FULLY RELATIVISTIC APPROACH TO THE THEORY OF SLOW AND PLASMA ECRF WAVES

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The new exact integral form for the fully relativistic plasma dielectric tensor in the ECRF range is presented. This form is suitable for numerical applications for arbitrary wave numbers.

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INTRODUCTION

The theoretical study of ECRF waves in plasma requires accurate accounting relativistic effects [1, 2]. The ground for studying waves in plasmas is the exact evaluation of the fully relativistic plasma dielectric tensor. This direction in the theory of plasma waves is known as the fully relativistic approach. Two original equivalent integral forms of this tensor were given in [3]. However, their applicability was rather limited. Then gradually there appeared the non-relativistic approximation [4] and the weakly relativistic one [5].

Later, the limitations were weakened due to introducing the fully relativistic plasma dielectric tensor with the exact plasma dispersion functions (PDFs) for two branches: $0 \le N_{||} < 1$ and $1 < N_{||} < +\infty$. This tensor is given as double series in the EC harmonics and in the exact PDFs with coefficients of the Taylor expansion of the functions $A_n(\lambda) = e^{-\lambda} I_n(\lambda)$ in the parameter λ [6].

The condition λ<<1 is fulfilled in the applications of this tensor for the fast EC waves, when series in λ converges rapidly. However, for the slow or plasma EC waves λ can reach values, $\lambda \sim 1$, and λ>>1. Then the series in λ converges slower, which can cause difficulties in summation, even in the weakly relativistic case [7] and in the fully relativistic one as well [8]. Then the key task is looking for the alternative form of tensor, suitable for exact applications.

1. FUNCTIONS GENERATING FULLY RELATIVISTIC DIELECTRIC TENSOR ELEMENTS

Let us demonstrate that $\mathrm{Im}(D_{ij})$ and $\mathrm{Re}(D_{ij})$ [6] for arbitrary λ can be expressed in terms of 1-D integrals, suitable for exact applications. Really, in the case of unfavorable (for summation) value of λ for element of the tensor [6] and harmonic number n the summation for the slow convergent series can be reduced on the base the theory of Cauchy type integrals to a two-step procedure. The 1st step introduces the function generating the anti-Hermitian part of this series in λ, and then a numerical calculation of the 1-D integral of this function leads to the anti-Hermitian part of the series. The 2nd step consists of calculating numerically the principal value of the integral of Cauchy of the anti-Hermitian part, which leads to the Hemitian part of the series. Then, both parts together give a value of the whole series for the tensor element.

Let us begin this two-step procedure for the tensor element D_{11}. From the first Trubnikov integral

$$D_{11} = \frac{\mu}{2K_2(\mu)} \int_{-\infty}^{+\infty} d\bar{p}_{||} \int_0^{+\infty} \bar{p}_\perp d\bar{p}_\perp \frac{e^{-\mu\gamma}}{\gamma} \sum_{n=-\infty}^{+\infty} \frac{\frac{n^2}{\nu_\perp^2} J_n^2(\nu_\perp \bar{p}_\perp)}{\gamma - N_{||}\bar{p}_{||} - \frac{n\Omega_c}{\omega}}. \quad (1)$$

Here $K_2(\mu)$ is the Macdonald function, $\bar{p} = p/(mc)$ is normalized momentum, $\gamma = \sqrt{1+\bar{p}^2}$, $\nu_\perp^2 = \mu\lambda$. Using the last definition, (1) can be converted into

$$D_{11} = \frac{1}{2K_2(\mu)} \sum_{n=-\infty}^{+\infty} \frac{n^2}{\lambda} \int_{-\infty}^{+\infty} d\bar{p}_{||} \int_0^{+\infty} \bar{p}_\perp d\bar{p}_\perp \frac{e^{-\mu\gamma}}{\gamma} \frac{J_n^2\left(\sqrt{\lambda\mu\,\bar{p}_\perp^2}\right)}{\gamma - N_{||}\bar{p}_{||} - \frac{n\Omega_c}{\omega}}. \quad (2)$$

After a change $\bar{p}_\perp \to \gamma$ in (2) with Jacobian $d\bar{p}_\perp / d\gamma = \gamma / \bar{p}_\perp$, accounting $\bar{p}_\perp^2 = \gamma^2 - (1+\bar{p}_{||}^2)$, we have

$$D_{11} = \frac{1}{2K_2(\mu)} \sum_{n=-\infty}^{+\infty} \frac{n^2}{\lambda} \int_{-\infty}^{+\infty} d\bar{p}_{||} \int_1^{+\infty} d\gamma e^{-\mu\gamma} \frac{J_n^2\left(\sqrt{\lambda\mu[\gamma^2 - (1+\bar{p}_{||}^2)]}\right)}{\gamma - N_{||}\bar{p}_{||} - \frac{n\Omega_c}{\omega}}. \quad (3)$$

