TRANSFORMATION OF THE LINEAR DIFFERENCE EQUATION INTO A SYSTEM OF THE FIRST ORDER DIFFERENCE EQUATIONS

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The transformation of the N-th-order linear difference equation into a system of the first order difference equations is presented. The proposed transformation opens possibility to obtain new forms of the N-dimensional system of the first order equations that can be useful for the analysis of solutions of the N-th-order difference equations. In particular for the third-order linear difference equation the nonlinear second-order difference equation that plays the same role as the Riccati equation for second-order linear difference equation is obtained. The new form of the N-dimensional system of first order equations can also be used to find the WKB solutions of the linear difference equation with coefficients that vary slowly with index

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INTRODUCTION

It is common knowledge that a difference equation of order N

$$\begin{aligned} & y_{k+N} + f_{N-1,k} y_{k+N-1} + f_{N-2,k} y_{k+N-2} + ... \\ & + f_{2,k} y_{k+2} + f_{1,k} y_{k+1} + f_{0,k} y_k + f_k = 0 \end{aligned} \quad (1)$$

may be transformed in a standard way to a system of the N first-order difference equations. To obtain such transformation we introduce a number of new variables (see, for example, [1, 2])

$$x_k^{(i)} = y_{k+i-1},\ i = 1,2,...,N. \quad (2)$$

The difference equation (1) can be rewritten as

$$X_{k+1} = T_k X_k + F_k, \quad (3)$$

where $X_k = (x_{k+N-1}, x_{k+N-1}, ..., x_k)^T$, $F_k = (f_k, 0, ..., 0)^T$, and the companion matrix of (1) is

$$T_k = \begin{pmatrix} -f_{N-1,k} & -f_{N-2,k} & ... -f_{1,k} & -f_{0,k} \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ \\ 0 & 0 & ... & 1 & 0 \end{pmatrix}. \quad (4)$$

In fact we do not introduce new variables[1], we only do re-designations and still work with the elements of the same sequence y_k.

There is another kind of transformation[2] that consists of representation of the solution y_k of the equation (1) as the sum of the N new unknown grid functions [3 -5]. By introducing N new unknowns, instead of the one, we can impose ($N-1$) additional conditions. Such approach gives new form of the N-dimensional system of first order equations, equivalent to the equation (1). In this article some generalization of the proposed transformation [3] is given. Analysis of literature shows that it apparently has not been described earlier.

[1]In the case of transformation of differential equations we do introduce new variables $x^{(i)} = d^{i-1}y / dt^{i-1},\ i = 1,2,...,N$.

[2]For the case of second-order differential equations it was used in [6].

1. TRANSFORMATION THE N-th-ORDER LINEAR DIFFERENCE EQUATION

We represent the solution of the difference equation (1) as the sum of new grid functions

$$y_k = \sum_{n=1}^{N} y_{n,k}. \quad (5)$$

By introducing N new unknowns $y_{n,k}$ instead of the one y_k, we can impose additional conditions. These conditions we write in the form

$$\begin{aligned} y_{k+1} &= \sum_{n=1}^{N} g_{1,n,k} y_{n,k}, \\ y_{k+2} &= \sum_{n=1}^{N} g_{2,n,k} y_{n,k}, \\ &.... \\ y_{k+N-1} &= \sum_{n=1}^{N} g_{N-1,n,k} y_{n,k}, \end{aligned} \quad (6)$$

where $g_{m,n,k}$ $(1 \le n \le N,\ 1 \le m \le N-1)$ are the arbitrary sequences.

If

$$\det \begin{pmatrix} 1 & 1 & .. & 1 \\ g_{1,1,k} & g_{1,2,k} & ... & g_{1,N,k} \\ ... & ... & ... & ... \\ g_{N-1,1,k} & g_{N-1,2,k} & ... & g_{N-1,N,k} \end{pmatrix} \ne 0, \quad (7)$$

then the representation (5), (6) is unique. Indeed, from (5) and (6) we can uniquely find $y_{n,k}$ as a linear combination of y_k. Using (1), (5), and (6) we can write such system of equations

$$\begin{aligned} y_{k+1} &= \sum_{n=1}^{N} y_{n,k+1} = \sum_{n=1}^{N} g_{1,n,k} y_{n,k}, \\ y_{k+2} &= \sum_{n=1}^{N} g_{1,n,k+1} y_{n,k+1} = \sum_{n=1}^{N} g_{2,n,k} y_{n,k}, \\ &.... \\ y_{k+N-1} &= \sum_{n=1}^{N} g_{N-2,n,k+1} y_{n,k+1} = \sum_{n=1}^{N} g_{N-1,n,k} y_{n,k}, \\ &\sum_{n=1}^{N} g_{N-1,n,k+1} y_{n,k+1} = \\ &= -\sum_{n=1}^{N} \left(\sum_{m=1}^{N-1} f_{N-m,k} g_{N-m,n,k} + f_{0,k} \right) y_{n,k} - f_k. \end{aligned}$$

