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Mean-field descriptions for the viscoelastic response of thermorheologically complex reinforced solids

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ABSTRACT

Mean-field descriptions for the thermo-viscoelastic response of reinforced solids undergoing small deformations but large temperature changes are presented. The descriptions follow from an approximate homogenization scheme that identifies macroscopic internal variables with low-order statistics of the microscopic internal variable fields as a result of a variational model reduction. Unlike descriptions based on the correspondence principle, these descriptions can account for thermorheologically complex constitutive laws with multiple internal times and/or temperature-dependent stiffnesses, and can provide information not only on the macroscopic response but also on the statistics of the microscopic mechanical fields; it is further demonstrated that the descriptions can even accommodate a special class of hereditary laws often employed for polymeric materials. Simple expressions for material systems with elastically rigid but thermally dilatant reinforcements are provided. By way of example, reduced descriptions are presented for a special class of material systems for which the thermomechanical response can be computed exactly by means of the correspondence principle. In the case of isotropic particle-reinforced solids under hydrostatic loadings, the reduced descriptions can reproduce the exact response identically; in the case of transversely isotropic fiber-reinforced solids subject to monotonic coolings, the reduced and exact descriptions provide indistinguishable macroscopic strains and stresses for the entire range of temperature drops and cooling rates considered.

1. Introduction

During their consolidation, reinforced polymers develop internal stresses due to the pronounced mismatch between the thermomechanical properties of the matrix and the reinforcement (e.g., Parlevliet et al., 2006). These so-called thermal residual stresses can significantly reduce the mechanical strength and fatigue endurance of the composite material, and can induce undesired dimensional changes of the structural element (e.g., Parlevliet et al., 2007). This has motivated numerous attempts to correlate the magnitude of residual stresses with material and process parameters such as matrix rheology, reinforcement shape and distribution, and cooling rate (e.g., Lu et al., 2008; Ding et al., 2015b,a; Chen and Zhang, 2019; Wu et al., 2020). In the absence of chemical changes – as in amorphous thermoplastics –, the problem reduces to that of estimating the macroscopic behavior and microscopic field

statistics of a viscoelastic composite in terms of the thermomechanical properties and microgeometrical arrangement of its constituents. The use of mean-field descriptions derived by homogenization methods thus constitutes an effectual approach. However, given that consolidation processes often vary the temperature of the sample across the glass transition temperature of the thermoplastic matrix, the method of choice should account for the strong coupling between elastic and viscous deformation mechanisms along with the strong variation of matrix rheology with temperature. This hinders the use of most mean-field descriptions commonly employed for describing isothermal processes, especially if the matrix response is thermorheologically complex in the sense of Harper and Weitsman (1985). The sections that follow present mean-field descriptions adequate for such situations.

The descriptions follow from a homogenization procedure proposed by Lahellec and Suquet (2007a) for composites undergoing isothermal

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deformations, recast as in [Idiart et al. \(2020a\)](#) and generalized to non-isothermal processes. The thermo-viscoelastic response of the constituent phases is described in Section 2 via generalized standard models in terms of viscous deformations that play the role of microscopic internal variables. Given that the homogenized thermo-viscoelastic response of the composite depends on the entire spatial distribution of these microscopic internal variables over a representative volume element, a variational model reduction is then employed in Section 3 to generate an approximate homogenized description in terms of a finite number of macroscopic internal variables identified with low-order statistics of the microscopic internal variable fields. Unlike descriptions based on the correspondence principle (e.g., [Hashin, 1965](#); [Schapery, 1967](#)), the reduced-order mean-field descriptions thus obtained can account for thermorheologically complex constitutive laws with multiple internal times – or even none – and/or temperature-dependent stiffnesses, and can provide information not only on the macroscopic response but also on statistics of the microscopic mechanical fields up to second order. Furthermore, it is demonstrated that these descriptions can even accommodate a special class of hereditary laws often employed for polymeric materials. Simple expressions for material systems with elastically rigid but thermally dilatant reinforcements are provided in Section 4. In order to assess the capabilities of the reduced descriptions, results are generated in Section 5 for a special class of rigidly reinforced solids whose effective response can be determined exactly by means of the correspondence principle. The reduced and exact descriptions for this reinforced solids are confronted and discussed in Section 6. The presentation concludes with final comments elaborated in Section 7.

2. Material description

2.1. Microscopic description

We consider a representative volume element of a composite material made up of N constituent phases, and denote by Ω and $\Omega^{(r)}$ ($r = 1, \dots, N$) the domains occupied by the element and the phases within it, respectively, so that $\Omega = \bigcup_{r=1}^N \Omega^{(r)}$. Also, we denote by $\chi^{(r)}(\mathbf{x})$ the characteristic function of each subdomain $\Omega^{(r)}$. The local viscoelastic response is described within the framework of generalized standard materials by constitutive relations of the form ([Germain et al., 1983](#))

$$\sigma = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\mathbf{x}, \boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \theta), \quad \eta = -\frac{\partial w}{\partial \theta}(\mathbf{x}, \boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \theta) \quad \text{and} \quad \frac{\partial w}{\partial \boldsymbol{\alpha}}(\mathbf{x}, \boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \theta) + \frac{\partial \varphi}{\partial \dot{\boldsymbol{\alpha}}}(\mathbf{x}, \dot{\boldsymbol{\alpha}}, \theta) = \mathbf{0}, \quad (1)$$

where

$$w(\mathbf{x}, \boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \theta) = \sum_{r=1}^N \chi^{(r)}(\mathbf{x}) w^{(r)}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \theta) \quad \text{and} \quad \varphi(\mathbf{x}, \dot{\boldsymbol{\alpha}}, \theta) = \sum_{r=1}^N \chi^{(r)}(\mathbf{x}) \varphi^{(r)}(\dot{\boldsymbol{\alpha}}, \theta). \quad (2)$$

In these expressions, $\boldsymbol{\varepsilon}$ and θ denote the strain and temperature change relative to a stress-free reference configuration, $\boldsymbol{\alpha}$ is a collection of internal variables describing viscous effects, σ and η are the Cauchy stress and entropy density, the dot over a variable denotes a time derivative, and the functions $w^{(r)}$ and $\varphi^{(r)}$ denote the Helmholtz free-energy density and the dissipation potential of phase r , respectively. These potentials are convex functions of the mechanical fields and are bounded from below. The dissipation potentials are, at the same time, positive functions vanishing at zero rates.

For our purposes it suffices to consider viscoelastic phases characterized by generalized Maxwell rheologies with an arbitrary number of viscoelastic units.² To each unit we associate a viscous strain tensor $\boldsymbol{\alpha}^{(u)}$

($u = 1, \dots, U$) and write the thermodynamic potentials as

$$w^{(r)}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \theta) = \sum_{u=1}^U \frac{1}{2} (\boldsymbol{\varepsilon} - \boldsymbol{\beta}^{(r)}(\theta) - \boldsymbol{\alpha}^{(u)}) \cdot \mathbb{L}^{(r,u)}(\theta) (\boldsymbol{\varepsilon} - \boldsymbol{\beta}^{(r)}(\theta) - \boldsymbol{\alpha}^{(u)}) + f^{(r)}(\theta) \quad \text{and} \quad (3)$$

$$\varphi^{(r)}(\dot{\boldsymbol{\alpha}}, \theta) = \sum_{u=1}^U \frac{1}{2} \dot{\boldsymbol{\alpha}}^{(u)} \cdot \mathbb{M}^{(r,u)}(\theta) \dot{\boldsymbol{\alpha}}^{(u)}, \quad (4)$$

where $\mathbb{L}^{(r,u)}$ and $\mathbb{M}^{(r,u)}$ denote positive-definite tensors of elastic and viscous moduli, respectively, characterizing the u th unit of the rheological model for phase r , $\boldsymbol{\beta}^{(r)}$ is the corresponding thermal straining, and $f^{(r)}$ is a function of temperature characterizing the specific heat capacity of the solid.³ The form of these potentials is motivated by a rheological model corresponding to U units of the Maxwellian type connected in parallel, along with a thermal unit connected in series, as depicted in [Fig. 1](#). This model can accommodate any variation of material properties with instant temperature change, provided it satisfies the requirements imposed on the thermodynamic potentials. It generates mechanical relations and evolution equations of the form

$$\boldsymbol{\sigma} = \sum_{u=1}^U \mathbb{L}^{(r,u)}(\theta) (\boldsymbol{\varepsilon} - \boldsymbol{\beta}^{(r)}(\theta) - \boldsymbol{\alpha}^{(u)}) \quad (5)$$

and

$$\mathbb{M}^{(r,u)}(\theta) \dot{\boldsymbol{\alpha}}^{(u)} + \mathbb{L}^{(r,u)}(\theta) \boldsymbol{\alpha}^{(u)} = \mathbb{L}^{(r,u)}(\theta) (\boldsymbol{\varepsilon} - \boldsymbol{\beta}^{(r)}(\theta)). \quad (6)$$

The instantaneous elastic stiffness of the material is thus

$$\mathbb{L}^{(r)}(\theta) = \sum_{u=1}^U \mathbb{L}^{(r,u)}(\theta). \quad (7)$$

When the elastic moduli are insensitive to temperature changes, the corresponding entropy density is given by

$$\eta = \sum_{u=1}^U (\boldsymbol{\varepsilon} - \boldsymbol{\beta}^{(r)}(\theta) - \boldsymbol{\alpha}^{(u)}) \cdot \mathbb{L}^{(r,u)} \frac{\partial \boldsymbol{\beta}^{(r)}}{\partial \theta}(\theta) - \frac{\partial f^{(r)}}{\partial \theta}(\theta) = \boldsymbol{\sigma} \cdot \frac{\partial \boldsymbol{\beta}^{(r)}}{\partial \theta}(\theta) - \frac{\partial f^{(r)}}{\partial \theta}(\theta); \quad (8)$$

when the elastic moduli are sensitive to temperature changes, additional terms arise, but are omitted for conciseness.

