Parameter estimation of discretely observed interacting particle systems

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Abstract

In this paper, we consider the problem of joint parameter estimation for drift and diffusion coefficients of a stochastic McKean-Vlasov equation and for the associated system of interacting particles. The analysis is provided in a general framework, as both coefficients depend on the solution and on the law of the solution itself. Starting from discrete observations of the interacting particle system over a fixed interval [0, T], we propose a contrast function based on a pseudo likelihood approach. We show that the associated estimator is consistent when the discretization step (Δ_n) and the number of particles (N) satisfy $\Delta_n \to 0$ and $N \to \infty$, and asymptotically normal when additionally the condition $\Delta_n N \to 0$ holds.

Keywords: Asymptotic normality, consistency, interacting particle systems, McKean-Vlasov equation, nonlinear diffusion, parameter estimation.

AMS 2010 subject classifications: 62F12, 62E20, 62M05, 60G07, 60H10.

1 Introduction

In this paper we focus on parametric estimation of interacting particle system of the form

$$\begin{cases}
dX_t^{\theta,i,N} = b(\theta_1, X_t^{\theta,i,N}, \mu_t^{\theta,N}) dt + a(\theta_2, X_t^{\theta,i,N}, \mu_t^{\theta,N}) dW_t^i, & i = 1, ..., N, \quad t \in [0, T], \\
\mathcal{L}(X_0^{\theta,1,N}, ..., X_0^{\theta,N,N}) := \mu_0 \times ... \times \mu_0.
\end{cases}$$
(1)

Here the unknown parameter $\theta := (\theta_1, \theta_2)$ belongs to the set $\Theta := \Theta_1 \times \Theta_2$, where $\Theta_j \subset \mathbb{R}^{p_j}$, j = 1, 2, are compact and convex sets; we set $p := p_1 + p_2$. The processes $(W_t^i)_{t \in [0,T]}$, $i = 1, \ldots, N$, are independent \mathbb{R} -valued Brownian motions, independent of the initial value $(X_0^{\theta,1,N}, \ldots, X_0^{\theta,N,N})$ of the system and $\mu_t^{\theta,N}$ is the empirical measure of the system at time t, i.e.

$$\mu_t^{\theta,N} := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{\theta,i,N}}.$$

The model coefficients are functions $b: U_1 \times \mathbb{R} \times \mathcal{P}_2 \to \mathbb{R}$ and $a: U_2 \times \mathbb{R} \times \mathcal{P}_2 \to \mathbb{R}$, where U_1 and U_2 are two open sets containing Θ_1 and Θ_2 , respectively, and \mathcal{P}_2 denotes the set of probability measures on \mathbb{R} with a finite second moment, endowed with the Wasserstein 2-metric

$$W_2(\mu, \nu) := \left(\inf_{m \in \Gamma(\mu, \nu)} \int_{\mathbb{R}^2} |x - y|^2 m(dx, dy)\right)^{\frac{1}{2}},\tag{2}$$

The authors gratefully acknowledge financial support of ERC Consolidator Grant 815703 "STAMFORD: Statistical Methods for High Dimensional Diffusions".

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and $\Gamma(\mu,\nu)$ denotes the set of probability measures on \mathbb{R}^2 with marginals μ and ν . The underlying observations are

 $(X_{t_{j,n}}^{\theta,i,N})_{j=1,\dots,n}^{i=1,\dots,N},$

where $t_{j,n} := Tj/n$ and $\Delta_n := T/n$ is the discretization step. We assume that the time horizon T is fixed, and $N, n \to \infty$.

The interacting particle system is naturally associated to its mean field equation as $N \to \infty$. The latter is described by the 1-dimensional McKean-Vlasov SDE

$$d\bar{X}_t^{\theta} = b(\theta_1, \bar{X}_t^{\theta}, \bar{\mu}_t^{\theta})dt + a(\theta_2, \bar{X}_t^{\theta}, \bar{\mu}_t^{\theta})dW_t, \quad t \in [0, T],$$
(3)

where $\bar{\mu}_t^{\theta}$ is the law of \bar{X}_t^{θ} and $(W_t)_{t \in [0,T]}$ is a standard Brownian motion, independent of the initial value \bar{X}_0^{θ} having the law $\bar{\mu}_0^{\theta} := \mu_0$. This equation is non-linear in the sense of McKean, see e.g. [47, 48, 57]. It means, in particular, that the coefficients depend not only on the current state but also on the current distribution of the solution. It is well known that, under appropriate assumptions on the coefficients a and b, it is possible to obtain a phenomenon commonly named propagation of chaos (see e.g. [57]). It implies that the empirical law $\mu_t^{\theta,N}$ weakly converges to $\bar{\mu}_t^{\theta}$ as $N \to \infty$. The McKean-Vlasov SDE in (3) links to a non-linear non-local partial differential equation on the space of probability measures (see e.g. [10]), which naturally arises in several applications in statistical physics. Indeed, stochastic systems of interacting particles and the associated McKean non-linear Markov processes have been introduced in 1966 in [47] starting from statistical physics, to model the dynamics of plasma. Their importance has increased in time, and a huge number of probabilistic tools have been progressively developed in this context (see [10, 24, 45, 49], just to name a few).

On the other hand, however, statistical inference in this framework remained out of reach for many years (except for the early work of Kasonga in [41]), mainly as microscopic particle systems derived from statistical physics are not directly observable. Later on, McKean-Vlasov models found applications in several other fields, in which the data is observable. Nowadays, these models are used in finance (smile calibration in [36]; systemic risk in [27]) as well as social sciences (opinion dynamics in [12]) or mean-field games (see e.g. [9, 21, 33]). Moreover, some applications in neuroscience and population dynamics can be found respectively in [4] and [50]. At the same time, the interest in analysis of statistical models related to PDEs has gradually increased. A clear illustration of that is provided by the works on nonparametric Bayes and uncertainty quantification for inverse problems, as in [1, 53, 54].

Motivated by the increasing interest in statistical inference for McKean-Vlasov processes, we aim at estimating jointly the parameters θ_1, θ_2 starting from the discrete observations of the interacting particle systems (1) over a fixed time interval [0,T]. Despite recent interest in the study of the McKean-Vlasov SDEs, the problem of parameter estimation for this class has received relatively little attention. In [59] the authors established asymptotic consistency and normality of the maximum likelihood estimator for a class of McKean-Vlasov SDEs with constant diffusion coefficient, based on the continuous observation of the trajectory. This has been extended to the path dependent case in [44]. The mean field regime has been firstly considered by Kasonga in [41], who studied a system of interacting diffusion processes depending linearly in the drift coefficient on some unknown parameter. Starting from continuous observation of the system over a fixed time interval [0,T], he showed that the MLE is consistent and asymptotically normal as $N \to \infty$. This has been extended in [55] to the case where the parametrisation is not linear, while Bishwal [6] extended it to the case where only discrete observations of the system are available and the parameter to be estimated is a function of time. In [33] the authors develop an asymptotic inference approach based on the approximation of the likelihood function for mean-fields models of large interacting financial systems. Moreover, Chen [13] has established the optimal convergence rate for the MLE in the large N and large T case. Even in this work the drift coefficient is linear and the diffusion coefficient is constant.

Let us also mention the works [31, 32], where parametric inference for a particular class of nonlinear self-stabilizing SDEs is studied, starting from continuous observation of the non-linear diffusion. Some different asymptotic regimes are considered, such as the small noise and the long time horizon. The problem of the semiparametric estimation of the drift coefficient starting from the observation of the particle system at time T, for $T \to \infty$ is studied in [5], while [17] considers non-parametric estimation of the drift term in a McKean-Vlasov SDE, based on the continuous observation of the associated interacting particle system over a fixed time horizon.

None of these works, however, consider the problem of the joint estimation of the drift and diffusion coefficients. Moreover, not only we are not aware of any work about parameter estimation for interacting particle system where the diffusion coefficient can depend on the solution and on the law of the solution itself, but in the majority of the above mentioned work the diffusion coefficient is directly assumed to be constant. We consider a more general model, as in (1), motivated by several applications in which the diffusion coefficient depends on the law. For example, this is the case in mathematical finance for the calibration of local and stochastic volatility models, with applications connected to the Dupire's local volatility function (see [7, 37, 43]). Moreover, they are used to capture the diversity of a financial market, as in [51].

We underline that the joint estimation of the two parameters introduces some significant difficulties: since the drift and the diffusion coefficient parameters are not estimated at the same rate, we have to deal with asymptotic properties in two different regimes. Another challenge comes from the fact that both coefficients depend on the empirical law of the process. This introduces some complexity compared to the case where a is constant.

A natural approach to estimation of unknown parameters in our context would be to use a maximum likelihood estimation. However, the likelihood function based on the discrete sample is not tractable in this setting, since it depends on the transition densities of the process, which are not explicitly known. To overcome this difficulty several methods have been developed, in the case of high frequency estimation for discretely observed classical SDEs. A widely-used method is to consider a pseudo likelihood function, for instance based on the high frequency approximation of the dynamic of the process by the dynamic of the Euler scheme, see for example [25, 42, 60].

Our statistical analysis is based upon minimisation of a contrast function, which is similar in spirit to the methods [25, 42, 60] that have been proposed in the setting of classical SDEs. The main result of the paper is the consistency and asymptotic normality of the resulting estimator, which is showed by using a central limit theorem for martingale difference triangular arrays. The convergence rates for estimation of the two parameters are different, which leads us to the study of the asymptotic properties of the contrast function in two different asymptotic schemes. Moreover, to illustrate our main results, we present numerical experiments for two models of interacting particle systems. Specifically, the first model is linear, while the second is a stochastic opinion dynamics model. While it is feasible to express the estimator explicitly for the linear model, the estimator for the stochastic opinion dynamics model is implicit and can only be obtained numerically. Our results show that the proposed estimators perform well in both cases.

We emphasize that our inference is made on the time horizon [0,T] with T being fixed. It is well known that it is impossible to estimate the drift parameter of a classical SDE on a finite time horizon. However, due to increasing number of particles, we are able to consistently estimate the drift even when T is fixed. Moreover, it is worth remarking that our results apply to the system of N independent copies of a diffusion process as a special case. Non-parametric statistical inference for this type of system can be found for example in [14, 46, 20] (see also references therein). Closer to the purpose of our work, [16, 19] discuss parameter estimation from discrete observations of independent copies of a diffusion process with mixed (or fixed) effects. Specifically, joint estimation of a fixed effect in the diffusion coefficient and parameters

of the special distribution of a random effect (or a fixed effect) in the drift coefficient of the SDE is shown possible with the same rates of convergence in the same asymptotic framework as ours. Interested readers can find further references about SDEs with random effects in the aforementioned papers.

The outline of the paper is as follows. In Section 2 we present the estimation approach, list the required assumptions and demonstrate some examples. Section 3 is devoted to main results of the paper, which include consistency and asymptotic normality of the estimator. Section 4 is devoted to numerical experiments. In Section 5 we provide the technical lemmas we will use in order to show our main results. The proofs of the main results are collected in Section 6 while the technical results are shown in Section 7.

Notation

Throughout the paper all positive constants are denoted by C or C_q if they depend on an external parameter q. All vectors are row vectors, $\|\cdot\|$ denotes the Euclidean norm for vectors. We write $f(\theta) = f(\theta_1, \theta_2)$ for $\theta = (\theta_1, \theta_2)$. For $r = 0, 1, \ldots$, we denote by $C^r(X; \mathbb{R})$ the set of r times continuously differentiable functions $f: X \to \mathbb{R}$. We denote by $\partial_x f$ the partial derivative of a function $f(x, y, \ldots)$ with respect to x. We denote by $\nabla_{\theta_j} f$ the vector $(\partial_{\theta_{j,1}} f, \ldots, \partial_{\theta_{j,p_j}} f)$, j = 1, 2, and $\nabla_{\theta} f = (\nabla_{\theta_1} f, \nabla_{\theta_2} f)$. We say that a function $f: \mathbb{R} \times \mathcal{P}_l \to \mathbb{R}$ has polynomial growth if

$$|f(x,\mu)| \le C(1+|x|^k + W_2^l(\mu,\delta_0)) \tag{4}$$

for some $k, l = 0, 1, \ldots$ and all $(x, \mu) \in \mathbb{R} \times \mathcal{P}_l$, where \mathcal{P}_l denotes the set of probability measures on \mathbb{R} with a finite l-th absolute moment. For $p \in [1, \infty)$, the Wasserstein p-metric between two probability measures μ and ν in \mathcal{P}_p is given as

$$W_p(\mu,\nu) := \left(\inf_{m \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^2} |x - y|^p m(dx, dy)\right)^{\frac{1}{p}};$$

where $\Gamma(\mu,\nu)$ denotes the set of probability measures on \mathbb{R}^2 with marginals μ and ν . Finally, we suppress the dependence of several objects on the true parameter θ_0 . In particular, we write $\mathbb{P}:=\mathbb{P}^{\theta_0}, \mathbb{E}:=\mathbb{E}^{\theta_0}, X_t^{i,N}:=X_t^{\theta_0,i,N}, \bar{X}_t:=\bar{X}_t^{\theta_0}, \mu_t^N:=\mu_t^{\theta_0,N}$ and $\bar{\mu}_t:=\bar{\mu}_t^{\theta_0}$. Furthermore, we denote by $\xrightarrow{\mathbb{P}}$, $\xrightarrow{\mathcal{L}}$, $\xrightarrow{L^p}$ the convergence in probability, in law, in L^p respectively. We also denote the value $a^2(\theta_2,x,\mu)$ as $c(\theta_2,x,\mu)$.

2 Minimal contrast estimator, assumptions and examples

We aim at estimating the unknown parameter $\theta_0 = (\theta_{0,1}, \theta_{0,2}) \in \Theta^{\circ}$ given equidistant discrete observations of the system introduced in (1). We study the asymptotic regime $N, n \to \infty$.

The estimator we propose is based upon a contrast function, which originates from the Gaussian quasi-likelihood. Starting from discrete observations of the model there are difficulties due to the fact that the transition density of the process is unknown. A common way to overcome this issue is to base the inference on a discretization of the continuous likelihood (see for example [29], [42] and [60] where classic SDEs are considered). This motivates us to consider the following contrast function:

$$S_n^N(\theta) := \sum_{i=1}^N \sum_{j=1}^n \left\{ \frac{\left(X_{t_{j,n}}^{i,N} - X_{t_{j-1,n}}^{i,N} - \Delta_n b\left(\theta_1, X_{t_{j-1,n}}^{i,N}, \mu_{t_{j-1,n}}^N\right)\right)^2}{\Delta_n c\left(\theta_2, X_{t_{j-1,n}}^{i,N}, \mu_{t_{j-1,n}}^N\right)} + \log c\left(\theta_2, X_{t_{j-1,n}}^{i,N}, \mu_{t_{j-1,n}}^N\right) \right\},$$

$$(5)$$

for $\theta = (\theta_1, \theta_2)$. The estimator $\hat{\theta}_n^N = (\theta_{n,1}^N, \theta_{n,2}^N)$ of θ_0 is obtained as

$$\hat{\theta}_n^N \in \operatorname*{arg\,min}_{\theta \in \Theta} S_n^N(\theta).$$

Comparing $S_n^N(\theta)$ with the contrast function for parameter estimation for classical SDEs, the main difference consists in the fact that we have now an extra sum over the number of interacting diffusion processes. The interaction depends on the empirical measure of the system. The dependence of the drift and diffusion coefficients on the measure can take a general form. In order to meet this challenge and prove some asymptotic properties for $\hat{\theta}_n^N$ we need to introduce a set of assumptions. The first two assumptions ensure the system's existence and uniqueness, while the next two impose additional regularity conditions on the coefficients a and b.

A1. (Boundedness of moments) For all $k \geq 1$,

$$\int_{\mathbb{R}} |x|^k \mu_0(dx) \le C_k.$$

A2. (Lipschitz condition) The drift and diffusion coefficients are Lipschitz continuous in (x, μ) , i.e. for all θ there exists C such that for all $(x, \mu), (y, \nu) \in \mathbb{R} \times \mathcal{P}_2$,

$$|b(\theta_1, x, \mu) - b(\theta_1, y, \nu)| + |a(\theta_2, x, \mu) - a(\theta_2, y, \nu)| \le C(|x - y| + W_2(\mu, \nu)).$$

A3. (Regularity of the diffusion coefficient) The diffusion coefficient is uniformly bounded away from 0:

$$\inf_{(\theta_2, x, \mu) \in \Theta_2 \times \mathbb{R} \times \mathcal{P}_2} c(\theta_2, x, \mu) > 0.$$

- **A4.** (Regularity of the derivatives) (I) For all (x,μ) , the functions $b(\cdot,x,\mu)$, $a(\cdot,x,\mu)$ are in $C^3(U_1;\mathbb{R})$, $C^3(U_2;\mathbb{R})$ respectively. Furthermore, all their partial derivatives up to order three have polynomial growth, in the sense of (4), uniformly in θ .
- (II) The first and second order derivatives in θ are locally Lipschitz in (x, μ) with polynomial weights, i.e. for all θ there exists C > 0, $k, l = 0, 1, \ldots$ such that for all $r_1 + r_2 = 1, 2, h_1, h_2 = 1, \ldots, p_1, \tilde{h}_1, \tilde{h}_2 = 1, \ldots, p_2, (x, \mu), (y, \nu) \in \mathbb{R} \times \mathcal{P}_2$,

$$\begin{aligned} \left| \partial_{\theta_{1,h_{1}}}^{r_{1}} \partial_{\theta_{1,h_{2}}}^{r_{2}} b(\theta_{1},x,\mu) - \partial_{\theta_{1,h_{1}}}^{r_{1}} \partial_{\theta_{1,h_{2}}}^{r_{2}} b(\theta_{1},y,\nu) \right| + \left| \partial_{\theta_{2,\tilde{h}_{1}}}^{r_{1}} \partial_{\theta_{2,\tilde{h}_{2}}}^{r_{2}} a(\theta_{2},x,\mu) - \partial_{\theta_{2,\tilde{h}_{1}}}^{r_{1}} \partial_{\theta_{2,\tilde{h}_{2}}}^{r_{2}} a(\theta_{2},y,\nu) \right| \\ & \leq C(|x-y| + W_{2}(\mu,\nu)) \left(1 + |x|^{k} + |y|^{k} + W_{2}^{l}(\mu,\delta_{0}) + W_{2}^{l}(\nu,\delta_{0}) \right). \end{aligned}$$

- **Remark 2.1.** (i) It is possible to relax assumption A2 on the drift coefficient to allow for a locally Lipschitz condition in x with polynomial weights, cf. [22, Assumption 2.1]. In this setting the boundedness of moments shown in our Lemma 5.1 can be replaced by [23, Theorem 3.3] and the propagation of chaos needed in order to prove Lemma 5.2 would follow from [22, Proposition 3.1]. As a consequence the main results of this paper still hold.
- (ii) **A4(I)** is sufficient to show consistency of the estimator $\hat{\theta}_n^N$. We require the additional condition (II) of **A4** to prove the asymptotic normality.