In matrix form

$$M_{k+1}Y_{k+1} = H_{k+1}Y_k + F_{k+1}, \quad (8)$$

where $Y_k = \left(y_{1,k}, y_{2,k}, ..., y_{N,k}\right)^T$, $F_k = \left(0, 0..., -f_k\right)^T$,

$$M_k = \begin{pmatrix} 1 & 1 & .. & 1 \\ g_{1,1,k} & g_{1,2,k} & \cdots & g_{1,N,k} \\ \cdots & \cdots & \cdots & \cdots \\ g_{N-1,1,k} & g_{N-1,2,k} & \cdots & g_{N-1,N,k} \end{pmatrix}, \quad (9)$$

$$H_{k+1} = \begin{pmatrix} g_{1,1,k} & g_{1,2,k} & \cdots & g_{1,N,k} \\ \cdots & \cdots & \cdots & \cdots \\ g_{N-1,1,k} & g_{N-1,2,k} & \cdots & g_{N-1,N,k} \\ A_{1,k-1} & A_{2,k-1} & \cdots & A_{N,k-1} \end{pmatrix}, \quad (10)$$

$$A_{n,k} = -\left(\sum_{m=1}^{N-1} f_{N-m,k} g_{N-m,n,k} + f_{0,k}\right). \quad (11)$$

And finally, we have the equation

$$Y_{k+1} = T_{k+1}Y_k + \bar{F}_{k+1}, \quad (12)$$

where $T_k = M_k^{-1}H_k$, $\bar{F}_k = M_k^{-1}F_k$.

We would like to emphasize that the sequences $g_{m,n,k}$ are the arbitrary ones, and we do not impose a condition that the new grid functions $y_{m,k}$ are the solutions of the equation (1).

As $g_{m,n,k}$ are the arbitrary sequences we can try to find such sequences that result in diagonal matrix T_k. If it can be done, we easily find the solution of the system (12) and the initial difference equation (1). It can be shown that in this case $y_{m,k}$ are the linearly independent solutions of the equation (1).

2. TRANSFORMATION THE SECOND-ORDER LINEAR DIFFERENCE EQUATION

Following the section 2 we represent the solution of the linear second-order equation

$$y_{k+2} + f_{1,k}y_{k+1} + f_{0,k}y_k + f_k = 0 \quad (13)$$

as the sum of the two new grid functions

$$y_k = y_{1,k} + y_{2,k}. \quad (14)$$

We write an additional condition as

$$y_{k+1} = g_{1,k}y_{1,k} + g_{2,k}y_{2,k}, \quad (15)$$

where $g_{n,k}$ ($1 \le n \le 2$) are the arbitrary sequences.

Applying transformations from section 2, we can write such system of equations

$$\begin{pmatrix} y_{1,k+1} \\ y_{2,k+1} \end{pmatrix} = T_{k+1}\begin{pmatrix} y_{1,k} \\ y_{2,k} \end{pmatrix} + F_{k+1}, \quad (16)$$

where

$$T_{k+1} = \begin{pmatrix} -\dfrac{f_{0,k} + g_{1,k}\left(g_{2,k+1} + f_{1,k}\right)}{g_{1,1,k+1} - g_{1,2,k+1}} & -\dfrac{f_{0,k} + g_{2,k}\left(g_{2,k+1} + f_{1,k}\right)}{g_{1,k+1} - g_{2,k+1}} \\ \dfrac{f_{0,k} + g_{1,k}\left(g_{1,k+1} + f_{1,k}\right)}{g_{1,k+1} - g_{2,k+1}} & \dfrac{f_{0,k} + g_{2,k}\left(g_{1,k+1} + f_{1,k}\right)}{g_{1,k+1} - g_{2,k+1}} \end{pmatrix}, \quad (17)$$

$$F_{k+1} = \begin{pmatrix} -\dfrac{f_{k+1}}{g_{1,k+1} - g_{2,k+1}} \\ \dfrac{f_{k+1}}{g_{1,k+1} - g_{2,k+1}} \end{pmatrix} = \begin{pmatrix} -\bar{f}_{k+1} \\ \bar{f}_{k+1} \end{pmatrix}. \quad (18)$$

We will consider the homogeneous difference equations ($f_k = 0$). The normal system of difference equations (16) can be transformed into the known ones.

From (17) it follows that we can choose the sequences $g_{(1,2),k}$ in such a way that matrix T_k will be triangular or even diagonal one. It is realized by setting $T_{k,12} = 0$ and $T_{k,21} = 0$.

These conditions give the non-linear second-order rational difference equation (Riccaty type [3, 7 - 9]) for the sequences $g_{(1,2),k}$

$$f_{0,k} + g_{(1,2),k}\left(g_{(1,2),k+1} + f_{1,k}\right) = 0. \quad (19)$$

In this case the matrix T_k is a diagonal one and the system (16) takes the form:

$$y_{(1,2),k+1} = g_{(1,2),k}y_{(1,2),k}. \quad (20)$$

Solutions $y_{1,k}, y_{2,k}$ are linearly independent.