2.2. Macroscopic description

The homogenized response relates the macroscopic stress $\bar{\boldsymbol{\sigma}}$ and entropy density $\bar{\eta}$ to the macroscopic strain $\bar{\boldsymbol{\varepsilon}}$ and temperature change $\bar{\theta}$, which are the averages of the local stress, entropy density, strain and temperature fields over the representative volume element. This relation can be written in terms of the macroscopic free-energy density and dissipation potential as (e.g., [Germain et al., 1983](#); [Suquet, 1985](#))

$$\bar{\boldsymbol{\sigma}} = \frac{\partial \bar{w}}{\partial \bar{\boldsymbol{\varepsilon}}}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\alpha}, \bar{\theta}), \quad \bar{\eta} = -\frac{\partial \bar{w}}{\partial \bar{\theta}}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\alpha}, \bar{\theta}) \quad \text{and} \quad \frac{\partial \bar{w}}{\partial \boldsymbol{\alpha}(\mathbf{x})}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\alpha}, \bar{\theta}) + \frac{\partial \bar{\varphi}}{\partial \dot{\boldsymbol{\alpha}}(\mathbf{x})}(\dot{\boldsymbol{\alpha}}, \bar{\theta}) = \mathbf{0}, \quad (9)$$

where

$$\bar{w}(\bar{\boldsymbol{\varepsilon}}, \boldsymbol{\alpha}, \bar{\theta}) = \inf_{\boldsymbol{\varepsilon} \in \mathcal{K}(\bar{\boldsymbol{\varepsilon}})} \langle w(\mathbf{x}, \boldsymbol{\varepsilon}, \boldsymbol{\alpha}, \bar{\theta}) \rangle \quad \text{and} \quad \bar{\varphi}(\dot{\boldsymbol{\alpha}}, \bar{\theta}) = \langle \varphi(\mathbf{x}, \dot{\boldsymbol{\alpha}}, \bar{\theta}) \rangle. \quad (10)$$

In these expressions, $\mathcal{K}(\bar{\boldsymbol{\varepsilon}})$ is the set of kinematically admissible strain fields with average $\bar{\boldsymbol{\varepsilon}}$, $\langle \cdot \rangle$ denotes volume averaging over the representative volume element, and the δ operator denotes a functional

² It is recalled that this class of rheologies can be put into correspondence with generalized Kelvin–Voigt rheologies (see for instance [Gutierrez-Lemini, 2014](#)).

³ A possible choice is $f^{(r)}(\theta) = c_\sigma^{(r)} \left[\theta - (T_0 + \theta) \ln \frac{T_0 + \theta}{T_0} \right] - \eta_0^{(r)}(T_0 + \theta) + e_0^{(r)}$ with T_0 , $\eta_0^{(r)}$ and $e_0^{(r)}$ representing reference temperature, entropy and internal energy levels, respectively, and $c_\sigma^{(r)}$ representing the heat capacity at fixed stress.

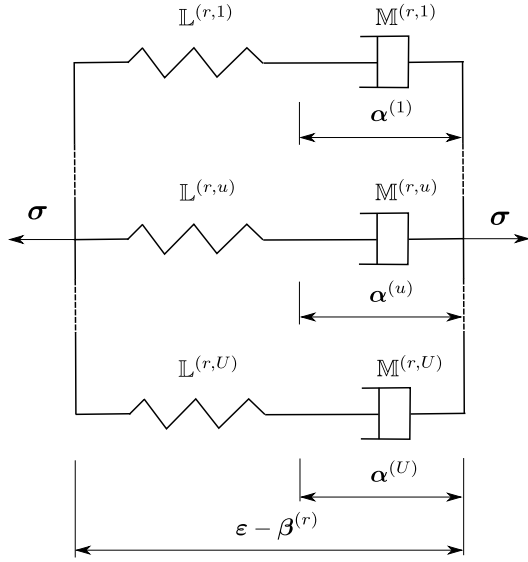


Fig. 1. Generalized Maxwell rheological model composed of U viscoelastic units in parallel, along with the thermal strain in series.

derivative. That the temperature change within the entire representative volume element is uniformly given by the macroscopic temperature change $\bar{\theta}$ has been found in multiple works with varying degrees of rigor and generality (e.g., Francfort, 1983; Yu and Fish, 2002; Amasov et al., 2013; Chatzigeorgiou et al., 2016) provided the local thermodynamic potentials satisfy certain mathematical properties that are assumed to hold in this work.

It is observed that the macroscopic free-energy density and dissipation potential are the volume averages of their microscopic counterparts, and are therefore functionals of the microscopic viscous strain fields and their rates. In the case of the generalized Maxwellian rheologies considered in this work, they take the forms

$$\bar{w}(\bar{\epsilon}, \alpha, \bar{\theta}) = \inf_{\epsilon \in \mathcal{K}(\bar{\epsilon})} \sum_{r=1}^N \sum_{u=1}^U c^{(r)} \left\langle \frac{1}{2} \left(\epsilon - \beta^{(r)}(\bar{\theta}) - \alpha^{(u)} \right) \cdot \mathbb{L}^{(r,u)}(\bar{\theta}) \left(\epsilon - \beta^{(r)}(\bar{\theta}) - \alpha^{(u)} \right) \right\rangle^{(r)} + \sum_{r=1}^N c^{(r)} f^{(r)}(\bar{\theta}) \quad (11)$$

and

$$\bar{\varphi}(\dot{\alpha}, \bar{\theta}) = \sum_{r=1}^N \sum_{u=1}^U c^{(r)} \left\langle \frac{1}{2} \dot{\alpha}^{(u)} \cdot \mathbb{M}^{(r,u)}(\bar{\theta}) \dot{\alpha}^{(u)} \right\rangle^{(r)}, \quad (12)$$

where the symbol $c^{(r)} = \langle \chi^{(r)} \rangle$ has been introduced to denote the volume fraction of phase r . These functionals inherit the convexity of the local potentials. Thus, homogenization preserves the generalized standard structure of the local response, with the microscopic viscous strain fields playing the role of macroscopic internal variables albeit of *infinite* dimension. The purpose of the approximate scheme presented in the next section is to reduce the dimensionality of the macroscopic internal variables to a *finite* number.

2.3. Extension to a class of hereditary responses

Thermal strains in polymers near the glass transition temperature are time dependent and can exhibit significant differences depending on whether heating or cooling occurs. Hereditary laws are often employed to describe such behavior (e.g., Knauss and Emri, 1987; Petermann and DeSimone, 2018). While the formalism presented above along with the model reduction presented below are cast in terms of internal-variable laws, the ensuing description for the mechanical response remains valid *ut stat* for more general constitutive laws of the

form (5)–(6) with the dependence on temperature change θ replaced by a dependence on temperature change history θ^t up to time t as defined by $\theta^t(\tau) = \theta(t-\tau)$ for $0 \leq \tau < \infty$, provided the replacement maintains the validity of the homogenization procedure leading to the macroscopic description (9)–(10). This is because the equilibrium conditions and the evolution laws for the internal variables within such representative volume element still follow from the macroscopic potentials (11) and (12) —now functionals of the macroscopic temperature change histories $\bar{\theta}^t$ —, and the incremental statement (14) still constitutes a valid time discretization of those evolution laws. On the other hand, the thermodynamic status of the potentials and the ensuing thermal response will be altered (cf., e.g., Gurtin and Hrusa, 1991; Boussaa, 2011).

3. Reduced-order description

3.1. Model reduction

Central to our purposes is the observation that the model reduction for viscoelastic composites proposed by Lhellec and Suquet (2007a) carries over to thermo-viscoelastic composites unaltered, regardless of the variation of mechanical properties with temperature. This model reduction relies on an implicit Euler discretization in time of the macroscopic evolution law (9)₂, so that the internal variables at the current instant are solution to the algebraic equation

$$\frac{\delta \bar{w}}{\delta \alpha(\mathbf{x})}(\bar{\epsilon}, \alpha, \bar{\theta}) + \frac{\delta \bar{\varphi}}{\delta \dot{\alpha}(\mathbf{x})} \left(\frac{\alpha - \alpha_n}{\Delta t}, \bar{\theta} \right) = \mathbf{0}, \quad (13)$$

where α_n are the values of the internal variables in the previous instant and Δt is the time step. In view of the convexity of the macroscopic potentials, this equation is the optimality condition of the variational problem

$$\inf \left[\bar{w}(\bar{\epsilon}, \alpha, \bar{\theta}) + \Delta t \bar{\varphi} \left(\frac{\alpha - \alpha_n}{\Delta t}, \bar{\theta} \right) \right]. \quad (14)$$

As recently elicited by Idiart et al. (2020a), the model reduction of Lhellec and Suquet (2007a) is effected upon estimating each phase average in the discretized dissipation (12) by Cauchy–Schwarz lower bounds. In practice, these bounds are applied on suitably chosen projections by expressing the constitutive tensors as

$$\mathbb{L}^{(r,u)}(\theta) = \sum_{b=1}^B \ell^{(r,u,b)}(\theta) \mathbb{B}^{(r,u,b)} \quad \text{and} \quad \mathbb{M}^{(r,u)}(\theta) = \sum_{b=1}^B m^{(r,u,b)}(\theta) \mathbb{B}^{(r,u,b)}, \quad (15)$$

where $\mathbb{B}^{(r,u,b)}$ are fourth-order symmetric tensors such that $\mathbb{B}^{(r,u,b)} \mathbb{B}^{(r,u,d)} = 0$ if $b \neq d$ and the moduli $\ell^{(r,u,b)}$ and $m^{(r,u,b)}$ are all positive functions of temperature (e.g., Walpole, 1984). For instance, within an isotropic phase use can be made of the decomposition $\mathbb{B}^{(r,u,1)} = \mathbb{J}$ and $\mathbb{B}^{(r,u,2)} = \mathbb{K}$ with $B = 2$, where \mathbb{J} and \mathbb{K} are the standard fourth-order, isotropic, hydrostatic and shear projection tensors, respectively. The macroscopic potentials can then be written as

$$\bar{w}(\bar{\epsilon}, \alpha, \bar{\theta}) = \inf_{\epsilon \in \mathcal{K}(\bar{\epsilon})} \sum_{r=1}^N \sum_{u=1}^U \sum_{b=1}^B c^{(r)} \ell^{(r,u,b)}(\bar{\theta}) \left\langle \frac{1}{2} \left(\epsilon - \beta^{(r)}(\bar{\theta}) - \alpha^{(u)} \right) \cdot \mathbb{B}^{(r,u,b)} \left(\epsilon - \beta^{(r)}(\bar{\theta}) - \alpha^{(u)} \right) \right\rangle^{(r)} + \sum_{r=1}^N c^{(r)} f^{(r)}(\bar{\theta}) \quad (16)$$

and

$$\bar{\varphi} \left(\frac{\alpha - \alpha_n}{\Delta t}, \bar{\theta} \right) = \sum_{r=1}^N \sum_{u=1}^U \sum_{b=1}^B c^{(r)} m^{(r,u,b)}(\bar{\theta}) \left\langle \frac{1}{2} \frac{\alpha^{(u)} - \alpha_n^{(u)}}{\Delta t} \cdot \mathbb{B}^{(r,u,b)} \frac{\alpha^{(u)} - \alpha_n^{(u)}}{\Delta t} \right\rangle^{(r)}. \quad (17)$$

