On the uniform Besov regularity of local times of general processes

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October 22, 2021

Abstract

Our main purpose is to use a new condition, *α-local nondeterminism*, which is an alternative to the classical *local nondeterminism* usually utilized in the Gaussian framework, in order to investigate Besov regularity, in the time variable t uniformly in the space variable x, for local times $L(x,t)$ of a class of continuous processes. We also extend the classical Adler's theorem [1, Theorem 8.7.1] to the Besov spaces case. These results are then exploited to study the Besov irregularity of the sample paths of the underlying processes. Based on similar known results in the case of the bifractional Brownian motion, we believe that our results are sharp. As applications, we get sharp Besov regularity results for some classical Gaussian processes and the solutions of systems of non-linear stochastic heat equations. The Besov regularity of their corresponding local times is also obtained.

Keywords: Local times ; Besov spaces ; Sample path properties ; Fourier analysis ; Adler's theorem

Mathematics Subject Classification (2010): 60G17 ; 60J55 ; 30H25

1 Introduction and main results

The local times of d-dimensional paths have gotten much interest during the last few decades by the analytical and probabilistic communities. The occupation measure basically measures the amount of time the path spends in a given set. The local time is defined as the Radon-Nikodym derivative of the occupation measure with respect to the Lebesgue measure. It is of importance in both theory and applications to investigate sample path properties of stochastic processes. One way of studying the irregularity properties of the sample paths of stochastic processes is by analyzing the smoothness of their local times. S. Berman has initiated this approach in a series of papers [3, 4, 5] by using Fourier analytic methods to Gaussian processes. Furthermore, he has introduced the concept of local nondeterminism (LND) for Gaussian processes to investigate the existence of jointly continuous local times. Since then, there have been a wide variety of extensions of the notion of local nondeterminism for Gaussian and stable processes, e.g. [21, 14, 20]. In the Gaussian case, the exponential form of the characteristic function allows expressing the LND property in terms of a condition on the variance. Nevertheless, the unknown form of the characteristic functions of general processes leads to difficulties in extending the LND condition beyond the Gaussian framework. Consequently, the LND property used in the Gaussian context should be replaced, for general processes, by fine estimations on the characteristic function of the increments. Recently, based on the conditional Malliavin calculus, Lou and Ouyang [19] have established an upper bound of Gaussian type for the partial derivatives of the n-point joint density of the solution to a stochastic differential equation driven by fractional Brownian motion. They have used this result as an alternative to the LND condition. Due to this, the authors in [19] have shown the existence and regularity of the local times of stochastic differential equations driven by fractional Brownian motions. In [11], a new condition, called α-local nondeterminism (α-LND for short), has been introduced to investigate the joint regularity of the local times of the solutions to systems of non-linear stochastic heat equations – which are neither Gaussian nor stable processes. In the Gaussian context, the α-LND property is particularly seen as a weaker condition than the classical LND (see Remark 2.9). Generally, the proof of the α-LND condition relies on the technique of integration by parts derived from the Malliavin calculus. Roughly speaking, we believe that the approach presented in [11] can be used to establish the α-LND for a class of adapted stochastic processes that are smooth in the Malliavin sense.

Consider $(X_t)_{t\in[0,1]}$ an $\mathbb{R}^d$-valued continuous stochastic process, such that $X(0)=0$. Assume that X satisfies the α-LND with $\alpha\in(0,1)$, see Definition 2.8. Let us state the following hypothesis on X:

H There exists $p_0>\frac{1}{\alpha}$ and $K>0$ such that for all $0\le s\le t\le 1$,

$$\mathbb{E}[\|X_t-X_s\|^{p_0}]\le K|t-s|^{p_0\alpha}. \tag{1.1}$$

One can get by the same calculations as in [11] (see [11, Remark 5.6]) that, when $\alpha<\frac{1}{d}$, X has a jointly continuous version of the local time, $L(x,t)$, satisfying almost surely a Hölder condition of order $\gamma<1-d\alpha$, in the time variable t uniformly in the space variable x. One of the main aims of this article is to improve this uniform Hölder continuity of $L(x,\bullet)$ to regularity in the Besov spaces – we refer to Subsection 2.1 for some notions on Besov spaces. There are well-known Besov regularity results, in the space variable x for fixed t, of the local times $L(x,t)$ of some classical Gaussian and stable processes, e.g. [13, 12]. However, to the best of our knowledge, there is no work in the literature treating the Besov regularity of $L(x,\bullet)$ even for the Gaussian or stable processes. To fill this gap, we use the α-LND condition to investigate the Besov regularity, in the time variable t uniformly in the space variable x, for local times of general processes. Our first main result is:

Theorem 1.1. *Let $X=(X_t)_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued continuous stochastic process which is α-LND with $\alpha\in(0,\frac{1}{d})$. Assume also that X verifies* **H**. *Denote by $L(x,t)$ the jointly continuous version of the local time of X. Then, almost surely, for any $1\le p<\infty$,*

$$\sup_{0<t\le 1} t^{-(1-d\alpha)}\sup_{|h|\le t}\|r\mapsto \sup_{x\in\mathbb{R}^d}|L(x,r+h)-L(x,r)|\|_{L^p(I(h);\mathbb{R})}<\infty. \tag{1.2}$$

In particular, we have the following Besov regularity

$$\mathbb{P}\left[L(x,\bullet)\in \mathbf{B}^{1-d\alpha}_{p,\infty}(I;\mathbb{R}),\quad \text{for all } x\in\mathbb{R}^d \text{ and } p\in[1,\infty)\right]=1, \tag{1.3}$$

where $I=[0,1]$, $I(h)=\{x\in I\,;\ x+h\in I\}$, and $L(x,\bullet):t\in I\mapsto L(x,t)$.

As mentioned above, we know that the regularity of the local time is linked to the irregular behavior of the sample paths of the corresponding process. Recall that Adler's theorem, [1, Theorem 8.7.1], establishes a connection between the Hölder continuity, in the time variable t uniformly on the space variable x, of the local time and the Hölder irregularity of its underlying function, as follows:

Theorem 1.2 (Adler). *Let $(f(t))_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued continuous function possessing a local time, $L(x,t)$, satisfying: there exist c and ρ positive and finite constants, such that for all $t, t+h \in [0,1]$ and all $|h| < \rho$,*

$$\sup_{x\in\mathbb{R}^d} |L(x,t+h) - L(x,t)| \leq c|h|^{\mu}, \tag{1.4}$$

for $0 < \mu < 1$. Then, all coordinate functions of f are nowhere Hölder continuous of order greater than $(1-\mu)/d$.

One of the goals of this article is to provide a similar theorem to that of Adler in the case of Besov spaces. We have the following theorem:

Theorem 1.3. *Let $(f(t))_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued continuous function possessing a local time, $L(x,t)$, satisfying for some $\mu \in (0,1)$ and $p \in (d/(1-\mu), \infty)$,*

$$\sup_{0<t\leq 1} t^{-\mu/d} \sup_{|h|\leq t} \|s \mapsto \sup_{x\in\mathbb{R}^d} |L(x,s+h) - L(x,s)|^{\frac{1}{d}}\|_{L^{\frac{p}{(p-1)}}(I(h);\mathbb{R})} < \infty. \tag{1.5}$$

Then, the function f does not belong to the Besov space $\mathbf{B}^{(1-\mu)/d,0}_{p,\infty}(I,\mathbb{R}^d)$, where $I = [0,1]$ and $I(h) = \{x \in I\,;\ x+h \in I\}$.

Based on the above results, we note that the Besov regularity, in the time variable t uniformly on the space variable x, of the local times $L(x,t)$ associated with α-LND stochastic processes is valuable, as this knowledge can be applied towards Besov irregularity of the underlying processes. Therefore, as a consequence of Theorem 1.1 and 1.3, we will obtain the following theorem:

Theorem 1.4. *Let $(X_t)_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued continuous stochastic process, $X(0) = 0$, which is α-LND with $\alpha \in (0, \frac{1}{d})$. Assume also that X verifies* **H**. Then

$$\mathbb{P}\left[X(\bullet) \in \mathbf{B}^{\alpha,0}_{p,\infty}(I,\mathbb{R}^d)\,, \quad \text{for some } p \in (1/\alpha, \infty)\right] = 0, \tag{1.6}$$

where $I = [0,1]$ and $X(\bullet) : t \in I \mapsto X_t \in \mathbb{R}^d$.

According to the following continuous injections

$$\mathbf{B}^{\alpha}_{p,q}(I,\mathbb{R}^d) \hookrightarrow \mathbf{B}^{\alpha,0}_{p,\infty}(I,\mathbb{R}^d), \qquad 1 \leq q < \infty,$$

we get:

Corollary 1.5. *Let $(X_t)_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued continuous stochastic process, $X(0)=0$, which is α-LND with $\alpha\in(0,\frac{1}{d})$. Assume also that X verifies* **H**. Then

$$\mathbb{P}\left[X(\bullet)\in\mathbf{B}^{\alpha}_{p,q}(I,\mathbb{R}^d)\,,\ \text{ for some } p\in(1/\alpha,\infty) \text{ and } q\in[1,\infty)\right]=0, \quad (1.7)$$

where $I=[0,1]$ and $X(\bullet): t\in I\mapsto X_t\in\mathbb{R}^d$.

It appears that Theorem 1.4, for $d=1$, covers some well-known results in Roynette [22] and Boufoussi and Nachit [9, 10]. We illustrate Theorem 1.4 and 1.1 with some examples. We infer the uniform Besov regularity of the local times, as well as the Besov irregularity, of the following α-LND processes: First, we consider an $\mathbb{R}^d$-valued Gaussian process $(Y_t)_{t\in[0,1]}$, with $Y_t=(Y^1,\dots,Y_t^d)$, where $Y^1,\dots,Y_t^d$ are independent copies of a real-valued LND Gaussian process $(Y_t^0)_{t\in[0,1]}$, satisfying for some $\alpha\in(0,1)$ and finite constants $c,C>0$,

$$c(t-s)^{2\alpha}\leq \operatorname{Var}\left(Y_t^0-Y_s^0\right)\leq C(t-s)^{2\alpha}, \quad (1.8)$$

for every $0\leq s<t\leq 1$. Notice that the d-dimensional bifractional Brownian motion verifies the above conditions. Secondly, as an example of non-Gaussian and non-stable processes, we regard systems of non-linear stochastic heat equations.