We now state an assumption on the identifiability of the model and some further conditions that are required to prove the asymptotic normality. For this purpose we define the functions $I: \Theta \to \mathbb{R}, J: \Theta_2 \to \mathbb{R}$ as

$$I(\theta) := \int_0^T \int_{\mathbb{R}} \frac{(b(\theta_1, x, \bar{\mu}_t) - b(\theta_{0,1}x, \bar{\mu}_t))^2}{c(\theta_2, x, \bar{\mu}_t)} \bar{\mu}_t(dx)dt, \tag{6}$$

$$J(\theta_2) := \int_0^T \int_{\mathbb{R}} \left(\frac{c(\theta_{0,2}, x, \bar{\mu}_t)}{c(\theta_2, x, \bar{\mu}_t)} + \log c(\theta_2, x, \bar{\mu}_t) \right) \bar{\mu}_t(dx) dt, \tag{7}$$

where recall that $\bar{\mu}_t$ stands for $\bar{\mu}_t^{\theta_0}$. The next set of conditions are the following assumptions.

A5. (Identifiability) The functions I, J defined above satisfy that for every $\varepsilon > 0$,

$$\inf_{\theta \in \Theta: \|\theta_1 - \theta_{0,1}\| \ge \varepsilon} I(\theta) > 0 \qquad and \quad \inf_{\theta_2 \in \Theta_2: \|\theta_2 - \theta_{0,2}\| \ge \varepsilon} (J(\theta_2) - J(\theta_{0,2})) > 0.$$

A6. (Invertibility) We define a $p \times p$ block diagonal matrix $\Sigma(\theta_0)$: = diag($\Sigma^{(1)}(\theta_0), \Sigma^{(2)}(\theta_0)$) whose main-diagonal blocks $\Sigma^{(j)}(\theta_0) = (\Sigma_{kl}^{(j)}(\theta_0))$ are defined via

$$\Sigma_{kl}^{(j)}(\theta_0) := \begin{cases} 2 \int_0^T \int_{\mathbb{R}} \frac{\partial_{\theta_{1,k}} b(\theta_{0,1}, x, \bar{\mu}_t) \, \partial_{\theta_{1,l}} b(\theta_{0,1}, x, \bar{\mu}_t)}{c(\theta_{0,2}, x, \bar{\mu}_t)} \bar{\mu}_t(dx) dt, & j = 1, k, l = 1, \dots, p_1, \\ \int_0^T \int_{\mathbb{R}} \frac{\partial_{\theta_{2,k}} c(\theta_{0,2}, x, \bar{\mu}_t) \, \partial_{\theta_{2,l}} c(\theta_{0,2}, x, \bar{\mu}_t)}{c^2(\theta_{0,2}, x, \bar{\mu}_t)} \bar{\mu}_t(dx) dt, & j = 2, k, l = 1, \dots, p_2. \end{cases}$$

We assume that $det(\Sigma^{(j)}(\theta_0)) \neq 0, j = 1, 2$.

A7. (Integral condition on the diffusion coefficient) At $\theta_{0,2}$ for all (x,μ) the diffusion coefficient takes the form

$$a(\theta_{0,2},x,\mu) := \tilde{a}\Big(x,\int_{\mathbb{D}}K(x,y)\mu(dy)\Big)$$

for some functions $\tilde{a}, K \in C^2(\mathbb{R}^2; \mathbb{R})$, which satisfy $|\partial_x^{r_1} \partial_y^{r_2} \tilde{a}(x, y)| + |\partial_x^{r_1} \partial_y^{r_2} K(x, y)| \leq C(1 + |x|^k + |y|^l)$ for some $k, l = 0, 1, \ldots$ and all $r_1 + r_2 = 1, 2, (x, y) \in \mathbb{R}^2$.

Assumptions A1-A5 are required to prove the consistency of our estimator and are relatively standard in the literature for statistics of random processes. However, Assumption A5 deserves some extra attention, as the quantities $I(\theta)$ and $J(\theta)$ are not at all explicit due to the presence of $\bar{\mu}_t$. Hence, it may be difficult to check Assumption A5 in practice and the identifiability of all parameters may not always be possible. In order to delve deeper into the topic, we refer to Section 2.4 in [18], where the authors have provided a thorough analysis. More specifically, for estimating the drift from continuous observations, they have identified explicit criteria that enable obtaining both identifiability and non-degeneracy of the Fisher information matrix. Notably, for a certain type of likelihood, they have established a connection between global identifiability and non-degeneracy of the Fisher information, which is highlighted in [18, Proposition 16]. It could be interesting to understand if it possible to prove an analogous proposition in our context, even if this is out of the purpose of the paper and it is therefore left for further investigation.

The additional conditions $\mathbf{A6}$ - $\mathbf{A7}$ are needed to obtain the central limit theorem, even if they are not of the same type. Indeed, $\mathbf{A6}$ is an invertibility condition which is always required when one wants to prove asymptotic normality. In $\mathbf{A6}$, note that $\partial_{\theta_{1,k}}b(\theta_{0,1},x,\bar{\mu}_t)$ and $\partial_{\theta_{2,k}}c(\theta_{0,2},x,\bar{\mu}_t)$ are respectively $\partial_{\theta_{1,k}}b(\theta_{0,1},x,\mu)|_{\mu=\bar{\mu}_t}$ and $\partial_{\theta_{2,k}}c(\theta_{0,2},x,\mu)|_{\mu=\bar{\mu}_t}$, whereas $\bar{\mu}_t$ stands for $\bar{\mu}_t^{\theta_0}$. On the other hand, $\mathbf{A7}$ is a technical condition needed in order to obtain the first statement of Lemma 5.3. We shed light to the fact that the bounds in Lemma 5.3 are stated for θ_0 and similarly we ask to $\mathbf{A7}$ to be valid exclusively for the true parameter value $\theta_{0,2}$. Naturally, both \tilde{a} and K in $\mathbf{A7}$ can be functions on $\Theta_2 \times \mathbb{R}^2$ with the first argument fixed at $\theta_{0,2}$.

We also remark that, in the case where the unknown parameter θ appears only in the drift coefficient, there is no need to add a further assumption on the derivatives of the diffusion coefficient to estimate it, even if the diffusion coefficient still depends on the law of the process.

Example 2.2. A number of interacting particle models (and associated mean field equations) have been analyzed in the literature. We highlight a few here to illustrate the scope of our paper. We start by considering some examples where the diffusion coefficient is a constant on a compact set that does not include the origin. This case has several applications (see (i) and (ii)). After that, some more general examples are presented.

(i) The Kuramoto model is the most classical model for synchronization phenomena in large

populations of coupled oscillators such as a clapping crowd, a population of fireflies or a system of neurons (see Section 5.2 of [11] and references therein). Let N oscillators be defined by N angles $X_t^{i,N}$, $i=1,\ldots,N$ (defined modulo 2π , in this way they can actually be considered as elements of the circle), evolving in $t \in [0,T]$ according to

$$dX_t^{i,N} = -\frac{\theta_{0,1}}{N} \sum_{j=1}^{N} \sin\left(X_t^{i,N} - X_t^{j,N}\right) dt + \theta_{0,2} dW_t^i.$$

This variant of the model satisfies our assumptions.

(ii) A popular model for opinion dynamics (see e.g. [12, 52]) takes the form

$$dX_t^{i,N} = -\frac{1}{N} \sum_{j=1}^{N} \varphi_{\theta_{0,1}} (|X_t^{i,N} - X_t^{j,N}|) (X_t^{i,N} - X_t^{j,N}) dt + \theta_{0,2} dW_t^{i}$$

for $i=1,\ldots,N,\ t\in[0,T]$, where $\varphi_{\theta_{0,1}}(x):=\theta_{0,1,1}\mathbbm{1}_{[0,\theta_{0,1,2}]}(x),\ x\in\mathbb{R}$, is the influence function which acts on the "difference of opinions" between agents. To have our regularity assumptions hold true in practice we can replace the function $\varphi_{\theta_{0,1}}$ by its infinitely differentiable approximation as it is done in Section 5.2 of [55]. In [55] we also note that the proxy of $\varphi_{\theta_{0,1}}$ depends non-linearly on the parameter $\theta_{0,1,2}$.

(iii) Another example is

$$dX_t^{i,N} = \left(\theta_{0,1,1} + \frac{\theta_{0,1,2}}{N} \sum_{i=1}^{N} X_t^{j,N} - \theta_{0,1,3} X_t^{i,N}\right) dt + \theta_{0,2} \sqrt{1 + \left(X_t^{i,N}\right)^2} dW_t^i$$

for i = 1, ..., N, $t \in [0, T]$. We note that in the case $\theta_{0,1,2} = 0$ the interacting particle system reduces to N independent samples of a special case of the Pearson diffusion, which has applications in finance, see [26] and references therein.

(iv) We consider the dynamic of the system

$$dX_t^{i,N} = \left(\theta_{0,1,1} + \frac{\theta_{0,1,2}}{N} \sum_{j=1}^{N} X_t^{j,N} - \theta_{0,1,3} X_t^{i,N}\right) dt + \left(\theta_{0,2,1} + \theta_{0,2,2} \sqrt{\frac{1}{N} \sum_{j=1}^{N} \left(X_t^{j,N}\right)^2}\right) dW_t^{i}$$

for i = 1, ..., N in $t \in [0, T]$, where both the coefficients b and a depend on the law argument. We remark that the mean field limit of the above interacting particle system is a time-inhomogeneous Ornstein-Uhlenbeck process. See [41] for the case $\theta_{0,1,1} = \theta_{0,2,2} = 0$.

Some remarks are in order. Example (iv), where $\theta_{0,2,2} = 0$, has been thoroughly discussed in Section 4.1 of [18], specifically, with regard to the restrictions on μ_0 and $\theta_{0,1}$ that ensure the latter parameter satisfies A5, A6. In examples (i), (iii) and (iv), where either $\theta_{0,2,1}$ or $\theta_{0,2,2}$ is set to 0, it is obvious that A5, A6 hold for $\theta_{0,2} \neq 0$. Finally, we note that in examples (i), (iii), and (iv), where either $\theta_{0,2,1}$ or $\theta_{0,2,2}$ is set to 0, the drift and diffusion coefficients are respectively linear and multiplicative functions of θ , which allows us to solve our estimator in closed form.

3 Main results

Our main results demonstrate the consistency and the asymptotic normality of the estimator $\hat{\theta}_n^N$.

Theorem 3.1. (Consistency) Assume that A1-A5 hold, with only condition (I) in A4. Then the estimator $\hat{\theta}_n^N$ is consistent in probability:

$$\hat{\theta}_n^N \xrightarrow{\mathbb{P}} \theta_0 \quad as \ n, N \to \infty.$$

In order to obtain the asymptotic normality of our estimator we need to add an assumption on the relation between the rates N and Δ_n . In particular, we require that $N\Delta_n \to 0$ as $N, n \to \infty$.

Theorem 3.2. (Asymptotic normality) Assume that A1-A7 hold. If $N\Delta_n \to 0$ then

$$\left(\sqrt{N}(\hat{\theta}_{n,1}^N - \theta_{0,1}), \sqrt{N/\Delta_n}(\hat{\theta}_{n,2}^N - \theta_{0,2})\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, 2(\Sigma(\theta_0))^{-1}\right) \quad as \ n, N \to \infty,$$

where

$$2(\Sigma(\theta_0))^{-1} := 2 \operatorname{diag} ((\Sigma^{(1)}(\theta_0))^{-1}, (\Sigma^{(2)}(\theta_0))^{-1})$$

with $\Sigma^{(j)}(\theta_0)$, j = 1, 2, being defined in $\mathbf{A6}$.

As common in the literature on contrast function based methods, understanding the asymptotic behaviour of $S_n^N(\theta_1, \theta_2)$ and its derivatives is key to obtain the statements of Theorems 3.1 and 3.2. In particular, we show that, under proper normalisation, the first derivative of $S_n^N(\theta_1, \theta_2)$ converges to a Gaussian law with mean 0 and covariance matrix $2\Sigma(\theta_0)$ (see Proposition 6.2), while the second derivative converges in probability to the matrix $\Sigma(\theta_0)$ defined in **A6** (see Proposition 6.3). These results lead to the statement of Theorem 3.2.

The condition on the rate, at which the discretization step Δ_n converges to 0, has been discussed in detail in the framework of classical SDEs. In this context, one disposes discrete observations of the trajectory of only one particle up to a time $T := n\Delta_n \to \infty$. In [25] the corresponding condition was $T\Delta_n = n\Delta_n^2 \to 0$ as $n \to \infty$, which has been later improved to $n\Delta_n^3 \to 0$ in [60] thanks to a correction introduced in the contrast function. Finally, Kessler [42] proposed a contrast function based on a Gaussian approximation of the transition density, which allowed him to consider a weaker condition $n\Delta_n^p \to 0$ for an arbitrary integer p. Similar developments have been made in the setting of classical SDEs with jumps in [2, 3, 34, 56].

One may wonder if it possible to weaken the condition on the discretization step in the context of interacting particle systems. For a system of independent copies of a diffusion process with random and/or fixed effects, [15, 16, 19] require it in the same asymptotic framework as ours. In [16] also the rates of convergence of the estimators towards the parameters θ_1 of the distribution of a random effect in the drift coefficient, and the fixed effect θ_2 in the diffusion coefficient, are shown to be the same as ours. On the one hand, the condition $N\Delta_n \to 0$ allows us to approximate the derivative of the contrast function with a triangular array of martingale increments, as it is the case for classical SDEs. For this step, higher order approximations, similar to those in [42], could potentially help us relax this condition. On the other hand, we need it because of the correlation between particles and higher order approximations do not seem to solve this issue. Thus, we leave this investigation for future research.

A recent paper [18] establishes the *LAN property* for drift estimation in *d*-dimensional McKean-Vlasov models under continuous observations and with diffusion coefficient being a function of (t, \bar{X}_t) only. The authors show that the Fisher information matrix is given as

$$\left(\int_{0}^{T} \int_{\mathbb{R}^{d}} \partial_{\theta_{1,k}} (c^{-\frac{1}{2}}b)(\theta_{0,1}, t, x, \bar{\mu}_{t})^{\top} \partial_{\theta_{1,l}} (c^{-\frac{1}{2}}b)(\theta_{0,1}, t, x, \bar{\mu}_{t}) \bar{\mu}_{t}(dx) dt\right)_{1 \leq k, l \leq p_{1}}$$
(8)

(cf. [55] where the diffusion coefficient is an identity matrix). This is consistent with our Theorem 3.2 when restricted to drift estimation. In other words, our drift estimator is asymptotically efficient. When considering joint estimation of the drift and diffusion coefficients, the LAN

property has not yet been shown, although the results of Gobet [35] in the classical diffusion setting give some hope. Indeed, Gobet [35] has shown that for classical SDEs, in the ergodic case, the Fisher information for the drift parameter is given by

$$(\Gamma_b^{\theta_0})_{k,l} = \int_{\mathbb{R}} \frac{\partial_{\theta_{1,k}} b(\theta_{0,1}, x) \, \partial_{\theta_{1,l}} b(\theta_{0,1}, x)}{c(\theta_{0,2}, x)} \pi(dx)$$

for $k, l = 1, ..., p_1$, while the one for the diffusion parameter is given by

$$(\Gamma_a^{\theta_0})_{k,l} = \int_{\mathbb{R}} \frac{\partial_{\theta_{2,k}} c(\theta_{0,2}, x) \, \partial_{\theta_{2,l}} c(\theta_{0,2}, x)}{c^2(\theta_{0,2}, x)} \pi(dx)$$

for $k, l = 1, ..., p_2$, where π is the invariant density associated to the diffusion. As $\Gamma_b^{\theta_0}$ modifies to (8) for McKean-Vlasov models, one could expect that $\Gamma_a^{\theta_0}$ modifies to our asymptotic variance as well. This is left for further investigation.

4 Numerical examples

We will now examine the finite-sample performance of the introduced estimator $\hat{\theta}_n^N$ on two examples of interacting particle systems.

4.1 Linear model

Consider an interacting particle system of the form:

$$dX_t^{i,N} = -\left(\theta_{1,1}X_t^{i,N} + \frac{\theta_{1,2}}{N}\sum_{i=1}^N (X_t^{i,N} - X_t^{j,N})\right)dt + \sqrt{\theta_2}dW_t^i,\tag{9}$$

where $i=1,...,N,\ t\in[0,T]$, for some $\theta_1=(\theta_{1,1},\theta_{1,2})\in\mathbb{R}^2,\ \theta_{1,1}\neq 0,\ \theta_{1,1}+\theta_{1,2}\neq 0,\ \theta_2>0$ and $\int_{\mathbb{R}}x\mu_0(dx)\neq 0$. In this model, the parameter $\theta_{1,1}$ determines the intensity of attraction of each individual particle towards zero, while $\theta_{1,2}$ governs the degree of interaction, which is the attraction of each individual particle towards the empirical mean. Notably, for $\theta_{1,2}=0$, the processes $(X_t^{i,N})_{t\in[0,T]},\ i=1,\ldots,N$, are independent.

Recall that for $\theta_2 = 1$, estimation of the parameter θ_1 from a continuous observation of the system has been studied in [41, 55]. Since the drift and squared diffusion coefficients in (9) are linear in $\theta := (\theta_1, \theta_2)$, it is possible to find our estimator $\hat{\theta}_n^N$ in the closed form similarly as in [41, 55]:

$$\hat{\theta}_{n,1,1}^{N} = \frac{A_n^N - B_n^N}{D_n^N - C_n^N}, \qquad \hat{\theta}_{n,1,2}^N = \frac{A_n^N D_n^N - B_n^N C_n^N}{(C_n^N)^2 - C_n^N D_n^N}, \tag{10}$$

where

$$A_{n}^{N} := \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} (X_{t_{j-1,n}}^{i,N} - \bar{X}_{t_{j-1,n}}^{N})(X_{t_{j,n}}^{i,N} - X_{t_{j-1,n}}^{i,N}), \quad B_{n}^{N} := \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} X_{t_{j-1,n}}^{i,N}(X_{t_{j,n}}^{i,N} - X_{t_{j-1,n}}^{i,N}), \quad C_{n}^{N} := \frac{\Delta_{n}}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} (X_{t_{j-1,n}}^{i,N} - \bar{X}_{t_{j-1,n}}^{N})^{2}, \quad D_{n}^{N} := \frac{\Delta_{n}}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} (X_{t_{j-1,n}}^{i,N})^{2}$$

with $\bar{X}_{t_{j-1,n}}^N := N^{-1} \sum_{k=1}^N X_{t_{j-1,n}}^{k,N}$, and then

$$\hat{\theta}_{n,2}^{N} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{n} \left(X_{t_{j,n}}^{i,N} - X_{t_{j-1,n}}^{i,N} + \Delta_n \left(\hat{\theta}_{n,1,1}^{N} X_{t_{j-1,n}}^{i,N} + \frac{\hat{\theta}_{n,1,2}^{N}}{N} \sum_{j=1}^{N} (X_{t_{j-1,n}}^{i,N} - X_{t_{j-1,n}}^{j,N}) \right) \right)^2.$$
(11)

To illustrate the finite sample performance of $\hat{\theta}_n^N$, we choose $\theta = (\theta_{1,1}, \theta_{1,2}, \theta_2) = (0.5, 1, 1)$ and $\mu_0 = \delta_1$ as in [55]. We simulate 1000 solutions of the system given by (9) using the Euler method with a step size of 0.01. We obtain observations of the system — data sets for all possible combinations of $T = 50, 100, \Delta_n = 0.1, 0.05, 0.01$ and N = 50, 100. Table 3 presents the effect of N, Δ_n , T on the performance of $\hat{\theta}_n^N$. As N or T increases, the sample RMSE and bias of $\hat{\theta}_{n,1}^N$ decrease, whereas that of $\hat{\theta}_{n,2}^N$ do not change significantly. However, as Δ_n gets smaller, the performance of $\hat{\theta}_{n,2}$ improves, as well as that of $\hat{\theta}_{n,1,2}^N$.

