### **QUASI-CRITICAL FLUCTUATIONS FOR** 2*d* **DIRECTED POLYMERS**

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Abstract. We study the 2*d* directed polymer in random environment in a novel *quasicritical regime*, which interpolates between the much studied sub-critical and critical regimes. We prove Edwards-Wilkinson fluctuations throughout the quasi-critical regime, showing that the diffusively rescaled partition functions are asymptotically Gaussian, under a rescaling which diverges arbitrarily slowly as criticality is approached. A key challenge is the lack of hypercontractivity, which we overcome deriving new sharp moment estimates.

#### **1. Introduction**

We consider the partition functions of the 2*d* directed polymer in random environment:

<span id="page-0-0"></span>
$$
Z_{N,\beta}^{\omega}(z) := \mathbf{E} \big[ e^{\sum_{n=1}^{N} \{\beta \omega(n, S_n) - \lambda(\beta)\}} \big| \, S_0 = z \big],\tag{1.1}
$$

where  $N \in \mathbb{N}$  is the system size,  $\beta \geq 0$  is the disorder strength,  $z \in \mathbb{Z}^2$  is the starting point, and we have two independent sources of randomness:

- $S = (S_n)_{n \geq 0}$  is the simple random walk on  $\mathbb{Z}^2$  with law P and expectation E;
- $\bullet \ \omega = (\omega(n,z))_{n \in \mathbb{N}, z \in \mathbb{Z}^2}$  are i.i.d. random variables with law  $\mathbb{P}$ , independent of *S*, with

<span id="page-0-2"></span>
$$
\mathbb{E}[\omega] = 0, \qquad \mathbb{E}[\omega^2] = 1, \qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta \omega}] < \infty \quad \text{for } \beta > 0. \tag{1.2}
$$

The factor  $\lambda(\beta)$  in [\(1.1\)](#page-0-0) has the effect to normalise the expectation:

<span id="page-0-3"></span>
$$
\mathbb{E}\big[Z_{N,\beta}^{\omega}(z)\big] = 1\,. \tag{1.3}
$$

Note that  $(Z_{N,\beta}^{\omega}(z))_{z\in\mathbb{Z}^2}$  is a family of (correlated) positive random variables, depending on the random variables  $\omega$  which play the role of *disorder* (or *random environment*).

In this paper we investigate the *diffusively rescaled* partition functions  $Z_{N,\beta}^{\omega}([\sqrt{N}x]),$ where  $\lfloor \cdot \rfloor$  denotes the integer part. For an integrable test function  $\varphi : \mathbb{R}^2 \to \mathbb{R}$  we set

<span id="page-0-1"></span>
$$
Z_{N,\beta}^{\omega}(\varphi) := \int_{\mathbb{R}^2} Z_{N,\beta}^{\omega}([\sqrt{N}x]) \, \varphi(x) \, dx = \frac{1}{N} \sum_{z \in \mathbb{Z}^2} Z_{N,\beta}^{\omega}(z) \, \varphi_N(z) \,, \tag{1.4}
$$

where for  $R > 0$  we define  $\varphi_R : \mathbb{Z}^2 \to \mathbb{R}$  by

<span id="page-0-4"></span>
$$
\varphi_R(z) := \int\limits_{[z,z+(1,1))} \varphi\left(\frac{y}{\sqrt{R}}\right) dy \tag{1.5}
$$

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(note that  $\varphi_R(z) \approx \varphi\left(\frac{z}{\sqrt{R}}\right)$  if  $\varphi$  is continuous). We look for the convergence in distribution of  $Z_{N,\beta}^{\omega}(\varphi)$  as  $N \to \infty$ , under an appropriate rescaling of the disorder strength  $\beta = \beta_N$ .

NOTATION. We denote by  $\varphi \in C_c(\mathbb{R}^2)$  the space of functions  $\varphi : \mathbb{R}^2 \to \mathbb{R}$  that are continuous and compactly supported. We write  $a_N \ll b_N$ ,  $a_N \sim b_N$ ,  $a_N \gg b_N$  to mean that the ratio  $a_N/b_N$  converges respectively to 0, 1,  $\infty$  as  $N \to \infty$ .

**1.1. The phase transition.** It is known since [\[CSZ17b\]](#page-35-0) that the partition functions undergo a *phase transition* on the scale  $\beta^2 = \beta_N^2 = O(\frac{1}{\log n})$  $\frac{1}{\log N}$ , that we now recall.

Let  $R_N$  be the *expected replica overlap* of two independent simple random walks  $S, S'$ :

<span id="page-1-3"></span>
$$
R_N := \mathcal{E}^{\otimes 2} \bigg[ \sum_{n=1}^N \mathbb{1}_{\{S_n = S'_n\}} \bigg] = \sum_{n=1}^N \mathcal{P}(S_{2n} = 0) = \frac{\log N}{\pi} + O(1), \tag{1.6}
$$

see the local limit theorem [\(3.8\)](#page-7-0). Using the more convenient parameter

<span id="page-1-6"></span>
$$
\sigma_{\beta}^{2} := \mathbb{V}\text{ar}[e^{\beta\omega - \lambda(\beta)}] = e^{\lambda(2\beta) - 2\lambda(\beta)} - 1 \tag{1.7}
$$

(note that  $\sigma_{\beta} \sim \beta$  as  $\beta \downarrow 0$ , since  $\lambda(\beta) \sim \frac{1}{2}\beta^2$ ), we can rescale  $\beta = \beta_N$  as follows:

<span id="page-1-4"></span>
$$
\sigma_{\beta_N}^2 = \frac{\hat{\beta}^2}{R_N} \sim \frac{\hat{\beta}^2 \pi}{\log N}, \quad \text{with} \quad \hat{\beta} \in (0, \infty). \tag{1.8}
$$

Let us recall some key results on the scaling limit of  $Z_{N,\beta}^{\omega}(\varphi)$  from [\(1.4\)](#page-0-1) for  $\beta = \beta_N$ .

• In the *sub-critical regime*  $\hat{\beta} < 1$ , after centering and rescaling by  $\sqrt{\log N}$ , the averaged partition function  $Z^{\omega}_{N,\beta_N}(\varphi)$  is asymptotically Gaussian, see [\[CSZ17b\]](#page-35-0):<sup>[†](#page-1-0)</sup>

<span id="page-1-1"></span>
$$
\hat{\beta} \in (0,1): \qquad \sqrt{\log N} \left\{ Z_{N,\beta_N}^{\omega}(\varphi) - \mathbb{E} [Z_{N,\beta_N}^{\omega}(\varphi)] \right\} \xrightarrow[N \to \infty]{d} \mathcal{N} \left( 0, \sigma_{\varphi,\hat{\beta}}^2 \right), \qquad (1.9)
$$

for an explicit limiting variance  $\sigma_{\varphi,\hat{\beta}}^2 \in (0,\infty)$  (which *diverges* as  $\hat{\beta} \uparrow 1$ ).

• In the *critical regime*  $\hat{\beta} = 1$ , actually in the *critical window*  $\hat{\beta}^2 = 1 + O(\frac{1}{\log n})$  $\frac{1}{\log N}$ ), the averaged partition function  $Z_{N,\beta_N}^{\omega}(\varphi)$  is asymptotically *non Gaussian*, see [\[CSZ23\]](#page-35-1):

<span id="page-1-2"></span>
$$
\hat{\beta} = 1 + O\left(\frac{1}{\log N}\right) : \qquad Z_{N,\beta_N}^{\omega}(\varphi) \xrightarrow[N \to \infty]{d} \mathscr{Z}(\varphi) = \int_{\mathbb{R}^2} \varphi(x) \mathscr{Z}(\mathrm{d}x), \tag{1.10}
$$

where  $\mathscr{Z}(\mathrm{d}x)$  is a non-trivial random measure on  $\mathbb{R}^2$  called the Stochastic Heat Flow.

Note that the sub-critical convergence [\(1.9\)](#page-1-1) involves a rescaling factor  $\sqrt{\log N}$ , while *no rescaling is needed for the critical convergence* [\(1.10\)](#page-1-2). In view of this discrepancy, it is natural to investigate the transition between these regimes.

**1.2. Main result.** To interpolate between the sub-critical regime  $\hat{\beta}$  < 1 and the critical regime  $\hat{\beta} = 1$ , we consider a *quasi-critical regime* in which  $\hat{\beta} \uparrow 1$  *but slower than the critical window*  $\hat{\beta}^2 = 1 + O(\frac{1}{\log n})$  $\frac{1}{\log N}$ ). Recalling [\(1.6\)](#page-1-3) and [\(1.8\)](#page-1-4), we fix  $\beta = \beta_N$  such that

<span id="page-1-5"></span>
$$
\sigma_{\beta_N}^2 = \frac{1}{R_N} \left( 1 - \frac{\vartheta_N}{\log N} \right) \qquad \text{for some} \quad 1 \ll \vartheta_N \ll \log N \,. \tag{1.11}
$$

<span id="page-1-0"></span><sup>&</sup>lt;sup>†</sup>The result proved in [\[CSZ17b,](#page-35-0) Theorem 2.13] actually involves a space-time average, but the same result for the space average as in [\(1.4\)](#page-0-1) follows by similar arguments, see [\[CSZ20\]](#page-35-2).

(Note that  $\vartheta_N = O(1)$  would correspond to the critical window, while  $\vartheta_N = (1 - \hat{\beta}^2) \log N$ with  $\hat{\beta} \in (0, 1)$  would correspond to the sub-critical regime.)

Our main result shows that the averaged partition function  $Z_{N,\beta_N}^{\omega}(\varphi)$  has Gaussian fluctuations *throughout the quasi-critical regime* [\(1.11\)](#page-1-5), after centering and rescaling by the factor  $\sqrt{\vartheta_N}$ , whose rate of divergence can be arbitrarily slow. This shows that *non-Gaussian behavior does not appear before the critical regime*. We call this result *Edwards-Wilkinson fluctuations* in view of its link with stochastic PDEs, that we discuss in Subsection [1.3.](#page-2-0)

<span id="page-2-1"></span>**Theorem 1.1 (Quasi-critical Edwards-Wilkinson fluctuations).** Let  $Z_{N,\beta}^{\omega}(\varphi)$  denote *the diffusively rescaled and averaged partition function of the* 2*d directed polymer model, see* [\(1.1\)](#page-0-0) and [\(1.4\)](#page-0-1), for disorder variables  $\omega$  which satisfy [\(1.2\)](#page-0-2). Then, for  $(\beta_N)_{N\in\mathbb{N}}$  in the quasi*critical regime, see* [\(1.7\)](#page-1-6) *and* [\(1.11\)](#page-1-5)*, we have the convergence in distribution*

<span id="page-2-3"></span>
$$
\forall \varphi \in C_c(\mathbb{R}^2): \qquad \sqrt{\vartheta_N} \left\{ Z_{N,\beta_N}^{\omega}(\varphi) - \mathbb{E}[Z_{N,\beta_N}^{\omega}(\varphi)] \right\} \xrightarrow[N \to \infty]{d} \mathcal{N}\left(0, \sigma_{\varphi}^2\right), \tag{1.12}
$$

*where the limiting variance is given by*

<span id="page-2-2"></span>
$$
\sigma_{\varphi}^{2} := \int_{\mathbb{R}^{2} \times \mathbb{R}^{2}} \varphi(x) K(x, x') \varphi(x') dx dx' \quad \text{with} \quad K(x, x') := \int_{0}^{1} \frac{1}{2u} e^{-\frac{|x - x'|^{2}}{2u}} du. \quad (1.13)
$$

Our strategy to prove Theorem [1.1](#page-2-1) is inspired by the recent paper [\[CC22\]](#page-35-3): we apply a Central Limit Theorem under a Lyapunov condition, which requires to estimate moments of the partition function of order higher than two (see Section [2](#page-4-0) for a detailed explanation). A key point is [\[CC22\]](#page-35-3) is to bound such high moments exploiting the hypercontractivity of polynomial chaos expansions in the sub-critical regime  $\hat{\beta}$  < 1. Crucially, *this fails in the quasi-critical regime* [\(1.11\)](#page-1-5), because the main contribution to the partition function *no longer comes from a finite number of chaotic components* (see Section [3\)](#page-6-0).

This is the key technical difficulty that we face in this paper, for which we need to use model-specific arguments to estimate high moments. To this purpose, we exploit and extend the strategy developed in [\[GQT21,](#page-36-0) [CSZ23,](#page-35-1) [LZ21+\]](#page-36-1), deriving *novel quantitative estimates* which are essential for our approach (see Sections [4](#page-11-0) and [5\)](#page-20-0). We believe that these estimates will find several applications in future research.

<span id="page-2-0"></span>**1.3. Relevant context and future perspectives.** The Gaussian fluctuations for  $Z_{N,\beta}^{\omega}(\varphi)$ in Theorem [1.1](#page-2-1) are closely connected to a stochastic PDE, the *Edwards-Wilkinson equation*, also known as Stochastic Heat Equation with *additive* noise:

<span id="page-2-4"></span>
$$
\partial_t v^{(\mathsf{s}, \mathsf{c})}(t, x) = \frac{\mathsf{s}}{2} \Delta_x v^{(\mathsf{s}, \mathsf{c})}(t, x) + \mathsf{c} \dot{W}(t, x), \qquad (1.14)
$$

where  $s, c > 0$  are fixed parameters and  $W(t, x)$  is space-time white noise. This equation is well-posed in any spatial dimension  $d \geq 1$ : its solution is the Gaussian process

$$
v^{(\mathsf{s},\mathsf{c})}(t,x) \,=\, v^{(\mathsf{s},\mathsf{c})}(0,x) \,+\, \mathsf{c}\,\int_0^t\int_{\mathbb{R}^d} g_{\mathsf{s}(t-u)}(x-z)\,\dot{W}(u,z)\,\mathrm{d} u\,\mathrm{d} z\,,
$$

where  $g_t(x) := (2\pi t)^{-d/2} e^{-\frac{|x|^2}{2t}}$  $\frac{x_1}{2t}$  is the heat kernel on  $\mathbb{R}^d$ . It is known that  $x \mapsto v^{(\mathsf{s},\mathsf{c})}(t,x)$  is a (random) function only for  $d = 1$ , while for  $d \ge 2$  it is a genuine distribution.

Henceforth we focus on  $d = 2$ . The solution  $v^{(\mathsf{s},\mathsf{c})}(t,\cdot)$  with initial condition  $v^{(\mathsf{s},\mathsf{c})}(0,\cdot) \equiv 0$ , averaged on test functions  $\varphi \in C_c(\mathbb{R}^2)$ , is the centered Gaussian process with covariance

$$
\mathbb{E}\big[v^{(\mathsf{s},\mathsf{c})}(t,\varphi)\,v^{(\mathsf{s},\mathsf{c})}(t,\psi)\big] = \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\,K_t^{(\mathsf{s},\mathsf{c})}(x,y)\,\psi(y)\,\mathrm{d}x\,\mathrm{d}y\,,
$$

where we set

$$
K_t^{(\mathsf{s},\mathsf{c})}(x,y) \coloneqq \mathsf{c}^2 \int_0^t g_{2\mathsf{s}u}(x-y) \, \mathrm{d}u = \frac{\mathsf{c}^2}{2\mathsf{s}} \int_0^{2\mathsf{s}t} \frac{1}{2\pi u} \mathsf{e}^{-\frac{|x-y|^2}{2u}} \, \mathrm{d}u \,. \tag{1.15}
$$

Comparing with [\(1.13\)](#page-2-2), we can rephrase our main result [\(1.12\)](#page-2-3): for any  $\varphi \in C_c(\mathbb{R}^2)$ 

$$
\sqrt{\vartheta_N} \left\{ Z_{N,\beta_N}^{\omega}(\varphi) - \mathbb{E}[Z_{N,\beta_N}^{\omega}(\varphi)] \right\} \xrightarrow[N \to \infty]{d} v^{(\mathsf{s},\mathsf{c})} (1,\varphi) \quad \text{with } \begin{cases} \mathsf{s} = \frac{1}{2}, \\ \mathsf{c} = \sqrt{\pi}. \end{cases} (1.16)
$$

In other term, *the diffusively rescaled partition functions in the quasi-critical regime converge, after centering and rescaling, to the solution of the Edwards-Wilkinson equation*.

<span id="page-3-1"></span>**Remark 1.2.** *Also relation* [\(1.9\)](#page-1-1)*, in the sub-critical regime*  $\hat{\beta} \in (0,1)$ *, can be rephrased as a* convergence to the Edwards-Wilkinson solution  $v^{(s,\hat{c})}(1,\varphi)$  with  $\hat{c} = \sqrt{\pi} \hat{\beta}/\sqrt{1-\hat{\beta}^2}$ .

The reason why stochastic PDEs emerge naturally in the study of directed polymers is that, by the Markov property of simple random walk, the diffusively rescaled partition function  $u_N(t, x) := Z^{\omega}_{[Nt], \beta}([\sqrt{N}x])$  solves (up to a time reversal) a *discretized version* of the Stochastic Heat Equation with *multiplicative noise*:

<span id="page-3-0"></span>
$$
\partial_t u(t,x) = \frac{1}{2} \Delta_x u(t,x) + \beta \dot{W}(t,x) u(t,x), \qquad (1.17)
$$

with initial condition  $u(0, x) = 1$ . This gives a hint how the Edwards-Wilkinson equation [\(1.14\)](#page-2-4) may arise in the scaling limit of directed polymer partition functions: intuitively, the singular product  $\dot{W}(t, x) u(t, x)$  in [\(1.17\)](#page-3-0) for  $u(t, x) = u_N(t, x)$  converges to an independent white noise as  $N \to \infty$  (see [\[CC22,](#page-35-3) Theorem 3.4] in the sub-critical regime).

Edwards-Wilkinson fluctuations were recently proved also for a *non-linear* Stochastic Heat Equation, see [\[DG22,](#page-36-2) [T22+\]](#page-36-3), always in the sub-critical regime. It would be interesting to extend these results in the quasi-critical regime, generalizing our Theorem [1.1.](#page-2-1)

**Remark 1.3.** *The multiplicative Stochastic Heat Equation* [\(1.17\)](#page-3-0) *in the continuum is wellposed in one space dimension*  $d = 1$ , *e.g. by classical Ito-Walsh stochastic integration, but* it is ill-defined in higher dimensions  $d \geq 2$ . For this reason, directed polymer partition *functions can provide precious insight on the equation* [\(1.17\)](#page-3-0)*. In particular, for*  $d = 2$ , *their scaling limit in the critical regime was obtained in* [\[CSZ23\]](#page-35-1) *and called the critical* 2*d Stochastic Heat Flow, see* [\(1.10\)](#page-1-2)*, as a natural candidate for the ill-defined solution of* [\(1.17\)](#page-3-0)*.*

In the same spirit, the log-partition function  $h_N(t, x) := \log Z_{[Nt], \beta}^{\omega}(\sqrt{N}x)$  provides a discretized approximation for the *Kardar-Parisi-Zhang (KPZ) equation* [\[KPZ86\]](#page-36-4):

$$
\partial_t h(t,x) = \frac{1}{2} \Delta_x h(t,x) + \frac{1}{2} |\nabla h(t,x)|^2 + \beta \dot{W}(t,x) \,,
$$

with initial condition  $h(0, x) = 0$ . This equation too, in the continuum, is only fully understood in one space-dimension  $d = 1$ , via recent breakthrough techniques of regularity

structures [\[H14\]](#page-36-5) or paracontrolled distributions [\[GIP15,](#page-36-6) [GP17\]](#page-36-7); see also [\[GJ14,](#page-36-8) [K16\]](#page-36-9). Sim-ilar to [\(1.9\)](#page-1-1), Edwards-Wilkinson fluctuations have been proved for  $h<sub>N</sub>(t, x)$  in the entire sub-critical regime [\(1.8\)](#page-1-4) with  $\hat{\beta} \in (0, 1)$  [\[CSZ20,](#page-35-2) [G20,](#page-36-10) [CD20\]](#page-35-4): for  $\varphi \in C_c(\mathbb{R}^2)$ 

<span id="page-4-1"></span>
$$
\sqrt{\log N} \left\{ \log Z^{\omega}_{N,\beta_N}(\varphi) - \mathbb{E}[\log Z^{\omega}_{N,\beta_N}(\varphi)] \right\} \xrightarrow[N \to \infty]{d} v^{(\mathsf{s},\hat{\mathsf{c}})}(1,\varphi), \tag{1.18}
$$

with  $s$ ,  $\hat{c}$  as in Remark [1.2.](#page-3-1) This was recently extended in [\[NN23\]](#page-36-11), which focuses on a mollification (rather than discretization) of the Stochastic Heat Equation [\(1.17\)](#page-3-0): phrased in our setting, the results of [\[NN23\]](#page-36-11) prove Gaussian fluctuations in the sub-critical regime for general transformations  $F(Z_{N,\beta_N}^{\omega})$ , besides  $F(z) = \log z$ , with general initial conditions.

It would be very interesting to extend [\(1.18\)](#page-4-1) to the quasi-critical regime [\(1.11\)](#page-1-5), namely to prove an analogue of our Theorem [1.18](#page-4-1) for  $\log Z_{N,\beta_N}^{\omega}(\varphi)$ , which we expect to hold. A natural strategy would be to generalize the linearization procedure established in [\[CSZ20\]](#page-35-2) to handle the logarithm. This requires estimating *negative moments* of the partition function, which is a challenge in the quasi-critical regime (since  $Z_{N,\beta_N}^{\omega}(z) \to 0$  for fixed  $z \in \mathbb{Z}^2$ ).

Local averages on *sub-diffusive scales* have also been investigated for the mollified KPZ solution in the sub-critical regime, see [\[C23,](#page-35-5) [T23+\]](#page-36-12). Similar results can be expected for the mollified solution of the Stochastic Heat Equation [\(1.17\)](#page-3-0), or for the directed polymer partition function, which should be obtainable in the sub-critical regime as in [\[CSZ17b\]](#page-35-0). It would be natural to study such local averages also in the quasi-critical regime.

We finally mention that Edwards-Wilkinson fluctuations like [\(1.9\)](#page-1-1) and [\(1.18\)](#page-4-1) have also been obtained in higher dimensions  $d \geq 3$ , in the so-called  $L^2$ -weak disorder phase where the partition function has bounded second moment [\[CN21,](#page-35-6) [LZ22,](#page-36-13) [CNN22,](#page-35-7) [CCM21+\]](#page-35-8), see also the previous works [\[MU18,](#page-36-14) [GRZ18,](#page-36-15) [CCM20,](#page-35-9) [DGRZ20\]](#page-36-16). Unlike the two-dimensional setting, for  $d \geq 3$  the partition function admits a non-zero limit also *beyond the*  $L^2$ -weak *disorder phase*: see [\[J22,](#page-36-17) [J22+\]](#page-36-18) for recent results in this challenging regime. It would be natural to investigate whether our approach can also be applied in higher dimensions  $d \geq 3$ , in order to prove Gaussian fluctuations *slightly beyond* the *L* 2 -weak disorder phase.

**1.4. Organization of the paper.** The paper is structured as follows.

- ' In Section [2](#page-4-0) we present the structure of the proof of Theorem [1.1](#page-2-1) based on two key steps, formulated as Propositions [2.1](#page-5-0) and [2.2.](#page-6-1)
- In Section [3](#page-6-0) we prove Proposition [2.1.](#page-5-0)
- ' In Section [4](#page-11-0) we derive upper bounds on the moments of the partition functions.
- In Section [5](#page-20-0) we prove Proposition [2.2.](#page-6-1)
- ' Finally, some technical points are deferred to Appendix [A.](#page-24-0)

<span id="page-4-0"></span>**Acknowledgements.** We gratefully acknowledge the support of INdAM/GNAMPA.

# **2. Proof of Theorem [1.1](#page-2-1)**

Let us call  $X_N$  the LHS of [\(1.12\)](#page-2-3): recalling [\(1.4\)](#page-0-1) and [\(1.3\)](#page-0-3), we can write

$$
X_N := \sqrt{\vartheta_N} \left\{ Z_{N,\beta_N}^{\omega}(\varphi) - \mathbb{E}[Z_{N,\beta_N}^{\omega}(\varphi)] \right\}
$$
  
= 
$$
\frac{\sqrt{\vartheta_N}}{N} \sum_{z \in \mathbb{Z}^2} \left\{ Z_{N,\beta_N}^{\omega}(z) - 1 \right\} \varphi_N\left(\frac{z}{\sqrt{N}}\right),
$$
 (2.1)

<span id="page-4-2"></span>with  $\varphi_N$  as in [\(1.5\)](#page-0-4). In this section, we prove Theorem [1.1](#page-2-1) via the following two main steps:

- (1) we first approximate  $X_N$  in  $L^2$  by a sum  $\sum_{i=1}^M X_{N,M}^{(i)}$  of *independent* random variables, for  $M = M_N \rightarrow \infty$  slowly enough;
- (2) we then show that the random variables  $(X_{N,M}^{(i)})_{1\leq i\leq M}$  for  $M = M_N$  satisfy the assumptions of the classical *Central Limit Theorem* for triangular arrays.