In this article, to deal with Besov spaces, we are based on representations of the Besov norms in terms of dyadic expansion coefficients of a given function. These descriptions of the Besov norms are derived from [18, 3.b.9 Corollary]. To the best of our knowledge, the treatise of König [18] has been used for the first time to investigate Besov regularity of stochastic processes by Hytönen and Veraar [17].

The rest of the paper is organized as follows. In the second section, we write some preliminary results on Besov spaces and local times. The third section is devoted to the proofs of the main results. In the fourth section, we give some examples.

Finally, we point out that constants in our proofs may change from line to line. For a process $X=(X_t)_{t\in[0,1]}$, sometimes if necessary, we write $X(t)$ instead of X_t.

2 Preliminaries

2.1 Besov spaces

For the definition of the real-valued Besov spaces, we refer to [23], and for the vector-valued Besov spaces, we suggest the treatise [18]. Let $I = [0,1]$, for any $h \in \mathbb{R}$, we put $I(h) = \{x \in I\,;\ x + h \in I\}$. Let $1 \le p < \infty$ and $\nu \in (0,1)$, the modulus of continuity of a function $f \in L^p(I;\mathbb{R}^d)$ is defined by

$$\omega_p(f,t) = \sup_{|h|\le t} \|x \mapsto f(x+h) - f(x)\|_{L^p(I(h);\mathbb{R}^d)}. \tag{2.1}$$

We define the vector-valued Besov space $\mathbf{B}^{\nu}_{p,\infty}(I;\mathbb{R}^d)$ as the space of all functions $f \in L^p(I;\mathbb{R}^d)$ such that the seminorm $\mathcal{N}_{\nu,p}(f) := \sup_{0<t\le 1} t^{-\nu}\omega_p(f,t)$ is finite. $\mathbf{B}^{\nu}_{p,\infty}(I;\mathbb{R}^d)$ endowed with the sum of the L^p-norm and the seminorm $\mathcal{N}_{\nu,p}$ is a Banach space. Let $\mathbf{B}^{\nu,0}_{p,\infty}(I,\mathbb{R}^d)$ be the space of all functions $f \in \mathbf{B}^{\nu}_{p,\infty}(I;\mathbb{R}^d)$ for which $\lim_{t\to 0^+} t^{-\nu}\omega_p(f,t) = 0$. Using a dyadic approximation argument (see the lemma in page 173 and Corollary 3.b.9 in [18]) one has the following theorem:

Theorem 2.1. *Let $1 \le p < \infty$ and $\nu \in (0,1)$. We have*

(i) *The seminorm $\mathcal{N}_{\nu,p}$ is equivalent to*

$$\|f\|_{\nu,p} := \sup_{j\ge 0} 2^{j\nu} \|x \mapsto f(x+2^{-j}) - f(x)\|_{L^p(I(2^{-j});\mathbb{R}^d)}. \tag{2.2}$$

(ii) *Let $f \in L^p(I;\mathbb{R}^d)$. Then f is in $\mathbf{B}^{\nu,0}_{p,\infty}(I,\mathbb{R}^d)$ if and only if*

$$\lim_{j\to\infty} 2^{j\nu} \|x \mapsto f(x+2^{-j}) - f(x)\|_{L^p(I(2^{-j});\mathbb{R}^d)} = 0. \tag{2.3}$$

(iii) *Let $1 \le p < \infty$ and $0 < \nu < 1$. Then there exists a positive and finite constant c such that for all g from $\mathbb{R}^d \times [0,1]$ to $\mathbb{R}$ jointly continuous function with compact support,*

$$\begin{aligned} &c^{-1} \sup_{0<t\le 1} t^{-\nu} \sup_{|h|\le t} \|r \mapsto \sup_{x\in\mathbb{R}^d} |g(x,r+h) - g(x,r)|\|_{L^p(I(h);\mathbb{R})} \\ &\le \sup_{j\ge 0} 2^{j\nu} \|r \mapsto \sup_{x\in\mathbb{R}^d} |g(x,r+2^{-j}) - g(x,r)|\|_{L^p(I(2^{-j});\mathbb{R})} \\ &\le c \sup_{0<t\le 1} t^{-\nu} \sup_{|h|\le t} \|r \mapsto \sup_{x\in\mathbb{R}^d} |g(x,r+h) - g(x,r)|\|_{L^p(I(h);\mathbb{R})}. \end{aligned} \tag{2.4}$$

(iv) *Let $1 \le p < \infty$, $0 < \nu < 1$, and f be an $\mathbb{R}^d$-valued continuous function. Then*

$$\lim_{t\to 0^+} t^{-\nu} \sup_{|h|\le t} \|x \mapsto \sup_{y,z\in[0,1]} \|f(hz+x) - f(hy+x)\|\|_{L^p(I(h),\mathbb{R})} = 0,$$

if and only if

$$\lim_{j\to\infty} 2^{j\nu} \|x \mapsto \sup_{y,z\in[0,1]} \|f(2^{-j}z+x) - f(2^{-j}y+x)\|\|_{L^p(I(2^{-j});\mathbb{R})} = 0.$$

Now we will introduce the spaces $\mathbf{b}^{\nu,0}_{p,\infty}(I;\mathbb{R}^d)$, which will play a key role in the proof of Theorem 1.3. Let $I = [0,1]$, $1 \le p < \infty$, and $\nu \in (0,1)$, $\mathbf{b}^{\nu,0}_{p,\infty}(I;\mathbb{R}^d)$ is defined as the space of all functions $f \in C(I;\mathbb{R}^d)$, where $C(I;\mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued continuous functions, such that

$$\lim_{j\to\infty} 2^{j\nu} \|x \mapsto \sup_{y,z\in[0,1]} \|f(2^{-j}z+x) - f(2^{-j}y+x)\|\|_{L^p(I(2^{-j});\mathbb{R})} = 0, \quad (2.5)$$

here $I(2^{-j}) = \{x \in I\,;\ x + 2^{-j} \in I\}$. In the below theorem, we will give the relation between the spaces $\mathbf{b}^{\nu,0}_{p,\infty}(I;\mathbb{R}^d)$ and the classical spaces $\mathbf{B}^{\nu,0}_{p,\infty}(I,\mathbb{R}^d)$.

Theorem 2.2. *Let $I = [0,1]$, $\nu \in (0,1)$, and $\frac{1}{\nu} < p < \infty$. Then*

$$\mathbf{b}^{\nu,0}_{p,\infty}(I;\mathbb{R}^d) = C(I;\mathbb{R}^d) \cap \mathbf{B}^{\nu,0}_{p,\infty}(I,\mathbb{R}^d).$$

At this point, to prove the above theorem, we need the Garsia-Rodemich-Rumsey inequality [15].

Lemma 2.3. *Let $\Psi(u)$ and $p(u)$ be non-negative even functions respectively on $\mathbb{R}$ and $[-1,1]$ with $p(0) = 0$ and $\Psi(\infty) = \infty$. Assume that $p(u)$ and $\Psi(u)$ are non decreasing for $u \ge 0$ and $p(u)$ is continuous. Let $g(x)$ be continuous on $[0,1]$ and suppose that*

$$\int_0^1 \int_0^1 \Psi\left(\frac{g(v)-g(w)}{p(v-w)}\right) dv\, dw \le B < \infty.$$

Then, for all $z, y \in [0,1]$,

$$|g(z) - g(y)| \le 8 \int_0^{|z-y|} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u).$$

Proof of Theorem 2.2. We just need to prove that $C(I;\mathbb{R}^d)\cap \mathbf{B}^{\nu,0}_{p,\infty}(I,\mathbb{R}^d)\subseteq \mathbf{b}^{\nu,0}_{p,\infty}(I;\mathbb{R}^d)$, since the second inclusion is trivial. Let $f=(f_1,\dots,f_d)\in C(I;\mathbb{R}^d)\cap\mathbf{B}^{\nu,0}_{p,\infty}(I,\mathbb{R}^d)$, $\Psi(u)=|u|^p$, and $p(u)=|u|^{\nu+\frac{\beta}{p}}$, where $\beta\in[0,1)$ such that $\nu p-1>1-\beta$. Hence by Fubini's theorem, changes of variables, and the fact that $f\in\mathbf{B}^{\nu}_{p,\infty}(I,\mathbb{R}^d)$, we have for any $1\le i\le d$,

$$B(i,j,x):=\int_0^1\int_0^1\frac{|f_i(2^{-j}v+x)-f_i(2^{-j}w+x)|^p}{|v-w|^{\nu p+\beta}}dv\,dw<\infty. \qquad (2.6)$$

Therefore, by Lemma 2.3 we get for all $1\le i\le d$ and $z,y\in[0,1]$,

$$\begin{aligned}&|f_i(2^{-j}z+x)-f_i(2^{-j}y+x)|^p\\&\le C_{\nu,\beta,p}|z-y|^{\nu p+\beta-2}\int_0^1\int_0^1\frac{|f_i(2^{-j}v+x)-f_i(2^{-j}w+x)|^p}{|v-w|^{\nu p+\beta}}dv\,dw\\&\le C_{\nu,\beta,p}\int_0^1\int_0^1\frac{|f_i(2^{-j}v+x)-f_i(2^{-j}w+x)|^p}{|v-w|^{\nu p+\beta}}dv\,dw,\end{aligned}$$

where $C_{\nu,\beta,p}=8^p4p^p/(\beta+p\nu-2)^p$. Hence, for all $1\le i\le d$,

$$\begin{aligned}&\sup_{z,y\in[0,1]}|f_i(2^{-j}z+x)-f_i(2^{-j}y+x)|^p\\&\le C_{\nu,\beta,p}\int_0^1\int_0^1\frac{|f_i(2^{-j}v+x)-f_i(2^{-j}w+x)|^p}{|v-w|^{\nu p+\beta}}dv\,dw,\end{aligned}$$

