{\widehat{\ }} \mathcal{S}[A] \longrightarrow \mathcal{S}[\mathbb{R}^2]. \tag{40}$$

Stacking a square on top of the other (and rescaling) defines an associative product on $\mathcal{S}[S_n]$. Similarly, the operation of putting an annulus inside another defines an associative product on $\mathcal{S}[A]$, which is commutative, as one can "slide" the diagram contained in the inner annulus under the other one by using the Reidemeister moves II and III, thus exchanging the two diagrams. To identify the algebra $\mathcal{S}[S_n]$, we recall the following definition:

Definition 5.23. — *The* Hecke algebra $\mathcal{H}_n$ *(of type A_n) is the associative $\mathbb{Z}[s, s^{-1}]$-algebra (with unit), defined by a set of generators $S = \{S_1, \cdots, S_{n-1}\}$, subject to the relations:*

$$S_i S_j = S_j S_i \text{ if } |i-j| \geq 2, \tag{41}$$

$$S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}, \tag{42}$$

$$(S_i - s)(S_i + s^{-1}) = S_i^2 - (s - s^{-1})S_i - 1 = 0. \tag{43}$$

Remark 5.24. — *The specialization $s = 1$ gives the group algebra of the symmetric group, with S_i corresponding to the transposition $(i, i+1)$. Hence, the Hecke algebra can be considered a one-parameter deformation of this group algebra.*

Remark 5.25. — *The standard set of generators $S = \{\sigma_1, \cdots, \sigma_{n-1}\}$ of the braid group on n strands $\mathscr{B}_n$ verifies Eqs.* (41) *and* (42)*, hence $\mathcal{H}_n$ is the quotient of the group algebra of $\mathscr{B}_n$ obtained imposing the relation Eq.* (43)*. In particular, a braid defines an element of $\mathcal{H}_n$.*

It is easily seen that Eq. (43) is just a rewriting of Eq. (37). We will often extend the coefficients of $\mathcal{H}_n$ to Λ. The resulting algebra will still be denoted $\mathcal{H}_n$.

Theorem 5.26. —
1. *The skein algebra $\mathcal{S}[S_n]$ is isomorphic to the Hecke algebra $\mathcal{H}_n$ with coefficients in Λ. This isomorphism sends a braid $\gamma \in \mathscr{B}_n$, thought of as a diagram in S_n, to its class in $\mathcal{H}_n$ (see Remark 5.25).*
2. *Set $\mathcal{C}_n := \widehat{\mathcal{S}[S_n]} \subset \mathcal{S}[A]$, where, as in Remark 5.22, $\widehat{\ }$ denotes the* closure *map. There is a graded isomorphism*

$$\tau : \mathcal{C} \to \mathbb{S}^{\bullet}_{\Lambda} \tag{44}$$

between the graded subalgebra $\mathcal{C} \subset \mathcal{S}[A]$ generated by $\bigcup_n \mathcal{C}_n$, and the graded algebra $\mathbb{S}^{\bullet}_{\Lambda}$ of symmetric functions in infinitely many variables with coefficients in Λ. It *is proved in* [42] *that* $\mathcal{C}$ *is freely generated as a polynomial algebra by the elements* $A_m := \widehat{\sigma_{m-1} \cdots \sigma_1}$*, placed in degree* m*, where, as above, the* σ_i*'s are the standard generators of* $\mathscr{B}_m$*.*

3. *In* $\mathbb{R}^2$ *every diagram can be represented uniquely as a multiple of the empty diagram. In other words, there is a canonical isomorphism:*

$$\langle\ \rangle : \mathcal{S}[\mathbb{R}^2] \longrightarrow \Lambda, \tag{45}$$

the framed HOMFLY-PT polynomial*, differing from the HOMFLY-PT polynomial by the multiplicative term* $v^{-w(\Delta_\mathcal{L})}$*, accounting for the framing.*

Remark 5.27. — *A particularly significant basis of* $\mathcal{S}[S_n]$ *(as a* Λ*-module) is given by* positive permutation braids*. Given a permutation* $\pi = (\pi(1), \cdots, \pi(n))$ *we consider the unique braid* ω_π *associated with* π *(i.e. the* i*-th point at the bottom joins the* $\pi(j)$*-th point at the top) in which each pair of strands cross at most once with positive sign (a useful way to visualize these braids is to imagine them disposed in layers, with the first strand at the very back and the last at the front. Notice that the standard generator* σ_i *of* $\mathscr{B}_n$ *are the positive permutation braids of the transpositions* $(i, i+1)$*.*

5.2.3. *The idempotents of Gyoja.* — By Maschke's Theorem, the complex group algebra $\mathbb{C}[G]$ of a finite group decomposes into a direct product of matrix algebras indexed by the irreducible representations: For each such irreducible one can choose a primitive idempotent, giving the projection on a copy of the irreducible representation inside $\mathbb{C}[G]$.

Let $G = \mathscr{S}_n$ be the symmetric group: the irreducible representations are indexed by the set of partitions of n, which we will identify with their associated Young diagrams, and explicit formulas for these idempotents, depending on the choice of a standard tableau of shape λ, are given by the Young symmetrizers $\{e_\lambda\}_{\lambda \vdash n}$ (see [11, §4.1]). It is known that if s is not a root of unity, then the Hecke algebra specialized at s is semisimple and isomorphic to the group algebra of $\mathscr{S}_n$. In [12] Gyoja defines primitive idempotents, which we will still denote $e_\lambda \in \mathcal{H}_n$, specializing to the Young symmetrizers when $s \to 1$. These idempotents are studied as elements of the skein $\mathcal{S}[S_n]$ in [2], which also provides a vivid three-dimensional description of them as linear combination of positive permutation braids. The following two propositions, characterizing the two maps $\mathcal{S}[S_n] \longrightarrow \mathcal{S}[A]$ and $\mathcal{S}[A] \longrightarrow \mathcal{S}[\mathbb{R}^2]$, are important from the computational point of view (see [23, §I.3] for the definition of the Schur functions):

Proposition 5.28. — [2, 20] *Let* $\lambda \vdash n$ *be a partition of* n*, and* $e_\lambda \in \mathcal{H}_n$ *be the corresponding Gyoja idempotent. The composition*

$$\mathcal{H}_n \xrightarrow{\simeq} \mathcal{S}[S_n] \xrightarrow{\widehat{\ }} \mathcal{S}[A] \xrightarrow{\tau} \mathbb{S}^\bullet_\Lambda \tag{46}$$

sends e_λ *to the Schur function* s_λ*.*

Let $Q_\lambda = \widehat{e_\lambda} \in \mathcal{S}[A]$ be the closure of the Gyoja idempotent $e_\lambda \in \mathcal{H}_n \simeq \mathcal{S}[S_n]$. By Proposition 5.28, Q_λ corresponds to s_λ under the isomorphism $\mathcal{C} \simeq \mathbb{S}^\bullet_\Lambda$, but we prefer to keep a separate notation for the two objects. The family $\{Q_\lambda\}_{\lambda \vdash n}$ is a basis for $\mathcal{C}_n$.