The characteristic equation of the difference equation (13) is

$$\rho_k^2 + f_{1,k}\rho_k + f_{0,k} = 0. \quad (21)$$

Let $g_{(1,2),k} = \rho_k^{(1,2)}$, where $\rho_k^{(1,2)}$ are the solutions of the characteristic equation (21)

$$\rho_k^{(1,2)} = -\frac{f_{1,k}}{2} \pm \frac{1}{2}\sqrt{f_{1,k}^2 - 4f_{0,k}}. \quad (22)$$

The matrix T_k takes the form

$$T_{k+1} = \begin{pmatrix} \rho_k^{(1)}\dfrac{\rho_k^{(1)} - \rho_{k+1}^{(2)}}{\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}} & \rho_k^{(2)}\dfrac{\rho_k^{(2)} - \rho_{k+1}^{(2)}}{\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}} \\ \rho_k^{(1)}\dfrac{\rho_{k+1}^{(1)} - \rho_k^{(1)}}{\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}} & \rho_k^{(2)}\dfrac{\rho_{k+1}^{(1)} - \rho_k^{(2)}}{\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}} \end{pmatrix}. \quad (23)$$

If sequences $f_{(0,1),k}$ vary sufficiently slowly with k ($f_{(0,1),k} = f_{(0,1)}(\varepsilon k)$, $0 \le \varepsilon \ll 1$), then the differences $\left(\rho_{k+1}^{(1,2)} - \rho_k^{(1,2)}\right)$ are small and we can neglect the non-diagonal terms in the matrix T_k. This gives

$$y_{1,k+1} = \rho_k^{(1)}\left(1 - \frac{\rho_{k+1}^{(1)} - \rho_k^{(1)}}{\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}}\right)y_{1,k},$$
$$y_{2,k+1} = \rho_k^{(2)}\left(1 + \frac{\rho_{k+1}^{(2)} - \rho_k^{(2)}}{\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}}\right)y_{2,k}. \quad (24)$$

It can be shown that these equations coincide with the equations of the discrete WKB approach (see, for example, [10, 11]). Indeed, from (20) it follows that the discrete WKB equations can be obtained by using an approximate solutions of the Riccati equation (19) under assumption that $f_{0,k}$ and $f_{1,k}$ vary sufficiently slowly with k. The Riccati equations can be transformed by an iteration procedure into the quadratic equations

$$\left(g_{(1,2),k}\right)^2 + g_{(1,2),k}f_{1,k} + f_{0,k} + \rho_{k+1}^{(1,2)}\left(\rho_{k+2}^{(1,2)} - \rho_{k+1}^{(1,2)}\right) = 0. \quad (25)$$

It is one of the possible forms of the quadratic equation (compare with [10, 11]) that can be obtained at the second iteration. Its solutions differ from the ones that were obtained in [10, 11] by an amount of order ε^2. The approximate solutions of this equations with error of $O(\varepsilon^2)$ are

$$g_{(1,2),k} \approx \rho_k^{(1,2)} \left(1 \mp \frac{\rho_{k+1}^{(1,2)} - \rho_k^{(1,2)}}{\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}} \right). \qquad (26)$$

Comparison (24) and (26) shows that these two different approaches give the same result, and the equations (24) and (20) coincide.

The solutions of the equations (24) at $k > k_0$ can be written as

$$y_k^{(1,2)} = \prod_{s=k_0+1}^{k} T_{s,11,22} y_{k_0}^{(1)} = y_{k_0}^{(1,2)} \exp\left(\sum_{s=k_0+1}^{k} \ln\rho_{s-1}^{(1,2)} - \right.$$
$$\left. - \sum_{s=k_0+1}^{k} \frac{\sqrt{f_{1,s}^2 - 4f_{0.s}} - \sqrt{f_{1,s-1}^2 - 4f_{0,s-1}}}{2\sqrt{f_{1,s}^2 - 4f_{0,s}}} \pm \sum_{s=k_0+1}^{k} \frac{f_{1,s} - f_{1,s-1}}{2\sqrt{f_{1,s}^2 - 4f_{0,s}}} \right) \sim \qquad (27)$$
$$\sim \frac{1}{\left(f_{1,k}^2 - 4f_{0.k}\right)^{1/4}} \exp\left(\sum_{s=k_0+1}^{k} \ln\rho_{s-1}^{(1,2)} \pm \sum_{s=k_0+1}^{k} \frac{f_{1,s} - f_{1,s-1}}{2\sqrt{f_{1,s}^2 - 4f_{0,s}}} \right).$$

Comparison of this formula with that obtained by directly finding an approximate solution from the equation (1) [12] gives some difference. The formula (27) contains additional sum in the exponent (the second sum).

