Physica A 620 (2023) 128780

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Hydrodynamic fluctuations in the presence of one parameter Mittag-Leffler friction

Evangelos Bakalis *, Francesco Zerbetto

Dipartimento di Chimica "G. Ciamician", Università di Bologna, V. F. Selmi 2, 40126, Bologna, Italy

ARTICLE INFO

Article history:
Received 20 February 2023
Received in revised form 7 April 2023
Available online 24 April 2023

Keywords:
Anomalous diffusion
Colored noises
Boussinesq–Basset force
Mittag-Leffler noise
Fractional generalized Langevin equation

ABSTRACT

The effects of hydrodynamic fluctuations on the subdiffusive motion of a particle subject to one parameter Mittag-Leffler friction are examined by means of the fractional Langevin equation. The particle experiences an overall additive colored noise formed by, on the one hand, the hydrodynamic back flow effects and, on the other hand, an additional contribution predicted by fluctuation dissipation relation. Particle motion may or may not be subject to a restoring force. All possible combinations of forces exerted on the test particle are being studied, and for each of them the generalized response function in terms of multinomial Mittag-Leffler functions is provided. Mean square displacement, normalized velocity and position auto-correlation functions are furnished as special cases of the generalized response function, and their short and long time limits are analytically given. In addition, for the same measures analytical expressions valid for time windows much broader than the usual asymptotic limit are provided, and can be used to fit real life data. We demonstrate that normalized velocity and position auto-correlation functions are the main sources providing information on the effect of hydrodynamic fluctuations on particle motion. Actually, they oppose to friction and to restoring force, and smooth out the anti-persistent character of the motion.

1. Introduction

In complex environments the erratic motion of particles departs from Brownian motion, and many factors can be responsible for that. Approaching a system phenomenologically, the origin of such a behavior can be sought either on a time- dependent friction coefficient, or on the presence of hydrodynamic fluctuations, or on conservative forces acting on the particle, or on any combination of some or all of them. [1–4] The effect of a time-dependent friction coefficient or of a time-dependent diffusion coefficient has been extensively investigated usually through an internal noise of various forms, [5–16] which establishes, in the long time limit, a link with the friction via the dissipation fluctuation theorem (FDT) considering that the system is within the linear response regime and thus any conservative forces do not affect this link. [17]. On the other hand, analytical investigation of the role of hydrodynamic fluctuations on the dynamics of a diffusing particle has attracted less interest [18–21] even though numerical simulations and experimental evidence have shown that hydrodynamic fluctuations effectuate on the diffusional particle and give rise to interesting dynamics. [22–27]

Hydrodynamic fluctuations are triggered and become important when the ratio $\chi = \rho_f/\rho_p$ is not negligible, ρ_f and ρ_p being the densities of the fluid and of the diffusing particle respectively. For an incompressible fluid, as χ departs from zero values, the flow is not any more steady and the friction coefficient cannot be considered as constant, see [28] where

* Corresponding author.
E-mail addresses: evangelos.bakalis2@unibo.it (E. Bakalis), francesco.zerbetto@unibo.it (F. Zerbetto).

https://doi.org/10.1016/j.physa.2023.128780
0378-4371/

also a discussion is made on how the research in this field has been evolved.[29] Actually, the source of the hydrodynamic fluctuations is the momentum of the displaced fluid molecules due to particle movements back to it during the course of its motion, and it is well known in the literature as Boussinesq–Basset force(BB). [30,31]

The following fractional generalized Langevin Equation (fGLE), [17]

$$m_p\dot{u}_i(t) = -\underbrace{\int_0^t \gamma(t-t')u_i(t')dt'}_{F_S(t)} - \underbrace{\frac{m_f}{2}\dot{u}_i(t) - \int_{t_0}^t \zeta(t-t')\dot{u}_i(t')dt'}_{F_{BB}(t)} - \underbrace{\frac{\partial V(x_i)}{\partial x_i}}_{F_C(t)} + \underbrace{\xi_i(t)}_{F_R(t)} \tag{1}$$

models the motion of a probe particle in the presence of hydrodynamic fluctuations and contains deterministic and stochastic terms. The deterministic part is local in time, only the close neighbors of the particle affect it. Instead, there are contributions arising from non-local sources, which means that an effect originated at time t' is retarded and persists for times $t > t'$. In Eq. (1), a particle of mass m_p, radius r, and density ρ_p moves within a fluid of density ρ_f, and it is subject to: (i) generalized Stokes friction force $F_S = \int_0^t \gamma(t-t')u(t')dt'$, which reduces to $F_S = \gamma_0 u(t)$ for constant friction coefficient $\gamma_0 = 6\pi\eta r$, and η being the dynamic viscosity of the medium. (ii) hydrodynamic fluctuations or Boussinesq–Basset (BB) force, $F_{BB}(t) = \frac{m_f}{2}\dot{u}(t) + \int_{t_0}^t \zeta(t-t')\dot{u}(t')dt'$. (iii) conservative forces $F_C = -\frac{\partial V(x_i)}{\partial x_i}$, and (iv) random force, F_R, whose mean is zero and its correlation remains to be specified since it defines the statistical properties of the random force. Any other conservative external force and/or periodic force can also be exerted on the probe particle acting or not as perturbation. Note that in Eq. (1) $u_i = \frac{dx_i}{dt}$ is the particle velocity along the ith direction.

In Eq. (1), the memory kernel appears in the generalized Stokes force, it has meaning in distributional sense, and it is equal to zero for $t < 0$, (casuality). For constant drag coefficient, the memory kernel is expressed through the Dirac Delta function, $\gamma(t-t') = 2\gamma_0\delta(t-t')$. The BB force consists of two terms; the first one describes the changes of the velocity caused by the molecules of the liquid being displaced by the tracer particle, m_f is the mass of the fluid element with a volume equal to that of the particle, and the second one expresses the convolution of a time dependent friction coefficient, $\zeta(t) = \gamma_0\sqrt{\frac{\tau_\nu}{\pi}}t^{-\frac{1}{2}}$, weighted by the vorticity time, $\tau_\nu = \frac{r^2\rho_f}{\eta}$, with the acceleration. The lower bound of the convolution integral in BB force t_0 can be set to $-\infty$, and thus the contribution $\int_{-\infty}^0 \zeta(t-t')\dot{u}(t')dt'$ corresponds to a colored noise caused by BB force, whose correlation goes as $\langle\xi_{BB}(t)\xi_{BB}(0)\rangle \sim t^{-\frac{3}{2}}$ [32,33]. The conservative force is expressed as the gradient of the potential energy. The overall random force contains at least thermal fluctuations which can be additively, multiplicatively, or convolutively modified by random forces reflecting internal processes evolving at much faster time scales than the random motion itself.

In Eq. (1), we assume that the noise term, $\xi(t)$, is internal and it may be the additive effect of all noise sources. In the presence of just one noise source the fluctuation dissipation theorem (FDT) states

$$\langle\xi(0)\xi(\tau_L)\rangle = k_B T\gamma(\tau_L) \tag{2}$$

where k_BT is the Boltzmann's constant times the temperature of the bath, T, and τ_L is the lag time whose minimum value is equal to the difference between two consecutive time moments. Yet, in the realm of a FDT the complexity of the initial set of equations may be reduced and lead to simplified drift terms.[34] For experimental measurements, its minimum value is the time difference between two consecutive recordings and it depends on the used machinery. Eq. (2) holds true when the system reaches equilibrium, or for large values of τ_L. It means that we can use Eq. (2) to access properties of the system, when the observation time is orders of magnitude larger than the time scale where the random collisions take place between tracer particle and molecules/obstacles of the solution, actual scale. Given that the actual scale is of order of picoseconds and the time scale of a routinely carried out experiment is from micro to milliseconds it is obvious that the system has already reached equilibrium at the minimum experimental time lag. Eq. (2) makes the connection between dissipation and friction given that both have common origin [17]. If more than one noises act cumulatively onto the probe particle, Eq. (2) still holds true, that is, instead of $\gamma(\tau_L)$ appears the generalized friction term $\gamma_G(\tau_L)$ which is the sum of the contributions of all noises, [21] see also Eq. (A.4) in Appendix A for an alternative derivation.

To the best of our knowledge, Eq. (1) has been solved for constant friction coefficient, [28], and slowly decaying friction term [19,20]. In the present work, which is a generalization of many works up to today, a one parameter Mittag-Leffler (ML) function, $E_a(z) = \sum_{n=0}^{\infty}\frac{z^a}{\Gamma(an+1)}$ describes the friction coefficient, and our choice assures that it goes to zero for $t \to 0$ and it behaves as power law for large times [35]. Mittag-Leffler functions have found many applications in areas unrelated to the present one such as pharmacokinetics [36]. We solve Eq. (1) and obtain closed analytical solutions through the multinomial ML function [37] for the generalized response function. VAF and PAF (velocity and position auto-correlation respectively) functions as well the MSD (mean-square displacement) are close related to response functions, and their early time behavior as well as their asymptotic solutions are provided. In addition to that, we provide closed analytical solutions, expressed in terms of the two parameter ML function, amenable to be used in fitting experimental evidence. Such solutions are valid for much broader time windows than the usual asymptotic limit and are given for very first time.

2. Generalized fractional Langevin equation and response functions

We set $M = m_p + \frac{m_f}{2}$, and $\frac{\partial V(x)}{\partial x} = kx$ (restoring force), in Eq. (1), and by taking the Laplace pair, $L\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t)dt$, we find

$$x(s) = \frac{x(0)}{s} + u(0)\left(1 + \frac{\zeta(s)}{M}\right)G(s) - \omega^2 x(0)I(s) + \frac{\xi(s)G(s)}{M} \tag{3}$$

and

$$u(s) = u(0)\left(1 + \frac{\zeta(s)}{M}\right)g(s) - \omega^2 x(0)G(s) + \frac{\xi(s)g(s)}{M} \tag{4}$$

where

$$G(s) = \left(s^2 + \frac{s}{M}(\gamma(s) + s\zeta(s)) + \omega^2\right)^{-1} \tag{5}$$

$I(s) = G(s)/s$ and $g(s) = sG(s)$. In Eqs. (3–5) the term $\tau_h = \omega^{-1} = \sqrt{M/k}$ denotes the period of oscillation and ω is thus the frequency of oscillation. At the same set of equations and other time scales exist, namely, $\tau_d = \frac{M}{\gamma_0}$ delivering how fast the effect of friction force is attenuated, $\tau_\nu = \frac{r^2 \rho_f}{\eta}$ is the vorticity time providing the loss of hydrodynamic memory, and $\tau_k = \gamma/k$ yielding the antagonism between friction coefficient and the stiffness of the harmonic potential and defines the trapping time.

Inversion of Eqs. (3) and (4) return the position and the velocity as functions of time

$$\begin{aligned} x(t) = x(0) + \left(G(t) + \frac{1}{M}\int_0^t \zeta(t-t')G(t')dt'\right)u(0) - \\ \omega^2 x(0)I(t) + \frac{1}{M}\int_0^t \xi(t-t')G(t')dt' \end{aligned} \tag{6}$$

and

$$\begin{aligned} u(t) = \left(g(t) + \frac{1}{M}\int_0^t \zeta(t-t')g(t')dt'\right)u(0) - \omega^2 x(0)G(t) \\ + \frac{1}{M}\int_0^t \xi(t-t')g(t')dt' \end{aligned} \tag{7}$$

Given that $\langle \xi(t) \rangle = 0$, the mean of position and velocity are obtained by Eqs. (6) and (7) and are $\langle x(t) \rangle = x(0) + G(t)u(0) + \frac{u(0)}{M}\int_0^\infty \zeta(t-t')G(t')dt' - \omega^2 x(0)I(t)$ and $\langle u(t) \rangle = g(t)u(0) + \frac{u(0)}{M}\int_0^\infty \zeta(t-t')g(t')dt' - \omega^2 x(0)G(t)$.