The next proposition describes the composition $\mathcal{C} \longrightarrow \mathcal{S}[\mathbb{R}^2] \xrightarrow{\langle\ \rangle} \Lambda$ in terms of the elements of the basis Q_λ by giving their framed HOMFLY-PT polynomials:

Proposition 5.29. — [22] *Let $\lambda \vdash n$ be a partition, and, for any box $\square$ in its Young diagram, let $c(\square)$ and $h(\square)$ denote its content*[2] *and its hook-length*[3] *respectively. Then:*

$$\langle Q_\lambda \rangle = \prod_{\square \in \lambda} \frac{v^{-1}s^{c(\square)} - vs^{-c(\square)}}{s^{h(\square)} - s^{-h(\square)}}. \tag{47}$$

Remark 5.30. — *It follows from Eq.* (47) *that, setting*

$$\langle Q_\lambda \rangle^{\mathrm{low}} := \prod_{\square \in \lambda} \frac{s^{c(\square)}}{s^{h(\square)} - s^{-h(\square)}},$$

we have

$$\langle Q_\lambda \rangle = v^{-|\lambda|}\left(\langle Q_\lambda \rangle^{\mathrm{low}} + vO(v)\right) \tag{48}$$

where $O(v)$ denotes a function with no poles at $v = 0$.

More generally, we can give the following

Definition 5.31. — *Given $X = \sum_{\gamma \vdash m} c_\gamma(v,s) Q_\gamma \in \mathcal{C}_m$, we set*

$$\langle X \rangle^{\mathrm{low}} := v^{m-A}\langle X \rangle_{|v=0},$$

where $A = \min_\gamma \mathrm{ord}_{v=0}\, c_\gamma(v,s)$. One always has $\langle X \rangle = v^{A-m}\left(\langle X \rangle^{\mathrm{low}} + vO(v)\right)$ as in Eq. (48)*.*

5.2.4. *Satellites.* — The diagrams in $\mathcal{S}[A]$ may be used as decorating patterns for links: given a framed link $\mathcal{L}$, with components $K_1, \cdots K_r$, we have, for every i, the annulus A_{K_i}, bounded by K_i and its parallel curve. Choose diffeomorphisms $A \xrightarrow{\simeq} A_{K_i}$. Given the diagrams $Q_1, \cdots Q_r$ in the standard annulus A, the (framed) link $\mathcal{L} * (Q_1, \cdots Q_r)$, called a *satellite* of $\mathcal{L}$, is obtained transplanting, for every $i = 1, \cdots r$, the diagram Q_i in A_{K_i} with the help of the diffeomorphism above.

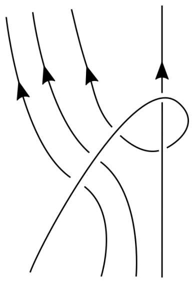
Example 5.32. — *The link $\mathcal{L}$ of an irreducible curve singularity, with Puiseux development as in Eq.* (3)*, is represented by the diagram*

$$\mathcal{L} = \widehat{\beta^{q_0}_{p_0,\looparrowright}} * \left(\widehat{\beta^{q_1}_{p_1,\looparrowright}} * \left(\cdots * \left(\widehat{\beta^{q_s}_{p_s,\looparrowright}}\right)\right)\right) \in \mathcal{S}[A], \tag{49}$$

2. if the box being considered is in the i-th row and j-th column, then $c(\square) = j - i$.
3. the hook length $h(\square)$ of a box in a Young diagram is defined as $a + b + 1$ where a is the number of boxes lying at its right and b the number of boxes lying below.

where $\beta^n_{m,\looparrowright}$ is the n-th power of the diagram $\beta_{m,\looparrowright} \in \mathcal{S}[R_m]$ defined in Remark 5.16, and $\widehat{\ }$ denotes the closure operation.

REMARK 5.33. — *In order to describe the link of a general (i.e. not necessarily irreducible) plane curve singularity as an iterated satellite construction, one also needs to consider satellites of the closure of $\gamma^n_{m,\looparrowright}$, the n−th power of the diagram $\gamma_{m,\looparrowright} \in \mathcal{S}[R_{m+1}]$ obtained adding an extra strand, linked to the curl, to $\beta_{m,\looparrowright}$ (*[24, Eq. (11)]*). This diagram may be thought of as obtained by smashing down on the plane the braid wrapping around the outside of a solid torus, along with a single strand running through its core.*



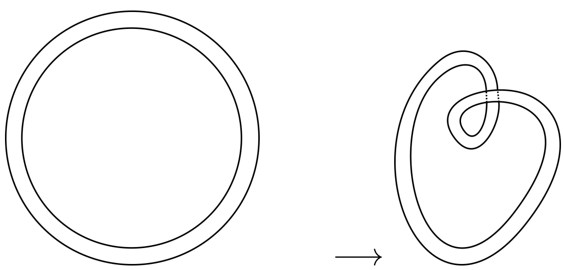
$\gamma_{3,\looparrowright}$

5.2.5. *Gyoja idempotents and the framing operator.* — In this section we discuss a theorem which relates the operation of taking a satellite of a toral knot with the framing operator. Although this result is not needed for the formulation of the conjecture of Diaconescu, Hua and Soibelman, it plays an important role in its proof (see Section 6.2.1).