**2.1. First step.** In order to define the random variables  $X_{N,M}^{(i)}$ , for  $M \in \mathbb{N}$  and  $1 \leq i \leq M$ , we introduce a variation of [\(1.1\)](#page-0-0), for  $-\infty < A < B < \infty$ :

<span id="page-5-5"></span>
$$
Z_{(A,B],\beta}^{\omega}(z) := \mathbb{E}\big[e^{\sum_{n\in(A,B]\cap\mathbb{N}}\{\beta\omega(n,S_n)-\lambda(\beta)\}}\big|\,S_0=z\big].\tag{2.2}
$$

We then define  $X_{N,M}^{(i)}$  replacing  $Z_{N,\beta}^{\omega}$  by  $Z_{(i)}^{\omega}$  $\frac{d^2}{dx^2}N, \frac{i}{M}N$ , *β* in the definition [\(2.1\)](#page-4-2) of  $X_N$ :

$$
X_{N,M}^{(i)} = \frac{\sqrt{\vartheta_N}}{N} \sum_{z \in \mathbb{Z}^2} \left\{ Z_{\left(\frac{i-1}{M}N, \frac{i}{M}N\right], \beta_N}^{ \omega}(z) - 1 \right\} \varphi_N\left(\frac{z}{\sqrt{N}}\right). \tag{2.3}
$$

<span id="page-5-6"></span>Note that  $Z^{\omega}_{(A,B],\beta}(z)$  only depends on  $\omega(n,x)$  for  $A < n \leq B$ , moreover  $\mathbb{E}[Z^{\omega}_{(A,B],\beta}(z)] = 1$ . As a consequence,  $X_{N,M}^{(i)}$  for  $1 \leq i \leq M$  are *independent* and *centered* random variables.

The core of this first step is the following approximation result, proved in Section [3.](#page-6-0)

<span id="page-5-0"></span>**Proposition 2.1** ( $L^2$  approximation). For  $(\beta_N)_{N \in \mathbb{N}}$  in the quasi-critical regime, see [\(1.7\)](#page-1-6) *and* [\(1.11\)](#page-1-5)*, the following relations hold for any*  $\varphi \in C_c(\mathbb{R}^2)$ *, with*  $\sigma_{\varphi}^2$  *as in* [\(1.13\)](#page-2-2)*:* 

<span id="page-5-1"></span>
$$
\lim_{N \to \infty} \mathbb{E}[X_N^2] = \sigma_\varphi^2, \qquad \forall M \in \mathbb{N}: \quad \lim_{N \to \infty} \left\| X_N - \sum_{i=1}^M X_{N,M}^{(i)} \right\|_{L^2} = 0. \tag{2.4}
$$

From the second relation in [\(2.4\)](#page-5-1) it follows that, for any  $(M_N)_{N\in\mathbb{N}}$  with  $M_N\to\infty$  slowly enough as  $N \to \infty$  (see [\[CC22,](#page-35-3) Remark 4.2]),

<span id="page-5-2"></span>
$$
\lim_{N \to \infty} \left\| X_N - \sum_{i=1}^{M_N} X_{N,M_N}^{(i)} \right\|_{L^2} = 0, \tag{2.5}
$$

that is we approximate  $X_N$  in  $L^2$  by a sum of independent and centered random variables. We then obtain, by the first relation in [\(2.4\)](#page-5-1),

<span id="page-5-3"></span>
$$
\lim_{N \to \infty} \mathbb{E}\left[\left(\sum_{i=1}^{M_N} X_{N,M_N}^{(i)}\right)^2\right] = \lim_{N \to \infty} \sum_{i=1}^{M_N} \mathbb{E}\left[\left(X_{N,M_N}^{(i)}\right)^2\right] = \sigma_\varphi^2.
$$
\n(2.6)

**2.2. Second step.** Recalling [\(2.1\)](#page-4-2), we can rephrase our goal [\(1.12\)](#page-2-3) as  $X_N \stackrel{d}{\rightarrow} \mathcal{N}(0, \sigma_{\varphi}^2)$ . In view of [\(2.5\)](#page-5-2), this follows if we prove the convergence in distribution

$$
\sum_{i=1}^{M_N} X_{N,M_N}^{(i)} \xrightarrow[N \to \infty]{d} \mathcal{N}\left(0, \sigma_\varphi^2\right). \tag{2.7}
$$

Since  $(X_{N,M_N}^{(i)})_{1\leq i\leq M_N}$  are independent and centered, we apply the classical Central Limit Theorem for triangular arrays, see e.g. [\[Bil95,](#page-35-10) Theorem 27.3]: since we have convergence of the variance by [\(2.6\)](#page-5-3), it is enough to check the Lyapunov condition

<span id="page-5-4"></span>for some 
$$
p > 2
$$
: 
$$
\lim_{N \to \infty} \sum_{i=1}^{M_N} \mathbb{E} \Big[ |X_{N,M_N}^{(i)}|^p \Big] = 0.
$$
 (2.8)

<span id="page-6-1"></span>This follows from the next result, proved in Section [4,](#page-11-0) where we focus on the case  $p = 4$ .

**Proposition 2.2 (Fourth moment bound).** For  $(\beta_N)_{N \in \mathbb{N}}$  in the quasi-critical regime, *see* [\(1.7\)](#page-1-6) *and* [\(1.11\)](#page-1-5)*, and for any*  $\varphi \in C_c(\mathbb{R}^2)$ *, there is a constant*  $C < \infty$  *such that* 

<span id="page-6-2"></span>
$$
\forall M \in \mathbb{N}, \quad \forall 1 \leq i \leq M: \qquad \limsup_{N \to \infty} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^4\Big] \leq \frac{C}{M^2}.
$$
 (2.9)

Since the estimate [\(2.9\)](#page-6-2) holds for any fixed *M*, it follows that we can let  $M_N \to \infty$  slowly enough as  $N \to \infty$  so that

$$
\mathbb{E}\Big[\big(X_{N,M_N}^{(i)}\big)^4\Big] \leqslant \frac{2C}{M_N^2} \quad \forall i=1,\ldots,M_N\,.
$$

This shows that [\(2.8\)](#page-5-4) holds with  $p = 4$  (the sum therein is  $\leq 2C/M_N \to 0$  as  $N \to \infty$ ).

<span id="page-6-0"></span>The proof of Theorem [1.1](#page-2-1) is then completed once we prove Propositions [2.1](#page-5-0) and [2.2.](#page-6-1) The next sections are devoted to these tasks.

## **3. Second moment bounds: proof of Proposition [2.1](#page-5-0)**

In this section we prove Proposition [2.1](#page-5-0) exploiting a polynomial chaos expansion of the partition function. We fix  $(\beta_N)_{N \in \mathbb{N}}$  in the quasi-critical regime, see [\(1.7\)](#page-1-6) and [\(1.11\)](#page-1-5), and  $\varphi \in C_c(\mathbb{R}^2)$ . We denote by  $C, C', \ldots$  generic constants that may vary from place to place.

**3.1. Polynomial chaos expansion.** The partition function admits a key polynomial chaos expansion [\[CSZ17a\]](#page-35-11). Let us define, for  $\beta > 0$ ,

<span id="page-6-4"></span>
$$
\xi_{\beta}(n,x) := e^{\beta \omega(n,x) - \lambda(\beta)} - 1, \quad \text{for } n \in \mathbb{N}, \ x \in \mathbb{Z}^2. \tag{3.1}
$$

Recalling [\(1.7\)](#page-1-6), we note that  $(\xi_{\beta}(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^2}$  are independent random variables with

<span id="page-6-7"></span>
$$
\mathbb{E}[\xi_{\beta}] = 0, \qquad \mathbb{E}[\xi_{\beta}^{2}] = \sigma_{\beta}^{2}, \qquad \mathbb{E}[[\xi_{\beta}]^{k}] \leq C_{k} \sigma_{\beta}^{k} \quad \forall k \geq 3, \tag{3.2}
$$

for some  $C_k < \infty$  (for the bound on  $\mathbb{E}[[\xi_\beta]^k]$  see, e.g., [\[CSZ17a,](#page-35-11) eq. (6.7)]).

We denote by  $q_n(x)$  the random walk transition kernel:

<span id="page-6-5"></span>
$$
q_n(x) := P(S_n = x \, | \, S_0 = 0). \tag{3.3}
$$

Then, writing  $e^{\sum_n {\beta \omega(n,x)-\lambda(\beta)}}$  =  $\prod_n (1 + {\xi_\beta(n,x)})$  and expanding the product, we can write  $Z^{\omega}_{(A,B],\beta}(z)$  in [\(2.2\)](#page-5-5) as the following polynomial chaos expansion:

$$
Z_{(A,B],\beta}^{\omega}(z) = 1 + \sum_{k=1}^{\infty} \sum_{\substack{A < n_1 < \dots < n_k \le B \\ x_1, \dots, x_k \in \mathbb{Z}^2}} q_{n_1}(x_1 - z) \xi_{\beta}(n_1, x_1) \times \sum_{j=2}^k q_{n_j - n_{j-1}}(x_j - x_{j-1}) \xi_{\beta}(n_j, x_j), \tag{3.4}
$$

<span id="page-6-3"></span>where we agree that the time variables  $n_1 < \ldots < n_k$  are summed in the set  $(A, B] \cap \mathbb{Z}$  (in particular, the seemingly infinite sum over *k* can be stopped at  $B - A$ .

Plugging  $(3.4)$  into  $(2.1)$ , we obtain a corresponding polynomial chaos expansion for  $X_N$ , recall [\(2.1\)](#page-4-2) and [\(1.5\)](#page-0-4): if we define the averaged random walk transition kernel

<span id="page-6-6"></span>
$$
q_n^f(x) := \sum_{z \in \mathbb{Z}^2} q_n(x - z) f(z), \quad \text{for } f: \mathbb{Z}^2 \to \mathbb{R},
$$
 (3.5)

we obtain

<span id="page-7-1"></span>
$$
X_N = \frac{\sqrt{\vartheta_N}}{N} \sum_{k=1}^{\infty} \sum_{\substack{0 < n_1 < \dots < n_k \le N \\ x_1, \dots, x_k \in \mathbb{Z}^2}} q_{n_1}^{\varphi_N}(x_1) \, \xi_{\beta_N}(n_1, x_1) \prod_{j=2}^k q_{n_j - n_{j-1}}(x_j - x_{j-1}) \, \xi_{\beta_N}(n_j, x_j). \tag{3.6}
$$

The analogous polynomial chaos expansion for the random variables  $X_{N,M}^{(i)}$ , see [\(2.3\)](#page-5-6), is obtained from [\(3.6\)](#page-7-1) restricting the sum to  $\frac{i-1}{M}N < n_1 < \ldots < n_k \leq \frac{i}{M}N$ :

<span id="page-7-2"></span>
$$
X_{N,M}^{(i)} = \frac{\sqrt{\vartheta_N}}{N} \sum_{k=1}^{\infty} \sum_{\substack{i=1 \ i \text{ odd} \\ x_1, \dots, x_k \in \mathbb{Z}^2}} q_{n_1}^{\varphi_N}(x_1) \xi_{\beta_N}(n_1, x_1) \times \sum_{x_1, \dots, x_k \in \mathbb{Z}^2} q_{n_1}^{\varphi_N}(x_1) \xi_{\beta_N}(n_1, x_1) \times \prod_{j=2}^k q_{n_j - n_{j-1}}(x_j - x_{j-1}) \xi_{\beta_N}(n_j, x_j).
$$
\n(3.7)

<span id="page-7-4"></span>**Remark 3.1.** *Since the random variables*  $(\xi_{\beta}(n,x))_{n\in\mathbb{N},x\in\mathbb{Z}^2}$  *are independent and centered,* see [\(3.1\)](#page-6-4), the terms in the polynomial chaos [\(3.4\)](#page-6-3), [\(3.6\)](#page-7-1), [\(3.7\)](#page-7-2) are orthogonal in  $L^2$ .

We finally recall the local limit theorem for the simple random walk on  $\mathbb{Z}^2$ , see [\[LL10,](#page-36-19) Theorem 2.1.3]: as  $n \to \infty$ , uniformly for  $x \in \mathbb{Z}^2$  we have<sup>[†](#page-7-3)</sup>

<span id="page-7-0"></span>
$$
q_n(x) = \frac{1}{n/2} \left( g\left(\frac{x}{\sqrt{n/2}}\right) + o(1) \right) 2 \mathbb{1}_{(n,x) \in \mathbb{Z}_{\text{even}}^3}, \quad \text{where} \quad g(y) := \frac{e^{-\frac{1}{2}|y|^2}}{2\pi}, \quad (3.8)
$$

and we set  $\mathbb{Z}_{even}^3 := \{ y = (y_1, y_2, y_3) \in \mathbb{Z}^3 : y_1 + y_2 + y_3 \in 2\mathbb{Z} \}.$ 

**3.2. Proof of Proposition [2.1.](#page-5-0)** Note that  $\sum_{i=1}^{M} X_{N,M}^{(i)}$  is a polynomial chaos where all time variables  $n_1 < \ldots < n_k$  belong to *one of the intervals*  $\left(\frac{i-1}{M}N, \frac{i}{M}N\right]$ , see [\(3.7\)](#page-7-2). It follows that  $X_N$  is a *larger polynomial chaos* than  $\sum_{i=1}^M X_{N,M}^{(i)}$ , i.e. it contains more terms, hence the difference  $X_N - \sum_{i=1}^M X_{N,M}^{(i)}$  is orthogonal in  $L^2$  to  $\sum_{i=1}^M X_{N,M}^{(i)}$  (see Remark [3.1\)](#page-7-4):

$$
\left\| X_N - \sum_{i=1}^M X_{N,M}^{(i)} \right\|_{L^2}^2 = \left\| X_N \right\|_{L^2}^2 - \left\| \sum_{i=1}^M X_{N,M}^{(i)} \right\|_{L^2}^2 = \left\| X_N \right\|_{L^2}^2 - \sum_{i=1}^M \left\| X_{N,M}^{(i)} \right\|_{L^2}^2.
$$

As a consequence, to prove our goal [\(2.4\)](#page-5-1) it is enough to show that

<span id="page-7-5"></span>
$$
\lim_{N \to \infty} \mathbb{E}[X_N^2] = \sigma_\varphi^2, \qquad \forall M \in \mathbb{N}: \quad \lim_{N \to \infty} \sum_{i=1}^M \mathbb{E}\Big[ \big(X_{N,M}^{(i)}\big)^2 \Big] = \sigma_\varphi^2, \tag{3.9}
$$

where we recall that  $\sigma_{\varphi}^2$  is defined in [\(1.13\)](#page-2-2). The first relation in [\(3.9\)](#page-7-5) follows from the second one, because  $X_N = X_{N,1}^{(1)}$ . Then the proof is completed by the next result.  $\Box$ 

<span id="page-7-3"></span><sup>&</sup>lt;sup>†</sup>The scaling factor in [\(3.8\)](#page-7-0) is  $n/2$  because the covariance matrix of the simple random walk on  $\mathbb{Z}^2$  is  $\frac{1}{2}I$ , while the factor  $2\mathbb{1}_{(m,z)\in\mathbb{Z}_{\text{even}}^3}$  is due to periodicity.

**Lemma 3.2 (Quasi-critical variance).** *Fix*  $(\beta_N)_{N \in \mathbb{N}}$  *in the quasi-critical regime, see*  $(1.7)$  and  $(1.11)$ *, and*  $\varphi \in C_c(\mathbb{R}^2)$ *. For any*  $M \in \mathbb{N}$ *, the following holds for all*  $i = 1, ..., M$ *:* 

<span id="page-8-0"></span>
$$
\lim_{N \to \infty} \mathbb{E}\big[\big(X_{N,M}^{(i)}\big)^2\big] = \sigma_{\varphi,\,(\frac{i-1}{M},\frac{i}{M}]}^2 := \int_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x) \,\varphi(x') \left(\int_{\frac{i-1}{M}}^{\frac{i}{M}} \frac{1}{2u} \,\mathrm{e}^{-\frac{|x-x'|^2}{2u}} \,\mathrm{d}u\right) \mathrm{d}x \,\mathrm{d}x' \,. \tag{3.10}
$$

**Proof.** Let us fix  $M \in \mathbb{N}$  and  $1 \leq i \leq M$ . We split the proof of [\(3.10\)](#page-8-0) in the two bounds

<span id="page-8-1"></span>
$$
\limsup_{N \to \infty} \mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] \leq \sigma_{\varphi,\left(\frac{i-1}{M},\frac{i}{M}\right)}^2\tag{3.11}
$$

and

<span id="page-8-5"></span>
$$
\liminf_{N \to \infty} \mathbb{E}\Big[ \big(X_{N,M}^{(i)}\big)^2 \Big] \geq \sigma_{\varphi, \, \left(\frac{i-1}{M}, \frac{i}{M}\right]}^2. \tag{3.12}
$$

We first obtain an exact expression for the second moment of  $X_{N,M}^{(i)}$  by [\(3.7\)](#page-7-2): since the random variables  $\xi_{\beta}(n, x)$  are independent with zero mean and variance  $\sigma_{\beta}^2$ , we have

$$
\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] = \frac{\vartheta_N}{N^2} \sum_{k=1}^{\infty} (\sigma_{\beta_N}^2)^k \sum_{\substack{i=1 \ \text{if } N < n_1 < \dots < n_k \leqslant \frac{i}{M} \ N \\ x_1, \dots, x_k \in \mathbb{Z}^2}} q_{n_1}^{\varphi_N}(x_1)^2 \prod_{j=2}^k q_{n_j - n_{j-1}} (x_j - x_{j-1})^2.
$$

We can sum the space variables  $x_k, x_{k-1}, \ldots, x_2$  because  $\sum_{x \in \mathbb{Z}^2} q_n(x)^2 = q_{2n}(0)$ , see [\(3.3\)](#page-6-5), while to handle the sum over  $x_1$  we note that, recalling  $(3.5)$ ,

<span id="page-8-4"></span>
$$
\sum_{x \in \mathbb{Z}^2} q_n^f(x)^2 = q_{2n}^{f,f} \qquad \text{where we set} \quad q_m^{f,f} := \sum_{z,z' \in \mathbb{Z}^2} q_m(z-z') f(z) f(z'). \tag{3.13}
$$

We then obtain

<span id="page-8-6"></span>
$$
\mathbb{E}\Big[\big(X_{N,M}^{(i)}\big)^2\Big] = \vartheta_N \sum_{k=1}^{\infty} (\sigma_{\beta_N}^2)^k \sum_{\frac{i-1}{M}N < n_1 < \ldots < n_k \leq \frac{i}{M}N} \frac{q_{2n_1}^{\varphi_N, \varphi_N}}{N^2} \prod_{j=2}^k q_{2(n_j - n_{j-1})}(0) \,. \tag{3.14}
$$

We then prove the upper bound [\(3.11\)](#page-8-1). We rename  $n_1 = n$  and enlarge the sum over the other time variables  $n_2, \ldots, n_k$ , by letting each increment  $m_j := n_j - n_{j-1}$  for  $j = 2, \ldots, k$ vary in the whole interval  $(0, N]$ : since  $\sum_{m=1}^{N} q_{2m}(0) = R_N$ , see [\(1.6\)](#page-1-3), we obtain

$$
\mathbb{E}\left[\left(X_{N,M}^{(i)}\right)^{2}\right] \leq \vartheta_{N} \sum_{\frac{i-1}{M}N < n \leq \frac{i}{M}N} \frac{q_{2n}^{\varphi_{N}, \varphi_{N}}}{N^{2}} \sum_{k=1}^{\infty} (\sigma_{\beta_{N}}^{2})^{k} (R_{N})^{k-1}
$$
\n
$$
= \vartheta_{N} \left\{\sum_{\frac{i-1}{M}N < n \leq \frac{i}{M}N} \frac{q_{2n}^{\varphi_{N}, \varphi_{N}}}{N^{2}}\right\} \frac{\sigma_{\beta_{N}}^{2}}{1 - \sigma_{\beta_{N}}^{2} R_{N}},\tag{3.15}
$$

<span id="page-8-2"></span>where we summed the geometric series since  $\sigma_{\beta_N}^2 R_N = 1 - \frac{\vartheta_N}{\log N} < 1$  for large *N*, by [\(1.11\)](#page-1-5). We will prove the following Riemann sum approximation, for any given  $0 \le a < b \le 1$ :

<span id="page-8-3"></span>
$$
\lim_{N \to \infty} \sum_{aN < n \le bN} \frac{q_{2n}^{\varphi_N, \varphi_N}}{N^2} = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x) \varphi(x') \left( \int_a^b \frac{1}{u} g\left(\frac{x - x'}{\sqrt{u}}\right) \mathrm{d}u \right) \mathrm{d}x \, \mathrm{d}x',\tag{3.16}
$$

where  $g(y) = \frac{1}{2\pi} e^{-\frac{1}{2}|y|^2}$  is the standard Gaussian density on  $\mathbb{R}^2$ , see [\(3.8\)](#page-7-0). Plugging this into [\(3.15\)](#page-8-2), since  $1 - \sigma_{\beta_N}^2 R_N = \frac{\vartheta_N}{\log N}$  and  $\sigma_{\beta_N}^2 \sim \frac{1}{R_N} \sim \frac{\pi}{\log N}$  as  $N \to \infty$  by [\(1.11\)](#page-1-5) and [\(1.6\)](#page-1-3), we obtain precisely the upper bound [\(3.11\)](#page-8-1) (note that  $\pi \frac{1}{u}$  $\frac{1}{u} g(\frac{x-x'}{\sqrt{u}}) = \frac{1}{2u} \exp(-\frac{|x-x'|^2}{2u})$  $\frac{-x}{2u}$ ).

Let us now prove [\(3.16\)](#page-8-3). This is based on the local limit theorem [\(3.8\)](#page-7-0) as  $n \to \infty$ , hence the case  $a = 0$  could be delicate, as the sum in [\(3.16\)](#page-8-3) starts from  $n = 1$  and, therefore, *n* needs not be large. For this reason, we first show that small values of *n* are negligible for [\(3.16\)](#page-8-3). Since  $\varphi$  is compactly supported, when we plug  $f = \varphi_N$  into  $q_{2n}^{f,f}$  $_{2n}^{J,J}$ , see [\(3.13\)](#page-8-4), we can restrict the sums to  $|z'| \le C\sqrt{N}$ , which yields the following *uniform bound*:

<span id="page-9-1"></span>
$$
\forall m \in \mathbb{N}: \qquad |q_m^{\varphi_N, \varphi_N}| \leqslant \|\varphi\|_{\infty}^2 \sum_{|z'| \leqslant C\sqrt{N}} \sum_{z \in \mathbb{Z}^2} q_m(z - z') \leqslant C' \|\varphi\|_{\infty}^2 N. \tag{3.17}
$$

In particular, the contribution of  $n \leq \varepsilon N$  to the LHS of [\(3.16\)](#page-8-3) is  $O(\varepsilon)$ . As a consequence, it is enough to prove  $(3.16)$  when  $a > 0$ , which we assume henceforth.