3. Friction coefficient as one parameter Mittag-Leffler function, and analytical solutions of the fractional generalized langevin equation

We consider the friction term $\gamma(t) = \bar{\gamma}E_a(-(t/\mu)^a)$, $0 < a < 1$, where $\bar{\gamma} = \frac{\gamma_0}{\mu^\alpha}$, $E_a(-(t/\mu)^a)$ being the one parameter ML function, μ reflects the time memory of the friction, and γ_0 is the friction coefficient expressed in proper units. The function $E_a(-(t/\mu)^a)$ behaves as a stretched exponential for short times and as inverse power law for long times. [35]. Yet, it encapsulates both power law and delta function representations for specific limits of μ and α. As $\mu \to 0$ and for $a \neq 1$ the friction force reads $\gamma(t) = \gamma_0 \frac{t^{-a}}{\Gamma(1-a)}$,[1] while if $\alpha = 1$ it returns $\gamma(t) = \gamma_0\delta(t)$,[2] memory-less friction and thus standard Brownian motion. And yet, it vanishes as time goes to zero and to infinity, in contrast with the divergence of power law friction when time goes to zero. $L\{E_a(-(t/\mu)^a)\}(s) = \frac{s^{a-1}}{s^a + \mu^{-a}}$ is the Laplace pair of the one-parameter ML function. Substituting in Eq. (5) the Laplace pairs of both friction and BB force we end up with the generalized response function $R(s)$, Eq. (8).

$$R(s) = \frac{s^\delta}{s^2 + \frac{\bar{\gamma}}{M}\frac{s^a}{(s^a + \mu^{-a})} + \frac{\gamma_0}{M}\sqrt{\tau_\nu}s^{3/2} + \omega^2} \tag{8}$$

In Eq. (8), the exponent δ takes three values; $\delta = 1$ returns the response function $g(s)$, $\delta = 0$ provides the response function $G(s)$, and $\delta = -1$ gives the response function $I(s)$, see below for their connection with frequent used statistical measures. The function $R(s) = \frac{s^\delta}{s^2 + \frac{\bar{\gamma}}{M}\frac{s^a}{(s^a + \mu^{-a})} + \frac{\gamma_0}{M}\sqrt{\tau_\nu}s^{3/2} + \omega^2}$, is split into two terms $R_1(s) = \frac{s^{\delta-2}}{R_0(s)}$ and $R_2(s) = \frac{\nu_2 s^{\delta-a-2}}{R_0(s)}$, where

[1] For $\mu \to 0$ the argument of $E_a(-(t/\mu)^a)$ goes to infinity for every value of t, so using eq.(C.4) for $n = 1$ and $b = 1$, we end up with $\gamma(t) = \gamma_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$

[2] Given that $E_1(-(t/\mu)) = e^{(-(t/\mu))}$, then the expression, $\lim_{\mu\to 0} \frac{1}{2\mu} e^{-\frac{t}{\mu}}$ is the limit representation of delta function.

$R_0(s) = 1+\nu_1 s^{-1/2}+\nu_2 s^{-a}+\nu_3 s^{-\frac{1}{2}-a}+\nu_4 s^{-2}+\nu_5 s^{-2-a}$, with $\nu_1 = \frac{\gamma_0}{M}\sqrt{\tau_\nu}$, $\nu_2 = \mu^{-a}$, $\nu_3 = \frac{\gamma_0}{M}\mu^{-a}\sqrt{\tau_\nu} = \nu_1\nu_2$, $\nu_4 = \nu_{4,1}+\nu_{4,2}$ with $\nu_{4,1} = \frac{\bar{\gamma}}{M}$, $\nu_{4,2} = \omega^2$, and $\nu_5 = \mu^{-a}\omega^2$. The inversion of $R_{1,2}(s)$ in the time domain is made through the multinomial ML function [37,38] and reads

$$R_1(t) = t^{1-\delta}E_{(2+a,2,\frac{1}{2}+a,a,\frac{1}{2}),2-\delta}(-\nu_5 t^{2+a}, -\nu_4 t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^a, -\nu_1 t^{\frac{1}{2}}) \tag{9}$$

and

$$R_2(t) = t^{1+a-\delta}E_{(2+a,2,\frac{1}{2}+a,a,\frac{1}{2}),2+a-\delta}(-\nu_5 t^{2+a}, -\nu_4 t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^a, -\nu_1 t^{\frac{1}{2}}) \tag{10}$$

See eq. (C.9), Appendix C, for the definition of the multinomial ML function. For $t \to 0$, the overall response function $R(t) = R_1(t) + R_2(t)$ is expanded and the first member of the expansion is equal to $\frac{t^{1-\delta}}{\Gamma(2-\delta)}$ and consequently $g(t)$ goes to 1 and $G(t)$, $I(t)$ are zero, ($\delta = 1, 0, -1$). Relaxing the condition $t \to 0$, expansion of $R(t)$ in the short time limit returns

$$R(t) = t^{1-\delta}\{\frac{1}{\Gamma(2-\delta)} - \frac{\nu_1}{\Gamma(\frac{5}{2}-\delta)}t^{\frac{1}{2}} + \frac{\nu_1^2}{\Gamma(3-\delta)}t - \frac{\nu_1^3}{\Gamma(\frac{7}{2}-\delta)}t^{\frac{3}{2}} + \frac{(\nu_1^4-\nu_4)}{\Gamma(4-\delta)}t^2\} \tag{11}$$

It is worth mentioning that Eq. (11) contains only terms due to hydrodynamic fluctuations up to $t^{\frac{3}{2}}$ term. Eqs. (9) and (10) can be further simplified following the steps provided in Appendix C, and $R(t)$ for $\frac{\nu_4}{\nu_3}t^{\frac{3}{2}-\alpha} > 1$ reads

$$R(t) = \frac{t^{-1-\delta}}{\nu_4}\{E_{\alpha,-\delta}(-\frac{\nu_5}{\nu_4}t^\alpha) + \nu_2 t^\alpha E_{\alpha,\alpha-\delta}(-\frac{\nu_5}{\nu_4}t^\alpha)\} \tag{12}$$

For $t > (\frac{\nu_4}{\nu_5})^{\frac{1}{\alpha}}$, the two parameter ML function $E_{a,b}(z)$ goes asymptotically as $-\sum_{n=1}^{\infty}\frac{z^{-n}}{\Gamma(b-an)}$ [39] and the asymptotic behavior of Eq. (12) takes the form

$$R(t) = \frac{\nu_2}{\nu_5}t^{-1-\delta}\{\frac{1}{\Gamma(-\delta)} - (\frac{1}{\nu_2} - \frac{\nu_4}{\nu_5})\frac{t^{-a}}{\Gamma(-a-\delta)} + \frac{\nu_4}{\nu_5}(\frac{1}{\nu_2} - \frac{\nu_4}{\nu_5})\frac{t^{-2a}}{\Gamma(-2a-\delta)}\} \tag{13}$$

As $t \to \infty$, Eq. (13) returns $g(t) \to 0$, $G(t) \to 0$, and $I(t) \to \frac{1}{\omega^2}$ indicating the trapping of the particle as we will discuss below. Both short and long time limit are listed in Table 2.

Eqs. (11) and (13) provide the early and the long time behavior of the most frequent used statistical measures like MSD, and NVAF/NPAF (the normalized velocity and position autocorrelation function respectively) in characterizing the dynamics of diffusing particles. Eqs. (B.13) and (B.14), see Appendix B, provide the detailed forms of MSD and of NVAF respectively as functions of the actual time moments t, t'. The time evolution of NPAF is given by Eq. (16) and will be treated separately, see below. Given that recording techniques have time lags, τ_L, much larger that the actual time, it makes sense to take $t, t' \to \infty$ and $|t-t'| = \tau_L$ finite, which simplifies a lot the forms of MSD, $\langle\Delta x^2\rangle(\tau_L) = \frac{2k_BT}{M}I(\tau_L)$, and of NVAF $\frac{C_u(\tau_L)}{C_u(0)} = \frac{m}{M}g(\tau_L)$, with $I(\tau_L) = R(\tau_L)|_{\delta=-1}$, and $g(\tau_L) = R(\tau_L)|_{\delta=1}$. Of note that $C_u(0) = \frac{k_BT}{m_p}$, and it is worth mentioning that in Eq. (B.14) the term $\frac{k_BT}{M}$ is coming from equi- partition in the long time limit. At time $t = 0$, and assuming thermal equilibrium the corresponding term is $\frac{k_BT}{m_p}$ which differs in the mass term when hydrodynamic effects are coming into play.

All forces described by Eq. (1) are not necessarily applied on a diffusing particle, a part of them may be non-active. Response functions for such cases are of interest because of they may describe experimental processes. We consider four different scenarios with common element a friction of the form $\gamma(t) = \bar{\gamma}E_a(-(t/\mu)^a)$, $0 < a < 1$, and we obtain analytical solutions for (i) $\zeta(t) = 0$, and $\omega = 0$, (ii) $\zeta(t) = 0$, and $\omega \neq 0$, (iii) $\zeta(t) \neq 0$, and $\omega = 0$. The fourth scenario has already been presented, eqs(9–13), for $\zeta(t) \neq 0$, and $\omega \neq 0$

Table D.1, see Appendix D, contains the response functions for each scenario. These functions are valid for all time moments, however, in presence of hydrodynamic fluctuations their validity holds true for $t \gg r/c$, where r is the radius of the particle and c the velocity of the sound in the medium. For $t < r/c$ compressibility effects should be taken into account. [40] Table 1 shows simplified forms of these functions which apply for $t > t_c$, where t_c is the minimum time for which the multinomial ML function can be expressed by a two-parameter ML function. The latter can be used directly for graphical illustration or can be used as trial functions for fitting real life data. [41] Table 2 contains the short time limit and the asymptotic behavior of the aforementioned response functions.

4. Discussion

Eq. (11) provides the short time limit of MSD and NVAF for $\delta = -1$ and $\delta = 1$ respectively, and their detailed form is given by Eqs. (14) and (15). Note that the mass M is replaced by the mass of the diffusing particle m_p when hydrodynamic fluctuations are not present.

$$\langle\Delta x^2\rangle(\tau_L) = \frac{2k_BT}{M}\tau_L^2\{\frac{1}{2} - \frac{8\nu_1}{15\sqrt{\pi}}\tau_L^{\frac{1}{2}} + \frac{\nu_1^2}{6}\tau_L - \frac{16\nu_1^3}{105\sqrt{\pi}}\tau_L^{\frac{3}{2}} + \frac{(\nu_1^4-\nu_4)}{24}\tau_L^2\} \tag{14}$$

Table 1
The response function, $R(t)$ of a diffusing particle valid for $t > t_c$.

	$R(t)$
$\zeta(t)=0$ $\omega=0$	$t^{1-\delta}\{\frac{1}{\Gamma(2-\delta)} - \frac{\nu_{4,1}}{\nu_2}t^{2-\alpha}E_{2-\alpha,\,4-\delta-\alpha}(-\frac{\nu_{4,1}}{\nu_2}t^{2-\alpha})\},\ t > (\frac{1}{\nu_2})^{\frac{1}{\alpha}}$
$\zeta(t)=0$ $\omega\neq 0$	$\frac{t^{-1-\delta}}{\nu_4}\{E_{\alpha,\,-\delta}(-\frac{\nu_5}{\nu_4}t^{a}) + \nu_2 t^{\alpha}E_{\alpha,\alpha-\delta}(-\frac{\nu_5}{\nu_4}t^{a})\},\ t > (\frac{\nu_2}{\nu_4})^{\frac{1}{2-\alpha}}$
$\zeta(t)\neq 0,$ $\omega=0$	$\frac{t^{\frac{1}{2}-\alpha-\delta}}{\nu_3}\{E_{\frac{3}{2}-\alpha,\,\frac{3}{2}-\delta-\alpha}(-\frac{\nu_{4,1}}{\nu_3}t^{\frac{3}{2}-\alpha}) + \nu_2 t^{\alpha}E_{\frac{3}{2}-\alpha,\,\frac{3}{2}-\delta}(-\frac{\nu_{4,1}}{\nu_3}t^{\frac{3}{2}-\alpha})\},\ t > (\frac{\nu_1}{\nu_3})^{\frac{1}{\alpha}}$
$\zeta(t)\neq 0,$ $\omega\neq 0$	$\frac{t^{-1-\delta}}{\nu_4}\{E_{\alpha,-\delta}(-\frac{\nu_5}{\nu_4}t^{\alpha}) + \nu_2 t^{\alpha}E_{\alpha,\alpha-\delta}(-\frac{\nu_5}{\nu_4}t^{\alpha})\}, t > (\frac{\nu_3}{\nu_4})^{(\frac{3}{2}-\alpha)^{-1}}$

Table 2
The response function, $R(t)$, in the short and long time limit.