DEFINITION 5.34. — *Let $\widehat{\beta_{1,\looparrowright}}$ be the closure of the braid with one strand and one curl (Remark 5.16). The operator*

$$\Phi : \mathcal{S}[A] \longrightarrow \mathcal{S}[A], \text{ defined as } \Phi(X) = \widehat{\beta_{1,\looparrowright}} * X \tag{50}$$

is called the framing *operator. It corresponds to a total twist of all the strands of a diagram in $\mathcal{S}[A]$.*



the framing operator Φ

Notice that $\Phi(\mathcal{C}_m) \subseteq \mathcal{C}_m$ for every m. One important property of the Q_λ's is that they give a basis of eigenvectors for Φ:

Proposition 5.35. — [2, Thm. 17] *Let* $\lambda = \{\lambda^{(1)} \geq \cdots \geq \lambda^{(\ell)} > 0\}$. *Then:*

$$\Phi(Q_\lambda) = s^{\kappa_\lambda} v^{-|\lambda|} Q_\lambda \tag{51}$$

with $|\lambda| = \sum_j \lambda^{(j)}$ *and* $\kappa_\lambda = 2\sum_{\square\in\lambda} c(\square)$, *where* $c(\square)$ *is the content of the box in the Young tableau associated with* λ.

Example 5.36. — *For* $n = 2$ *there are only two partitions,* (2) *and* 1^2, *and the corresponding elements, expressed in the basis introduced in Theorem 5.26, are*

$$Q_2 = \frac{1}{s^2+1}(sA_2 + A_1^2), \qquad Q_{1^2} = \frac{s}{s^2+1}(-A_2 + sA_1^2). \tag{52}$$

The framing operator acts as

$$\Phi(Q_2) = v^{-2}s^2 Q_2, \qquad \Phi(Q_{1^2}) = v^{-2}s^{-2}Q_{1^2}. \tag{53}$$

Proposition 5.29 gives:

$$\langle Q_2\rangle = \frac{(v^{-1}-v)(v^{-1}s - vs^{-1})}{(s-s^{-1})(s^2-s^{-2})}, \qquad \langle Q_{1^2}\rangle = \frac{(v^{-1}-v)(v^{-1}s^{-1} - vs)}{(s-s^{-1})(s^2-s^{-2})}. \tag{54}$$

Given a partition $\lambda \vdash r$, let $Q_\lambda[p_m] \in \mathcal{C}_{rm}$ be the element corresponding to the symmetric function $s_\lambda(z_1^m, z_2^m, \cdots)$ under the isomorphism τ. Since $\{Q_\nu\}_{\nu\vdash rm}$ is a basis for $\mathcal{C}_{rm}$ we have an expression

$$Q_\lambda[p_m] = \sum_{\nu\vdash rm} a_\lambda^\nu(m) Q_\nu. \tag{55}$$

We define

$$\Phi^{\frac{n}{m}}(Q_\lambda[p_m]) := \sum_{\nu\vdash rm} a_\lambda^\nu(m)\left(s^{\kappa_\nu} v^{-|\nu|}\right)^{\frac{n}{m}} Q_\nu. \tag{56}$$

Theorem 5.37. — [26] *Let* m, n *be coprime. Then, for every partition* λ*:*

$$\widehat{\beta^n_{m,\looparrowright}} * Q_\lambda = \Phi^{\frac{n}{m}}(Q_\lambda[p_m]). \tag{57}$$

Remark 5.38. — 1. *If* m *and* n *have a common factor* d, *then*

$$\widehat{\beta^n_{m,\looparrowright}} * (Q_{\lambda_1}, \cdots Q_{\lambda_d}) = \widehat{\beta^{\frac{n}{d}}_{\frac{m}{d},\looparrowright}} * \left(\prod_1^d Q_{\lambda_i}\right), \tag{58}$$

where the product on the right hand side is computed in $\mathcal{C}$.

2. *A formula analogous to the one of Theorem 5.37 holds for* $\widehat{\gamma^n_{m,\looparrowright}} * (Q_\mu, Q_\lambda)$ *(*[24, Lemma 3.2]*), where* $\gamma_{m,\looparrowright}$ *is the diagram defined in Remark 5.33, and* Q_λ *is the decoration of the extra strand.*

In principle, Theorem 5.37 and the analogous formulæ for the reducible case (Remark 5.38) allow one to compute the (colored) HOMFLY-PT polynomial of any algebraic knot.

Example 5.39. — *We apply Theorem 5.37 to compute* $\widehat{\beta^3_{2,\looparrowright}} * Q_1$*, which corresponds to the toric knot* $(2,3)$ *(the trefoil).* $Q_1[2]$ *corresponds to the sum of squares, which equals* $h_2 - e_2$*, where* h_2 *is the complete symmetric function of order two, also equal to* s_2*, and* e_2 *the elementary symmetric function of order two, which equals* s_{1^2}*. Thus* $Q_1[2] = Q_2 - Q_{1^2}$*. It follows from Eq.* (53) *that*

$$\widehat{\beta^3_{2,\looparrowright}} * Q_1 = \Phi^{\frac{3}{2}}(Q_1[2]) = \Phi^{\frac{3}{2}}(Q_2 - Q_{1^2}) = v^{-3}\left(s^3 Q_2 - s^{-3} Q_{1^2}\right). \tag{59}$$

Applying Eq. (54) *we find*

$$\langle \widehat{\beta^3_{2,\looparrowright}} * Q_1 \rangle = \frac{v - v^{-1}}{v^6(s - s^{-1})}\left(v^2(s^2 + s^{-2}) - v^4\right), \tag{60}$$

which coincides, up to a monomial normalization, with the HOMFLY-PT polynomial of the trefoil (Eq. (13)*).*

Finally, we define the meridian operator as follows: We consider the Hopf link $H \in \mathcal{S}[A]$ by choosing the first component to be homotopic to zero, and choosing a positive generator of the fundamental group of A as the second component. Then, given $X, Y \in \mathcal{S}[A]$, we set $M_X(Y) = H * (X, Y)$. It is proved in [22] that also this operator is diagonalized by the basis $\{Q_\mu\}$, namely, for every partition μ we have $M_X(Q_\mu) = t_\mu(X) Q_\mu$ for some $t_\mu(X) \in \Lambda$.

5.2.6. *The colored HOMFLY-PT polynomial and the Diaconescu-Hua-Soibelman conjecture.* — The operation $*$ can be extended by Λ-linearity. Given the framed link $\mathcal{L} = \bigcup_{i=1}^N \mathcal{L}_i$, decorated with the partition $\overrightarrow{\lambda} = (\lambda_1, \cdots, \lambda_N)$, let, as above, Q_{λ_i} be the closure of the Gyoja idempotent $e_{\lambda_i} \in \mathcal{H}_n \simeq \mathcal{S}[S_n]$ for $i = 1, \cdots, N$.