Recalling  $(3.13)$  and applying  $(3.8)$ , we can write the LHS of  $(3.16)$  as follows:

$$
\sum_{aN < n \leq bN} \frac{q_{2n}^{\varphi_N, \varphi_N}}{N^2} = \frac{1}{N^2} \sum_{aN < n \leq bN} \sum_{\substack{z, z' \in \mathbb{Z}^2:\\(n, z - z') \in \mathbb{Z}^3_{\text{even}}}} \frac{2}{n} \left( g\left(\frac{z - z'}{\sqrt{n}}\right) + o(1) \right) \varphi\left(\frac{z}{\sqrt{N}}\right) \varphi\left(\frac{z'}{\sqrt{N}}\right),
$$

where  $o(1) \rightarrow 0$  as  $N \rightarrow \infty$  (because  $n > aN \rightarrow \infty$  and we assume  $a > 0$ ). The additive term *o*(1) gives a vanishing contribution as  $N \to \infty$ , because we can bound  $\frac{2}{n} \leq \frac{2}{aN}$  and  $|\varphi(\cdot)| \leqslant |\varphi|_{\infty}$ , and the sums contain  $O(N^3)$  terms (since  $|z|, |z'| \leqslant C\sqrt{N}$ ). Introducing the rescaled variables  $u := \frac{n}{N}$  $\frac{n}{N}$  and  $x := \frac{z}{\sqrt{N}}, x' := \frac{z'}{\sqrt{N}}$ , we can then rewrite the RHS as

$$
\frac{1}{N^3} \sum_{u \in (a,b] \cap \frac{\mathbb{N}}{N}} \sum_{\substack{x,x' \in \frac{\mathbb{Z}^2}{\sqrt{N}}:\\(Nu,\sqrt{N}(x-x')) \in \mathbb{Z}^3_{\text{even}}}} \frac{2}{u} \left( g\left(\frac{x-x'}{\sqrt{u}}\right) \right) \varphi(x) \varphi(x') + o(1),
$$

which is a Riemann sum for the integral in the RHS of [\(3.16\)](#page-8-3). Note that the restriction  $(Nu, \sqrt{N(x-x')}) \in \mathbb{Z}_{\text{even}}^3$  effectively *halves* the range of the sum: indeed, for any given *u* and *x*, the sum over  $x' = \frac{z'}{\sqrt{N}} \in \frac{\mathbb{Z}^2}{\sqrt{N}}$  $\frac{\mathbb{Z}^2}{\sqrt{N}}$  is restricted to points  $z' \in \mathbb{Z}^2$  with a fixed parity (even or odd, depending on  $u, x$ ). This restriction is compensated by the multiplicative factor 2, which disappears as we let  $N \to \infty$ . This completes the proof of [\(3.16\)](#page-8-3).

We finally prove the lower bound [\(3.12\)](#page-8-5). We fix  $\varepsilon > 0$  small enough and we bound the RHS of [\(3.14\)](#page-8-6) from below as follows:

- we rename  $n = n_1$  and we restrict its sum to the interval  $\left(\frac{i-1}{M}N, (1-\varepsilon)\frac{i}{M}N\right]$ ;
- for  $k \ge 2$ , we introduce the "displacements"  $m_j := n_j n_1$  from  $n_1$ , for  $j = 2, ..., k$ , and we restrict the sum over  $n_2, \ldots, n_k$  to the set  $0 < m_2 < \ldots < m_k \leqslant \varepsilon \frac{i}{M}N$ .

We thus obtain by [\(3.14\)](#page-8-6)

*ϕ<sup>N</sup> ,ϕ<sup>N</sup>*

<span id="page-9-0"></span>
$$
\mathbb{E}\left[\left(X_{N,M}^{(i)}\right)^{2}\right] \geq \vartheta_{N} \sum_{\frac{i-1}{M}N < n \leq (1-\varepsilon)\frac{i}{M}N} \frac{q_{2n}^{\varphi_{N},\varphi_{N}}}{N^{2}} \times \left(\sigma_{\beta_{N}}^{2} + \sum_{k=2}^{\infty} (\sigma_{\beta_{N}}^{2})^{k} \sum_{0 < m_{2} < \dots < m_{k} \leq \varepsilon\frac{i}{M}N} q_{2m_{2}}(0) \prod_{j=3}^{k} q_{2(m_{j}-m_{j-1})}(0)\right).
$$
\n(3.18)

We now give a probabilistic interpretation to the sum over  $m_2, \ldots, m_k$ : following [\[CSZ19a\]](#page-35-12) and recalling [\(1.6\)](#page-1-3), given  $N \in \mathbb{N}$  we define i.i.d. random variables  $(T_i^{(N)})_{i \in \mathbb{N}}$  with distribution

<span id="page-10-1"></span>
$$
P(T_i^{(N)} = n) = \frac{q_{2n}(0)}{R_N} 1_{\{1,\dots,N\}}(n),
$$
\n(3.19)

so that the second line of [\(3.18\)](#page-9-0) can be written, renaming  $\ell = k - 1$ , as

$$
\sigma_{\beta_N}^2 \left( 1 + \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^{\ell} P \left( T_1^{(N)} + \dots + T_{\ell}^{(N)} \leq \varepsilon \frac{i}{M} N \right) \right)
$$
\n
$$
= \sigma_{\beta_N}^2 \left( \frac{1}{1 - \sigma_{\beta_N}^2 R_N} - \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^{\ell} P \left( T_1^{(N)} + \dots + T_{\ell}^{(N)} > \varepsilon \frac{i}{M} N \right) \right).
$$
\n(3.20)

Plugging this into [\(3.18\)](#page-9-0) and recalling [\(3.17\)](#page-9-1), we obtain

<span id="page-10-0"></span>
$$
\mathbb{E}\left[\left(X_{N,M}^{(i)}\right)^{2}\right] \geq \vartheta_{N} \left\{\sum_{\frac{i-1}{M}N < n \leqslant (1-\varepsilon)\frac{i}{M}N} \frac{q_{2n}^{\varphi_{N},\varphi_{N}}}{N^{2}}\right\} \frac{\sigma_{\beta_{N}}^{2}}{1-\sigma_{\beta_{N}}^{2}R_{N}} - \left(C'\|\varphi\|_{\infty}^{2}\right)\vartheta_{N}\sigma_{\beta_{N}}^{2} \sum_{\ell=1}^{\infty} (\sigma_{\beta_{N}}^{2}R_{N})^{\ell} P\left(T_{1}^{(N)} + \ldots + T_{\ell}^{(N)} > \frac{\varepsilon}{M}N\right). \tag{3.21}
$$

The first term in the RHS is similar to [\(3.15\)](#page-8-2), just with  $(1-\varepsilon)\frac{i}{M}$  instead of  $\frac{i}{M}$ , therefore *we already proved that it converges to*  $\sigma_{\alpha}^2$  $\frac{2}{\varphi}$ ,  $(\frac{i-1}{M}, (1-\varepsilon)\frac{i}{M}]$  *as*  $N \to \infty$ , see [\(3.16\)](#page-8-3) and the following lines (recall also [\(3.10\)](#page-8-0)). Letting  $\varepsilon \downarrow 0$  after  $N \to \infty$  we recover  $\sigma_{\varphi}^2$  $\varphi$ ,  $\left(\frac{i-1}{M}, \frac{i}{M}\right]$ , hence to prove  $(3.12)$  we just need to show that the second term in the RHS of  $(3.21)$  is negligible:

<span id="page-10-2"></span>
$$
\lim_{N \to \infty} \vartheta_N \sigma_{\beta_N}^2 \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^{\ell} \mathcal{P} \Big( T_1^{(N)} + \ldots + T_{\ell}^{(N)} > \frac{\varepsilon}{M} N \Big) = 0 \,. \tag{3.22}
$$

*.*

Recall that the random variables  $(T_i^{(N)})_{i\in\mathbb{N}}$  are i.i.d. with distribution [\(3.19\)](#page-10-1). Since  $q_{2n}(0) \leq \frac{C}{n}$  by the local limit theorem [\(3.8\)](#page-7-0), we have  $E[T_i^{(N)}] = \frac{1}{R_N} \sum_{n=1}^{N} n q_{2n}(0) \leq C \frac{N}{R_N}$ *R<sup>N</sup>* and, by Markov's inequality, we can bound

$$
P(T_1^{(N)} + \ldots + T_\ell^{(N)} > \frac{\varepsilon}{M}N) \le \frac{E[T_1^{(N)} + \ldots + T_\ell^{(N)}]}{\frac{\varepsilon}{M}N} \le \frac{C\ell}{\frac{\varepsilon}{M}R_N}
$$

Since  $\sum_{\ell=1}^{\infty} \ell x^{\ell} = \frac{x}{(1-x)^{\ell}}$  $\frac{x}{(1-x)^2}$ , we obtain

$$
\vartheta_N \sigma_{\beta_N}^2 \sum_{\ell=1}^{\infty} (\sigma_{\beta_N}^2 R_N)^{\ell} P\left(T_1^{(N)} + \dots + T_{\ell}^{(N)} > \frac{\varepsilon}{M} N\right) \leq \vartheta_N \sigma_{\beta_N}^2 \frac{C}{\frac{\varepsilon}{M} R_N} \frac{\sigma_{\beta_N}^2 R_N}{\left(1 - \sigma_{\beta_N}^2 R_N\right)^2}
$$

$$
= \frac{C M}{\varepsilon} \frac{\vartheta_N (\sigma_{\beta_N}^2)^2}{\left(1 - \sigma_{\beta_N}^2 R_N\right)^2}.
$$

Note that  $1 - \sigma_{\beta_N}^2 R_N = \frac{\vartheta_N}{\log N}$  and  $\sigma_{\beta_N}^2 \sim \frac{1}{R_1}$  $\frac{1}{R_N} \sim \frac{\pi}{\log n}$  $\frac{\pi}{\log N}$  by [\(1.11\)](#page-1-5) and [\(1.6\)](#page-1-3), hence the last term is asymptotically equivalent to *CM ε π* 2  $\frac{\pi}{\vartheta_N} \to 0$  as  $N \to \infty$ , since  $\vartheta_N \to \infty$ , see [\(1.11\)](#page-1-5). This shows that [\(3.22\)](#page-10-2) holds and completes the proof of Proposition [2.1.](#page-5-0)  $\Box$ 

## **4. General moment bounds**

<span id="page-11-0"></span>In this section we estimate the *moments of the partition function*  $Z_{L,\beta}^{\omega}$  through a re-finement of the operator approach from [\[CSZ23,](#page-35-1) Theorem 6.1] and  $[LZ21+,$  Theorem 1.3] (inspired by [\[GQT21\]](#page-36-0)). We point out that these papers deal with the critical and sub-critical regimes, while we are interested the quasi-critical regime [\(1.11\)](#page-1-5).

For transparency, and in view of future applications, we develop in this section a *non asymptotic approach which is independent of the regime of β*: we obtain bounds with explicit constants which hold for any given system size  $L$  and disorder strength  $\beta$ . Some novelties with respect to  $[CSZ23, LZ21+]$  $[CSZ23, LZ21+]$  are described in Remarks [4.4,](#page-13-0) [4.7,](#page-13-1) [4.9.](#page-14-0) These bounds will be crucially applied in Section [5](#page-20-0) to prove Proposition [2.2.](#page-6-1)

The section is organised as follows:

- ' in Subsection [4.1](#page-11-1) we give an *exact expansion* for the moments, see Theorem [4.5,](#page-13-2) in terms of suitable operators linked to the random walk and the disorder;
- ' Subsection [4.2](#page-14-1) we deduce *upper bounds* for the moments, see Theorems [4.8](#page-14-2) and [4.11,](#page-15-0) which depend on two pairs of quantities, that we call *boundary terms* and *bulk terms*;
- ' in Subsection [4.3](#page-16-0) we state some basic random walk bounds needed in our analysis (we consider general symmetric random walks with sub-Gaussian tails);
- ' in Subsections [4.4](#page-17-0) and [4.5](#page-19-0) we obtain explicit estimates on the boundary terms and bulk terms, which plugged in Theorem [4.11](#page-15-0) yield explicit bounds on the moments.

<span id="page-11-1"></span>**4.1. Moment expansion.** The partition function  $Z^{\omega}_{(A,B],\beta}(z)$  in [\(2.2\)](#page-5-5) is called "pointto-plane", since random walk paths start at  $S_0 = z$  but have no constrained endpoint. We introduce a "point-to-point" version, for simplicity when  $(A, B] = (0, L]$  for  $L \in \mathbb{N}$ , restricting to random walk paths with a fixed endpoint  $S_L = w$ :

<span id="page-11-6"></span>
$$
Z_{L,\beta}^{\omega}(z,w) := \mathbf{E}\Big[e^{\sum_{n=1}^{L-1}\{\beta\omega(n,S_n) - \lambda(\beta)\}}\mathbf{1}_{\{S_L=w\}}\Big|S_0=z\Big]
$$
(4.1)

(we stop the sum at  $n = L - 1$  for later convenience).

Given two "boundary conditions"  $f, g : \mathbb{Z}^2 \to \mathbb{R}$ , we define the averaged version

<span id="page-11-2"></span>
$$
\mathcal{Z}_{L,\beta}^{\omega}(f,g) := \sum_{z,w \in \mathbb{Z}^2} f(z) \, Z_{L,\beta}^{\omega}(z,w) \, g(w) \,, \tag{4.2}
$$

where we use a different font to avoid confusions with the diffusively rescaled average  $(1.4)$ . We focus on the *centred moments* of  $Z_{L,\beta}^{\omega}(f,g)$ , that we denote by

<span id="page-11-3"></span>
$$
\mathcal{M}^h_{L,\beta}(f,g) := \mathbb{E}\Big[\Big(\mathcal{Z}^\omega_{L,\beta}(f,g) - \mathbb{E}\big[\mathcal{Z}^\omega_{L,\beta}(f,g)\big]\Big)^h\Big] \qquad \text{for } h \in \mathbb{N} \,.
$$
 (4.3)

<span id="page-11-5"></span>**Remark 4.1.** *Recalling* [\(2.2\)](#page-5-5)*,* [\(2.3\)](#page-5-6) *and* [\(1.5\)](#page-0-4)*,* [\(3.5\)](#page-6-6)*, by translation invariance we have*

<span id="page-11-4"></span>
$$
\mathbb{E}\left[\left(X_{N,M}^{(i)}\right)^4\right] = \frac{\vartheta_N^2}{N^4} \mathcal{M}_{\frac{N}{M},\beta_N}^4(f_i,g), \qquad \text{where} \quad \begin{cases} f_i(z) := q_{\frac{i-1}{M}N}^{\varphi_N}(z), \\ g(w) := 1. \end{cases} \tag{4.4}
$$

*To prove Proposition [2.2,](#page-6-1) in Section [5](#page-20-0)* we will focus on  $\mathcal{M}_{L,\beta}^4(f,g)$ .

Henceforth we fix  $h \in \mathbb{N}$  with  $h \geq 2$  (the interesting case is  $h \geq 3$ ). We are going to give an *exact expression* for  $\mathcal{M}_{L,\beta}^h(f,g)$ , see Theorem [4.5.](#page-13-2) We first need some notation.

We denote by  $I \vdash \{1, \ldots, h\}$  a *partition of*  $\{1, \ldots, h\}$ , i.e. a family  $I = \{I^1, \ldots, I^m\}$  of non empty disjoint subsets  $I^j \subseteq \{1, ..., h\}$  with  $I^1 \cup ... \cup I^m = \{1, ..., h\}$ . We single out:

- the unique partition  $I = * := \{\{1\}, \{2\}, \ldots, \{h\}\}\)$  composed by all singletons;
- $\bullet$  the  $\binom{h}{2}$  $\binom{h}{2}$  partitions of the form  $I = \{\{a, b\}, \{c\} : c \neq a, c \neq b\}$ , that we call *pairs*.

**Example 4.2 (Cases**  $h = 2, 3, 4$ ). All partitions  $I \vdash \{1, 2\}$  are  $I = *$  and  $I = \{\{1, 2\}\}.$ *All partitions*  $I \vdash \{1, 2, 3\}$  *are*  $I = *$ , *three* pairs  $I = \{\{a, b\}, \{c\}\}$  *and*  $I = \{\{1, 2, 3\}\}.$ 

*All partitions*  $I \vdash \{1, 2, 3, 4\}$  *are*  $I = *,$  *six* pairs  $I = \{\{a, b\}, \{c\}, \{d\}\},$  *six* double pairs  $I = \{\{a, b\}, \{c, d\}\}\$ , four triples  $I = \{\{a, b, c\}, \{d\}\}\$  and the quadruple  $I = \{\{1, 2, 3, 4\}\}\$ .

Given a partition  $I = \{I^1, ..., I^m\} \mapsto \{1, ..., h\}$ , we define for  $\mathbf{x} = (x^1, ..., x^h) \in (\mathbb{Z}^2)^h$ 

<span id="page-12-4"></span>
$$
\mathbf{x} \sim I \quad \text{if and only if } \begin{cases} x^a = x^b & \text{if } a, b \in I^i \text{ for some } i, \\ x^a \neq x^b & \text{if } a \in I^i, b \in I^j \text{ for some } i \neq j \text{ with } |I^i|, |I^j| \ge 2. \end{cases} \tag{4.5}
$$

For instance  $\mathbf{x} \sim \{\{1, 2\}, \{3\}, \{4\}\}\$  means  $x^1 = x^2$ , while  $\mathbf{x} \sim \{\{1, 2\}, \{3, 4\}\}\$  means  $x^1 = x^2$ and  $x^3 = x^4$  with  $x^1 \neq x^3$ . Note that  $\mathbf{x} \sim *$  imposes no constraint. We also define

<span id="page-12-3"></span>
$$
(\mathbb{Z}^2)_I^h := \{ \mathbf{x} \in (\mathbb{Z}^2)^h : \mathbf{x} = (x^1, \dots, x^h) \sim I \},
$$
\n(4.6)

which is essentially a copy of  $(\mathbb{Z}^2)^m$  embedded in  $(\mathbb{Z}^2)^h$ .

A family  $I_1, \ldots, I_r$  of partitions  $I_i = \{I_i^1, \ldots, I_i^{m_i}\} \mapsto \{1, \ldots, h\}$  is said to have *full support* if any  $a \in \{1, ..., h\}$  belongs to some partition  $I_i$  not as a singleton, i.e.  $a \in I_i^j$  with  $|I_i^j|$  $\left|\frac{j}{i}\right|\geqslant 2.$ 

**Example 4.3 (Full support for**  $h = 4$ ). *A single partition*  $I_1 \vdash \{1, 2, 3, 4\}$  with full *support is either the quadruple*  $I_1 = \{\{1, 2, 3, 4\}\}\$  *or a double pair*  $I_1 = \{\{a, b\}, \{c, d\}\}.$ *There are many families of two partitions*  $I_1, I_2 \vdash \{1, 2, 3, 4\}$  *with full support, for instance two non overlapping pairs such as*  $I_1 = \{\{1,3\},\{2\},\{4\}\}, I_2 = \{\{2,4\},\{1\},\{3\}\}.$ 

We now introduce *h*-fold analogues of the random walk transition kernel [\(3.3\)](#page-6-5) and of its averaged version [\(3.5\)](#page-6-6): given partitions  $I, J \vdash \{1, \ldots, h\}$ , we define for  $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h$ 

<span id="page-12-0"></span>
$$
\mathbf{Q}_n^{I,J}(\mathbf{z}, \mathbf{x}) := \mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}} \prod_{i=1}^h q_n(x^i - z^i), \qquad \mathbf{q}_n^{f,J}(\mathbf{x}) := \mathbb{1}_{\{\mathbf{x} \sim J\}} \prod_{i=1}^h q_n^f(x^i).
$$
 (4.7)

Given  $m \in \mathbb{N}_0$  and  $J \vdash \{1, ..., h\} \neq *,$  we define for  $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h$  the weighted *Green's kernel*

<span id="page-12-1"></span>
$$
\mathsf{U}_{m,\beta}^{J}(\mathbf{z},\mathbf{x}) := \begin{cases} \sum_{k=1}^{\infty} \mathbb{E}[\xi_{\beta}^{J}]^{k} \sum_{0 = :n_{0} < n_{1} < \cdots < n_{k} := m \\ \sum_{\substack{\mathbf{y}_{1}, \ldots, \mathbf{y}_{k-1} \in (\mathbb{Z}^{2})^{h} \\ \mathbf{y}_{0} := \mathbf{z}, \ \mathbf{y}_{k} := \mathbf{x}}} \prod_{i=1}^{k} \mathsf{Q}_{n_{i} - n_{i-1}}^{J, J}(\mathbf{y}_{i-1}, \mathbf{y}_{i}) & \text{if } m \geq 1, \\ \sum_{\substack{\mathbf{y}_{0} := \mathbf{z}, \ \mathbf{y}_{k} := \mathbf{x}}} \mathbb{E}[\xi_{\beta}^{J}]^{k} & \text{if } m = 0, \end{cases} \tag{4.8}
$$

where for  $J = \{J^1, \ldots, J^m\}$  with  $J \neq *$  we define

<span id="page-12-2"></span>
$$
\mathbb{E}[\xi_{\beta}^{J}] := \prod_{i: |J^{i}| \geqslant 2} \mathbb{E}[\xi_{\beta}^{|J^{i}|}]. \tag{4.9}
$$

When *J* is a pair, this reduces to  $\mathbb{E}[\xi_{\beta}^{J}] = \mathbb{E}[\xi_{\beta}^{2}] = \sigma_{\beta}^{2}$ , see [\(3.2\)](#page-6-7).

<span id="page-13-0"></span>**Remark 4.4 (On the definition of**  $U^J$ ). We point out that  $U^J$  was only defined in [\[CSZ23,](#page-35-1) [LZ21+\]](#page-36-1) when *J* is a pair. Defining  $\mathsf{U}^J$  for any partition *J* makes formulas simpler, *as it avoids to distinguish between pairs and non-pairs in the sums* [\(4.12\)](#page-13-3) *and* [\(4.18\)](#page-14-3)*.*

*For a pair*  $J = \{\{a, b\}, \{c\} : c \neq a, b\}$ , by Chapman-Kolmogorov we can express

<span id="page-13-4"></span>
$$
\mathsf{U}_{m,\beta}^{J}(\mathbf{z},\mathbf{x}) = U_{m,\beta}(x^{a} - z^{a}) \, \mathbb{1}_{\{x^{b} = x^{a}, z^{b} = z^{a}\}} \prod_{c \neq a,b} q_{m}(x^{c} - z^{c}), \tag{4.10}
$$

*where we define for*  $x \in \mathbb{Z}^2$ 

<span id="page-13-5"></span>
$$
U_{m,\beta}(x) := \sum_{k=1}^{\infty} (\sigma_{\beta}^{2})^{k} \sum_{\substack{0 =: n_{0} < n_{1} < \dots < n_{k} := m \\ x_{0} = 0, \ x_{1}, \dots, x_{k-1} \in \mathbb{Z}^{2}, \ x_{k} := x}} \prod_{i=1}^{k} q_{n_{i} - n_{i-1}} (x_{i} - x_{i-1})^{2}.
$$
 (4.11)

Given two functions  $q^f(\mathbf{x})$ ,  $q^g(\mathbf{x})$  and a family of matrices  $U_i(\mathbf{z}, \mathbf{x})$ ,  $Q_i(\mathbf{z}, \mathbf{x})$  for  $\mathbf{x}, \mathbf{z} \in \mathbb{T}$ , where  $\mathbb T$  is a countable set, we use the standard notation

$$
\left\langle \mathsf{q}^f,\, \mathsf{U}_1\left\{ \prod_{i=2}^r \mathsf{Q}_i \, \mathsf{U}_i\right\} \mathsf{q}^g \right\rangle := \sum_{\substack{\mathbf{z}_1,\ldots,\mathbf{z}_r\in \mathbb{T}\\ \mathbf{z}_1',\ldots,\mathbf{z}_r'\in \mathbb{T}}} \mathsf{q}^f(\mathbf{z}_1)\, \mathsf{U}_1(\mathbf{z}_1,\mathbf{z}_1')\left\{ \prod_{i=2}^r \mathsf{Q}_i(\mathbf{z}_{i-1}',\mathbf{z}_i)\, \mathsf{U}_i(\mathbf{z}_i,\mathbf{z}_i')\right\} \mathsf{q}^g(\mathbf{z}_r')\,.
$$

<span id="page-13-2"></span>We can now give the announced expansion for  $\mathcal{M}_{L,\beta}^h(f,g)$ , that we prove in Appendix [A.](#page-24-0)

**Theorem 4.5 (Moment expansion).** Let  $\mathcal{Z}_{L,\beta}^{\omega}(f,g)$  be the averaged partition function *in* [\(4.2\)](#page-11-2) *with centred moments*  $\mathcal{M}_{L,\beta}^h(f,g)$ *, see* [\(4.3\)](#page-11-3)*. For any*  $h \in \mathbb{N}$  *with*  $h \geq 2$  *we have* 

<span id="page-13-3"></span>
$$
\mathcal{M}_{L,\beta}^{h}(f,g) = \sum_{r=1}^{\infty} \sum_{0 < n_1 \le m_1 < \dots < n_r \le m_r < L} \sum_{\substack{I_1, \dots, I_r \vdash \{1, \dots, h\} \\ \text{with full support} \\ \text{and } I_i \ne I_{i-1}, \ I_i \ne * \ \forall i}} \left\{ \prod_{i=1}^r \mathbb{E}[\xi_{\beta}^{I_i}] \right\} \times \left\{ \sum_{i=1}^r \mathbb{E}[\xi_{\beta}^{I_i}] \right\} \times \left\
$$