Now, each term in this last expression for the discretized dissipation potential can be bounded from below as (Idiart et al., 2020a)

$$\left\langle \left(\frac{\alpha^{(u)} - \alpha_n^{(u)}}{\Delta t} \right) \cdot \mathbb{B}^{(r,u,b)} \left(\frac{\alpha^{(u)} - \alpha_n^{(u)}}{\Delta t} \right) \right\rangle^{(r)} \geq \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} \cdot \mathbb{B}^{(r,u,b)} \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} + \left(\frac{C_\alpha^{(r,u,b)1/2} - C_{\alpha_n}^{(r,u,b)1/2}}{\Delta t} \right)^2, \quad (18)$$

where

$$C_\alpha^{(r,u,b)} = \langle (\alpha^{(u)} - \langle \alpha^{(u)} \rangle^{(r)}) \cdot \mathbb{B}^{(r,u,b)} (\alpha^{(u)} - \langle \alpha^{(u)} \rangle^{(r)}) \rangle^{(r)} \quad (19)$$

and $C_\alpha^{(r,u,b)}$ are traces of the intraphase fluctuations of the viscous strain fields $\alpha^{(u)}$ and $\alpha_n^{(u)}$, respectively. The discretized dissipation functional is thus approximated by the discretized dissipation function

$$\begin{aligned} \bar{\varphi} \left(\frac{\alpha - \alpha_n}{\Delta t}, \bar{\theta} \right) &\approx \hat{\varphi} \left(\left\{ \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} \right\}, \left\{ \frac{C_\alpha^{(r,u,b)1/2} - C_{\alpha_n}^{(r,u,b)1/2}}{\Delta t} \right\}, \bar{\theta} \right) \\ &= \frac{1}{2} \sum_{r=1}^N \sum_{u=1}^U c^{(r)} \left[\frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} \cdot \mathbb{M}^{(r,u)}(\bar{\theta}) \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} + \sum_{b=1}^B m^{(r,u,b)}(\bar{\theta}) \left(\frac{C_\alpha^{(r,u,b)1/2} - C_{\alpha_n}^{(r,u,b)1/2}}{\Delta t} \right)^2 \right], \quad (20) \end{aligned}$$

which depends on the viscous strain fields only through their phase averages and intraphase fluctuations. Here, $\{\cdot\}$ refers to an entire collection of variables for all $r = 1, \dots, N$, $u = 1, \dots, U$, and $b = 1, \dots, B$. Making use of this estimate in the discretized evolution law (14) generates the variational approximation

$$\inf_{\alpha} \left[\bar{w}(\bar{\varepsilon}, \alpha, \bar{\theta}) + \Delta t \hat{\varphi} \left(\left\{ \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} \right\}, \left\{ \frac{C_\alpha^{(r,u,b)1/2} - C_{\alpha_n}^{(r,u,b)1/2}}{\Delta t} \right\}, \bar{\theta} \right) \right], \quad (21)$$

which, upon partitioning the infimum problem, can be written as

$$\inf_{\bar{\alpha}^{(r,u)} \geq 0} \left[\hat{w}(\bar{\varepsilon}, \{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\}, \bar{\theta}) + \Delta t \hat{\varphi} \left(\left\{ \frac{\langle \alpha^{(u)} \rangle^{(u)} - \langle \alpha_n^{(u)} \rangle^{(u)}}{\Delta t} \right\}, \left\{ \frac{C_\alpha^{(r,u,b)1/2} - C_{\alpha_n}^{(r,u,b)1/2}}{\Delta t} \right\}, \bar{\theta} \right) \right], \quad (22)$$

where

$$\hat{w}(\bar{\varepsilon}, \{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\}, \bar{\theta}) = \inf_{\alpha \in \mathcal{J}(\{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\})} \bar{w}(\bar{\varepsilon}, \alpha, \bar{\theta}) \quad (23)$$

and

$$\mathcal{J}(\{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\}) = \left\{ \alpha : \langle \alpha^{(u)} \rangle^{(r)} = \bar{\alpha}^{(r,u)} \text{ and } C_\alpha^{(r,u,b)1/2} = \tilde{\alpha}^{(r,u,b)} \right\}, \quad (24)$$

for $r = 1, \dots, N$, $u = 1, \dots, U$, $b = 1, \dots, B$. For later reference, we note that the internal variable fields $\alpha^{(u)}(\mathbf{x})$ minimizing the free-energy density (23) can be shown to be

$$\left(\alpha^{(u)}(\mathbf{x}) - \bar{\alpha}^{(r,u)} \right) = \sum_{b=1}^B \frac{\tilde{\alpha}^{(r,u,b)}}{C_\alpha^{(r,u,b)1/2}} \mathbb{B}^{(r,u,b)} (\varepsilon(\mathbf{x}) - \langle \varepsilon \rangle^{(r)}) \text{ in } \Omega^{(r)}, \quad (25)$$

where use has been made of the decomposition (15), and the symbols $C_\varepsilon^{(r,u,b)} = \mathbb{B}^{(r,u,b)} \cdot \mathbb{C}_\varepsilon^{(r)}$ have been introduced to denote traces of the fourth-order covariance tensors of the strain field within phase r as given by

$$\mathbb{C}_\varepsilon^{(r)} = \langle (\varepsilon - \langle \varepsilon \rangle^{(r)}) \otimes (\varepsilon - \langle \varepsilon \rangle^{(r)}) \rangle^{(r)}. \quad (26)$$

The infimum problem (22) is now seen to constitute an implicit Euler discretization of the continuous evolution laws

$$\frac{\partial \hat{w}}{\partial \alpha^{(r,u)}} \left(\bar{\varepsilon}, \{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\}, \bar{\theta} \right) + \frac{\partial \hat{\varphi}}{\partial \alpha^{(r,u)}} \left(\left\{ \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} \right\}, \left\{ \frac{C_\alpha^{(r,u,b)1/2} - C_{\alpha_n}^{(r,u,b)1/2}}{\Delta t} \right\}, \bar{\theta} \right) = \mathbf{0}, \quad (27)$$

$$\frac{\partial \hat{w}}{\partial \tilde{\alpha}^{(r,u,b)}} \left(\bar{\varepsilon}, \{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\}, \bar{\theta} \right) + \frac{\partial \hat{\varphi}}{\partial \tilde{\alpha}^{(r,u,b)}} \left(\left\{ \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} \right\}, \left\{ \frac{C_\alpha^{(r,u,b)1/2} - C_{\alpha_n}^{(r,u,b)1/2}}{\Delta t} \right\}, \bar{\theta} \right) = 0, \quad (28)$$

for $r = 1, \dots, N$, $u = 1, \dots, U$, and $b = 1, \dots, B$, where

$$\begin{aligned} \hat{w}(\bar{\varepsilon}, \{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\}, \bar{\theta}) &= \inf_{\varepsilon \in \mathcal{K}(\bar{\varepsilon})} \sum_{r=1}^N \sum_{u=1}^U \frac{c^{(r)}}{2} \times \\ &\left[\left(\langle \varepsilon \rangle^{(r)} - \beta^{(r)}(\bar{\theta}) - \bar{\alpha}^{(r,u)} \right) \cdot \mathbb{L}^{(r,u)}(\bar{\theta}) \left(\langle \varepsilon \rangle^{(r)} - \beta^{(r)}(\bar{\theta}) - \bar{\alpha}^{(r,u)} \right) \right. \\ &\left. + \sum_{b=1}^B \ell^{(r,u,b)}(\bar{\theta}) \left(C_\varepsilon^{(r,u,b)1/2} - \tilde{\alpha}^{(r,u,b)} \right)^2 \right] + \sum_{r=1}^N c^{(r)} f^{(r)}(\bar{\theta}) \quad (29) \end{aligned}$$

and

$$\begin{aligned} \hat{\varphi} \left(\left\{ \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} \right\}, \left\{ \frac{C_\alpha^{(r,u,b)1/2} - C_{\alpha_n}^{(r,u,b)1/2}}{\Delta t} \right\}, \bar{\theta} \right) &= \sum_{r=1}^N \sum_{u=1}^U \frac{c^{(r)}}{2} \times \\ &\left[\frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} \cdot \mathbb{M}^{(r,u)}(\bar{\theta}) \frac{\langle \alpha^{(u)} \rangle^{(r)} - \langle \alpha_n^{(u)} \rangle^{(r)}}{\Delta t} + \sum_{b=1}^B m^{(r,u,b)}(\bar{\theta}) \tilde{\alpha}^{(r,u,b)2} \right]. \quad (30) \end{aligned}$$

Expression (29) follows from expression (23) evaluated at the minimizing internal variable fields (25), while expression (30) follows from expression (20) evaluated at a vanishing time step $\Delta t \rightarrow 0$ (see Idiart et al., 2020a, for details). The main observation in the context of these expressions is that the system of Eqs. (27)–(28) provides a reduced-order description of the macroscopic thermo-viscoelastic evolution problem in terms of a finite set of effective internal variables identified with the first moments of the viscous strains over each phase $\bar{\alpha}^{(r,u)}$ and the second moments of their intraphase fluctuations $\tilde{\alpha}^{(r,u,b)}$, and a pair of reduced effective potentials identified with a free-energy density \hat{w} and a dissipation potential $\hat{\varphi}$; in addition, the reduced free-energy density provides the estimates

$$\begin{aligned} \bar{\sigma} &= \frac{\partial \hat{w}}{\partial \bar{\varepsilon}} \left(\bar{\varepsilon}, \{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\}, \bar{\theta} \right) \quad \text{and} \\ \bar{\eta} &= - \frac{\partial \hat{w}}{\partial \bar{\theta}} \left(\bar{\varepsilon}, \{\bar{\alpha}^{(r,u)}\}, \{\tilde{\alpha}^{(r,u,b)}\}, \bar{\theta} \right) \quad (31) \end{aligned}$$

for the macroscopic stress and entropy density.

Evaluating the various derivatives in these expressions we obtain the reduced mechanical relation

$$\bar{\sigma} = \sum_{r=1}^N \sum_{u=1}^U c^{(r)} \mathbb{L}^{(r,u)}(\bar{\theta}) \left(\bar{\varepsilon}^{(r)} - \beta^{(r)}(\bar{\theta}) - \bar{\alpha}^{(r,u)} \right) \quad (32)$$

along with the reduced evolution equations

$$\mathbb{M}^{(r,u)}(\bar{\theta}) \dot{\bar{\alpha}}^{(r,u)} + \mathbb{L}^{(r,u)}(\bar{\theta}) \bar{\alpha}^{(r,u)} = \mathbb{L}^{(r,u)}(\bar{\theta}) \left(\bar{\varepsilon}^{(r)} - \beta^{(r)}(\bar{\theta}) \right) \quad (33)$$

$$m^{(r,u,b)}(\bar{\theta}) \dot{\tilde{\alpha}}^{(r,u,b)} + \ell^{(r,u,b)}(\bar{\theta}) \tilde{\alpha}^{(r,u,b)} = \ell^{(r,u,b)}(\bar{\theta}) \tilde{\varepsilon}^{(r,u,b)} \quad (34)$$

for $r = 1, \dots, N$, $u = 1, \dots, U$, and $b = 1, \dots, B$, where $\bar{\varepsilon}^{(r)} = \langle \varepsilon \rangle^{(r)}$ and $\tilde{\varepsilon}^{(r,u,b)} = C_\varepsilon^{(r,u,b)1/2}$ are the phase averages and intraphase fluctuations of the strain field that minimizes the reduced free-energy density (29) at the prescribed macroscopic strain $\bar{\varepsilon}$ and temperature change $\bar{\theta}$. The Euler–Lagrange equations for the minimizing strain field, along with the minimizing internal variable fields (25), also imply the intraphase reduced stress statistics

$$\bar{\sigma}^{(r)} = \langle \sigma \rangle^{(r)} = \sum_{u=1}^U \mathbb{L}^{(r,u)}(\bar{\theta}) \left(\bar{\varepsilon}^{(r)} - \beta^{(r)}(\bar{\theta}) - \bar{\alpha}^{(r,u)} \right) \quad (35)$$

and

$$\begin{aligned} \bar{\sigma}^{(r,b)} &= \sum_{u=1}^U \langle (\sigma - \langle \sigma \rangle^{(r)}) \cdot \mathbb{B}^{(r,u,b)} (\sigma - \langle \sigma \rangle^{(r)}) \rangle^{(r)1/2} \\ &= \left| \sum_{u=1}^U \ell^{(r,u,b)}(\bar{\theta}) \left(\tilde{\varepsilon}^{(r,u,b)} - \tilde{\alpha}^{(r,u,b)} \right) \right|, \quad (36) \end{aligned}$$

which will serve to quantify the amount of residual stresses present within each phase r of the composite. Finally, the reduced entropic relation can be similarly obtained. When the elastic moduli are insensitive

to temperature changes, the relation can be written as

$$\bar{\eta} = \sum_{r=1}^N c^{(r)} \left[\bar{\sigma}^{(r)} \cdot \frac{\partial \beta^{(r)}}{\partial \bar{\theta}}(\bar{\theta}) - \frac{\partial f^{(r)}}{\partial \bar{\theta}}(\bar{\theta}) \right]. \quad (37)$$