We construct the *satellite*

$$\mathcal{L} * Q_{\overrightarrow{\lambda}} := \mathcal{L} * (Q_{\lambda_1}, \cdots, Q_{\lambda_N}), \tag{61}$$

and set

Definition 5.40. — *The colored HOMFLY-PT polynomial of* $\mathcal{L}$ *with the coloring* $\overrightarrow{\lambda}$ *is*

$$W(\mathcal{L}, \overrightarrow{\lambda}, v, s) = v^{j(\mathcal{L}, \overrightarrow{\lambda})} s^{k(\mathcal{L}, \overrightarrow{\lambda})} \langle \mathcal{L} * Q_{\overrightarrow{\lambda}} \rangle \tag{62}$$

where $j(\mathcal{L}, \overrightarrow{\lambda})$ *and* $k(\mathcal{L}, \overrightarrow{\lambda})$ *are integers, depending on* $\overrightarrow{\lambda}$ *and* $w(\mathcal{L})$*, (see* [24, §3.2] *for the exact expression) making the polynomial independent of the choice of the framing of* $\mathcal{L}$*.*

Remark that when $\overrightarrow{\lambda} = (1)$, we recover the original definition of the HOMFLY-PT polynomial, up to a normalization by a monomial.

We are finally ready to state the refined version of the conjecture of Oblomkov and Shende, due to Diaconescu, Hua and Soibelman [8], proved by D. Maulik in [24]:

Theorem 5.41. — *There exist integers a, b and a sign ϵ, all depending on C and $\overrightarrow{\mu}$, such that the following equality holds:*

$$Z'(Y, C, \overrightarrow{\mu}; s^2, -v^2) = \epsilon v^a s^b W(\mathcal{L}, {}^t\overrightarrow{\mu}; v, s), \tag{63}$$

where ${}^t\overrightarrow{\mu} := ({}^t\mu_1, \cdots, {}^t\mu_N)$.

As a special case, picking $\overrightarrow{\mu} = ((1))$, and applying Remark 5.11, we have the original statement conjectured by Oblomkov and Shende, Theorem 4.1.

6. A SKETCH OF THE PROOF

The line of the proof of Theorem 5.41 is quite direct: both sides of Eq. (63) are shown to change in the same way after a blow-up. By the theorem on embedded resolution of singularities, one is reduced to checking the equality in the case of a smooth point (an unknot) or a node (a Hopf link), where a direct verification is possible. The single steps in the proof, however, are technically involved, and a detailed exposition is impossible here for reasons of space. We will therefore limit ourselves to a summary of the main points of the proof, in the hope that it may help the reading of the original paper.

We start by introducing some notation for the blow-up: we are in the set-up of Section 5.1.2 and Assumption 5.1 holds. We have the flop $\phi : Y \dashrightarrow Y'$. The proper transform $C' = \overline{\phi(C \setminus E)}$ of C is the blow-up of C at p, and we set $C' \cap E' = \{p_1, \cdots, p_\ell\}$. As recalled in Section 1, these points correspond to the lines in the tangent cone of C at p. For $k = 1, \cdots, \ell$, we denote by B_k the singularity of C' at p_k, and by D_k the *reduced* singularities of $C' \cup E'$ at p_k, namely $D_k = B_k \cup E'$. For each $k = 1, \cdots, \ell$, the array $\overrightarrow{\mu}$ defines an array of partitions $\overrightarrow{\mu}[k]$, corresponding to the components of C' meeting E' at p_k, which may be used to decorate B_k. Given a partition λ, we may color $D_k = B_k \cup E'$ with the partition $(\overrightarrow{\mu}[k], \lambda)$, attaching the partitions in the array $\overrightarrow{\mu}[k]$ to the irreducible components of B_k and the partition λ to the component E'.

Finally, $C'_{\overrightarrow{\mu}}$ denotes the scheme theoretic closure of $\phi(C_{\overrightarrow{\mu}} \setminus E)$. Clearly $\left(C'_{\overrightarrow{\mu}}\right)_{\mathrm{red}} = C'$. As explained in Section 5.1.1, there exist moduli spaces $\mathcal{P}(Y', C', \overrightarrow{\mu}, r, n)$, with the associated generating function $Z'(Y', C', \overrightarrow{\mu}; q, Q)$, and, for each $k = 1, \cdots, \ell$, moduli spaces $\mathcal{P}(Y, D_k, (\overrightarrow{\mu}[k], \lambda), r, n)$, with generating functions $Z'(Y, D_k, (\overrightarrow{\mu}[k], \lambda); q, Q)$. Similarly, on the link side, we have the link $\mathcal{L}_C$, decorated with ${}^t\overrightarrow{\mu}$, the links $\mathcal{L}_{B_k}$, decorated with ${}^t\mu[k]$, and the links $\mathcal{L}_{D_k}$ decorated with $({}^t\mu[k], {}^t\lambda)$.

Notation 6.1. — *Most of the results in this section are identities between rational functions in two variables. For the sake of simplicity we will state some of the main theorems in the form of identities which hold up to a monomial which we will not specify and adopt the notation "$\approx$" to indicate this. These monomials are computed explicitly in* [24].

6.1. The blow-up relation for framed stable pairs

6.1.1. *The flop invariance theorem.* — The hardest technical step in [24] is probably the proof of Theorem 6.3 below. The argument relies on the wall-crossing results due to Bridgeland ([4, 5]) and Calabrese ([6]), and on the comparison between Donaldson-Thomas invariants and stable pairs invariants, properly adapted to the set-up of framed stable pairs, see [24, §2.4]. In order to rely on the available results on wall-crossing quoted above, Maulik compactifies the set-up, introducing projective Calabi-Yau varieties X_+, X_- with maps

$$\begin{tikzcd} X_+ \arrow[rr, dotted, "\phi"] \arrow[dr, "\pi"'] & & X_- \arrow[dl, "\pi'"] \\ & X_0 & \end{tikzcd} \quad , \tag{64}$$

where π (resp. π') contracts the curve $E_+ \simeq \mathbb{P}^1(\mathbb{C})$ (resp. $E_- \simeq \mathbb{P}^1(\mathbb{C})$), both with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and ϕ is the flop along E and E', and containing surfaces S_+, S_- such that the formal completions of $X_\pm$ along $S_\pm \cup E_\pm$ are isomorphic to the formal completions of Y, Y' along $\mathbb{A}^2(\mathbb{C}) \cup E$ and $\widetilde{\mathbb{A}^2}(\mathbb{C}) \cup E'$ respectively.