**Remark 4.6 (Sanity check).** *In case*  $h = 2$ , *the conditions*  $I_i \neq I_{i-1}$  *and*  $I_i \neq *$  *in* [\(4.12\)](#page-13-3) *force*  $r = 1$  *and*  $I_1 = \{\{1, 2\}\}\$ . *Then, recalling* [\(4.10\)](#page-13-4)*-*[\(4.11\)](#page-13-5)*, formula* [\(4.12\)](#page-13-3) *reduces to* 

$$
\mathcal{M}_{L,\beta}^2(f,g) = \mathbb{V}\ar[\mathcal{Z}_{L,\beta}^{\omega}(f,g)] = \sigma_{\beta}^2 \sum_{\substack{0 < n \leq m < L \\ z, x \in \mathbb{Z}^2}} q_n^f(z) U_{m-n,\beta}(x-z) q_{L-m}^g(x),
$$

<span id="page-13-1"></span>*which is a classical expansion for the variance, see e.g.* [\[CSZ23,](#page-35-1) eq. (3.51)]*.*

**Remark 4.7 (Boundary conditions).** In [\[CSZ23,](#page-35-1) [LZ21+\]](#page-36-1), the quantity  $q_{n_1}^{f,I_1}$  in [\(4.12\)](#page-13-3) is expanded as  $Q_{n_1}^{I_1,*} f^{\otimes h}$  (recall [\(4.7\)](#page-12-0) and [\(3.5\)](#page-6-6)); similarly for  $q_{L-i}^{g,I_r}$  $L_{m_r}$ . We keep these quantities *unexpanded in order to derive tailored estimates, see Subsection [4.4,](#page-17-0) which could not be derived by simply applying operator norm bounds on*  $Q_{n_1}^{I_1,*}$  *as in* [\[CSZ23,](#page-35-1) [LZ21+\]](#page-36-1)*.* 

<span id="page-14-1"></span>**4.2. Moment upper bounds.** We next obtain upper bounds from [\(4.12\)](#page-13-3). For  $L \in \mathbb{N}$  we define the summed kernels

<span id="page-14-6"></span>
$$
\widehat{\mathbf{Q}}_L^{I,J}(\mathbf{z}, \mathbf{x}) := \sum_{n=1}^L \mathbf{Q}_n^{I,J}(\mathbf{z}, \mathbf{x}), \qquad \widehat{\mathbf{q}}_L^{f,I}(\mathbf{x}) := \sum_{n=1}^L \mathbf{q}_n^{f,I}(\mathbf{x}). \tag{4.13}
$$

Recalling [\(4.8\)](#page-12-1) and [\(4.9\)](#page-12-2) we set, with some abuse of notation,

<span id="page-14-7"></span>
$$
|\mathsf{U}|_{m-n,\beta}^{J}(\mathbf{z},\mathbf{x}) := \mathsf{U}_{m-n,\beta}^{J}(\mathbf{z},\mathbf{x}) \text{ from (4.8) with } \mathbb{E}[\xi_{\beta}^{J}] \text{ replaced by } |\mathbb{E}[\xi_{\beta}^{J}]|.
$$
 (4.14)

Then, for  $L \in \mathbb{N}$  and  $\lambda \geq 0$ , we define the Laplace sum

<span id="page-14-8"></span>
$$
|\widehat{\mathbf{U}}|_{L,\lambda,\beta}^{J}(\mathbf{z},\mathbf{x}) := \mathbb{1}_{\{\mathbf{z} = \mathbf{x} \sim J\}} + \sum_{m=1}^{L} e^{-\lambda m} |\mathbf{U}|_{m,\beta}^{J}(\mathbf{z},\mathbf{x}). \tag{4.15}
$$

Finally, we introduce a *uniform bound* on the right boundary function  $q_{L}^{g,I_r}$  $_{L-m_r}^{g,I_r}$  in [\(4.12\)](#page-13-3):

<span id="page-14-5"></span>
$$
\overline{\mathbf{q}}_L^{g,J}(\mathbf{z}) := \max_{1 \le n \le L} \mathbf{q}_n^{g,J}(\mathbf{z}). \tag{4.16}
$$

<span id="page-14-2"></span>We can now state our first moment upper bound.

**Theorem 4.8 (Moment upper bound, I).** Let  $\mathcal{Z}_{L,\beta}^{\omega}(f,g)$  denote the averaged partition *function in* [\(4.2\)](#page-11-2) *with centred moment*  $\mathcal{M}_{L,\beta}^h(f,g)$ *, see* [\(4.3\)](#page-11-3)*, for*  $h \in \mathbb{N}$  *with*  $h \geq 2$ *. For any λ* ě 0 *we have the upper bound*

<span id="page-14-4"></span>
$$
\left|\mathcal{M}_{L,\beta}^h(f,g)\right| \leq e^{\lambda L} \sum_{r=1}^{\infty} \Xi(r) \tag{4.17}
$$

*with*

<span id="page-14-3"></span>
$$
\Xi(r) := \sum_{\substack{I_1,\ldots,I_r\vdash\{1,\ldots,h\}\\ \text{with full support}\\ \text{and } I_i \neq I_{i-1}, \ I_i \neq * \forall i}} \left\{ \prod_{i=1}^r \left| \mathbb{E}[\xi_{\beta}^{I_i}] \right| \right\} \left\langle \hat{\mathsf{q}}_L^{|f|,I_1}, |\hat{\mathsf{U}}|_{L,\lambda,\beta}^{I_1} \left\{ \prod_{i=2}^r \hat{\mathsf{Q}}_L^{I_{i-1},I_i} |\hat{\mathsf{U}}|_{L,\lambda,\beta}^{I_i} \right\} \overline{\mathsf{q}}_L^{|g|,I_r} \right\rangle. (4.18)
$$

**Proof.** Replacing  $\mathbb{E}[\xi_{\beta}^{I_i}], f, g, \mathsf{U}$  in [\(4.12\)](#page-13-3) respectively by  $|\mathbb{E}[\xi_{\beta}^{I_i}]|, |f|, |g|, |\mathsf{U}|$ , every term becomes non-negative. We next replace  $q_{L-r}^{|g|,I}$  $\left[\frac{|g|, I}{L - m_r}\right]$  by the uniform bound  $\overline{\mathsf{q}}_{L}^{[g], I}$  and then enlarge the sum in [\(4.12\)](#page-13-3), allowing increments  $n_i - m_{i-1}$  and  $m_i - n_i$  to vary freely in  $\{1, \ldots, L\}$ . Plugging  $1 \le e^{\lambda L} e^{-\lambda m_r} \le e^{\lambda L} e^{-\lambda \sum_{i=1}^r (m_i - n_i)}$ , we obtain [\(4.17\)](#page-14-4).

<span id="page-14-0"></span>**Remark 4.9 (On the right boundary condition).** The function  $q_{L-1}^{g,I_r}$  $_{L-m_r}^{g,r_r}$  *in* [\(4.12\)](#page-13-3) *is controlled in*  $[CSZ23, LZ21+]$  $[CSZ23, LZ21+]$  *by introducing an average over L, which forces the function q to be estimated in*  $\ell^{\infty}$ . Our approach avoids such averaging, via the quantity  $\bar{\mathbf{q}}_L^{g,J}$ *L from* [\(4.16\)](#page-14-5): this lets us estimate the function g in  $\ell^q$  also for  $q < \infty$  (see Proposition [4.21\)](#page-19-1).

We next bound  $\Xi(r)$  in [\(4.18\)](#page-14-3), starting from the scalar product. Let us recall some functional analysis: given a countable set  $\mathbb T$  and a function  $f : \mathbb T \to \mathbb R$ , we define

$$
||f||_{\ell^{p}(\mathbb{T})} = ||f||_{\ell^{p}} := \left(\sum_{z \in \mathbb{T}} |f(z)|^{p}\right)^{\frac{1}{p}} \quad \text{for } p \in [1, \infty).
$$
 (4.19)

For a linear operator  $A: \ell^q(\mathbb{T}) \to \ell^q(\mathbb{T}')$ , with  $p, q \in (1, \infty)$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$
\|A\|_{\ell^{q}\to\ell^{q}} := \sup_{g\neq 0} \frac{\|A g\|_{\ell^{q}(\mathbb{T}')}}{\|g\|_{\ell^{q}(\mathbb{T})}} = \sup_{\|f\|_{\ell^{p}(\mathbb{T}')} \leq 1, \|g\|_{\ell^{q}(\mathbb{T})} \leq 1} \langle f, A g \rangle.
$$
 (4.20)

*.*

By Hölder's inequality  $|\langle g, h \rangle| \leq \|g\|_{\ell^p} \|h\|_{\ell^q}$ , so the scalar product in [\(4.18\)](#page-14-3) is bounded by

<span id="page-15-1"></span>
$$
\|\widehat{\mathbf{q}}_L^{[f],I_1}\|_{\ell^p} \|\widehat{\mathbf{U}}|_{L,\lambda,\beta}^{I_1}\|_{\ell^q \to \ell^q} \left\{ \prod_{i=2}^r \|\widehat{\mathbf{Q}}_L^{I_{i-1},I_i}\|_{\ell^q \to \ell^q} \|\widehat{\mathbf{U}}|_{L,\lambda,\beta}^{I_i}\|_{\ell^q \to \ell^q} \right\} \|\overline{\mathbf{q}}_L^{[g],I_r}\|_{\ell^q}.
$$
 (4.21)

**Remark 4.10 (Restricted**  $l^q$  spaces). Due to the constraint  $\mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}}$  in [\(4.7\)](#page-12-0), we *may regard*  $\widehat{Q}_L^{I,J}$  *as a linear operator from*  $\ell^q((\mathbb{Z}^2)^h_J)$  *to*  $\ell^q((\mathbb{Z}^2)^h_I)$ *, see* [\(4.6\)](#page-12-3)*. Similarly, we may view*  $|\hat{\mathbf{U}}|_{L,\lambda,\beta}^J$  *as a linear operator from*  $\ell^q((\mathbb{Z}^2)^h_J)$  *to itself.* 

To make the bound [\(4.21\)](#page-15-1) more useful, we introduce a *weight*  $W : (\mathbb{Z}^2)^h \to (0, \infty)$ , that we also identify with the diagonal operator  $W(\mathbf{x}) \mathbb{1}_{\{\mathbf{x}=\mathbf{y}\}}$ , so that in particular

$$
(\mathcal{W} \mathsf{A} \frac{1}{\mathcal{W}})(\mathbf{x}, \mathbf{y}) := \mathcal{W}(\mathbf{x}) \mathsf{A}(\mathbf{x}, \mathbf{y}) \frac{1}{\mathcal{W}(\mathbf{y})}
$$

Inserting  $(W\frac{1}{W})$  between each pair of adjacent operators in [\(4.17\)](#page-14-4), we improve [\(4.21\)](#page-15-1) to

$$
\|\hat{\mathbf{q}}_L^{|f|, I_1} \frac{1}{\mathcal{W}}\|_{\ell^p} \|\mathcal{W}|\hat{\mathbf{U}}|_{L,\lambda,\beta}^{I_1} \frac{1}{\mathcal{W}}\|_{\ell^q \to \ell^q} \times \times \left\{\prod_{i=2}^r \|\mathcal{W}\hat{\mathbf{Q}}_L^{I_{i-1},I_i} \frac{1}{\mathcal{W}}\|_{\ell^q \to \ell^q} \|\mathcal{W}|\hat{\mathbf{U}}|_{L,\lambda,\beta}^{I_i} \frac{1}{\mathcal{W}}\|_{\ell^q \to \ell^q}\right\} \|\mathcal{W}\overline{\mathbf{q}}_L^{|g|,I_r}\|_{\ell^q}.
$$
\n
$$
(4.22)
$$

<span id="page-15-0"></span>In view of [\(4.17\)](#page-14-4)-[\(4.18\)](#page-14-3), this leads directly to our second moment upper bound.

**Theorem 4.11 (Moment upper bound, II).** Let  $\mathcal{Z}_{L,\beta}^{\omega}(f,g)$  be the averaged partition *function in* [\(4.2\)](#page-11-2) *with centred moment*  $\mathcal{M}_{L,\beta}^h(f,g) \leq e^{\lambda L} \sum_{r=1}^{\infty} \Xi(r)$ *, see* [\(4.3\)](#page-11-3) *and* [\(4.17\)](#page-14-4)*,*  $for \lambda \geq 0$  and  $h \geq 2$ . For any weight  $W : (\mathbb{Z}^2)^h \to (0, \infty)$  and for  $p, q \in (1, \infty)$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , we have the following upper bound on  $\Xi(r)$  from [\(4.18\)](#page-14-3)*:* 

$$
\Xi(r) \leqslant \left(\max_{I \neq \ast} \|\widehat{\mathsf{q}}_L^{|f|, I} \frac{1}{\mathcal{W}}\|_{\ell^p}\right) \left(\max_{J \neq \ast} \|\mathcal{W}\overline{\mathsf{q}}_L^{|g|, J}\|_{\ell^q}\right) \Xi^{\text{bulk}}(r) \tag{4.23}
$$

<span id="page-15-2"></span>*with*

<span id="page-15-3"></span>
$$
\Xi^{\text{bulk}}(r) := \sum_{\substack{I_1,\ldots,I_r\vdash\{1,\ldots,h\}\\ \text{with full support}\\ \text{and } I_i \neq I_{i-1}, \ I_i \neq * \ \forall i}} \left\{ \prod_{i=1}^r \left| \mathbb{E}[\xi_{\beta}^{I_i}] \right| \right\} \left( C_L^{Q,\ell^q} \right)^{r-1} \left( C_{L,\lambda,\beta}^{U,\ell^q} \right)^r, \tag{4.24}
$$

*where we set for short*

<span id="page-15-5"></span>
$$
\mathsf{C}_{L}^{\mathsf{Q},\ell^{q}} := \max_{\substack{I,J \neq * \\ I \neq J}} \|\mathcal{W}\,\widehat{\mathsf{Q}}_{L}^{I,J}\,\frac{1}{\mathcal{W}}\|_{\ell^{q} \to \ell^{q}}\,, \qquad \mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^{q}} := \max_{I \neq *} \|\mathcal{W}\,|\widehat{\mathsf{U}}|_{L,\lambda,\beta}^{I}\,\frac{1}{\mathcal{W}}\|_{\ell^{q} \to \ell^{q}}\,. \tag{4.25}
$$

Note that the bound  $(4.23)-(4.24)$  $(4.23)-(4.24)$  $(4.23)-(4.24)$  depends on two pairs of quantities, that we call

<span id="page-15-4"></span>*boundary terms* 
$$
\begin{cases} \|\widehat{\mathsf{q}}_{L}^{[f],I} \frac{1}{\mathcal{W}}\|_{\ell^{p}} & \text{and} \qquad bulk \ terms \ \begin{cases} \|\mathcal{W} \widehat{\mathsf{Q}}_{L}^{I,J} \frac{1}{\mathcal{W}}\|_{\ell^{q} \to \ell^{q}} \\ \|\mathcal{W} \overline{\mathsf{q}}_{L}^{[g],J}\|_{\ell^{q}} & \end{cases} \tag{4.26}
$$

We will estimate these terms in Subsections [4.4](#page-17-0) and [4.5](#page-19-0) respectively, exploiting some basic random walk bounds that we collect in Subsection [4.3.](#page-16-0)

**Remark 4.12 (Choice of the weight).** We will choose a weight  $W = W_t : (\mathbb{Z}^2)^h \to$  $p(0, \infty)$  which is exponential of rate  $t \geq 0$ , that is for  $\mathbf{x} = (x^1, \dots, x^h) \in (\mathbb{Z}^2)^h$ 

<span id="page-16-4"></span>
$$
\mathcal{W}_t(\mathbf{x}) := \prod_{i=1}^h w_t(x^i) \qquad \text{where} \qquad w_t(x) := e^{-t|x|} \text{ for } x \in \mathbb{Z}^2. \tag{4.27}
$$

*Note that by the triangle inequality we can bound, for all*  $\mathbf{x}, \mathbf{z} \in (\mathbb{Z}^2)^h$ ,

<span id="page-16-6"></span>
$$
\frac{\mathcal{W}_t(\mathbf{z})}{\mathcal{W}_t(\mathbf{x})} \leq \prod_{i=1}^h e^{t|z^i - x^i|}.
$$
\n(4.28)

We will later need to consider an additional weight  $V_s^I$ , see [\(4.42\)](#page-18-0) below.

We finally bound the product  $\left\{ \prod_{i=1}^r \right\}$  $\left| \mathbb{E}[\xi_{\beta}^{I_i}] \right|$  in [\(4.24\)](#page-15-3). Recall  $\sigma_{\beta}$  from [\(1.7\)](#page-1-6) and [\(3.2\)](#page-6-7) and note that  $\lim_{\beta\downarrow 0} \sigma_\beta = 0$ .

<span id="page-16-5"></span>**Proposition 4.13 (Moments of disorder).** *For any*  $h \in \mathbb{N}$  *there are*  $\beta_0(h) > 0$  *and*  $C(h) < \infty$  (which depend on the disorder distribution) such that for  $\beta < \beta_0(h)$  we have

<span id="page-16-1"></span>
$$
\left|\mathbb{E}[\xi_{\beta}^{I}]\right| \leq \begin{cases} \sigma_{\beta}^{2} & \text{if } I = \{\{a,b\},\{c\} \colon c \neq a,b\} \text{ is a pair} \\ C(h) \sigma_{\beta}^{3} & \text{if } I \neq * \text{ is not a pair} \end{cases} \leq \sigma_{\beta}^{2} \quad \text{if } I \neq *.
$$
 (4.29)

*Moreover*

<span id="page-16-2"></span>if 
$$
I_1, ..., I_r \vdash \{1, ..., h\}
$$
 have full support: 
$$
\prod_{i=1}^r |\mathbb{E}[\xi_\beta^{I_i}]| \leq C(h)^r \sigma_\beta^{\max\{2r, h\}}.
$$
 (4.30)

**Proof.** We have  $|\mathbb{E}[\xi_{\beta}^{I}]| = \sigma_{\beta}^{2}$  if *I* is a pair, while  $|\mathbb{E}[\xi_{\beta}^{I}]| = O(\sigma_{\beta}^{3})$  if  $I \neq *$  is not a pair. Indeed, if  $||I|| := \sum_{i=1}^m |I^i| 1\!\!1_{\{|I^i|\geq 2\}}$  denotes the number of  $a \in \{1, \ldots, h\}$  which are not singletons in  $I = \{I^1, ..., I^m\} \mapsto \{1, ..., h\}$ , by [\(3.2\)](#page-6-7) and [\(4.9\)](#page-12-2) we have  $|\mathbb{E}[\xi_\beta^I]| = O(\sigma_\beta^{\|I\|})$ (note that  $||I|| = 2$  if *I* is a pair while  $||I|| \ge 3$  if  $I \ne *$  is not a pair).

Since  $\lim_{\beta\downarrow 0} \sigma_\beta = 0$ , we see that  $(4.29)$  holds for  $\beta > 0$  for small enough, depending on *h* (it suffices that  $\mathbb{E}[\xi_{\beta}^k] \leq \mathbb{E}[\xi_{\beta}^2] = \sigma_{\beta}^2 \leq 1$  for all  $k \in \{3, ..., h\}$ , see [\(3.2\)](#page-6-7)). Finally, if  $I_1, ..., I_r$ have full support, then each  $a \in \{1, \ldots, h\}$  is a non-trivial element (i.e. not a singleton) of some partition  $I_i$ , hence  $||I_1|| + \ldots + ||I_r|| \geq h$  which yields  $\prod_{i=1}^r$  $\left| \mathbb{E}[\xi_{\beta}^{I_i}] \right| = O(\sigma_{\beta}^h)$ . This proves [\(4.30\)](#page-16-2) because  $\prod_{i=1}^{r}$  $\left| \mathbb{E}[\xi_{\beta}^{I_i}] \right| = O(\sigma_{\beta}^{2r})$  by [\(4.29\)](#page-16-1).

<span id="page-16-0"></span>**4.3. Random walk bounds.** In this subsection we collect some useful random walk bounds, stated in Lemmas [4.16,](#page-17-1) [4.17](#page-17-2) and [4.18.](#page-17-3) The proofs are deferred to Appendix [B.](#page-26-0)

Instead of sticking to the simple random walk on  $\mathbb{Z}^2$ , we can allow for *any symmetric random walk with sub-Gaussian tails*, in the following sense.

<span id="page-16-3"></span>**Assumption 4.14 (Random walk).** We consider a random walk  $S = (S_n)_{n \geq 0}$  on  $\mathbb{Z}^2$ with a symmetric distribution, i.e.  $q_1(x) = P(S_1 = x) = q_1(-x)$  for any  $x \in \mathbb{Z}^2$ , and with *sub-Gaussian tails, i.e. for some*  $c > 0$  *we have, writing*  $x = (x^1, x^2)$ ,

<span id="page-17-4"></span>
$$
\forall t \in \mathbb{R}, \ \forall a = 1, 2: \qquad \mathcal{E}\left[e^{t S_1^a}\right] = \sum_{x \in \mathbb{Z}^2} e^{t x^a} q_1(x) \leqslant e^{c \frac{t^2}{2}}.
$$
 (4.31)

**Remark 4.15.** *The simple random walk on*  $\mathbb{Z}^2$  *satisfies* [\(4.31\)](#page-17-4) *with*  $c = 1$ *: indeed, we can*  $\text{compute } \sum_{x \in \mathbb{Z}^2} e^{tx^a} q_1(x) = \frac{1}{2} (1 + \cosh(t)) \leq \exp(t^2/2) \text{ (because } \cosh(t) \leq \exp(t^2/2).$ 

<span id="page-17-1"></span>We derive useful bounds for the random walk transition kernel  $q_n(x) = P(S_n = x)$ .

**Lemma 4.16 (Random walk bounds).** Let Assumption [4.14](#page-16-3) hold. There is  $c \in [1, \infty)$ *such that for all*  $t \geq 0$  *and*  $n \in \mathbb{N}$ 

$$
\forall a = 1, 2: \qquad \sum_{x \in \mathbb{Z}^2} e^{tx^a} q_n(x) \leq e^{c \frac{t^2}{2} n}, \qquad \sum_{x \in \mathbb{Z}^2} e^{tx^a} \frac{q_n(x)^2}{q_{2n}(0)} \leq e^{c \frac{t^2}{2} n}. \tag{4.32}
$$

*Moreover, recalling*  $w_t(x) = e^{-t|x|}$  *from* [\(4.27\)](#page-16-4)*, we can bound* 

<span id="page-17-5"></span>
$$
\left\|\frac{q_n}{w_t}\right\|_{\ell^1} = \sum_{x \in \mathbb{Z}^2} e^{t|x|} q_n(x) \leqslant c e^{2ct^2 n}, \qquad \left\|\frac{q_n}{w_t}\right\|_{\ell^\infty} = \sup_{x \in \mathbb{Z}^2} \left\{ e^{t|x|} q_n(x) \right\} \leqslant \frac{c e^{2ct^2 n}}{n}. \tag{4.33}
$$

We next extend the bounds in [\(4.33\)](#page-17-5) to the *averaged* random walk transition kernel  $q_n^f(x)$ , see [\(3.5\)](#page-6-6), for any  $f: \mathbb{Z}^2 \to \mathbb{R}$ . Let us agree that  $a^{\frac{1}{\infty}} := 1$  for any  $a > 0$ .

<span id="page-17-2"></span>**Lemma 4.17 (Averaged random walk bounds).** *Let Assumption [4.14](#page-16-3) hold and let* c *be the constant from Lemma [4.16.](#page-17-1) For any*  $t \ge 0$  *and*  $n \in \mathbb{N}$  *we have, with*  $w_t(x) = e^{-t|x|}$ ,

$$
\forall p \in [1, \infty]: \qquad \left\| \frac{q_n^f}{w_t} \right\|_{\ell^p} \leqslant c e^{2c t^2 n} \left\| \frac{f}{w_t} \right\|_{\ell^p}, \qquad \left\| \frac{q_n^f}{w_t} \right\|_{\ell^\infty} \leqslant \frac{c e^{2c t^2 n}}{n^{\frac{1}{p}}} \left\| \frac{f}{w_t} \right\|_{\ell^p}.
$$
 (4.34)

We finally consider the *maximal* averaged random walk transition kernel  $\overline{q}^f_I$  $L^f: \mathbb{Z}^2 \to \mathbb{R}$ 

<span id="page-17-10"></span><span id="page-17-8"></span><span id="page-17-7"></span>
$$
\overline{q}^f_L(x) := \max_{1 \le n \le L} q^f_n(x). \tag{4.35}
$$

<span id="page-17-3"></span>We prove a variant of Hardy-Littlewood maximal inequality, see Appendix [B](#page-26-0) for the details.