	$R(t\to 0)$	$R(t\to\infty)$
$\zeta(t)=0$ $\omega=0$	$t^{1-\delta}\{\frac{1}{\Gamma(2-\delta)} - \frac{\nu_{4,1}}{\Gamma(4-\delta)}t^2 + \frac{\nu_2\nu_{4,1}}{\Gamma(4+\alpha-\delta)}t^{2+\alpha}\}$	$\frac{t^{-1-\delta}}{\nu_{4,1}}\{\frac{1}{\Gamma(-\delta)} + \nu_2\frac{t^{\alpha}}{\Gamma(\alpha-\delta)}\}$
$\zeta(t)=0$ $\omega\neq 0$	$t^{1-\delta}\{\frac{1}{\Gamma(2-\delta)} - \frac{\nu_4}{\Gamma(4-\delta)}t^2 + \frac{(\nu_2\nu_4-\nu_5)}{\Gamma(4+\alpha-\delta)}t^{2+\alpha}\}$	$\frac{\nu_2}{\nu_5}t^{-1-\delta}\{\frac{1}{\Gamma(-\delta)} - \nu_4\frac{t^{-\alpha}}{\Gamma(-\alpha-\delta)}\}$
$\zeta(t)\neq 0,$ $\omega=0$	$t^{1-\delta}\{\frac{1}{\Gamma(2-\delta)} - \frac{\nu_1}{\Gamma(\frac{5}{2}-\delta)}t^{\frac{1}{2}} - \frac{\nu_1^2}{\Gamma(3-\delta)}t - \frac{\nu_1^3}{\Gamma(\frac{7}{2}-\delta)}t^{\frac{3}{2}} + \frac{(\nu_1^4-\nu_{4,1})}{\Gamma(4-\delta)}t^2\}$	$\frac{t^{-1-\delta}}{\nu_{4,1}}\{\frac{1}{\Gamma(-\delta)} + \nu_2\frac{t^{\alpha}}{\Gamma(\alpha-\delta)}\}$
$\zeta(t)\neq 0,$ $\omega\neq 0$	$t^{1-\delta}\{\frac{1}{\Gamma(2-\delta)} - \frac{\nu_1}{\Gamma(\frac{5}{2}-\delta)}t^{\frac{1}{2}} - \frac{\nu_1^2}{\Gamma(3-\delta)}t - \frac{\nu_1^3}{\Gamma(\frac{7}{2}-\delta)}t^{\frac{3}{2}} + \frac{(\nu_1^4-\nu_4)}{\Gamma(4-\delta)}t^2\}$	$\frac{\nu_2}{\nu_5}t^{-1-\delta}\{\frac{1}{\Gamma(-\delta)} - (\frac{\nu_4}{\nu_5} - \frac{1}{\nu_2})\frac{t^{-\alpha}}{\Gamma(-\alpha-\delta)} + \frac{\nu_4}{\nu_5}(\frac{\nu_4}{\nu_5} - \frac{1}{\nu_2})\frac{t^{-2\alpha}}{\Gamma(-2\alpha-\delta)}\}$

and

$$\frac{C_u(\tau_L)}{C_u(0)} = \frac{m_p}{M}\{1 - \frac{2\nu_1}{\sqrt{\pi}}\tau_L^{1/2} + \nu_1^2\tau_L - \frac{4\nu_1^3}{3\sqrt{\pi}}\tau_L^{\frac{3}{2}} + \frac{(\nu_1^4-\nu_4)}{2}\tau_L^2\} \tag{15}$$

Eq. (14) states that the MSD is ballistic, $\sim \tau_L^2$, at early time moments, and in absence of hydrodynamic fluctuations the first correction term is proportional to τ_L^4 accompanied by a factor containing information about the presence, ν_4, or not, $\nu_{4,1}$, of the restoring force. In this limit, hydrodynamic fluctuations change significantly the time dependence of the MSD. Three correction terms appear before the fourth order term, and all contain the influence of the BB force and their scaling goes from $\sqrt{\tau_L}$, to τ_L and up to $\tau_L^{\frac{3}{2}}$. Eq. (15) provides the early behavior of the NVAF, which always starts with the value $\frac{m_p}{M}$. As in the case of MSD, the first correction term is quadratic reducing its value, when BB forces are not present, and its cofactor depends on the forces exerted on the diffusing particle. On the contrary, BB forces abruptly reduce the value of the NVAF because of the existence of three terms before the quadratic one. Solution of exactly the same structure has been reported, namely, $C_u(\tau_L)/C_u(0) = 1 - \frac{2}{\sqrt{\pi}}\frac{\sqrt{\tau_v}}{\tau_d}t^{\frac{1}{2}} + \frac{(\tau_v-\tau_d)}{\tau_d^2}t + \frac{4}{3\sqrt{\pi}}\frac{\sqrt{\tau_v}(2\tau_d-\tau_v)}{\tau_d^3}t^{\frac{3}{2}}$ describing the dynamics of diffusing particle under restoring force and hydrodynamic fluctuations by assuming constant friction [42].

The solutions presented so far are general, can access particle dynamics in complex environments, and furthermore, distinguish the nature of the forces acting on the tagged particle. In the following, we demonstrate how these relations can reveal insights about a system dynamics given that experimental evidence is available. We will use, as an example, two systems which have already been studied and presented elsewhere, and for which we use only some numerical values concerning properties of the particle (radius), the environment (viscosity, density), and the restoring force (stiffness) [25]. We consider all possible combinations of forces and we comment on their effect on particle dynamics.

Fig. 1 shows the NVAF (left panel) and the MSD for a barium titanate microsphere executing random motion in acetone. Fig. 1a–1d are based on the analytical expressions listed in Table 1 (valid for $t > t_c$), and Fig. 1e–1h are based on numerical inversion of Laplace transform of Eq. (8). [43] Friction obeys a one parameter ML function in all four different models examined in this work. All graphs displayed in Fig. 1 have been drawn for $a = 0.9$. The NVAF for zero restoring force is displayed in Figs. 1a and 1e. Red for $z(t) = 0$ where both the analytical expression and the numerical inversion of Eq. (8) for $\nu_1 = \nu_3 = \nu_5 = 0$ and $\nu_{4,1} \neq 0$ present a nice agreement. At the same graphs the presence of hydrodynamic fluctuations, green lines, change the decline rate of NVAF. The analytical expression, Fig. 1a, works well for $t > \tau_\nu$, which covers a much broader time window than the usual asymptotic limit. The MSD for the same combination of forces is illustrated in Figs. 1c and 1g, where in addition a straight line (magenta) stands for Brownian motion. For $z(t) = k = 0$ (red lines), analytical expression and numerical inversion present a nice agreement, MSD starts balistically and gradually the slope decreases reaching the value of α for long times, see similarity between MSD and magenta straight line ($\alpha = 1$). The effects of the hydrodynamic fluctuations, for $k = 0$ (green lines), with respect to the red lines are mainly reflected on the smoother change in slope of MSD from its initial value of 2 to its final value of 0.9, and MSD rises slower under

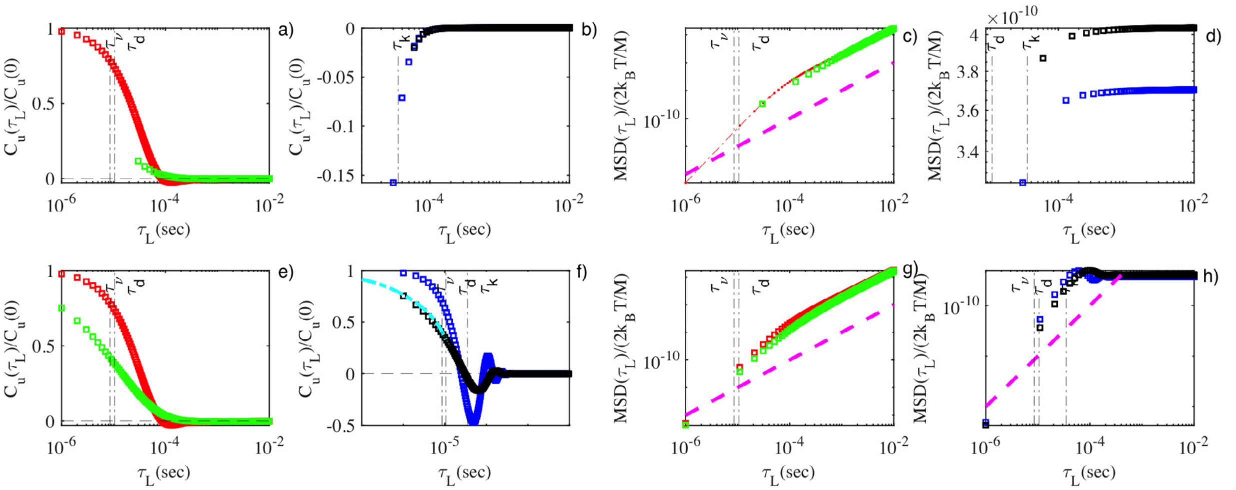

Fig. 1. It displays the NVAF (left panel) and the MSD (right panel) for various combinations of forces for a barium titanate microsphere in acetone with $r = 1.86 \pm 0.03$ μm, $k = 3.2 \pm 0.2$ x 10^{-4} N/m, $\tau_\nu = 8.5$ μs, see Fig.2 of [25]. Color code: red for $z(t) = k = 0$, green for $k = 0, z(t) \neq 0$, blue for $k \neq 0, z(t) = 0$, black for $z(t) \neq 0, k \neq 0$; magenta shows the time evolution of MSD for a truly Brownian motion, and cyan illustrates Eq. (15). Vertical lines indicate time scales, namely, vorticity time, t_ν, diffusion time, t_d, and trapping time, t_k.

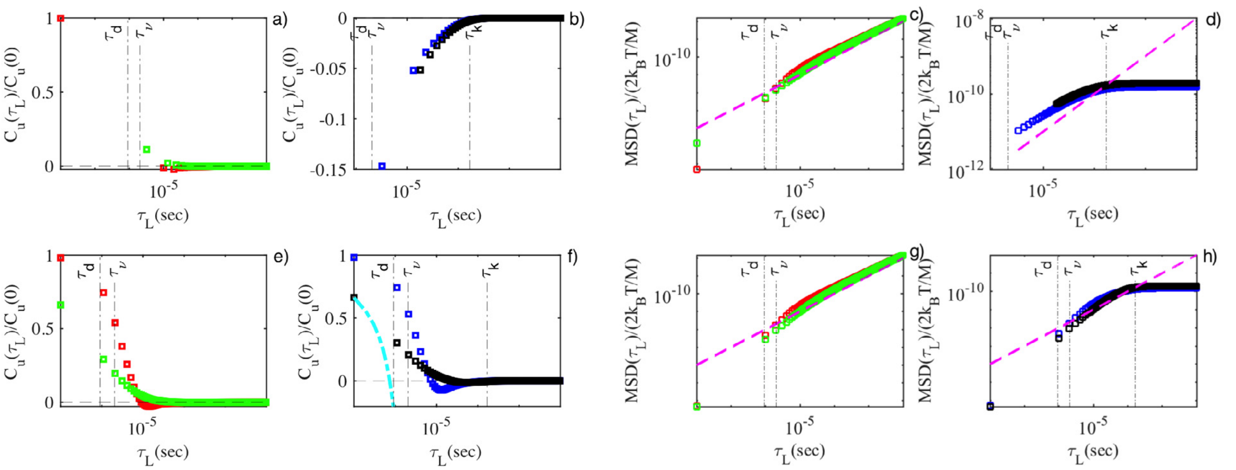

Fig. 2. It displays the NVAF and the MSD for various combinations of forces for a silica microsphere in water with $r = 1.43 \pm 0.015$ μm, $k = 1.6 \pm 0.3$ x 10^{-4} N/m, $\tau_\nu = 2$ μs, see Fig.2 of [25]. Color code: red for $z(t) = k = 0$, green for $z(t) \neq 0, k = 0$, blue for $z(t) \neq 0$ and $k = 0$, black for $z(t) \neq 0$ and $k \neq 0$, magenta illustrates a truly Brownian motion, and cyan displays Eq. (15). Vertical lines indicate the three time scales; vorticity time, t_{nu}, diffusion time, t_d, and trapping time, t_k.

this influence. Again, the analytical expression works well for a big time window and also covers the smooth change of the slope.

When restoring force is present, Figs. 1b and 1f display the NVAF and Figs. 1d and 1h show the MSD. The analytical solutions in the absence of hydrodynamic fluctuations (blue), Fig. 1b, as well as in their presence (black), describe only how the NVAF goes to zero. Instead, numerical inversion, which covers the whole time window, highlights that BB forces lead to smoother changes of NVAF, and to smaller negative minimum, a concept associated to anti-persistent motion. Additionally, the short time behavior of NVAF, Eq. (15), has been drawn (cyan line), which seats well on the numerical solution. The MSD does not differ significantly in the presence and absence of hydrodynamic fluctuations. The presence of a restoring force leads to a plateau, the value of which is similar in both cases.