3. TRANSFORMATION THE THIRD-ORDER LINEAR DIFFERENCE EQUATION

Let's represent the solution of the linear third-order equation

$$y_{k+3} + f_{2,k} y_{k+2} + f_{1,k} y_{k+1} + f_{0,k} y_k + f_k = 0 \qquad (28)$$

as the sum of the three new functions

$$y_k = y_{1,k} + y_{2,k} + y_{3,k}. \qquad (29)$$

We write additional conditions in the form

$$\begin{aligned} y_{k+1} &= g_{1,1,k} y_{1,k} + g_{1,2,k} y_{2,k} + g_{1,3,k} y_{3,k}, \\ y_{k+2} &= g_{2,1,k} y_{1,k} + g_{2,2,k} y_{2,k} + g_{2,3,k} y_{3,k}. \end{aligned} \qquad (30)$$

Applying the transformations that are given in section 2, we obtain a system of the first-order linear difference equations

$$\begin{aligned} &y_{1,k+1} D_{k+1} = g_{1,1,k} y_{1,k} D_{k+1} + \\ &+y_{1,k} \left[\begin{array}{l} g_{1,1,k} \left\{ \begin{array}{l} \left(g_{1,1,k+1} - g_{1,1,k}\right)\left(g_{2,3,k+1} - g_{2,2,k+1}\right) + \\ +\left(g_{2,1,k+1} - g_{2,1,k}\right)\left(g_{1,2,k+1} - g_{1,3,k+1}\right) \end{array} \right\} + \\ +x_{1,k}\left(g_{2,3,k+1} - g_{2,2,k+1}\right) + \left(g_{1,2,k+1} - g_{1,3,k+1}\right) x_{4,k} \end{array} \right] \\ &+y_{2,k} \left[\begin{array}{l} g_{1,2,k} \left\{ \begin{array}{l} \left(g_{1,2,k+1} - g_{1,2,k}\right)\left(g_{2,3,k+1} - g_{2,2,k+1}\right) + \\ +\left(g_{2,2,k+1} - g_{2,2,k}\right)\left(g_{1,2,k+1} - g_{1,3,k+1}\right) \end{array} \right\} + \\ +x_{2,k}\left(g_{2,3,k+1} - g_{2,2,k+1}\right) + \left(g_{1,2,k+1} - g_{1,3,k+1}\right) x_{5,k} \end{array} \right] \\ &+y_{3,k} \left[\begin{array}{l} g_{1,3,k} \left\{ \begin{array}{l} \left(g_{1,3,k+1} - g_{1,3,k}\right)\left(g_{2,3,k+1} - g_{2,2,k+1}\right) + \\ +\left(g_{2,3,k+1} - g_{2,3,k}\right)\left(g_{1,2,k+1} - g_{1,3,k+1}\right) \end{array} \right\} + \\ +x_{3,k}\left(g_{2,3,k+1} - g_{2,2,k+1}\right) + \left(g_{1,2,k+1} - g_{1,3,k+1}\right) x_{6,k} \end{array} \right] \\ &-f_k\left(g_{1,3,k+1} - g_{1,2,k+1}\right), \end{aligned} \qquad (31)$$

$$\begin{aligned} &y_{2,k+1} D_{k+1} = y_{2,k} g_{1,2,k} D_{k+1} + \\ &+y_{1,k} \left[\begin{array}{l} g_{1,1,k} \left\{ \begin{array}{l} \left(g_{1,1,k+1} - g_{1,1,k}\right)\left(g_{2,1,k+1} - g_{2,3,k+1}\right) + \\ +\left(g_{2,1,k+1} - g_{2,1,k}\right)\left(g_{1,3,k+1} - g_{1,1,k+1}\right) \end{array} \right\} + \\ +x_{1,k}\left(g_{2,1,k+1} - g_{2,3,k+1}\right) + \left(g_{1,3,k+1} - g_{1,1,k+1}\right) x_{4,k} \end{array} \right] \\ &+y_{2,k} \left[\begin{array}{l} g_{1,2,k} \left\{ \begin{array}{l} \left(g_{1,2,k+1} - g_{1,2,k}\right)\left(g_{2,1,k+1} - g_{2,3,k+1}\right) + \\ +\left(g_{2,2,k+1} - g_{2,2,k}\right)\left(g_{1,3,k+1} - g_{1,1,k+1}\right) \end{array} \right\} + \\ +x_{2,k}\left(g_{2,1,k+1} - g_{2,3,k+1}\right) + \left(g_{1,3,k+1} - g_{1,1,k+1}\right) x_{5,k} \end{array} \right] \\ &+y_{3,k} \left[\begin{array}{l} g_{1,3,k} \left\{ \begin{array}{l} \left(g_{1,3,k+1} - g_{1,3,k}\right)\left(g_{2,1,k+1} - g_{2,3,k+1}\right) + \\ +\left(g_{2,3,k+1} - g_{2,3,k}\right)\left(g_{1,3,k+1} - g_{1,1,k+1}\right) \end{array} \right\} + \\ +x_{3,k}\left(g_{2,1,k+1} - g_{2,3,k+1}\right) + \left(g_{1,3,k+1} - g_{1,1,k+1}\right) x_{6,k} \end{array} \right] \\ &-f_k\left(g_{1,1,k+1} - g_{1,3,k+1}\right), \end{aligned} \qquad (32)$$