N =	50	100	50	100	
$(\Delta_n, T) =$	(0.1, 50)	(0.1, 50)	(0.1, 100)	(0.1, 100)	
$\begin{array}{c} \widehat{\theta}_{n,1,1}^{N} \\ \widehat{\theta}_{n,1,2}^{N} \\ \widehat{\theta}_{n,2}^{N} \end{array}$	0.10 (0.00)	0.08 (0.00)	0.08 (0.00)	0.07 (0.00)	
$\hat{ heta}_{n,1,2}^N$	0.15 (-0.10)	0.13 (-0.10)	0.13 (-0.10)	0.12 (-0.10)	
$\hat{ heta}_{n,2}^N$	0.12 (-0.12)	0.12 (-0.12)	0.12 (-0.12)	0.12 (-0.12)	
$(\Delta_n, T) =$	(0.05, 50)	(0.05, 50)	(0.05, 100)	(0.05, 100)	
$\hat{\theta}_{n,1,1}^N$	0.10 (0.01)	0.08 (0.01)	0.08 (0.01)	0.07 (0.00)	
$\hat{ heta}_{n,1,2}^N$	0.12 (-0.05)	0.10 (-0.05)	0.10 (-0.05)	0.09 (-0.05)	
$\begin{array}{c} \hat{\theta}_{n,1,1}^N \\ \hat{\theta}_{n,1,2}^N \\ \hat{\theta}_{n,2}^N \end{array}$	0.06 (-0.06)	0.06 (-0.06)	0.06 (-0.06)	0.06 (-0.06)	
,					
$(\Delta_n, T) =$	(0.01, 50)	(0.01, 50)	(0.01, 100)	(0.01, 100)	
$\hat{\theta}_{n,1,1}^N$	0.11 (0.01)	0.08 (0.01)	0.09 (0.01)	0.07 (0.01)	
$\hat{ heta}_{n,1,2}^N$	0.11 (-0.02)	0.09 (-0.01)	0.09 (-0.01)	0.07 (-0.01)	
$\begin{array}{c} \widehat{\theta}_{n,1,1}^{N} \\ \widehat{\theta}_{n,1,2}^{N} \\ \widehat{\theta}_{n,2}^{N} \end{array}$	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)	

Table 1: Sample RMSE (and bias in brackets) of $\hat{\theta}_n^N$ for $\theta = (0.5, 1, 1)$ and different values of N, Δ_n , T. The number of replications is 1000.

We note that the numerical results presented above for $\Delta_n = 0.01$ can be viewed as the maximum likelihood estimation. Indeed, our contrast function up to a negative constant is the log-likelihood function for the Euler approximation with the same step Δ_n . Therefore, it is difficult to improve upon the estimation provided in the last lines of Table 1. Interestingly, the performance of our estimator for $\Delta_n = 0.1$ and $\Delta_n = 0.05$ is quite similar to that of $\Delta_n = 0.01$, particularly with respect to the RMSE for the estimation of $\hat{\theta}_{n,1,1}^N$ and $\hat{\theta}_{n,1,2}^N$.

One possible application of our Theorem 3.2 is to test the hypothesis of noninteraction of particles similarly as in [41]. Consider the null hypothesis $H_0: \theta_{1,2} = 0$ and the alternative $H_1: \theta_{1,2} \neq 0$. According to Theorem 3.2, if $N\Delta_n \to 0$, then

$$\sqrt{N}(\hat{\theta}_{n,1,2}^N - \theta_{1,2}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V(\theta)),$$

where

$$V(\theta) := 2\Sigma_{11}^{(1)}(\theta)/(\Sigma_{11}^{(1)}(\theta)\Sigma_{22}^{(1)}(\theta) - \Sigma_{12}^{(1)}(\theta)\Sigma_{21}^{(1)}(\theta)),$$

and for all i, j = 1, 2,

$$\Sigma_{ij}^{(1)}(\theta) := \begin{cases} 2\theta_2^{-1} \int_0^T \int_{\mathbb{R}} x^2 \bar{\mu}_t(dx) dt, & i = j = 1, \\ 2\theta_2^{-1} \int_0^T \int_{\mathbb{R}} \left(x - \int_{\mathbb{R}} y \bar{\mu}_t(dy) \right)^2 \bar{\mu}_t(dx) dt, & \text{else,} \end{cases}$$

can be explicitly computed in terms of the model parameters, see [41, 55]. By using Lemma 5.2 and Theorem 3.1, we have that

$$V_n^N := \hat{\theta}_{n,2}^N D_n^N / ((D_n^N - C_n^N) C_n^N) \xrightarrow{\mathbb{P}} V(\theta)$$
 as $n, N \to \infty$.

Therefore, if $N\Delta_n \to 0$, under H_0 , we can conclude that

$$Z_n^N := \hat{\theta}_{n,1,2}^N \sqrt{N/V_n^N} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \text{ as } n, N \to \infty.$$

Thus, we reject H_0 if

$$|Z_n^N| > z_{\alpha/2},$$

where $\alpha \in (0,1)$ is the chosen level of significance and z_{α} denotes the α -quantile of the standard normal distribution.

Next, we examine the performance of the test statistic Z_n^N . We simulate 1000 solutions of the system given by (9) with $\mu_0 = \delta_1$, using the Euler method with a step size of 0.01. Table 2 reports the rejection rates of H_0 in favor of H_1 at a significance level of $\alpha = 5\%$ using Z_n^N for all possible combinations of $N, T = 50, 100, \ \Delta_n = 0.1$ and $\theta = (0.5, \theta_{1,2}, 1)$, where $\theta_{1,2} = 0, 0.1, 0.25, 0.5$, or 1. The empirical size is quite well observed. Rejection rates of incorrect H_0 increase with increasing $\theta_{1,2}$ or N and T.

$\theta_{1,2}$	(N,T) =	(50, 50)	(100, 50)	(50, 100)	(100, 100)
0		4.8	4.6	4.2	4.1
0.1		17.8	22.5	21.4	28.9
0.25		61.3	78.2	75.6	87.0
0.5		97.2	99.7	99.8	99.9
1		100.0	100.0	100.0	100.0

Table 2: Rejection rates (in %) of $H_0: \theta_{1,2} = 0$ vs. $H_1: \theta_{1,2} \neq 0$ at level $\alpha = 5\%$ with Z_n^N for $\theta = (0.5, \theta_{1,2}, 1), \Delta_n = 0.1$ and different values of N, T. The number of replications is 1000.

4.2 Stochastic opinion dynamics model

We now consider an interacting particle system that can model opinion dynamics:

$$dX_t^{i,N} = -\frac{1}{N} \sum_{j=1}^N \varphi_{\theta_1}(|X_t^{i,N} - X_t^{j,N}|)(X_t^{i,N} - X_t^{j,N})dt + \sqrt{\theta_2}dW_t^i, \tag{12}$$

where $i = 1, ..., N, t \in [0, T]$, and

$$\varphi_{\theta_1}(x) := \theta_{1,2} \exp\left(-\frac{0.01}{1 - (x - \theta_{1,1})^2}\right) \mathbb{1}_{[\theta_{1,1} - 1, \theta_{1,1} + 1]}(x), \quad x \in \mathbb{R},$$

for some $-1 < \theta_{1,1} \le 1$, $\theta_{1,2} > 0$, $\theta_2 > 0$. The interaction kernel $\varphi_{\theta_1}(x)$ provides an infinitely differentiable approximation to the scaled indicator function $\theta_{1,2}\mathbb{1}_{[0,\theta_{1,1}+1]}(x)$, $x \ge 0$. We interpret that $\theta_{1,1}$ governs the intensity of attraction of each individual particle towards the scaled empirical mean of all the others within a distance $\theta_{1,1} + 1$. The position of each particle represents its opinion, and over time, the opinions of particles merge into metastable "soft clusters". For further information on this stochastic opinion dynamics model, see [55] and references therein.

Note that the squared diffusion coefficient is a multiplicative function of θ_2 which enables us to express $\hat{\theta}_{n,2}^N$ in terms of $(\hat{\theta}_{n,1,1}^N, \hat{\theta}_{n,1,2}^N)$. However, the latter estimator is implicit and can only be found using a numerical method. To illustrate the performance of $\hat{\theta}_n^N = (\hat{\theta}_{n,1,1}^N, \hat{\theta}_{n,1,2}^N, \hat{\theta}_{n,2}^N)$ we choose the parameter $\theta = (\theta_{1,1}, \theta_{1,2}, \theta_2) = (-0.5, 2, 0.04)$ as in [55], and the initial distribution $\mu_0 = \mathcal{N}(0,1)$ for each individual particle. We simulate 1000 solutions of the system given by (12) using the Euler method with a step size of 0.01. We obtain 1000 data sets for $\Delta_n = 0.1$ and all possible combinations of N, T = 50, 100 as in the previous subsection. Table 3 presents the effect of N, T on the performance of $\hat{\theta}_n^N$. As N increases, the sample RMSE and bias of $\hat{\theta}_n^N$ decrease, whereas they do not change that much with increasing T. We can also see that $\hat{\theta}_{n,1,1}^N$ is more accurate than $\hat{\theta}_{n,1,2}^N$.

(N,T) =	(50, 50)		(100, 50)		(50, 100)		(100	(100, 100)	
$\hat{\theta}_{n,1,1}^N$	0.0340	(0.0159)	0.0263	(0.0145)	0.0280	(0.0154)	0.0206	(0.0137)	
$\hat{ heta}_{n,1,2}^{N} \ \hat{ heta}_{n,2}^{N}$	0.1652	(-0.1378)	0.1503	(-0.1347)	0.1526	(-0.1420)	0.1472	(-0.1416)	
$\hat{ heta}_{n,2}^{N}$	0.0027	(-0.0026)	0.0026	(-0.0025)	0.0033	(-0.0032)	0.0033	(-0.0033)	

Table 3: Sample RMSE (and bias in brackets) of $\hat{\theta}_n^N$ for $\theta = (-0.5, 2, 0.04)$, $\Delta_n = 0.1$ and different values of N, T. The number of replications is 1000.

5 Technical lemmas

Before proving the main statistical results stated in previous section, we need to introduce some additional notations and to state some lemmas which will be useful in the sequel.

Define $\mathcal{F}_t^N := \sigma\{(W_u^k)_{u \in [0,t]}, X_0^{k,N}; k = 1,...,N\}$ and $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t^N]$. For a set $(Y_{t,n}^{i,N})$ of random variables and $\delta \geq 0$, the notation

$$Y_{t,n}^{i,N} = R_t^i(\Delta_n^\delta)$$

means that $Y_{t,n}^{i,N}$ is \mathcal{F}_t^N -measurable and the set $(Y_{t,n}^{i,N}/\Delta_n^{\delta})$ is bounded in L^q for all $q\geq 1$, uniformly in t,i,n,N. That is

$$\mathbb{E}\left[\left|Y_{t,n}^{i,N}/\Delta_n^{\delta}\right|^q\right]^{1/q} \le C_q$$

for all $t, i, n, N, q \ge 1$.

We will repeatedly use some moment inequalities gathered in the following lemma.

Lemma 5.1. Assume **A1-A2**. Then, for all $p \ge 1$, $0 \le s < t \le T$ such that $t - s \le 1$, $i \in \{1, ..., N\}$, $N \in \mathbb{N}$, the following hold true.

- 1. $\sup_{t \in [0,T]} \mathbb{E}[|X_t^{i,N}|^p] < C$, moreover, $\sup_{t \in [0,T]} \mathbb{E}[W_p^q(\mu_t^N, \delta_0)] < C$ for $p \leq q$.
- 2. $\mathbb{E}[|X_t^{i,N} X_s^{i,N}|^p] \le C(t-s)^{\frac{p}{2}}$.
- 3. $\mathbb{E}_s[|X_t^{i,N} X_s^{i,N}|^p] \le C(t-s)^{\frac{p}{2}}R_s^i(1)$.
- 4. $\mathbb{E}[W_2^p(\mu_t^N, \mu_s^N)] \le C(t-s)^{\frac{p}{2}}$.
- 5. $\mathbb{E}_s[W_2^p(\mu_t^N, \mu_s^N)] \le C(t-s)^{\frac{p}{2}}R_s(1)$.

The asymptotic properties of the estimator are deduced by the asymptotic behaviour of our contrast function. To study it, the following lemma will be useful.

Lemma 5.2. Assume **A1-A2**. Let $f : \mathbb{R} \times \mathcal{P}_l \to \mathbb{R}$ satisfy for some C > 0, k, l = 0, 1, ... and all $(x, \mu), (y, \nu) \in \mathbb{R} \times \mathcal{P}_l$,

$$|f(x,\mu) - f(y,\nu)| \le C(|x-y| + W_2(\mu,\nu))(1 + |x|^k + |y|^k + W_l^l(\mu,\delta_0) + W_l^l(\nu,\delta_0)).$$
 (13)

Moreover, let the mapping $(x,t) \mapsto f(x,\bar{\mu}_t)$ be integrable with respect to $\bar{\mu}_t(dx)dt$ over $\mathbb{R} \times [0,T]$. Then

$$\frac{\Delta_n}{N} \sum_{i=1}^N \sum_{j=1}^n f(X_{t_{j-1,n}}^{i,N}, \mu_{t_{j-1,n}}^N) \xrightarrow{\mathbb{P}} \int_0^T \int_{\mathbb{R}} f(x, \bar{\mu}_t) \bar{\mu}_t(dx) dt \quad as \ n, N \to \infty.$$

It is worth underlining that the boundedness of the moments and the convergence of the Riemann sums, which are obtained almost for free in the classical SDE case, are more complex in our setting. In particular, the proof of Lemma 5.2 consists now in three steps, the first

deals with the convergence of the proper Riemann sums, in the second step we move from the interacting particle system to the iid system though the propagation of chaos property, while the third step is an application of the law of large numbers.

Another challenge compared to the classical SDE case is gathered in next lemma. Indeed, our main results heavily rely on the study of derivatives of our contrast function and so on the moment bounds of its numerator. To accomplish this, we need to use Itô's lemma on the squared diffusion coefficient as a function of the particle system's state. Therefore, we must understand how to express derivatives of a with respect to the measure argument. That is the purpose of the extra hypothesis A7, thanks to which the problem reduces to study the derivatives of K. We recall that, in the sequel, we will denote by $c(\theta_2, x, \mu)$ the value $a^2(\theta_2, x, \mu)$.

Lemma 5.3. Assume A1-A2. Then, the following hold true.

1. If also **A7** is satisfied, then
$$\mathbb{E}_{t_{j,n}}[(X_{t_{j+1,n}}^{i,N} - X_{t_{j,n}}^{i,N} - \Delta_n b(\theta_{0,1}, X_{t_{j,n}}^{i,N}, \mu_{t_{j,n}}^N))^2] = \Delta_n c(\theta_{0,2}, X_{t_{j,n}}^{i,N}, \mu_{t_{j,n}}^N) + R_{t_{j,n}}^i(\Delta_n^2).$$

2.
$$\mathbb{E}_{t_{j,n}}[(X_{t_{j+1,n}}^{i,N} - X_{t_{j,n}}^{i,N} - \Delta_n b(\theta_{0,1}, X_{t_{j,n}}^{i,N}, \mu_{t_{j,n}}^N))^4] = 3\Delta_n^2 c^2(\theta_{0,2}, X_{t_{j,n}}^{i,N}, \mu_{t_{j,n}}^N) + R_{t_{j,n}}^i(\Delta_n^{\frac{5}{2}}).$$

3.
$$|\mathbb{E}_{t_{j,n}}[X_{t_{j+1,n}}^{i,N} - X_{t_{j,n}}^{i,N} - \Delta_n b(\theta_{0,1}, X_{t_{j,n}}^{i,N}, \mu_{t_{j,n}}^N)]| = R_{t_{j,n}}^i(\Delta_n^{\frac{3}{2}}).$$

We underline that **A7** is needed in order to prove that the size of the remainder function in the first point is Δ_n^2 . Without it, the size of the rest function would have been $\Delta_n^{\frac{3}{2}}$, which would not have been enough to obtain the asymptotic normality as in Proposition 6.2 (see the proof of (36)). The proof of the lemmas stated in this section can be found in Section 7.

6 Proofs

6.1 Consistency

Let us prove the (asymptotic) consistency of $\hat{\theta}_n^N = (\hat{\theta}_{n,1}^N, \hat{\theta}_{n,2}^N)$ component-wise. Our approach is similar to that taken in the proof of [58, Theorem 5.7]. In particular, we consider a criterion function $\theta \mapsto S_n^N(\theta)$ as a random element taking values in $(C(\Theta; \mathbb{R}), \|\cdot\|_{\infty})$. The uniform convergence of criterion functions is proved in the following lemma.

Lemma 6.1. Assume A1-A3, A4(I), A5. Then as $N, n \to \infty$,

$$\sup_{(\theta_1,\theta_2)\in\Theta} \left| \frac{\Delta_n}{N} S_n^N(\theta_1,\theta_2) - J(\theta_2) \right| \xrightarrow{\mathbb{P}} 0, \tag{14}$$

$$\sup_{(\theta_1, \theta_2) \in \Theta} \left| \frac{1}{N} (S_n^N(\theta_1, \theta_2) - S_n^N(\theta_{0,1}, \theta_2)) - I(\theta_1, \theta_2) \right| \xrightarrow{\mathbb{P}} 0, \tag{15}$$

where the functions I, J are defined in (6), (7) respectively.

Proof. It suffices to show the following steps:

- 1. $\frac{\Delta_n}{N} S_n^N(\theta_1, \theta_2) \xrightarrow{\mathbb{P}} J(\theta_2)$ for every $(\theta_1, \theta_2) \in \Theta$.
- 2. The sequence $(\theta_1, \theta_2) \mapsto \frac{\Delta_n}{N} S_n^N(\theta_1, \theta_2)$ is tight in $(C(\Theta; \mathbb{R}), \|\cdot\|_{\infty})$.
- 3. $\frac{1}{N}(S_n^N(\theta_1, \theta_2) S_n^N(\theta_{0,1}, \theta_2)) \xrightarrow{\mathbb{P}} I(\theta_1, \theta_2)$ for every $(\theta_1, \theta_2) \in \Theta$,
- 4. The sequence $(\theta_1, \theta_2) \mapsto \frac{1}{N} (S_n^N(\theta_1, \theta_2) S_n^N(\theta_{0,1}, \theta_2))$ is tight in $(C(\Theta; \mathbb{R}), \|\cdot\|_{\infty})$.

Let us omit the notation for dependence on N, n, in particular, write X_t^i for $X_t^{i,N}$, μ_t for μ_t^N , t_j for $t_{j,n}$. Denote $f(\cdot, X_t^i, \mu_t)$ by $f_t^i(\cdot)$ for a function f, for example equal to h or g defined as

$$h(\theta, x, \mu) = \frac{(b(\theta_{0,1}, x, \mu) - b(\theta_1, x, \mu))^2}{c(\theta_2, x, \mu)}, \qquad g(\theta, x, \mu) = \frac{b(\theta_{0,1}, x, \mu) - b(\theta_1, x, \mu)}{c(\theta_2, x, \mu)}$$
(16)

for all $\theta = (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 = \Theta$, $x \in \mathbb{R}$, $\mu \in \mathcal{P}_2$.