3.2. Mean-field homogenization

Instrumental to the above order reduction is the fact that the reduced constitutive relation and evolution Eqs. (32)–(34) depend on the strain field only through its first and second moments over each phase, and that these moments can be evaluated via linear homogenization techniques. This is evident by noting that, even though the minimization problem (29) for the strain field is *nonlinear*, the associated Euler–Lagrange equations are those of a *linear* thermoelastic comparison solid with the same microstructure as the thermo-viscoelastic composite but with piecewise uniform stiffness tensor and eigenstress field given by

$$\begin{aligned} \mathbb{L}_0^{(r)} &= \sum_{u=1}^U \sum_{b=1}^B \ell_0^{(r,u,b)} \mathbb{B}^{(r,u,b)} \quad \text{and} \\ \tau_0^{(r)} &= \sum_{u=1}^U \mathbb{L}^{(r,u)}(\bar{\theta}) \left(\bar{\varepsilon}^{(r)} - \beta^{(r)}(\bar{\theta}) - \bar{\alpha}^{(r,u)} \right) - \mathbb{L}_0^{(r)} \bar{\varepsilon}^{(r)}. \end{aligned} \quad (38)$$

where

$$\ell_0^{(r,u,b)} = \ell^{(r,u,b)}(\bar{\theta}) \frac{\tilde{\varepsilon}^{(r,u,b)} - \tilde{\alpha}^{(r,u,b)}}{\tilde{\varepsilon}^{(r,u,b)}}, \quad (39)$$

Therefore, the minimizing strain field in (29) is exactly identical to that of a linear comparison problem

$$\hat{w}_0(\bar{\varepsilon}, \bar{\theta}) = \text{stat}_{\varepsilon \in \mathcal{K}(\bar{\varepsilon})} \sum_{r=1}^N c^{(r)} \left\langle \frac{1}{2} \varepsilon \cdot \mathbb{L}_0^{(r)} \varepsilon + \tau_0^{(r)} \cdot \varepsilon \right\rangle^{(r)} \quad (40)$$

with constitutive tensors given self-consistently by (38). The stationary rather than extremal character of the variational problem (40) is due to the fact that the tensors $\mathbb{L}_0^{(r)}$ are not necessarily positive definite. In any event, this comparison energy density can be expressed as

$$\hat{w}_0(\bar{\varepsilon}, \bar{\theta}) = \frac{1}{2} \bar{\varepsilon} \cdot \tilde{\mathbb{L}}_0 \bar{\varepsilon} + \tilde{\tau}_0 \cdot \bar{\varepsilon} + \tilde{g}_0, \quad (41)$$

where $\tilde{\mathbb{L}}_0$, $\tilde{\tau}_0$ and \tilde{g}_0 are effective properties that can be determined with any suitable mean-field homogenization technique for N -phase linear thermoelastic solids (see, for instance, Ponte Castañeda and Suquet, 1998). The first and second moments of the strain field within each phase can then be determined from this comparison energy by evaluating the derivatives

$$\langle \varepsilon \rangle^{(r)} = \frac{1}{c^{(r)}} \frac{\partial \hat{w}_0}{\partial \tau_0^{(r)}}(\bar{\varepsilon}, \bar{\theta}) \quad \text{and} \quad \langle \varepsilon \otimes \varepsilon \rangle^{(r)} = \frac{2}{c^{(r)}} \frac{\partial \hat{w}_0}{\partial \mathbb{L}_0^{(r)}}(\bar{\varepsilon}, \bar{\theta}) \quad (42)$$

for $r = 1, \dots, N$, which follow from well-known relations for field statistics in linear heterogeneous media and, together with relations (38), constitute a set of algebraic non-linear equations for those moments. Whenever these equations exhibit multiple roots, the root giving the minimum value of (29) with positive-definite phase covariances of the strain field must be selected. In any event, the reduced free-energy density (23) and ensuing relations (32)–(34) are completely determined by the linear homogenization scheme of choice.

Since the stress fields associated with both variational problems agree exactly as well, the reduced constitutive relation (32) can also be obtained from the comparison energy density as

$$\bar{\sigma} = \frac{\partial \hat{w}_0}{\partial \bar{\varepsilon}}(\bar{\varepsilon}, \bar{\theta}) = \tilde{\mathbb{L}}_0 \bar{\varepsilon} + \tilde{\tau}_0, \quad (43)$$

where the partial derivative has been taken with comparison properties $\mathbb{L}_0^{(r)}$ and $\tau_0^{(r)}$ held fixed. Finally, relations (42) provide the first- and second-order intraphase statistics of the underlying strain field, and the corresponding stress statistics follow from the identities

$$\bar{\sigma}^{(r)} = \mathbb{L}_0^{(r)} \bar{\varepsilon}^{(r)} + \tau_0^{(r)} \quad \text{and} \quad \mathbb{C}_\sigma^{(r)} = \mathbb{L}_0^{(r)} \mathbb{C}_\varepsilon^{(r)} \mathbb{L}_0^{(r)}, \quad (44)$$

where $\mathbb{C}_\varepsilon^{(r)}$ and $\mathbb{C}_\sigma^{(r)}$ are the fourth-order covariance tensors of the strain and the stress fields, respectively, within phase r . When the elastic moduli are insensitive to temperature changes, the macroscopic entropy density can then be obtained by introducing the relations (42) into (37).

4. Specialization to rigidly reinforced solids with isotropic constituents

Viscoelastic solids containing elastically rigid reinforcements are of particular practical interest. We consider reinforced systems ($N = 2$) composed of a viscoelastic matrix phase ($r = 1$) characterized by isotropic constitutive tensors of the form

$$\begin{aligned} \beta^{(1)}(\theta) &= \beta^{(1)}(\theta) \mathbf{I}, \quad \mathbb{L}^{(1,u)}(\theta) = \mathbb{L}^{(u)}, \quad \mathbb{M}^{(1,u)}(\theta) = \tau^{(u)}(\theta) \mathbb{L}^{(u)} \quad \text{with} \\ \mathbb{L}^{(u)} &= 3\kappa^{(u)} \mathbb{J} + 2\mu^{(u)} \mathbb{K} \end{aligned} \quad (45)$$

$u = 1, \dots, U$, where \mathbf{I} denotes the second-order identity tensor, \mathbb{J} and \mathbb{K} denote the standard fourth-order isotropic bulk and shear projection tensors, respectively, $\kappa^{(u)}$ and $\mu^{(u)}$ represent bulk and shear elastic moduli, respectively, and the $\tau^{(u)}$ represent temperature-dependent relaxation times. These constitutive assumptions are typically employed to model the response of amorphous polymers (e.g., Christensen, 1982). In turn, the corresponding tensors in the reinforcement phase ($r = 2$) are all set to infinity. Thus, the reinforcements undergo purely thermal deformations given by

$$\beta^{(2)}(\theta) = \beta^{(2)}(\theta) \mathbf{I}. \quad (46)$$

Results reported below make use of the simplest possible choice of basis tensors given by

$$\begin{aligned} \mathbb{L}^{(u)} &= \ell^{(u)} \mathbb{B}^{(u)} \quad \text{where} \quad \ell^{(u)2} = 9\kappa^{(u)2} + 20\mu^{(u)2} \quad \text{and} \\ \mathbb{B}^{(u)} &= \frac{3\kappa^{(u)}}{\ell^{(u)}} \mathbb{J} + \frac{2\mu^{(u)}}{\ell^{(u)}} \mathbb{K}, \end{aligned} \quad (47)$$

so that $\mathbb{B}^{(u)} \cdot \mathbb{B}^{(u)} = 1$. The reduced potentials (29) and (30) can then be written as

$$\begin{aligned} \hat{w}(\bar{\varepsilon}, \{\bar{\alpha}^{(u)}\}, \{\tilde{\alpha}^{(u)}\}, \bar{\theta}) &= \inf_{\varepsilon \in \mathcal{K}_*(\bar{\varepsilon} - \beta^{(2)}(\bar{\theta}))} \frac{1-c}{2} \sum_{u=1}^U \left[\left(\frac{\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})}{1-c} - \bar{\alpha}^{(u)} \right) \right. \\ &\quad \left. \cdot \mathbb{L}^{(u)} \left(\frac{\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})}{1-c} - \bar{\alpha}^{(u)} \right) + \ell^{(u)} (\tilde{\varepsilon}^{(u)} - \tilde{\alpha}^{(u)})^2 \right] + \sum_{r=1}^2 c^{(r)} f^{(r)}(\bar{\theta}) \end{aligned} \quad (48)$$

and

$$\hat{\varphi}(\{\bar{\alpha}^{(u)}\}, \{\tilde{\alpha}^{(u)}\}, \bar{\theta}) = \frac{1-c}{2} \sum_{u=1}^U \tau^{(u)}(\bar{\theta}) \left[\bar{\alpha}^{(u)} \cdot \mathbb{L}^{(u)} \bar{\alpha}^{(u)} + \ell^{(u)} \tilde{\alpha}^{(u)2} \right], \quad (49)$$

where $c = c^{(2)}$ is the reinforcement content, $\bar{\beta} = c^{(1)} \beta^{(1)} + c^{(2)} \beta^{(2)}$ is the average thermal strain, $\bar{\alpha}^{(u)} \equiv \bar{\alpha}^{(1,u)}$ and $\tilde{\alpha}^{(u)} \equiv \tilde{\alpha}^{(1,u,1)}$ are the effective internal variables associated with the viscoelastic matrix, $\tilde{\varepsilon}^{(u)} \equiv \tilde{\varepsilon}^{(1,u,1)}$ is a measure of the strain fluctuations within the matrix phase given by

$$\tilde{\varepsilon}^{(u)} = \sqrt{\mathbb{B}^{(u)} \cdot \mathbb{C}_\varepsilon^{(1)}} = \sqrt{\frac{2\mu^{(u)} \tilde{\varepsilon}_d^{(1)2}}{\ell^{(u)}} + \frac{9\kappa^{(u)} \tilde{\varepsilon}_m^{(1)2}}{\ell^{(u)}}} \quad (50)$$

in terms of the fluctuations of the mean and deviatoric parts of the strain field $\tilde{\varepsilon}_m^{(1)2} = \mathbb{J} \cdot \mathbb{C}_\varepsilon^{(1)}/3$ and $\tilde{\varepsilon}_d^{(1)2} = \mathbb{K} \cdot \mathbb{C}_\varepsilon^{(1)}$, respectively, and $\mathcal{K}_*(\bar{\varepsilon} - \beta^{(2)}(\bar{\theta}))$ is the set of kinematically admissible strain fields with average $\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})$ that vanish identically within the reinforcing phase. In deriving this expression for the free-energy density, use has been made of the change of variables $\varepsilon \rightarrow \varepsilon + \beta^{(2)}(\bar{\theta})$ for convenience; the strain fluctuations (50) are insensitive to this change.