**Lemma 4.18 (Maximal random walk bounds).** *Let Assumption [4.14](#page-16-3) hold and let* c *be the constant from Lemma [4.16.](#page-17-1) For any*  $t \ge 0$  *and*  $L \in \mathbb{N}$  *we have, with*  $w_t(x) = e^{-t|x|}$ ,

<span id="page-17-9"></span>
$$
\forall p \in (1, \infty]: \qquad \left\| \overline{q}_L^f w_t \right\|_{\ell^p} \leq \frac{p}{p-1} 25^{\frac{1}{p}} \mathsf{C} \left\| f w_t \right\|_{\ell^p} \qquad with \quad \mathsf{C} := 200\pi \mathsf{c}^2 \mathsf{e}^{4\mathsf{c}t^2 L} \tag{4.36}
$$
\n
$$
(with \frac{\infty}{\infty - 1} := 1).
$$

<span id="page-17-0"></span>**4.4. Boundary terms.** In this section we estimate the *boundary terms* appearing in [\(4.23\)](#page-15-2), see [\(4.26\)](#page-15-4). The proofs are deferred to Appendix [C.](#page-29-0)

We recall that the weight  $W_t: (\mathbb{Z}^2)^h \to (0, \infty)$  is defined in [\(4.27\)](#page-16-4) for  $t \ge 0$ . Our estimates contain the following constants (with c from Lemma [4.16\)](#page-17-1):

<span id="page-17-6"></span>
$$
\mathscr{C} := \mathsf{c} e^{2\mathsf{c} t^2 L}, \qquad \overline{\mathscr{C}} := 5000 \pi \mathsf{c}^2 e^{4\mathsf{c} t^2 L}, \tag{4.37}
$$

where  $L$  is the "time horizon", see  $(4.26)$ . We anticipate that we will take

<span id="page-18-4"></span>
$$
t = \frac{1}{\sqrt{N}} \quad \text{with} \quad N \geqslant L. \tag{4.38}
$$

hence the constants  $\mathscr C$  and  $\overline{\mathscr C}$  are uniformly bounded in this regime.

We start estimating the *left boundary term* which involves  $\hat{\mathbf{q}}_L^{[f],I}$  (see [\(4.13\)](#page-14-6) and [\(4.7\)](#page-12-0)). It was proved<sup>[†](#page-18-1)</sup> in [\[LZ21+,](#page-36-1) Proposition 3.4], extending [\[CSZ23,](#page-35-1) Proposition 6.6], that for any  $h \ge 2$  there is  $C = C(h) < \infty$  such that, for any  $p \in (1, \infty)$ ,

<span id="page-18-2"></span>
$$
\max_{I \neq *} \left\| \widehat{\mathsf{q}}_L^{|f|, I} \frac{1}{\mathcal{W}_t} \right\|_{\ell^p} \leq \frac{p}{p-1} C L^{1-\frac{1}{p}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^h. \tag{4.39}
$$

For our goals it will be fundamental to have a *linear dependence in L*, which would amount to take  $p = \infty$  in [\(4.39\)](#page-18-2), but this is not allowed by our approach. To solve this problem, we improve the estimate [\(4.39\)](#page-18-2), showing that for  $p \in (0, \infty)$  we can still have a linear dependence in *L* in the RHS, provided we replace one factor  $\Vert \frac{f}{w} \Vert$  $\frac{f}{w_t}$  || $_{\ell}$ <sup>*p*</sup> by || $\frac{f}{w}$  $\frac{J}{w_t}\Vert_{\ell^\infty}.$ 

<span id="page-18-8"></span>**Proposition 4.19 (Left boundary term, I).** Recall the weights  $W_t$  and  $w_t$  from [\(4.27\)](#page-16-4). *For any*  $h \ge 2$ ,  $t \ge 0$ ,  $L \in \mathbb{N}$  *we have, for any*  $p \in (1, \infty)$  *and*  $\mathscr{C}$  *as in* [\(4.37\)](#page-17-6)*,* 

$$
\max_{I \neq *} \left\| \widehat{\mathsf{q}}_L^{|f|, I} \frac{1}{\mathcal{W}_t} \right\|_{\ell^p} \leq 4 \, \mathscr{C}^h \, L \left\| \frac{f}{w_t} \right\|_{\ell^\infty} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-1} . \tag{4.40}
$$

<span id="page-18-5"></span><span id="page-18-3"></span>*More generally, for any*  $r \in [1, \infty]$  *we have (with*  $\frac{1}{0} := \infty$ ,  $\frac{\infty}{\infty - 1} := 1$ )

$$
\max_{I \neq *} \left\| \widehat{\mathsf{q}}_L^{|f|, I} \frac{1}{\mathcal{W}_t} \right\|_{\ell^p} \leq 4 \, \mathscr{C}^h \, \min\{\frac{r}{r-1}, \frac{p}{p-1}\} \, L^{1-\frac{1}{r}} \left\| \frac{f}{w_t} \right\|_{\ell^r} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-1} . \tag{4.41}
$$

We further improve the bound [\(4.40\)](#page-18-3) through a *restricted weight*  $V_s^I : (\mathbb{Z}^2)^h \to (0, \infty)$ , defined for a *pair*  $I \vdash \{1, ..., h\}$  and  $s \geq 0$  by

<span id="page-18-0"></span>
$$
\mathcal{V}_s^I(\mathbf{x}) := w_s(x^a - x^b) = e^{-s|x^a - x^b|} \quad \text{for } I = \{\{a, b\}, \{c\} \colon c \neq a, b\}.
$$
 (4.42)

Note that  $||z^a - z^b| - |x^a - x^b|| \le |z^a - x^a| + |x^b - z^b|$ , therefore we can estimate

<span id="page-18-10"></span>
$$
\frac{\mathcal{V}_s^I(\mathbf{z})}{\mathcal{V}_s^I(\mathbf{x})} \leqslant e^{s|z^a - x^a| + s|z^b - x^b|} \,. \tag{4.43}
$$

In analogy with [\(4.38\)](#page-18-4), we anticipate that we will take

<span id="page-18-7"></span>
$$
s = \frac{1}{\sqrt{L}}.\tag{4.44}
$$

<span id="page-18-9"></span>**Proposition 4.20 (Left boundary term, II).** *For any*  $h \geq 3$ ,  $t \geq 0$ ,  $s \in (0,1]$ ,  $L \in \mathbb{N}$ *we have, for any*  $p \in (1, \infty)$  *and*  $\mathscr C$  *as in* [\(4.37\)](#page-17-6)*,* 

<span id="page-18-6"></span>
$$
\max_{\substack{J \,\, pair \\ I \neq *}} \left\| \widehat{\mathsf{q}}_{L}^{[f],I} \frac{\mathsf{y}_{s}^{J}}{\mathsf{W}_{t}} \right\|_{\ell^{p}} \leq 36^{\frac{1}{p}} \, \mathscr{C}^{h} \, \frac{L}{s^{\frac{2}{p}}} \left\| \frac{f}{w_{t}} \right\|_{\ell^{\infty}}^{2} \left\| \frac{f}{w_{t}} \right\|_{\ell^{p}}^{h-2},\tag{4.45}
$$

where  $I \nightharpoonup J$ , for  $I = \{I^1, ..., I^m\}$  and  $J = \{\{a, b\}, \{c\} : c \neq a, b\}$ , means  $I^j \nightharpoonup \{a, b\} \ \forall j$ .

<span id="page-18-1"></span><sup>&</sup>lt;sup>†</sup>The factor  $q = \frac{p}{p-1}$  in the RHS of [\(4.39\)](#page-18-2), first identified in [\[LZ21+\]](#page-36-1), is essential to allow for *p* which can vary with the system size *L*.

We next estimate the *right boundary term* which involves  $\bar{\mathbf{q}}_L^{[g],J}$ , see [\(4.16\)](#page-14-5) and [\(4.7\)](#page-12-0), obtaining estimates analogous to [\(4.41\)](#page-18-5) and [\(4.45\)](#page-18-6).

<span id="page-19-1"></span>**Proposition 4.21 (Right boundary term).** For any  $h \ge 2$ ,  $t \ge 0$ ,  $L \in \mathbb{N}$  we have, for  $any \ q \in (1, \infty) \ and \ \overline{\mathscr{C}} \ as \ in \ (4.37),$  $any \ q \in (1, \infty) \ and \ \overline{\mathscr{C}} \ as \ in \ (4.37),$  $any \ q \in (1, \infty) \ and \ \overline{\mathscr{C}} \ as \ in \ (4.37),$ 

$$
\max_{J \neq *} \|\overline{\mathbf{q}}_L^{[g],J} \mathcal{W}_t\|_{\ell^q} \le (\frac{q}{q-1} \overline{\mathcal{C}})^h \|g \, w_t\|_{\ell^{2q}}^2 \|g \, w_t\|_{\ell^q}^{h-2} \le (\frac{q}{q-1} \overline{\mathcal{C}})^h \|g \, w_t\|_{\ell^\infty} \|g \, w_t\|_{\ell^q}^{h-1} .
$$
\n(4.46)

<span id="page-19-3"></span><span id="page-19-2"></span>*Moreover, for any*  $h \geq 3$ ,  $s \in (0,1]$  *we have, for*  $\overline{\mathscr{C}}$  *as in* [\(4.37\)](#page-17-6)*,* 

$$
\max_{\substack{I \text{ pair} \\ J \neq *, J \supsetneq I}} \|\overline{\mathsf{q}}_L^{[g], J} \mathcal{W}_t \mathcal{V}_s^I\|_{\ell^q} \le (\frac{q}{q-1} \overline{\mathscr{C}})^h \frac{1}{s^{\frac{2}{q}}} \|g \, w_t\|_{\ell^\infty}^2 \|g \, w_t\|_{\ell^q}^{h-2} \,. \tag{4.47}
$$

where  $J \nightharpoonup I$ , for  $J = \{J^1, \ldots, J^m\}$  and  $I = \{\{a, b\}, \{c\} : c \neq a, b\}$ , means  $J^i \nightharpoonup \{a, b\} \forall i$ .

**Remark 4.22.** We can bound  $||g w_t||_{\ell^{\infty}} \le ||g||_{\ell^{\infty}} ||w_t||_{\ell^{\infty}}$  and  $||g w_t||_{\ell^q} \le ||g||_{\ell^{\infty}} ||w_t||_{\ell^q}$ . By a *direct computation, see* [\(C.16\)](#page-31-0)*, we have*

<span id="page-19-9"></span>
$$
||w_t||_{\ell^{\infty}} = 1, \qquad ||w_t||_{\ell^q} = \left(\sum_{z \in \mathbb{Z}^2} e^{-qt|z|}\right)^{\frac{1}{q}} \leq \frac{36^{\frac{1}{q}}}{t^{\frac{2}{q}}}, \qquad (4.48)
$$

<span id="page-19-5"></span>*therefore we obtain from* [\(4.46\)](#page-19-2)

$$
\max_{J \neq *} \|\bar{\mathsf{q}}_L^{|g|, J} \mathcal{W}_t\|_{\ell^q} \le (\frac{q}{q-1} 36^{\frac{1}{q}} \overline{\mathscr{C}})^h \frac{\|g\|_{\ell^\infty}^h}{t^{\frac{2}{q}(h-1)}}. \tag{4.49}
$$

*h*

*Similarly, from* [\(4.47\)](#page-19-3) *we deduce that*

<span id="page-19-7"></span>
$$
\max_{\substack{I \,\,\text{pair} \\ J \neq \ast\,,\, J \,\neq I}} \|\overline{\mathsf{q}}_L^{|g|,J} \,\mathcal{W}_t \,\mathcal{V}_s^I\|_{\ell^q} \leqslant \left(\frac{q}{q-1} \,36^{\frac{1}{q}} \,\overline{\mathscr{C}}\right)^h \frac{\|g\|_{\ell^\infty}^n}{s^{\frac{2}{q}} \,t^{\frac{2}{q}(h-2)}}\,. \tag{4.50}
$$

<span id="page-19-0"></span>**4.5. Bulk terms.** In this section we estimate the the *bulk terms* appearing in [\(4.24\)](#page-15-3), i.e. the constants  $C_L^{Q,\ell^q}$  $L^{\mathbf{Q},\ell^q}$  and  $\mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^q}$  from [\(4.25\)](#page-15-5). The proofs are also given in Appendix [C.](#page-29-0)

We recall the weights  $W_t$  and  $V_s^I$ , see [\(4.27\)](#page-16-4) and [\(4.42\)](#page-18-0). We will choose the parameters  $t, s = O(\frac{1}{\sqrt{2}})$  $L_L$ ), see [\(4.38\)](#page-18-4) and [\(4.44\)](#page-18-7), hence *the following constants are uniformly bounded*:

<span id="page-19-4"></span>
$$
\widehat{\mathscr{C}} := 4000 \, \mathsf{c}^2 \, \mathsf{e}^{8 \mathsf{c} \, t^2 L}, \qquad \widehat{\widehat{\mathscr{C}}} := 4000 \, \mathsf{c}^2 \, \mathsf{e}^{8 \mathsf{c} \, (t+2s)^2 L},
$$
\n
$$
\widetilde{\mathscr{C}} := 2 \, \mathsf{e}^{4 \mathsf{c} \, t^2 L}, \qquad \widetilde{\mathscr{C}} := 2 \, \mathsf{e}^{4 \mathsf{c} \, (t+s)^2 L}.
$$
\n
$$
(4.51)
$$

We first estimate the "bulk random walk term"  $C_L^{Q,\ell^q}$  which involves  $\hat{Q}_L^{I,J}$ , see [\(4.25\)](#page-15-5).

<span id="page-19-8"></span>**Proposition 4.23 (Bulk random walk term).** *For any*  $h \ge 2$ ,  $t \ge 0$ ,  $L \in \mathbb{N}$  *we have, for any*  $q \in (1, \infty)$  *and*  $\widehat{\mathscr{C}}$  *from* [\(4.51\)](#page-19-4)*,* 

<span id="page-19-6"></span>
$$
\mathsf{C}_{L}^{\mathsf{Q},\ell^{q}} := \max_{I,J \neq \mathbf{I},J \neq J} \|\mathcal{W}_{t}\,\widehat{\mathsf{Q}}_{L}^{I,J}\frac{1}{\mathcal{W}_{t}}\|_{\ell^{q}\to\ell^{q}} \leq h!\,\widehat{\mathscr{C}}^{h}\,q\,\frac{q}{q-1} \,. \tag{4.52}
$$

*Moreover, for*  $s \geq 0$  *and*  $\widehat{\mathscr{C}}$  *from* [\(4.51\)](#page-19-4)*,* 

<span id="page-20-4"></span>
$$
\max_{I,J \text{ pairs}, I \neq J} \|\frac{\mathcal{W}_t}{\mathcal{V}_s^J} \widehat{\mathsf{Q}}_L^{I,J} \frac{1}{\mathcal{W}_t \mathcal{V}_s^I} \|_{\ell^q \to \ell^q} \leq h! \widehat{\mathscr{C}}^h q \frac{q}{q-1} \,. \tag{4.53}
$$

(note that the weights  $V_s^J$ ,  $V_s^I$  appear in the denominator on both sides).

We next focus on the quantity  $C_{L,\lambda,\beta}^{U,\ell^q}$  in [\(4.25\)](#page-15-5), which depends on the operator  $|\hat{\mathbf{U}}|_{L,\lambda,\beta}^I$ , see [\(4.8\)](#page-12-1) and [\(4.14\)](#page-14-7). Recalling  $R_N$  from [\(1.6\)](#page-1-3) and  $q_n(x)$  from [\(3.3\)](#page-6-5), we define

<span id="page-20-7"></span>
$$
R_N^{(\lambda)} := \sum_{n=1}^N e^{-\lambda n} q_{2n}(0), \qquad (4.54)
$$

which reduces to  $R_N$  for  $\lambda = 0$ . In the next result we are going to assume that  $|\mathbb{E}[\xi_\beta^I]| \leq \sigma_\beta^2$ for any partition  $I \neq *$ , which holds for  $\beta > 0$  small enough (see Proposition [4.13\)](#page-16-5).

<span id="page-20-6"></span> $\text{Proposition 4.24 (Bulk interacting term).}$  *Let*  $\beta > 0$  *satisfy*  $\max_{I \neq *} |\mathbb{E}[\xi_{\beta}^{I}]| \leq \sigma_{\beta}^{2}$ . *For any*  $h \ge 2$ ,  $t \ge 0$ ,  $L \in \mathbb{N}$ ,  $\lambda \ge 0$  such that  $\sigma_\beta^2 R_L^{(\lambda)} < 1$  we have, for any  $q \in (1, \infty)$  and  $\check{\mathscr{C}}$  from [\(4.51\)](#page-19-4).

$$
\mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^q} := \max_{I \neq \ast} \, \|\, \mathcal{W}_t \, |\, \widehat{\mathsf{U}}|_{L,\lambda,\beta}^I \, \frac{1}{\mathcal{W}_t} \, \|\,_{\ell^q \to \ell^q} \leq 1 + \widecheck{\mathscr{C}}^h \, \frac{\sigma_\beta^2 \, R_L^{(\lambda)}}{1 - \sigma_\beta^2 \, R_L^{(\lambda)}} \,. \tag{4.55}
$$

<span id="page-20-5"></span><span id="page-20-3"></span>*Moreover, for any*  $s \geq 0$  *we have, for*  $a \in \{+1, -1\}$  and  $\check{\mathscr{C}}$  from [\(4.51\)](#page-19-4),

$$
\max_{\substack{J \,\text{pair} \\ I \neq *}} \| (\mathcal{V}_s^J)^a \,\mathcal{W}_t \, |\hat{\mathsf{U}}|_{L,\lambda,\beta}^I \, \frac{1}{\mathcal{W}_t \, (\mathcal{V}_s^J)^a} \, \|_{\ell^q \to \ell^q} \leq 1 + \widecheck{\mathscr{C}}^h \, \frac{\sigma_\beta^2 \, R_L^{(\lambda)}}{1 - \sigma_\beta^2 \, R_L^{(\lambda)}} \,. \tag{4.56}
$$

## **5. Proof of Proposition [2.2](#page-6-1)**

<span id="page-20-0"></span>In this section we prove Proposition [2.2.](#page-6-1) The key difficulty is that our goal [\(2.9\)](#page-6-2) involves *the (optimal)*  $1/M^2$  *dependence* on the width of the time interval  $(\frac{i-1}{M}N, \frac{i}{M}N]$  (recall the definition [\(3.7\)](#page-7-2) of the random variable  $X_{N,M}^{(i)}$ ). This requires sharp ad hoc estimates.

**5.1. Setup.** By formula [\(4.4\)](#page-11-4) from Remark [4.1,](#page-11-5) for  $l = 1, \ldots, M$  we can write

$$
\mathbb{E}\left[\left(X_{N,M}^{(l)}\right)^4\right] = \frac{\vartheta_N^2}{N^4} \mathcal{M}_{L,\beta}^4(f,g) \tag{5.1}
$$

where  $L, \beta, f, g$  are given as follows:

<span id="page-20-2"></span>
$$
L = \frac{N}{M}, \qquad \beta = \beta_N \text{ in (1.11)}, \qquad f(\cdot) = q_{\frac{l-1}{M}N}^{\varphi_N}(\cdot) \text{ in (1.5)-(3.5)}, \qquad g(\cdot) \equiv 1. \tag{5.2}
$$

We can bound  $\mathcal{M}_{\frac{N}{M},\beta_N}^4(f,g)$  exploiting [\(4.17\)](#page-14-4) for  $h = 4$  and  $\lambda = 0$ , which yields

<span id="page-20-1"></span>
$$
\mathbb{E}\left[\left(X_{N,M}^{(l)}\right)^4\right] \leq \frac{\vartheta_N^2}{N^4} \left(\Xi(1) + \Xi(2) + \sum_{r=3}^{\infty} \Xi(r)\right),\tag{5.3}
$$

where  $\Xi(r)$  is defined in [\(4.18\)](#page-14-3). We show that the only non-negligible term in [\(5.3\)](#page-20-1) is  $\Xi(2)$ : more precisely, we will prove that there is  $C < \infty$  such that, for any  $M \in \mathbb{N}$ ,

$$
\limsup_{N \to \infty} \frac{\vartheta_N^2}{N^4} \,\Xi(2) \leqslant \frac{C}{M^2},\tag{5.4}
$$

<span id="page-21-1"></span>while

<span id="page-21-0"></span>
$$
\lim_{N \to \infty} \frac{\vartheta_N^2}{N^4} \Xi(1) = 0 \quad \text{and} \quad \lim_{N \to \infty} \frac{\vartheta_N^2}{N^4} \sum_{r=3}^{\infty} \Xi(r) = 0. \tag{5.5}
$$

This will complete the proof of Proposition [2.2.](#page-6-1)

We estimate  $\Xi(r)$  exploiting the bound [\(4.23\)](#page-15-2)-[\(4.24\)](#page-15-3) with the choice

$$
p=q=2\,.
$$

We need to control the *boundary terms* and the *bulk terms*, see [\(4.26\)](#page-15-4). We recall that the weights  $W_t$  and  $V_s^I$  are defined in [\(4.27\)](#page-16-4) and [\(4.42\)](#page-18-0), and we fix

$$
t = \frac{1}{\sqrt{N}}, \qquad s = \frac{1}{\sqrt{L}} = \sqrt{\frac{M}{N}}.
$$
\n
$$
(5.6)
$$

*.*

For notational lightness, we write  $a \leq b$  whenever  $a \leq C b$  for some constant  $0 < C < \infty$ . We also denote by  $\|\varphi\|_p := \left(\int_{\mathbb{R}^2} \varphi(x)^p dx\right)^{1/p}$  the usual  $L^p$  norm of a function  $\varphi : \mathbb{R}^2 \to \mathbb{R}$ .

**5.2. Boundary terms.** We estimate the *left boundary term*  $\|\hat{\mathbf{q}}_L^{[f],I} \mathbf{1}_{\mathcal{W}_t}\|_{\ell^2}$  applying [\(4.40\)](#page-18-3). We recall from [\(5.2\)](#page-20-2) that  $f(\cdot) = q^{\varphi_N}_{\frac{l-1}{M}N}(\cdot)$  for  $1 \leq l \leq M$ . Let us estimate  $\|\frac{f}{w}\|$  $\frac{f}{w_t}$ <sup> $\parallel$ </sup> $\ell^{\infty}$  and  $\parallel \frac{f}{w}$  $\frac{J}{w_t}\big\|_{\ell^2},$ starting from the former. By [\(4.34\)](#page-17-7), for  $l \leq M$  and  $t = \frac{1}{\sqrt{l}}$  $\frac{1}{N}$  we have

$$
\left\|\frac{f}{w_t}\right\|_{\ell^\infty}\leqslant \mathsf{c}\,\mathrm{e}^{2\mathsf{c}\,t^2\frac{l-1}{M}N}\left\|\frac{\varphi_N}{w_t}\right\|_{\ell^\infty}\leqslant \mathsf{c}\,\mathrm{e}^{2\mathsf{c}}\left\|\frac{\varphi_N}{w_t}\right\|_{\ell^\infty}
$$

Since  $\varphi$  is compactly supported, say in a ball  $B(0, R)$ , we have that  $\varphi_N$  is supported in  $B(0, R\sqrt{N} + \sqrt{2}) \subseteq B(0, 2R\sqrt{N})$ , see [\(1.5\)](#page-0-4). By  $w_t(x) = e^{-t|x|}$ , we then obtain

$$
\left\|\frac{\varphi_N}{w_t}\right\|_{\ell^\infty} \leqslant e^{t 2R\sqrt{N}} \left\|\varphi_N\right\|_{\ell^\infty} \leqslant e^{2R} \left\|\varphi\right\|_{\infty} \leqslant 1, \qquad \text{hence} \qquad \left\|\frac{f}{w_t}\right\|_{\ell^\infty} \leqslant 1, \tag{5.7}
$$

<span id="page-21-2"></span>because  $\|\varphi_N\|_{\ell^\infty} \leq \|\varphi\|_{\infty}$ . We next estimate  $\|\frac{f}{w}\|_{\ell^\infty}$  $\frac{J}{w_t}$   $\|_{\ell^2}$ . By a Riemann sum approximation, we see from [\(1.5\)](#page-0-4) that  $\|\varphi_N\|_{\ell^2} \lesssim$  $\sqrt{N} \|\varphi\|_2$ , hence by [\(4.34\)](#page-17-7) we obtain

<span id="page-21-3"></span>
$$
\left\| \frac{f}{w_t} \right\|_{\ell^2} \leqslant c e^{2c} \left\| \frac{\varphi_N}{w_t} \right\|_{\ell^2} \leqslant c e^{2c} e^{2R} \left\| \varphi_N \right\|_{\ell^2} \lesssim \sqrt{N} \,. \tag{5.8}
$$

We can finally apply the estimate [\(4.40\)](#page-18-3) for  $p = 2$  and  $h = 4$  to get, since  $L = \frac{N}{M}$ ,

<span id="page-21-4"></span>
$$
\max_{I \neq *} \left\| \widehat{\mathbf{q}}_L^{|f|, I} \frac{1}{\mathcal{W}_t} \right\|_{\ell^2} \leqslant 4 \, \mathscr{C}^h \, L \left\| \frac{f}{w_t} \right\|_{\ell^\infty} \left\| \frac{f}{w_t} \right\|_{\ell^2}^3 \leqslant \frac{N^{\frac{5}{2}}}{M} \,. \tag{5.9}
$$

We now estimate the *right boundary term*  $\|\overline{\mathsf{q}}_L^{\{g\},J}\mathcal{W}_t\|_{\ell^2}$ : applying [\(4.49\)](#page-19-5) for  $q=2$  and  $h = 4$ , since  $g \equiv 1$  and  $t = \frac{1}{\sqrt{l}}$  $\frac{1}{N}$ , we obtain

<span id="page-21-5"></span>
$$
\max_{J \neq *} \|\overline{\mathsf{q}}_L^{|g|, J} \mathcal{W}_t\|_{\ell^2} \leq (12\overline{\mathscr{C}})^4 \frac{\|g\|_{\infty}^4}{t^3} \lesssim N^{\frac{3}{2}}.
$$
\n(5.10)

Overall, we have shown that

<span id="page-22-1"></span>
$$
\left(\max_{I\neq *}\|\widehat{\mathsf{q}}_L^{|f|,I}\frac{1}{W}\|_{\ell^p}\right)\left(\max_{J\neq *}\|\mathcal{W}\overline{\mathsf{q}}_L^{|g|,J}\|_{\ell^q}\right) \lesssim \frac{N^4}{M}.
$$
\n(5.11)

In view of [\(4.23\)](#page-15-2), it remains to estimate  $\Xi^{\text{bulk}}(r)$  defined in [\(4.24\)](#page-15-3).