$$\begin{aligned} &y_{3,k+1} D_{k+1} = y_{3,k} g_{1,3,k} D_{k+1} + \\ &+y_{1,k} \left[\begin{array}{l} g_{1,1,k} \left\{ \begin{array}{l} \left(g_{1,1,k+1} - g_{1,1,k}\right)\left(g_{2,2,k+1} - g_{2,1,k+1}\right) + \\ +\left(g_{2,1,k+1} - g_{2,1,k}\right)\left(g_{1,1,k+1} - g_{1,2,k+1}\right) \end{array} \right\} + \\ +x_{1,k}\left(g_{2,2,k+1} - g_{2,1,k+1}\right) + \left(g_{1,1,k+1} - g_{1,2,k+1}\right) x_{4,k} \end{array} \right] \\ &+y_{2,k} \left[\begin{array}{l} g_{1,2,k} \left\{ \begin{array}{l} \left(g_{1,2,k+1} - g_{1,2,k}\right)\left(g_{2,2,k+1} - g_{2,1,k+1}\right) + \\ +\left(g_{2,2,k+1} - g_{2,2,k}\right)\left(g_{1,1,k+1} - g_{1,2,k+1}\right) + \end{array} \right\} \\ +x_{2,k}\left(g_{2,2,k+1} - g_{2,1,k+1}\right) + \left(g_{1,1,k+1} - g_{1,2,k+1}\right) x_{5,k} \end{array} \right] \\ &+y_{3,k} \left[\begin{array}{l} g_{1,3,k} \left\{ \begin{array}{l} \left(g_{1,3,k+1} - g_{1,3,k}\right)\left(g_{2,2,k+1} - g_{2,1,k+1}\right) + \\ +\left(g_{2,3,k+1} - g_{2,3,k}\right)\left(g_{1,1,k+1} - g_{1,2,k+1}\right) \end{array} \right\} + \\ +x_{3,k}\left(g_{2,2,k+1} - g_{2,1,k+1}\right) + \left(g_{1,1,k+1} - g_{1,2,k+1}\right) x_{6,k} \end{array} \right] \\ &-f_k\left(g_{1,2,k+1} - g_{1,1,k+1}\right). \end{aligned} \qquad (33)$$

where the following notations were introduced

$$\begin{aligned} x_{1,k} &= \left(g_{1,1,k}\right)^2 - g_{2,1,k}, \\ x_{2,k} &= \left(g_{1,2,k}\right)^2 - g_{2,2,k}, \\ x_{3,k} &= \left(g_{1,3,k}\right)^2 - g_{2,3,k}, \\ x_{4,k} &= g_{1,1,k} g_{2,1,k} + f_{2,k} g_{2,1,k} + f_{1,k} g_{1,1,k} + f_{0,k}, \\ x_{5,k} &= g_{1,2,k} g_{2,2,k} + f_{2,k} g_{2,2,k} + f_{1,k} g_{1,2,k} + f_{0,k}, \\ x_{6,k} &= g_{1,3,k} g_{2,3,k} + f_{2,k} g_{2,3,k} + f_{1,k} g_{1,3,k} + f_{0,k}. \end{aligned} \qquad (34)$$

If we choose

$$g_{1,n,k} = \rho_k^{(n)},\ g_{2,n,k} = \rho_k^{(n)2},\ n = 1,2,3, \qquad (35)$$

where $\rho_k^{(n)}$ are the solutions of the equation

$$\rho_k^3 + f_{2,k}\rho_k^2 + f_{1,k}\rho_k + f_{0,k} = 0, \qquad (36)$$

then $x_{i,k} = 0,\ i = 1,...,6$ and the system (31) - (33) takes the form

$$\begin{aligned} &y_{1,k+1} = y_{1,k}\rho_k^{(1)} + \\ &y_{1,k}\rho_k^{(1)}\left(\rho_{k+1}^{(1)} - \rho_k^{(1)}\right)\left(\rho_{k+1}^{(3)} - \rho_{k+1}^{(2)}\right)\left[\left(\rho_{k+1}^{(3)} + \rho_{k+1}^{(2)}\right) - \left(\rho_{k+1}^{(1)} + \rho_k^{(1)}\right)\right] / D_{k+1} + \\ &y_{2,k}\rho_k^{(2)}\left(\rho_{k+1}^{(2)} - \rho_k^{2)}\right)\left(\rho_{k+1}^{(3)} - \rho_{k+1}^{(2)}\right)\left[\left(\rho_{k+1}^{(3)} + \rho_{k+1}^{(2)}\right) - \left(\rho_{k+1}^{(2)} + \rho_k^{(2)}\right)\right] / D_{k+1} \\ &y_{3,k}\rho_k^{(3)}\left(\rho_{k+1}^{(3)} - \rho_k^{(3)}\right)\left(\rho_{k+1}^{(3)} - \rho_{k+1}^{(2)}\right)\left[\left(\rho_{k+1}^{(3)} + \rho_{k+1}^{(2)}\right) - \left(\rho_{k+1}^{(3)} + \rho_k^{(3)}\right)\right] / D_{k+1} \\ &-f_k\left(\rho_{k+1}^{(3)} - \rho_{k+1}^{(2)}\right) / D_{k+1}, \end{aligned} \qquad (37)$$