• Step 3. We start proving that for every $\theta = (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2 = \Theta$,

$$\frac{1}{N}(S_n^N(\theta_1, \theta_2) - S_n^N(\theta_{0,1}, \theta_2)) \xrightarrow{\mathbb{P}} I(\theta) = \int_0^T \int_{\mathbb{R}} h(\theta, x, \bar{\mu}_t) \bar{\mu}_t(dx) dt.$$

Let us first decompose the left hand side as a sum of a main term and remainder. We have

$$S_n^N(\theta_1, \theta_2) = \sum_{i=1}^N \sum_{j=1}^n \frac{(H_j^i + \Delta_n(b_{t_{j-1}}^i(\theta_{0,1}) - b_{t_{j-1}}^i(\theta_1)))^2}{\Delta_n c_{t_{j-1}}^i(\theta_2)} + (\log c)_{t_{j-1}}^i(\theta_2),$$

where $H_{j}^{i} = X_{t_{j-1}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n} b_{t_{j-1}}^{i}(\theta_{0,1})$ for all i, j. We decompose

$$\frac{1}{N}(S_n^N(\theta_1, \theta_2) - S_n^N(\theta_{0,1}, \theta_2)) = I_n^N(\theta) + 2\rho_n^N(\theta), \tag{17}$$

where

$$I_n^N(\theta) = \frac{\Delta_n}{N} \sum_{i=1}^N \sum_{j=1}^n h_{t_{j-1}}^i(\theta), \qquad \rho_n^N(\theta) = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^n g_{t_{j-1}}^i(\theta) H_j^i.$$
 (18)

Then

$$I_n^N(\theta) \xrightarrow{\mathbb{P}} I(\theta)$$

follows from Lemma 5.2 if the function $h(\theta,\cdot)$ is locally Lipschitz with polynomial growth. To check this assumption we note that the functions $b(\theta_{0,1},\cdot) - b(\theta_1,\cdot)$, $a(\theta_2,\cdot)$ are Lipschitz continuous and have linear growth by **A2**. We also recall that $\inf_{x,\mu} c(\theta_2,x,\mu) > 0$ by **A3**. Hence, $h(\theta,\cdot)$ satisfies the assumption of Lemma 5.2.

It remains to show that

$$\rho_n^N(\theta) \xrightarrow{\mathbb{P}} 0.$$
(19)

With $H_j^i = B_j^i + A_j^i$, where

$$B_j^i = \int_{t_{j-1}}^{t_j} (b_s^i(\theta_{0,1}) - b_{t_{j-1}}^i(\theta_{0,1})) ds, \qquad A_j^i = \int_{t_{j-1}}^{t_j} a_s^i(\theta_{0,2}) dW_s^i,$$

for all i, j, let us further decompose

$$\rho_n^N(\theta) = \rho_{n,1}^N(\theta) + \rho_{n,2}^N(\theta), \tag{20}$$

where

$$\rho_{n,1}^N(\theta) = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^n g_{t_{j-1}}^i(\theta) B_j^i, \qquad \rho_{n,2}^N(\theta) = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^n g_{t_{j-1}}^i(\theta) A_j^i.$$

It is enough to show that

$$\rho_{n,k}^N(\theta) \xrightarrow{L^k} 0, \qquad k = 1, 2.$$
(21)

First, let us show (21) in case k=2. Note that for all $i_1=i_2$ and $j_1\neq j_2$,

$$\mathbb{E}[g_{t_{j_1-1}}^{i_1}(\theta)A_{j_1}^{i_1}g_{t_{j_2-1}}^{i_2}(\theta)A_{j_2}^{i_2}] = 0$$
(22)

follows from $\mathbb{E}_{t_{j_1-1}}[A_{j_1}^{i_1}]=0$, whereas independence of Brownian motions implies (22) for all $i_1 \neq i_2$ and j_1, j_2 . We conclude that

$$\mathbb{E}[(\rho_{n,2}^N(\theta))^2] = \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^n \mathbb{E}[(g_{t_{j-1}}^i(\theta)A_j^i)^2]. \tag{23}$$

Next, the Itô isometry gives

$$\mathbb{E}[(g_{t_{j-1}}^i(\theta)A_j^i)^2] = \int_{t_{i-1}}^{t_j} \mathbb{E}[(g^2)_{t_{j-1}}^i(\theta)c_s^i(\theta_{0,2})]ds,$$

where $\mathbb{E}[(g^2)^i_{t_{j-1}}(\theta)c^i_s(\theta_{0,2})] = O(1)$ uniformly in $t_{j-1} \leq s \leq t_j, j, i$ thanks to $\inf_{x,\mu} c(\theta_2, x, \mu) > 0$ by **A3**, linear growth of $a(\theta_{0,2},\cdot)$, $b(\theta_1,\cdot)$ by **A2** and moment bounds in Lemma 5.1(1). We conclude that $\mathbb{E}[(g^i_{t_{j-1}}(\theta)A^i_j)^2] = O(\Delta_n)$ uniformly in i,j, which in turn implies

$$\mathbb{E}[(\rho_{n,2}^N(\theta))^2] = O(N^{-1}).$$

Finally, let us show (21) in case k = 1. For this purpose, use

$$\mathbb{E}[|g_{t_{j-1}}^i(\theta)B_j^i|] \le \int_{t_{i-1}}^{t_j} \mathbb{E}[|g_{t_{j-1}}^i(\theta)(b_s^i(\theta_{0,1}) - b_{t_{j-1}}^i(\theta_{0,1}))|] ds$$

and then the Cauchy–Schwarz inequality. Note $\mathbb{E}[(g^2)_{t_{j-1}}^i(\theta)] = O(1)$ uniformly in j,i follows in the same way as above. Lipschitz continuity of $b(\theta_{0,1},\cdot)$ by **A2** and moment bounds in Lemma 5.1(2) and (4) imply $\mathbb{E}[(b_s^i(\theta_{0,1}) - b_{t_{j-1}}^i(\theta_{0,1}))^2] = O(\Delta_n)$ uniformly in $t_{j-1} \leq s \leq t_j, j, i$. We conclude that

$$\mathbb{E}[|\rho_{n,1}^N|] = O(\Delta_n^{\frac{1}{2}}).$$

This completes the proof of Step 3.

• Step 4. Recall the decomposition (17), (20). It is enough to show tightness of

$$\theta \mapsto I_n^N(\theta), \qquad \theta \mapsto \rho_{n,k}^N(\theta), \qquad k = 1, 2.$$

Our approach to showing tightness of both sequences are based upon [40, Theorem 14.5]. We need to show that for all N, n:

$$\mathbb{E}\left[\sup_{\theta} \|\nabla_{\theta} I_{n}^{N}(\theta)\|\right] \leq C, \qquad \mathbb{E}\left[\sup_{\theta} \|\nabla_{\theta} \rho_{n,1}^{N}(\theta)\|\right] \leq C. \tag{24}$$

The above bounds follow if for all N, n, and $i, j, t_{j-1} \le s \le t_j$,

$$\mathbb{E}\left[\sup_{a} \|\nabla_{\theta} h_{t_{j-1}}^{i}(\theta)\|\right] \leq C, \qquad \mathbb{E}\left[|b_{s}^{i}(\theta_{0,1})|\sup_{a} \|\nabla_{\theta} g_{t_{j-1}}^{i}(\theta)\|\right] \leq C, \tag{25}$$

where $h, g: \Theta \times \mathbb{R} \times \mathcal{P}_2 \to \mathbb{R}$ are defined by (16). In $\nabla_{\theta_k} h, \nabla_{\theta_k} g, k = 1, 2$, we note $\nabla_{\theta_1} (b(\theta_{0,1}, \cdot) - b(\theta_1, \cdot)) = -\nabla_{\theta_1} b(\theta_1, \cdot)$. Moreover, by the mean value theorem, $|b(\theta_{0,1}, \cdot) - b(\theta_1, \cdot)| \le C \sup_{\theta_1} ||\nabla_{\theta_1} b(\theta_1, \cdot)||$ for all $\theta_1 \in \Theta_1$, since Θ_1 is convex, bounded. Additionally using $\inf_{\theta_2, x, \mu} c(\theta_2, x, \mu) > 0$ by **A3**, we get

$$\|\nabla_{\theta_1}g(\theta,\cdot)\| \leq C \sup_{\theta_1} \|\nabla_{\theta_1}b(\theta_1,\cdot)\|, \qquad \|\nabla_{\theta_2}g(\theta,\cdot)\| \leq C \sup_{\theta_1} \|\nabla_{\theta_1}b(\theta_1,\cdot)\| \sup_{\theta_2} \|\nabla_{\theta_2}a(\theta_2,\cdot)\|,$$

and

$$\|\nabla_{\theta_1}h(\theta,\cdot)\| \leq C \sup_{\theta_1} \|\nabla_{\theta_1}b(\theta_1,\cdot)\|^2, \qquad \|\nabla_{\theta_2}h(\theta,\cdot)\| \leq C \sup_{\theta_1} \|\nabla_{\theta_1}b(\theta_1,\cdot)\|^2 \sup_{\theta_2} \|\nabla_{\theta_2}a(\theta_2,\cdot)\|$$

for all θ . We have the polynomial growth of $\sup_{\theta_1} \|\nabla_{\theta_1} b(\theta_1, \cdot)\|$, $\sup_{\theta_2} \|\nabla_{\theta_2} a(\theta_2, \cdot)\|$ thanks to assumption **A4** and linear growth of $b(\theta_{0,1}, \cdot)$ thanks to **A2**. The Cauchy-Schwarz inequality and moment bounds in Lemma 5.1(1) yield (25) and so (24).

Following the approach of [39, Theorem 20 in Appendix 1], we want to show that for all N, n and $\theta, \theta' \in \Theta$,

$$\mathbb{E}[|\rho_{n,2}^N(\theta)|^2] \le C, \qquad \mathbb{E}[|\rho_{n,2}^N(\theta) - \rho_{n,2}^N(\theta')|^2] \le C||\theta - \theta'||_2^2.$$

We note that the second relation implies the first one because $\rho_{n,2}^N(\theta) = 0$ with $\theta_1 = \theta_{0,1}$ and Θ_2 is bounded. In the same way as in (23) we get

$$\mathbb{E}[|\rho_{n,2}^N(\theta) - \rho_{n,2}^N(\theta')|^2] = \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^n \mathbb{E}[|(g_{t_{j-1}}^i(\theta) - g_{t_{j-1}}^i(\theta'))A_j^i|^2],$$

where the Itô isometry gives

$$\mathbb{E}[|(g_{t_{j-1}}^i(\theta) - g_{t_{j-1}}^i(\theta'))A_j^i|^2] = \int_{t_{j-1}}^{t_j} \mathbb{E}[(g_{t_{j-1}}^i(\theta) - g_{t_{j-1}}^i(\theta'))^2 c_s^i(\theta_{0,2})] ds.$$

By the mean value theorem,

$$|g(\theta,\cdot) - g(\theta',\cdot)| \le \|\theta - \theta'\| \sup_{\theta} \|\nabla_{\theta} g(\theta,\cdot)\|$$

since Θ is convex. Then

$$\mathbb{E}\left[\sup_{\theta} \|\nabla_{\theta} g_{t_{j-1}}^{i}(\theta)\|^{2} c_{s}^{i}(\theta_{0,2})\right] \leq C$$

for all $t_{j-1} \leq s \leq t_j$, j, i and N, n follows in a similar way as the second bound in (25) does using, in addition, linear growth of $a(\theta_{0,2}, \cdot)$, which follows from its Lipschitz continuity by **A2**.

• Step 1. We want to prove that for every $\theta \in \Theta$,

$$\frac{\Delta_n}{N} S_n^N(\theta) \xrightarrow{\mathbb{P}} J(\theta_2) = \int_0^T \int_{\mathbb{R}} f(\theta_2, x, \bar{\mu}_t) \bar{\mu}_t(dx) dt, \tag{26}$$

where

$$f(\theta_2, x, \mu) = \frac{c(\theta_{0,2}, x, \mu)}{c(\theta_2, x, \mu)} + \log c(\theta_2, x, \mu)$$

for every $(\theta_2, x, \mu) \in \Theta_2 \times \mathbb{R} \times \mathcal{P}_2$. For this purpose, in $\Delta_n S_n^N(\theta)$ let us decompose every term as

$$\frac{(X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_1))^2}{c_{t_{j-1}}^i(\theta_2)} + \Delta_n(\log c)_{t_{j-1}}^i(\theta_2) = \Delta_n f_{t_{j-1}}^i(\theta_2) + r_j^i.$$
 (27)

We can decompose r_j^i further with

$$X_{t_i}^i - X_{t_{i-1}}^i - \Delta_n b_{t_{i-1}}^i(\theta_1) = B_i^i(\theta_1) + A_i^i, \tag{28}$$

where

$$B_j^i(\theta_1) = \int_{t_{j-1}}^{t_j} b_s^i(\theta_{0,1}) ds - \Delta_n b_{t_{j-1}}^i(\theta_1), \qquad A_j^i = \int_{t_{j-1}}^{t_j} a_s^i(\theta_{0,2}) dW_s^i, \tag{29}$$

note

$$\mathbb{E}_{t_{j-1}}[(A_j^i)^2] = \int_{t_{j-1}}^{t_j} c_s^i(\theta_{0,2}) ds.$$

We get

$$r_j^i = \sum_{k=0}^2 r_{j,k}^i, \quad \text{where } r_{j,k}^i = \frac{H_{j,k}^i}{c_{t_{j-1}}^i(\theta_2)}, \quad k = 0, 1, 2,$$
 (30)

and

$$H_{j,2}^{i} = (A_{j}^{i})^{2} - \mathbb{E}_{t_{j-1}}[(A_{j}^{i})^{2}], \qquad H_{j,1}^{i} = 2A_{j}^{i}B_{j}^{i}(\theta_{1}) + (B_{j}^{i}(\theta_{1}))^{2},$$

$$H_{j,0}^{i} = \mathbb{E}_{t_{j-1}}[(A_{j}^{i})^{2}] - \Delta_{n}c_{t_{j-1}}^{i}(\theta_{0,2}).$$

Our proof of (26) consists of the following steps:

$$\frac{\Delta_n}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} f_{t_{j-1}}^i(\theta_2) \xrightarrow{\mathbb{P}} J(\theta_2), \qquad \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} r_{j,k}^i \xrightarrow{L^1} 0, \quad k = 0, 1, 2.$$
 (31)

Let us start from the convergence in (31) for k=2. It is enough to show that $\sup_i \mathbb{E}[(\sum_j r_{j,2}^i)^2] = o(1)$. We note that $\mathbb{E}[r_{j_1,2}^i r_{j_2,2}^i] = 0$, $j_1 \neq j_2$, since $\mathbb{E}_{t_{j-1}}[r_{j,2}^i] = 0$. We are left to show that $\sup_i \sum_j \mathbb{E}[(r_{j,2}^i)^2] = o(1)$. Thanks to assumption **A3** it reduces to showing $\sup_i \sum_j \mathbb{E}[(H_{j,2}^i)^2] = o(1)$, where $\mathbb{E}_{t_{j-1}}[(H_{j,2}^i)^2] = \mathbb{E}_{t_{j-1}}[(A_j^i)^4] - (\mathbb{E}_{t_{j-1}}[(A_j^i)^2])^2$ leads to $\mathbb{E}[(H_{j,2}^i)^2] \leq \mathbb{E}[(A_j^i)^4]$ for all i,j. Furthermore, by the Burkholder-Davis-Gundy inequality and Jensen's inequality,

$$\mathbb{E}[(A_j^i)^4] \le C \mathbb{E}\Big[\Big(\int_{t_{j-1}}^{t_j} c_s^i(\theta_{0,2}) ds\Big)^2\Big] \le C \Delta_n \int_{t_{j-1}}^{t_j} \mathbb{E}[(c^2)_s^i(\theta_{0,2})] ds = O(\Delta_n^2)$$
(32)

uniformly in i, j, where the last relation follows thanks to linear growth of $a(\theta_{0,2}, \cdot)$ by **A2** and moment bounds in Lemma 5.1(1). We conclude that $\sup_{i,j} \mathbb{E}[(R_{j,2}^i)^2] = O(\Delta_n^2)$.

We now turn to the convergence in (31) for k=1. It is enough to show that $n \sup_{i,j} \mathbb{E}[|r_{j,1}^i|] = o(1)$. Assumption **A3** implies $\mathbb{E}[|r_{j,1}^i|] \leq C\mathbb{E}[|H_{j,1}^i|]$ for all i,j, where $\sup_{i,j} \mathbb{E}[(A_j^i)^2] = O(\Delta_n)$ follows from (32). Moreover, by Jensen's inequality,

$$\mathbb{E}[(B_j^i(\theta_1))^2] \le 2\Delta_n \int_{t_{j-1}}^{t_j} \mathbb{E}[(b_s^i(\theta_{0,1}))^2] ds + 2\Delta_n^2 \mathbb{E}[(b_{t_{j-1}}^i(\theta_1))^2] = O(\Delta_n^2)$$

uniformly in i, j, where the last relation follows thanks to linear growth of $b(\theta_1, \cdot)$ for every θ_1 by **A2** and moment bounds in Lemma 5.1(1). We conclude that $\sup_{i,j} \mathbb{E}[|r_{j,1}^i|] = O(\Delta_n^{\frac{3}{2}})$.

Next, we consider the convergence in (31) for k=0. It is enough to show that $n \sup_{i,j} \mathbb{E}[|r_{j,0}^i|] = o(1)$. Assumption **A3** implies $\mathbb{E}[|r_{j,0}^i|] \leq C\mathbb{E}[|H_{j,0}^i|]$, where

$$\mathbb{E}[|H^i_{j,0}|] \leq \int_{t_{i-1}}^{t_j} \mathbb{E}[|c^i_s(\theta_{0,2}) - c^i_{t_{j-1}}(\theta_{0,2})|] ds.$$

Lipschitz continuity of $a(\theta_{0,2},\cdot)$ and Lemma 5.1(2) and (4) imply $\mathbb{E}[(a_s^i(\theta_{0,2}) - a_{t_{j-1}}^i(\theta_{0,2}))^2] = O(\Delta_n)$ uniformly in $t_{j-1} \leq s \leq t_j, j, i$. Finally, linear growth of $a(\theta_{0,2},\cdot)$ and moment bounds in Lemma 5.1(1) guarantee $\mathbb{E}[(a_s^i(\theta_{0,2}) + a_{t_{j-1}}^i(\theta_{0,2}))^2] = O(1)$ uniformly in $t_{j-1} \leq s \leq t_j, j, i$.

We conclude by Cauchy-Schwarz inequality that $\mathbb{E}[|c_s^i(\theta_{0,2}) - c_{t_{j-1}}^i(\theta_{0,2})|] = O(\Delta_n^{\frac{1}{2}})$ uniformly in $t_{j-1} \leq s \leq t_j, j, i$, whence $\sup_{i,j} \mathbb{E}[|r_{j,0}^i|] = O(\Delta_n^{\frac{3}{2}})$.

The first relation in (31) follows from Lemma 5.2 if the function $f(\theta_2, \cdot)$ is locally Lipschitz with polynomial growth. To check this assumption, use $|\log y_1 - \log y_2| \le |y_1 - y_2|/\min(y_1, y_2)$ for $y_1, y_2 > 0$ and assumption **A3**. Note $b(\theta_1, \cdot)$, $a(\theta_2, \cdot)$ are Lipschitz continuous and have linear growth by **A2**. Hence, the function $f(\theta_2, \cdot)$ satisfies the assumption of Lemma 5.2.