**5.3. Bulk terms.** We next estimate the *bulk terms*, see [\(4.25\)](#page-15-5). For the first term  $C_L^{Q,\ell^2}$  $L^{\mathcal{U},\ell}$ , we apply directly the estimate  $(4.52)$  with  $q = 2$  and  $h = 4$  to get

<span id="page-22-3"></span>
$$
\mathsf{C}_{L}^{\mathsf{Q},\ell^{2}} = \max_{I,J \neq \ast, I \neq J} \|\mathcal{W}_{t}\,\widehat{\mathsf{Q}}_{L}^{I,J}\frac{1}{\mathcal{W}_{t}}\|_{\ell^{2}\to\ell^{2}} \leq 4!\,\widehat{\mathscr{C}}^{4}\,4 \lesssim 1. \tag{5.12}
$$

(Also note that  $C_L^{Q,\ell^2} \geq \mathcal{W}_t(0) \,\hat{Q}_L^{I,J}(0,0) \frac{1}{\mathcal{W}_t(0)} \geq Q_2(0,0) \geq 1.$ )

We then focus on the second term  $C_{L,\lambda,\beta}^{U,\ell^2}$ . For  $L = \frac{N}{M} \le N$  and  $\beta = \beta_N$  as in [\(1.11\)](#page-1-5)

$$
1 - \sigma_{\beta_N}^2 R_L \ge 1 - \sigma_{\beta_N}^2 R_N \ge \frac{\vartheta_N}{\log N} > 0, \quad \text{in particular } \sigma_{\beta_N}^2 R_L < 1. \tag{5.13}
$$

Then by [\(4.55\)](#page-20-3) with  $\lambda = 0$  (so that  $R_N^{(\lambda)} = R_N$ ) we obtain, recalling that  $\vartheta_N \ll \log N$ ,

<span id="page-22-2"></span>
$$
\mathsf{C}_{L,\lambda,\beta}^{\mathsf{U},\ell^2} := \max_{I \neq \ast} \|\mathcal{W}_t \|\widehat{\mathsf{U}}\|_{L,\lambda,\beta}^I \frac{1}{\mathcal{W}_t} \|\mathsf{L}_{\ell^2 \to \ell^2} \leq 1 + \widecheck{\mathscr{C}}^4 \frac{\sigma_{\beta_N}^2 R_L}{1 - \sigma_{\beta_N}^2 R_L} \lesssim \frac{\log N}{\vartheta_N}.
$$
 (5.14)

Since  $\beta_N \to 0$ , we can apply [\(4.29\)](#page-16-1) which ensures that  $|\mathbb{E}[\xi_{\beta_N}^I]| \leq \sigma_{\beta_N}^2 \leq \frac{1}{R_I}$  $\frac{1}{R_N} = O(\frac{1}{\log n})$  $\frac{1}{\log N}$ for any  $I \neq *$  and *N* large, therefore there is  $C < \infty$  such that

<span id="page-22-0"></span>
$$
\left(\max_{I \neq *} \left| \mathbb{E}[\xi_{\beta_N}^I] \right| \right) \mathsf{C}_{L}^{Q,\ell^2} \mathsf{C}_{L,\lambda,\beta}^{U,\ell^2} \leq \frac{C}{\vartheta_N}.
$$
\n(5.15)

**5.4. Terms**  $r \geq 3$ . We are ready to prove the second relation in [\(5.5\)](#page-21-0), which shows that the terms  $r \ge 3$  give a negligible contributions to  $\mathbb{E}[(X_{N,M}^{(l)})^4]$ , recall [\(5.3\)](#page-20-1).

Let us denote by  $c(h) \in \mathbb{N}$  the number of partitions  $I \vdash \{1, \ldots, h\}$  with  $I \neq *$ . Then by [\(4.24\)](#page-15-3) we have the geometric bound

$$
\Xi^{\text{bulk}}(r) \leq (C_L^{Q,\ell^2})^{-1} \left\{ c(h) \left( \max_{I \neq *} |\mathbb{E}[\xi_{\beta_N}^I]| \right) C_L^{Q,\ell^2} C_{L,\lambda,\beta}^{U,\ell^2} \right\}^r,
$$

and note that the term in brackets is  $\lt \frac{1}{2}$  $\frac{1}{2}$  for large *N*, by [\(5.15\)](#page-22-0) and  $\vartheta_N \to \infty$ , therefore

$$
\sum_{r=3}^{\infty} \Xi^{\text{bulk}}(r) \lesssim \Xi^{\text{bulk}}(3) \lesssim \frac{1}{\vartheta_N^3}.
$$

Applying  $(4.23)$  and  $(5.11)$ , we then obtain the second relation in  $(5.5)$ :

$$
\frac{\vartheta_N^2}{N^4} \sum_{r=3}^{\infty} \Xi(r) \leq \frac{\vartheta_N^2}{M} \sum_{r=3}^{\infty} \Xi^{\text{bulk}}(r) \lesssim \frac{1}{M \vartheta_N} \xrightarrow[N \to \infty]{} 0.
$$

**Remark 5.1.** *The same arguments can be applied to show that in the quasi-critical regime, the contribution of the terms*  $r > \left\lfloor \frac{h}{2} \right\rfloor$  $\frac{h}{2}$  for the *h*-th moment of  $X_{N,M}^{(l)}$  is negligible as  $N \to \infty$ .

**5.5. Term**  $r = 1$ . We now prove the first relation in [\(5.5\)](#page-21-0). A partition  $I \vdash \{1, 2, 3, 4\}$ with full support is either a double pair  $I = \{\{a, b\}, \{c, d\}\}\$  or the quadruple  $I = \{1, 2, 3, 4\},\$ hence  $\mathbb{E}[\xi_{\beta_N}^I] \lesssim \sigma_{\beta_N}^4$  for large *N*, by [\(4.9\)](#page-12-2) and [\(3.2\)](#page-6-7) (see also Proposition [4.13\)](#page-16-5). Then, by  $(4.24),$  $(4.24),$ 

$$
\Xi^{\text{bulk}}(1) \, = \sum_{\substack{I \vdash \{1,\ldots,h\} \\ \text{with full support}}} \big| \mathbb{E}[\xi_{\beta_N}^I] \big| \, \mathsf{C}^{ \mathsf{U}, \ell^q}_{L,\lambda,\beta} \, \lesssim \, \sigma_{\beta_N}^4 \, \mathsf{C}^{ \mathsf{U}, \ell^2}_{L,\lambda,\beta} \, \lesssim \, \frac{1}{(\log N) \, \vartheta_N}
$$

*,*

*,*

where we applied [\(5.14\)](#page-22-2) and  $\sigma_{\beta_N}^2 \leq \frac{1}{R_N} = O(\frac{1}{\log n})$  $\frac{1}{\log N}$ ). Applying [\(4.23\)](#page-15-2) and [\(5.11\)](#page-22-1), and recalling that  $\vartheta_N \ll \log N$ , we obtain the first relation in [\(5.5\)](#page-21-0):

$$
\frac{\vartheta_N^2}{N^4} \,\Xi(1) \leqslant \frac{\vartheta_N^2}{M} \,\Xi^{\text{bulk}}(1) \lesssim \frac{\vartheta_N}{M\,\log N} \xrightarrow[N \to \infty]{} 0\,.
$$

**5.6. Term**  $r = 2$ . We finally prove  $(5.4)$ , which completes the proof of Proposition [2.2.](#page-6-1) We recall that  $\Xi(2)$ , defined by [\(4.18\)](#page-14-3), is a sum over two partitions  $I_1, I_2 \vdash \{1, ..., h\}$  with  $I_1 \neq *, I_2 \neq *$  and  $I_1 \neq I_2$ . We then split  $\Xi(2) = \Xi_{\text{pairs}}(2) + \Xi_{\text{others}}(2)$  where:

- $\Xi_{\text{pairs}}(2)$  is the contribution to [\(4.18\)](#page-14-3) when both  $I_1, I_2$  are pairs;
- $\Xi_{\text{others}}(2)$  is the complementary contribution when *I*<sub>1</sub> and/or *I*<sub>2</sub> is not a pair.

We first focus on  $\Xi_{\text{others}}(2)$  and on the corresponding quantity  $\Xi_{\text{others}}^{\text{bulk}}(2)$ , see [\(4.24\)](#page-15-3). If either *I*<sub>1</sub> or *I*<sub>2</sub> is not a pair, by Proposition [4.13](#page-16-5) we can bound  $\left| \mathbb{E}[\xi_{\beta_1}^{I_1}] \right|$  $\frac{I_1}{\beta_N}$ ]  $\mathbb{E}[\xi_{\beta_I}^{I_2}]$  $\left[\frac{I_2}{\beta_N}\right]$ |  $\leq \sigma_{\beta_N}^5$ , hence

$$
\Xi_{\rm others}^{\rm bulk}(2) \lesssim \sigma_{\beta_N}^5\, \mathsf{C}_{L}^{\mathsf{Q}, \ell^q}\, \big(\mathsf{C}_{L, \lambda, \beta}^{\mathsf{U}, \ell^q}\big)^2 \lesssim \frac{1}{\big(\!\log N\big)^{5/2}} \left(\frac{\log N}{\vartheta_N}\right)^2 \lesssim \frac{1}{\vartheta_N^2\, \sqrt{\log N}}
$$

where we applied [\(5.12\)](#page-22-3), [\(5.14\)](#page-22-2) and  $\sigma_{\beta_N}^2 \leq \frac{1}{R_N} = O(\frac{1}{\log n})$  $\frac{1}{\log N}$ ). Then, by [\(4.23\)](#page-15-2) and [\(5.11\)](#page-22-1),

$$
\frac{\vartheta_N^2}{N^4} \,\Xi_{\text{others}}(2) \leqslant \frac{\vartheta_N^2}{M} \,\Xi_{\text{others}}^{\text{bulk}}(2) \lesssim \frac{1}{M \sqrt{\log N}} \xrightarrow[N \to \infty]{} 0\,,
$$

which shows that the contribution of  $\Xi_{\text{others}}(2)$  to [\(5.4\)](#page-21-1) is negligible.

It only remains to focus on  $\Xi_{\text{pairs}}(2)$ : since  $\mathbb{E}[\xi_{\beta}^{I}] = \sigma_{\beta}^{2}$  when *I* is a pair, we can write

$$
\Xi_{\text{pairs}}(2) := \sum_{\substack{I_1 \neq I_2 \vdash \{1,\ldots,h\} \\ \text{pairs with full support}}} \sigma_\beta^4 \left\langle \widehat{\mathsf{q}}_{L}^{[f],I_1}\,,\, |\widehat{\mathsf{U}}|_{L,\lambda,\beta}^{I_1}\, \, \widehat{\mathsf{Q}}_{L}^{I_1,I_2}\,\, |\widehat{\mathsf{U}}|_{L,\lambda,\beta}^{I_2}\, \, \overline{\mathsf{q}}_{L}^{[g],I_r} \right\rangle.
$$

Besides inserting  $\frac{1}{\mathcal{W}_t} \mathcal{W}_t$  as above, we also insert  $\mathcal{V}_s^{\mathcal{I}_2} \frac{1}{\mathcal{V}_s^{\mathcal{I}}}$  $\frac{1}{\mathcal{V}_s^{I_2}}$  on the left of  $\hat{\mathsf{Q}}_L^{I_1,I_2}$ while we insert  $\frac{1}{\mathcal{V}_s^{I_1}} \mathcal{V}_s^{I_1}$  on the right of  $\hat{\mathsf{Q}}_L^{I_1,I_2}$  and  $|\hat{\mathsf{U}}|_{L,\lambda}^{I_2}$  $L^{I_1,I_2}$  and  $|\hat{\mathsf{U}}|_{L,\lambda,\beta}^{I_1}$ ,  $\mathcal{V}_{s}^{I_1}$  on the right of  $\hat{\mathsf{Q}}_L^{I_1,I_2}$  $L^{I_1,I_2}$  and  $|\hat{\mathsf{U}}|_{L,\lambda,\beta}^{I_2}$  (recall [\(4.42\)](#page-18-0)): we thus obtain

<span id="page-23-0"></span>
$$
\Xi_{\text{pairs}}(2) \leq \sum_{\substack{I_1 \neq I_2 \vdash \{1, \ldots, h\} \\ \text{pairs with full support}}} \sigma_{\beta}^4 \left\| \widehat{\mathbf{q}}_L^{[f], I_1} \frac{\mathcal{V}_{s^2}^{I_2}}{\mathcal{W}_t} \right\|_{\ell^p} \left\| \frac{\mathcal{W}_t}{\mathcal{V}_s^{I_2}} \right\| \widehat{\mathbf{U}}_{L,\lambda,\beta}^{I_1} \frac{\mathcal{V}_{s^2}^{I_2}}{\mathcal{W}_t} \left\|_{\ell^q \to \ell^q} \right\| \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right)^{I_2} \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right)^{I_3} \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right)^{I_4} \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right)^{I_5} \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right)^{I_6} \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right)^{I_7} \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right)^{I_8} \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right)^{I_9} \left( \frac{\mathcal{W}_t}{\mathcal{W}_t} \right
$$

It remains to estimate these norms. Let us recall that  $h = 4$ ,  $p = q = 2$  and  $t = \frac{1}{\sqrt{l}}$  $\frac{1}{N}$ ,  $s = \frac{1}{\sqrt{2}}$  $\frac{L}{L}$ , where  $L = \frac{M}{N}$  $\frac{M}{N}$ . We start with the boundary terms:

• applying the estimate  $(4.45)$ , in view of  $(5.7)-(5.8)$  $(5.7)-(5.8)$ , we improve the estimate  $(5.9)$ :

$$
\max_{\substack{I,J \text{ pairs} \\ I \neq J}} \left\| \hat{\mathsf{q}}_{L}^{[f],I} \frac{\mathcal{V}_s^J}{\mathcal{W}_t} \right\|_{\ell^2} \leq 6 \, \mathscr{C}^4 \, \frac{L}{s} \left\| \frac{f}{w_t} \right\|_{\ell^\infty}^2 \left\| \frac{f}{w_t} \right\|_{\ell^2}^2 \lesssim L^{\frac{3}{2}} \, N \lesssim \frac{N^{\frac{5}{2}}}{M^{\frac{3}{2}}};\tag{5.17}
$$

• applying the estimate [\(4.50\)](#page-19-7), since  $g \equiv 1$ , we improve the estimate [\(5.10\)](#page-21-5):

$$
\max_{\substack{I,J \text{ pairs} \\ I \neq J}} \|\mathcal{W}_t \mathcal{V}_s^I \overline{\mathsf{q}}_L^{|g|,J}\|_{\ell^2} \leq (12\overline{\mathscr{C}})^4 \frac{\|g\|_{\ell^\infty}^4}{st^2} \lesssim \sqrt{L} N \lesssim \frac{N^{\frac{3}{2}}}{\sqrt{M}}.\tag{5.18}
$$

Overall, the product of the two boundary terms is  $\lesssim \frac{N^4}{M^2}$  $\frac{N}{M^2}$ , which improves on the previous estimates by an essential factor  $\frac{1}{M}$ , thanks to the use of the restricted weight  $\mathcal{V}_s^I$ .

We next estimate the bulk terms:

• applying [\(4.53\)](#page-20-4) with  $p = q = 2$  and  $h = 4$ , we obtain an analogue of [\(5.12\)](#page-22-3):

$$
\max_{\substack{I,J \text{ pairs} \\ I \neq J}} \|\frac{\mathcal{W}_t}{\mathcal{V}_s^J} \widehat{\mathsf{Q}}_L^{I,J} \frac{1}{\mathcal{W}_t \mathcal{V}_s^I} \|_{\ell^2 \to \ell^2} \leq 4! \widehat{\mathscr{C}}^4 \, 4 \lesssim 1 ;\tag{5.19}
$$

*,*

• applying [\(4.56\)](#page-20-5) for both  $a = +1$  and  $a = -1$ , we obtain an analogue of [\(5.14\)](#page-22-2):

$$
\max_{I,J \text{ pairs}} \left\| (\mathcal{V}_s^J)^a \mathcal{W}_t \left| \hat{\mathsf{U}} \right|_{L,\lambda,\beta}^I \frac{1}{\mathcal{W}_t (\mathcal{V}_s^J)^a} \left\|_{\ell^2 \to \ell^2} \leq 1 + \widetilde{\check{\mathscr{C}}}^4 \frac{\sigma_{\beta_N}^2 R_L}{1 - \sigma_{\beta_N}^2 R_L} \lesssim \frac{\log N}{\vartheta_N} . \right. \tag{5.20}
$$

Plugging the previous estimates into [\(5.16\)](#page-23-0), since  $\sigma_{\beta_N}^2 \leq \frac{1}{R_l}$  $\frac{1}{R_N} = O(\frac{1}{\log n})$  $\frac{1}{\log N}$ , we finally obtain

$$
\Xi_{\rm pairs}(2) \lesssim \frac{1}{(\log N)^2} \, \frac{N^{\frac{5}{2}}}{M^{\frac{3}{2}}} \left(\frac{\log N}{\vartheta_N}\right)^2 \frac{N^{\frac{3}{2}}}{\sqrt{M}} = \frac{N^4}{M^2 \, \vartheta_N^2}
$$

<span id="page-24-0"></span>which completes the proof of  $(5.4)$ , hence of Proposition [2.2.](#page-6-1)

## **Appendix A. Some technical proofs**

We give the proof of Theorem [4.5.](#page-13-2) We recall that the averaged partition function  $\mathcal{Z}_{L,\beta}^{\omega}(f,g)$  is defined in [\(4.1\)](#page-11-6)-[\(4.2\)](#page-11-2). In analogy with [\(3.4\)](#page-6-3) and [\(3.6\)](#page-7-1), by (4.1)-(4.2) we can write

$$
\mathcal{Z}_{L,\beta}^{\omega}(f,g) - \mathbb{E}[\mathcal{Z}_{L,\beta}^{\omega}(f,g)] = \sum_{k=1}^{\infty} \sum_{\substack{0 < n_1 < \ldots < n_k < L \\ x_1, \ldots, x_k \in \mathbb{Z}^2}} q_{n_1}^f(x_1) \xi_{\beta}(n_1, x_1) \times \left\{ \prod_{j=2}^k q_{n_j - n_{j-1}}(x_j - x_{j-1}) \xi_{\beta}(n_j, x_j) \right\} q_{L-n_k}^g(x_k), \tag{A.1}
$$

where we recall the random walk kernels  $(3.3)$  and  $(3.5)$ . Recalling  $(4.3)$ , we obtain

<span id="page-24-1"></span>
$$
\mathcal{M}_{L,\beta}^{h}(f,g) = \mathbb{E}\Bigg[ \Bigg( \sum_{k=1}^{\infty} \sum_{\substack{0 < n_1 < \dots < n_k < L \\ x_1, \dots, x_k \in \mathbb{Z}^2}} q_{n_1}^f(x_1) \xi_{\beta}(n_1, x_1) \times \\ \times \Bigg\{ \prod_{j=2}^k q_{n_j - n_{j-1}}(x_j - x_{j-1}) \xi_{\beta}(n_j, x_j) \Bigg\} q_{L-n_k}^g(x_k) \Bigg)^h \Bigg]. \tag{A.2}
$$

When we expand the *h*-th power, we obtain a sum over *h* families of space-time points  $A_i := \{(n_1^i, x_1^i), \ldots, (n_{k_i}^i, x_{k_i}^i)\}\$  for  $i = 1, \ldots, h$ . These points must match at least in pairs, i.e. any point  $(n_{\ell}^i, x_{\ell}^i)$  in any family  $A_i$  must coincide with at least another point  $(n_m^j, x_m^j)$ in a different family  $A_j$  for  $j \neq i$ , otherwise the expectation vanishes (since  $\xi_\beta(n, x)$ ) are independent and centered). In order to handle this constraint, following [\[CSZ23,](#page-35-1) Theorem 6.1], we rewrite [\(A.2\)](#page-24-1) by first *summing over the set of all space-time points*

$$
A := \bigcup_{i=1}^{h} A_i = \bigcup_{i=1}^{h} \{ (n_1^i, x_1^i), \dots, (n_{k_i}^i, x_{k_i}^i) \} \subseteq \mathbb{N} \times \mathbb{Z}^2
$$

and then specifying *which families* each point  $(n, x) \in A$  belongs to.

Let us fix the *time coordinates*  $n_1 < \ldots < n_r$  of the points in *A*. For each such time  $n \in \{n_1, \ldots, n_r\}$ , we have  $(n, x) \in A$  for one or more  $x \in \mathbb{Z}^2$  (there are at most  $h/2$  such *x*, by the matching constraint described above). We then make the following observations:

• if  $(n, x) = (n_j^i, x_j^i)$  belongs to the family  $A_i$ , then we have in [\(A.2\)](#page-24-1) the product of a random walk kernel "entering"  $(n, x)$  and another one "exiting"  $(n, x)$ :

$$
q_{n-n^i_{j-1}}(x-x^i_{j-1})\cdot q_{n^i_{j+1}-n}(x^i_{j+1}-x)\,;
$$

• if  $(n, x)$  does *not* belong to the family  $A_i$ , then we have in  $(A.2)$  a random walk kernel "jumping over time n", say  $q_{n_j^i - n_{j-1}^i} (x_j - x_{j-1})$  with  $n_{j-1}^i < n < n_j^i$ : we can split this kernel at time *n* by Chapman-Kolmogorov, writing

<span id="page-25-0"></span>
$$
q_{n_j^i - n_{j-1}^i}(x_j^i - x_{j-1}^i) = \sum_{z \in \mathbb{Z}^2} q_{n - n_{j-1}^i}(z - x_{j-1}^i) \cdot q_{n_j^i - n}(x_j^i - z).
$$
 (A.3)

Then, to each time  $n \in \{n_1, \ldots, n_r\}$ , we can associate a vector  $\mathbf{y} = (y^1, \ldots, y^h) \in (\mathbb{Z}^2)^h$ with *h* space coordinates, where  $y^i = x$  if the family  $A^i$  contains  $(n, x)$  and  $y^i = z$  from [\(A.3\)](#page-25-0) otherwise. The constraint that a point  $(n, x) \in A$  belongs to two families  $A^i$  and  $A^{i'}$ means that the corresponding coordinates of the vector **y** must coincide:  $y^i = y^{i'}$ . In order to specify which families  $A^i$  share the same points, we assign a *partition*  $I \vdash \{1, \ldots, h\}$  to each time  $n \in \{n_1, \ldots, n_r\}$  and we require that  $\mathbf{y} \sim I$ , see [\(4.5\)](#page-12-4).