Remark 6.2. — *It follows from the main result in* [4], *that, in this case, there is an equivalence of (unbounded) derived categories of coherent sheaves* $\mathrm{D}(X) \simeq \mathrm{D}(X')$ *extending the natural identification* $\mathrm{D}(X \setminus E) \simeq \mathrm{D}(X' \setminus E')$.

Theorem 6.3. — [24, Prop. 2.2] *In the set-up above*

$$Z'(Y, C, \overrightarrow{\mu}; q, Q^{-1}) \approx Z'(Y', C', \overrightarrow{\mu}; q, Q). \tag{65}$$

6.1.2. *Localization.* — The second step consists in splitting the right hand side of Theorem 6.3 into local contributions corresponding to the points in $C' \cap E'$. This is done by considering the $\mathbb{C}^\times$-action on Y' which fixes the proper transform of $\mathbb{A}^2(\mathbb{C})$ and scales its normal bundle. Since the Euler characteristic of a one-dimensional $\mathbb{C}^\times$-orbit vanishes, we have $\chi_{\text{top}}(\mathcal{P}(Y, C, \overrightarrow{\mu}, r, n)) = \chi_{\text{top}}\left(\mathcal{P}(Y, C, \overrightarrow{\mu}, r, n)^{\mathbb{C}^*}\right)$, so that the computation of $Z'(Y', C', \overrightarrow{\mu}; q, Q)$ reduces to the study of the contributions of the fixed points of this action on the various moduli spaces $\mathcal{P}(Y', C', \overrightarrow{\mu}, r, n)$. These fixed points correspond to framed stable pairs supported on a $\mathbb{C}^\times$-invariant Cohen-Macaulay scheme. In particular, the nonreduced structure along E' is given by a monomial ideal associated with a partition λ of r.

Proposition 6.4. — [24, Prop 2.6] *There exists function* $U_\lambda(q, Q)$ *(explicitly determined in* [24]*) such that*

$$Z'(Y', C', \overrightarrow{\mu}; q, Q) \approx \sum_\lambda U_\lambda(q, Q) \prod_{k=1}^{\ell} Z'(Y, D_k, (\overrightarrow{\mu}[k], \lambda); q, 0) \tag{66}$$

where the sum is extended to all partitions λ.

REMARK 6.5. — *As already noticed, taking* $Q = 0$ *in* $Z'(Y, D_k, (\overrightarrow{\mu}[k], \lambda); q, Q)$ *amounts to considering sheaves supported on* D_k *i.e. not containing the projective line* E *in their support.*

Finally, joining Proposition 6.4 with Theorem 6.3 we obtain the description of the behaviour of $Z'(Y, C, \overrightarrow{\mu}; q, Q)$ under blow-up, which will be compared with the analogous formula for the colored HOMFLY-PT polynomial (Section 6.2).

THEOREM 6.6. — *We have:*

$$Z'(Y, C, \overrightarrow{\mu}; q, Q^{-1}) \approx \sum_{\lambda} U_\lambda(q, Q) \prod_{k=1}^{\ell} Z'(Y, D_K, (\overrightarrow{\mu}[k], \lambda); q, 0). \tag{67}$$

6.2. Blow-up relation for colored HOMFLY-PT polynomials

The next goal is to prove an identity analogous to Eq. (67) for the HOMFLY-PT polynomials of algebraic links. The starting point is the presentation of $\mathcal{L}_C$ or of $\mathcal{L}_C * Q_{\overrightarrow{\lambda}}$ as an iteration of the operations $\widehat{\beta^n_{m,\looparrowright}} * (\,)$ and $\widehat{\gamma^n_{m,\looparrowright}} * (\,)$ (see Example 5.32 and Remark 5.33).

6.2.1. *The effect of blowing up on the link.* — The second step is relating the colored HOMFLY-PT polynomials of the links of C, B_k, D_k. Instead of giving the complete statement, which is quite involved, we will limit ourselves to give an idea of what happens in the case when C has a single branch, with a Puiseux parameterization

$$y(x) = x^{\frac{q_0}{p_0}}(a_0 + x^{\frac{q_1}{p_0 p_1}}(a_1 + x^{\frac{q_2}{p_0 p_1 p_2}}(a_2 + \dots)))$$

as in Eq. (3). As in Example 5.32, the associated knot has a presentation

$$\mathcal{L}_C = \widehat{\beta^{q_0}_{p_0,\looparrowright}} * \left(\widehat{\beta^{q_1}_{p_1,\looparrowright}} * \left(\cdots * \left(\widehat{\beta^{q_s}_{p_s,\looparrowright}}\right)\right)\right) \in \mathcal{C} \subset \mathcal{S}[A].$$

Assume $\frac{q_0}{p_0} > 1$: blowing up the singular point amounts to the change of variables $y = xw$, and the Puiseux parameterization for C' is:

$$w(x) = x^{\frac{q_0}{p_0}-1}(a_0 + x^{\frac{q_1}{p_0 p_1}}(a_1 + x^{\frac{q_2}{p_0 p_1 p_2}}(a_2 + \dots))),$$

from which we see that only the first Puiseux pair changes, and the class in $\mathcal{C}$ of the link for C' is, by Example 5.32,

$$\mathcal{L}_{C'} = \widehat{\beta^{q_0-p_0}_{p_0,\looparrowright}} * \left(\widehat{\beta^{q_1}_{p_1,\looparrowright}} * \left(\cdots * \left(\widehat{\beta^{q_s}_{p_s,\looparrowright}}\right)\right)\right).$$