After one more change in (3) $\gamma \to x$, with $x = \gamma - \sqrt{1+\bar{p}_{||}^2}$ and Jacobian $d\gamma / dx = 1$, and accounting the relation $x = x\left(x + 2\sqrt{1+\bar{p}_{||}^2}\right) = \gamma^2 - \left(1+\bar{p}_{||}^2\right)$, we obtain

$$D_{11} = \frac{1}{2K_2(\mu)} \sum_{n=-\infty}^{+\infty} \frac{n^2}{\lambda} \int_{-\infty}^{+\infty} d\bar{p}_{||} e^{-\mu\sqrt{1+\bar{p}_{||}^2}} \int_0^{+\infty} dx e^{-\mu x} \frac{J_n^2\left(\sqrt{\lambda\mu x(x + 2\sqrt{1+\bar{p}_{||}^2})}\right)}{x + \sqrt{1+\bar{p}_{||}^2} - N_{||}\bar{p}_{||} - \frac{n\Omega_c}{\omega}}. \quad (4)$$

The integral over x in (4), with the multiplier $\exp(-\mu\sqrt{1+\bar{p}_{||}^2})$, is one of the Cauchy type with real density, satisfying the Hölder condition of continuity, and tending to 0 when $x \to 0$ and $x \to \infty$. It is known from complex analysis that the integral of the Cauchy-type

$$F(z) = \frac{1}{2\pi i} \int_0^{+\infty} \frac{\varphi(\tau) d\tau}{\tau - z} \quad (5)$$

with density $\varphi(\tau)$ satisfying the former conditions at the contour, is defined at the contour itself by the formulas of Sokhotskii-Plemelj

$$F^{\pm}(z) = \pm\frac{\varphi(z)}{2} + \frac{1}{2\pi i} P\int_0^{+\infty} \frac{\varphi(\tau)d\tau}{\tau - z}, \quad (z \geq 0). \qquad (6)$$

Here, the functions $F^{+}(z)$, $F^{-}(z)$ are the boundary values of integral (6) when z tends to the contour from the right or from the left-hand side with respect to the integration direction, respectively. The letter P before integral denotes its principal value in the Cauchy sense. At the real axis, out of the contour, the integral (6) is not singular and is, consequently,

$$F^{\pm}(z) = \pm\frac{\varphi(z)}{2} + \frac{1}{2\pi i} P\int_0^{+\infty} \frac{\varphi(\tau)d\tau}{\tau - z}, \quad (z < 0). \qquad (7)$$

Thus, for the case $\sqrt{1+\bar{p}_{\parallel}^2} - N_{\parallel}\bar{p}_{\parallel} - n\Omega_c/\omega \leq 0$ the anti-Hermitian part of the integral over x in (4), with the multiplier $\exp(-\mu\sqrt{1+\bar{p}_{\parallel}^2})$, can be obtained by substituting the anti-Hermitian part $-\pi i\varphi(-\sqrt{1+\bar{p}_{\parallel}^2} + N_{\parallel}\bar{p}_{\parallel} + n\Omega_c/\omega)$ of the second of the formulae (7), times $2\pi i$, which corresponds to the Landau rule for passing the pole, instead of the Cauchy integral in (4). In this way

$$e^{-\mu\sqrt{1+\bar{p}_{\parallel}^2}} \int_0^{+\infty} dx e^{-\mu x} \frac{J_n^2\left(\sqrt{\lambda\mu x\left(x + 2\sqrt{1+\bar{p}_{\parallel}^2}\right)}\right)}{x + \sqrt{1+\bar{p}_{\parallel}^2} - N_{\parallel}\bar{p}_{\parallel} - n\Omega_c/\omega} = Hermitian \quad part -$$
$$\pi i\, e^{-\mu(N_{\parallel}\bar{p}_{\parallel} + n\Omega_c/\omega)} J_n^2\left[\sqrt{\lambda\mu\left((N_{\parallel}\bar{p}_{\parallel} + n\Omega_c/\omega)^2 - 1 - \bar{p}_{\parallel}^2\right)}\right]. \qquad (8)$$