Fig. 2 shows the NVAF (left panel) and the MSD (right panel) for a silica microsphere executing random motion in water. For this system the vorticity time is larger than the diffusive time, the opposite is true for BTG particle. Assuming zero restoring force, the pattern behavior is the same as for the BTG particle; the NVAF of silica microsphere becomes slightly negative when only the friction force is present, while, the presence of hydrodynamic fluctuations washes out the anti- persistency of particle movements and NVAF is always positive. Furthermore, the MSD follows similar trends for both cases, however, the transition from the ballistic to diffusive regime is smoother in presence of BB forces. The dynamics change in presence of restoring force. Analytical expressions satisfactorily capture the NVAF for times much longer than the vorticity time, Fig. 2b. In Fig. 2f, numerical inversion of Eq. (8) for the corresponding parameters show that hydrodynamic fluctuations smooth anti-persistence and impose a kind of persistence. Figs. 2d and 2h display the MSD in both cases, hydrodynamic fluctuations likely lead the MSD to plateau regime a bit later than in their absence. It is worth mentioning that Eq. (15) (cyan line in Fig. 2f), except the very first few points, does not fit well the curve of the NVAF; similar behavior has been depicted in the original work.[25]

PAF or its normalized version (NPAF) accesses particle dynamics, however, its handling is more cumbersome than VAF because of usually behaves as two point correlator. From Eqs. (B.1) and (B.6), we get

$$\langle x(t)x(t')\rangle = \langle x^2(0)\rangle - \frac{k_BT}{M}I(|t-t'|) + \left(\frac{k_BT}{M} - \omega^2\langle x^2(0)\rangle\right)(I(t)+I(t')-\omega^2 I(t)I(t')) \tag{16}$$

Eq. (16) describes the PAF of a stationary process in wide sense, only when a restoring force is present, otherwise, it does not depend only on the time lag, τ_L where $\tau_L = t' - t$ but also on the actual time t and reads $\langle x(t)x(t+\tau_L)\rangle = \langle x^2(0)\rangle + \frac{k_BT}{M}\{I(t+\tau_L)+I(t)-I(|\tau_L|)\}$. For diffusing motion within a harmonic potential and assuming thermal initial conditions, $\langle \xi(t)x(0)\rangle = 0$, $< x^2(0) >= \frac{k_BT}{m\omega^2}$, and $\langle x(0)u(0)\rangle = 0$, Eq. (16) takes the simpler form

$$\frac{C_x(\tau_L)}{C_x(0)} = 1 - \omega^2 I(\tau_L) \tag{17}$$

which behaves for short and long times as follows

$$\frac{C_x(\tau_L)}{C_x(0)} = \begin{cases} 1 - \frac{\omega^2}{2}\tau_L^2 + \frac{8\omega^2\nu_1}{15\sqrt{\pi}}\tau_L^{\frac{5}{2}}, & \tau_L \to 0 \\ \frac{\gamma_0}{k}\frac{\tau_L^{-\alpha}}{\Gamma(1-\alpha)} - \frac{\gamma_0}{k}\left(\frac{\gamma_0}{k} + \frac{1}{\nu_2}\right)\frac{\tau_L^{-2\alpha}}{\Gamma(1-2\alpha)}, & \tau_L \to \infty \end{cases} \tag{18}$$

where the expansion, $\lim_{t\to\infty} I(t) = \frac{1}{\omega^2} - \frac{(\nu_2\nu_4-\nu_5)}{\nu_5^2}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\nu_4}{\nu_5}\frac{(\nu_2\nu_4-\nu_5)}{\nu_5^2}\frac{t^{-2\alpha}}{\Gamma(1-2\alpha)}$, has been used. Eq. (18) can access both the analytic form of the friction expressed by the one parameter ML function as well the stiffness of the restoring force. And yet, Eq. (18) can also provide the autocorrelation of the restoring force (FAC), Eq. (19), since $\lim_{t\to\infty}\langle F(t)F(t+\tau_L)\rangle = \lim_{t\to\infty} k^2\langle x(t)x(t+\tau_L)\rangle$, with k being the stiffness.

$$\lim_{t\to\infty}\frac{\langle F(t)F(t+\tau_L)\rangle}{k^2\langle x^2(0)\rangle} = 1 - \frac{\nu_5}{\nu_2\nu_4}\{E_\alpha(-\frac{\nu_5}{\nu_4}\tau_L^\alpha) + \nu_2\tau_L^\alpha E_{\alpha,\alpha+1}(-\frac{\nu_5}{\nu_4}\tau_L^\alpha)\} \tag{19}$$

The asymptotic behavior of Eq. (19) is given by Eq. (18).

Fig. 3a displays the NPAF for BTG in acetone in absence (red) and in presence (black) of hydrodynamic fluctuations as it is predicted by the numerical inversion of Eq. (8). Restoring force and friction (one parameter ML function) are always present, and the value of α has been set to 0.9. Fig. 3b provides the behavior of PACF as it is predicted by Eq. (17) under the assumption of thermal initial conditions and it valid for $t > (\nu_3/\nu_4)^{1/(1.5-a)}$. It states that for time larger than the trapping time the position loses any kind of correlation and NPAF goes to zero. However, this argument is challenged by Fig. 3a, where, For $t > t_k$, NPAF becomes negative,higher the minimum in absence of BB forces, and then it returns to zero. Zooming in this regime, Fig. 3c, it becomes apparent that hydrodynamic fluctuations smooth out the oscillatory behavior of the NPAF around zero. On the other hand, for Silica in water, Eq. (17) returns quasi similar behavior in absence/presence (green/magenta) of hydrodynamic fluctuations, Fig. 3d. However, according to numerical inversion of Eq. (8), Fig. 3g, this is true in long times. There is a short time window where hydrodynamic fluctuations differentiate the behavior of NPAF with respect to their absence, Fig. 3e which is zoom in of Fig. 3g. Notice, that for silica in water, NPAF decreases monotonically, and furthermore, in the short time limit, Eq. (19), fits well on the curve (cyan).

5. Conclusions

The model presented in this article investigates the features of the sub-diffusive motion of a particle in the presence of hydrodynamic fluctuations and harmonic restoring force under the influence of friction modeled by a one-parameter ML function. The model is a generalization of other investigations and distinct from them due to one-parameter ML friction term. The use of the latter is of great importance since it ensures a smooth transition of friction from zero at $t = 0$ to a power law behavior at long times. It also fits better the friction properties of then environment where a diffusion motion evolves. We provide the solution in Laplace space. We then express the generalized response function in real space in terms of the multinomial ML function. The asymptotic limits are extracted, and, in addition to that, we obtain closed analytical solutions for time windows much broader than the usual long time limit. These solutions are given as combinations of a two-parameter ML functions, and its value is due to the fact that it can be easily used to access system dynamics through a fitting process [41].

And yet, the general model presented in this work may describe a number of different situations where some or all of the force/s exerted on the tagged particle, e.g. hydrodynamic fluctuations and/or restoring force, is/are not active. This

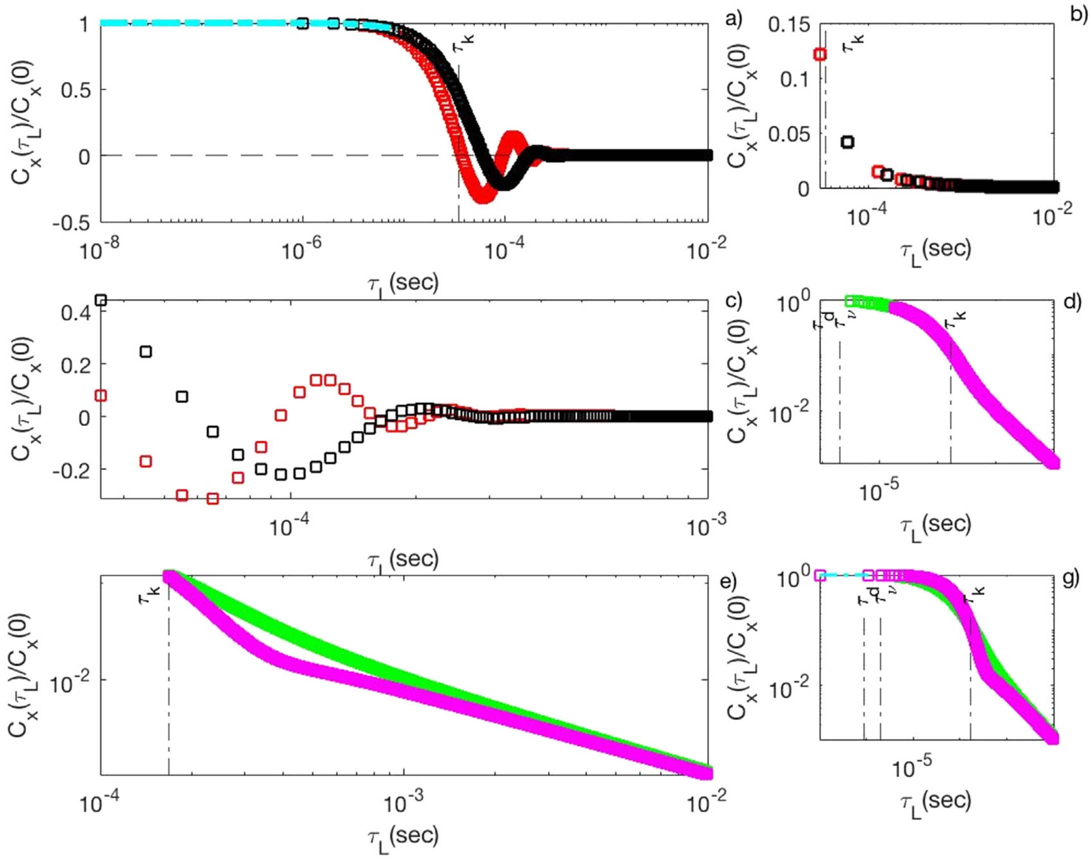

Fig. 3. It shows the NPAF of BTG in acetone and of Silica in water. Color code: red and black for BTG, green and magenta for silica. Restoring force is present in all cases displayed herein. Black and magenta for activated hydrodynamic fluctuations, while, red and green for the opposite. Figs. 2b and 2d have been drawn based on Eq. (17), where the rest are based on numerical inversion of Eq. (8).

can be done with a slight modification of the general solution in Laplace space, deactivation of the corresponding term/s, which return/s different multinomial ML functions as solutions of the generalized response function. In this context, we present solutions when both hydrodynamic fluctuations and restoring forces are not present, as well when only one of the two is active. For each case, the generalized response function is obtained and closed analytical expressions for large time windows are extracted. Furthermore, the most frequent measures in both statistical treatment and experimental processes such as MSD, NVAF, and NPAF are provided. Their analytical forms are listed in Table 1, and their short and long time behavior is listed in Table 2. All analytical expressions listed in Table 1 are combinations of two-parameter ML functions and can be used to fit experimental evidence, when available, and thus leading to uncover features of the system dynamics. Notice that the numerical simulations reported here concern only the inversion of the Laplace transform. The simulations were carried out to demonstrate the equivalence of the analytical solutions and numerical inversion for time moments $t > t_c$. Indeed, the aim of this work is to provide a number of analytical equations that can be used in support of experiments or of simulations. As such, it is not the intent of this work to further extend the use of numerical simulations.

An obvious extension of the present work is to consider the friction term as either a two or a three-parameter ML function. However, there is room, in the existing model, for searching further special features and dynamics of a diffusing particle. For example, it has been reported experimentally that hydrodynamic memory causes resonances in Brownian motion.[24] The present model enhanced with a fluctuating stiffness term can be used for analytical investigation of resonance effects, [44] and such a study may be supported by extensive numerical simulations.[45] It is worth mentioning that resonance effects due to fluctuating mass in the presence of a one-parameter ML friction and in the absence of hydrodynamic forces has been studied [46]. Another possibility is the investigation of system behavior when the scaling exponent α is very small and friction itself behaves as a restoring force. In addition, the present work can support the study of systems where the diffusing particle has a density that allows the activation of hydrodynamic effects, for example nanostructures/nanoparticles moving on surfaces under the stimulus of either electrons [47] or light [48], and for which analysis showed intriguing behavior.[49]. A safe route of testing and validating the present model is the study of experimental evidence coming from AFM experiments since they offer high resolution and usual long trajectories [50,51].

In a more practical summary of the work one can say that Eq. (1) has not been solved before in the presence of one-parameter ML friction. Under this general framework, a number of sub-cases, active and non-active forces, have been studied, and for each case analytical solutions valid for large time windows have been obtained. Through comparative analysis we provided qualitative evidence that hydrodynamic fluctuations smooth out the anti-persistent character of particle motion. The friction, always present, imposes a sub-diffusive character, which entails that NVAF becomes negative and its absolute minimum becomes larger as the sub-diffusion becomes stronger. Smoothing out anti-persistence is visually conspicuous from the time evolution of NVAF, whose value of the absolute minimum is reduced and accordingly never takes negative values. In addition to that, an active harmonic restoring force enhances the sub-diffusive character of the motion, which however, is attenuated by hydrodynamic fluctuations. NPAF confirms the way hydrodynamic fluctuations alter particle motion, and it is more pronounced when the diffusion time, τ_d, is larger than the vorticity time τ_ν. On the other hand, MSD does not greatly contribute to distinguishing the effects hydrodynamic fluctuations have on motion. Actually, it starts ballistically and scales as t^α, in the long time limit, when a restoring force does not exist, opposed to complete trapping when it is present. For intermediate times, some slight differentiation exists in its form in the presence/absence of hydrodynamic fluctuations. The same analysis can be carried out for any combination of the scaling exponent, α, and all existing time scales (diffusion, vorticity, trapping).