$$\begin{aligned} &y_{2,k+1} = y_{2,k}\rho_k^{(2)} + \\ &y_{1,k}\rho_k^{(1)}\left(\rho_{k+1}^{(1)} - \rho_k^{(1)}\right)\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(3)}\right)\left[\left(\rho_{k+1}^{(1)} + \rho_{k+1}^{(3)}\right) - \left(\rho_{k+1}^{(1)} + \rho_k^{(1)}\right)\right] / D_{k+1} + \\ &y_{2,k}\rho_k^{(2)}\left(\rho_{k+1}^{(2)} - \rho_k^{(2)}\right)\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(3)}\right)\left[\left(\rho_{k+1}^{(1)} + \rho_{k+1}^{(3)}\right) - \left(\rho_{k+1}^{(2)} + \rho_k^{(2)}\right)\right] / D_{k+1} + \\ &y_{3,k}\rho_k^{(3)}\left(\rho_{k+1}^{(3)} - \rho_k^{(3)}\right)\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(3)}\right)\left[\left(\rho_{k+1}^{(1)} + \rho_{k+1}^{(3)}\right) - \left(\rho_{k+1}^{(3)} + \rho_k^{(3)}\right)\right] / D_{k+1} - \\ &-f_k\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(3)}\right) / D_{k+1}, \end{aligned} \qquad (38)$$

$$\begin{aligned} &y_{3,k+1} = y_{3,k}\rho_k^{(3)} + \\ &y_{1,k}\rho_k^{(1)}\left(\rho_{k+1}^{(1)} - \rho_k^{(1)}\right)\left(\rho_{k+1}^{(2)} - \rho_{k+1}^{(1)}\right)\left[\left(\rho_{k+1}^{(2)} + \rho_{k+1}^{(1)}\right) - \left(\rho_{k+1}^{(1)} + \rho_k^{(1)}\right)\right] / D_{k+1} + \\ &y_{2,k}\rho_k^{(2)}\left(\rho_{k+1}^{(2)} - \rho_k^{(2)}\right)\left(\rho_{k+1}^{(2)} - \rho_{k+1}^{(1)}\right)\left[\left(\rho_{k+1}^{(2)} + \rho_{k+1}^{(1)}\right) - \left(\rho_{k+1}^{(2)} + \rho_k^{(2)}\right)\right] / D_{k+1} + \\ &y_{3,k}\rho_k^{(3)}\left(\rho_{k+1}^{(3)} - \rho_{k+1}^{(3)}\right)\left(\rho_{k+1}^{(2)} - \rho_{k+1}^{(1)}\right)\left[\left(\rho_{k+1}^{(2)} + \rho_{k+1}^{(1)}\right) - \left(\rho_{k+1}^{(3)} + \rho_k^{(3)}\right)\right] / D_{k+1} - \\ &-f_k\left(\rho_{k+1}^{(2)} - \rho_{k+1}^{(1)}\right) / D_{k+1}, \end{aligned} \qquad (39)$$

where $D_{k+1} = \left(\rho_{k+1}^{(2)} - \rho_{k+1}^{(1)}\right)\left(\rho_{k+1}^{(3)} - \rho_{k+1}^{(1)}\right)\left(\rho_{k+1}^{(3)} - \rho_{k+1}^{(2)}\right)$ – the Vandermonde determinant.

If the sequences $f_{0,k}$, $f_{1,k}$, $f_{2,k}$ vary sufficiently slowly with k ($f_{0,k} = f_0(\varepsilon k)$, $f_{1,k} = f_1(\varepsilon k)$, $f_{2,k} = f_2(\varepsilon k)$, $0 \le \varepsilon \ll 1$), then the differences $\left(\rho_{k+1}^{(n)} - \rho_k^{(n)}\right)$ are the small values and we can neglect the non-diagonal terms in the matrix T_k. This gives the WKB approximation

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$$y_{1,k+1} \approx y_{1,k}\rho_k^{(1)} -$$
$$-y_{1,k}\rho_k^{(1)}\left(\rho_{k+1}^{(1)}-\rho_k^{(1)}\right)\left[\frac{1}{\left(\rho_{k+1}^{(1)}-\rho_{k+1}^{(2)}\right)}+\frac{1}{\left(\rho_{k+1}^{(1)}-\rho_{k+1}^{(3)}\right)}\right]-f_k\frac{\left(\rho_{k+1}^{(3)}-\rho_{k+1}^{(2)}\right)}{D_{k+1}}, \quad (40)$$
$$y_{2,k+1} \approx y_{2,k}\rho_k^{(2)} -$$
$$-y_{2,k}\rho_k^{(2)}\left(\rho_{k+1}^{(2)}-\rho_k^{(2)}\right)\left[\frac{1}{\left(\rho_{k+1}^{(2)}-\rho_{k+1}^{(3)}\right)}+\frac{1}{\left(\rho_{k+1}^{(2)}-\rho_{k+1}^{(1)}\right)}\right]-f_k\frac{\left(\rho_{k+1}^{(1)}-\rho_{k+1}^{(3)}\right)}{D_{k+1}}, \quad (41)$$
$$y_{3,k+1} \approx y_{3,k}\rho_k^{(3)} -$$
$$-y_{3,k}\rho_k^{(3)}\left(\rho_{k+1}^{(3)}-\rho_k^{(3)}\right)\left[\frac{1}{\left(\rho_{k+1}^{(3)}-\rho_{k+1}^{(1)}\right)}+\frac{1}{\left(\rho_{k+1}^{(3)}-\rho_{k+1}^{(2)}\right)}\right]-f_k\frac{\left(\rho_{k+1}^{(2)}-\rho_{k+1}^{(1)}\right)}{D_{k+1}}. \quad (42)$$