• Step 2. We want to prove that the sequence $\frac{\Delta_n}{N}S_n^N(\theta)$ in $(C(\Theta;\mathbb{R}),\|\cdot\|_{\infty})$ is tight. So we have to show that for all N, n,

$$\frac{\Delta_n}{N} \mathbb{E} \Big[\sup_{\theta} \sum_{k=1}^2 \| \nabla_{\theta_k} S_n^N(\theta) \| \Big] \le C.$$

We have

$$\nabla_{\theta_k} S_n^N(\theta) = \sum_{i=1}^N \sum_{j=1}^n \zeta_{j,k}^i(\theta), \qquad k = 1, 2,$$

where

$$\zeta_{j,1}^{i}(\theta) = -\frac{2(X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n}b_{t_{j-1}}^{i}(\theta_{1}))}{c_{t_{j-1}}^{i}(\theta_{2})} \nabla_{\theta_{1}}b_{t_{j-1}}^{i}(\theta_{1}),$$

$$\zeta_{j,2}^{i}(\theta) = -\frac{(X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n}b_{t_{j-1}}^{i}(\theta_{1}))^{2}}{\Delta_{n}(c^{2})_{t_{j-1}}^{i}(\theta_{2})} \nabla_{\theta_{2}}c_{t_{j-1}}^{i}(\theta_{2}) + \frac{1}{c_{t_{j-1}}^{i}(\theta_{2})} \nabla_{\theta_{2}}c_{t_{j-1}}^{i}(\theta_{2}).$$

It suffices to show that for all N, n and i, j,

$$\mathbb{E}\left[\sup_{\theta} \|\zeta_{j,k}^{i}(\theta)\|\right] \le C, \qquad k = 1, 2. \tag{33}$$

Using A3 and the Cauchy-Schwarz inequality, we get

$$\mathbb{E}\left[\sup_{\theta} \|\zeta_{j,1}^{i}(\theta)\|\right] \leq C\left(\mathbb{E}\left[\sup_{\theta_{1}} |X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n} b_{t_{j-1}}^{i}(\theta_{1})|^{2}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\sup_{\theta_{1}} \|\nabla_{\theta_{1}} b_{t_{j-1}}^{i}(\theta_{1})\|^{2}\right]\right)^{\frac{1}{2}},$$

$$\mathbb{E}\left[\sup_{\theta} \|\zeta_{j,2}^{i}(\theta)\|\right] \leq \frac{C}{\Delta_{n}} \left(\mathbb{E}\left[\sup_{\theta_{1}} |X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n} b_{t_{j-1}}^{i}(\theta_{1})|^{4}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\sup_{\theta_{2}} \|\nabla_{\theta_{2}} a_{t_{j-1}}^{i}(\theta_{2})\|^{2}\right]\right)^{\frac{1}{2}} + C\mathbb{E}\left[\sup_{\theta_{2}} \|\nabla_{\theta_{2}} a_{t_{j-1}}^{i}(\theta_{2})\|\right].$$

We use polynomial growth of $\sup_{\theta_1} \|\nabla_{\theta_1} b(\theta_1, \cdot)\|$, $\sup_{\theta_2} \|\nabla_{\theta_2} a(\theta_2, \cdot)\|$ and moment bounds in Lemma 5.1(1). Moreover, Lemma 5.1(2) gives $\sup_{i,j} \mathbb{E}[|X_{t_j}^i - X_{t_{j-1}}^i|^4] = O(\Delta_n^2)$. Finally, $b(\theta_{0,1}, \cdot)$ has a linear growth and the mean value theorem implies $b(\theta_1, \cdot) - b(\theta_{0,1}, \cdot) = \int_0^1 \nabla_{\theta_1} b(\theta_{0,1} + (\theta_1 - \theta_{0,1})u, \cdot)du \cdot (\theta_1 - \theta_{0,1})$ for all θ_1 in Θ_1 , where Θ_1 is convex, bounded and we recall that $\sup_{\theta_1} \|\nabla_{\theta_1} b(\theta_1, \cdot)\|$ has polynomial growth. The moment bounds in Lemma 5.1(1) imply $\mathbb{E}[\sup_{\theta_1} |b_{t_{j-1}}^i(\theta_1)|^4] \leq C$, completing the proof of (33).

6.1.1 Proof of Theorem 3.1

Proof. Assumption **A5** implies that for every $\varepsilon > 0$ there exists $\eta > 0$ such that $J(\theta_2) - J(\theta_{0,2}) > \eta$ for every θ_2 with $\|\theta_2 - \theta_{0,2}\| \ge \varepsilon$. Thus $\{\|\hat{\theta}_{n,2}^N - \theta_{0,2}\| \ge \varepsilon\} \subseteq \{J(\hat{\theta}_{n,2}^N) - J(\theta_{0,2}) > \eta\}$. The probability of the latter event converges to 0 in view of

$$J(\hat{\theta}_{n,2}^N) - J(\theta_{0,2}) = J_{n,0}^N + J_{n,1}^N,$$

where the definition of $\hat{\theta}_n^N$ and (14) imply respectively

$$\begin{split} J_{n,0}^{N} &:= \frac{\Delta_{n}}{N} (S_{n}^{N}(\hat{\theta}_{n,1}^{N}, \hat{\theta}_{n,2}^{N}) - S_{n}^{N}(\hat{\theta}_{n,1}^{N}, \theta_{0,2})) \leq 0, \\ J_{n,1}^{N} &:= J(\hat{\theta}_{n,2}^{N}) - J(\theta_{0,2}) - J_{n,0}^{N} \leq 2 \sup_{(\theta_{1}, \theta_{2}) \in \Theta} \left| \frac{\Delta_{n}}{N} S_{n}^{N}(\theta_{1}, \theta_{2}) - J(\theta_{2}) \right| = o_{\mathbb{P}}(1). \end{split}$$

Consistency of $\hat{\theta}_{n,1}^N$ follows in a similar way. Assumption **A5** implies that for every $\varepsilon > 0$ there exists $\eta > 0$ such that $I(\theta_1, \theta_2) > \eta$ for every (θ_1, θ_2) with $\|\theta_1 - \theta_{0,1}\| \ge \varepsilon$. Thus $\{\|\hat{\theta}_{n,1}^N - \theta_{0,1}\| \ge \varepsilon\} \subseteq \{I(\hat{\theta}_{n,1}^N, \hat{\theta}_{n,2}^N) > \eta\}$. The probability of the latter event converges to 0 because

$$I(\hat{\theta}_{n,1}^N, \hat{\theta}_{n,2}^N) = I_{n,0}^N + I_{n,1}^N,$$

where the definition of $\hat{\theta}_n^N$ and (15) imply respectively

$$\begin{split} I_{n,0}^N &:= \frac{1}{N} (S_n^N(\hat{\theta}_{n,1}^N, \hat{\theta}_{n,2}^N) - S_n^N(\theta_{0,1}, \hat{\theta}_{n,2}^N)) \leq 0, \\ I_{n,1}^N &:= I(\hat{\theta}_{n,1}^N, \hat{\theta}_{n,2}^N) - I_{n,0}^N = o_{\mathbb{P}}(1). \end{split}$$

6.2 Asymptotic normality

The proof of the asymptotic normality of our estimator is obtained following a classical route. It consists in proving the asymptotic normality of the first derivative of the contrast function (5) (see for example [30, Section5a]). We introduce in particular the appropriate normalization matrix

$$M_n^N := \operatorname{diag}\left(\underbrace{\frac{1}{\sqrt{N}}, \dots, \frac{1}{\sqrt{N}}}_{p_1 \text{ times}}, \underbrace{\sqrt{\frac{\Delta_n}{N}}, \dots, \sqrt{\frac{\Delta_n}{N}}}_{p_2 \text{ times}}\right).$$

The proof of Theorem 3.2 is based on the following proposition.

Proposition 6.2. Assume A1-A4(I) and (II), A7. If $N\Delta_n \to 0$ then as $N, n \to \infty$,

$$\nabla_{\theta} S_n^N(\theta_0) M_n^N \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2\Sigma(\theta_0)),$$

where $\Sigma(\theta_0)$ is a $p \times p$ matrix defined in $\mathbf{A6}$.

We observe that, as $\nabla_{\theta} S_n^N(\hat{\theta}_n^N) = 0$, by Taylor's formula we obtain

$$(\hat{\theta}_n^N - \theta_0) \int_0^1 \nabla_{\theta}^2 S_n^N (\theta_0 + s(\hat{\theta}_n^N - \theta_0)) ds = -\nabla_{\theta} S_n^N (\theta_0).$$
 (34)

Multiplying the equation (34) by M_n^N , we obtain

$$(\hat{\theta}_n^N - \theta_0)(M_n^N)^{-1} \int_0^1 \Sigma_n^N (\theta_0 + s(\hat{\theta}_n^N - \theta_0)) ds = -\nabla_\theta S_n^N(\theta_0) M_n^N, \tag{35}$$

where

$$\Sigma_{n}^{N}(\theta) := M_{n}^{N} \nabla_{\theta}^{2} S_{n}^{N}(\theta) M_{n}^{N} = \begin{pmatrix} \Sigma_{n}^{N,(1)}(\theta) & \Sigma_{n}^{N,(12)}(\theta) \\ \Sigma_{n}^{N,(21)}(\theta) & \Sigma_{n}^{N,(2)}(\theta) \end{pmatrix}$$

with

$$\begin{split} \Sigma_n^{N,(1)}(\theta) &= (1/N) \nabla_{\theta_1}^2 S_n^N(\theta), \\ \Sigma_n^{N,(21)}(\theta) &= (\sqrt{\Delta_n}/N) \nabla_{\theta_2} \nabla_{\theta_1} S_n^N(\theta), \\ \Sigma_n^{N,(21)}(\theta) &= (\sqrt{\Delta_n}/N) \nabla_{\theta_2} \nabla_{\theta_1} S_n^N(\theta), \\ \end{split}$$

The analysis of the second derivatives of the contrast function is gathered in the following proposition, which will be proven at the end of this section.

Proposition 6.3. Assume A1-A5 with both (I) and (II) in A4. Then as $N, n \to \infty$,

- 1. $\Sigma_n^N(\theta_0) \xrightarrow{\mathbb{P}} \Sigma(\theta_0)$
- 2. $\sup_{s \in [0,1]} \|\Sigma_n^N(\theta_0 + s(\hat{\theta}_n^N \theta_0)) \Sigma_n^N(\theta_0)\| \xrightarrow{\mathbb{P}} 0$, where $\|\cdot\|$ refers to the operator norm on the space of $p \times p$ matrices induced by the Euclidean norm for vectors.

By Proposition 6.3 assumption **A6** implies that the probability that $\int_0^1 \Sigma_n^N (\theta_0 + s(\hat{\theta}_n^N - \theta_0)) ds$ is invertible tends to 1. Applying its inverse to the equation (35), by Proposition 6.2 and the continuous mapping theorem, we get

$$\left(\sqrt{N}(\hat{\theta}_{n,1}^{N} - \theta_{0,1}), \sqrt{N/\Delta_{n}}(\hat{\theta}_{n,2}^{N} - \theta_{0,2})\right) = (\hat{\theta}_{n}^{N} - \theta_{0})(M_{n}^{N})^{-1} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 2(\Sigma(\theta_{0}))^{-1}).$$

6.3 Proof of Proposition 6.2

Proof. As in the proof of consistency, we omit the notation for dependence on N, n. In particular, we write X_t^i for $X_t^{i,N}$, μ_t for μ_t^N , t_j for $t_{j,n}$. Denote by $f_{t_{j-1}}^i(\theta)$ the values of $f(\theta, X_{t_{j-1}}^i, \mu_{t_{j-1}})$. We note that $-\nabla_\theta S_n^N(\theta) M_n^N$ consists of $-\partial_{\theta_{1,h}} S_n^N(\theta)/\sqrt{N} =: \sum_{j=1}^n \xi_{j,h}^{(1)}(\theta)$ and $-\sqrt{\Delta_n/N} \partial_{\theta_{2,\bar{h}}} S_n^N(\theta) =: \sum_{j=1}^n \xi_{j,\bar{h}}^{(2)}(\theta)$, where

$$\xi_{j,h}^{(1)}(\theta) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} 2 \frac{\partial_{\theta_{1,h}} b_{t_{j-1}}^{i}(\theta_{1})}{c_{t_{j-1}}^{i}(\theta_{2})} (X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n} b_{t_{j-1}}^{i}(\theta_{1})),$$

$$\xi_{j,\tilde{h}}^{(2)}(\theta) := \sqrt{\frac{\Delta_{n}}{N}} \sum_{i=1}^{N} \frac{\partial_{\theta_{2,\tilde{h}}} c_{t_{j-1}}^{i}(\theta_{2})}{\Delta_{n} (c_{t_{j-1}}^{i}(\theta_{2}))^{2}} (X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n} b_{t_{j-1}}^{i}(\theta_{1}))^{2} - \frac{\partial_{\theta_{2,\tilde{h}}} c_{t_{j-1}}^{i}(\theta_{2})}{c_{t_{j-1}}^{i}(\theta_{2})}$$

for $h = 1, ..., p_1$, $\tilde{h} = 1, ..., p_2$. To prove the asymptotic normality of $-\nabla_{\theta} S_n^N(\theta_0) M_n^N$ we want to use a central limit theorem for martingale difference arrays, in accordance with Theorems 3.2 and 3.4 of [38]. Approximation of $-\nabla_{\theta} S_n^N(\theta_0) M_n^N$ by a martingale array follows from

$$\sum_{j=1}^{n} \mathbb{E}_{t_{j-1}}[\xi_{j,h}^{(1)}(\theta_0)] \xrightarrow{\mathbb{P}} 0, \qquad \sum_{j=1}^{n} \mathbb{E}_{t_{j-1}}[\xi_{j,\tilde{h}}^{(2)}(\theta_0)] \xrightarrow{\mathbb{P}} 0$$
(36)

for $h = 1, ..., p_1$, $\tilde{h} = 1, ..., p_2$. Moreover, application of the central limit theorem requires that for some r > 0 the following convergences hold:

$$\sum_{j=1}^{n} \mathbb{E}_{t_{j-1}} \left[\xi_{j,h_1}^{(1)}(\theta_0) \xi_{j,h_2}^{(1)}(\theta_0) \right] \xrightarrow{\mathbb{P}} 4 \int_0^T \int_{\mathbb{R}} \frac{\partial_{\theta_{1,h_1}} b(\theta_{0,1}, x, \bar{\mu}_t) \partial_{\theta_{1,h_2}} b(\theta_{0,1}, x, \bar{\mu}_t)}{c(\theta_{0,2}, x, \bar{\mu}_t)} \bar{\mu}_t(dx) dt, \tag{37}$$

$$\sum_{j=1}^{n} \mathbb{E}_{t_{j-1}} [\xi_{j,\tilde{h}_{1}}^{(2)}(\theta_{0}) \xi_{j,\tilde{h}_{2}}^{(2)}(\theta_{0})] \xrightarrow{\mathbb{P}} 2 \int_{0}^{T} \int_{\mathbb{R}} \frac{\partial_{\theta_{2,\tilde{h}_{1}}} c(\theta_{0,2}, x, \bar{\mu}_{t}) \partial_{\theta_{2,\tilde{h}_{2}}} c(\theta_{0,2}, x, \bar{\mu}_{t})}{c^{2}(\theta_{0,2}, x, \bar{\mu}_{t})} \bar{\mu}_{t}(dx) dt,$$
(38)

$$\sum_{j=1}^{n} \mathbb{E}_{t_{j-1}} \left[\xi_{j,h}^{(1)}(\theta_0) \xi_{j,\tilde{h}}^{(2)}(\theta_0) \right] \xrightarrow{\mathbb{P}} 0, \tag{39}$$

$$\sum_{j=1}^{n} \mathbb{E}_{t_{j-1}}[|\xi_{j,h}^{(1)}(\theta_0)|^{2+r}] \xrightarrow{\mathbb{P}} 0, \qquad \sum_{j=1}^{n} \mathbb{E}_{t_{j-1}}[|\xi_{j,\tilde{h}}^{(2)}(\theta_0)|^{2+r}] \xrightarrow{\mathbb{P}} 0, \tag{40}$$

where $h, h_1, h_2 = 1, \dots, p_1, \tilde{h}, \tilde{h}_1, \tilde{h}_2 = 1, \dots, p_2$.

• Proof of (36).

Assumptions **A3** and **A4(I)** imply that $F^i_{j,h} := 2\partial_{\theta_{1,h}} b^i_{t_{j-1}}(\theta_{0,1}) (c^i_{t_{j-1}}(\theta_{0,2}))^{-1}$ satisfies $|F^i_{j,h}| \le C(1+|X^i_{t_{j-1}}|^{k_1}+W^{l_1}_2(\mu_{t_{j-1}},\delta_0))$. Hence, from Lemma 5.1(1) it is easy to see that $F^i_{j,h} = R^i_{t_{j-1}}(1)$. If $N\Delta_n \to 0$ then Lemma 5.3(3) implies

$$\sum_{j=1}^{n} \mathbb{E}_{t_{j-1}}[\xi_{j,h}^{(1)}(\theta_0)] = \frac{1}{N^{\frac{1}{2}}} \sum_{i=1}^{N} \sum_{j=1}^{n} R_{t_{j-1}}^{i}(1) R_{t_{j-1}}^{i}(\Delta_n^{\frac{3}{2}}) \xrightarrow{L^1} 0$$

and so the convergence in probability. In a similar way, using Lemma 5.3(1), we obtain

$$\begin{split} \sum_{j=1}^{n} \mathbb{E}_{t_{j-1}}[\xi_{j,\tilde{h}}^{(2)}(\theta_{0})] &= \left(\frac{\Delta_{n}}{N}\right)^{\frac{1}{2}} \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\partial_{\theta_{2,\tilde{h}}} c_{t_{j-1}}^{i}(\theta_{0,2})}{\Delta_{n} (c_{t_{j-1}}^{i}(\theta_{0,2}))^{2}} (\Delta_{n} c_{t_{j-1}}^{i}(\theta_{0,2}) + R_{t_{j-1}}^{i}(\Delta_{n}^{2})) - \frac{\partial_{\theta_{2,\tilde{h}}} c_{t_{j-1}}^{i}(\theta_{0,2})}{c_{t_{j-1}}^{i}(\theta_{0,2})} \\ &= \left(\frac{\Delta_{n}}{N}\right)^{\frac{1}{2}} \sum_{i=1}^{N} \sum_{j=1}^{n} \frac{\partial_{\theta_{2,\tilde{h}}} c_{t_{j-1}}^{i}(\theta_{0,2})}{\Delta_{n} (c_{t_{j-1}}^{i}(\theta_{0,2}))^{2}} R_{t_{j-1}}^{i}(\Delta_{n}^{2}) \\ &= \left(\frac{\Delta_{n}}{N}\right)^{\frac{1}{2}} \sum_{i=1}^{N} \sum_{j=1}^{n} R_{t_{j-1}}^{i}(\Delta_{n}), \end{split}$$

which converges to 0 in L^1 and so in probability if $N\Delta_n \to 0$.

• Proof of (37).