We are now ready to provide a convenient rewriting of [\(A.2\)](#page-24-1) by first summing over the number  $r \geq 1$  and the time coordinates  $n_1 < \ldots < n_r$ , then on the corresponding space coordinates  $\mathbf{y}_1, \ldots, \mathbf{y}_r$  and partitions  $I_1, \ldots, I_r \vdash \{1, \ldots, h\}$  with  $\mathbf{y}_i \sim I_i$ . Recalling the definitions of  $Q_n^{I,J}$  and  $q_n^{f,J}$  from [\(4.7\)](#page-12-0), we can rewrite [\(A.2\)](#page-24-1) as follows:

<span id="page-25-1"></span>
$$
\mathcal{M}_{L,\beta}^{h}(f,g) = \sum_{r=1}^{\infty} \sum_{\substack{0 < n_1 < \cdots < n_r < L \\ \mathbf{y}_1, \ldots, \mathbf{y}_r \in (\mathbb{Z}^2)^h}} \sum_{\substack{I_1, \ldots, I_r \vdash \{1, \ldots, h\} \\ \text{with full support} \\ \text{and } I_i \neq * \forall i}} \mathsf{q}_{n_1}^{f, I_1}(\mathbf{y}_1) \mathbb{E}[\xi_{\beta}^{I_1}] \times \left(\mathbf{A}.4\right)
$$
\n
$$
\times \left\{ \prod_{i=2}^r \mathsf{Q}_{n_i - n_{i-1}}^{I_{i-1}, I_i}(\mathbf{y}_{i-1}, \mathbf{y}_i) \mathbb{E}[\xi_{\beta}^{I_i}] \right\} \mathsf{q}_{L-n_r}^{g, I_r}(\mathbf{y}_r).
$$
\n(A.4)

Finally, formula [\(4.12\)](#page-13-3) follows from [\(A.4\)](#page-25-1) grouping together stretches of *consecutive repeated partitions*, i.e. when  $I_i = J$  for consecutive indexes *i*. The kernel  $\mathsf{U}_{m-n,\beta}^J(\mathbf{z},\mathbf{x})$ from [\(4.8\)](#page-12-1) does exactly this job, which leads to [\(4.12\)](#page-13-3).

**Remark A.1.** *Formula* [\(4.12\)](#page-13-3) *still contains the product of*  $\mathbb{E}[\xi_{\beta}^{I_i}]$  *because these factors from* [\(A.4\)](#page-25-1) *are only partially absorbed in*  $\bigcup_{m=n,\beta}^{J} (\mathbf{z},\mathbf{x})$ : *indeed, in* [\(4.8\)](#page-12-1) *we have*  $k+1$ *points*  $n_0 < n_1 < \ldots < n_k$ , but the factor  $\mathbb{E}[\xi_\beta^J]$  therein is only raised to the power *k*.

# **Appendix B. Random walk bounds**

<span id="page-26-0"></span>In this section we prove the random walk bounds from Lemmas [4.16,](#page-17-1) [4.17](#page-17-2) and [4.18.](#page-17-3) We also prove a heat kernel bound, see Lemma [B.1](#page-29-1) below.

**B.1. Proof of Lemma [4.16.](#page-17-1)** We prove each of the four bounds in [\(4.32\)](#page-17-8)-[\(4.33\)](#page-17-5) for a different constant c (it then suffices to take the maximal value).

The first bound in [\(4.32\)](#page-17-8) with  $c = c$  follows by [\(4.31\)](#page-17-4), thanks to the independence of the increments of the random walk. This directly implies the first bound in [\(4.33\)](#page-17-5): it suffices to estimate  $\sum_{x \in \mathbb{Z}^2} e^{t|x|} q_n(x) \le \sum_{x \in \mathbb{Z}^2} e^{2t|x^1|} q_n(x)$  (by  $|x| \le |x^1| + |x^2|$ , Cauchy-Scwharz and symmetry) and then  $e^{|z|} \leq e^z + e^{-z}$ , hence  $\sum_{x \in \mathbb{Z}^2} e^{t|x|} q_n(x) \leq 2 e^{2ct^2 n}$ .

To get the second bound in [\(4.33\)](#page-17-5), we fix  $\ell \leq n$  and write  $q_n(x) = \sum_{y \in \mathbb{Z}^2} q_\ell(y) q_{n-\ell}(x-y)$ by Chapman-Kolmogorov. We next decompose the sum in the two parts  $\langle y, x \rangle > \frac{1}{2}|x|^2$  and  $\langle y, x \rangle \leq \frac{1}{2}|x|^2$ : renaming *y* as  $x - y$  in the second part, we obtain

<span id="page-26-1"></span>
$$
q_n(x) \leq \sum_{y \in \mathbb{Z}^2 : \langle y, x \rangle \geq \frac{1}{2}|x|^2} \left\{ q_\ell(y) \, q_{n-\ell}(x-y) + q_{n-\ell}(y) \, q_\ell(x-y) \right\}.
$$
 (B.1)

We can bound  $q_k(x-y) \leq \sup_{z \in \mathbb{Z}^2} q_k(z) \leq \frac{c}{k}$  by the local limit theorem (any random walk satysfying Assumption [4.14](#page-16-3) is in  $L^2$  with zero mean). We next observe that  $\langle y, x \rangle \geq \frac{1}{2}|x|^2$ implies  $|x| \le 2|y|$  by Cauchy-Schwarz, therefore the first bound in [\(4.33\)](#page-17-5) yields

$$
\forall x \in \mathbb{Z}^2: \qquad e^{t|x|} q_n(x) \leqslant \mathsf{c} \sum_{y \in \mathbb{Z}^2} e^{2t|y|} \left\{ \frac{q_\ell(y)}{n-\ell} + \frac{q_{n-\ell}(y)}{\ell} \right\} \leqslant \frac{2\mathsf{c} e^{8\mathsf{c} t^2 n}}{\min\{n-\ell,\ell\}}.
$$

If we choose  $\ell = \left\lfloor \frac{n}{2} \right\rfloor$  $\frac{n}{2}$ , we obtain the second bound in [\(4.33\)](#page-17-5) renaming c.

It remains to prove the second bound in [\(4.32\)](#page-17-8). We first note that  $q_n(x)^2/q_{2n}(0) \leq c q_n(x)$ for some  $\mathbf{c} \in [1, \infty)$ , because  $q_n(x)^2 \leq \|q_n\|_{\ell^\infty} q_n(x)$  and  $\|q_n\|_{\ell^\infty} \leq c q_{2n}(0)$  by the local limit theorem. Since  $q_n(x) = q_n(-x)$ , we get

$$
\sum_{x \in \mathbb{Z}^2} e^{tx^a} \frac{q_n(x)^2}{q_{2n}(0)} - 1 = \sum_{x \in \mathbb{Z}^2} \left( \frac{e^{tx^a} + e^{-tx^a}}{2} - 1 \right) \frac{q_n(x)^2}{q_{2n}(0)} \le c \sum_{x \in \mathbb{Z}^2} \left( \frac{e^{tx^a} + e^{-tx^a}}{2} - 1 \right) q_n(x)
$$
  

$$
\le c \left( e^{ct^2 - n} - 1 \right) = c \sum_{k=1}^{\infty} \frac{1}{k!} \left( c \frac{t^2}{2} n \right)^k \le \sum_{k=1}^{\infty} \frac{1}{k!} \left( c^2 \frac{t^2}{2} n \right)^k = e^{ct^2 \frac{t^2}{2} n} - 1,
$$

which proves the second bound in [\(4.32\)](#page-17-8) if we rename  $c^2$  as c.

**B.2. Proof of Lemma [4.17.](#page-17-2)** For any  $y \in \mathbb{Z}^2$  and  $p \in [1, \infty]$  we can write, recalling [\(3.5\)](#page-6-6),

$$
\frac{q_n^f(y)}{w_t(y)} = q_n^f(y) e^{t|y|} \leq \sum_{z \in \mathbb{Z}^2} e^{t|z|} |f(z)| \left\{ e^{t|y-z|} q_n(y-z) \right\} \leq \left\| \frac{f}{w_t} \right\|_{\ell^p} \left\| \frac{q_n}{w_t} \right\|_{\ell^q},
$$

where  $q \in [1, \infty]$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Since  $\|\frac{q_n}{w_t}\|$  $\frac{q_n}{w_t}$  $\|_{\ell}^q$  $\frac{q}{\ell^q} \leqslant \left\| \frac{q_n}{w_t} \right\|_{\ell^\infty}^{q-1} \left\| \frac{q_n}{w_t} \right\|$  $\frac{q_n}{w_t}$ <sup> $\parallel$ </sup><sub> $\ell$ </sub><sup>1</sup>, it suffices to apply the bounds in [\(4.33\)](#page-17-5) to obtain the second bound in [\(4.34\)](#page-17-7).

We next prove the first bound in [\(4.34\)](#page-17-7), assuming  $p \in [1, \infty)$ ; we have, by Jensen,

$$
\left\| \frac{q_n^f}{w_t} \right\|_{\ell^p}^p = \sum_{x \in \mathbb{Z}^2} e^{tp|x|} \left| \sum_{z \in \mathbb{Z}^2} f(z) q_n(x - z) \right|^p \le \sum_{x \in \mathbb{Z}^2} \left| \sum_{z \in \mathbb{Z}^2} e^{t|z|} f(z) e^{t|x - z|} q_n(x - z) \right|^p
$$
  

$$
\le \sum_{x \in \mathbb{Z}^2} \left( \sum_{z \in \mathbb{Z}^2} |e^{t|z|} f(z)|^p e^{t|x - z|} q_n(x - z) \right) \left\{ \sum_{z \in \mathbb{Z}^2} e^{t|x - z|} q_n(x - z) \right\}^{p-1},
$$

and the sum in the last brackets is at most  $ce^{2ct^2n}$ , by the first bound in [\(4.33\)](#page-17-5). Bringing the sum over *x* inside the parenthesis and applying again [\(4.33\)](#page-17-5), the proof is completed.  $\square$ 

**B.3. Proof of Lemma [4.18.](#page-17-3)** We state two key bounds, from which our goal [\(4.36\)](#page-17-9) follows. Let  $B(x,r) := \{y \in \mathbb{R}^2 : |y-x| \leq r\}$  denote the Euclidean ball and let  $B(x,r) := B(x,r) \cap \mathbb{Z}^2$ be its restriction to  $\mathbb{Z}^2$ . For  $g: \mathbb{Z}^2 \to \mathbb{R}$ , define the *maximal function*  $\mathcal{M}^g: \mathbb{Z}^2 \to [0, \infty]$  by

<span id="page-27-2"></span>
$$
\mathcal{M}^g(x) := \sup_{0 < r < \infty} \left\{ \frac{1}{|\mathcal{B}(x,r)|} \sum_{y \in \mathcal{B}(x,r)} |g(y)| \right\}.
$$
\n(B.2)

Setting  $\{\mathcal{M}^g > t\} := \{y \in \mathbb{Z}^2 : \mathcal{M}^g(y) > t\}$  for short, we are going to prove the following discrete version of *Hardy-Littlewood maximal inequality*:

<span id="page-27-0"></span>
$$
\forall \lambda > 0: \qquad |\{\mathcal{M}^g > \lambda\}| \leq 25 \frac{\|g\|_{\ell^1}}{\lambda} \,. \tag{B.3}
$$

We are also going to prove the following *upper bound on*  $\overline{q}_I^f$  $L<sup>J</sup>$ , defined in  $(4.35)$ :

<span id="page-27-1"></span>
$$
\forall L \in \mathbb{N}, \ \forall x \in \mathbb{Z}^2 : \qquad \left| \overline{q}_L^f(x) \, w_t(x) \right| \leq C \left| \mathcal{M}^{fw_t}(x) \right| \qquad \text{with} \quad \mathsf{C} := 200 \pi \, \mathsf{c}^2 \, \mathsf{e}^{4 \mathsf{c} \, t^2 L}. \tag{B.4}
$$

Since  $\|\mathcal{M}^g\|_{\ell^\infty} \leq \|g\|_{\ell^\infty}$ , this implies  $\|\overline{q}_L^f w_t\|_{\ell^\infty} \leq C \|f w_t\|_{\ell^\infty}$ , which is our goal [\(4.36\)](#page-17-9) for  $p = \infty$ . Also note that combining [\(B.3\)](#page-27-0) and [\(B.4\)](#page-27-1) we obtain

$$
\forall \lambda > 0: \qquad \left| \left\{ \overline{q}_L^f w_t > \lambda \right\} \right| \leqslant 25 \,\mathrm{C} \,\frac{\left\| f w_t \right\|_{\ell^1}}{\lambda} \,,
$$

hence our goal [\(4.36\)](#page-17-9) for  $p \in (1, \infty)$  follows by *Marcinkiewicz's Interpolation Theorem*, see e.g. [\[Gra14,](#page-36-20) Theorem 1.3.2 and Exercise 1.3.3(a)]. It remains to prove  $(B.3)$  and  $(B.4)$ .

*Proof of* [\(B.3\)](#page-27-0)*.* We follow closely the classical proof of the Hardy-Littlewood maximal inequality, see [\[Gra14,](#page-36-20) Theorem 2.1.6], which is stated on  $\mathbb{R}^d$  instead of  $\mathbb{Z}^d$ . By definition of  $\mathcal{M}^g$ , see [\(B.2\)](#page-27-2), for every point  $x \in {\mathcal{M}^g > \lambda}$  there is  $r_x > 0$  such that

<span id="page-27-3"></span>
$$
\sum_{y \in \mathcal{B}(x, r_x)} |g(y)| > \lambda |\mathcal{B}(x, r_x)|.
$$
\n(B.5)

It suffices to fix any *finite* set  $K \subseteq \{M^g > \lambda\}$  and prove that [\(B.3\)](#page-27-0) holds with the LHS replaced by |K|. From the family of balls  $\mathcal{F} := \{ \mathcal{B}(x, r_x) : x \in K \}$  we extract a *disjoint*  $sub-family \mathcal{F}' := \{ \mathcal{B}(z, r_z) : z \in K' \}$  with  $K' \subseteq K$  by the greedy algorithm, see [\[Gra14,](#page-36-20) Lemma 2.1.5]: we first pick the ball of largest radius, then we select the ball of largest radius among the remaining ones *which do not intersect the balls that have already been picked*, and so on. By construction, if a ball  $\mathcal{B}(x, r_x)$  is *not* included in  $\mathcal{F}'$ , then it must overlap

with some ball  $\mathcal{B}(z, r_z)$  of larger radius  $r_z \geq r_x$ , which implies that  $\mathcal{B}(x, r_x) \subseteq \mathcal{B}(z, 3r_z)$ . In other terms, tripling the radii of the balls in  $\mathcal{F}'$  we cover all the balls in  $\mathcal{F}$ , hence

$$
|K| \leqslant \sum_{z \in K'} |\mathcal{B}(z, 3r_z)| \leqslant c \sum_{z \in K'} |\mathcal{B}(z, r_z)| \leqslant \frac{c}{\lambda} \sum_{z \in K'} \sum_{y \in \mathcal{B}(z, r_z)} |g(y)| \leqslant \frac{c}{\lambda} \|g\|_{\ell^1},
$$

where we estimated  $|\mathcal{B}(z, 3r)| \leq c |\mathcal{B}(z, r)|$  (see below), we applied [\(B.5\)](#page-27-3) and we bounded ř  $z \in K'$   $\sum_{y \in \mathcal{B}(z,r_z)} |g(y)| \le ||g||_{\ell^1}$ , because the balls  $\mathcal{B}(z,r_z)$  for  $z \in K'$  are disjoint. To com-plete the proof of [\(B.3\)](#page-27-0), we claim that we can take  $c = 25$ , i.e.  $|\mathcal{B}(z, 3r)| \leq 25 |\mathcal{B}(z, r)|$ .

Note that for  $0 < r < 1$  the Euclidean ball  $B(x, r)$  contains just the point x, while  $B(x, 3r)$  contains at most 25 integer points, i.e.  $x \pm (a, b)$  with  $-2 \leq a, b \leq +2$  (all these points are inside  $B(x, 3r)$  when *r* is close to 1). Next we note that each integer point  $y = (y^1, y^2) \in B(x, r)$  is the center of a square with vertices  $y^i \pm \frac{1}{2}$  $\frac{1}{2}$ : the union of these squares covers the Euclidean ball  $B(x, r - \frac{\sqrt{2}}{2})$  $\frac{\sqrt{2}}{2}$ ) and is included in  $B(x, r + \frac{\sqrt{2}}{2})$  $\frac{\sqrt{2}}{2}$ . Denoting by  $m(\cdot)$  the Euclidean area (i.e. the 2-dimensional Lebesgue measure), we obtain

$$
|\mathcal{B}(x,3r)| \leq m\big(B(x,3r+\frac{\sqrt{2}}{2})\big) = \big(3+\frac{\sqrt{2}}{r}\big)^2 m\big(B(x,r-\frac{\sqrt{2}}{2})\big) \leq \big(3+\frac{\sqrt{2}}{r}\big)^2 |\mathcal{B}(x,r)|,
$$

hence also for  $r \geq 1$  we have  $\left(3 + \frac{\sqrt{2}}{r}\right)$  $\left(\frac{\sqrt{2}}{r}\right)^2 \leqslant (3 + \sqrt{2})^2 \leqslant 25$  as claimed.

*Proof of* [\(B.4\)](#page-27-1). We claim that for all  $1 \le n \le L$  and  $x \in \mathbb{Z}^2$ 

<span id="page-28-0"></span>
$$
q_n(x) e^{t|x|} \leq \tilde{q}_n(x) := \frac{C'}{n} e^{-\frac{|x|^2}{16\epsilon n}} \quad \text{where} \quad C' := 6c e^{4c t^2 L}. \tag{B.6}
$$

Indeed, we prove in Lemma [B.1](#page-29-1) below that  $q_n(x) \leq \frac{6c}{n} e^{-\frac{|x|^2}{8cn}}$  $\overline{{}^{\overline{8\mathbf{c}}n}}$ , see [\(B.7\)](#page-29-2), therefore

$$
q_n(x) e^{t|x|} \leq \frac{6c}{n} e^{t|x| - \frac{|x|^2}{8cn}} \leq \frac{6c}{n} e^{-\frac{|x|^2}{16cn}} \cdot \left(\sup_{\gamma \geq 0} e^{t\gamma - \frac{\gamma^2}{16cn}}\right) = \frac{6c}{n} e^{-\frac{|x|^2}{16cn}} e^{4c t^2 n},
$$

which shows that [\(B.6\)](#page-28-0) holds for  $n \leq L$ .

Let us now deduce [\(B.4\)](#page-27-1) from [\(B.6\)](#page-28-0). Since  $\frac{w_t(x)}{w_t(x)}$  $\frac{w_t(x)}{w_t(z)} \leqslant e^{t|x-z|}$ , by [\(3.5\)](#page-6-6) we can estimate

$$
|q_n^f(x) w_t(x)| \leq \sum_{z \in \mathbb{Z}^2} |f(z) w_t(z)| q_n(x-z) e^{t|x-z|} \leq \sum_{z \in \mathbb{Z}^2} |f(z) w_t(z)| \tilde{q}_n(x-z),
$$

hence, writing  $\tilde{q}_n(y) = \int_0^\infty \mathbb{1}_{\{s \leq \tilde{q}_n(y)\}} ds$ , we obtain

$$
\left| q_n^f(x) w_t(x) \right| \leqslant \int_0^\infty ds \sum_{z \in \mathbb{Z}^2 : \tilde{q}_n(x-z) \geqslant s} \left| f(z) w_t(z) \right|.
$$

Since  $x \mapsto \tilde{q}_n(x)$  is radially decreasing, the set  $\{\tilde{q}_n(\cdot) \geq s\} = \{z \in \mathbb{Z}^2 : \tilde{q}_n(x-z) \geq s\}$  is a ball  $\mathcal{B}(x, r)$  of suitable radius  $r = r(n, s)$ . Recalling [\(B.3\)](#page-27-0), we then obtain

$$
\left|\max_{1\leqslant n\leqslant L} q_n^f(x)\,w_t(x)\right| \leqslant \mathcal{M}^{fw_t}(x) \cdot \max_{1\leqslant n\leqslant L} \int_0^\infty ds \left|\{\tilde{q}_n(\cdot)\geqslant s\}\right| = \mathcal{M}^{fw_t}(x) \cdot \max_{1\leqslant n\leqslant L} \|\tilde{q}_n\|_{\ell^1},
$$

where the equality holds because  $\int_0^\infty ds \left| \{ \tilde{q}_n(\cdot) \geq s \} \right| = \sum_{y \in \mathbb{Z}^2} \int_0^\infty ds \, 1_{\{ s \leq \tilde{q}_n(y) \}} = \sum_{y \in \mathbb{Z}^2} \tilde{q}_n(y)$ . It remains to evaluate  $\|\tilde{q}_n\|_{\ell^1}$ : by monotonicity we can bound

$$
\sum_{a \in \mathbb{Z}} e^{-\frac{a^2}{16c n}} \leq 1 + \int_{\mathbb{R}} e^{-\frac{x^2}{16c n}} dx = 1 + \sqrt{16\pi c n},
$$

hence writing  $x = (a, b)$ , so that  $|x|^2 = a^2 + b^2$ , we obtain

$$
\|\tilde{q}_n\|_{\ell^1} = \sum_{x \in \mathbb{Z}^2} \tilde{q}_n(x) = \frac{C'}{n} \bigg( \sum_{a \in \mathbb{Z}} e^{-\frac{a^2}{16c\,n}} \bigg)^2 \leq C' \frac{2(1 + 16\pi\,c\,n)}{n} \leq (2 + 32\pi)\,c\,C' \leq C\,,
$$

where the two last inequalities hod since  $n \geq 1$  and  $c \geq 1$ , hence  $1 + 16\pi c n \leq (1 + 16\pi)c n$ , and  $2 + 32\pi \leq 33\pi \leq \frac{200}{6}$  $\frac{00}{6}\pi$ , recalling the definition [\(B.4\)](#page-27-1) of C. The proof is completed.  $\Box$ 

<span id="page-29-1"></span>**Lemma B.1 (Heat kernel bound).** *Let Assumption [4.14](#page-16-3) hold and let* c *be the constant from Lemma [4.16.](#page-17-1) Then for every*  $n \in \mathbb{N}$  and  $x \in \mathbb{Z}^2$  we have

<span id="page-29-2"></span>
$$
q_n(x) \leq \frac{6c}{n} e^{-\frac{|x|^2}{8cn}}.
$$
\n(B.7)

**Proof.** We assume that  $n \geq 2$ , since the case  $n = 1$  is easier. Let us apply the formula [\(B.1\)](#page-26-1) with  $\ell = \left\lfloor \frac{n}{2} \right\rfloor$  $\frac{n}{2}$ , so that  $\frac{n}{3} \leq \ell \leq \frac{n}{2}$  $\frac{n}{2}$ : by [\(4.33\)](#page-17-5) (with  $t = 0$ ) we have  $q_k(x - y) \leq \frac{c}{k} \leq \frac{3c}{n}$ for both  $k = \ell$  and  $k = n - \ell$ , therefore for any  $\ell \geq 0$ 

<span id="page-29-3"></span>
$$
q_n(x) \leq \frac{3c}{n} e^{-\varrho|x|} \sum_{y \in \mathbb{Z}^2 : \langle y, x \rangle \geq \frac{1}{2}|x|^2} e^{2\varrho \langle y, \frac{x}{|x|} \rangle} \{ q_\ell(y) + q_{n-\ell}(y) \}, \tag{B.8}
$$

where we bounded  $1 \leq e^{-\varrho |x|} e^{2\varrho \langle y, \frac{x}{|x|} \rangle}$  because  $\langle y, x \rangle \geq \frac{1}{2}|x|^2$  (with  $\frac{x}{|x|} := 0$  for  $x = 0$ ). For any  $w = (w^1, w^2) \in \mathbb{R}^2$ , by [\(4.32\)](#page-17-8) and Cauchy-Schwarz we can bound

$$
\sum_{y \in \mathbb{Z}^2} e^{\langle y, w \rangle} q_\ell(y) \leqslant \sqrt{\sum_{y \in \mathbb{Z}^2} e^{2y^1 w^1} q_\ell(y) \cdot \sum_{y \in \mathbb{Z}^2} e^{2y^2 w^2} q_\ell(y)} \leqslant e^{c |w|^2 \ell},
$$

and similarly for  $q_{n-\ell}(\cdot)$ , therefore for  $\max\{\ell, n-\ell\} \leq \frac{n}{2}$  we obtain by  $(B.8)$ 

$$
q_n(x) \leq \frac{6c}{n} e^{-\varrho |x| + 2c \varrho^2 n}.
$$

<span id="page-29-0"></span>Optimising over  $\varrho$  leads us to choose  $\varrho = \frac{|x|}{4cn}$ , which yields [\(B.7\)](#page-29-2).