Therefore, for all $1\le i\le d$,

$$\begin{aligned}&2^{jp\nu}\int_{I(2^{-j})}\sup_{z,y\in[0,1]}|f_i(2^{-j}z+x)-f_i(2^{-j}y+x)|^pdx\\&\le C_{\nu,\beta,p}\int_{[0,1]^2}\frac{2^{jp\nu}}{|v-w|^{\nu p+\beta}}\int_0^{1-2^{-j}}|f_i(2^{-j}v+x)-f_i(2^{-j}w+x)|^pdx\,dv\,dw\\&=C_{\nu,\beta,p}\int_{[0,1]^2}\frac{2^{jp\nu}}{|v-w|^{\nu p+\beta}}\int_{2^{-j}w}^{1-2^{-j}+2^{-j}w}|f_i(x+2^{-j}(v-w))-f_i(x)|^pdx\,dv\,dw\\&\le C_{\nu,\beta,p}\int_{[0,1]^2}\frac{2^{jp\nu}}{|v-w|^{\nu p+\beta}}\int_{I(2^{-j}(v-w))}|f_i(x+2^{-j}(v-w))-f_i(x)|^pdx\,dv\,dw,\end{aligned} \qquad (2.7)$$

where $I(2^{-j}(v-w))=\{x\in[0,1]\,;\;x+2^{-j}(v-w)\in[0,1]\}$. By the definition of $f\in\mathbf{B}^{\nu,0}_{p,\infty}(I,\mathbb{R}^d)$, we have for all $v,w\in[0,1]$ with $v\ne w$,

$$\lim_{j\to\infty}\frac{2^{jp\nu}}{|v-w|^{\nu p+\beta}}\int_{I(2^{-j}(v-w))}|f_i(x+2^{-j}(v-w))-f_i(x)|^pdx=0.$$

Therefore, By Lebesgue's dominated convergence theorem the right hand side of (2.7) converges to 0. Which concludes the proof of Theorem 2.2. $\square$

Remark 2.4. Let $\mathbf{B}^{\nu}_{p,q}(I,\mathbb{R}^d)$, for $1 \le p,q < \infty$ and $\nu \in (0,1)$, be the Besov spaces defined as follows:

$$\mathbf{B}^{\nu}_{p,q}(I,\mathbb{R}^d) := \{f \in L^p(I,\mathbb{R}^d)\,;\ \|f\|_{\nu,p,q} < \infty\},$$

where by [18, 3.b.9 Corollary] we have

$$\|f\|_{\nu,p,q} := \left\{\sum_{j\ge 0} 2^{jq\nu}\|x \mapsto f(x+2^{-j}) - f(x)\|^q_{L^p(I(2^{-j});\mathbb{R}^d)}\right\}^{\frac{1}{q}}.$$

Therefore, we have the following continuous injection: for all $1 \le p,q < \infty$ and $\nu \in (0,1)$,

$$\mathbf{B}^{\nu}_{p,q}(I,\mathbb{R}^d) \hookrightarrow \mathbf{B}^{\nu,0}_{p,\infty}(I,\mathbb{R}^d). \tag{2.8}$$

2.2 The local times

This section is devoted to give some aspects of the theory of local times. For more details on the subject, we refer to the survey of Geman and Horowitz [16].

Let $(\varphi_t)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued Borel function. For any Borel set $B \subseteq [0,T]$, the occupation measure of φ on B is given by the following measure on $\mathbb{R}^d$:

$$\nu_B(\bullet) = \lambda\{t \in B\,;\ \varphi_t \in \bullet\},$$

where λ is the Lebesgue measure. When ν_B is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, λ_d, we say that the local time of φ on B exists and it is defined as the Radon-Nikodym derivative of ν_B with respect to λ_d, i.e., for almost every x,

$$L(x,B) = \frac{d\nu_B}{d\lambda_d}(x).$$

In the above, we call B the time variable and x the space variable. We write $L(x,t)$ and $L(x)$ instead of respectively $L(x,[0,t])$ and $L(x,[0,T])$.

The local time fulfills the following occupation formula: for any Borel set $B \subseteq [0,T]$, and for every measurable bounded function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_B f(\varphi_s)ds = \int_{\mathbb{R}^d} f(x)L(x,B)dx. \tag{2.9}$$

The deterministic function φ can be chosen to be the sample path of a separable stochastic process $(X_t)_{t\in[0,T]}$ with $X(0) = 0$ a.s. In this regard, we state that the process X has a local time (resp. square integrable local time) if for almost all ω, the trajectory $t \mapsto X_t(\omega)$ has a local time (resp. square integrable local time).

We study the local time through Berman's approach. The idea is to derive properties of the local time, $L(\bullet, B)$, from the integrability properties of the Fourier transform of the sample paths of the process X.

Let us state the following hypotheses:

A1

$$\int_{\mathbb{R}^d} \int_0^T \int_0^T \mathbb{E}\left[e^{i\langle u, X_t - X_s\rangle}\right] dt\, ds\, du < \infty,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^d$.

A2 For every even integer $m \geq 2$,

$$\int_{(\mathbb{R}^d)^m} \int_{[0,T]^m} \left| \mathbb{E}\left[\exp\left(i \sum_{j=1}^m \langle u_j, X_{t_j}\rangle\right)\right]\right| \prod_{j=1}^m dt_j \prod_{j=1}^m du_j < \infty.$$

Recall the following essential result that we can find in [3]:

Theorem 2.5. *Assume* **A1**. Hence the process X has a square integrable local time. Furthermore, we have almost surely, for all Borel set $B \subseteq [0,T]$, and for almost every x,

$$L(x, B) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u, x\rangle} \int_B e^{i\langle u, X_t\rangle} dt\, du. \tag{2.10}$$

Remark that $L(x, B)$, given by (2.10), is not a stochastic process. We will follow Berman [4] to create a version of the local time, which is a stochastic process. The following theorem is given in Berman [4, Theorem 4.1] for $d = 1$ and $m = 2$, so we will omit its proof.

Theorem 2.6. *Assume* **A1** and **A2**. Put for all integer $N \geq 1$,

$$L_N(x, t) = \frac{1}{(2\pi)^d} \int_{[-N,N]^d} e^{-i\langle u, x\rangle} \int_0^t e^{i\langle u, X_s\rangle} ds\, du.$$

Therefore, there exists a stochastic process $\tilde{L}(x, t)$ separable in the x-variable, such that for each even integer $m \geq 2$,

$$\lim_{N\to\infty} \sup_{(x,t)\in\mathbb{R}^d\times[0,T]} \mathbb{E}\left[|L_N(x,t) - \tilde{L}(x,t)|^m\right] = 0. \tag{2.11}$$

Theorem 2.7 (Theorem 4.3 in [4])**.** *Let $\tilde{L}(x,t)$ be given by* (2.11)*. If the stochastic process $\{\tilde{L}(x,t),\, x \in \mathbb{R}^d\}$ is almost surely continuous, hence it is a continuous (in the x-variable) version of the local time on* $[0,t]$.

In order to prove Theorem 1.1, we will need to estimate the moments of the increments of $\tilde{L}$. For this end, we have by (2.11), for all $x, y \in \mathbb{R}^d$, $t, h \in [0,T]$ such that $t+h \in [0,T]$, and even integer $m \geq 2$,

$$\begin{aligned}
&\mathbb{E}[\tilde{L}(x+y,t+h) - \tilde{L}(x,t+h) - \tilde{L}(x+y,t) + \tilde{L}(x,t)]^m = \frac{1}{(2\pi)^{md}} \\
&\times \int_{(\mathbb{R}^d)^m} \int_{[t,t+h]^m} \prod_{j=1}^m \left(e^{-i\langle u_j, x+y\rangle} - e^{-i\langle u_j, x\rangle}\right) \mathbb{E}\left[e^{i\sum_{j=1}^m \langle u_j, X_{t_j}\rangle}\right] \prod_{j=1}^m dt_j \prod_{j=1}^m du_j \\
&= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{[t,t+h]^m} \prod_{j=1}^m \left(e^{-i\langle v_j - v_{j+1}, x+y\rangle} - e^{-i\langle v_j - v_{j+1}, x\rangle}\right) \\
&\qquad\qquad \times \mathbb{E}\left[e^{i\sum_{j=1}^m \langle v_j, X_{t_j} - X_{t_{j-1}}\rangle}\right] \prod_{j=1}^m dt_j \prod_{j=1}^m dv_j,
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\mathbb{E}[\tilde{L}(x,t+h) - \tilde{L}(x,t)]^m \\
&= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{[t,t+h]^m} e^{-i\sum_{j=1}^m \langle u_j, x\rangle} \mathbb{E}\left[e^{i\sum_{j=1}^m \langle u_j, X_{t_j}\rangle}\right] \prod_{j=1}^m dt_j \prod_{j=1}^m du_j \\
&= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{[t,t+h]^m} e^{-i\langle v_1, x\rangle} \mathbb{E}\left[e^{i\sum_{j=1}^m \langle v_j, X_{t_j} - X_{t_{j-1}}\rangle}\right] \prod_{j=1}^m dt_j \prod_{j=1}^m dv_j,
\end{aligned} \tag{2.13}$$

where $t_0 = 0$, and the last equality in (2.12) (resp. (2.13)) holds by the following changes of variables:

$$u_j = v_j - v_{j+1}, \qquad j = 1, \cdots, m, \qquad \text{with} \qquad v_{m+1} = 0.$$

In order to estimate (2.12) and (2.13), we need first to manage the characteristic function $\mathbb{E}\left[e^{i\sum_{j=1}^m \langle v_j, X_{t_j} - X_{t_{j-1}}\rangle}\right]$. Therefore, we will use the following condition called α-local nondeterminism (α-LND), which was introduced for the first time in [11].

Definition 2.8. Let $X = (X_t)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued stochastic process, J a subinterval of $[0,T]$ and $\alpha \in (0,1)$. X is said to satisfy the α-LND property on J, if for every non-negative integers $m \geq 2$, and $k_{j,l}$, for $j = 1, \cdots, m$, $l = 1, \cdots, d$, there exist positive constants c and ε, both may depend on m and $k_{j,l}$, such that

$$\left|\mathbb{E}\left[e^{i\sum_{j=1}^m \langle v_j, X_{t_j} - X_{t_{j-1}}\rangle}\right]\right| \leq \frac{c}{\prod_{j=1}^m \prod_{l=1}^d |v_{j,l}|^{k_{j,l}} (t_j - t_{j-1})^{\alpha k_{j,l}}}, \tag{2.14}$$

for all $v_j = (v_{j,l}\,;\ 1 \leq l \leq d) \in (\mathbb{R} \setminus \{0\})^d$, for $j = 1, \cdots, m$, and for every ordered points $t_1 < \cdots < t_m$ in J with $t_m - t_1 < \varepsilon$ and $t_0 = 0$.