We have

$$\mathbb{E}_{t_{j-1}}[\xi_{j,h_1}^{(1)}(\theta_0)\xi_{j,h_2}^{(1)}(\theta_0)] = \frac{1}{N} \sum_{i_1,i_2=1}^{N} \mathbb{E}_{t_{j-1}}[(A_j^{i_1} + B_j^{i_1})(A_j^{i_2} + B_j^{i_2})]F_{j,h_1}^{i_1}F_{j,h_2}^{i_2}, \tag{41}$$

where

$$F_{j,h}^{i} := 2 \frac{\partial_{\theta_{1,h}} b_{t_{j-1}}^{i}(\theta_{0,1})}{c_{t_{j-1}}^{i}(\theta_{0,2})} = R_{t_{j-1}}^{i}(1),$$

and

$$B_j^i := \int_{t_{j-1}}^{t_j} (b_s^i(\theta_{0,1}) - b_{t_{j-1}}^i(\theta_{0,1})) ds, \qquad A_j^i := \int_{t_{j-1}}^{t_j} a_s^i(\theta_{0,2}) dW_s^i. \tag{42}$$

We have $\mathbb{E}_{t_{j-1}}[(B_j^i)^2] = R_{t_{j-1}}^i(\Delta_n^3)$ and $\mathbb{E}_{t_{j-1}}[(A_j^i)^2] = R_{t_{j-1}}^i(\Delta_n)$, whereas if $i_1 \neq i_2$ then $\mathbb{E}_{t_{j-1}}[A_j^{i_1}A_j^{i_2}] = 0$ because of the independence of Brownian motions. Hence, by the Cauchy-Schwarz inequality,

$$\mathbb{E}_{t_{j-1}}[(A_j^{i_1} + B_j^{i_1})(A_j^{i_2} + B_j^{i_2})] = \mathbb{E}_{t_{j-1}}[(A_j^{i_1})^2]\mathbf{1}(i_1 = i_2) + R_{t_{j-1}}^{i_1, i_2}(\Delta_n^2).$$

We get

$$\sum_{j=1}^{n} \mathbb{E}_{t_{j-1}}[\xi_{j,h_1}^{(1)}(\theta_0)\xi_{j,h_2}^{(1)}(\theta_0)] = \frac{1}{N} \sum_{j=1}^{n} \sum_{i=1}^{N} \mathbb{E}_{t_{j-1}}[(A_j^i)^2] F_{j,h_1}^i F_{j,h_2}^i + \frac{1}{N} \sum_{j=1}^{n} \sum_{i_1,i_2=1}^{N} R_{t_{j-1}}^{i_1,i_2}(\Delta_n^2),$$

where the last sum converges to 0 in L^1 and so in probability if $N\Delta_n \to 0$. We can therefore focus on the first sum. We decompose the term $\mathbb{E}_{t_{j-1}}[(A_j^i)^2]$ into $\Delta_n c_{t_{j-1}}^i(\theta_{0,2})$ and

$$\mathbb{E}_{t_{j-1}}[(A_j^i)^2] - \Delta_n c_{t_{j-1}}^i(\theta_{0,2}) = \int_{t_{j-1}}^{t_j} \mathbb{E}_{t_{j-1}}[c_s^i(\theta_{0,2}) - c_{t_{j-1}}^i(\theta_{0,2})] ds = R_{t_{j-1}}^i(\Delta_n^{\frac{3}{2}}).$$

The result follows from $\Delta_n \to 0$ and application of Lemma 5.2.

• Proof of (40), first convergence.

We want to show (40) with r=2. We use the same notation as in (41) and consider the terms

$$\mathbb{E}_{t_{j-1}}[(A_j^{i_1} + B_j^{i_1})(A_j^{i_2} + B_j^{i_2})(A_j^{i_3} + B_j^{i_3})(A_j^{i_4} + B_j^{i_4})]F_{j,h}^{i_1}F_{j,h}^{i_2}F_{j,h}^{i_3}F_{j,h}^{i_4}. \tag{43}$$

We have $F_j^i = R_{t_{j-1}}^i(1)$, moreover, $\mathbb{E}_{t_{j-1}}[(A_j^i)^4] = R_{t_{j-1}}^i(\Delta_n^2)$, $\mathbb{E}_{t_{j-1}}[(B_j^i)^4] = R_{t_{j-1}}^i(\Delta_n^6)$ and so $\mathbb{E}_{t_{j-1}}[(A_j^i + B_j^i)^4] = R_{t_{j-1}}^i(\Delta_n^2)$. Application of the Cauchy-Schwarz inequality shows that the

term in (43) is also $R_{t_{j-1}}^{i_1,i_2,i_3,i_4}(\Delta_n^2)$. In case where i_1,i_2,i_3,i_4 are pairwise distinct we decompose A_i^i into

$$A_{j,2}^{i} := \int_{t_{j-1}}^{t_{j}} (a_{s}^{i}(\theta_{0,2}) - a_{t_{j-1}}^{i}(\theta_{0,2})) dW_{s}^{i}, \qquad A_{j,1}^{i} := \int_{t_{j-1}}^{t_{j}} a_{t_{j-1}}^{i}(\theta_{0,2}) dW_{s}^{i}, \tag{44}$$

which satisfy $\mathbb{E}_{t_{j-1}}[(A_{j,k}^i)^4] = R_{t_{j-1}}^i(\Delta_n^{2k}), k = 1, 2$. In particular the independence of the Brownian motions implies

$$\mathbb{E}_{t_{j-1}}[A_{j,1}^{i_1}A_{j,1}^{i_2}A_{j,1}^{i_3}A_{j,k}^{i_4}]F_{j,h}^{i_1}F_{j,h}^{i_2}F_{j,h}^{i_3}F_{j,h}^{i_4}=0$$

for k = 1, 2. The term converging to 0 at the slowest rate in (43) is then, up to a permutation of the indices i_1, i_2, i_3, i_4 ,

$$\mathbb{E}_{t_{j-1}}[A_{j,1}^{i_1}A_{j,1}^{i_2}A_{j,2}^{i_3}A_{j,2}^{i_4} + A_{j,1}^{i_1}A_{j,1}^{i_2}A_{j,1}^{i_3}B_j^{i_4}]F_{j,h}^{i_1}F_{j,h}^{i_2}F_{j,h}^{i_3}F_{j,h}^{i_4} = R_{t_{j-1}}^{i_1,i_2,i_3,i_4}(\Delta_n^3).$$

We get

$$\sum_{i=1}^{n} \mathbb{E}_{t_{j-1}}[(\xi_{j,h}^{(1)}(\theta_0))^4] = \frac{1}{N^2} \sum_{i=1}^{n} \left(\sum_{i \in I} R_{t_{j-1}}^i(\Delta_n^3) + \sum_{i \in I^c} R_{t_{j-1}}^i(\Delta_n^2) \right), \tag{45}$$

where I denotes a set of all $i = (i_1, i_2, i_3, i_4) \in \{1, \dots, N\}^4$ such that i_1, i_2, i_3, i_4 are pairwise distinct. We note that $\operatorname{card}(I) = O(N^4)$ and $\operatorname{card}(I^c) = O(N^3)$. We conclude that (45) converges to 0 in L^1 and so in probability if $N\Delta_n \to 0$.

• Proof of (38).

We rewrite the left hand side of (38) as

$$\frac{\Delta_n}{N} \sum_{j=1}^n \sum_{i_1, i_2=1}^N \Delta_n^{-2} C_{j, \tilde{h}_1}^{i_1} C_{j, \tilde{h}_2}^{i_2} \mathbb{E}_{t_{j-1}} [D_j^{i_1} D_j^{i_2}], \tag{46}$$

where

$$C_{j,\tilde{h}}^{i} := \frac{\partial_{\theta_{2,\tilde{h}}} c_{t_{j-1}}^{i}(\theta_{0,2})}{(c_{t_{j-1}}^{i}(\theta_{0,2}))^{2}} = R_{t_{j-1}}^{i}(1), \qquad D_{j}^{i} := (X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n} b_{t_{j-1}}^{i}(\theta_{0,1}))^{2} - \Delta_{n} c_{t_{j-1}}^{i}(\theta_{0,2}).$$

We consider the term $\mathbb{E}_{t_{j-1}}[D_j^{i_1}D_j^{i_2}]$ in (46). By Lemma 5.3(1) it equals

$$\mathbb{E}_{t_{j-1}}[(X_{t_{j}}^{i_{1}} - X_{t_{j-1}}^{i_{1}} - \Delta_{n}b_{t_{j-1}}^{i_{1}}(\theta_{0,1}))^{2}(X_{t_{j}}^{i_{2}} - X_{t_{j-1}}^{i_{2}} - \Delta_{n}b_{t_{j-1}}^{i_{2}}(\theta_{0,1}))^{2}] - \Delta_{n}c_{t_{j-1}}^{i_{1}}(\theta_{0,2})\Delta_{n}c_{t_{j-1}}^{i_{2}}(\theta_{0,2}) + R_{t_{j-1}}^{i_{1},i_{2}}(\Delta_{n}^{3}).$$

$$(47)$$

If $i_1 = i_2$ then Lemma 5.3(2) implies

$$\mathbb{E}_{t_{j-1}}[(X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_{0,1}))^4] = 3\Delta_n^2 (c_{t_{j-1}}^i(\theta_{0,2}))^2 + R_{t_{j-1}}^i(\Delta_n^{\frac{5}{2}}),$$

whence

$$\mathbb{E}_{t_{i-1}}[(D_i^i)^2] = 2\Delta_n^2 (c_{t_{i-1}}^i(\theta_{0,2}))^2 + R_{t_{i-1}}^i(\Delta_n^{\frac{5}{2}}). \tag{48}$$

If $i_1 \neq i_2$ then to deal with the term in (47) we decompose

$$X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_{0,1}) = A_{j,1}^i + A_{j,2}^i + B_j^i$$

as in (42), (44), where $\mathbb{E}_{t_{j-1}}[(A_{j,k}^i)^4] = R_{t_{j-1}}^i(\Delta_n^{2k})$, k = 1, 2, and $\mathbb{E}_{t_{j-1}}[(B_j^i)^4] = R_{t_{j-1}}^i(\Delta_n^6)$. We note that

$$\mathbb{E}_{t_{j-1}}[(A_{j,1}^{i_1})^2(A_{j,1}^{i_2})^2] = \Delta_n c_j^{i_1}(\theta_{0,2}) \Delta_n c_j^{i_2}(\theta_{0,2}).$$

Moreover, we have

$$\mathbb{E}_{t_{j-1}}[(A_{j,1}^{i_1})^2 A_{j,1}^{i_2} A_{j,2}^{i_2}] = c_{t_{j-1}}^{i_1}(\theta_{0,2}) a_{t_{j-1}}^{i_2}(\theta_{0,2}) V_j^{i_1,i_2},$$

where independence of Brownian motions together with Itô isometry implies

$$V_{j}^{i_{1},i_{2}} := \mathbb{E}_{t_{j-1}} \left[(W_{t_{j}}^{i_{1}} - W_{t_{j-1}}^{i_{1}})^{2} \int_{t_{j-1}}^{t_{j}} dW_{s}^{i_{2}} \int_{t_{j-1}}^{t_{j}} (a_{s}^{i_{2}}(\theta_{0,2}) - a_{t_{j-1}}^{i_{2}}(\theta_{0,2})) dW_{s}^{i_{2}} \right]$$

$$= \int_{t_{j-1}}^{t_{j}} \mathbb{E}_{t_{j-1}} \left[(W_{t_{j}}^{i_{1}} - W_{t_{j-1}}^{i_{1}})^{2} (a_{t}^{i_{2}}(\theta_{0,2}) - a_{t_{j-1}}^{i_{2}}(\theta_{0,2})) \right] dt. \tag{49}$$

Assumption A7 allows us to apply Itô's lemma to $a_t^{i_2}(\theta_{0,2})$. We get that the conditional expectation in (49) equals

$$\begin{split} \mathbb{E}_{t_{j-1}} \Big[(W_{t_{j}}^{i_{1}} - W_{t_{j-1}}^{i_{1}})^{2} \int_{t_{j-1}}^{t} \sum_{k=1}^{N} \Big(b_{s}^{k}(\theta_{0,1}) \partial_{x_{k}} a_{s}^{i_{2}}(\theta_{0,2}) + \frac{1}{2} c_{s}^{k}(\theta_{0,2}) \partial_{x_{k}}^{2} a_{s}^{i_{2}}(\theta_{0,2}) \Big) ds \Big] \\ + \mathbb{E}_{t_{j-1}} \Big[(W_{t_{j}}^{i_{1}} - W_{t_{j-1}}^{i_{1}})^{2} \int_{t_{j-1}}^{t} \sum_{k=1}^{N} a_{s}^{k}(\theta_{0,2}) \partial_{x_{k}} a_{s}^{i_{2}}(\theta_{0,2}) dW_{s}^{k} \Big]. \end{split}$$

The first term is clearly a $R_{t_{j-1}}^{i_1,i_2}(\Delta_n^2)$ function. Regarding the second one, for $k \neq i_1$, the independence of the Brownian motions makes it directly equal to 0. For $k = i_1$, instead, we have

$$\mathbb{E}_{t_{j-1}} \Big[(W_{t_j}^{i_1} - W_{t_{j-1}}^{i_1})^2 \int_{t_{j-1}}^t a_s^{i_1}(\theta_{0,2}) \partial_{x_{i_1}} a_s^{i_2}(\theta_{0,2}) dW_s^{i_1} \Big],$$

where under A7 we obtain

$$\partial_{x_{i_1}} a_s^{i_2}(\theta_{0,2}) := \partial_y \tilde{a}\left(X_s^{i_2}, \frac{1}{N} \sum_{l=1}^N K(X_s^{i_2}, X_s^l)\right) \frac{1}{N} \partial_y K(X_s^{i_2}, X_s^{i_1})$$

with $\partial_y \tilde{a}$, $\partial_y K$ having polynomial growth. Using the Cauchy-Schwarz inequality, it follows that the above quantity is upper bounded by

$$\begin{split} \left(3\Delta_n^2 \mathbb{E}_{t_{j-1}} \left[\left(\int_{t_{j-1}}^t a_s^{i_1}(\theta_{0,2}) \partial_{x_{i_1}} a_s^{i_2}(\theta_{0,2}) dW_s^{i_1} \right)^2 \right] \right)^{\frac{1}{2}} \\ &= \left(3\Delta_n^2 \int_{t_{j-1}}^t \mathbb{E}_{t_{j-1}} [(a_s^{i_1}(\theta_{0,2}) \partial_{x_{i_1}} a_s^{i_2}(\theta_{0,2}))^2] ds \right)^{\frac{1}{2}} = \frac{1}{N} R_{t_{j-1}}^{i_1, i_2} (\Delta_n^{\frac{3}{2}}). \end{split}$$

It implies

$$\mathbb{E}_{t_{j-1}}[(A_{j,1}^{i_1})^2 A_{j,1}^{i_2} A_{j,2}^{i_2}] = R_{t_{j-1}}^{i_1,i_2}(\Delta_n^3) + \frac{1}{N} R_{t_{j-1}}^{i_1,i_2}(\Delta_n^{\frac{5}{2}}). \tag{50}$$

We conclude that

$$\begin{split} \mathbb{E}_{t_{j-1}}[(X_{t_{j}}^{i_{1}}-X_{t_{j-1}}^{i_{1}}-\Delta_{n}b_{t_{j-1}}^{i_{1}}(\theta_{0,1}))^{2}(X_{t_{j}}^{i_{2}}-X_{t_{j-1}}^{i_{2}}-\Delta_{n}b_{t_{j-1}}^{i_{2}}(\theta_{0,1}))^{2}]\\ &=\Delta_{n}c_{j}^{i_{1}}(\theta_{0,2})\Delta_{n}c_{j}^{i_{2}}(\theta_{0,2})+R_{t_{j-1}}^{i_{1},i_{2}}(\Delta_{n}^{3})+\frac{1}{N}R_{t_{j-1}}^{i_{1},i_{2}}(\Delta_{n}^{\frac{5}{2}}), \end{split}$$

whence

$$\mathbb{E}_{t_{j-1}}[D_j^{i_1}D_j^{i_2}] = R_{t_{j-1}}^{i_1,i_2}(\Delta_n^3) + \frac{1}{N}R_{t_{j-1}}^{i_1,i_2}(\Delta_n^{\frac{5}{2}})$$
(51)

if $i_1 \neq i_2$. Finally, we plug (48), (51) back into (46), where use of the conditions $N\Delta_n \to 0$, $\Delta_n \to 0$ and Lemma 5.2 completes the proof of the convergence in (38).

• Proof of (40), second convergence.

We prove it for r = 2. We use the same notation as in (46) and rewrite the left hand side of (40) as

$$\frac{\Delta_n^2}{N^2} \sum_{j=1}^n \sum_{i_1, i_2, i_3, i_4=1}^N \Delta_n^{-4} C_{j,\tilde{h}}^{i_1} C_{j,\tilde{h}}^{i_2} C_{j,\tilde{h}}^{i_3} C_{j,\tilde{h}}^{i_4} \mathbb{E}_{t_{j-1}} [D_j^{i_1} D_j^{i_2} D_j^{i_3} D_j^{i_4}]. \tag{52}$$

We have $\mathbb{E}_{t_{j-1}}[(D_j^i)^4] = R_{t_{j-1}}^i(\Delta_n^4)$ and $\operatorname{card}(I^c) = O(N^3)$, where I denotes a set of all $i = (i_1, i_2, i_3, i_4) \in \{1, \dots, N\}^4$ such that i_1, i_2, i_3, i_4 are pairwise distinct. In (52) the sum over $i \in I^c$ converges to 0 in L^1 and so in probability since $N\Delta_n \to 0$. In case $i \in I$ we use the decomposition

$$D_{j}^{i} = (A_{j,1}^{i} + A_{j,2}^{i} + B_{j}^{i})^{2} - \Delta_{n} c_{t_{j-1}}^{i}(\theta_{0,2})$$

= $(A_{j,2}^{i} + B_{j}^{i})(2A_{j,1}^{i} + A_{j,2}^{i} + B_{j}^{i}) + (A_{j,1}^{i})^{2} - \Delta_{n} c_{t_{j-1}}^{i}(\theta_{0,2}).$

We note that

$$\mathbb{E}_{t_{j-1}}[((A_{j,1}^i)^2 - \Delta_n c_{t_{j-1}}^i(\theta_{0,2}))^4] = R_{t_{j-1}}^i(\Delta_n^4),$$

$$\mathbb{E}_{t_{j-1}}[(A_{j,k}^i)^8] = R_{t_{j-1}}^i(\Delta_n^{4k}), \ k = 1, 2, \qquad \mathbb{E}_{t_{j-1}}[(B_j^i)^8] = R_{t_{j-1}}^i(\Delta_n^{12}).$$

Moreover, because of the independence of Brownian motions, we have

$$\mathbb{E}_{t_{j-1}} \left[\prod_{k=1}^{4} ((A_{j,1}^{i_k})^2 - \Delta_n c_{t_{j-1}}^{i_k}(\theta_{0,2})) \right] = 0$$

and in a similar manner as in (50) under A7 we have

$$\begin{split} \mathbb{E}_{t_{j-1}} \Big[A_{j,2}^{i_1} A_{j,1}^{i_1} \prod_{k=2}^4 ((A_{j,1}^{i_k})^2 - \Delta_n c_{t_{j-1}}^{i_k}(\theta_{0,2})) \Big] \\ &= a_{t_{j-1}}^{i_1} (\theta_{0,2}) \prod_{k=2}^4 c_{t_{j-1}}^{i_k} (\theta_{0,2}) \int_{t_{j-1}}^{t_j} \mathbb{E}_{t_{j-1}} \Big[(a_s^{i_1}(\theta_{0,2}) - a_{t_{j-1}}^{i_1}(\theta_{0,2})) \prod_{l=2}^4 ((W_{t_j}^{i_l} - W_{t_{j-1}}^{i_l})^2 - \Delta_n) \Big] ds \\ &= R_{t_{j-1}}^{i_1, i_2, i_3, i_4} (\Delta_n^5) + \frac{1}{N} R_{t_{j-1}}^{i_1, i_2, i_3, i_4} (\Delta_n^{\frac{9}{2}}), \end{split}$$

whence it follows

$$\mathbb{E}_{t_{j-1}}[D_j^{i_1}D_j^{i_2}D_j^{i_3}D_j^{i_4}] = R_{t_{j-1}}^{i_1,i_2,i_3,i_4}(\Delta_n^5) + \frac{1}{N}R_{t_{j-1}}^{i_1,i_2,i_3,i_4}(\Delta_n^{\frac{9}{2}}).$$

We recall that $\operatorname{card}(I) = O(N^4)$. Since $N\Delta_n \to 0$, $\Delta_n \to 0$, the sum over $i \in I$ in (52) converges to 0 in L^1 and so in probability.