Following the arguments provided in the previous subsection, and invoking well-known relations for two-phase composites derived by Levin (1967), the effective constitutive relations and intraphase strain

statistics generated by (48) can be conveniently computed from the comparison energy density

$$\hat{w}_0(\bar{\varepsilon}, \bar{\theta}) = \frac{1}{2}(\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})) \cdot \tilde{\mathbb{L}}_0(\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})) + \tau_0 \cdot (\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})), \quad (51)$$

where $\tilde{\mathbb{L}}_0$ is the effective elasticity tensor of a rigidly reinforced solid with the same microstructural morphology as the reinforced solid of interest but with matrix properties

$$\begin{aligned} \mathbb{L}_0 &= 3\kappa_0 \mathbb{J} + 2\mu_0 \mathbb{K} \equiv \sum_{u=1}^U \ell_0^{(u)} \mathbb{B}^{(u)} \quad \text{and} \\ \tau_0 &= \sum_{u=1}^U \mathbb{L}^{(u)} \left(\frac{\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})}{1-c} - \bar{\alpha}^{(u)} \right) - \mathbb{L}_0 \frac{\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})}{1-c} \end{aligned} \quad (52)$$

given in terms of the comparison moduli $\ell_0^{(u)}$ defined by

$$\ell_0^{(u)} = \ell^{(u)} \frac{\tilde{\varepsilon}^{(u)} - \tilde{\alpha}^{(u)}}{\tilde{\varepsilon}^{(u)}} \quad u = 1, \dots, U, \quad (53)$$

and determined self-consistently by the accompanying identities

$$\begin{aligned} \tilde{\varepsilon}_d^{(1)} &= \frac{1}{1-c} \sqrt{(\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})) \cdot \left[(1-c) \frac{\partial \tilde{\mathbb{L}}_0}{\partial (2\mu_0)} - \mathbb{K} \right] (\bar{\varepsilon} - \beta^{(2)}(\bar{\theta}))} \quad \text{and} \\ \tilde{\varepsilon}_m^{(1)} &= \frac{1}{1-c} \sqrt{(\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})) \cdot \left[(1-c) \frac{\partial \tilde{\mathbb{L}}_0}{\partial (9\kappa_0)} - \frac{1}{3} \mathbb{J} \right] (\bar{\varepsilon} - \beta^{(2)}(\bar{\theta}))}. \end{aligned} \quad (54)$$

Note that expressions (47)₃ and (52)₁ imply the relations

$$\kappa_0 = \sum_{u=1}^U \frac{\ell_0^{(u)}}{\ell^{(u)}} \kappa^{(u)} \quad \text{and} \quad \mu_0 = \sum_{u=1}^U \frac{\ell_0^{(u)}}{\ell^{(u)}} \mu^{(u)}, \quad (55)$$

which should be invoked to express the strain fluctuations (54) in terms of the comparison moduli $\ell_0^{(u)}$, and with (53), in terms of the strain fluctuations $\tilde{\varepsilon}^{(u)}$. Evaluation of the reduced free-energy density (48) requires introducing the resulting expressions into (50) to generate a set of equations for the quantities $\tilde{\varepsilon}^{(u)}$ to be solved for given temperature change $\bar{\theta}$, macroscopic deformation $\bar{\varepsilon}$ and internal variables $\tilde{\alpha}^{(u)}$ ($u = 1, \dots, U$). In general, the set of equations requires numerical treatment.

Differentiation of (51) with respect to the macroscopic deformation generates the effective constitutive relations

$$\bar{\sigma} = \tilde{\mathbb{L}}_0(\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})) + \tau_0, \quad (56)$$

while differentiation of the reduced potentials (48) and (49) with respect to the internal variables generates the evolution laws

$$\tau^{(u)}(\bar{\theta}) \dot{\bar{\alpha}}^{(u)} + \bar{\alpha}^{(u)} = \frac{\bar{\varepsilon} - \beta^{(2)}(\bar{\theta})}{1-c} \quad \text{and} \quad \tau^{(u)}(\bar{\theta}) \dot{\tilde{\alpha}}^{(u)} + \tilde{\alpha}^{(u)} = \tilde{\varepsilon}^{(u)} \quad u = 1, \dots, U. \quad (57)$$

Finally, the macroscopic entropy density (37) reduces to

$$\bar{\eta} = - \sum_{r=1}^2 c^{(r)} \left[\bar{p}^{(r)} \beta^{(r)'}(\bar{\theta}) + f^{(r)'}(\bar{\theta}) \right], \quad (58)$$

where $\bar{p}^{(r)} = -\text{tr} \bar{\sigma}^{(r)}$ is the average hydrostatic pressure over phase r and the prime denotes differentiation with respect to $\bar{\theta}$.

5. Exact results for a class of rigidly reinforced solids with isotropic constituents

In order to assess the capabilities of the reduced-order mean-field description proposed above, results are generated in this section for a special class of rigidly reinforced solids whose effective response can be determined exactly by means of the correspondence principle. The viscoelastic matrix is characterized by constitutive tensors of the form (45) with relaxation times (e.g., Christensen, 1982)

$$\tau^{(u)}(\theta) = \tau_0^{(u)} a_T(\theta) \quad u = 1, \dots, U. \quad (59)$$

Thus, each rheological unit u can exhibit a viscosity characterized by a different relaxation time constant $\tau_0^{(u)}$ but the same temperature dependence through the common shift function $a_T(\theta)$; the ratio between relaxation times of different units is insensitive to temperature changes. For a given thermomechanical loading program, the constitutive relation and evolution laws within the matrix phase are then given by

$$\begin{aligned} \sigma(t) &= \sum_{u=1}^U \mathbb{L}^{(u)} \left(\varepsilon(t) - \beta^{(1)}(\bar{\theta}(t)) - \alpha^{(u)}(t) \right) \quad \text{and} \\ \tau_0^{(u)} a_T(\bar{\theta}(t)) \dot{\alpha}^{(u)}(t) + \alpha^{(u)}(t) &= \varepsilon(t) - \beta^{(1)}(\bar{\theta}(t)) \end{aligned}$$

for $u = 1, \dots, U$. This form of evolution laws permits the introduction of an ‘‘internal’’ time variable defined as

$$\xi = \int_0^t \frac{d\tau}{a_T(\bar{\theta}(\tau))}, \quad (60)$$

so that, with a slight abuse of notation, the constitutive relations are commonly rewritten as

$$\begin{aligned} \sigma(\xi) &= \sum_{u=1}^U \mathbb{L}^{(u)} \left(\varepsilon(\xi) - \beta^{(1)}(\xi) - \alpha^{(u)}(\xi) \right) \quad \text{and} \\ \tau_0^{(u)} \dot{\alpha}^{(u)}(\xi) + \alpha^{(u)}(\xi) &= \varepsilon(\xi) - \beta^{(1)}(\xi), \end{aligned} \quad (61)$$

where $(\dot{\cdot})$ denotes differentiation with respect to ξ . Application of the Laplace–Carson transform to these expressions yields

$$\sigma^*(p) = \mathbb{L}^*(p) \left(\varepsilon^*(p) - \beta^{(1)*}(p) \right) \quad \text{with} \quad \mathbb{L}^*(p) = \sum_{u=1}^U \frac{p}{p + \tau_0^{(u)-1}} \mathbb{L}^{(u)}, \quad (62)$$

where $(\cdot)^*$ refers to a transformed function. The correspondence principle together with the Levin relations then imply the effective response

$$\bar{\sigma}^*(p) = \tilde{\mathbb{L}}^*(p) \left(\bar{\varepsilon}^*(p) - \beta^{(2)*}(p) \right) + \mathbb{L}^*(p) \left(\beta^{(2)*}(p) - \beta^{(1)*}(p) \right) \quad (63)$$

along with intraphase stress averages

$$\bar{\sigma}^{(1)*}(p) = \mathbb{B}^{(1)*}(p) \bar{\sigma}^*(p) \quad \text{and} \quad \bar{\sigma}^{(2)*}(p) = \mathbb{B}^{(2)*}(p) \bar{\sigma}^*(p) \quad (64)$$

with $\mathbb{B}^{(1)*}(p) = \mathbb{L}^*(p) [(1-c) \tilde{\mathbb{L}}^*(p)]^{-1}$ and $\mathbb{B}^{(2)*}(p)$ such that $(1-c) \mathbb{B}^{(1)*}(p) + c \mathbb{B}^{(2)*}(p) = \mathbb{I}$, where $\tilde{\mathbb{L}}^*(p)$ depends on the microstructural morphology of the composite. When this dependence is of the form

$$\tilde{\mathbb{L}}^*(p) = \sum_{\rho=1}^R \frac{p}{p + \tau_*^{(\rho)-1}} \tilde{\mathbb{A}}^{(\rho)}, \quad (65)$$

the inverse transform of (63) for time-continuous loadings yields

$$\begin{aligned} \bar{\sigma}(\xi) &= \sum_{\rho=1}^R \tilde{\mathbb{A}}^{(\rho)} \int_0^\xi \left(\bar{\varepsilon}(s) - \beta^{(2)}(s) \right) e^{-\frac{\xi-s}{\tau_*^{(\rho)}}} ds \\ &+ \sum_{u=1}^U \mathbb{L}^{(u)} \int_0^\xi \left(\beta^{(2)}(s) - \beta^{(1)}(s) \right) e^{-\frac{\xi-s}{\tau_0^{(u)}}} ds. \end{aligned} \quad (66)$$

Two particular cases are considered next.