CRediT authorship contribution statement

Evangelos Bakalis: Conceptualization, Methodology, Software, Formal analysis, Writing original draft - review & editing. **Francesco Zerbetto:** Conceptualization, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Derivation of fluctuation dissipation relations

Fluctuation dissipation relations (FDR) are not affected by the presence of a conservative force when the latter is assumed within the linear response regime. [17,52]. For zero conservative forces, and by using the notation $M = m_p + \frac{m_f}{2}$, Eq. (1), for $t = 0$, returns $\xi(0) = M\dot{u}_i(0)$, a result that is used to obtain the mean average of the noise auto-correlation, $\Xi(t) = \langle\xi(t)\xi(0)\rangle$

$$\Xi(t) = M^2\langle\dot{u}(t)\dot{u}(0)\rangle + M\int_0^t \gamma(t-t')\langle u(t')\dot{u}(0)\rangle dt' + M\int_0^t \zeta(t-t')\langle\dot{u}(t')\dot{u}(0)\rangle dt' \tag{A.1}$$

Assuming velocity at least wide sense stationary process (mean and autocorrelation are time invariant), its autocorrelation function is defined as, $C_u(t_1, t_2) = \langle u(t_1)u(t_2)\rangle = \langle u(t)u(0)\rangle$, with $t = t_1 - t_2$. By making use of the identity, $\langle\dot{u}(t_1)u(t_2)\rangle + \langle u(t_1)\dot{u}(t_2)\rangle = 0$, Eq. (A.1) is transformed into a second order stochastic differential equation with respect to $C_u(t)$, and reads

$$-M^2\ddot{C}_u(t) - M\int_0^t \gamma(t-t')\dot{C}_u(t')dt' - M\int_0^t \zeta(t-t')\ddot{C}_u(t')dt' = \Xi(t) \tag{A.2}$$

Eq. (A.2) interrelates velocity and noise autocorrelation functions. The Laplace pair of Eq. (A.2) returns

$$\Xi(s) = M(sC_u(s) - C_u(0))(-Ms - \gamma(s) - s\zeta(s)) \tag{A.3}$$

For $t \to \infty$, or equivalently in Laplace space for $s \to 0$, the term $sC_u(s)$ should go to zero, as we demand equilibrium conditions. It remains, $\Xi(s) = MC_u(0)(\gamma(s) + s\zeta(s))$, and assuming equilibrium conditions, $Mu^2 = k_BT$ at $t = 0$, we end up with

$$\Xi(s) = k_BT\{\gamma(s) + s\zeta(s)\} \tag{A.4}$$

Eq. (A.4) is the fluctuation dissipation relation (FDR), also known as fluctuation dissipation theorem (FDT), for the system described by Eq. (1) [18]. Of note, in Eq. (A.4) the term originated from the BB force reminds in Laplace space a quantity derived as the time derivative of the BB force, $L\{\dot{f}(t)\}(s) = sf(s) - f(0)$, however the latter at least suffers at $t \to 0$ since it diverges. [53] One can define a generalized friction term, $\gamma_G(s) = \gamma(s) + s\zeta(s)$, and Eq. (A.4) takes the familiar form of FDT's $\langle\xi(s)\rangle = k_B T\gamma_G(s)$. Because of stationary conditions, only the time difference is important, we define the noise autocorrelation for two distinct time moments t_1, and t_2 as $\langle\xi(t_1)\xi(t_2)\rangle = \Xi(|t_1-t_2|)$. By taking a double Laplace transform, s for t_1, and s' for t_2, of $\Xi(|t_1-t_2|)$, [6] we write

$$\langle\xi(s)\xi(s')\rangle = k_B T\frac{\gamma_G(s)+\gamma_G(s')}{s+s'} \tag{A.5}$$

Eq. (A.5) is the general form of the FDT and it is in line with previous results extracted by more sophisticated techniques. [21]

Appendix B. Observables and response functions

The mean square displacement, $\langle(x(t)-x(t'))^2\rangle = \langle x(t)^2\rangle - 2\langle x(t)x(t')\rangle + \langle x(t')^2\rangle$, and $C_{u,u}(t,t') = \langle u(t)u(t')\rangle$ the velocity autocorrelation function, are quantities directly evaluated in an experiment. Towards assessing the dynamics of a tracer particle the following response functions are also of interest; position covariance $\sigma_{x,x}(t,t') = (x(t)-\langle x\rangle)(x(t')-\langle x\rangle)$, velocity covariance $\sigma_{u,u}(t,t') = (u(t)-\langle u\rangle)(u(t')-\langle u\rangle)$, and $\sigma_{x,u}(t,t') = (x(t)-\langle x\rangle)(u(t')-\langle u\rangle)$ the position velocity covariance. Common in all mentioned measures is either the term $\langle x(t)x(t')\rangle$ or the term $\langle u(t)u(t')\rangle$, which can be easily constructed in Laplace space by using Eqs. (3) and (4)

$$\langle x(s)x(s')\rangle = A(s)A(s') + \frac{1}{M^2}\langle\xi(s)\xi(s')\rangle G(s)G(s') \tag{B.1}$$

and

$$\langle u(s)u(s')\rangle = B(s)B(s') + \frac{1}{M^2}\langle\xi(s)\xi(s')\rangle g(s)g(s') \tag{B.2}$$

where, $A(s) = \frac{x(0)}{s} + (1+M^{-1}\zeta(s))u(0)G(s) - \omega^2 x(0)I(s)$, $B(s) = (1+M^{-1}\zeta(s))u(0)g(s) - \omega^2 x(0)I(s)$, $G(s)$ is given by Eq. (5), and $I(s) = G(s)/s$, $g(s) = sG(s)$. The cross terms are zero because of $\langle\xi(s)\rangle = 0$. Furthermore, in Laplace space the covariance functions read

$$\sigma_{x,x}(s,s') = \frac{\langle\xi(s)\xi(s')\rangle G(s)G(s')}{M^2} \tag{B.3}$$

$$\sigma_{u,u}(s,s') = \frac{\langle\xi(s)\xi(s')\rangle g(s)g(s')}{M^2} \tag{B.4}$$

$$\sigma_{x,u}(s,s') = \frac{\langle\xi(s)\xi(s')\rangle G(s)g(s')}{M^2} \tag{B.5}$$

By using Eq. (A.5), the second term at the r.h.s of Eq. (B.2) , after some algebra [10], is rearranged as

$$\begin{aligned}\frac{\langle\xi(s)\xi(s')\rangle G(s)G(s')}{M^2} &= \frac{k_B T}{M}\{\frac{I(s')}{s} + \frac{I(s)}{s'} - \frac{I(s)+I(s')}{s+s'} \\ &\quad -G(s)G(s') - \omega^2 I(s)I(s')\}\end{aligned} \tag{B.6}$$

and the second term at the r.h.s of Eq. (B.2) as

$$\frac{1}{M^2}\langle\xi(s)\xi(s')\rangle g(s)g(s') = \frac{k_B T}{M}\{\frac{g(s)+g(s')}{s+s'} - g(s)g(s') - \omega^2 G(s)G(s')\} \tag{B.7}$$

Position and velocity covariance, Eqs. (B.3) and Eq. (B.4), are expressed through Eq. (B.7) and Eq. (B.8), whose inverse Laplace transform deliver

$$\sigma_{x,x}(t,t') = \frac{k_B T}{M}(I(t) + I(t') - I(|t-t'|) - G(t)G(t') - \omega^2 I(t)I(t')) \tag{B.8}$$

and

$$\sigma_{u,u}(t,t') = \frac{k_B T}{M}(g(|t-t'|) - g(t)g(t') - \omega^2 G(t)G(t')) \tag{B.9}$$

Eqs. (B.4) and (B.5) can be used to define variances, when we set $t = t'$, [5,9,54]

$$\sigma_x(t) = \frac{k_B T}{M}(2I(t) - I(0) - G^2(t) - \omega^2 I^2(t)) \tag{B.10}$$

$$\sigma_u(t) = \frac{k_B T}{M}(g(0) - g^2(t) - \omega^2 G^2(t)) \tag{B.11}$$

and

$$\sigma_{x,u}(t) = \frac{1}{2}\frac{d\sigma_{x,x}(t)}{dt} = \frac{2k_B T}{M}G(t)(1 - g(t) - \omega^2 I(t)) \tag{B.12}$$

Eqs. (B.8)–(B.10) can provide probability density function, at least in the long time limit where the system reaches equilibrium and obeys a quasi Gaussian distribution, $P(y, y_0, t) = \frac{1}{2\sqrt{\pi}\sigma_y(t)} e^{-\frac{1}{2}\frac{(y-y_0)^2}{\sigma_y^2(t)}}$, $y = x, u$.

By using Eq. (B.1) and Eq. (B.2) and after some algebra the mean square displacement reads

$$\begin{aligned}\langle (x(t)-x(t'))^2\rangle = &\frac{2k_BT}{M}I(|t-t'|)+(u^2(0)-\frac{k_BT}{M})(G(t)-G(t'))^2\\ &+\omega^2(\omega^2x_0^2-\frac{k_BT}{M})(I(t)-I(t'))^2+\frac{u^2(0)}{m^2}(\phi(t)-\phi(t'))^2\\ &+\frac{2u^2(0)}{M}(G(t)-G(t'))(\phi(t)-\phi(t'))\\ &-\frac{2\omega^2x(0)u(0)}{M}(I(t)-I(t'))(\phi(t)-\phi(t'))\end{aligned} \tag{B.13}$$

and the velocity autocorrelation function takes the form

$$\begin{aligned}&\langle u(t)u(t')\rangle = \frac{k_BT}{M}g(|t-t'|)+(u(0)^2-\frac{k_BT}{M})g(t)g(t')\\ &\omega^2(\omega^2x^2(0)-\frac{k_BT}{M})G(t)G(t')+\frac{u(0)^2}{M^2}\lambda(t)\lambda(t')-\\ &\omega^2x(0)u(0)(G(t)g(t')+G(t')g(t))+\frac{u(0)^2}{M}(g(t)\lambda(t')+g(t')\lambda(t))\\ &\frac{\omega^2x(0)u(0)}{M}(G(t)\lambda(t')+G(t')\lambda(t))\end{aligned} \tag{B.14}$$

where, $\phi(t) = \int_0^t \zeta(t-y)G(y)dy$ and $\lambda(t) = \int_0^t \zeta(t-y)g(y)dy$, notice also that $\phi(t) = \int_0^t \lambda(t-x)dx$.

Finally there is a direct relation between the diffusion coefficient, D, and the velocity autocorrelation function. We start from the mean square displacement, $< \Delta x^2 > (t) = \langle (x(t)-x_0)^2\rangle = \int_0^t\int_0^t \langle u(t')u(t'')\rangle dt'dt''$, and assuming that $u(t)$ is stationary at least in wide-sense, we write in Laplace domain

$$\langle \Delta x^2\rangle(s) = \frac{2}{s^2}C_u(s) \tag{B.15}$$

and for the diffusion coefficient

$$D(s) = \frac{C_u(s)}{s} \tag{B.16}$$

Appendix C. Mittag-Leffler functions

At the beginning of the 20th century the Swedish mathematician Magnus Gustaf Mittag-Leffler introduced a function related with summation of divergent series,[55,56] which got his name (ML) and reads

$$E_a(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(an+1)} \tag{C.1}$$

where $a > 0$. A two parameter ML function is defined as follows

$$E_{a,b}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(an+b)} \tag{C.2}$$

where $a > 0$ and $b \in C$. For $b = 1$, Eq. (C.2) returns Eq. (C.1). The three parameter ML function, which is also known as Prabhakar function, [57] reads

$$E_{a,b}^n(z) = \frac{1}{\Gamma(n)}\sum_{k=0}^{\infty}\frac{\Gamma(k+n)}{\Gamma(ak+b)}\frac{(z)^k}{k!} \tag{C.3}$$

where $Re(a) > 0$, $Re(b) > 0$, and $n > 0$. For $n = 1$, Eq. (C.3) becomes the two parameter ML function. Several other functions of the same type has been proposed, see for example [58]. The importance of the ML functions is that they play the role of the exponential functions in the integer order calculus. For $|z| > 1$, Eq. (C.3) reads [39,41,59]

$$E_{a,b}^n(z) = \frac{(-z)^{-n}}{\Gamma(n)}\sum_{k=0}^{\infty}\frac{\Gamma(k+n)}{\Gamma(b-a(k+n))}\frac{z^{-k}}{k!} \tag{C.4}$$

The third parameter in Eq. (C.4) can be reduced using either the formula [57]

$$anE^{n+1}_{a,b}(z) = E^{n}_{a,b-1}(z) + (1 - b + an)E^{n}_{a,b}(z) \tag{C.5}$$

or the formula [60]

$$anzE^{n+1}_{a,b}(z) = E^{n}_{a,b-a-1}(z) + (1 - b + a)E^{n}_{a,b-a}(z) \tag{C.6}$$