In (8), it is convenient to use the arguments: $x = \bar{p}_{\parallel}\sqrt{\mu/2}$, $z = \mu(1 - n\Omega_c/\omega)$, $a = \mu N_{\parallel}^2/2$. Then (8) is

$$Hermitian \quad part -$$
$$\pi i\, e^{-2\sqrt{a}x + z - \mu} J_n^2\left\{\sqrt{-2\lambda[z - 2\sqrt{a}x + x^2 - (z - 2\sqrt{a}x)^2/(2\mu)]}\right\}. \qquad (9)$$

From (4) and (9) for the case $0 \leq N_{\parallel} < 1$ it follows

$$\operatorname{Im} D_{11} = \begin{cases} -\dfrac{1}{\lambda}\displaystyle\sum_{n=-\infty}^{+\infty} n^2 \frac{\pi e^{z-\mu}}{\sqrt{2\mu}K_2(\mu)} \int_{x^-}^{x^+} dx e^{-2\sqrt{a}x} \times \\ J_n^2\left\{\sqrt{-2\lambda[z - 2\sqrt{a}x + x^2 - (z - 2\sqrt{a}x)^2/(2\mu)]}\right\} & z < a^* \\ 0, & z \geq a^*. \end{cases} \qquad (10)$$

Here were used the notation $a^* = \mu(1 - \sqrt{1 - N_{\parallel}^2})$ and the integration limits follow from the condition that the pole must appear inside the integral of expression (9), which is equivalent to the equation $\sqrt{1 + 2x^2/\mu} - 2\sqrt{a}x/\mu + z/\mu - 1 = 0$. Consequently the limits are $x^{\pm} = \beta[\sqrt{a}(1 - z/\mu) \pm (a - z + z^2/(2\mu))]$, where $\beta = 1/(1 - N_{\parallel}^2)$. Thus, we obtain an alternative expression to (1) for the cyclotron harmonic n and for the anti-Hermitian part of the element D_{11}. For this expression, the summation of the series in λ is reduced to the evaluation of the one-dimensional integral (10). We perform now the change $x \to t$ in accordance to $t = x - \beta\sqrt{a}(1 - z/\mu)$, which leads to the symmetry of the new limits of integration $t^{\pm} = \pm\beta\sqrt{a - z + z^2/(2\mu)}$ respectively 0. Then, one more change $t \to u$ with $t = u\beta\sqrt{a - z + z^2/(2\mu)}$ leads to the normalization of integration limits to $u^{\pm} = \pm 1$

$$\operatorname{Im} D_{11} = \begin{cases} -\dfrac{\pi}{\sqrt{2\mu\lambda}} \displaystyle\sum_{n=-\infty}^{+\infty} n^2 \frac{e^{\beta(z-2a)-\mu}}{K_2(\mu)} K \int_{-1}^{1} du\, e^{-2\sqrt{a}Ku} \times \\ J_n^2\left\{K\sqrt{2\lambda(1-u^2)/\beta}\right\}, & z < a^* \\ 0, & z \geq a^*, \end{cases} \qquad (11)$$

where we used $K = \beta\sqrt{a - z + z^2/(2\mu)}$ for brevity.

For the case $1 < N_{\parallel} < +\infty$ from (10) for arbitrary z in the similar way it follows

$$\operatorname{Im} D_{11} = -\frac{\pi}{\sqrt{2\mu\lambda}} \sum_{n=-\infty}^{+\infty} n^2 \frac{e^{\beta(z-2a)-\mu}}{K_2(\mu)} K^* \int_1^{+\infty} du\, e^{-2\sqrt{a}K^* u} \times$$
$$J_n^2\left\{K^*\sqrt{2\lambda(1-u^2)/(-\beta)}\right\}, \qquad (12)$$

where was used $K^* = (-\beta)\sqrt{a - z + z^2/(2\mu)}$. The 1-D integrals (11) and (12) are not singular, and can be exactly calculated for arbitrary λ. Their calculation completes the 1st step of the two-step numerical procedure mentioned above.

The hermitian part of D_{11} can be evaluated from the imaginary parts (11) and (12) using one of Kramers-Kronig formulas, linking Hermitian and anti-Hermitian parts of Cauchy type integral defined on the real axis and obeying the Landau rule of passing the pole [1],

$$\operatorname{Re}\, D_{11}(a, z, \mu) = \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\operatorname{Im} D_{11}(a, t, \mu)dt}{t - z}. \qquad (13)$$

After inputting (11) into (13) one will have

$$\operatorname{Re}\, D_{11}(a, z, \mu) = P\int_{-\infty}^{a^*} \frac{\operatorname{Im} D_{11}(a, t, \mu)dt}{\pi(t - z)} = -P\int_0^{+\infty} \frac{\operatorname{Im} D_{11}(a, a^* - u, \mu)du}{\pi(u - (a^* - z))}, \qquad (14)$$

after the change $t = a^* - u$, where the contour of integration passes above the pole, obeying the Landau rule. To move in (14) to the standard coordinates it is necessary to change variables $z = 2\sqrt{a}z_n$. Thus, to calculate D_{11} for a cyclotron harmonic n, the evaluation of the series in λ can be reduced to the 1-D integral (11) and then to the principal value of the integral (14).