5.1. Particulate composites under hydrostatic loadings

We consider particulate composites with isotropic microstructural statistics such that the effective bulk modulus is given exactly by the Hashin–Shtrikman lower bound; thus (e.g., Willis, 1982)

$$3\tilde{\kappa}^*(p) = \sum_{u=1}^U \frac{p}{p + \tau_0^{(u)-1}} \frac{3\kappa^{(u)} + 4c\mu^{(u)}}{1-c}, \quad (67)$$

which is of the form (65) with $R = U$ and $\tau_*^{(u)} = \tau_o^{(u)}$. The macroscopic response (66) for spherical deformations $\bar{\epsilon}(\xi) = \bar{\epsilon}_m(\xi)\mathbf{I}$ is then given by

$$\bar{\sigma}_m(\xi) = \sum_{u=1}^U \int_0^\xi \left[\frac{3\kappa^{(u)} + 4c\mu^{(u)}}{1-c} \left(\dot{\bar{\epsilon}}_m(s) - \dot{\beta}^{(2)}(s) \right) + 3\kappa^{(u)} \left(\dot{\beta}^{(2)}(s) - \dot{\beta}^{(1)}(s) \right) \right] e^{-\frac{\xi-s}{\tau_o^{(u)}}} ds. \quad (68)$$

Corresponding results in the absence of thermal strains were given by Hashin (1965) and Ricaud and Masson (2009).

5.2. Fiber-reinforced composites under thermal loadings

We now consider fiber-reinforced composites with transversely isotropic microstructural statistics such that the effective elasticity tensor is given exactly by the Hashin–Shtrikman lower bound; thus (e.g., Willis, 1982)

$$\begin{aligned} \tilde{\mathbb{L}}^*(p) = & \sum_{u=1}^U \frac{p}{p + \tau_o^{(u)-1}} \left[\frac{6\kappa^{(u)} + 2(1+3c)\mu^{(u)}}{3(1-c)} \mathbb{E}_1 \right. \\ & + \frac{(1-3c)3\kappa^{(u)} - (1+3c)2\mu^{(u)}}{3(1-c)} (\mathbb{E}_2 + \mathbb{E}_3) \\ & + 2\mu^{(u)} \frac{\sum_{v=1}^U (p + \tau_o^{(v)-1})^{-1} [3(1+c)\kappa^{(v)} + (7+c)\mu^{(v)}]}{(1-c) \sum_{v=1}^U (p + \tau_o^{(v)-1})^{-1} [3\kappa^{(v)} + 7\mu^{(v)}]} \mathbb{E}_5 \\ & \left. + 2\mu^{(u)} \frac{1+c}{1-c} \mathbb{E}_6 \right], \quad (69) \end{aligned}$$

where the set of tensors \mathbb{E}_i corresponds to the basis for transversely isotropic tensors of Walpole (1984). This expression is of the form (65) with R depending on U . For a given number of units U , the parameters $\tau_*^{(p)}$ and corresponding tensors $\tilde{\mathbb{A}}^{(p)}$ ($p = 1, \dots, R$) are obtained by identifying the zeros of the rational polynomials in this expression. Alternatively, they can be obtained by means of the methodology presented in Jalocha et al. (2015) for frequency dependent responses. This procedure combines a highly accurate approximation of the discrete time spectra derived from the Nevanlinna Pick interpolation problem (Byrnes and Lindquist, 2000) with a conventional mean-squares procedure for the computation of the associated moduli distributions (see Suarez-Afanador et al., 2022).

6. Sample comparisons and discussion

6.1. Particulate composites under hydrostatic loadings

We consider the particulate composites of Section 5.1 subject to spherical deformations $\bar{\epsilon} = \bar{\epsilon}_m\mathbf{I}$. The reduced description based on effective internal variables require the effective bulk modulus of the comparison solid, which is given by (e.g., Willis, 1982)

$$3\tilde{\kappa}_0 = \frac{3\kappa_0 + 4c\mu_0}{1-c}. \quad (70)$$

Expression (54) can then be easily evaluated, and upon introducing this expression into (50) we obtain the expression

$$\tilde{\epsilon}^{(u)} = \sqrt{\frac{12c\mu^{(u)}}{\rho^{(u)}} \frac{|\bar{\epsilon}_m - \beta_m^{(2)}(\bar{\theta})|}{1-c}} \quad (71)$$

for the intraphase strain fluctuations. In turn, symmetry implies that $\bar{\alpha}^{(u)} = \bar{\alpha}^{(u)}\mathbf{I}$ for all $u = 1, \dots, U$ throughout the deformation history, provided all internal variables depart from zero. The reduced effective potentials (48) and (49) are thus given by

$$\hat{w}(\bar{\epsilon}_m\mathbf{I}, \{\bar{\alpha}^{(u)}\}, \{\tilde{\alpha}^{(u)}\}, \bar{\theta}) = \frac{1-c}{2} \sum_{u=1}^U \left[9\kappa^{(u)} \left(\frac{\bar{\epsilon}_m - \bar{\beta}(\bar{\theta})}{1-c} - \bar{\alpha}^{(u)} \right)^2 + \right.$$

$$\left. \rho^{(u)} \left(\sqrt{\frac{12c\mu^{(u)}}{\rho^{(u)}} \frac{|\bar{\epsilon}_m - \beta^{(2)}(\bar{\theta})|}{1-c}} - \tilde{\alpha}^{(u)} \right)^2 \right] + \sum_{r=1}^2 c^{(r)} f^{(r)}(\bar{\theta}) \quad \text{and} \quad (72)$$

$$\hat{\phi}(\{\bar{\alpha}^{(u)}\}, \{\tilde{\alpha}^{(u)}\}, \bar{\theta}) = \frac{1-c}{2} \sum_{u=1}^U \tau^{(u)}(\bar{\theta}) \left[9\kappa^{(u)} \bar{\alpha}^{(u)2} + \rho^{(u)} \tilde{\alpha}^{(u)2} \right], \quad (73)$$

where $\bar{\beta} = (1-c)\beta^{(1)} + c\beta^{(2)}$. These reduced potentials are explicit and completely characterize the overall response of the composite under any hydrostatic thermomechanical loading conditions. However, while the reduced dissipation potential is a convex function of the internal variable rates, the reduced free-energy density is *not* a convex function of the macroscopic deformation and internal variables conjointly.⁴ This non-convexity is introduced by the model reduction performed in Section 3.1 for reasons already elicited by Idiart et al. (2020a) in the context of isothermal viscoelasticity. The negative consequences on ensuing predictions for the macroscopic response have been assessed in depth by Idiart et al. (2020b). However, it was also found in that work that an efficacious convexification of this reduced free-energy could be carried out in some simple cases like the one considered here. This convexification amounts to eliminating the absolute value in the last term of (72) and adopting the function

$$\begin{aligned} \hat{w}(\bar{\epsilon}_m\mathbf{I}, \{\bar{\alpha}^{(u)}\}, \{\tilde{\alpha}^{(u)}\}, \bar{\theta}) = & \frac{1-c}{2} \sum_{u=1}^U \left[9\kappa^{(u)} \left(\frac{\bar{\epsilon}_m - \bar{\beta}(\bar{\theta})}{1-c} - \bar{\alpha}^{(u)} \right)^2 + \right. \\ & \left. \rho^{(u)} \left(\sqrt{\frac{12c\mu^{(u)}}{\rho^{(u)}} \frac{\bar{\epsilon}_m - \beta^{(2)}(\bar{\theta})}{1-c}} - \tilde{\alpha}^{(u)} \right)^2 \right] \\ & + \sum_{r=1}^2 c^{(r)} f^{(r)}(\bar{\theta}) \quad (74) \end{aligned}$$

as the reduced free-energy density. This function is now convex. Upon differentiating these reduced potentials we obtain the effective constitutive relation

$$\begin{aligned} \bar{\sigma}_m = & \sum_{u=1}^U \left[3\kappa^{(u)} \left(\frac{\bar{\epsilon}_m - \bar{\beta}(\bar{\theta})}{1-c} - \bar{\alpha}^{(u)} \right) \right. \\ & \left. + \sqrt{\frac{4c\mu^{(u)}\rho^{(u)}}{3}} \left(\sqrt{\frac{12c\mu^{(u)}}{\rho^{(u)}} \frac{\bar{\epsilon}_m - \beta^{(2)}(\bar{\theta})}{1-c}} - \tilde{\alpha}^{(u)} \right) \right] \quad (75) \end{aligned}$$

along with the evolution laws for the internal variables

$$\begin{aligned} \tau^{(u)}(\bar{\theta}) \dot{\bar{\alpha}}^{(u)} + \bar{\alpha}^{(u)} &= \frac{\bar{\epsilon}_m - \bar{\beta}(\bar{\theta})}{1-c} \quad \text{and} \\ \tau^{(u)}(\bar{\theta}) \dot{\tilde{\alpha}}^{(u)} + \tilde{\alpha}^{(u)} &= \sqrt{\frac{12c\mu^{(u)}}{\rho^{(u)}} \frac{\bar{\epsilon}_m - \beta^{(2)}(\bar{\theta})}{1-c}}. \quad (76) \end{aligned}$$

The reduced description (75)–(76) admits any choice of relaxation times $\tau^{(u)}(\bar{\theta})$. Now, it is easy to show that for the special choice (59), it reproduces the exact response (68). Indeed, in that case the Laplace–Carson transform of the evolution laws (76) expressed in terms of the internal time (60) yields

$$\bar{\alpha}^{(u)*}(p) = \frac{1}{1 + \tau_o^{(u)} p} \frac{\bar{\epsilon}_m^*(p) - \bar{\beta}^*(p)}{1-c} \quad \text{and}$$

⁴ It is emphasized that the reduced-order description remains thermodynamically consistent, notwithstanding, in view of the convexity of the reduced dissipation potential.

$$\tilde{\alpha}^{(u)*}(p) = \frac{1}{1 + \tau_o^{(u)} p} \sqrt{\frac{12c\mu^{(u)} \bar{\epsilon}_m^*(p) - \beta^{(2)*}(p)}{\ell^{(u)} (1-c)}}, \quad (77)$$

and introducing these expressions into the corresponding Laplace-Carson transform of (75) yields

$$\bar{\sigma}_m^*(p) = \sum_{u=1}^U \frac{p}{p + \tau_o^{(u)-1}} \left[\frac{3\kappa^{(u)} + 4c\mu^{(u)}}{1-c} (\bar{\epsilon}_m^*(p) - \beta^{(2)*}(p)) + 3\kappa^{(u)} (\beta^{(2)*}(p) - \beta^{(1)*}(p)) \right], \quad (78)$$

whose inverse transform is precisely (68). This remarkable result is in line with earlier results already reported in the context of isothermal viscoelasticity with simpler rheologies by Idiart et al. (2020b) and Cotelto et al. (2020). It shows that the variational model reduction performed in Section 3.1 has the capability of reproducing exact thermomechanical responses for composite materials with complex local responses and microstructural morphologies, even in the presence of multiple relaxation mechanisms.