The Laplace pair of the three parameter ML function is given by [61,62]

$$L\{t^{b-1}E^{n}_{a,b}(\pm\lambda t^{a})\}(s) = \frac{s^{an-b}}{(s^{a} \mp \lambda)^{n}} \tag{C.7}$$

In many problems, the response function is of the form $G(s) = s^{-b}(1-\sum_{i=1}^{k}(-\lambda_i)s^{a_i})^{-1}$, where the exponents α_i satisfy the condition $\alpha_1 > \alpha_2 > \cdots . > \alpha_k$. The inverse Laplace of $G(s)$ is expressed through the multinomial Mittag-Leffler function, which has been introduced as the fundamental solution of the ordinary differential equation of fractional discrete distributed order, [37] and reads

$$G(t) = t^{b-1}E_{(a_1,a_2,a_3,\ldots,a_n),b}(z_1, z_2, \ldots, z_n) \tag{C.8}$$

where $z_i = -\lambda_i t^{\alpha_i}$ and

$$E_{(a_1,a_2,a_3,\ldots,a_n),b}(z_1, z_2, \ldots, z_n) = \sum_{k=0}^{\infty}\sum^{m_1+m_2+..+m_n=k} \frac{k!}{m_1!m_2!\ldots m_n!}\frac{\prod_{i=1}^{n} z_i^{m_i}}{\Gamma(b+\sum_{i=1}^{n} a_i m_i)} \tag{C.9}$$

Simplification of Eq. (C.9): Let $E_{(a_1,a_2),b}(-\lambda_1 t^{a_1}, -\lambda_2 t^{a_2})$ with $\lambda_1 > \lambda_2$ is the function of interest. According to Eq. (C.9) we write

$$E_{(a_1,a_2),b}(-\lambda_1 t^{a_1}, -\lambda_2 t^{a_2}) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{(-\lambda_1 t^{a_1})^{n-k}(-\lambda_2 t^{a_2})^{k}}{\Gamma(b+(n-k)a_1+ka_2)} \tag{C.10}$$

By setting $n = n + k$, Eq. (C.9) reads

$$E_{(a_1,a_2),b}(-\lambda_1 t^{a_1}, -\lambda_2 t^{a_2}) = \sum_{n=0}^{\infty}(-\lambda_2 t^{a_2})^{n}\sum_{k=0}^{\infty} \frac{(n+k)!}{k!n!}\frac{(-\lambda_1 t^{a_1})^{k}}{\Gamma(ka_1+b+na_2)} \tag{C.11}$$

Given that $E^{n+1}_{a,b}(-zt^{a}) = \sum_{k=0}^{\infty}\frac{(n+k)!}{n!k!}\frac{(-zt^{a})^{k}}{\Gamma(ak+b)}$, Eq. (C.10) reads

$$E_{(a_1,a_2),b}(-\lambda_1 t^{a_1}, -\lambda_2 t^{a_2}) = \sum_{n=0}^{\infty}(-\lambda_1 t^{a_1})^{n}E^{n+1}_{a_2,b+n\alpha_1}(-\lambda_2 t^{a_2}) \tag{C.12}$$

For $\lambda_2 t^{a_2} > 1$, we use Eq. (C.4) and we write $E^{n+1}_{(a_2,b+n\alpha_1)}(-\lambda_2 t^{a_2}) \sim \frac{t^{-(n+1)a_2}}{\lambda_2^{n+1}\Gamma(b-a_2+n(a_1-a_2))}$, substituting the latter into Eq. (C.11) we end up with

$$\begin{aligned} E_{(a_1,a_2),b}(-\lambda_1 t^{a_1}, -\lambda_2 t^{a_2}) &= \frac{t^{-a_2}}{\lambda_2}\sum_{n=0}^{\infty}\frac{(-\frac{\lambda_2}{\lambda_1}t^{(a_1-a_2)})^{n}}{\Gamma(n(a_1-a_2)+b-a_2)} \\ &= \frac{t^{-a_2}}{\lambda_2}E_{a_1-a_2,b-a_2}(-\frac{\lambda_1}{\lambda_2}t^{(a_1-a_2)}),\ \lambda_2 t^{a_2} > 1 \end{aligned} \tag{C.13}$$

Eq. (C.13) can be further expanded in the long time limit, $\frac{\lambda_1}{\lambda_2}t^{a_1-a_2} \gg 1$, as it holds true that $E_{a_1-a_2,b-a_2}(-\frac{\lambda_1}{\lambda_2}t^{(a_1-a_2)}) \sim \frac{\lambda_2}{\lambda_1}\frac{t^{-a_1+a_2}}{\Gamma(b-a_1)}$ in this limit, see also Eq. (C.4) for $n = 1$. Substitution of it into Eq. (C.13) yields

$$E_{(a_1,a_2),b}(-\lambda_1 t^{a_1}, -\lambda_2 t^{a_2}) = \frac{t^{-a_1}}{\lambda_1\Gamma(b-a_1)},\ \frac{\lambda_1}{\lambda_2}t^{a_1-a_2} \gg 1 \tag{C.14}$$

We proceed with the analytical treatment of a multinomial Mittag-Leffler function with five arguments. In the same way analysis can be conducted for less or more arguments. We introduce the shorthand function $MMF_5 = E_{(a_1,a_2,a_3,a_4,a_5),b}(-\lambda_1 t^{a_1}, -\lambda_2 t^{a_2}, -\lambda_3 t^{a_3}, -\lambda_4 t^{a_4}, -\lambda_5 t^{a_5})$, with $\alpha_1 > \alpha_2 > \cdots . > \alpha_5$, and according to Eq. (C.9) reads

$$\begin{aligned} MMF_5 = &\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{m=0}^{n-k-l}\sum_{j=0}^{n-k-l-m}\frac{n!}{k!l!m!j!(n-k-l-m-j)!} \\ &\frac{(-\lambda_1 t^{a_1})^{n-k-l-m-j}(-\lambda_2 t^{a_2})^{k}(-\lambda_3 t^{a_3})^{l}(-\lambda_4 t^{a_4})^{m}(-\lambda_5 t^{a_5})^{j}}{\Gamma(b+(n-k-l-m-j)a_1+ka_2+la_3+ma_4+ja_5)} \end{aligned} \tag{C.15}$$

Step 1: Change of variables, $n = n + j$, so the variable j goes now from 0 to ∞. Furthermore, the factorial term reads $\frac{n!}{k!l!m!(n-k-l-m)!}\frac{(n+j)!}{n!j!}$. Eq. (C.15) takes the form

$$MMF_5 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{m=0}^{n-k-l} f(n,k,l,m)E^{n+1}_{\alpha_5,\beta}(-\lambda_5 t^{\alpha_5}) \tag{C.16}$$

$f(n,k,l,m) = \frac{n!}{k!l!m!(n-k-l-m)!}(-\lambda_1 t^{a_1})^k(-\lambda_2 t^{a_2})^l(-\lambda_3 t^{a_3})^m(-\lambda_5 t^{a_5})^{n-k-l-m}$, and $\beta = b+(n-k-l-m)a_1+ka_2+la_3+ma_4$.

Eq. (C.16) is valid for all time moments. For $\lambda_5 t^{\alpha_5} > 1$, the function $E^{n+1}_{a_5,\beta}(-\lambda_5 t^{a_5})$ can be replaced by $\frac{t^{-a_5(n+1)}}{\lambda_5^{n+1}\Gamma(\beta-(n+1)\alpha_5)}$, Eq. (C.4), and substitution of it into Eq. (C.16), delivers

$$MMF_5 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\sum_{m=0}^{n-k-l} f(n,k,l,m)\frac{t^{-a_5(n+1)}}{\lambda_5^{n+1}\Gamma(\beta-(n+1)\alpha_5)},\ \lambda_5 t^{\alpha_5} > 1 \tag{C.17}$$

Step 2: We change the variable n to $n = n + m$, so m now goes from 0 to ∞. By rearranging the terms in Eq. (C.17) and by using the definition of three parameter Mittag-Leffler function we write

$$MMF_5 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{t^{-a_5(n+1)}}{\lambda_5^{n+1}}f(n,k,l)E^{n+1}_{a_5-a_4,\beta_1}(-\frac{\lambda_4}{\lambda_5}t^{a_4-a_5}),\ \lambda_5 t^{\alpha_5} > 1 \tag{C.18}$$

where $\beta_1 = b+(n-k-l)a_1+ka_2+la_3-a_5(n+1)$ and $f(n,k,l) = \frac{n!}{k!l!(n-k-l)!}(-\lambda_1 t^{a_1})^{n-k-l}(-\lambda_2 t^{a_2})^k(-\lambda_3 t^{a_5})^l$. For $\frac{\lambda_4}{\lambda_5}t^{a_4-a_5} > 1$, we write the expansion of the $E^{n+1}_{a_4-a_5,\beta_1}(-\frac{\lambda_4}{\lambda_5}t^{a_4-a_5})$ which goes as $\frac{t^{-(n+1)(a_4-a_5)}}{(\frac{\lambda_4}{\lambda_5})^{n+1}\Gamma(\beta_1-(n+1)(\alpha_4-\alpha_5))}$. Substituting the latter into Eq. (C.18) we end up with

$$MMF_5 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{f(n,k,l)t^{-a_4(n+1)}}{\lambda_4^{n+1}\Gamma(\beta_1-(n+1)(\alpha_4-\alpha_5))},\ \frac{\lambda_4}{\lambda_5}t^{a_4-a_5} > 1 \tag{C.19}$$

Step 3: We change the variable n to $n = n + l$ so l now goes from 0 to ∞. By rearranging the terms in Eq. (C.19) we write

$$MMF_5 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{t^{-a_4(n+1)}}{\lambda_4^{n+1}}f(n,k)E^{n+1}_{a_3-a_4,\beta_2}(-\frac{\lambda_3}{\lambda_4}t^{a_3-a_4}) \tag{C.20}$$

where $f(n,k) = \frac{n!}{k!(n-k)!}(-\lambda_1 t^{a_1})^{n-k}(-\lambda_5 t^{a_5})^k$, and $\beta_2 = b+(n-k)a_1+ka_2-a_4(n+1)$. Expansion of $E^{n+1}_{a_3-a_4,\beta_2}(-\frac{\lambda_3}{\lambda_4}t^{a_3-a_4})$ returns $\frac{t^{-(n+1)(a_3-a_4)}}{(\frac{\lambda_3}{\lambda_4}\Gamma(\beta_2-(n+1)(a_3-a_4)))}$, valid for $\frac{\lambda_3}{\lambda_4}t^{a_3-a_4} > 1$. Substituting these into Eq. (C.20) we write

$$MMF_5 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{f(n,k)t^{-a_3(n+1)}}{\lambda_3^{n+1}\Gamma(\beta_2-(n+1)(a_3-a_4))},\ \frac{\lambda_3}{\lambda_4}t^{a_3-a_4} > 1 \tag{C.21}$$

Step 4: Change the variable n to $n = n + k$, and k goes from 0 to ∞. We following the same steps as above and after some trivial calculus we end up with

$$MMF_5 = \frac{1}{\lambda_3 t^{\alpha_3}}\sum_{n=0}^{\infty}(-\frac{\lambda_1}{\lambda_3}t^{\alpha_1-\alpha_3})^n E^{n+1}_{\alpha_2-\alpha_3,\beta_3}(-\frac{\lambda_2}{\lambda_3}t^{\alpha_2-\alpha_3}),\ \frac{\lambda_3}{\lambda_4}t^{a_3-a_4} > 1 \tag{C.22}$$

where $\beta_3 = b + n\alpha_1 - \alpha_3(n+1)$.

Step 5: For $\frac{\lambda_2}{\lambda_3}t^{a_2-a_1} > 1$, expansion of the three parameter Mittag-Leffler function in Eq. (C.22) returns $\frac{t^{-(n+1)(a_2-a_3)}}{(\frac{\lambda_2}{\lambda_3}\Gamma(\beta_3-(n+1)(a_2-a_3)))}$, valid for $\frac{\lambda_2}{\lambda_3}t^{a_2-a_3} > 1$. Substitute the latter into Eq. (C.22) we end up with

$$MMF_5 = \frac{1}{\lambda_2 t^{\alpha_2}}E_{\alpha_1-\alpha_2,b-\alpha_2}(-\frac{\lambda_1}{\lambda_2}t^{\alpha_1-\alpha_2}),\ \frac{\lambda_2}{\lambda_3}t^{a_2-a_3} > 1 \tag{C.23}$$

Appendix D. Possible combinations of forces according to eq. (1) exerted on the diffusion particle.

The following Table D.1 summarizes the explicit solutions, expressed through the multinomial ML function, for all the cases examined in this work.