Remark 2.9.

1. It is well-known that the concept of local nondeterminism in the Gaussian framework means that "the value of the process at a given time point is relatively unpredictable based on a finite set of observations from the immediate past". In the Gaussian context, Berman uses conditional variance to express this. But unfortunately, he can't use the conditional variance outside the Gaussian case because in a general framework the conditional variance is not deterministic. So, Berman has introduced the local g-nondeterminism concept for general processes by replacing the incremental variance, which is a measure of local unpredictability, by a measure of local predictability, namely, the value of the incremental density function at the origin, see [6, Definition 5.1]. Therefore, the local g-nondeterminism concept reflects well his name. By the Fourier inversion theorem, it is easy to see that the condition in Definition 2.8 implies the local g-nondeterminism condition. On the other hand, Nolan has introduced the notion of *characteristic function locally approximately independent increments* (see [20, Definition 3.1]), which is equivalent in the Gaussian and stable context to the classical LND condition. The condition in Definition 2.8 ($d = 1$) is an extension of Nolan's notion by replacing the characteristic functions $\left|\mathbb{E}\left[e^{ic_m u_j (X(t_j) - X(t_{j-1}))}\right]\right|$ in the right-hand side of [20, Ineq. (3.3)] by $c\,|u_j|^{-k_j}(t_j - t_{j-1})^{-\alpha k_j}$. For all these reasons, we choose to call the condition in Definition 2.8 by α-local nondeterminism (α-LND).

2. Let $Y^0 = (Y_t^0)_{t\in[0,T]}$ be a real-valued centred Gaussian process satisfying the classical local nondeterminism (LND) property on $J \subseteq [0,T]$ (see [7, Lemma 2.3]). Assume also that there exists a positive constant K, such that for every $s, t \in J$ with $s < t$,

$$K(t-s)^{2\alpha} \leq \mathrm{Var}\,(Y_t - Y_s)\,.$$

Define $Y_t = (Y_t^1, \cdots, Y_t^d)$, where $Y^1, \cdots, Y^d$ are independent copies of Y^0. Then Y is α-LND on J.

3. The question of whether or not the α-LND is strictly weaker than the classical local nondeterminism in the Gaussian framework is an open problem. We know from [8] that the process $(f(t)W_t)_{t\in(0,\delta)}$, where $f(t)$ is the Weierstrass function and $(W_t)_{t\in[0,1]}$ is a standard Brownian motion, is not LND; however, we have no idea whether $(f(t)W_t)_{t\in(0,\delta)}$ is α-LND or not.

3 Proof of the main results

Our aim in this section is to prove Theorem 1.1, 1.4, and 1.3. We will need first the following preliminary lemmas.

Lemma 3.1. *Let $1 \le p < \infty$, $0 < \nu < 1$, and $I = [0,1]$. Then, for all $\mathbb{R}$-valued jointly continuous function g with compact support defined on $\mathbb{R}^d \times [0,1]$,*

$$\begin{aligned}
&\sup_{j\ge 0} 2^{jp\nu} \| r \mapsto \sup_{x\in\mathbb{R}^d} |g(x, r+2^{-j}) - g(x,r)| \|^p_{L^p(I(2^{-j});\mathbb{R})} \\
&= \sup_{j\ge 0} 2^{jp\nu - j} \int_0^1 \sum_{k=1}^{2^j-1} \sup_{x\in\mathbb{R}^d} |g(x, 2^{-j}(s+k)) - g(x, 2^{-j}(s+k-1))|^p ds.
\end{aligned} \tag{3.1}$$

Proof. Denote

$$Z_j = 2^{jp\nu} \| r \mapsto \sup_{x\in\mathbb{R}^d} |g(x, r+2^{-j}) - g(x,r)| \|^p_{L^p(I(2^{-j});\mathbb{R})},$$

here $I(2^{-j}) = \{t \in [0,1]\,;\ t + 2^{-j} \in [0,1]\}$. We have

$$\begin{aligned}
Z_j &= 2^{jp\nu} \int_0^{1-2^{-j}} \sup_{x\in\mathbb{R}^d} |g(x, r+2^{-j}) - g(x,r)|^p dr \\
&= 2^{jp\nu} \sum_{k=1}^{2^j-1} \int_{(k-1)2^{-j}}^{k2^{-j}} \sup_{x\in\mathbb{R}^d} |g(x, r+2^{-j}) - g(x,r)|^p dr \\
&= 2^{jp\nu - j} \int_0^1 \sum_{k=1}^{2^j-1} \sup_{x\in\mathbb{R}^d} |g(x, 2^{-j}(s+k)) - g(x, 2^{-j}(s+k-1))|^p ds,
\end{aligned}$$

where we have used the change of variables $s = 2^j(r - 2^{-j}(k-1))$. Which finishes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Let $X = (X_t)_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued continuous stochastic process which is α-LND with $\alpha \in (0, \frac{1}{d})$. Denote by $L(x,t)$ the jointly continuous version of the local time of X, then*

- *For any positive integer q, there exists a positive constant $K = K(q,d,\alpha)$ such that for all integer $j \geq 1$ and $x \in \mathbb{R}^d$,*

$$\mathbb{P}\left[\sum_{k=1}^{2^j-1} |L(x + X(d_{j,k}), D_{j,k})|^q \geq 2^{-jq(1-d\alpha)+j}\right] \leq K 2^{-jd\alpha}; \tag{3.2}$$

- *Let a and q be positive integers and $0 < \theta < \{(\frac{1}{\alpha} - d)/2\} \wedge 1$, then there exists a positive constant $K = K(q,a,d,\alpha,\theta)$, such that for all integers $j, h \geq 1$ and any $x, y \in \mathbb{R}^d$ and $\gamma > 0$,*

$$\mathbb{P}\left[\sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q \geq 2^{-jq(1-d\alpha-\theta\alpha)+j}\|x-y\|^{q\theta} 2^{\gamma h}\right] \leq K 2^{-jd\alpha} 2^{-2^a \gamma h}, \tag{3.3}$$

where $d_{j,k} = 2^{-j}(k-1)$, $D_{j,k} = [2^{-j}(k-1), 2^{-j}(k+1)]$, and

$$A_{j,k,x,y} = L(x + X(d_{j,k}), D_{j,k}) - L(y + X(d_{j,k}), D_{j,k}). \tag{3.4}$$

Proof. We only prove (3.3). We have by Hölder's inequality, for all positive integer a,

$$\begin{aligned}\mathbb{E}\left[\left\{\sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q\right\}^{2^a}\right] &= \sum_{k_1,\ldots,k_{2^a}=1}^{2^j-1} \mathbb{E}\left[\prod_{i=1}^{2^a} |A_{j,k_i,x,y}|^q\right] \\ &\leq \left\{\sum_{k=1}^{2^j-1} \mathbb{E}\left[|A_{j,k,x,y}|^{q2^a}\right]^{1/2^a}\right\}^{2^a}.\end{aligned} \tag{3.5}$$

On the other hand, let $Y_t = X(t) - X(d_{j,k})$. The occupation measure of Y is just the occupation measure of X translated by the (random) constant $X(d_{j,k})$. Since the occupation measure of X has a jointly continuous density, the occupation measure of Y has also a jointly continuous density given by $L_Y(x,t) = L_X(x + X(d_{j,k}), t)$. Put $m = q2^a$, hence by (3.4) and (2.12) we obtain

$$\begin{aligned}
&\mathbb{E}\left[|A_{j,k,x,y}|^m\right] \\
&= E\left[|L(x + X(d_{j,k}), D_{j,k}) - L(y + X(d_{j,k}), D_{j,k})|^m\right] \\
&= E\left[|L_Y(x, D_{j,k}) - L_Y(y, D_{j,k})|^m\right] \\
&= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{(D_{j,k})^m} \prod_{n=1}^{m} \left(e^{-i\langle v_n - v_{n+1}, x\rangle} - e^{-i\langle v_n - v_{n+1}, y\rangle}\right) \\
&\qquad\qquad \times \mathbb{E}\left[e^{i\sum_{n=1}^m \left\langle v_n, X_{t_n} - X_{t_{n-1}}\right\rangle}\right] \prod_{n=1}^{m} dt_n \prod_{n=1}^{m} dv_n,
\end{aligned} \tag{3.6}$$

By (3.6) and the elementary inequality $|1 - e^{i\rho}| \leq 2^{1-\theta}|\rho|^\theta$ for any $0 < \theta < 1$ and $\rho \in \mathbb{R}$, we get

$$\mathbb{E}\left[|A_{j,k,x,y}|^m\right] \leq 2^{-m(d+\theta-1)}\pi^{-md}\|x - y\|^{m\theta}\mathcal{J}(m,\theta), \tag{3.7}$$

where

$$\begin{aligned}
&\mathcal{J}(m,\theta) \\
&= \int_{(D_{j,k})^m} \int_{(\mathbb{R}^d)^m} \prod_{n=1}^{m} \|v_n - v_{n+1}\|^\theta \left|\mathbb{E}\left[e^{i\sum_{n=1}^m \left\langle v_n, X_{t_n} - X_{t_{n-1}}\right\rangle}\right]\right| \prod_{n=1}^{m} dv_n \prod_{n=1}^{m} dt_n.
\end{aligned}$$