Under stress-free conditions, the reduced description (75)–(76) predicts a macroscopic strain

$$\bar{\epsilon}_m = \bar{\beta}(\bar{\theta}) + \frac{4c\mu}{3\tilde{\kappa}} (\beta^{(2)}(\bar{\theta}) - \beta^{(1)}(\bar{\theta})) + \sum_{u=1}^U \left[\frac{\kappa^{(u)}}{\tilde{\kappa}} \bar{\alpha}^{(u)} + \sqrt{\frac{4c\mu^{(u)} \ell^{(u)}}{27\tilde{\kappa}^2}} \tilde{\alpha}^{(u)} \right] \quad (79)$$

and evolution laws

$$\begin{aligned} \tau^{(u)}(\bar{\theta}) \dot{\bar{\alpha}}^{(u)} + \bar{\alpha}^{(u)} - \sum_{v=1}^U \left[\frac{\kappa^{(v)}}{\tilde{\kappa}} \frac{\bar{\alpha}^{(v)}}{1-c} + \sqrt{\frac{4c\mu^{(v)} \ell^{(v)}}{3}} \frac{\tilde{\alpha}^{(v)}}{1-c} \right] \\ = \frac{c}{1-c} \frac{4\mu}{3\tilde{\kappa}} (\beta^{(2)}(\bar{\theta}) - \beta^{(1)}(\bar{\theta})) \end{aligned} \quad (80)$$

$$\begin{aligned} \tau^{(u)}(\bar{\theta}) \dot{\tilde{\alpha}}^{(u)} + \tilde{\alpha}^{(u)} - \sum_{v=1}^U \left[\frac{\kappa^{(v)}}{\tilde{\kappa}} \frac{\bar{\alpha}^{(v)}}{1-c} + \sqrt{\frac{4c\mu^{(v)} \ell^{(v)}}{3}} \frac{\tilde{\alpha}^{(v)}}{1-c} \right] \\ = \frac{c^{3/2}}{1-c} \frac{12\mu^{(u)}}{\ell^{(u)}} (\beta^{(2)}(\bar{\theta}) - \beta^{(1)}(\bar{\theta})). \end{aligned} \quad (81)$$

Casting expressions in this form makes it plain that the source of viscous deformations within the matrix phase is solely the mismatch between microscopic thermal strains, which consequently generate a dependence of the macroscopic thermal strain on cooling or heating rate even when the microscopic thermal strains are rate insensitive.

6.2. Fiber-reinforced composites under monotonic cooling

We now consider the unidirectional fiber-reinforced composites of Section 5.2. But for illustrative purposes, however, we adopt thermal strains within the matrix phase characterized by a hereditary law as given by a functional rather than a function of temperature; the functional is (e.g., Knauss and Emri, 1987; Pettermann and DeSimone, 2018)

$$\beta^{(1)}(\theta^\xi) = \int_0^\xi \alpha^{(1)}(\xi - s) \dot{\theta}(s) ds, \quad (82)$$

where ξ is the internal time defined by (60), θ^ξ is the temperature change history up to the internal instant ξ with notation introduced in Section 2.3, and $\alpha^{(1)}$ is a thermal creep function given by

$$\alpha^{(1)}(\xi) = \alpha_0^{(1)} - \sum_{v=1}^V \alpha^{(1,v)} e^{-\xi/\tau_{th}^{(v)}}. \quad (83)$$

In this expression, $\tau_{th}^{(v)}$ and $\alpha^{(1,v)}$ represent thermal relaxation times and thermal expansion coefficients, respectively, and $\alpha_0^{(1)}$ represents a thermal expansion coefficient at high temperature. With this choice, the mechanical relations (5)–(6) become hereditary laws of the single-integral type like the ones studied in Gurtin and Hrusa (1991). On the other hand, the thermal relation (8) can no longer be described by an expression of that form, which is inherent to thermal strain functions. We do not dwell on this issue given that the focus here is on mechanical

responses under prescribed temperature histories.⁵ In any event, this model takes into account the influence of heating and cooling rates on thermal expansion. On the other hand, the thermal strain within the fibers is chosen as

$$\beta^{(2)}(\theta) = \alpha^{(2)}\theta, \quad (85)$$

where $\alpha^{(2)}$ is a thermal expansion coefficient.

The composites are taken as representative of a lamina situated in the bulk of a symmetric cross-ply laminate undergoing monotonic cooling at constant pressure as in a consolidation process. The presence of contiguous orthogonal laminae is mimicked by subjecting the composite lamina to mixed boundary conditions of the form

$$\bar{\epsilon}_{11} = \bar{\epsilon}_{33} = \beta^{(2)}(\bar{\theta}), \quad \bar{\sigma}_{22} = 0, \quad \bar{\epsilon}_{12} = \bar{\epsilon}_{13} = \bar{\epsilon}_{23} = 0, \quad (86)$$

where tensor components are referred to a basis with directions 2 and 3 aligned with the directions of lamination and of the lamina fibers, respectively. Under these circumstances, the exact mechanical response is given by suitable specialization of expression (66) with (69). It is noted that despite their derivation in terms of thermal strain *functions*, those expressions remain valid for thermal strain *functionals* like (82). The numerical calculations required to evaluate the resulting expressions have been carried out following the strategy of Suarez-Afanador et al. (2022). In turn, the reduced description for the reinforced lamina depends on the effective in-plane bulk and shear moduli of the comparison solid, which are given by (e.g., Willis, 1982)

$$\tilde{\kappa}_0 = \frac{2\kappa_0 + (2/3)(1 + 3c)\mu_0}{1-c} \quad \text{and} \quad \tilde{m}_0 = \frac{3(1+c)\kappa_0 + (7+c)\mu_0}{3\kappa_0 + 7\mu_0} \frac{\mu_0}{1-c}, \quad (87)$$

respectively. Expressions (54) for the intraphase strain fluctuations then become

$$\tilde{\epsilon}_m^{(1)} = \frac{1}{1-c} \sqrt{\frac{1-c}{9} \left(\frac{\partial \tilde{\kappa}_0}{\partial (2\kappa_0)} + \frac{\partial \tilde{m}_0}{\partial \kappa_0} \right) - \frac{1}{9} |\bar{\epsilon}_{22} - \beta^{(2)}(\bar{\theta})|} \quad \text{and} \quad (88)$$

$$\tilde{\epsilon}_d^{(1)} = \frac{1}{1-c} \sqrt{\frac{1-c}{2} \left(\frac{\partial \tilde{\kappa}_0}{\partial (2\mu_0)} + \frac{\partial \tilde{m}_0}{\partial \mu_0} \right) - \frac{2}{3} |\bar{\epsilon}_{22} - \beta^{(2)}(\bar{\theta})|}, \quad (89)$$

which should be introduced in (50) to generate the set of equations for the strain fluctuations $\tilde{\epsilon}^{(u)}$ required to evaluate the reduced free-energy density for the boundary conditions (86). For a given cooling program $\bar{\theta}(t)$, the evolution laws (57) are discretized in time following an implicit Euler scheme and taken as additional equations which, together with the previous set of equations, are solved numerically for the quantities $\bar{\alpha}^{(u)}$, $\tilde{\alpha}^{(u)}$, $\tilde{\epsilon}^{(u)}$, and $\bar{\epsilon}_{22}$. Once these quantities are determined, the macroscopic stress component $\bar{\sigma}_{11}$ is computed from (56), while the statistics of the stress field within the matrix phase are computed from (35) and (36).

Specific results are reported below for matrix responses described with nineteen viscoelastic units ($U = 19$) characterized by constitutive tensors of the form (45) with relaxation times of the form (59). Fig. 2 shows the choice of elastic moduli and relaxation times ($\kappa^{(u)}$, $\mu^{(u)}$, $\tau_o^{(u)}$) along with the choice of shift factor $a_T(\theta)$. In turn, we adopt only one term ($V = 1$) in (83) with $\alpha_0^{(1)} = 1.02 \times 10^{-4} \text{ }^\circ\text{C}^{-1}$, $\alpha^{(1,1)} = 4.22 \times 10^{-5} \text{ }^\circ\text{C}^{-1}$ and $\tau_{th}^{(1)} = 0.2017 \text{ s}$. This set of material parameters is taken to represent the thermomechanical response of an amorphous thermoplastic like polyetherimide (see Suarez-Afanador et al., 2022).

⁵ A possible expression for the entropy density compatible with (82)–(83) under the proviso that $V \leq U$ is

$$\eta(\xi) = \int_0^\xi \sum_{u=1}^U 3\kappa^{(u)} \tilde{\alpha}^{(u)} e^{-\frac{\xi-s}{\tau_o^{(u)}}} (\tilde{\epsilon}(s) - \tilde{\alpha}^{(u)} \dot{\theta}(s)) ds - f^{(1)'}(\theta), \quad (84)$$

where $\tilde{\alpha}^{(u)}$ are suitably chosen parameters.

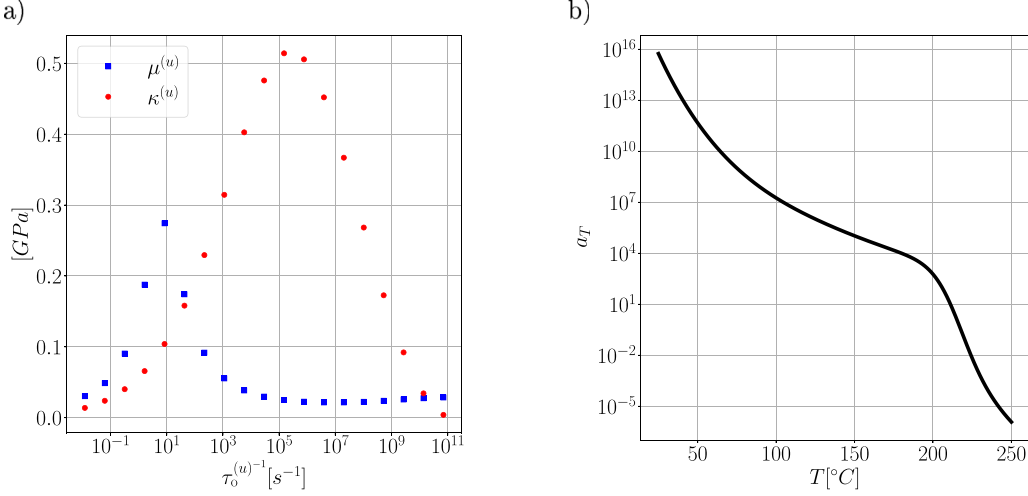


Fig. 2. Material parameters adopted for the viscoelastic matrix: (a) elastic moduli $\mu^{(u)}$ and $\kappa^{(u)}$ as a function of the relaxation times $\tau_o^{(u)}$ and (b) shift factor function a_T versus temperature.