A: Free particle subject to friction force given by the one parameter ML function, $\zeta(t) = 0$ and $\omega = 0$. We set $\nu_1 = \nu_3 = \nu_5 = \nu_{4,2} = 0$, $\nu_2 = \mu^{-\alpha}$, and $\nu_{4,1} = \frac{\bar{\gamma}}{m}$ in Eqs. (9) and (10). The inverse Laplace pair returns the multinomial ML function and the generalized response function reads

$$R(t) = t^{1-\delta}\{E_{(2,\alpha),2-\delta}(-\nu_{4,1}t^2, -\nu_2 t^\alpha) + \nu_2 t^\alpha E_{(2,\alpha),2+\alpha-\delta}(-\nu_{4,1}t^2, -\nu_2 t^\alpha)\} \tag{D.1}$$

Table D.1
The response function, $R(t) = R_1(t) + R_2(t)$ of a diffusing particle subject to $\gamma(t) = \bar{\gamma} E_a(-(t/\mu)^a)$, $0 < a < 1$ friction term, and to (i) $\zeta(t) = 0$, and $\omega = 0$, (ii) $\zeta(t) = 0$, and $\omega \neq 0$, (iii) $\zeta(t) \neq 0$, and $\omega = 0$, and (iv) $\zeta(t) \neq 0$, and $\omega \neq 0$.

	$R(t)$
$\zeta(t) = 0$ $\omega = 0$	$\frac{t^{1-\delta}}{\Gamma(2-\delta)} + t^{1-\delta} \sum_{n=1}^{\infty} (-\nu_{4,1} t^2)^n E^n_{a,\, 2n+2-\delta}(-\nu_2 t^a)$
$\zeta(t) = 0$ $\omega \neq 0$	$t^{1-\delta}\{E_{(2+\alpha,\,2,\,a),\,2-\delta}(-\nu_5 t^{2+\alpha}, -\nu_4 t^2, -\nu_2 t^\alpha) +$ $\nu_2 t^\alpha E_{(2+\alpha,\,2,\,a),2+\alpha-\delta}(-\nu_5 t^{2+\alpha}, -\nu_4 t^2, -\nu_2 t^\alpha)\}$
$\zeta(t) \neq 0$, $\omega = 0$	$t^{1-\delta}\{E_{(2,\,\frac{1}{2}+a,\,a,\,\frac{1}{2}),\,2-\delta}(-\nu_{4,1} t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^\alpha, -\nu_1 t^{\frac{1}{2}}) +$ $\nu_2 t^\alpha E_{(2,\,\frac{1}{2}+a,\,a,\,\frac{1}{2}),\,2+\alpha-\delta}(-\nu_{4,1} t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^\alpha, -\nu_1 t^{\frac{1}{2}})\}$
$\zeta(t) \neq 0$, $\omega \neq 0$	$t^{1-\delta}\{E_{(2+a,\,2,\,\frac{1}{2}+a,\,a,\,\frac{1}{2}),\,2-\delta}(-\nu_5 t^{2+a}, -\nu_4 t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^a, -\nu_1 t^{\frac{1}{2}}) +$ $t^a E_{(2+a,2,\,\frac{1}{2}+a,\,a,\,\frac{1}{2}),\,2+a-\delta}(-\nu_5 t^{2+a}, -\nu_4 t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^a, -\nu_1 t^{\frac{1}{2}})\}$

We simplify Eq. (D.1) by using arguments as they are described by Eq. (C.16), and we write

$$R(t) = t^{1-\delta}\{\sum_{n=0}^{\infty}(-\nu_{4,1}t^2)^n\{E^{n+1}_{\alpha,2+2n-\delta}(-\nu_2 t^\alpha) + \nu_2 t^\alpha E^{n+1}_{\alpha,2+2n+\alpha-\delta}(-\nu_2 t^\alpha)\}\} \tag{D.2}$$

Eq. (D.2) can be further simplified by using the reduction formulas in the third parameter of the ML function, see Eq. (C.5) and Eq. (C.6), and reads[3]

$$R(t) = \frac{t^{1-\delta}}{\Gamma(2-\delta)} + t^{1-\delta}\sum_{n=1}^{\infty}(-\nu_{4,1}t^2)^n E^n_{a,2n+2-\delta}(-\nu_2 t^a) \tag{D.3}$$

For $t \to 0$, Eq. (D.1) returns, $R(t) = \frac{t^{1-\delta}}{\Gamma(2-\delta)}$, and thus $g(t) \to 1$ ($\delta = 1$), and $G(t), I(t) = 0$ for $\delta = 0, -1$ respectively. Furthermore, by changing $n- > n+1$ in Eq. (D.3), the sum at the r.h.s changes to $\sum_{n=0}^{\infty}(-\nu_{4,1}t^2)^{n+1}E^{n+1}_{a,2n+4-\delta}(-\nu_2 t^\alpha)$. In addition, for $\nu_2 t^\alpha > 1$, the term $E^{n+1}_{a,2n+4-\delta}(-\nu_2 t^\alpha)$ is replaced by its asymptotic expansion and the generalized response function reads

$$R(t) = t^{1-\delta}\{\frac{1}{\Gamma(2-\delta)} - \frac{\nu_{4,1}}{\nu_2}t^{2-\alpha}E_{2-\alpha,4-\delta-\alpha}(-\frac{\nu_{4,1}}{\nu_2}t^{2-\alpha})\},\ \nu_2 t^\alpha > 1 \tag{D.4}$$

Eq. (D.4) can easily be used to fit experimental data either of normalized velocity autocorrelation function or of the MSD or of both, see for example [60]. The long time behavior of Eq. (D.4) holds true for $t > (\frac{\nu_2}{\nu_{4,1}})^{\frac{1}{2-\alpha}}$, and the response function reads

$$\lim_{t\to\infty} R(t) = \frac{\nu_2}{\nu_{4,1}}\frac{t^{\alpha-1-\delta}}{\Gamma(\alpha-\delta)} \tag{D.5}$$

B: Diffusing particle subject to one parameter ML type friction force, and to restoring force. In Eqs. (9) and (10) we set $\nu_1 = \nu_3 = 0$, $\nu_2 = \mu^{-\alpha}$, $\nu_4 = (\frac{\gamma}{m} + \omega^2)$, and $\nu_5 = \mu^{-\alpha}\omega^2$. Taking the inverse Laplace pair we end up with the overall response function $R(t) = R_1(t) + R_2(t)$, where

$$R_1(t) = t^{1-\delta}E_{(a+2,2,a),2-\delta}(-\nu_5 t^{2+a}, -\nu_4 t^2, -\nu_2 t^a) \tag{D.6}$$

and

$$R_2(t) = \nu_2 t^{1+a-\delta}E_{(a+2,2,a,),2+a-\delta}(-\nu_5 t^{2+a}, -\nu_4 t^2, -\nu_2 t^a) \tag{D.7}$$

From the definition of the multinomial ML function and by using Eq. (C.14) we write Eq. (D.6) and Eq. (D.7) as follows,

$$R_1(t) = t^{1-\delta}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-\nu_5 t^{2+\alpha})^{n-k}(-\nu_4 t^2)^k E^{n+1}_{a,b}(-\nu_2 t^a) \tag{D.8}$$

and

$$R_2(t) = \nu_2 t^{1+a-\delta}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-\nu_5 t^{2+\alpha})^{n-k}(-\nu_4 t^\alpha)^k E^{n+1}_{a,b+\alpha}(-\nu_2 t^a) \tag{D.9}$$

[3] Eq. (D.3) can be found from the direct inversion of Eq. (8), $R(s) = \frac{s^{\delta-2}}{1+\nu_{4,1}\frac{s^{\alpha-2}}{s^\alpha+\nu_2}}$ when $\tau_\nu = 0$ and $\omega = 0$

where $b = 2-\delta+(2+\alpha)(n-k)+2k$. Taking the sum of Eqs. (D.8) and (D.9)and by using Eqs. (C.5) and (C.6) we end up with the generalized response function $R(t)$

$$R(t) = t^{1-\delta}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(-\nu_5 t^{2+\alpha})^{n-k}(-\nu_4 t^2)^k E^n_{a,b}(-\nu_2 t^a) \tag{D.10}$$

For $\nu_2 t^\alpha > 1$, Eq. (D.10) can be further simplified. We make use of the expansion $\frac{(\nu_2 t^\alpha)^{-n}}{\Gamma((2-\alpha)n+2-\delta-ak)}$ and we end up with

$$R(t) = t^{1-\delta}\sum_{n=0}^{\infty}(-\frac{\nu_5}{\nu_2}t^2)^n E^{n+1}_{2-\alpha,2-\delta+2n}(-\frac{\nu_4}{\nu_2}t^{2-\alpha}),\ \nu_2 t^a > 1 \tag{D.11}$$

For $\frac{\nu_4}{\nu_2}t^{2-\alpha} > 1$, is further simplified and reads

$$R(t) = \frac{\nu_2}{\nu_4}t^{-1-\delta+\alpha}E_{\alpha,\alpha-\delta}(-\frac{\nu_5}{\nu_4}t^a),\ \frac{\nu_4}{\nu_2}t^{2-\alpha} > 1 \tag{D.12}$$

The asymptotic limit, as $t \to \infty$, of eq.(D.12) reads

$$R(t) = \frac{\nu_2}{\nu_5}\frac{t^{-1-\delta}}{\Gamma(-\delta)},\ \frac{\nu_5}{\nu_2}t^a > 1 \tag{D.13}$$

C: Diffusing particle subject to one parameter ML type friction force, and to hydrodynamic fluctuations. In Eqs. (9) and (10), we set $\nu_1 = \frac{\gamma_0}{M}\sqrt{\tau_f}$, $\nu_2 = \mu^{-\alpha}$, $\nu_3 = \nu_1\nu_2$, $\nu_{4,1} = \frac{\gamma}{M}$, and $\nu_5 = 0$. The inversion of $R_{1,2}(s)$ in time domain, Laplace pair, is made through the multinomial ML function. [38] The generalized response function $R(t) = R_1(t) + R_2$ takes the form

$$\begin{aligned}R(t) =\ & t^{1-\delta}\{E_{(2,\frac{1}{2}+a,a,\frac{1}{2}),2-\delta}(-\nu_{4,1}t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^\alpha, -\nu_1 t^{\frac{1}{2}})\\ &+\nu_2 t^\alpha E_{(2,\frac{1}{2}+a,a,\frac{1}{2}),2+\alpha-\delta}(-\nu_{4,1}t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^\alpha, -\nu_1 t^{\frac{1}{2}})\}\end{aligned} \tag{D.14}$$

By following the procedure analyzed in Appendix C, the kernel $E_{(2,\frac{1}{2}+a,a,\frac{1}{2}),b}(-\nu_{4,1}t^2, -\nu_3 t^{\frac{1}{2}+a}, -\nu_2 t^\alpha, -\nu_1 t^{\frac{1}{2}})$ of Eq. (D.14) is consecutive transformed to $\frac{1}{\nu_1 t^{\frac{1}{2}}}\sum_{n=0}^{\infty}\sum_{k=0}\frac{n!}{(n-k)!k!}\frac{(-\nu_{4,1}t^2)^{n-k}(-\nu_3 t^{\frac{1}{2}+\alpha})}{\nu_2 t^\alpha}E^{n+1}_{\alpha-\frac{1}{2},b_1}(-\frac{\nu_2}{\nu_1}t^{-\frac{1}{2}+\alpha})$ for $\nu_1 t^{\frac{1}{2}} > 1$, with $b_1 = b+2(n-k)+k(\alpha+\frac{1}{2})-\frac{n+1}{2}$, then for $\frac{\nu_2}{\nu_1}t^{\alpha-\frac{1}{2}} > 1$ it goes like $\frac{1}{\nu_2 t^\alpha}\sum_{n=0}^{\infty}(-\frac{\nu_{4,1}}{\nu_2}t^{2-\alpha})^n E^{n+1}_{\frac{1}{2},b+(2-\alpha)n-a}(-\frac{\nu_3}{\nu_2}t^{\frac{1}{2}})$, and finally for $\frac{\nu_3}{\nu_2}t^{\frac{1}{2}} > 1$ goes as $\frac{1}{\nu_3 t^{\alpha+\frac{1}{2}}}E_{\frac{3}{2},b-\alpha-\frac{1}{2}}(-\frac{\nu_{4,1}}{\nu_3}t^{\frac{3}{2}-\alpha})$. By substituting the latter expression into Eq. (D.14) we end up with

$$R(t) = \frac{t^{\frac{1}{2}-\alpha-\delta}}{\nu_3}E_{\frac{3}{2}-\alpha,\frac{3}{2}-\delta-\alpha}(-\frac{\nu_{4,1}}{\nu_3}t^{\frac{3}{2}-\alpha}) + \frac{\nu_2}{\nu_3}t^{\frac{1}{2}-\delta}E_{\frac{3}{2}-\alpha,\frac{3}{2}-\delta}(-\frac{\nu_{4,1}}{\nu_3}t^{\frac{3}{2}-\alpha}) \tag{D.15}$$