• Proof of (39).

We rewrite the left hand side of (39) as

$$\frac{\Delta_n^{\frac{1}{2}}}{N} \sum_{i=1}^n \sum_{j_1, j_2=1}^N \mathbb{E}_{t_{j-1}} [(A_{j,1}^{i_1} + A_{j,2}^{i_1} + B_j^{i_1}) D_j^{i_2}] \Delta_n^{-1} C_{j,\tilde{h}}^{i_2} F_{j,h}^{i_1}, \tag{53}$$

where

$$D_j^i = (A_{j,2}^i + B_j^i)(2A_{j,1}^i + A_{j,2}^i + B_j^i) + (A_{j,1}^i)^2 - \Delta_n c_{t_{j-1}}^i(\theta_{0,2})$$

with the notations introduced above. We recall that $F_{j,h}^i = R_{t_{j-1}}^i(1)$, $C_{j,\tilde{h}}^i = R_{t_{j-1}}^i(1)$, $\mathbb{E}_{t_{j-1}}[(B_j^i)^4] = R_{t_{j-1}}^i(\Delta_n^6)$, $\mathbb{E}_{t_{j-1}}[(A_{j,k}^i)^4] = R_{t_{j-1}}^i(\Delta_n^{2k})$, k = 1, 2, and so $\mathbb{E}_{t_{j-1}}[((A_{j,1}^i)^2 - \Delta_n c_{t_{j-1}}^i(\theta_{0,2}))^2] = R_{t_{j-1}}^i(\Delta_n^2)$. We note that

$$\mathbb{E}_{t_{j-1}}[A_{j,1}^{i_1}((A_{j,1}^{i_2})^2 - \Delta_n c_{t_{j-1}}^{i_2}(\theta_{0,2}))] = 0$$

for all i_1, i_2 . This is a consequence of the independence of the Brownian motions for $i_1 \neq i_2$, while for $i_1 = i_2$ it derives from the fact that the odd moments are centered. Hence, in case $i_1 = i_2 = i$ the term $\mathbb{E}_{t_{j-1}}[(A_{j,1}^i)^2 A_{j,2}^i]$ makes the main contribution to

$$\mathbb{E}_{t_{j-1}}[(A_{j,1}^i + A_{j,2}^i + B_j^i)D_j^i] = R_{t_{j-1}}^i(\Delta_n^2).$$

Now we can see that the sum over $i_1 = i_2$ in (53) converges to 0 in L^1 and so in probability. In case $i_1 \neq i_2$ we have

$$\mathbb{E}_{t_{i-1}}[A_{i,2}^{i_1}((A_{i,1}^{i_2})^2 - \Delta_n c_{t_{i-1}}^{i_2}(\theta_{0,2}))] = 0.$$

Moreover,

$$\mathbb{E}_{t_{j-1}}[A^{i_1}_{j,1}A^{i_2}_{j,1}A^{i_2}_{j,1}A^{i_2}_{j,2}] = a^{i_1}_{t_{j-1}}(\theta_{0,2})a^{i_2}_{t_{j-1}}(\theta_{0,2}) \int_{t_{j-1}}^{t_j} \mathbb{E}_{t_{j-1}}[(W^{i_1}_{t_j} - W^{i_1}_{t_{j-1}})(a^{i_2}_s(\theta_{0,2}) - a^{i_2}_{t_{j-1}}(\theta_{0,2})]ds.$$

The application of Itô's lemma to $a_s^{i_2}(\theta_{0,2})$ under **A7** similarly as in the proof of (50) provides

$$\mathbb{E}_{t_{j-1}}[A_{j,1}^{i_1}A_{j,1}^{i_2}A_{j,2}^{i_2}] = R_{t_{j-1}}^{i_1,i_2}(\Delta_n^{\frac{5}{2}}) + \frac{1}{N}R_{t_{j-1}}^{i_1,i_2}(\Delta_n^2).$$

We conclude that

$$\mathbb{E}_{t_{j-1}}[(A_{j,1}^{i_1} + A_{j,2}^{i_1} + B_j^{i_1})D_j^{i_2}] = R_{t_{j-1}}^{i_1,i_2}(\Delta_n^{\frac{5}{2}}) + \frac{1}{N}R_{t_{j-1}}^{i_1,i_2}(\Delta_n^2).$$

in case $i_1 \neq i_2$. Hence, the sum over $i_1 \neq i_2$ in (53) converges to 0 in L^1 and so in probability when $N\Delta_n \to 0$, $\Delta_n \to 0$. This concludes the proof of the asymptotic normality of $-\nabla_\theta S_n^N(\theta_0) M_n^N$.

6.4 Proof of Proposition 6.3

Proof. The proof relies on the computation of the second derivatives of the contrast function. We have that, for any $k, l = 1, ..., p_1$,

$$\begin{split} \partial_{\theta_{1,k}} \partial_{\theta_{1,l}} S_n^N(\theta) &= 2 \sum_{i=1}^N \sum_{j=1}^n \Big\{ \Delta_n \frac{\partial_{\theta_{1,k}} b_{t_{j-1}}^i(\theta_1) \partial_{\theta_{1,l}} b_{t_{j-1}}^i(\theta_1)}{c_{t_{j-1}}^i(\theta_2)} \\ &\qquad \qquad - \frac{\partial_{\theta_{1,k}} \partial_{\theta_{1,l}} b_{t_{j-1}}^i(\theta_1)}{c_{t_{j-1}}^i(\theta_2)} (X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_1)) \Big\}, \end{split}$$

where the last factor can further be decomposed into $\Delta_n(b^i_{t_{j-1}}(\theta_{0,1}) - b^i_{t_{j-1}}(\theta_1))$ and $X^i_{t_j} - X^i_{t_{j-1}} - \Delta_n b^i_{t_{j-1}}(\theta_{0,1})$. We can see that $\partial_{\theta_{1,k}} \partial_{\theta_{1,l}} S^N_n(\theta)/N$ converges to

$$\Sigma_{kl}^{(1)}(\theta) := 2 \int_{0}^{1} \int_{\mathbb{R}} \left\{ \frac{\partial_{\theta_{1,k}} b(\theta_{1}, x, \bar{\mu}_{t}) \partial_{\theta_{1,l}} b(\theta_{1}, x, \bar{\mu}_{t})}{c(\theta_{2}, x, \bar{\mu}_{t})} - \frac{\partial_{\theta_{1,k}} \partial_{\theta_{1,l}} b(\theta_{1}, x, \bar{\mu}_{t})}{c(\theta_{2}, x, \bar{\mu}_{t})} (b(\theta_{0,1}, x, \bar{\mu}_{t}) - b(\theta_{1}, x, \bar{\mu}_{t})) \right\} \bar{\mu}_{t}(dx) dt$$
(54)

uniformly in θ in probability. Indeed, the proof follows along the lines of the proof of (14). We refer to Steps 3, 4 of the proof of Lemma 6.1, where in (18) in $I_n^N(\theta)$, $\rho_n^N(\theta)$ it is enough to replace

the functions $h(\theta, \cdot)$ and $g(\theta, \cdot)$ with the integrand of (54) and $\partial_{\theta_{1,k}}\partial_{\theta_{1,l}}b(\theta_1, \cdot)/c(\theta_2, \cdot)$ respectively, and to check them for the respective conditions. We note that both functions have polynomial growth. Moreover, the integrand in (54) is locally Lipschitz continuous, which allows us to apply Lemma 5.2 and yields the convergence in probability of the sequence $\partial_{\theta_{1,k}}\partial_{\theta_{1,l}}S_n^N(\theta)/N$ for every θ . To get tightness in $(C(\Theta; \mathbb{R}), \|\cdot\|_{\infty})$, we use that uniformly in θ the partial derivatives with respect to $\theta_{i',j'}$, $j' = 1, \ldots, p_{i'}$, i' = 1, 2, of the two functions have polynomial growth.

In the same way as above we get that for any $k = 1, ..., p_1, l = 1, ..., p_2$, once multiplied by $\sqrt{\Delta_n}/N$,

$$\partial_{\theta_{1,k}}\partial_{\theta_{2,l}}S_n^N(\theta) = 2\sum_{i=1}^N \sum_{j=1}^n \frac{\partial_{\theta_{1,k}}b_{t_{j-1}}^i(\theta_1)\partial_{\theta_{2,l}}c_{t_{j-1}}^i(\theta_2)}{(c_{t_{j-1}}^i(\theta_2))^2} (X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_1)),$$

converges to 0 uniformly in θ in probability.

Finally, we have that for any $k, l = 1, \ldots, p_2$,

$$\begin{split} \partial_{\theta_{2,k}}\partial_{\theta_{2,l}}S_{n}^{N}(\theta) &= \sum_{i=1}^{N}\sum_{j=1}^{n} \Big\{ \frac{\partial_{\theta_{2,k}}\partial_{\theta_{2,l}}c_{t_{j-1}}^{i}(\theta_{2})c_{t_{j-1}}^{i}(\theta_{2}) - \partial_{\theta_{2,k}}c_{t_{j-1}}^{i}(\theta_{2})\partial_{\theta_{2,l}}c_{t_{j-1}}^{i}(\theta_{2})}{(c_{t_{j-1}}^{i}(\theta_{2}))^{2}} \\ &\quad + \frac{2\partial_{\theta_{2,k}}c_{t_{j-1}}^{i}(\theta_{2})\partial_{\theta_{2,l}}c_{t_{j-1}}^{i}(\theta_{2}) - \partial_{\theta_{2,k}}\partial_{\theta_{2,l}}c_{t_{j-1}}^{i}(\theta_{2})c_{t_{j-1}}^{i}(\theta_{2})}{\Delta_{n}(c_{t_{j-1}}^{i}(\theta_{2}))^{3}} \\ &\quad \times (X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n}b_{t_{j-1}}^{i}(\theta_{1}))^{2} \Big\}, \end{split}$$

where the last factor can further be decomposed into $(X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_1))^2 - \Delta_n c_{t_{j-1}}^i(\theta_{0,2})$ and $\Delta_n c_{t_{j-1}}^i(\theta_{0,2})$. We note that $(\Delta_n/N)\partial_{\theta_{2,k}}\partial_{\theta_{2,l}}S_n^N(\theta)$ converges to

$$\begin{split} \Sigma_{kl}^{(2)}(\theta) := \int_{0}^{T} \int_{\mathbb{R}} \Big\{ \frac{\partial_{\theta_{2,k}} \partial_{\theta_{2,l}} c(\theta_{2}, x, \bar{\mu}_{t}) c(\theta_{2}, x, \bar{\mu}_{t}) - \partial_{\theta_{2,k}} c(\theta_{2}, x, \bar{\mu}_{t}) \partial_{\theta_{2,l}} c(\theta_{2}, x, \bar{\mu}_{t})}{c(\theta_{2}, x, \bar{\mu}_{t})^{2}} \\ + \frac{2\partial_{\theta_{2,k}} c(\theta_{2}, x, \bar{\mu}_{t}) \partial_{\theta_{2,l}} c(\theta_{2}, x, \bar{\mu}_{t}) - \partial_{\theta_{2,k}} \partial_{\theta_{2,l}} c(\theta_{2}, x, \bar{\mu}_{t}) c(\theta_{2}, x, \bar{\mu}_{t})}{c(\theta_{2}, x, \bar{\mu}_{t})^{3}} \\ \times c(\theta_{0,2}, x, \bar{\mu}_{t}) \Big\} \bar{\mu}_{t}(dx) dt \end{split}$$

uniformly in θ in probability. We will prove the uniform in θ convergence to the second term of $\Sigma_{kl}^{(2)}(\theta)$ only:

$$\sum_{j=1}^{n} \chi_{n,j}^{N}(\theta) \xrightarrow{\mathbb{P}} \tilde{\Sigma}_{kl}^{(2)}(\theta) := \int_{0}^{T} \int_{\mathbb{R}} \tilde{f}(\theta_{2}, x, \bar{\mu}_{t}) c(\theta_{0,2}, x, \bar{\mu}_{t}) \bar{\mu}_{t}(dx) dt, \tag{55}$$

where

$$\chi_{n,j}^{N}(\theta) = \frac{1}{N} \sum_{i=1}^{N} \tilde{f}_{t_{j-1}}^{i}(\theta_{2}) (X_{t_{j}}^{i} - X_{t_{j-1}}^{i} - \Delta_{n} b_{t_{j-1}}^{i}(\theta_{1}))^{2}$$

and function $\tilde{f}: \Theta_2 \times \mathbb{R} \times \mathcal{P} \to \mathbb{R}$ is given by $(2(\partial_{\theta_{2,k}}c)(\partial_{\theta_{2,l}}c) - (\partial_{\theta_{2,k}}\partial_{\theta_{2,l}}c)c)/c^3$. For every θ the convergence in (55) follows from

$$\sum_{i=1}^{n} \mathbb{E}_{t_{j-1}}[\chi_{n,j}^{N}(\theta)] \xrightarrow{\mathbb{P}} \tilde{\Sigma}_{kl}^{(2)}(\theta), \qquad \sum_{i=1}^{n} \mathbb{E}_{t_{j-1}}[(\chi_{n,j}^{N}(\theta))^{2}] \xrightarrow{\mathbb{P}} 0$$

by [30, Lemma 9]. Indeed, the above relations hold, because by Lemma 5.3(1),

$$\mathbb{E}_{t_{j-1}}[\chi_{n,j}^N(\theta)] = \frac{1}{N} \sum_{i=1}^N \tilde{f}_{t_{j-1}}^i(\theta_2) (\Delta_n c_{t_{j-1}}^i(\theta_{0,2}) + R_{t_{j-1}}^i(\Delta_n^{3/2})),$$

by Jensen's inequality and Lemma 5.3(2),

$$\mathbb{E}_{t_{j-1}}[(\chi_{n,j}^N(\theta))^2] \le \frac{1}{N} \sum_{i=1}^N (\tilde{f}_{t_{j-1}}^i(\theta_2))^2 R_{t_{j-1}}^i(\Delta_n^2),$$

by polynomial growth of $\partial_{\theta_2,j'}^{i'}c(\theta_2,\cdot)$, $i'=0,1,2,\ j'=1,\ldots,p_2$, **A3** and Point 1. of Lemma 5.1,

$$(\tilde{f}_{t_{i-1}}^i(\theta_2))^2 = R_{t_{i-1}}^i(1).$$

The tightness in $(C(\Theta; \mathbb{R}), \|\cdot\|_{\infty})$ follows from $\mathbb{E}[\sup_{\theta} \|\nabla_{\theta} \sum_{j=1}^{n} \chi_{n,j}^{N}(\theta)\|] = O(1)$. Indeed, we have

$$\nabla_{\theta_1} \chi_{n,j}^N(\theta) = -2 \frac{\Delta_n}{N} \sum_{i=1}^N \nabla_{\theta_1} b_{t_{j-1}}^i(\theta_1) \tilde{f}_{t_{j-1}}^i(\theta_2) (X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_1)),$$

$$\nabla_{\theta_2} \chi_{n,j}^N(\theta) = \frac{1}{N} \sum_{i=1}^N \nabla_{\theta_2} \tilde{f}_{t_{j-1}}^i(\theta_2) (X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_1))^2,$$

where by polynomial growth of $\sup_{\theta_1} \|\nabla_{\theta_1} b(\theta_1, \cdot)\|$, $\sup_{\theta_2} |\partial_{\theta_{2,j'}}^{i'} c(\theta_2, \cdot)|$, $i' = 0, 1, 2, 3, j' = 1, \ldots, p_2$, and $\mathbf{A3}$,

$$\sup_{\theta} \|\nabla_{\theta_1} b_{t_{j-1}}^i(\theta_1) \tilde{f}_{t_{j-1}}^i(\theta_2)\| = R_{t_{j-1}}^i(1), \qquad \sup_{\theta_2} \|\nabla_{\theta_2} \tilde{f}_{t_{j-1}}^i(\theta_2)\| = R_{t_{j-1}}^i(1)$$

and

$$\sup_{\theta_1} |X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_1)| \le |X_{t_j}^i - X_{t_{j-1}}^i| + \Delta_n \sup_{\theta_1} |b_{t_{j-1}}^i(\theta_1)|$$

with $\sup_{\theta_1} |b_{t_{j-1}}^i(\theta_1)| = R_{t_{j-1}}^i(1)$. Finally, we have $\mathbb{E}[|X_{t_j}^i - X_{t_{j-1}}^i|^4] \le C\Delta_n^2$ uniformly in i, j and N, n by Lemma 5.1(2).

We conclude that the matrix $\Sigma_n^N(\theta)$ converges to $\Sigma(\theta) = \operatorname{diag}(\Sigma^{(1)}(\theta), \Sigma^{(2)}(\theta))$ uniformly in θ and so at $\theta = \theta_0$ in probability. Hence,

$$\|\Sigma_n^N(\theta_0 + s(\hat{\theta}_n^N - \theta_0)) - \Sigma_n^N(\theta_0)\| \le o_{\mathbb{P}}(1) + \|\Sigma(\theta_0 + s(\hat{\theta}_n^N - \theta_0)) - \Sigma(\theta_0)\|,$$

where the uniform convergence in probability (in s) of the last term to 0 follows from continuity of $\Sigma(\theta)$ at $\theta = \theta_0$ and consistency of the estimator sequence $\hat{\theta}_n^N$.

7 Proof of technical results

7.1 Proof of Lemma 5.1

Proof. Proof of Lemma 5.1(1).

We have, for any i = 1, ..., N, $0 \le t \le T$, $p \ge 2$,

$$\begin{split} \mathbb{E}[|X_t^i|^p] &\leq \mathbb{E}\Big[\Big|X_0^i + \int_0^t b_u^i(\theta_{0,1})du + \int_0^t a_u^i(\theta_{0,2})dW_u^i\Big|^p\Big] \\ &\leq C\Big(\mathbb{E}[|X_0^i|^p] + t^{p-1}\int_0^t \mathbb{E}[|b_u^i(\theta_{0,1})|^p]du + t^{\frac{p}{2}-1}\int_0^t \mathbb{E}[|a_u^i(\theta_{0,2})|^p]du\Big), \end{split}$$

where we have used the Burkholder-Davis-Gundy and Jensen inequalities. We observe that, as a consequence of the lipschitzianity gathered in $\mathbf{A2}$, for the true value of the parameter both coefficients are upper bounded by $C(1+|X_u^i|+W_2(\mu_u,\delta_0))$. Due to Jensen's inequality, we have

$$\mathbb{E}[W_2^p(\mu_u, \delta_0)] \le \frac{1}{N} \sum_{i=1}^N \mathbb{E}[|X_u^j|^p] = \mathbb{E}[|X_u^i|^p].$$

The last identity follows from the fact that the particles are equally distributed. We obtain

$$\mathbb{E}[|X_t^i|^p] \le C\Big(\mathbb{E}[|X_0^i|^p] + (t^{p-1} + t^{\frac{p}{2} - 1})\Big(t + 2\int_0^t \mathbb{E}[|X_u^i|^p]du\Big)\Big). \tag{56}$$

We infer by Gronwall's lemma that

$$\mathbb{E}[|X_t^i|^p] \le C(\mathbb{E}[|X_0^i|^p] + T^p + T^{\frac{p}{2}}) \exp(C'(T^p + T^{\frac{p}{2}})).$$

As the constants do not depend on $t \leq T$ and $\mathbb{E}[|X_0^i|^p] < \infty$ by **A1**, we have the wanted result for $p \geq 2$. Then, by a Jensen argument and the boundedness of the moments for $p \geq 2$, it follows the result also for p < 2.