We replace the integration over the domain $(D_{j,k})^m$ by the integration over the subset $\Lambda_{j,k} = \{2^{-j}(k-1) \leq t_1 < \cdots < t_m \leq 2^{-j}(k+1)\}$, hence we obtain

$$\begin{aligned}
&\mathcal{J}(m,\theta) \\
&= m! \int_{\Lambda_{j,k}} \int_{(\mathbb{R}^d)^m} \prod_{n=1}^{m} \|v_n - v_{n+1}\|^\theta \left|\mathbb{E}\left[e^{i\sum_{n=1}^m \left\langle v_n, X_{t_n} - X_{t_{n-1}}\right\rangle}\right]\right| \prod_{n=1}^{m} dv_n \prod_{n=1}^{m} dt_n,
\end{aligned}$$

where $t_0 = 0$ and $v_{m+1} = 0$. By the fact that $\|b - c\|^\theta \leq \|b\|^\theta + \|c\|^\theta$ for each $0 < \theta < 1$ and $b, c \in \mathbb{R}^d$, it follows that

$$\prod_{n=1}^{m} \|v_n - v_{n+1}\|^\theta \leq \prod_{n=1}^{m} \left(\|v_n\|^\theta + \|v_{n+1}\|^\theta\right). \tag{3.8}$$

Remark that the right side of this last inequality is at most equal to a finite sum of terms each of the form $\prod_{n=1}^m \|v_n\|^{\epsilon_n\theta}$, where $\epsilon_n = 0, 1$, or 2 and $\sum_{n=1}^m \epsilon_n = m$. Hence

$$\mathcal{J}(m,\theta) \le m! \sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} \int_{\Lambda_{j,k}} \int_{(\mathbb{R}^d)^m} \prod_{n=1}^m \|v_n\|^{\epsilon_n\theta} \times \left|\mathbb{E}\left[e^{i\sum_{n=1}^m \langle v_n, X_{t_n}-X_{t_{n-1}}\rangle}\right]\right| \prod_{n=1}^m dv_n \prod_{n=1}^m dt_n. \tag{3.9}$$

On the other hand, by the α-LND property of the process X, we get for every nonnegative integers $m \ge 2$, $k_{n,l}$, for $n = 1, \cdots, m$ and $l = 1, \cdots, d$, there exists a constant $c = c(m, k_{n,l})$ such that

$$\left|\mathbb{E}\left[e^{i\sum_{n=1}^m \langle v_n, X_{t_n}-X_{t_{n-1}}\rangle}\right]\right| \le \frac{c}{\prod_{n=1}^m \prod_{l=1}^d |v_{n,l}|^{k_{n,l}} (t_n - t_{n-1})^{\alpha k_{n,l}}}, \tag{3.10}$$

where $v_n = (v_{n,1}, \cdots, v_{n,d})$. Put $I_1^n = [-1/(t_n - t_{n-1})^\alpha, 1/(t_n - t_{n-1})^\alpha]$ and $I_2^n = \mathbb{R} \setminus I_1^n$, Therefore

$$(\mathbb{R}^d)^m = \bigcup_{\substack{i_{n,l}\in\{1,2\} \\ n=1,\cdots,m; l=1,\cdots,d}} \prod_{n=1}^m \prod_{l=1}^d I_{i_{n,l}}^n. \tag{3.11}$$

Set, for $n = 1, \cdots, m$ and $l = 1, \cdots, d$,

$$k_{n,l}(i_{n,l}) = \begin{cases} 0, & \text{if} \quad i_{n,l} = 1; \\ 4, & \text{if} \quad i_{n,l} = 2, \end{cases}$$

Hence, by (3.9)-(3.11), we obtain

$$\mathcal{J}(m,\theta) \le m!c \sum_{\substack{i_{n,l}\in\{1,2\} \\ n=1,\cdots,m; l=1,\cdots,d}} \sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} \int_{\Lambda_{j,k}} \int_{\prod_{n=1}^m \prod_{l=1}^d I_{i_{n,l}}^n} \times \frac{\prod_{n=1}^m \|v_n\|^{\epsilon_n\theta}}{\prod_{n=1}^m \prod_{l=1}^d |v_{n,l}|^{k_{n,l}(i_{n,l})} (t_n - t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} \prod_{n=1}^m dv_n \prod_{n=1}^m dt_n. \tag{3.12}$$

We remark that

$$\prod_{n=1}^m \|v_n\|^{\epsilon_n\theta} \le \prod_{n=1}^m \left(|v_{n,1}|^{\epsilon_n\theta} + \cdots + |v_{n,d}|^{\epsilon_n\theta}\right) = \sum_{l_1,\cdots,l_d\in\{1,\cdots,d\}} \prod_{n=1}^m |v_{n,l_n}|^{\epsilon_n\theta}.$$

Therefore

$$\mathcal{J}(m,\theta) \le m!c \sum_{l_1,\cdots,l_d\in\{1,\cdots,d\}} \sum_{\substack{i_{n,l}\in\{1,2\}\\ n=1,\cdots,m;l=1,\cdots,d}} \sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} \int_{\Lambda_{j,k}} \int_{\prod_{n=1}^m \prod_{l=1}^d I^n_{i_{n,l}}} \times \frac{\prod_{n=1}^m |v_{n,l_n}|^{\epsilon_n\theta}}{\prod_{n=1}^m \prod_{l=1}^d |v_{n,l}|^{k_{n,l}(i_{n,l})}(t_n-t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} \prod_{n=1}^m dv_n \prod_{n=1}^m dt_n.$$

By Fubini's theorem, the right side of the above expression is equal to

$$m!c \sum_{l_1,\cdots,l_d\in\{1,\cdots,d\}} \sum_{\substack{i_{n,l}\in\{1,2\}\\ n=1,\cdots,m;l=1,\cdots,d}} \sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} \int_{\Lambda_{j,k}} \prod_{n=1}^m \int_{\prod_{l=1}^d I^n_{i_{n,l}}} \times \frac{|v_{n,l_n}|^{\epsilon_n\theta}}{\prod_{l=1}^d |v_{n,l}|^{k_{n,l}(i_{n,l})}(t_n-t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} dv_n \prod_{n=1}^m dt_n.$$

$$= m!c \sum_{l_1,\cdots,l_d\in\{1,\cdots,d\}} \sum_{\substack{i_{n,l}\in\{1,2\}\\ n=1,\cdots,m;l=1,\cdots,d}} \sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} \times \int_{\Lambda_{j,k}} \prod_{n=1}^m \prod_{\substack{l=1\\ l\neq l_n}}^d \int_{I^n_{i_{n,l}}} \frac{1}{|v_{n,l}|^{k_{n,l}(i_{n,l})}(t_n-t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} dv_{n,l} \times \int_{I^n_{i_{n,l_n}}} \frac{1}{|v_{n,l_n}|^{k_{n,l_n}(i_{n,l_n})-\epsilon_n\theta}(t_n-t_{n-1})^{\alpha k_{n,l_n}(i_{n,l_n})}} dv_{n,l_n} \prod_{n=1}^m dt_n. \tag{3.13}$$

- If $i_{n,l}=1$ or 2 with $l\neq l_n$, then we have

  $$\int_{I^n_{i_{n,l}}} \frac{1}{|v_{n,l}|^{k_{n,l}(i_{n,l})}(t_n-t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} dv_{n,l} = \frac{K_1}{(t_n-t_{n-1})^{\alpha}},$$

  where the constant K_1 depends only on $i_{n,l}$.

- If $i_{n,l_n}=1$ or 2, then we get

  $$\int_{I^n_{i_{n,l_n}}} \frac{1}{|v_{n,l_n}|^{k_{n,l_n}(i_{n,l_n})-\epsilon_n\theta}(t_n-t_{n-1})^{\alpha k_{n,l_n}(i_{n,l_n})}} dv_{n,l_n} = \frac{K_2}{(t_n-t_{n-1})^{\alpha(1+\epsilon_n\theta)}},$$

  where the constant K_2 depends on i_{n,l_n}, θ, and ϵ_n such that $\sup_{\theta,\epsilon_n} K_2 < \infty$.

Combining the above discussion with (3.13), we obtain

$$\begin{aligned}
\mathcal{J}(m,\theta) &\le m!c_1 \sum_{l_1,\cdots,l_d\in\{1,\cdots,d\}} \sum_{\substack{i_{n,l}\in\{1,2\}\\ n=1,\cdots,m;l=1,\cdots,d}} \sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} \\
&\qquad \times \int_{\Lambda_{j,k}} \prod_{n=1}^m \frac{1}{(t_n-t_{n-1})^{\alpha(d+\epsilon_n\theta)}} \prod_{n=1}^m dt_n \\
&= m!c_2 \sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} \int_{\Lambda_{j,k}} \prod_{n=1}^m \frac{1}{(t_n-t_{n-1})^{\alpha(d+\epsilon_n\theta)}} \prod_{n=1}^m dt_n \\
&= m!c_2 \sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} \int_{a_{j,k}}^{b_{j,k}} dt_1 \int_{t_1}^{b_{j,k}} dt_2 \cdots \int_{t_{m-1}}^{b_{j,k}} dt_m \\
&\qquad \times \prod_{n=1}^m \frac{1}{(t_n-t_{n-1})^{\alpha(d+\epsilon_n\theta)}}, \qquad (3.14)
\end{aligned}$$

where $a_{j,k}=2^{-j}(k-1)$ and $b_{j,k}=2^{-j}(k+1)$. Let $0<\theta<\left\{(\frac{1}{\alpha}-d)/2\right\}\wedge 1$. We integrate in the order of $dt_m, dt_{m-1}, \ldots, dt_1$, and use changes of variables in each step to construct Beta functions. Hence, (3.14) is equal to

$$\sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} K_\epsilon \int_{a_{j,k}}^{b_{j,k}} (b_{j,k}-t_1)^{(m-1)(1-d\alpha)-\alpha\theta\sum_{i=0}^{m-2}\epsilon_{m-i}} t_1^{-\alpha(d+\epsilon_1\theta)} dt_1, \quad (3.15)$$

where $K_\epsilon = m!c_2 \frac{1}{1-\alpha(d+\epsilon_m\theta)} \frac{\Gamma(2-\alpha(d+\epsilon_m\theta))\prod_{i=1}^{m-2}\Gamma(1-\alpha(d+\epsilon_{m-i}\theta))}{\Gamma(1+(m-1)(1-d\alpha)-\theta\sum_{i=0}^{m-2}\epsilon_{m-i})}$ and $\epsilon=(\epsilon_1,\cdots,\epsilon_m)$. Recall that $\sum_{n=1}^m \epsilon_n = m$. Then

- If $k=1$. According to (3.7) and (3.15) we get

$$\mathbb{E}\left[|A_{j,1,x,y}|^m\right] \le K_1 \|x-y\|^{m\theta} 2^{-jm(1-d\alpha-\alpha\theta)}, \qquad (3.16)$$

  where $K_1 = c\sum_{\epsilon\in\{0,1,2\}^m} K_\epsilon \frac{\Gamma(1+(m-1)(1-d\alpha)-\alpha\theta\sum_{i=0}^{m-2}\epsilon_{m-i})\Gamma(1-\alpha(d+\epsilon_1\theta))}{\Gamma(1+m(1-d\alpha-\alpha\theta))}$ with $c=2^{-m(d+\theta-1)}\pi^{-md}2^{m(1-d\alpha-\alpha\theta)}$.