2. CALCULATION OF THE FULLY RELATIVISTIC PLASMA DIELECTRIC TENSOR ELEMENT ε_{11} FOR $\lambda = 3$

Since all elements of D_{ij} are of the same type and differ only in weighted multipliers, it is sufficient to describe in detail the computation of the tensor element

D_{11} for the harmonic $n=1$. Let us demonstrate applications of the formulae (11) for calculations of this element for the branch $N_{\|}<1$. From (11) one can numerically evaluate $\operatorname{Im} D_{11}$ with the desired accuracy using a standard program for computing a nonsingular 1-D integral over a finite interval for arbitrary value of the parameter λ.

The Hermitian part of the component D_{11} can also be evaluated numerically from the $\operatorname{Im} D_{11}$ using one of Kramers-Kronig formulas (17), linking Hermitian and anti-Hermitian parts of an integral of Cauchy type defined on the real axis and obeying the Landau rule of passing the pole above it [6]. To move to the standard coordinates in (17) it is necessary to make the change $z=2\sqrt{a}z_n$. Thus, calculation of the element D_{11}, which is equivalent to the evaluation of the series in λ in the expression (2), can be reduced to the calculation of the principal value of (17). Introducing in (17) the notations $-\operatorname{Im} D_{11}(a, a^*-u, \mu)/\pi = f(u)$ and $b=a^*-z$ one will have the following integral of Cauchy type

$$\operatorname{Re}\, D_{11}(a,z,\mu) = P\int_0^{+\infty} \frac{f(u)du}{u-b}. \tag{15}$$

For the evaluation of the singular integral (15) for the case $b>0$ it was used the following non-singular integral form which was used earlier for evaluating of exact relativistic PDFs for the case $N_{\|}<1$ [6]

$$P\int_0^{+\infty} \frac{f(u)du}{u-b} = -\int_{-\infty}^{0} \frac{f(2b-u)du}{u-b} + \int_0^{b} \frac{\left(f(u)-f(2b-u)\right)du}{u-b}. \tag{16}$$

Indeed, the first integral on the right-hand side is not singular, since the constant b does not belong to the integration interval. In the second integral in the same part, the constant b is the right-hand end of the integration interval $[0,b]$, but it can be shown that for $u \to b$ the integrand $\left(f(u)-f(2b-u)\right)/(u-b) \to 2f'(b)$, that is, it is finite at a point b, and hence there is no singularity at this point either.

For the case $b<0$, the point b also does not fall into the integration interval $[0,\infty]$ and hence the integral on the left-hand side of (6) is not singular and hence

$$P\int_0^{+\infty} \frac{f(u)du}{u-b} = \int_0^{+\infty} \frac{f(u)du}{u-b}. \tag{17}$$

It remains to consider the case $b=0$ in (17). But in this case, for $u \to 0$ the integrand function $f(u) \to 0$ as well, since $\operatorname{Im} D_{11}(a,a^*,\mu)=0$ (see (3)) and therefore $f(u)/u \to f''(0)$, according to Lompital rule, and the singularity at $u=0$ on the right in (17) is also absent.

Fig. 1 shows the graph of the anti-Hermitian and Hermitian parts of the element $D_{11}(\boldsymbol{k}, z_1, \mu)$ of the fully relativistic tensor obtained with the use of expressions (4), (5) with relative accuracy 10^{-10}. Calculations were carried out for the following plasma and wave parameters: electron temperature $T_e = 20\,keV$, $\lambda = 3$, $N_{\|} = 0.3$. These dependences are qualitatively very similar to the analogous dependences for the anti-Hermitian and Hermitian parts of the exact relativistic plasma dispersion function for plasma and waves with the same parameters.