In particular, it is noted that the choice of shift factor pertains to a material response that transitions from negligible elasticity at the higher temperatures to negligible viscosity at the lower temperatures. Finally, the choice $\alpha^{(2)} = 2.21 \times 10^{-5} \text{ }^\circ\text{C}^{-1}$ is adopted for the thermal expansion coefficient of the fibers. A composite with reinforcement content $c = 0.3$ is subject to monotonic cooling from a processing temperature of 250 °C down to the room temperature 25 °C, so that the total macroscopic temperature change relative to the initial temperature is $\bar{\theta}_0 = -225 \text{ }^\circ\text{C}$. Various cooling rates $\bar{\theta}$ in the wide range between $-225 \times 10^{-12} \text{ }^\circ\text{C/s}$ and $-225 \times 10^{20} \text{ }^\circ\text{C/s}$ are considered. These unrealistically extreme values have been chosen so that elastic deformations are negligible at the slowest rate while viscous deformations are negligible at the fastest rate. This last case is considered as representative of non-dissipative material models based on purely thermoelastic constitutive laws.

Fig. 3 displays predictions for the macroscopic stress ($\bar{\sigma}_{11}$) and strain ($\bar{\epsilon}_{22}$) versus temperature drop for the various cooling rates, normalized by the final stress and strain levels ($\bar{\sigma}_{11}^e$, $\bar{\epsilon}_{22}^e$) produced by the fastest cooling rate. The main observation in the context of this figure is that the reduced and exact predictions are quantitatively indistinguishable for the entire range of temperature drops and cooling rates considered. As a consequence of the choice of shift factor, the stresses and strains at the end of the cooling process do not relax but remain as residual instead. These residual stresses and deformations increase with increasing cooling rate, as expected. The reduced description is seen to provide accurate predictions for both of these quantities. In practice, cooling rates in the order of 1 °C/min (0.017 °C/s) are commonly employed. At such rates, the residual stress levels predicted by the thermo-viscoelastic model are in the order of 40% of those predicted by a purely thermoelastic model; the residual thermal shrinkages, in turn, are seen to be in the order of 80%. Also displayed in the figure are corresponding reduced predictions for the average values and fluctuations of the mean (σ_m) and deviatoric (σ_d) stresses within the matrix phase ($r = 1$). Exact results for stress fluctuations are not reported due to the well-known limitations of descriptions based on the correspondence principle to predict intraphase stress statistics of order higher than one. The reduced and exact predictions are seen to remain indistinguishable for the average stresses over the matrix phase. The predicted residual stress field within that phase is seen to exhibit fluctuation levels of about half of the average values. Once again, values for realistic cooling rates are seen to be well below the values for the fastest cooling rate. The associated evolution of the effective internal variables is provided for five representative units out of the nineteen, for the cooling rate 2.25 °C/s. It is recalled that the reduced variables $\tilde{\alpha}^{(u)}$ depend on traces of the intraphase fluctuations of the reduced internal variables $\alpha^{(u)}$ dictated by the choice of tensors $\mathbb{B}^{(u)}$ introduced in (47)₃. In all cases

the internal variables are seen evolve from the outset and up to certain temperature below which they remain constant. This is a manifestation of the viscous-to-elastic transition effected by the choice of shift factor. The saturation value and temperature change of each internal variable is dictated by the associated relaxation time $\tau_o^{(u)}$ shown in Fig. 2a. In any event, these results suggest that purely thermoelastic analyses which neglect all viscous mechanisms will, in general, provide quite inaccurate predictions for residual stresses resulting from consolidation processes of this sort.

We conclude this discussion by noting that the free-energy density of the fiber-reinforced composite employed by this reduced description is non-convex and cannot be convexified like the reduced free-energy density of the particulate composites considered in the previous subsection. Consequently, predictions will exhibit spurious transients for certain classes of non-monotonic thermomechanical programs for which the intraphase fluctuations of some thermodynamic forces may vanish (Idiart et al., 2020a,b). However, the non-convex reduced free-energy density (29) does agree exactly with its convexification whenever the deformation history is such that all reduced internal variables $\tilde{\alpha}^{(u)}$ increase monotonically (Idiart et al., 2020a). This is precisely what occurs in the above example and it is likely to occur during thermomechanical programs commonly employed in consolidation processes of practical interest. The reduced mean-field descriptions derived in this work are thus expected to provide a valuable tool to identify processing routes to lower residual stresses or use them to our advantage. The expectation should be confirmed by pursuing similar studies for material systems with other relevant microgeometries such as, for instance, discontinuous fiber-reinforced solids. In those cases, the linearly elastic properties required by the reduced mean-field descriptions are likely to introduce an additional degree of approximation. Given the complications associated with the use of the correspondence principle in those cases, the studies will certainly benefit from full-field simulations based on Fast Fourier Transform algorithms (e.g., Wicht et al., 2020; Krause and Böhlke, 2020; Suarez-Afanador et al., 2022).

7. Final comments

We have proposed a thermodynamically consistent framework for generating mean-field descriptions for the viscoelastic response of reinforced solids undergoing small deformations but large temperature changes. The main novelty of the work lies in the recognition that the variational model reduction of Labelle and Suquet (2007a) originally conceived for simple rheologies can also be exploited for a fairly large class of complex thermorheologies. In the absence of thermal effects, the mean-field descriptions generated in this work reduce to those previously generated by Idiart et al. (2020a). In all these descriptions

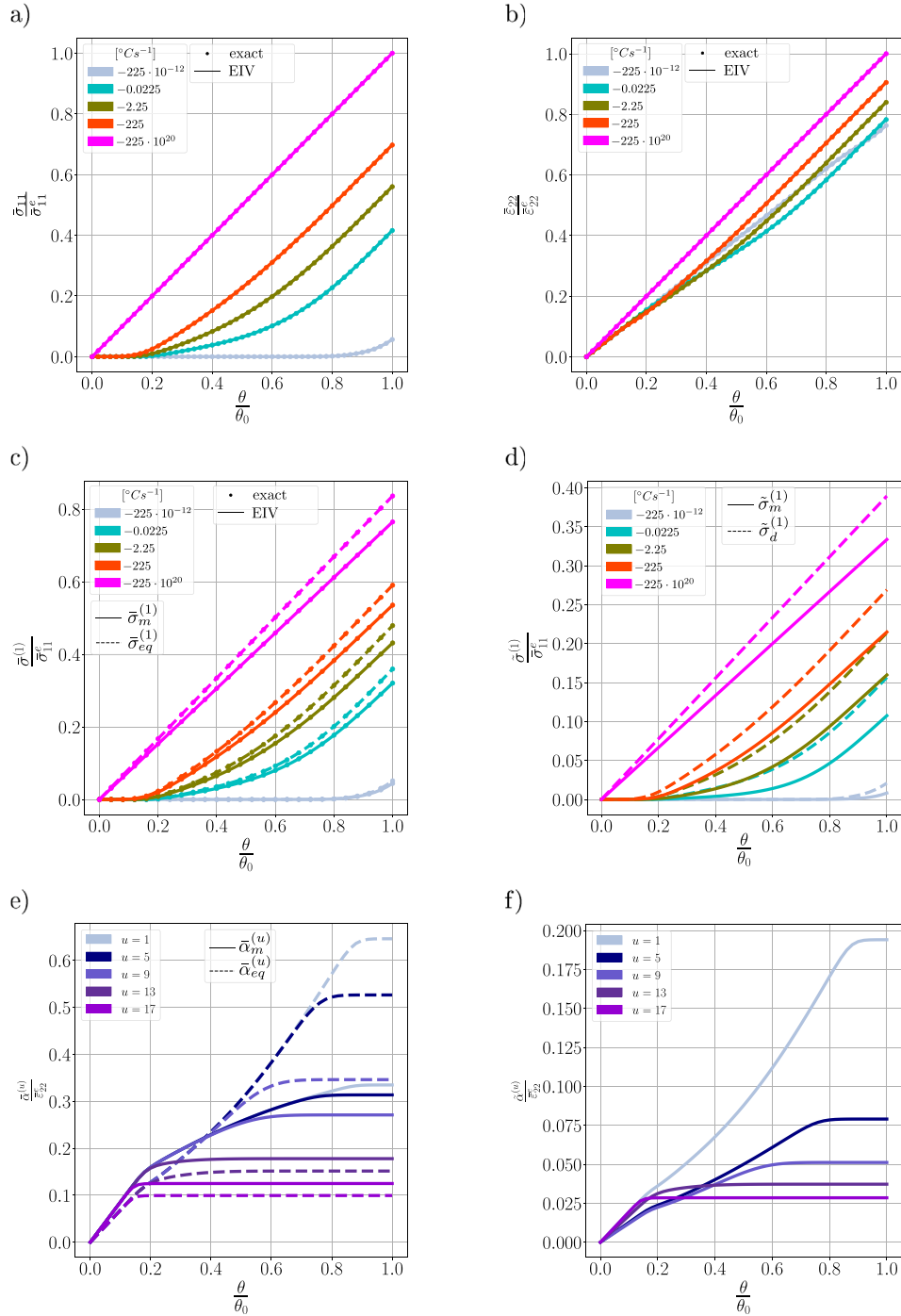


Fig. 3. Reduced-order estimates (EIV) and exact results versus temperature change for fiber-reinforced composites subject to monotonic cooling at various cooling rates: (a) macroscopic stress, (b) macroscopic strain, (c) and (d) average stress and stress fluctuations within the matrix phase, (e) and (f) evolution of effective internal variables for the cooling rate 2.25 °C/s. Stress and strain quantities are normalized, respectively, by the macroscopic residual stress and strain for the fastest cooling rate. The reinforcement content is $c = 0.3$.

the mechanical fields entering the potentials are the strains and internal variables. Dual mean-field descriptions wherein the mechanical fields entering the potentials are the stresses and thermodynamic forces have been proposed by [Lahellec and Suquet \(2013\)](#) and applied to viscoelastic composites and polycrystals in several studies ([Badulescu et al., 2015](#); [Agoras et al., 2016](#); [Cotelo et al., 2020](#); [Shuvrangsu and Ponte Castañeda, 2021](#)). The predictions generated by this alternative descriptions are different from those generated by the primal descriptions considered in this work, and their relative merits have been discussed by [Idiart and Lahellec \(2016\)](#) and [Lahellec et al. \(2021\)](#). This

last work casts the dual descriptions within an alternative formalism to that of [Lahellec and Suquet \(2013\)](#) which makes the generalization of dual descriptions to thermorheologically complex solids mathematically similar to the one carried out here for the primal description. A thorough comparison between the primal and dual versions of the reduced descriptions, along with approximate extensions of the correspondence principle to ageing behaviors ([Masson et al., 2012](#); [Vu et al., 2012](#)), will be reported elsewhere.

In any event, the mean-field descriptions provided in this work admit further generalizations with ease. For instance, they can incorporate nonlinear viscous mechanisms via non-quadratic dissipation potentials together with well-known linearization strategies (e.g., Lahelec and Suquet, 2007b; Idiart and Lahelec, 2016; Boudet et al., 2016). More importantly, they can incorporate curing deformations arising in thermosetting composites (e.g., Bogetti and Gillespie, 1992) via homogenization results for well-known curing kinetic laws (Amasov et al., 2013). They can even be employed to study the influence of other types of second-phase inclusions, including voids, on the overall behavior. Future efforts will be directed towards these problems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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