- If $2\le k\le 2^j-1$. Therefore (3.15) is less than or equal to

$$\sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} K_\epsilon (b_{j,k}-a_{j,k})^{(m-1)(1-d\alpha)-\alpha\theta\sum_{i=0}^{m-2}\epsilon_{m-i}} \int_{a_{j,k}}^{b_{j,k}} t_1^{-\alpha(d+\epsilon_1\theta)} dt_1 \qquad (3.17)$$

And this last term is equal to

$$\sum_{(\epsilon_1,\cdots,\epsilon_m)\in\{0,1,2\}^m} K_\epsilon 2^{(m-1)(1-d\alpha)-\alpha\theta\sum_{i=0}^{m-2}\epsilon_{m-i}} 2^{-jm(1-d\alpha-\alpha\theta)} \times \int_{k-1}^{k+1} u^{-\alpha(d+\epsilon_1\theta)}du$$

$$\leq \tilde{K}_2 2^{-jm(1-d\alpha-\alpha\theta)} \int_{k-1}^{k+1} u^{-\alpha d}du \leq \tilde{K}_2 2^{-jm(1-d\alpha-\alpha\theta)}(k-1)^{-\alpha d}. \tag{3.18}$$

Hence by (3.7) and (3.17), we have

$$\mathbb{E}\left[|A_{j,k,x,y}|^m\right] \leq K_2\|x-y\|^{m\theta}2^{-jm(1-d\alpha-\alpha\theta)}(k-1)^{-\alpha d}. \tag{3.19}$$

Now we return to estimate $\mathbb{E}\left[\left\{\sum_{k=1}^{2^j-1}|A_{j,k,x,y}|^q\right\}^{2^a}\right]$. Recall that $m = q2^a$. According to (3.5) and the convexity of the function $x \mapsto x^{2^a}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\{\sum_{k=1}^{2^j-1}|A_{j,k,x,y}|^q\right\}^{2^a}\right] \\
&\leq \left\{\sum_{k=1}^{2^j-1}\mathbb{E}\left[|A_{j,k,x,y}|^{q2^a}\right]^{1/2^a}\right\}^{2^a} \\
&\leq 2^{a-1}\left(\mathbb{E}\left[|A_{j,1,x,y}|^{q2^a}\right] + \left\{\sum_{k=2}^{2^j-1}\mathbb{E}\left[|A_{j,k,x,y}|^{q2^a}\right]^{1/2^a}\right\}^{2^a}\right).
\end{aligned}$$

Hence, using (3.16) and (3.19), we derive that this last term is less than or equal to

$$\begin{aligned}
&\tilde{K}\|x-y\|^{q2^a\theta}2^{-jq2^a(1-d\alpha-\alpha\theta)}\left(1+\left\{\sum_{k=2}^{2^j-1}(k-1)^{-\alpha d/2^a}\right\}^{2^a}\right) \\
&\leq \tilde{K}\|x-y\|^{q2^a\theta}2^{-jq2^a(1-d\alpha-\alpha\theta)}\left(1+\left\{\sum_{k=2}^{2^j-1}2\int_{k-\frac{3}{2}}^{k-1}x^{-d\alpha/2^a}dx\right\}^{2^a}\right)
\end{aligned}$$

Therefore, it is easy to see that the right-hand side of the above inequality is less than or equal to

$$\begin{aligned}&\tilde{K}\|x-y\|^{q2^a\theta}2^{-jq2^a(1-d\alpha-\alpha\theta)}\left(1+\left\{2\int_{\frac{1}{2}}^{2^j-2}x^{-d\alpha/2^a}dx\right\}^{2^a}\right)\\&\leq \hat{K}\|x-y\|^{q2^a\theta}2^{-jq2^a(1-d\alpha-\alpha\theta)}\left(1+2^{2^aj-jd\alpha}\right).\end{aligned}$$

Therefore

$$\mathbb{E}\left[\left\{\sum_{k=1}^{2^j-1}|A_{j,k,x,y}|^q\right\}^{2^a}\right]\leq K\|x-y\|^{q2^a\theta}2^{-jq2^a(1-d\alpha-\alpha\theta)}2^{j(2^a-\alpha d)}. \tag{3.20}$$

The remainder of the proof is by Chebyshev's inequality, i.e.

$$\begin{aligned}&\mathbb{P}\left[\sum_{k=1}^{2^j-1}|A_{j,k,x,y}|^q\geq 2^{-jq(1-d\alpha-\theta\alpha)+j}\|x-y\|^{q\theta}2^{\gamma h}\right]\\&\leq 2^{j2^aq(1-d\alpha-\theta\alpha)-2^aj}\|x-y\|^{-q2^a\theta}2^{-2^a\gamma h}\mathbb{E}\left[\left\{\sum_{k=1}^{2^j-1}|A_{j,k,x,y}|^q\right\}^{2^a}\right].\end{aligned}$$

Hence (3.20) concludes the proof of Lemma 3.2. □

Lemma 3.3. *Let* $(X_t)_{t\in[0,1]}$ *be an* $\mathbb{R}^d$*-valued continuous stochastic process that verifies* **H**. Then, for all $0<\delta<1$, almost surely there exists $j_1 = j_1(\omega,\delta)$ such that

$$\sup_{1\leq k\leq 2^j-1}\sup_{t\in D_{j,k}}\|X(t)-X(d_{j,k})\|\leq 2^{-j(\alpha-\frac{2-\delta}{p_0})}\qquad \text{for } j\geq j_1, \tag{3.21}$$

where $d_{j,k}$ and $D_{j,k}$ are as in Lemma 3.2.

Proof. According to (1.1), we have almost surely

$$B:=\int_0^1\int_0^1\frac{\|X(t)-X(s)\|^{p_0}}{|t-s|^{\alpha p_0+\gamma}}dt\,ds<\infty. \tag{3.22}$$

Then by Lemma 2.3 with $\Psi(u)=|u|^{p_0}$ and $p(u)=|u|^{\alpha+\frac{\gamma}{p_0}}$ where $\frac{1}{p_0}<\alpha$ and $2-\alpha p_0<\gamma<1$, we derive

$$\sup_{1\leq k\leq 2^j-1}\sup_{t\in D_{j,k}}\|X(t)-X(d_{j,k})\|^{p_0}\leq C_{\alpha,p_0,\gamma}B2^{-j(\alpha p_0+\gamma-2)}. \tag{3.23}$$

Therefore, by (1.1), (3.22), and (3.23) we get

$$\mathbb{E}\left[\sup_{1\le k\le 2^j-1}\sup_{t\in D_{j,k}}\|X(t)-X(d_{j,k})\|^{p_0}\right]\le K_{\alpha,p_0,\gamma}2^{-j(\alpha p_0+\gamma-2)}. \tag{3.24}$$

Hence, let $0<\delta<1$ and $2-\alpha p_0<\gamma<1$ such that $\delta<\gamma$, then by Chebyshev's inequality and (3.24) we write

$$\mathbb{P}\left[\sup_{1\le k\le 2^j-1}\sup_{t\in D_{j,k}}\|X(t)-X(d_{j,k})\|\ge 2^{-j(\alpha-\frac{2-\delta}{p_0})}\right]\le K_{\alpha,p_0,\gamma}2^{-j(\gamma-\delta)}. \tag{3.25}$$

Therefore the Borel–Cantelli lemma concludes the proof of Lemma 3.3. □

Lemma 3.4. *Let $(X_t)_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued continuous stochastic process which is α-LND with $\alpha\in(0,\frac{1}{d})$. Assume also that X verifies* **H**. Then for all positive integer q, almost surely we have

$$\sup_{j\ge1}2^{jq(1-d\alpha)-j}\int_0^1\sum_{k=1}^{2^j-1}\sup_{x\in\mathbb{R}^d}|L(x,2^{-j}(s+k))-L(x,2^{-j}(s+k-1))|^q ds<\infty. \tag{3.26}$$

Proof. Let $d_{j,k}=2^{-j}(k-1)$; $D_{j,k}=[2^{-j}(k-1),2^{-j}(k+1)]$ for $j\ge1$ and $1\le k\le 2^j-1$. It follows from Lemma 3.3 that for all $2-\alpha p_0<\delta<1$, almost surely there exists $j_1=j_1(\omega,\delta)$ such that

$$\sup_{1\le k\le 2^j-1}\sup_{t\in D_{j,k}}\|X(t)-X(d_{j,k})\|\le 2^{-j(\alpha-\frac{2-\delta}{p_0})}\qquad\text{for } j\ge j_1, \tag{3.27}$$

Let $\beta_j=2^{-j\alpha}$ and

$$G_j=\left\{x\in\mathbb{R}^d;\|x\|\le 2^{-j(\alpha-\frac{2-\delta}{p_0})},\quad x=\beta_j b\quad\text{for some } b\in\mathbb{Z}^d\right\},$$

where $\mathbb{Z}$ is the set of integers. The cardinality of G_j verifies

$$\#G_j\le\left(2\left[2^{j(\frac{2-\delta}{p_0})}\right]+1\right)^d\le 3^d2^{jd(\frac{2-\delta}{p_0})}, \tag{3.28}$$

where $\left[2^{j(\frac{2-\delta}{p_0})}\right]$ is the integral part of $2^{j(\frac{2-\delta}{p_0})}$. It follows from (3.28) and Lemma 3.2 that

$$\mathbb{P}\left[\sum_{k=1}^{2^j-1}|L(x+X(d_{j,k}),D_{j,k})|^q\ge 2^{-jq(1-d\alpha)+j}\quad\text{for some } x\in G_j\right]$$
$$\le\#G_jK2^{-jd\alpha}\le 3^dK2^{-jd(\alpha-\frac{2-\delta}{p_0})}. \tag{3.29}$$