By Theorem 5.37, relating the operator $\widehat{\beta^{q_0-p_0}_{p_0,\looparrowright}} * (\,)$ with the fractional power of the framing operator Φ, the equality

$$\mathcal{L}_{C'} = \Phi^{-1}(\mathcal{L}_C) \tag{68}$$

holds in $\mathcal{C}$, and clearly also holds if $\mathcal{L}_C$ and $\mathcal{L}_{C'}$ are decorated with a Q_λ. More generally, let us assume that $\frac{q_0}{p_0} > 1$ in the Puiseux development of every component of C. Since

$$\widehat{\gamma^n_{m,\looparrowright}} * (Q_\lambda, \Phi(Q_\mu)) = \Phi\left(\widehat{\gamma^{n-m}_{m,\looparrowright}} * (Q_\lambda, Q_\mu)\right), \tag{69}$$

there is an analogue of Eq. (68) in this case too (see Remark 5.38). A slight modification of the argument fixes the general case, with no assumption on the terms $\frac{q_0}{p_0}$ of the components, expressing the link $\mathcal{L}_C$ in term of the $\mathcal{L}_{B_i}$'s and the operations Φ and $\widehat{\gamma_{1,\ominus}}*$. Finally, the link of $D_k = B_k \cup E'$, colored with λ on E', can be expressed simply as $\mathcal{L}_{D_k} = M_\lambda(\mathcal{L}_{B_k})$, where M_λ is the meridian operator introduced at the end of Section 5.2.5.

6.2.2. *The inductive step.* — The main step is establishing the following identity in Λ, which is then applied to $X = \mathcal{L}_C$ (under the hypothesis that $x = 0$ is not a branch of the curve):

Theorem 6.7. — *Let $X = v^A \sum_{\gamma \vdash m} c_\gamma(s) Q_\gamma \in \mathcal{C}_m$. Then*

$$(-1)^m v^{m+A} \langle X \rangle = \frac{1}{\prod_k (1 - s^{2k} v^{-2})^k} \sum_\lambda (-v^{-2})^{|\lambda|} s^{-\kappa_\lambda} \langle Q_\lambda \rangle^{\mathrm{low}} \langle M_\lambda \Phi^{-1} X \rangle^{\mathrm{low}}. \tag{70}$$

Here M_λ and Φ^{-1} denote the meridian operator and the inverse of the framing operator respectively, whereas $\langle\ \rangle^{\mathrm{low}}$, a function only of s, is as in Definition 5.31.

Notice that in each summand the only term depending on v is $(-v^{-2})^{|\lambda|}$. Eq. (70) is first established for $X = Q_\mu$, where it follows from the "vertex flop" identity proved in [16], a combinatorial formula expressing $\langle Q_\mu \rangle$ (Eq. (47)) as a sum over the set of partitions of a product of Schur functions, after interpreting each term in this sum as $(-v^2)^{|\lambda|} s^{-\kappa_\lambda} \langle Q_\lambda \rangle^{\mathrm{low}} \langle M_\lambda \Phi^{-1} Q_\mu \rangle^{\mathrm{low}}$, using the knowledge of the eigenvalues of Φ (Eq. (51)) and those of M_λ (Theorem 4.4 in [22]), and Remark 5.30.

6.3. Conclusion of the proof

Finally one has to compare the identitites in Theorem 6.7 and Theorem 6.6. The proof of Theorem 5.41 is reduced to checking the lowest degree part of the identity by the following:

Proposition 6.8. — [24, Proposition 6.4] *Assume*

$$Z'(Y, D_k, (\mu[k], \lambda); s^2, 0) \approx \langle \mathcal{L}_{D_k} * (Q_{{}^t\mu[k]}, Q_{{}^t\lambda}) \rangle^{\mathrm{low}} \tag{71}$$

holds for every $k = 1, \cdots, \ell$, and for every partition λ. Then Theorem 5.41 holds for C decorated with $\overrightarrow{\mu}$.

Remark 6.9. — *In turn, Theorem 5.41 implies that*

$$Z'(Y, C, \overrightarrow{\mu}; s^2, 0) \approx \langle \mathcal{L}_C * Q_{{}^t\overrightarrow{\mu}} \rangle^{\mathrm{low}},$$

as soon as one proves that $\langle \mathcal{L}_C * Q_{{}^t\overrightarrow{\mu}} \rangle^{\mathrm{low}} \neq 0$.

Eq. (71) is verified for the Hopf link (a node) colored with any pair of partition. Recall that, by Eq. (67),

$$Z'(Y, C, \overrightarrow{\mu}; s^2, -v^{-2}) \approx \sum_\lambda U_\lambda(s^2, -v^2) \prod_{k=1}^{\ell} Z'(Y, D_k, (\overrightarrow{\mu}[k], \lambda); s^2, 0).$$

The proof of Proposition 6.8 consists in matching every term

$$U_\lambda(s^2,-v^2)\prod_{k=1}^{\ell} Z'(Y,D_k,(\overrightarrow{\mu}[k],\lambda);s^2,0) \tag{72}$$

in Eq. (67) with the corresponding term

$$(-v^2)^{|\lambda|}s^{-\kappa_{t_\lambda}}\langle Q_{t_\lambda}\rangle^{\mathrm{low}}\langle M_{t_\lambda}\Phi^{-1}\mathcal{L}_C * Q_{t\overrightarrow{\mu}}\rangle^{\mathrm{low}} \tag{73}$$

in Eq. (70). By hypothesis $Z'(Y,D_K,(\overrightarrow{\mu}[k],\lambda);s^2,0) \approx \langle \mathcal{L}_{D_k} * Q_{({}^t\overrightarrow{\mu}[k],{}^t\lambda)}\rangle^{\mathrm{low}}$. Using the results sketched in Section 6.2.1 on the relation between $\mathcal{L}_C$ and the $\mathcal{L}_{D_k}$'s, the term in Eq. (73) turns out to be $\approx \prod_k \langle \mathcal{L}_{D_k} * Q_{({}^t\overrightarrow{\mu}[k],{}^t\lambda)}\rangle^{\mathrm{low}}$. At the end the quantities in Eq. (72) and in Eq. (73) differ by a monomial, a priori depending on λ, and the last step is proving that in fact the monomial is the same for all the partitions λ. A byproduct of the proof of this last fact is the nonvanishing $\langle \mathcal{L}_C * Q_{{}^t\mu}\rangle^{\mathrm{low}} \neq 0$ for every C and $\overrightarrow{\mu}$, which completes the argument, see Remark 6.9.