Proof of Lemma 5.1(2).

We have for any $0 \le s < t \le T$, $p \ge 2$,

$$\begin{split} \mathbb{E}[|X_t^i - X_s^i|^p] &= \mathbb{E}\Big[\Big| \int_s^t b_u^i(\theta_{0,1}) du + \int_s^t a_u^i(\theta_{0,2}) dW_u^i\Big|^p\Big] \\ &\leq C\Big((t-s)^{p-1} \int_s^t \mathbb{E}[|b_u^i(\theta_{0,1})|^p] du + (t-s)^{\frac{p}{2}-1} \int_s^t \mathbb{E}[|a_u^i(\theta_{0,2})|^p] ds\Big), \end{split}$$

where we have used the Jensen and Burkholder-Davis-Gundy inequalities. Because of (4) and the just shown Lemma 5.1(1), the result follows letting $t - s \le 1$.

Proof of Lemma 5.1(3).

According to the definition of $R_s^i(1)$, we want to evaluate the L^q norm of $\mathbb{E}_s[|X_t^i - X_s^i|^p]$. For any $0 \le s < t \le T$ such that $t - s \le 1$ and $p \ge 2$, $q \ge 1$,

$$\mathbb{E}[|\mathbb{E}_{s}[|X_{t}^{i} - X_{s}^{i}|^{p}]|^{q}]^{\frac{1}{q}} \leq \mathbb{E}[|X_{t}^{i} - X_{s}^{i}|^{pq}]^{\frac{1}{q}} \leq C(t - s)^{\frac{p}{2}}$$

follows by conditional Jensen's inequality and Lemma 5.1(2).

Proof of Lemma 5.1(4).

This is a straightforward consequence of

$$W_2^p(\mu_t, \mu_s) \le \left(\frac{1}{N} \sum_{j=1}^N |X_t^j - X_s^j|^2\right)^{\frac{p}{2}} \le \frac{1}{N} \sum_{j=1}^N |X_t^j - X_s^j|^p$$
(57)

by Jensen's inequality for any $0 \le s < t \le T$ such that $t - s \le 1$, $p \ge 2$ and Lemma 5.1(2).

Proof of Lemma 5.1(5).

It follows directly from (57), where we use Minkowski's inequality as follows:

$$\mathbb{E}[|\mathbb{E}_{s}[W_{2}^{p}(\mu_{t}, \mu_{s})]|^{q}]^{\frac{1}{q}} \leq \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}[|\mathbb{E}_{s}[|X_{t}^{j} - X_{s}^{j}|^{pq}]|]^{\frac{1}{q}},$$

and then Lemma 5.1(3).

7.2 Proof of Lemma 5.2

Proof. Step 1. We prove that

$$\frac{\Delta_n}{N} \sum_{i=1}^N \sum_{j=1}^n f(X_{t_{j-1,n}}^{i,N}, \mu_{t_{j-1,n}}^N) - \frac{1}{N} \sum_{i=1}^N \int_0^T f(X_s^{i,N}, \mu_s^N) ds \xrightarrow{L^1} 0.$$

Here we note $\Delta_n = t_{j,n} - t_{j-1,n}$ and decompose the above integral into integrals over $[t_{j-1,n}, t_{j,n})$. We can see that the above convergence follows from

$$\sum_{j=1}^{n} \int_{t_{j-1,n}}^{t_{j,n}} \mathbb{E}[|f(X_{t_{j-1,n}}^{i,N}, \mu_{t_{j-1,n}}^{N}) - f(X_{s}^{i,N}, \mu_{s}^{N})|] ds \to 0, \qquad N, n \to \infty,$$

for fixed i, which in turn follows using the condition (13), Cauchy-Schwarz inequality and moment bounds in Lemma 5.1(1), (2) and (4). In particular, $\mathbb{E}[|X_{t_{j-1,n}}^{i,N} - X_s^{i,N}|^2] \leq C\Delta_n$ for all $t_{j-1,n} \leq s \leq t_{j,n}$, j and n, N.

Step 2. Next, let us prove that

$$\frac{1}{N}\sum_{i=1}^{N}\int_{0}^{T}f(X_{s}^{i,N},\mu_{s}^{N})ds - \frac{1}{N}\sum_{i=1}^{N}\int_{0}^{T}f(\bar{X}_{s}^{i},\bar{\mu}_{s})ds \xrightarrow{L^{1}}0, \qquad N \to \infty,$$

where each $(\bar{X}_t^i)_{t\in[0,T]}$ satisfies (3) with $(W_t)_{t\in[0,T]}=(W_t^i)_{t\in[0,T]}$ and $\bar{X}_0^i=X_0^{i,N}$. It suffices to prove

$$\int_0^T \mathbb{E}[|f(X_s^{i,N},\mu_s^N) - f(\bar{X}_s^i,\bar{\mu}_s)|]ds \to 0,$$

where i is fixed and the integral is over a bounded interval. For this purpose, let us use again the condition (13) and the Cauchy-Schwarz inequality. Following the same arguments as in the proof of Lemma 5.1(1) and Gronwall lemma, it is easy to show that for all p>0 there exists $C_p>0$ such that for all s,i,N it holds $\mathbb{E}[|\bar{X}_s^i|^p]< C_p$. Moreover we have

$$\mathbb{E}[|X_s^{i,N} - \bar{X}_s^i|^2] \le \frac{C}{\sqrt{N}}$$

for all $0 \le s \le T$ and i, N, thanks to Theorem 3.20 in [11], based on Theorem 1 of [28]. We remark that, from the boundedness of the moments, the quantity q appearing in the statement of Theorem 3.20 in [11] is larger than 4. Hence, the rate $N^{-(q-2)/q}$ is negligible compared to $N^{-1/2}$. The propagation of chaos stated above implies

$$\mathbb{E}[W_2^2(\mu_s^N, \bar{\mu}_s)] \le \frac{C}{\sqrt{N}}.$$

Indeed, to get the last relation, we introduce the empirical measure $\bar{\mu}_s^N = N^{-1} \sum_{i=1}^N \delta_{\bar{X}_s^i}$ of the independent particle system at time s and use the triangle inequality for W_2 . Then

$$\mathbb{E}[W_2^2(\mu_s^N, \bar{\mu}_s^N)] \le \frac{1}{N} \sum_{i=1}^N \mathbb{E}[|X_s^{i,N} - \bar{X}_s^i|^2] \le \frac{C}{\sqrt{N}},$$

whereas Theorem 1 of [28] implies

$$\mathbb{E}[W_2^2(\bar{\mu}_s^N, \bar{\mu}_s)] \le \frac{C}{\sqrt{N}}.$$

Step 3. Finally, the law of large numbers gives

$$\frac{1}{N} \sum_{i=1}^{N} \int_{0}^{T} f(\bar{X}_{s}^{i}, \bar{\mu}_{s}) ds \xrightarrow{\mathbb{P}} \mathbb{E} \Big[\int_{0}^{T} f(\bar{X}_{s}, \bar{\mu}_{s}) ds \Big], \qquad N \to \infty.$$

7.3 Proof of Lemma 5.3

Proof. We use the same notation as before.

Proof of Lemma 5.3(2). We decompose $X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_{0,1})$ into $A_{j,1}^i$ and $H_{j,2}^i := A_{j,2}^i + B_j^i$, where

$$A_{j,1}^{i} := \int_{t_{j-1}}^{t_{j}} a_{t_{j-1}}^{i}(\theta_{0,2}) dW_{s}^{i}, \qquad A_{j,2}^{i} := \int_{t_{j-1}}^{t_{j}} (a_{s}^{i}(\theta_{0,2}) - a_{t_{j-1}}^{i}(\theta_{0,2})) dW_{s}^{i},$$

$$B_{j}^{i} := \int_{t_{j-1}}^{t_{j}} (b_{s}^{i}(\theta_{0,1}) - b_{t_{j-1}}^{i}(\theta_{0,1})) ds,$$

$$(58)$$

are the same as in (42), (44).

Firstly, we will show that for any $p \geq 2$,

$$\mathbb{E}[|H_{i,2}^i|^p] \le C\Delta_n^p. \tag{59}$$

Using Jensen's inequality and Lipschitz continuity of $b(\theta_1, \cdot)$ we get

$$\mathbb{E}[|B_{j}^{i}|^{p}] \leq \mathbb{E}\left[\Delta_{n}^{p-1} \int_{t_{j-1}}^{t_{j}} |b_{s}^{i}(\theta_{0,1}) - b_{t_{j-1}}^{i}(\theta_{0,1})|^{p} ds\right]
\leq C\Delta_{n}^{p-1} \int_{t_{j-1}}^{t_{j}} (\mathbb{E}[|X_{s}^{i} - X_{t_{j-1}}^{i}|^{p}] + \mathbb{E}[W_{2}^{p}(\mu_{s}, \mu_{t_{j-1}})]) ds
\leq C\Delta_{n}^{p-1} \int_{t_{j-1}}^{t_{j}} (s - t_{j-1})^{\frac{p}{2}} ds = C\Delta_{n}^{\frac{3}{2}p},$$
(60)

where the last inequality follows from Lemma 5.1(2) and (4). Further use of the Burkholder-Davis-Gundy and Jensen inequalities gives

$$\mathbb{E}[|A_{j,2}^{i}|^{p}] \leq C \mathbb{E}\left[\left(\int_{t_{j-1}}^{t_{j}} |a_{s}^{i}(\theta_{0,2}) - a_{t_{j-1}}^{i}(\theta_{0,2})|^{2} ds\right)^{\frac{p}{2}}\right] \\
\leq C \Delta_{n}^{\frac{p}{2}-1} \int_{t_{j-1}}^{t_{j}} \mathbb{E}[|a_{s}^{i}(\theta_{0,2}) - a_{t_{j-1}}^{i}(\theta_{0,2})|^{p}] ds \\
\leq C \Delta_{n}^{p}, \tag{61}$$

where the last inequality follows from Lipschitz continuity of $a(\theta_2, \cdot)$ and Lemma 5.1(2) and (4) as so does (60). Hence, we have shown (59).

Next, we have

$$\mathbb{E}[|A_{j,1}^i|^p] = C\Delta_n^{\frac{p}{2}} \mathbb{E}[|a_{t_{j-1}}^i(\theta_{0,2})|^p] \le C\Delta_n^{\frac{p}{2}}$$
(62)

since we know the absolute moments of a centered normal distribution and have linear growth of $a(\theta_{0,2},\cdot)$, moment bounds in Lemma 5.1(1). In particular, we note

$$\mathbb{E}_{t_{i-1}}[(A_{i,1}^i)^4] = 3\Delta_n^2(c^2)_{t_{i-1}}^i(\theta_{0,2}).$$

Finally, we have

$$\mathbb{E}_{t_{j-1}}[(X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_{0,1}))^4] = 3\Delta_n^2(c^2)_{t_{j-1}}^i(\theta_{0,2}) + \sum_{k=0}^3 \binom{4}{k} \mathbb{E}_{t_{j-1}}[(A_{j,1}^i)^k (H_{j,2}^i)^{4-k}].$$
(63)

For any k = 0, 1, 2, 3 and $q \ge 1$, using Jensen's inequality for conditional expectation, we get

$$\mathbb{E}\left[\left|\mathbb{E}_{t_{i-1}}[(A_{i,1}^i)^k(H_{i,2}^i)^{4-k}]\right|^q\right] \le \mathbb{E}\left[\left|(A_{i,1}^i)^k(H_{i,2}^i)^{4-k}\right|^q\right] \le C\Delta_n^{(4-\frac{k}{2})q},$$

where the last inequality follows from (61), (59) using Cauchy-Schwarz inequality. Hence, the term converging to 0 in L^q at the slowest rate is the one for which k=3. We therefore obtain that the remaining sum on the right hand side of (63) is an $R_{t_{j-1}}^i(\Delta_n^{\frac{5}{2}})$ function.

Proof of Lemma 5.3(3). This follows directly from (60) by decomposing the dynamics of X^i as in (58) and remarking that the stochastic integral is centered.

Proof of Lemma 5.3(1). We decompose $X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_{0,1})$ into

$$A_j^i := A_{j,1}^i + A_{j,2}^i = \int_{t_{j-1}}^{t_j} a_s^i(\theta_{0,2}) dW_s^i,$$

and B_j^i satisfying respectively $\mathbb{E}[|A_j^i|^{2p}] \leq C\Delta_n^p$ and $\mathbb{E}[|B_j^i|^{2p}] \leq C\Delta_n^{3p}$, whence $\mathbb{E}[|A_j^iB_j^i|^p] \leq C\Delta_n^{2p}$ for any $p \geq 1$, see (60)-(62). We conclude that

$$\mathbb{E}_{t_{j-1}}[(X_{t_j}^i - X_{t_{j-1}}^i - \Delta_n b_{t_{j-1}}^i(\theta_{0,1}))^2] = \int_{t_{j-1}}^{t_j} \mathbb{E}_{t_{j-1}}[c_s^i(\theta_{0,2})]ds + R_{t_{j-1}}^i(\Delta_n^2).$$

We are left to show that we can replace $\mathbb{E}_{t_{j-1}}[c_s^i(\theta_{0,2})]$ with $c_{t_{j-1}}^i(\theta_{0,2})$ and that the remaining integral is an $R_{t_{j-1}}^i(\Delta_n^2)$ function.

Under A7 we have that for any i,

$$(x_1, \dots, x_N) \mapsto c\Big(\theta_{0,2}, x_i, \frac{1}{N} \sum_{j=1}^N \delta_{x_j}\Big) = \tilde{a}^2\Big(x_i, \frac{1}{N} \sum_{j=1}^N K(x_i, x_j)\Big) =: g^i(x_1, \dots, x_N)$$

is a twice continuously differentiable function from \mathbb{R}^N to \mathbb{R} . Given a vector $(X_s^1, \dots, X_s^N)_{s \in [0,T]}$ of processes, we denote

$$(\partial_{x_k}^l c)_s^i(\theta_{0,2}) := \partial_{x_k}^l g^i(X_s^1, \dots, X_s^N).$$

We apply the multidimensional Itô's formula to $g^i(X^1_s,\ldots,X^N_s)=c^i_s(\theta_{0,2})$ as follows:

$$c_s^i(\theta_{0,2}) - c_{t_{j-1}}^i(\theta_{0,2}) = \sum_{k=1}^N \int_{t_{j-1}}^s \left((\partial_{x_k} c)_u^i(\theta_{0,2}) b_u^k(\theta_{0,1}) + \frac{1}{2} (\partial_{x_k}^2 c)_u^i(\theta_{0,2}) c_u^k(\theta_{0,2}) \right) du + \sum_{k=1}^N \int_{t_{j-1}}^s (\partial_{x_k} c)_u^i(\theta_{0,2}) a_u^k(\theta_{0,2}) dW_u^k.$$

Since the driving $(W_u^1, \dots, W_u^N)_{u \in [t_{j-1}, s]}$ is independent of $\mathcal{F}_{t_{j-1}}^N$, it follows that

$$\mathbb{E}_{t_{j-1}}[c_s^i(\theta_{0,2})] - c_{t_{j-1}}^i(\theta_{0,2})$$

$$= \mathbb{E}_{t_{j-1}}\Big[\sum_{k=1}^N \int_{t_{j-1}}^s \Big((\partial_{x_k} c)_u^i(\theta_{0,2}) b_u^k(\theta_{0,1}) + \frac{1}{2} (\partial_{x_k}^2 c)_u^i(\theta_{0,2}) c_u^k(\theta_{0,2}) \Big) du \Big]. \tag{64}$$

To conclude, we need to bound each $(\partial_{x_k}^l c)_u^i(\theta_{0,2})$, l=1,2. To do that, we rely on the assumption about the dependence of the diffusion coefficient on the convolution with a probability measure gathered in **A7**. To compute the derivatives with respect to x_k we need to consider two different cases, depending on whether $k \neq i$ or k = i. When $k \neq i$ we have $(\partial_{x_k} c)_u^i(\theta_{0,2}) = 2a_u^i(\theta_{0,2})(\partial_{x_k} a)_u^i(\theta_{0,2})$, where

$$(\partial_{x_k} a)_u^i(\theta_{0,2}) := \partial_y \tilde{a} \left(X_u^i, \frac{1}{N} \sum_{j=1}^N K(X_u^i, X_u^j) \right) \frac{1}{N} \partial_y K(X_u^i, X_u^k), \tag{65}$$

while for k=i we have $(\partial_{x_i}c)^i_u(\theta_{0,2})=2a^i_u(\theta_{0,2})(\partial_{x_i}a)^i_u(\theta_{0,2})$, where

$$(\partial_{x_i}a)_u^i(\theta_{0,2}) := \partial_x \tilde{a}\left(X_u^i, \frac{1}{N} \sum_{j=1}^N K(X_u^i, X_u^j)\right) + \partial_y \tilde{a}\left(X_u^i, \frac{1}{N} \sum_{j=1}^N K(X_u^i, X_u^j)\right) \times \left(\frac{1}{N} \sum_{j=1}^N \partial_x K(X_u^i, X_u^j) + \frac{1}{N} \partial_y K(X_u^i, X_u^i)\right).$$

From polynomial growth of the l-th order partial derivatives of K, \tilde{a} for l = 0, 1, that of $b(\theta_{0,1}, \cdot)$, moment bounds in Lemma 5.1(1) applying Jensen's inequality it follows that $\sum_{k=1}^{N} (\partial_{x_k} c)_u^i(\theta_{0,2}) b_u^k(\theta_{0,1})$ is bounded in L^p for any $p \geq 1$ uniformly in u, i. We proceed similarly to compute $(\partial_{x_k}^2 c)_u^i(\theta_{0,2})$. Then from polynomial growth of the l-th order partial derivatives of K, \tilde{a} for l = 0, 1, 2, moment bounds in Lemma 5.1(1) applying Jensen's inequality it follows that $\sum_{k=1}^{N} (\partial_{x_k}^2 c)_u^i(\theta_{0,2}) c_u^k(\theta_{0,2})$ is bounded in L^p for any $p \geq 1$ uniformly in u, i. For any $p \geq 1$, $t_{j-1} \leq s \leq t_j$, repeatedly applying Jensen's inequality to (64) we get

$$\mathbb{E}\left[\left|\mathbb{E}_{t_{j-1}}[c_s^i(\theta_{0,2})] - c_{t_{j-1}}^i(\theta_{0,2})\right|^p\right] \le C(s - t_{j-1})^p,$$

whence

$$\mathbb{E}\left[\left|\int_{t_{j-1}}^{t_j} \left(\mathbb{E}_{t_{j-1}}[c_s^i(\theta_{0,2})] - c_{t_{j-1}}^i(\theta_{0,2})\right) ds\right|^p\right] \le C\Delta_n^{2p}.$$

which completes the proof.

Acknowledgements

We are grateful to two anonymous referees for truly helpful comments and suggestions.

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