Since $2-\alpha p_0<\delta<1$, it follows by the Borel-Cantelli lemma that almost surely there exists $j_2=j_2(\omega,\delta)$ such that

$$\sum_{k=1}^{2^j-1}\sup_{x\in G_j}|L(x+X(d_{j,k}),D_{j,k})|^q\le 2^{-jq(1-d\alpha)+j}\quad \text{for } j\ge j_2. \tag{3.30}$$

For any fixed integers $j,h\ge 1$ and any $x\in G_j$, define

$$F(j,h,x)=\left\{y\in\mathbb{R}^d;\ y=x+\beta_j\sum_{n=1}^{h}\epsilon_n 2^{-n}\quad \text{for } \epsilon_n\in\{0,1\}^d\right\}. \tag{3.31}$$

A pair of points $y_1,y_2\in F(j,h,x)$ is said to be linked if $y_2-y_1=\beta_j\epsilon 2^{-h}$ for some $\epsilon\in\{0,1\}^d$. Then by (3.3) with γ and a such that $\gamma<q\theta$ and $2^a>\frac{d}{\gamma}$, we have

$$\begin{aligned}\mathbb{P}\Bigg[&\sum_{k=1}^{2^j-1}|A_{j,k,y_1,y_2}|^q\ge 2^{-jq(1-d\alpha-\theta\alpha)+j}\|y_1-y_2\|^{q\theta}2^{\gamma h}\\ &\qquad \text{for some } x\in G_j, h\ge 1 \text{ and some linked pair } y_1,y_2\in F(j,h,x)\Bigg]\\ &\le \#G_j\sum_{h=1}^{\infty}2^{hd}K2^{-jd\alpha}2^{-2^a\gamma h}\le K3^d2^{-jd(\alpha-\frac{2-\delta}{p_0})}\sum_{h=1}^{\infty}2^{-h(2^a\gamma-d)}.\end{aligned}$$

As $2-\alpha p_0<\delta<1$ and $2^a>\frac{d}{\gamma}$, it follows by the Borel-Cantelli lemma that almost surely there exists $j_3=j_3(\omega,\delta,\gamma)$ such that for $j\ge j_3$

$$\sum_{k=1}^{2^j-1}|L(y_1+X(d_{j,k}),D_{j,k})-L(y_2+X(d_{j,k}),D_{j,k})|^q\le 2^{-jq(1-d\alpha-\theta\alpha)+j}\|y_1-y_2\|^{q\theta}2^{\gamma h}, \tag{3.32}$$

for all $x\in G_j$, $h\ge 1$ and any linked pair $y_1,y_2\in F(j,h,x)$. Let Ω_0 be the event that (3.27), (3.30) and (3.32) hold, hence $\mathbb{P}[\Omega_0]=1$. Let $j\ge j_4:=\max\{j_1,j_2,j_3\}$ be fixed. For any $y\in\mathbb{R}^d$ with $\|y\|\le 2^{-j(\alpha-\frac{2-\delta}{p_0})}$, we represent y in the form $y=\lim_{h\to\infty}y_h$, where

$$y_h=x+\beta_j\sum_{n=1}^{h}\epsilon_n2^{-n}\quad (y_0=x,\ \epsilon_n\in\{0,1\}^d),$$

for some $x \in G_j$. Then each pair y_{h-1}, y_h is linked, so by (3.32) and the continuity of $L(\cdot, D_{j,k})$ we have

$$\begin{aligned}
&\sum_{k=1}^{2^j-1} |L(y+X(d_{j,k}), D_{j,k}) - L(x+X(d_{j,k}), D_{j,k})|^q \\
&\le 2^{-jq(1-d\alpha-\theta\alpha)+j} \sum_{h=1}^{\infty} |\beta_j 2^{-h}|^{q\theta} 2^{\gamma h} \\
&= 2^{-jq(1-d\alpha)+j} \sum_{h=1}^{\infty} 2^{-h(q\theta-\gamma)}
\end{aligned}$$

Since $\gamma < q\theta$, we have almost surely for $j \ge j_4$,

$$\sum_{k=1}^{2^j-1} |L(y+X(d_{j,k}), D_{j,k}) - L(x+X(d_{j,k}), D_{j,k})|^q \le C 2^{-jq(1-d\alpha)+j}. \quad (3.33)$$

for all $y \in \mathbb{R}^d$ with $\|y\| \le 2^{-j(\alpha - \frac{2-\delta}{p_0})}$. It follows from (3.30) and (3.33) that almost surely for $j \ge j_4$,

$$\sum_{k=1}^{2^j-1} |L(y+X(d_{j,k}), D_{j,k})|^q \le C_1 2^{-jq(1-d\alpha)+j}, \quad (3.34)$$

for all $y \in \mathbb{R}^d$ with $\|y\| \le 2^{-j(\alpha - \frac{2-\delta}{p_0})}$. On the other hand, we have almost surely for $j \ge j_4$,

$$\begin{aligned}
&\int_0^1 \sum_{k=1}^{2^j-1} \sup_{x\in\mathbb{R}^d} |L(x, 2^{-j}(s+k)) - L(x, 2^{-j}(s+k-1))|^q ds \\
&\le \sum_{k=1}^{2^j-1} \sup_{x\in\mathbb{R}^d} |L(x, D_{j,k})|^q.
\end{aligned}$$

This last term is equal to

$$\begin{aligned}
\sum_{k=1}^{2^j-1} \sup_{x\in X(D_{j,k})} |L(x, D_{j,k})|^q &\le \sum_{k=1}^{2^j-1} \sup_{y\in V} |L(y+X(d_{j,k}), D_{j,k})|^q \\
&\le C_1 2^{-jq(1-d\alpha)+j},
\end{aligned}$$

where $V = \{y \in \mathbb{R}^d;\ \|y\| \le 2^{-j(\alpha-\frac{2-\delta}{p_0})}\}$. This completes the proof of Lemma 3.4. $\square$

Proof of Theorem 1.1. According to Lemma 3.4 and 3.1, and Theorem 2.1(iii) we conclude the proof. □

Now we provide the proof of Theorem 1.3, which clearly explains that if a functions local time, $L(x,t)$, is Besov regular, in t uniformly in x, then this has a significant effect on the Besov irregularity of the function itself.

Proof of Theorem 1.3. According to the occupation formula (2.9), we have for all $t \in (0,1]$, $0 < h \leq t$, and $s \in [0, 1-h]$,

$$\begin{aligned} h &= \int_{f([s,s+h])} L(x,[s,s+h])dx \\ &\leq \lambda_d\left(f([s,s+h])\right) \sup_{x\in\mathbb{R}^d} L(x,[s,s+h]) \\ &\leq \sup_{r,\tau\in[s,s+h]} \|f(r)-f(\tau)\|^d \sup_{x\in\mathbb{R}^d} L(x,[s,s+h]) \\ &= \sup_{r,\tau\in[0,1]} \|f(hr+s)-f(h\tau+s)\|^d \sup_{x\in\mathbb{R}^d} L(x,[s,s+h]). \end{aligned}$$

Hence

$$h^{1/d} \leq \sup_{r,\tau\in[0,1]} \|f(hr+s)-f(h\tau+s)\| \sup_{x\in\mathbb{R}^d} |L(x,[s,s+h])|^{1/d}.$$

Therefore by Hölder's inequality we derive that for all $t \in (0,1]$ and $0 < h \leq t$,

$$\begin{aligned} &(1-h)h^{1/d} \\ &\leq \int_0^{1-h} \sup_{r,\tau\in[0,1]} \|f(hr+s)-f(h\tau+s)\| \sup_{x\in\mathbb{R}^d} |L(x,[s,s+h])|^{1/d} ds \\ &\leq \|s \mapsto \sup_{r,\tau\in[0,1]} \|f(hr+s)-f(h\tau+s)\|\|_{L^p(I(h),\mathbb{R})} \\ &\qquad\qquad \times \left\{ \int_0^{1-h} \sup_{x\in\mathbb{R}^d} |L(x,[s,s+h])|^{\frac{p}{d(p-1)}} ds \right\}^{\frac{p-1}{p}}. \end{aligned} \tag{3.35}$$

By the same calculations as above we get for all $t \in (0,1]$ and $-t < h < 0$

$$\begin{aligned} &(1-|h|)|h|^{1/d} \\ &\leq \|s \mapsto \sup_{r,\tau\in[0,1]} \|f(hr+s)-f(h\tau+s)\|\|_{L^p(I(h),\mathbb{R})} \\ &\qquad\qquad \times \left\{ \int_{-h}^{1} \sup_{x\in\mathbb{R}^d} |L(x,s)-L(x,s+h)|^{\frac{p}{d(p-1)}} ds \right\}^{\frac{p-1}{p}}. \end{aligned} \tag{3.36}$$

According to (3.35), (3.36), and (1.5) we get for all $t \in (0, \frac{1}{2})$,

$$\frac{t^{1/d}}{2} \le c\, t^{\mu/d} \sup_{|h|\le t} \|s \mapsto \sup_{r,\tau\in[0,1]} \|f(hr+s) - f(h\tau+s)\|\|_{L^p(I(h),\mathbb{R})}. \tag{3.37}$$

Therefore,

$$0 < \lim_{t\to 0^+} t^{-(1-\mu)/d} \sup_{|h|\le t} \|s \mapsto \sup_{r,\tau\in[0,1]} \|f(hr+s) - f(h\tau+s)\|\|_{L^p(I(h),\mathbb{R})}.$$

Then Theorem 2.1 (iv) and Theorem 2.2 conclude the proof of Theorem 1.3. □

Proof of Theorem 1.4. For any $1 < p < \infty$, let $n = n(p)$ be a positive integer such that $d^n \ge \frac{p}{p-1}$. We have almost surely,

$$\begin{aligned}
&\sup_{0<t\le 1} t^{-(1-\alpha d)/d} \sup_{|h|\le t} \|s \mapsto \sup_{x\in\mathbb{R}^d} |L(x,s+h) - L(x,s)|^{\frac{1}{d}}\|_{L^{\frac{p}{(p-1)}}(I(h);\mathbb{R})} \\
&\le \sup_{0<t\le 1} t^{-(1-\alpha d)/d} \sup_{|h|\le t} \|s \mapsto \sup_{x\in\mathbb{R}^d} |L(x,s+h) - L(x,s)|\|^{\frac{1}{d}}_{L^{d^{n-1}}(I(h);\mathbb{R})}
\end{aligned}$$