For $t \to \infty$, or for $t > (\frac{\nu_{4,1}}{\nu_3})^{\frac{1}{\frac{3}{2}-\alpha}}$, Eq. (D.16) provides the long time behavior of the response function

$$R(t) = \frac{\nu_2}{\nu_{4,1}}\frac{t^{-1-\delta+\alpha}}{\Gamma(\alpha-\delta)} + \frac{1}{\nu_{4,1}}\frac{t^{-1-\delta}}{\Gamma(-\delta)} - \frac{\nu_3}{\nu_{4,1}^2}\frac{t^{\alpha-\delta-\frac{5}{2}}}{\Gamma(\alpha-\delta-\frac{3}{2})} \tag{D.16}$$

References

[1] F. Hofling, T. Franosch, Anomalous transport in the crowded world of biological cells, Rep. Progr. Phys. 76 (2013) 046602.
[2] M.A.F. dos Santos, Analytic approaches of the anomalous diffusion: A review, Chaos Solitons Fractals 124 (2019) 86–96.
[3] I. Eliazar, M. Shlesinger, Fractional motions, Phys. Rep. 527 (2013) 101–129.
[4] F.A. Oliveira, R.M.S. Ferreira, L. Lapas, M.H. Vainstein, Anomalous diffusion: A basic mechanism for the evolution of inhomogeneous systems, Front. Phys. 7 (2019) 00018.
[5] K.G. Wang, Long-time-correlation effects and biased anomalous diffusion, Phys. Rev. A 45 (1992) 833–837.
[6] N. Pottier, Aging properties of an anomalously diffusing particule, Physica A 317 (2003) 371–382.
[7] A.D. Viñales, M.A. Despósito, Anomalous diffusion: Exact solution of the generalized Langevin equation for harmonically bounded particle, Phys. Rev. E 73 (2006) 016111.
[8] T.A.M. Langlands, Solution of a modified fractional diffusion equation, Physica A 367 (2006) 136–144.
[9] A.D. Viñales, K.G. Wang, M.A. Despósito, Anomalous diffusive behavior of a harmonic oscillator driven by a Mittag-Leffler noise, Phys. Rev. E 80 (2009) 011101.
[10] M.A. Desposito, A.D. Viñales, Subdiffusive behavior in a trapping potential: Mean square displacement and velocity autocorrelation function, Phys. Rev. E 80 (2009) 021111.
[11] T. Sandev, R. Metzler, Z. Tomovski, Correlation functions for the fractional generalized Langevin equation in the presence of internal and external noise, J. Math. Phys. 55 (2014) 023301.
[12] T. Sandev, Z. Tomovski, Langevin equation for a free particle driven by power law type of noises, Phys. Lett. A 378 (2014) 1–9.

[13] E. Bakalis, S. Höfinger, A. Venturini, F. Zerbetto, Crossover of two power laws in the anomalous diffusion of a two lipid membrane, J. Chem. Phys. 142 (2015) 215102.
[14] H. Safdari, A.G. Cherstvy, A.A. Chechkin, F. Thiel, I.M. Sokolov, R. Metzler, Quantifying the non-ergodicity of scaled Brownian motion, J. Phys. A 48 (2015) 375002.
[15] H. Safdari, A.G. Cherstvy, A.A. Chechkin, A. Bodrova, R. Metzler, Aging underdamped scaled Brownian motion: Ensemble- and time-averaged particle displacements, nonergodicity, and the failure of the overdamping approximation, Phys. Rev. E 95 (2017) 012120.
[16] W. Wang, R. Metzler, A.G. Cherstvy, Anomalous diffusion, aging, and nonergodicity of scaled Brownian motion with fractional Gaussian noise: Overview of related experimental observations and models, Phys. Chem. Chem. Phys. 24 (2022) 18482–18504.
[17] R. Kubo, The fluctuation-dissipation theorem, Rep. Progr. Phys. 29 (1966) 255–284.
[18] V. Lisý, J. Tóthová, L. Glod, On the correlation properties of thermal noise in fluids, Int. J. Thermophys. 34 (2013) 629–641.
[19] D.S. Grebenkov, M. Vahabi, E. Bertseva, L. Forro, S. Jeney, Hydrodynamic and subdiffusive motion of tracers in a viscoelastic medium, Phys. Rev. E. 88 (2013) 040701, (R).
[20] D.S. Grebenkov, M. Vahabi, Analytical solution of the generalized Langevin equation with hydrodynamic interactions: Subdiffusion of heavy tracers, Phys. Rev. E. 89 (2014) 012130.
[21] Á. Fodor, D.S. Grebenkov, P. Visco, F. van Wijland, Generalized Langevin equation with hydrodynamic backflow: Equilibrium properties, Physica A 422 (2015) 107–112.
[22] P. Moreno-Casas, F. Bombardelli, Computation of the basset force: Recent advances and environmental flow applications, Environ. Fluid Mech. 16 (2015) 193–208.
[23] B. Lukić, S. Jeney, C. Tischer, A.J. Kulik, L. Forro, E. Florin, Direct observation of nondiffusive motion of a Brownian particle, Phys. Rev. Lett. 95 (2005) 160601.
[24] T. Franosch, M. Grimm, M. Belushkin, F.M. Mor, G. Foffi, L. Forro, S. Je, Resonances arising from hydrodynamic memory in Brownian motion, Nature 478 (2011) 85–88.
[25] S. Kheifets, A. Simha, K. Melin, T. Li, M.G. Raizen, Observation of Brownian motion in liquids at short times: Instantaneous velocity and memory loss, Science 343 (2014) 1493–1496.
[26] N. Makris, A rheological analog for Brownian motion with hydrodynamic memory, Phys. Fluids 33 (2021) 072014.
[27] H.-Y. Yu, D.M. Eckmann, P.S. Ayyaswamy, R. Radhakrishnan, Composite generalized Langevin equation for Brownian motion in different hydrodynamic and adhesion regimes, Phys. Rev. E 91 (2015) 052303.
[28] J. Tothova, G. Vasziiona, L. Glod, V. Lisy, Langevin theory of anomalous Brownian motion made simple, Eur. J. Phys. 32 (2011) 645.
[29] V. Vladimirsky, Y.A. Terletzky, Hydrodynamical theory of translational Brownian motion, Zh. Eksp. Teor. Fiz 15 (1945) 258–263.
[30] A.B. Boussinesq, Sur la resistance qu'oppose un fluide indefini en repos, sans pesanteur, au mouvement varie d'une sphere solide qu'il mouille sur toute sa surface, quand les vitesses restent bien continues et assez faibles pour que leurs carres et produits soient negligiables, C. R. Acad. Sci. III 100 (1885) 935–937.
[31] A.B. Basset, A Treatise in Hydrodynamics, University of Michigan Library, 1888.
[32] U.B. Felderhof, On the derivation of the fluctuation-dissipation theorem, J. Phys. A: Math. Gen. 11 (1978) 921–927.
[33] F. Mainardi, A. Mura, F. Tampieri, Brownian motion and anomalous diffusion revisited via a fractional Langevin equation, Mod. Probl. Stat. Phys. 8 (2009) 3–23.
[34] S. Lim Hoe, J. Wehr, M. Lewenstein, Homogenization for generalized Langevin equations with applications to anomalous diffusion, Ann. Henri Poincar e 21 (2020) 1813–1871.
[35] F. Mainardi, R. Gorenflo, On Mittag-Leffler-type functions in fractional evolution processes, J. Comput. Appl. Math. 118 (2000) 283–299.
[36] V.F. Morales-Delgado, M.A. Taneco-Hernández, C. Vargas-De-León, J.F. Gómez-Aguilar, Exact solutions to fractional pharmacokinetic models using multivariate Mittag-Leffler functions, Chaos Solitons Fractals 168 (2023) 113164.
[37] Y. Luchko, Operational method in fractional calculus, Fract. Calc. Appl. Anal. 2 (1999) 463–488.
[38] Y. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, Acta Math. Vietnam. 24 (1999) 207–233.
[39] R.K. Saxena, A.M. Mathai, H.J. Haubold, Unified fractional kinetic equation and a fractional diffusion equation, Astrophys. Space Sci. 290 (2004) 299–310.
[40] R. Zwanzig, M. Bixon, Compressibility effects in the hydrodynamic theory of Brownian motion, J. Fluid Mech. 69 (1975) 21–25.
[41] R. Garrappa, Numerical evaluation of two and three parameter Mittag-Leffler functions, SIAM J. Numer. Anal. 53 (2015) 1350–1369.
[42] H.J.H. Clercx, P.P.J.M. Schram, Brownian particles in shear flow and harmonic potentials: A study of long-time tails, Phys. Rev. A. 46 (1992) 1942–1950.
[43] T. McClure, Numerical Inverse Laplace Transform, MATLAB Central File Exchange, MATLAB, The MathWorks Inc, 2023.
[44] G. He, H. Liu, G. Tang, J. Cao, Resonance behavior for a generalized Mittag-Leffler fractional Langevin equation with hydrodynamic interactions, Internat. J. Modern Phys. B 34 (2020) 2050310.
[45] Y. Luo, C. Zeng, B. Li, A perfect probe: Resonance of underdamped scaled Brownian motion, Europhys. Lett. 137 (2022) 21002.
[46] S. Zhong, J. Yang, Z. Lu, H. Ma, M. Luo, Resonant behavior of a harmonic oscillator with fluctuating mass driven by a Mittag-Lefler noise, J. Stat. Mech. 2017 (2017) 023211.
[47] L.R. Parent, E. Bakalis, M. Proetto, Y. Li, C. Park, F. Zerbetto, N.C. Gianneschi, Tackling the challenges of dynamic experiments using liquid-cell transmission electron microscopy, Acc. Chem. Res. 51 (2018) 3–11.
[48] M. Guernelli, E. Bakalis, A. Mavridi-Printezi, V. Petropoulos, G. Cerulo, F. Zerbetto, M. Montalti, Photothermal motion: Effect of low-intensity irradiation on the thermal motion of organic nanoparticles, Nanoscale 14 (2022) 7233–7241.
[49] E. Bakalis, L.R. Parent, M. Vratsanos, C. Park, N.C. Gianneschi, F. Zerbetto, Complex nanoparticle diffusional motion in liquid-cell transmission electron microscopy, J. Phys. Chem. C 124 (2020) 14881–14890.
[50] E. Bakalis, V. Gavriil, A.-C. Cefalas, Z. Kollia, F. Zerbetto, E. Sarantopoulou, Viscoelasticity and noise properties reveal the formation of biomemory in cells, J. Phys. Chem. B 125 (2021) 10883–10892.
[51] F. Devaux, X. Li, D. Sluysmans, V. Maurizot, E. Bakalis, F. Zerbetto, I. Huc, A.-S. Duwez, Single-molecule mechanics of synthetic aromatic amide helices: Ultrafast and robust non-dissipative winding, Chem 7 (2021) 1333–1346.
[52] P. Reimann, A uniqueness-theorem for "linear" thermal baths, Chem. Phys. 268 (2001) 337–346.
[53] P. Siegle, I. Goychuk, P. Hanggi, Markovian embedding of fractional superdiffusion, Europhys. Lett. 93 (2011) 20002.
[54] M.A. Despósito, A.D. Viñales, Memory effects in the asymptotic diffusive behavior of a classical oscillator described by a generalized Langevin equation, Phys. Rev. E 77 (2008) 031123.
[55] M.G. Mittag-Leffler, Sur l'int'egrale de Laplace-Abel, C. R. Acad. Sci. Paris (Ser. II) 136 (1902) 937–939.
[56] M.G. Mitta4g-Leffler, Sopra la funzione eα (x), Rend. Accad. Lincei 13 (1904) 3–5.
[57] T.R. Prabhakar, A singular integral equation with a generalized Mittag–Leffler function in the kernel, Yokohama Math. J. 19 (1971) 7–15.
[58] R. Gorenflo, A.A. Kilbas, F. Mainardi, S. Rogosin, Fractional differential equations, in: Mittag-Leffler Functions, Related Topics and Applications, in: Springer Monographs in Mathematics, Springer, Berlin, Heidelberg, 2014.

[59] F. Mainardi, R. Garrappa, On complete monotonicity of the Prabhakar function and non-Debye relaxation in dielectrics, J. Comput. Phys. 293 (2015) 70–80.
[60] R. Garra, R. Garrappa, The prabhakar or three parameter Mittag–Leffler function: Theory and application, Commun. Nonlinear Sci. Numer. Simul. 56 (2018) 314–329.
[61] I. Podlubny, Fractional Differential Equations, Academic Press, 1999, URL https://www.britannica.com/science/noise-acoustics.
[62] H.J. Haubold, A.M. Mathai, R.K. Saxena, Mittag-Leffler functions and their applications, J. Appl. Math. (2010) 298628.