If we choose the sequences $g_{m.n,k}$ to be the solutions of the following equations

$$\begin{aligned}
&x_{1,k}=\left(g_{1,1,k}\right)^2-g_{2,1,k}=-g_{1,1,k}\left(g_{1,1,k+1}-g_{1,1,k}\right),\\
&x_{4,k}=g_{1,1,k}g_{2,1,k}+f_{2,k}g_{2,1,k}+f_{1,k}g_{1,1,k}+f_{0,k}=-g_{1,1,k}\left(g_{2,1,k+1}-g_{2,1,k}\right),\\
&x_{2,k}=\left(g_{1,2,k}\right)^2-g_{2,2,k}=-g_{1,2,k}\left(g_{1,2,k+1}-g_{1,2,k}\right),\\
&x_{5,k}=g_{1,2,k}g_{2,2,k}+f_{2,k}g_{2,2,k}+f_{1,k}g_{1,2,k}+f_{0,k}=-g_{1,2,k}\left(g_{2,2,k+1}-g_{2,2,k}\right),\\
&x_{3,k}=\left(g_{1,3,k}\right)^2-g_{2,3,k}=-g_{1,3,k}\left(g_{1,3,k+1}-g_{1,3,k}\right),\\
&x_{6,k}=g_{1,3,k}g_{2,3,k}+f_{2,k}g_{2,3,k}+f_{1,k}g_{1,3,k}+f_{0,k}=-g_{1,3,k}\left(g_{2,3,k+1}-g_{2,3,k}\right),
\end{aligned} \quad (43)$$

the system (33) takes form

$$y_{1,k+1}=y_{1,k}g_{1,1,k}-f_k\frac{\left(g_{1,3,k+1}-g_{1,2,k+1}\right)}{D_{k+1}}, \quad (44)$$

$$y_{2,k+1}=y_{2,k}g_{1,2,k}-f_k\frac{\left(g_{1,1,k+1}-g_{1,3,k+1}\right)}{D_{k+1}}, \quad (45)$$

$$y_{3,k+1}=y_{3,k}g_{1,3,k}-f_k\frac{\left(g_{1,2,k+1}-g_{1,1,k+1}\right)}{D_{k+1}}. \quad (46)$$

From (43) it follows that sequences $g_{m.n,k}$ are the three different solutions of the system of the first-order nonlinear difference equations

$$\begin{aligned}
&p_k^{(1)}\left(p_{k+1}^{(1)}-p_k^{(1)}\right)+p_k^{(1)2}-p_k^{(2)}=0,\\
&p_k^{(1)}\left(p_{k+1}^{(2)}-p_k^{(2)}\right)+f_{2,k}p_k^{(2)}+f_{1,k}p_k^{(1)}+f_{0,k}+p_k^{(2)}p_k^{(1)}=0.
\end{aligned} \quad (47)$$

This system can be written as the second-order nonlinear difference equation

$$p_{k+2}^{(1)}p_{k+1}^{(1)}p_k^{(1)}+f_{2,k}p_{k+1}^{(1)}p_k^{(1)}+f_{1,k}p_k^{(1)}+f_{0,k}=0. \quad (48)$$

For the third-order linear difference equation (28) the equation (48) (or system (47)) plays the same role as the Riccati equation for second-order linear difference equation.

The functions $y_{n,k}=\prod_k g_{1,n,k}$ are linear independent, and the general solution of the homogeneous equation (28) ($f_k=0$) is

$$y_k=\sum_{n=1}^{3}y_{n,k_0}\prod_{s=k_0}^{k-1}g_{1,n,s}. \quad (49)$$

There are other forms of the system of the first order equations that can be obtained from the system (33) by choosing different sequences $g_{m.n,k}$.

Finding the WKB solutions of the linear difference equation (28) with coefficients that vary sufficiently slowly with index k by finding the three iteration solutions of the equation (48) is not a simple procedure (compare with [10, 11]). So it seems preferable to use the approach that leads us to the WKB equations (40)-(42).