Hence Theorem 1.1 and 1.3 finish the proof of Theorem 1.4. □

Proof of Corollary 1.5. It is a consequence of Theorem 1.4 and the injections (2.8). □

4 Examples

4.1 The Gaussian case

Let $Y^0 = (Y^0_t)_{t\in[0,1]}$ be a real-valued continuous centred Gaussian process, with $Y(0) = 0$, that satisfies the classical local nondeterminism (LND) property on $[0,1]$. By [7, Lemma 2.3] we have for any $m \ge 2$, there exist two positive constants c_m and ε such that for every ordered points $0 = t_0 \le t_1 < \cdots < t_m \le 1$ with $t_m - t_1 < \varepsilon$, and $(v_1, \cdots, v_m) \in \mathbb{R}^m \setminus \{0\}$,

$$\operatorname{Var}\left(\sum_{j=1}^m v_j (Y^0_{t_j} - Y^0_{t_{j-1}})\right) \ge c_m \sum_{j=1}^m v_j^2 \operatorname{Var}\left(Y^0_{t_j} - Y^0_{t_{j-1}}\right). \tag{4.1}$$

Assume also that Y^0 verifies (1.8), with some $\alpha \in (0,1)$. Define $Y_t = (Y_t^1, \cdots, Y_t^d)$, where $Y^1, \cdots, Y^d$ are independent copies of Y^0. Then Y is α-LND on $[0,1]$. The following theorem is a consequence of Theorem 1.1 and 1.4, and Corollary 1.5.

Theorem 4.1. *Let $Y^0 = (Y_t^0)_{t\in[0,1]}$ be a real-valued continuous centered Gaussian process, with $Y(0) = 0$, that satisfies the classical local nondeterminism (LND) property on $[0,1]$ and inequalities (1.8) with $0 < \alpha < \frac{1}{d}$. Let $Y^1, \cdots, Y^d$ be independent copies of Y^0 and put $Y_t = (Y_t^1, \cdots, Y_t^d)$. Denote by $L(x,t)$ the jointly continuous version of the local time of Y, Therefore*

$$\mathbb{P}\left[L(x,\bullet) \in \mathbf{B}^{1-d\alpha}_{p,\infty}(I;\mathbb{R}), \quad \text{for all } x \in \mathbb{R}^d \text{ and } p \in [1,\infty)\right] = 1; \tag{4.2}$$

$$\mathbb{P}\left[Y(\bullet) \in \mathbf{B}^{\alpha,0}_{p,\infty}(I,\mathbb{R}^d), \quad \text{for some } p \in (1/\alpha,\infty)\right] = 0; \tag{4.3}$$

$$\mathbb{P}\left[Y(\bullet) \in \mathbf{B}^{\alpha}_{p,q}(I,\mathbb{R}^d), \quad \text{for some } p \in (1/\alpha,\infty) \text{ and } q \in [1,\infty)\right] = 0, \tag{4.4}$$

where $I = [0,1]$, $L(x,\bullet) : t \in I \mapsto L(x,t) \in \mathbb{R}$, and $Y(\bullet) : t \in I \mapsto Y_t \in \mathbb{R}^d$.

A particular example is $Y^0 = B^{H,K}$ a bifractional Brownian motion with $H \in (0,1)$ and $K \in (0,1]$; that is a real-valued centred Gaussian process, starting from zero, with covariance function

$$\mathbb{E}\left(B_t^{H,K}\, B_s^{H,K}\right) = \frac{1}{2^K}\left[(t^{2H} + s^{2H})^K - |t-s|^{2HK}\right].$$

Notice that the case $K = 1$ corresponds to the fractional Brownian motion with Hurst parameter $H \in (0,1)$. From [2, Lemma 3.3] we know that the bifractional Brownian motion is LND and by [2, Eq. (1)] the Hypothesis **H** holds with $\alpha = HK$. Therefore, The below corollary is a consequence of Theorem 4.1.

Corollary 4.2. *Let $(B_t^{H,K})_{t\in[0,1]}$ be a d-dimensional bifractional Brownian motion with $H \in (0,1)$ and $K \in (0,1]$, s.t. $HK < \frac{1}{d}$. Denote by $L(x,t)$ the jointly continuous version of the local time of $B^{H,K}$, Therefore*

$$\mathbb{P}\left[L(x,\bullet) \in \mathbf{B}^{1-dHK}_{p,\infty}(I;\mathbb{R}), \quad \text{for all } x \in \mathbb{R}^d \text{ and } p \in [1,\infty)\right] = 1; \tag{4.5}$$

$$\mathbb{P}\left[B^{H,K}(\bullet) \in \mathbf{B}^{HK,0}_{p,\infty}(I,\mathbb{R}^d), \quad \text{for some } p \in (1/HK,\infty)\right] = 0; \tag{4.6}$$

$$\mathbb{P}\left[B^{H,K}(\bullet) \in \mathbf{B}^{HK}_{p,q}(I,\mathbb{R}^d), \quad \text{for some } p \in (1/HK,\infty) \text{ and } q \in [1,\infty)\right] = 0,$$

where $I = [0,1]$, $L(x,\bullet) : t \in I \mapsto L(x,t) \in \mathbb{R}$, and $B^{H,K}(\bullet) : t \in I \mapsto B_t^{H,K} \in \mathbb{R}^d$.

Remark 4.3. A different approach, based on characterization of Besov spaces in terms of sequences spaces, has been used in [10] to investigate (4.6) for $d = 1$. However, to the best of our knowledge, the uniform Besov regularity results for the local times of Gaussian processes given by (4.2) and (4.5) are new and have never been considered in the literature.

4.2 The non-Gaussian case

Let us consider the following system of non-linear stochastic heat equations

$$\frac{\partial u_k}{\partial t}(t,x) = \frac{\partial^2 u_k}{\partial x^2}(t,x) + b_k(u(t,x)) + \sum_{l=1}^{d} \sigma_{k,l}(u(t,x))\dot{W}^l(t,x), \tag{4.7}$$

with Neumann boundary conditions

$$u_k(0,x) = 0, \qquad \frac{\partial u_k(t,0)}{\partial x} = \frac{\partial u_k(t,1)}{\partial x} = 0,$$

for $t \in [0,T]$, $1 \leq k \leq d$, $x \in [0,1]$, where $u := (u_1, \cdots, u_d)$. Let $\dot{W} = (\dot{W}^1, \cdots, \dot{W}^d)$ be a vector of d-independent space-time white noises on $[0,T] \times [0,1]$. Set $b = (b_k)_{1 \leq k \leq d}$ and $\sigma = (\sigma_{k,l})_{1 \leq k,l \leq d}$. We consider the below hypotheses on the coefficients $\sigma_{k,l}$ and b_k:
A1 For all $1 \leq k, l \leq d$, $\sigma_{k,l}$ and b_k are bounded and infinitely differentiable functions with their partial derivatives of all orders are bounded.
A2 The matrix σ is uniformly elliptic, i.e. there exists $\rho > 0$ such that for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ with $\|y\| = 1$, we get $\|\sigma(x)y\|^2 \geq \rho^2$ (where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$).

Following Walsh [24], a mild solution of (4.7) is a jointly measurable $\mathbb{R}^d$-valued process $u = (u_1, \cdots, u_d)$ such that for any $k \in \{1, \cdots, d\}$, $t \in [0,T]$, and $x \in [0,1]$,

$$\begin{aligned} u_k(t,x) = &\int_0^t \int_0^1 G_{t-r}(x,v) \sum_{l=1}^{d} \sigma_{k,l}(u(r,v)) W^l(dr,dv) \\ &+ \int_0^t \int_0^1 G_{t-r}(x,v) b_k(u(r,v)) dv\, dr. \end{aligned} \tag{4.8}$$

According to [11, Theorem 5.4], we know that the solution to the system of non-linear stochastic heat equations, (4.8), is $\frac{1}{4}$-LND, and by [11, Eq. (2.12)] the hypothesis **H** holds with $\alpha = \frac{1}{4}$. Hence, Theorem 1.1 and 1.4, and Corollary 1.5 give the following:

Theorem 4.4. *Let u be given by (4.8). Assume that $d \leq 3$ and denote by $L(\xi, t)$ the jointly continuous version of the local time of the process $\{u(t,x)\,,\ t \in [0,T]\}$ for x being fixed in $(0,1)$. Then*

$$\mathbb{P}\left[L(\xi,\bullet) \in \mathbf{B}^{1-d/4}_{p,\infty}(I;\mathbb{R})\,,\quad \text{for all } \xi \in \mathbb{R}^d \text{ and } p \in [1,\infty)\right] = 1; \tag{4.9}$$

$$\mathbb{P}\left[u(\bullet,x) \in \mathbf{B}^{1/4,0}_{p,\infty}(I,\mathbb{R}^d)\,,\quad \text{for some } p \in (4,\infty)\right] = 0; \tag{4.10}$$

$$\mathbb{P}\left[u(\bullet,x) \in \mathbf{B}^{1/4}_{p,q}(I,\mathbb{R}^d)\,,\quad \text{for some } p \in (4,\infty) \text{ and } q \in [1,\infty)\right] = 0, \tag{4.11}$$

where $I = [0,1]$, $L(\xi,\bullet) : t \in I \mapsto L(\xi,t) \in \mathbb{R}$, and $u(\bullet,x) : t \in I \mapsto u(t,x) \in \mathbb{R}^d$.

Remark 4.5. In [9], we have studied, for $d = 1$ and $\sigma = 1$, i.e., $u(t,x)$ is the solution to the linear stochastic heat equation, by a different method the following:

$$\mathbb{P}\left[u(\bullet,x) \in \mathbf{B}^{1/4,0}_{p,\infty}(I,\mathbb{R})\right] = 0.$$

However, (4.9) is new even in the linear case.

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