In this last step, an important role is played by the following evaluation of the term of lowest order in v and s in the expansion of $\mathcal{L} * Q_{\overrightarrow{\lambda}}$ in $\mathcal{C}$ in the basis $\{Q_\mu\}$:

Theorem 6.10. — *Assume $x=0$ is not a component of C, let m_i denote the number of strands in the annulus diagram of the i-th connected component of $\mathcal{L}_C$, and set $\mu^{\mathbf{m}} = \mu_1^{\cup m_1}\cup\cdots\cup\mu_\ell^{\cup m_\ell}$, where $\cup$ denotes the concatenation of partitions. Then, there exist exponents A and B, depending on the Puiseux pairs of C, such that the following holds in $\mathcal{C}$:*

$$\mathcal{L}_C * (Q_{\mu_1},\ldots,Q_{\mu_r}) = \pm v^A s^B\left(Q_{\mu^{\mathbf{m}}} + \sum_{\gamma\succ\mu^{\mathbf{m}}} c_\gamma(s)Q_\gamma\right)$$

where the function $c_\gamma(s)$ has no poles at $s=0$, and $\succ$ denotes the natural order on partitions.

7. A HOMOLOGICAL VERSION

Theorem 4.1 is an equality of Euler characteristics: this is evident for the left hand side $\sum_{\ell,n\geq 0} s^{2\ell}(-v^2)^n\chi_{\mathrm{top}}(C_p^{[\ell,\ell+n]})$, but, thanks to the work of Khovanov and Rozansky [15], the HOMFLY-PT polynomial turns out to be the Euler characteristic of a complex as well. More precisely, for every link $\mathcal{L}$, there are the triply graded *knot homology groups* $\mathcal{H}^{i,j,k}(\mathcal{L})$, such that, up to some normalization (see the discussion in [31] for the numbering conventions),

$$\sum_{i,j,k}(-1)^k v^i s^j \dim\mathcal{H}^{i,j,k}(\mathcal{L}) = \mathbf{P}(\mathcal{L},v,s).$$

It is natural to consider the "superpolynomial"

$$\mathcal{P}(v,s,t) = \sum_{i,j,k} t^k v^i s^j \dim\mathcal{H}^{i,j,k}(\mathcal{L}), \tag{74}$$

specializing to the HOMFLY-PT for $t=-1$, and wonder whether it has an algebraic geometric interpretation, in the spirit of Theorem 4.1.

EXAMPLE 7.1. — *If $\mathcal{L}$ is the trefoil knot, then*

$$\mathcal{P}(v,s,t)=\frac{vt+v^{-1}}{s^{-1}-s}\left(v^2s^{-2}+v^4t^3+v^2s^2t^2\right).$$

On the other hand, a polynomial invariant of an algebraic variety which is finer than the Euler characteristic is the *weight polynomial*: its definition is based on the theory of mixed Hodge structures, created by P. Deligne in the early '70's. The cohomology groups with rational coefficients, compactly supported or not, of a complex algebraic variety are endowed with an increasing filtration $W_\bullet$, the *weight filtration*, natural with respect to all the maps between cohomology groups induced by maps of the corresponding varieties. An important property of the weight filtration is that these maps between cohomology groups are *strict* with respect to it: if

$$0\to \mathrm{H}_1\to \mathrm{H}_2\to \mathrm{H}_3\to 0$$

is an exact sequence of mixed Hodge structures, then for every k

$$0\to \mathrm{Gr}_k^W\mathrm{H}_1\to \mathrm{Gr}_k^W\mathrm{H}_2\to \mathrm{Gr}_k^W\mathrm{H}_3\to 0$$

is exact. If X is a complex algebraic variety, we set

$$\mathfrak{w}(X):=\sum_{j,k}(-1)^{j+k}t^k\dim \mathrm{Gr}_k^W\mathrm{H}_c^j(X).$$

For instance $\mathfrak{w}(\mathbb{C})=t^2$, and $\mathfrak{w}(\mathbb{C}^\times)=t^2-1$, since $\mathrm{H}_c^1(\mathbb{C}^\times)$ has dimension one and weight zero. Note that if for every i we have $\mathrm{Gr}_i^W\mathrm{H}_c^i(X)=\mathrm{H}_c^i(X)$, as in the case of nonsingular proper varieties, then $\mathfrak{w}(X)=\sum_k t^k\dim\mathrm{H}_c^k(X)$ is just the Poincaré polynomial. The most important property of the weight polynomial is its additivity: if $Y\subset X$ is a closed algebraic subvariety of X, then $\mathfrak{w}(X)=\mathfrak{w}(Y)+\mathfrak{w}(X\setminus Y)$, which follows from the fact that the long exact sequence

$$\ldots\to \mathrm{H}_c^i(Y)\to \mathrm{H}_c^i(X)\to \mathrm{H}_c^i(X\setminus Y)\to \mathrm{H}_c^{i+1}(Y)\to\ldots$$

is an exact sequence of mixed Hodge structures, and the strictness property above.

In [31], Oblomkov, Rasmussen and Shende conjecture a "homological" refinement of Eq. (14), for which they provide some evidence, relating the superpolynomial of the algebraic link $\mathcal{L}$ with the generating function for the weight polynomials of nested Hilbert schemes:

CONJECTURE 7.2. — *Let $\mathcal{L}$ be the link of a plane curve singularity (C,p) Then:*

$$\mathcal{P}(\mathcal{L},v,s,t)=\left(\frac{v}{s}\right)^{\mu(f)-1}\sum_{l,m}s^{2l}v^{2m}t^{m^2}\mathfrak{w}(C_p^{[l,l+m]}). \tag{75}$$

Recent development and perspectives on the new emerging picture on algebraic links, stemming from this conjecture, are thoroughly discussed in the lecture notes [29, 30].

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REFERENCES

[1] A. Aiston, *Skein theoretic idempotents of Hecke algebras and quantum group invariants*, Ph.D. thesis, available at http://www.liverpool.ac.uk/ su14/paper/aistonthesis.pdf

[2] A. Aiston, H. Morton, *Idempotents of Hecke algebras of type A*, J. Knot Theory Ramifications 7 (1998), no. 4, 463-487.

[3] M. F. Atiyah, *On analytic surfaces with double points,* Proc. Roy. Soc. London. Ser. A 247 1958 237–244.

[4] T. Bridgeland, *Flops and derived categories*, Invent. Math. **147** (2002), no. 3, 613-632.

[5] T. Bridgeland, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. **24** (2011), 969-998.

[6] J. Calabrese, *Donaldson-Thomas Invariants and Flops*, J. Reine Angew. Math. **716** (2016), 103–145.