4. THE WKB APPROXIMATION FOR THE *N*-th-ORDER LINEAR DIFFERENCE EQUATION

Sections' 3 and 4 results show that WKB equations for the *N*-order linear difference equation with coefficients that vary sufficiently slowly with index can be obtained by choosing sequences $g_{m,n,k}=\left(\rho_k^{(n)}\right)^m$, where $\rho_k^{(n)}$ are the solutions of the characteristic equation

$$\begin{aligned}
&\rho_k^N+f_{N-1,k}\rho_k^{N-1}+f_{N-2,k}\rho_k^{N-2}+\ldots\\
&+f_{2,k}\rho_k^2y_{k+2}+f_{1,k}\rho_k+f_{0,k}=0
\end{aligned} \quad (50)$$

and taking into consideration only diagonal elements of the matrix T_k in the equation

$$Y_{k+1}=T_{k+1}Y_k=M_{k+1}^{-1}H_{k+1}Y_k. \quad (51)$$

If we chose $g_{m,n,k}=\left(\rho_k^{(n)}\right)^m$, the matrix M_k transforms into the Vandermonde matrix and we can find its inverse [13]

$$M_{k,i,j}^{-1}=\frac{\left(-1\right)^{j-1}\sigma_{k,i}^{(N-j)}}{\prod_{\substack{s=1\\ s\neq i}}^{N}\left(\rho_k^{(s)}-\rho_k^{(1)}\right)}, \quad (52)$$

where $\sigma_{k,i}^{(j)}=\sum_{1\le m_1<m_2<\ldots<m_j\le N}\prod_{s=1}^{j}\rho_k^{(m_s)}\left(1-\delta_{m_s,i}\right)$. The matrix H_{k+1} for such choice of sequences $g_{m,n,k}$ has the form

$$H_{k+1}=\begin{pmatrix}\rho_k^{(1)} & \rho_k^{(2)} & \ldots & \rho_k^{(N)}\\ \ldots & \ldots & \ldots & \ldots\\ \rho_k^{(1)N-1} & \rho_k^{(2)N-1} & \ldots & \rho_k^{(N)N-1}\\ \rho_k^{(1)N} & \rho_k^{(2)N} & \ldots & \rho_k^{(N)N}\end{pmatrix}. \quad (53)$$

In the WKB approximation we suppose that all elements of the matrix $M_{k+1}^{-1}H_{k+1}$ equal zero except the diagonal ones. In this case the system of equations (51) can be rewritten as

$$\begin{aligned}
&y_{i,k+1}\approx y_{i,k}\sum_{j=1}^{N}M_{k+1,i,j}^{-1}\rho_k^{(i)j}=\\
&=y_{i,k}\sum_{j=1}^{N}\left(\rho_k^{(i)j}\frac{\left(-1\right)^{j-1}\sigma_{k+1,i}^{(N-j)}}{\prod_{\substack{s=1\\ s\neq i}}^{N}\left(\rho_{k+1}^{(s)}-\rho_{k+1}^{(1)}\right)}\right).
\end{aligned} \quad (54)$$

These equations are generalization to the case of the of *N*-th-order difference equation the WKB solutions obtained for the second and third-order difference equations.

CONCLUSIONS

We presented transformations of the linear difference equation into a system of the first order difference equations. The proposed transformation gives possibility to get new forms of the N dimensional system of first order equations that can be useful for analysis of the solutions of the *N*-th-order difference equation. In particular, for the third-order linear difference equation the nonlinear second-order difference equation that plays the same role as the Riccati equation for second-order linear equation is obtained. The new form of the N dimensional system of first order equations can also be used for finding the WKB solutions of the linear difference equation with coefficients that vary sufficiently slowly with index.

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ТРАНСФОРМАЦИЯ ЛИНЕЙНОГО РАЗНОСТНОГО УРАВНЕНИЯ В СИСТЕМУ РАЗНОСТНЫХ УРАВНЕНИЙ ПЕРВОГО ПОРЯДКА

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Представлено преобразование линейного разностного уравнения N-го порядка в систему разностных уравнений первого порядка. Предложенное преобразование открывает возможность получения новых форм N-мерной системы уравнений первого порядка, которые могут быть полезны для анализа решений разностных уравнений N-го порядка. В частности, для линейного разностного уравнения третьего порядка получено нелинейное разностное уравнение второго порядка, которое играет ту же роль, что и уравнение Риккати для линейного разностного уравнения второго порядка. Новая форма N-мерной системы уравнений первого порядка также может быть использована для нахождения ВКБ-решений линейного разностного уравнения с коэффициентами, которые медленно меняются в зависимости от индекса.

ПЕРЕТВОРЕННЯ ЛІНІЙНОГО РІЗНИЦЕВОГО РІВНЯННЯ В СИСТЕМУ РІЗНИЦЕВИХ РІВНЯНЬ ПЕРШОГО ПОРЯДКУ

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Представлено перетворення лінійного різницевого рівняння N-го порядку в систему різницевих рівнянь першого порядку. Запропонована трансформація відкриває можливість отримання нових форм N-вимірної системи рівнянь першого порядку, які можуть бути корисними для аналізу рішень різницевих рівнянь N-го порядку. Зокрема, для лінійних різницевих рівнянь третього порядку отримано нелінійне різницеве рівняння другого порядку, яке відіграє ту ж саму роль, що й рівняння Ріккаті для лінійного різницевого рівняння другого порядку. Нова форма N-вимірної системи рівнянь першого порядку також може бути використана для пошуку ВКБ-рішень лінійного різницевого рівняння з коефіцієнтами, які повільно змінюються з індексом.