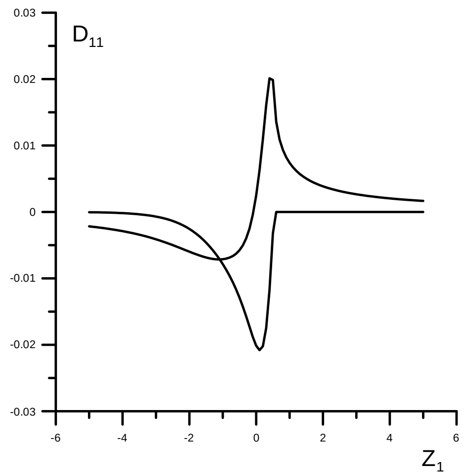

Fig. 1. The real and imaginary tensor components D_{11} for longitudinal refractive index, $N_{\|} = 0.3$ and temperature T_e = 20 keV

For the case $N_{\|}>1$ a scheme for calculation D_{11} of the relativistic plasma dielectric tensor remains very similar previous one except for some features. First, instead of integrals of Cauchy type, all the elements of the tensor are expressed in terms of Cauchy integrals and are smoother analytic functions that do not contain branch points as in the previous case. We will calculate the element D_{11} on the basis of the calculation technique used to calculate fully relativistic PDFs for the case $N_{\|}>1$ [6]. The 1-D integrals in the first formula of (10) are not singular, and therefore ones can be calculated with desired precision for arbitrary value of the parameter λ. Their calculation completes the first step of the two-step numerical procedure.

The Hermitian part of D_{11} can also be numerically evaluated from the imaginary part on the base (11) using one of Kramers-Kronig formulas [6]. A change to the standard coordinates, usual in the non-relativistic case, can be made in (7) through the relation $z=2\sqrt{a}z_n$. Introducing in the formula (17) the notations $\operatorname{Im} D_{11}(a,t,\mu)/\pi = \varphi(t)$ and $b=z$ one will have the following typical Cauchy integral

$$\operatorname{Re}\, D_{11}(a,z,\mu) = P\int_{-\infty}^{+\infty} \frac{\varphi(t)dt}{t-b}. \tag{18}$$

For the exact evaluation of the singular integral (18) it was used the following non-singular integral form which was used earlier for evaluating of exact relativistic PDFs for the case $N_{\|}>1$ [6]

$$\operatorname{Re}\, D_{11}(a,z,\mu) = P\int_0^{+\infty} \frac{f(u)du}{u-b} = \int_{-\infty}^{+\infty} \frac{f(2b-u)du}{u-b}, \tag{19}$$

where the contour of integration passes below the pole.

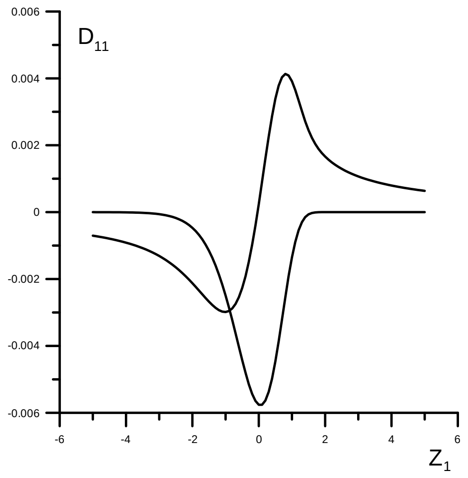

Fig. 2. The same as Fig. 1, but for $N_\|$=1.1 *and temperature* T_e *= 40 keV*

Fig. 2 shows the graph of the anti-Hermitian and Hermitian parts of the element D_{11} of the fully relativistic tensor obtained with the use of expressions (18), (19) with accuracy 10^{-10}. Calculations were carried out for the following plasma and wave parameters: electron temperature T_e = 40 keV, λ = 3, $N_\|$=1.1.

CONCLUSIONS

1. The 1-D integral form for the fully relativistic plasma dielectric tensor is presented. This form is suitable for exact numerical applications for the value of the parameter $\lambda>1$.

2. This form is interesting for applications to study electron Bernstein waves in the ECRF range in thermonuclear plasmas and to study arbitrary fast and slow ECRF waves in hot astrophysical plasmas.

3. In a similar way, it can be obtained the similar exact 1-D integral form of the fully relativistic plasma dielectric tensor for ions.

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ПОЛНОСТЬЮ РЕЛЯТИВИСТСКИЙ ПОДХОД К ТЕОРИИ МЕДЛЕННЫХ И ПЛАЗМЕННЫХ ЭЦР ВОЛН

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Предлагается новая точная одномерная интегральная форма для вычисления полностью релятивистского тензора диэлектрической проницаемости плазмы.

ПОВНІСТЮ РЕЛЯТИВІСТСЬКИЙ ПІДХІД ДО ТЕОРІЇ ПОВІЛЬНИХ І ПЛАЗМОВИХ ЕЦР ХВИЛЬ

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Пропонується нова точна одномірна інтегральна форма для обчислення повністю релятивістського тензора діелектричної проникності плазми.