[7] A. Campillo, F. Delgado, S.M. Gusein-Zade *The Alexander polynomial of a plane curve singularity via the ring of functions on it,* Duke Math. J. 117 (2003), no. 1, 125–156.

[8] E. Diaconescu, Z. Hua, Y. Soibelman, *HOMFLY polynomials, stable pairs and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. 6 (2012), no. 3, 517–600.

[9] E. Diaconescu, V. Shende, C. Vafa, *Large N duality, Lagrangian cycles, and algebraic knots*, Comm. Math Phys. **319**, (2013), no. 3, 813–863.

[10] D. Eisenbud, W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities.* Annals of Mathematics Studies, 110. Princeton University Press, Princeton, NJ, 1985.

[11] W. Fulton, J. Harris, *Representation Theory. A first course,* Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.

[12] A. Gyoja, *A q-analogue of Young symmetrizer.* Osaka J. Math. **23** (1986), no. 4, 841-852.

[13] J. Hoste, A. Ocneanu, K. Millett, P. Freyd, W. B. R. Lickorish and D. Yetter, *A new polynomial invariant of knots and links,* Bull. Amer. Math. Soc. 12 (1985) 239-246.

[14] C. Kassel, V. Turaev, *Braid Groups,* Graduate Texts in Mathematics, 247. Springer, New York, 2008.

[15] M. Khovanov, L. Rozansky, *Matrix factorizations and link homology, II,* Geom. Topol. **12** (2008), no. 3, 1387-1425.

[16] Y. Konishi, S. Minabe. *Flop invariance of the topological vertex.* Internat. J. Math. **19** (2008), no. 1, 27-45.

[17] J. Le Potier, *Systemes cohérents et structures de niveau* Astérisque 214, (1993).

[18] J. Le Potier, *Faisceaux semi-stables et systemes cohérents.* In: Vector Bundles in Algebraic Geometry, Durham, 1993. London Math. Soc. Lecture Note Ser., vol. 208, pp. 179–239. Cambridge Univ. Press, Cambridge (1995).

[19] W.B.R. Lickorish, *An introduction to the theory of knots,* Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997.

[20] S. Lukac, *Idempotents of the Hecke algebra become Schur functions in the skein of the annulus*, Math. Proc. Cambridge Philos. Soc. **138** (2005), no. 1, 79-96.

[21] S. Lukac, *HOMFLY skeins and the Hopf link*, Ph. D. thesis, available at http://www.liv.ac.uk/ su14/papers/lukacthesis.pdf.

[22] S. Lukac, H. Morton, *The HOMFLY polynomial of the decorated Hopf link*, J. Knot Theory Ramifications **12** (2003), no. 3, 395-416.

[23] I.G. Macdonald, *Symmetric functions and Hall polynomial*, Second edition, Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, 2015. xii+475 pp.

[24] D. Maulik, *Stable pairs and the HOMFLY polynomial.* Invent. Math. **204** (2016), no. 2, 787-831.

[25] J. Milnor, *Singular points of complex hypersurfaces.* Annals of Mathematics Studies, No. 61 Princeton University Press, Princeton, N.J.

[26] H. Morton, P. Manchón, *Geometrical relations and plethysms in the HOMFLY skein of the annulus.* J. Lond. Math. Soc. (2) **78** (2008), no. 2, 305-328.

[27] H. Nakajima, *Lectures on the Hilbert Schemes of points on surfaces,* University Lecture Series, vol.18, Amer.Math.Soc. Providence, RI,1988.

[28] W. Neumann, *Topology of hypersurface singularities.*, Erich Kaehler: Mathematische Werke, R. Berndt and O. Riemenschneider eds., (de Gruyter 2003), 727-736.

[29] S. Nawata, A. Oblomkov, *Lectures on knot homology,* Physics and mathematics of link homology 137–177, Contemp. Math., 680, Centre Rech. Math. Proc., Amer. Math. Soc., Providence, RI, 2016.

[30] A. Oblomkov, *Notes on matrix factorizations and knot homology,* Notes for the CIME Summer School June 2018, arXiv:1901.04052

[31] A. Oblomkov, J. Rasmussen, V. Shende, with appendix by E. Gorsky, *The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link.*, Geom. Topol. 22 (2018), no. 2, 645–691.

[32] A. Oblomkov, V. Shende, *The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link,* Duke Mathematical Journal **161**,7, (2012), 1277-1303.

[33] J. Przytycki and P. Traczyk, *Conway Algebras and Skein Equivalence of Links,* Proc. Amer. Math. Soc. 100 (1987) 744–748.

[34] R. Pandharipande, R. Thomas, *Curve counting via stable pairs in the derived category.* Invent. Math. **178** (2009), 407-447.

[35] R. Pandharipande, R. Thomas, *Stable pairs and BPS invariants,* J. Amer. Math. Soc. 23 (2010), no. 1, 267–297.

[36] R. Pandharipande, R. Thomas, *13/2 ways of counting curves.* in Moduli spaces, 282–333, London Math. Soc. Lecture Note Ser., 411, Cambridge Univ. Press, Cambridge, 2014.

[37] R. Pandharipande, R. Thomas, *The 3-fold vertex via stable pairs.* Geom. Topol. **13** (2009), no. 4, 1835-1876.

[38] N. Reshetikhin, V. Turaev, *Ribbon graphs and their invariants derived from quantum groups.* Comm. Math. Phys. **127** (1990), 1–26.

[39] N. Reshetikhin, N. Turaev, *Invariants of 3 -manifolds via link polynomials and quantum groups.* Invent. Math.**103**(1991), 547–597.

[40] D. Sullivan, *Combinatorial invariants of analytic spaces,* Proceedings of Liverpool Singularities Symposium I Lecture Notes in Mathematics Volume 192, 1971, 165-177.

[41] J. Stoppa, R. Thomas, *Hilbert schemes and stable pairs: GIT and derived category wall crossings,* Bull. SMF. **139**, 297-339, 2011.

[42] V. Turaev, *The Conway and Kauffman modules of a solid torus.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167, 1988, Issled. Topol. 6, 79–89 (Russian), English translation: J. Soviet Math. 52 (1990), 2799–2805.

[43] C.T.C. Wall, *Singular points of plane curves.* London Mathematical Society Student Texts, **63.** Cambridge University Press, Cambridge, 2004.

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