#### **Appendix C. Estimates on boundary and bulk terms**

In this section we prove the estimates on the boundary terms (Propositions [4.19](#page-18-8) and [4.20](#page-18-9) for the left boundary, Proposition [4.21](#page-19-1) for the right boundary) and on the bulk terms (Proposition [4.23](#page-19-8) and Proposition [4.24\)](#page-20-6).

**C.1. Proof of Propositions [4.19.](#page-18-8)** By the triangle inequality we can bound

<span id="page-29-4"></span>
$$
\left\| \frac{\hat{\mathbf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \sum_{n=1}^L \left\| \frac{\mathbf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} . \tag{C.1}
$$

Writing  $I = \{I^1, \ldots, I^m\}$  we can write

<span id="page-29-5"></span>
$$
\left\|\frac{q_n^{|f|,I}}{\mathcal{W}_t}\right\|_{\ell^p}^p = \sum_{\mathbf{x}\in(\mathbb{Z}^2)^h} \frac{q_n^{|f|,I}(\mathbf{x})^p}{\mathcal{W}_t(\mathbf{x})^p} \le \prod_{j=1}^m \left\{\sum_{y\in\mathbb{Z}^2} q_n^{|f|}(y)^{p|I^j|} e^{pt|I^j||y|} \right\} = \prod_{j=1}^m \left\|\frac{q_n^{|f|}}{w_t}\right\|_{\ell^{p|I^j|}}^{\ell^{p|I^j|}}.
$$
 (C.2)

 $\text{Since } \|\cdot\|_{\ell^{p^j}}^{pk}$  $\frac{pk}{\ell^{pk}} \leqslant \|\cdot\|_{\ell^{\infty}}^{p(k-1)}$  $\frac{p(k-1)}{\ell^\infty} \|\cdot\|^p_{\ell^2}$  $\sum_{j=1}^{p} |I^j| = h$  we get (raising to  $1/p$ ) › › › ›  $\mathsf{q}_n^{|f|, I}$  $\overline{\mathcal{W}_t}$  $\Big\|_{\ell^p} \leqslant$  $\begin{picture}(20,20) \put(0,0){\dashbox{0.5}(10,0){ }} \put(15,0){\dashbox{0.5}(10,0){ }} \put(15,0){\dashbox{$  $q_n^{|f|}$ *wt*  $\centering \centering \includegraphics[width=0.47\textwidth]{Figures/14-101-101}} \caption{The 3D (black) model for the 3D (black) model. The 3D (black) model is shown in Fig.~\ref{fig:14} and~\ref{fig:14}. The 3D (black) model is shown in Fig.~\ref{fig:14}. The$ *h*´*m ℓ* 8  $\centering \includegraphics[width=0.47\textwidth]{Figures/140191111.pdf} \includegraphics[width=0.47\textwidth]{Figures/140191111.pdf} \includegraphics[width=0.47\textwidth]{Figures/140191111.pdf} \includegraphics[width=0.47\textwidth]{Figures/140191111.pdf} \includegraphics[width=0.47\textwidth]{Figures/140191111.pdf} \includegraphics[width=0.47\textwidth]{Figures/140191111.pdf} \includegraphics[width=0.47\textwidth]{Figures/140191111.pdf} \includegraphics[width=0.47\textwidth]{Figures/1401$  $q_n^{|f|}$ *wt*  $\begin{picture}(20,20) \put(0,0){\dashbox{0.5}(10,0){ }} \put(15,0){\dashbox{0.5}(10,0){ }} \put(15,0){\dashbox{$ *m*  $e^{p}$ <sup> $\leq$ </sup> › › › ›  $q_n^{|f|}$ *wt*  $\Big\|_{\ell^\infty}$ › › › ›  $q_n^{|f|}$ *wt* › › › ›  $h-1$ *ℓ p ,* (C.3)

where the last inequality holds since  $m \leq h - 1$  for  $I \neq *$ . By [\(4.34\)](#page-17-7), for any  $r \in [1, \infty]$ ,

<span id="page-30-3"></span><span id="page-30-2"></span>
$$
\left\|\frac{q_n^{|f|}}{w_t}\right\|_{\ell^\infty} \leqslant \frac{\mathsf{c}\,\mathsf{e}^{2\mathsf{c}\,t^2 n}}{n^{\frac{1}{r}}} \left\|\frac{f}{w_t}\right\|_{\ell^r}, \qquad \left\|\frac{q_n^{|f|}}{w_t}\right\|_{\ell^p} \leqslant \mathsf{c}\,\mathsf{e}^{2\mathsf{c}\,t^2 n} \left\|\frac{f}{w_t}\right\|_{\ell^p},\tag{C.4}
$$

hence we obtain for  $n \leq L$ , recalling the definition of  $\mathscr{C}$  in [\(4.41\)](#page-18-5),

<span id="page-30-1"></span>
$$
\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \frac{\mathscr{C}^h}{n^{\frac{1}{r}}} \left\| \frac{f}{w_t} \right\|_{\ell^r} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-1} . \tag{C.5}
$$

Plugging this into [\(C.1\)](#page-29-4), since  $\sum_{n=1}^{L} \frac{1}{n^2}$  $\frac{1}{n^a} \leqslant \int_0^L$ 1  $\frac{1}{x^a} dx = \frac{L^{1-a}}{1-a}$  $\frac{b}{1-a}$ , we obtain

<span id="page-30-0"></span>
$$
\max_{I \neq *} \left\| \frac{\hat{\mathbf{q}}_L^{|f|, I}}{\mathcal{W}_t} \right\|_{\ell^p} \leq \frac{r}{r - 1} \mathcal{C}^h L^{1 - \frac{1}{r}} \left\| \frac{f}{w_t} \right\|_{\ell^r} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h - 1},\tag{C.6}
$$

which proves [\(4.41\)](#page-18-5) for  $r \geq p$  (so that  $\min\{\frac{r}{r-1}\}$  $\frac{r}{r-1}, \frac{p}{p-1}$  $\frac{p}{p-1}$ } =  $\frac{r}{r-1}$ ). More generally, if  $r \ge \frac{3p}{1+2}$  $\frac{3p}{1+2p}$ then  $\frac{r}{r-1} \leqslant 3\frac{p}{p-1}$  $\frac{p}{p-1}$  hence [\(C.6\)](#page-30-0) still proves [\(4.41\)](#page-18-5).

It remains to prove  $(4.41)$  for  $r \in \left[1, \frac{3p}{1+2}\right]$  $\left[\frac{3p}{1+2p}\right] \subseteq [1, p)$ . Let us obtain an estimate alternative to [\(C.5\)](#page-30-1). Since  $\| \cdot \|_{\ell^p}^p \le \| \cdot \|_{\ell^{\infty}}^{p-r} \| \cdot \|_{\ell^r}^r$  for  $r < p$ , by [\(4.34\)](#page-17-7) we obtain

$$
\left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^p} \leqslant \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^\infty}^{1 - \frac{r}{p}} \left\| \frac{q_n^{|f|}}{w_t} \right\|_{\ell^r}^{\frac{r}{p}} \leqslant \frac{\mathsf{c} \, \mathsf{e}^{2 \mathsf{c} \, t^2 n}}{n^{\frac{1}{r} - \frac{1}{p}}} \left\| \frac{f}{w_t} \right\|_{\ell^r},\tag{C.7}
$$

which we can use to estimate one factor of  $\|\frac{q_n^{|f|}}{w_t}\|_{\ell^p}$  appearing in [\(C.3\)](#page-30-2) (recall that  $h \ge 2$ ): applying again the first bound in  $(C.4)$ , for  $n \le L$  we obtain from  $(C.3)$ 

<span id="page-30-4"></span>
$$
\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leqslant \frac{\mathscr{C}^h}{n^{\gamma}} \left\| \frac{f}{w_t} \right\|_{\ell^r}^2 \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-2} \qquad \text{with} \quad \gamma := \frac{2}{r} - \frac{1}{p} = \frac{1}{r} + \frac{p-r}{pr}. \tag{C.8}
$$

The RHS of [\(C.8\)](#page-30-4) is smaller than the RHS of [\(C.5\)](#page-30-1) if and only if

<span id="page-30-5"></span>
$$
\frac{1}{n^{\gamma}} \left\| \frac{f}{w_t} \right\|_{\ell^r} < \frac{1}{n^{\frac{1}{r}}} \left\| \frac{f}{w_t} \right\|_{\ell^p} \qquad \Longleftrightarrow \qquad n > \tilde{n} := \left( \frac{\left\| \frac{f}{w_t} \right\|_{\ell^r}}{\left\| \frac{f}{w_t} \right\|_{\ell^p}} \right)^{\frac{pr}{p-r}}. \tag{C.9}
$$

Note that for  $r \in \left[1, \frac{3p}{1+2}\right]$  $\frac{3p}{1+2p}$  we have  $\gamma > 1$ , indeed  $\gamma - 1 \geq \frac{2(1+2p)}{3p} - \frac{1+p}{p} = \frac{p-1}{3p}$ . Then  $\sum_{n>\tilde{n}}^{\infty} \frac{1}{n^{\tilde{n}}}$  $\frac{1}{n^{\gamma}} \leqslant \int_{\tilde{n}}^{\infty}$ 1  $\frac{1}{x^{\gamma}} dx = \frac{1}{\gamma - \gamma}$  $\frac{1}{\gamma-1} \tilde{n}^{1-\gamma} \leqslant \frac{3p}{p-1}$  $\frac{3p}{p-1} \tilde{n}^{1-\gamma}$ , hence by [\(C.8\)](#page-30-4) we can bound

$$
\sum_{n>\bar{n}}\left\|\frac{\mathbf{q}_n^{|f|,I}}{\mathcal{W}_t}\right\|_{\ell^p}\leqslant \frac{3p}{p-1}\,\mathscr{C}^h\,\tilde{n}^{1-\gamma}\left\|\frac{f}{w_t}\right\|_{\ell^r}^2\left\|\frac{f}{w_t}\right\|_{\ell^p}^{h-2}=\frac{3p}{p-1}\,\mathscr{C}^h\left\|\frac{f}{w_t}\right\|_{\ell^r}^{\frac{r(p-1)}{p-r}}\left\|\frac{f}{w_t}\right\|_{\ell^p}^{h-\frac{r(p-1)}{p-r}}
$$

where the equality follows by the definitions of  $\tilde{n}$  in [\(C.9\)](#page-30-5) and  $\gamma$  in [\(C.8\)](#page-30-4). For the contribution of  $n \leq \tilde{n}$ , the previous bound [\(C.5\)](#page-30-1) with  $r = p$  yields, as in [\(C.6\)](#page-30-0),

$$
\sum_{n=1}^{\tilde{n}} \left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^2} \leq \frac{p}{p-1} \, \mathscr{C}^h \, \tilde{n}^{1-\frac{1}{p}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^h = \frac{p}{p-1} \, \mathscr{C}^h \left\| \frac{f}{w_t} \right\|_{\ell^r}^{\frac{r(p-1)}{p-r}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-\frac{r(p-1)}{p-r}},
$$

*,*

having used the definition of  $\tilde{n}$  in [\(C.9\)](#page-30-5). Overall, see [\(C.1\)](#page-29-4), for  $r \in [1, \frac{3p}{1+2}$  $\frac{3p}{1+2p}$  we have

$$
\max_{I \neq *} \left\| \frac{\hat{\mathbf{q}}_L^{|f|, I}}{\mathcal{W}_t} \right\|_{\ell^p} \leq \underbrace{\frac{4p}{p-1} \mathcal{C}^h}{\frac{4p}{p-1}} \left\| \frac{f}{w_t} \right\|_{\ell^r}^{\frac{1}{\alpha}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^{\frac{1}{\alpha}} \qquad \text{with } \alpha := \frac{p-r}{r(p-1)} \in (0, 1].
$$
 (C.10)

At the same time, we can apply again the previous bound  $(C.6)$  with  $r = p$  to estimate

$$
\max_{I \neq *} \left\| \frac{\hat{\mathbf{q}}_L^{|f|,I}}{\mathcal{W}_t} \right\|_{\ell^p} \leq \underbrace{\frac{p}{p-1} \mathcal{C}^h L^{1-\frac{1}{p}} \left\| \frac{f}{w_t} \right\|_{\ell^p}^h}_{B}.
$$
\n(C.11)

Combining these bounds we get  $\max_{I \neq *} \left\| \frac{\hat{q}_L^{[f],I}}{\mathcal{W}_t} \right\|_{\ell^2} \leq A^{\alpha} B^{1-\alpha}$ , hence

$$
\forall r \in \left[1, \frac{3p}{1+2p}\right]: \qquad \max_{I \neq *}\left\|\frac{\widehat{\mathbf{q}}_L^{|f|,I}}{\mathcal{W}_t}\right\|_{\ell^p} \leqslant \frac{4p}{p-1} \,\mathscr{C}^h \, L^{1-\frac{1}{r}} \left\|\frac{f}{w_t}\right\|_{\ell^r} \left\|\frac{f}{w_t}\right\|_{\ell^p}^{h-1},
$$

which coincides with our goal  $(4.41)$ , since  $\min\{\frac{r}{r-1}\}$  $\frac{r}{r-1}, \frac{p}{p-1}$  $\frac{p}{p-1}$ } =  $\frac{p}{p-1}$  for  $r < p$ .

**C.2. Proof of Proposition [4.20.](#page-18-9)** We follow the proof of Proposition [4.19.](#page-18-8) By the triangle inequality, as in  $(C.1)$ , it is enough to show that

<span id="page-31-2"></span>
$$
\left\| \frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t} \mathcal{V}_s^J \right\|_{\ell^p} \leqslant \frac{36^{\frac{1}{p}} \mathscr{C}^h}{s^{2/p}} \left\| \frac{f}{w_t} \right\|_{\ell^\infty}^2 \left\| \frac{f}{w_t} \right\|_{\ell^p}^{h-2} . \tag{C.12}
$$

We assume for ease of notation that  $J = \{\{1,2\}, \{3\}, \ldots, \{h\}\}\)$ . Let us fix a partition  $I = \{I^1, \ldots, I^m\}$  such that  $I \not\supseteq J$ , say  $1 \in I^1$  and  $2 \in I^2$ . In analogy with [\(C.2\)](#page-29-5), we have

<span id="page-31-1"></span>
$$
\left\| \frac{\mathsf{q}_n^{|f|, I}}{\mathcal{W}_t} \mathcal{V}_s^J \right\|_{\ell^p}^p \leq \hat{\Sigma}_n^{(1, 2)} \cdot \prod_{j = 3}^m \left\| \frac{\mathsf{q}_n^{|f|}}{w_t} \right\|_{\ell^{p|I^j|}}^{p|I^j|} . \tag{C.13}
$$

where

<span id="page-31-3"></span>
$$
\widehat{\Sigma}_n^{(1,2)} := \sum_{y^1, y^2 \in \mathbb{Z}^2} \left( q_n^{|f|} (y^1) e^{t|y^1|} \right)^{p|I^1|} \left( q_n^{|f|} (y^2) e^{t|y^2|} \right)^{p|I^2|} e^{-ps|y^1 - y^2|}.
$$
\n(C.14)

By a uniform bound, we can estimate

$$
\hat{\Sigma}_{n}^{(1,2)} \leq \left\| \frac{q_{n}^{|f|}}{w_{t}} \right\|_{\ell^{\infty}}^{p|I^{1}|} \sum_{y^{1}, y^{2} \in \mathbb{Z}^{2}} \left( \frac{q_{n}^{|f|}(y^{2})}{w_{t}(y^{2})} \right)^{p|I^{2}|} e^{-ps|y^{1}-y^{2}|}
$$
\n
$$
= \left\| \frac{q_{n}^{|f|}}{w_{t}} \right\|_{\ell^{\infty}}^{p|I^{1}|} \left\| \frac{q_{n}^{|f|}}{w_{t}} \right\|_{\ell^{p}}^{p|I^{2}|} \left( \sum_{y \in \mathbb{Z}^{2}} e^{-ps|y|} \right).
$$
\n(C.15)

<span id="page-31-4"></span>Since  $2|z| \ge |z^1| + |z^2|$  for  $z = (z^1, z^2) \in \mathbb{Z}^2$  and  $1 - e^{-x} \ge \frac{2}{3}x$  for  $0 \le x \le \frac{1}{2}$ , we can bound

<span id="page-31-0"></span>
$$
\sum_{z \in \mathbb{Z}^2} e^{-ps|z|} \le \sum_{z \in \mathbb{Z}^2} e^{-s|z|} \le \left(\sum_{x \in \mathbb{Z}} e^{-s\frac{|x|}{2}}\right)^2 \le \left(\frac{2}{1 - e^{-\frac{s}{2}}}\right)^2 \le \frac{36}{s^2}.
$$
 (C.16)

Plugging these estimates into [\(C.13\)](#page-31-1) and bounding  $\left\| \cdot \right\|_{\ell^{pi}}^{pk}$  $\frac{pk}{\ell^{pk}} \leqslant \left\| \cdot \right\|_{\ell^{\infty}}^{p(k-1)}$  $\frac{p(k-1)}{\ell^\infty}$   $\|\cdot\|_{\ell^2}^p$  $\sum_{j=1}^{p} |I^j|$  = *h* and  $m \leq h - 1$ , we obtain (raising to  $1/p$ )

$$
\left\|\frac{\mathsf{q}_n^{|f|,I}}{\mathcal{W}_t}\mathcal{V}^J_s\right\|_{\ell^p}\leqslant \frac{36^{\frac{1}{p}}}{s^{2/p}}\left\|\frac{q_n^{|f|}}{w_t}\right\|_{\ell^\infty}^{h-m+1}\left\|\frac{q_n^{|f|}}{w_t}\right\|_{\ell^p}^{m-1}\leqslant \frac{36^{\frac{1}{p}}}{s^{2/p}}\left\|\frac{q_n^{|f|}}{w_t}\right\|_{\ell^\infty}^2\left\|\frac{q_n^{|f|}}{w_t}\right\|_{\ell^p}^{h-2}.
$$

Applying the estimates in  $(C.4)$ , we obtain  $(C.12)$ .

**C.3. Proof of Proposition [4.21.](#page-19-1)** The second line of [\(4.46\)](#page-19-2) follows by the first line because  $\|\cdot\|_{\ell^{2q}}^2 \leqslant \|\cdot\|_{\ell^{q}}$ . Let us prove the first line of [\(4.46\)](#page-19-2). Writing  $J = \{J^1, \ldots, J^m\}$ and arguing as in [\(C.2\)](#page-29-5), we can write

$$
\|\overline{\mathbf{q}}_L^{[g],J} \mathcal{W}_t\|_{\ell^q}^q = \sum_{\mathbf{x}\in(\mathbb{Z}^2)^h} \overline{\mathbf{q}}_L^{[g],J}(\mathbf{x})^q \mathcal{W}_t(\mathbf{x})^q \leq \prod_{j=1}^m \left\{ \sum_{y\in\mathbb{Z}^2} \left( \overline{q}_L^{[g]}(y) w_t(y) \right)^{q|J^j|} \right\} = \prod_{j=1}^m \|\overline{q}_L^{[g]} w_t\|_{\ell^{q|J^j|}}^{q|J^j|},
$$

where  $\overline{q}^{[g]}_L(y) := \max_{1 \le n \le L} q^{[g]}_n(y)$ , see [\(4.35\)](#page-17-10). Since  $J \ne \ast$ , we have  $|J^j| \ge 2$  for at least one *j*, say for  $j = 1$ , hence for  $k = |J^1|$  we bound  $\|\cdot\|_{\ell^{qi}}^{qk}$  $\frac{qk}{\ell^{qk}} \leqslant \|\cdot\|_{\ell^{\infty}}^{q(k-2)}$  $\frac{q(k-2)}{\ell^\infty}$   $\|\cdot\|_{\ell^{2\ell}}^{2q}$  $\ell^{\frac{2q}{2q}}$ , while for all other  $k = |J^j| \geq 1$  we simply bound  $\|\cdot\|_{\ell^{qi}}^{qk}$  $\frac{qk}{\ell^{qk}} \leqslant \|\cdot\|_{\ell^{\infty}}^{q(k-1)}$  $\frac{q(k-1)}{\ell^\infty} \|\cdot\|_\ell^q$  $\sum_{j=1}^{q} |J^j| = h$ , we obtain

$$
\|\overline{\mathbf{q}}_{L}^{[g],J} \, \mathcal{W}_t\|_{\ell^q}^q \leqslant \|\overline{q}_{L}^{[g]}\, w_t\|_{\ell^{2q}}^{2q} \, \|\overline{q}_{L}^{[g]}\, w_t\|_{\ell^q}^{q(m-1)} \, \|\overline{q}_{L}^{[g]}\, w_t\|_{\ell^\infty}^{q(h-m-1)} \leqslant \|\overline{q}_{L}^{[g]}\, w_t\|_{\ell^{2q}}^{2q} \, \|\overline{q}_{L}^{[g]}\, w_t\|_{\ell^q}^{q(h-2)} \, ,
$$

because  $m \leq h - 1$  for  $J \neq *$ . In order to obtain the first line of [\(4.46\)](#page-19-2), it suffices to apply the estimate [\(4.36\)](#page-17-9), where we can bound  $\frac{2q}{2q-1} 25^{\frac{1}{2q}} \text{ C} \leqslant \frac{q}{q-1}$  $\frac{q}{q-1} 25^{\frac{1}{q}} C \leqslant \frac{q}{q-1}$  $\frac{q}{q-1}$  25 C =  $\frac{q}{q-1}$  $\frac{q}{q-1}\overline{\mathscr{C}}$ .

We next prove [\(4.47\)](#page-19-3). We may assume that  $I = \{\{1, 2\}, \{3\}, \ldots, \{h\}\}\)$ . Let us fix a partition  $J = \{J^1, \ldots, J^m\}$  with  $J \not\equiv I$ , say  $1 \in J^1$  and  $2 \in J^2$ . In analogy with [\(C.13\)](#page-31-1), we can write

$$
\max_{\substack{J\neq *\\J\not\equiv I}}\|\overline{\mathsf{q}}^{[g],J}_{L}\,\mathcal{W}_{t}\,\mathcal{V}_{s}^{I}\|_{\ell^{q}}^{q} \leqslant \overline{\Sigma}_{M}^{(1,2)}\cdot\prod_{j=3}^{m}\left\|\overline{q}^{[g]}_{L}\,w_{t}\right\|_{\ell^{q|J^{j}|}}^{q|J^{j}|},
$$

where, as in  $(C.14)-(C.15)$  $(C.14)-(C.15)$  $(C.14)-(C.15)$ , we have

$$
\overline{\Sigma}_{M}^{(1,2)} := \sum_{y^1, y^2 \in \mathbb{Z}^2} \left( \overline{q}_{L}^{|g|}(y^1) w_t(y^1) \right)^{q|J^1|} \left( \overline{q}_{L}^{|g|}(y^2) w_t(y^2) \right)^{q|J^2|} w_s(y^1 - y^2)^q
$$
\n
$$
\leq \|\overline{q}_{L}^{|g|} w_t \|_{\ell^\infty}^{q|J^1|} \|\overline{q}_{L}^{|g|} w_t \|_{\ell^q}^{q|J^2|} \|w_s\|_{\ell^q}^q.
$$

Bounding  $\|\overline{q}^{|g|}_{L} w_t\|_{\frac{q|J^j}{q}}^{q|J^j|}$  $\frac{q|J^j|}{\ell^{q|J^j|}} \leq \left\| \overline{q} \right\|_L^{g} w_t \right\|_{\ell^{\infty}}^{q(|J^j|-1)}$  $\frac{q(|J^{\jmath}| - 1)}{\ell^{\infty}} \left\| \overline{q}^{\lfloor g \rfloor}_L \, w_t \right\|_{\ell^{\jmath}}^q$  $\ell_q^q$  for  $j \geq 2$ , we then obtain

$$
\begin{aligned} \max_{J\neq\ast}\|\overline{\mathsf{q}}_{L}^{[g],J}\,\mathcal{W}_{t}\|_{\ell^{q}}^{q} &\leqslant \|\overline{q}_{L}^{[g]}\,w_{t}\|_{\ell^{\infty}}^{q(h-m+1)}\,\|\overline{q}_{L}^{[g]}\,w_{t}\|_{\ell^{q}}^{q(m-1)}\,\|w_{s}\|_{\ell^{q}}^{q} \\ &\leqslant \|\overline{q}_{L}^{[g]}\,w_{t}\|_{\ell^{\infty}}^{2q}\,\|\overline{q}_{L}^{[g]}\,w_{t}\|_{\ell^{q}}^{q(h-2)}\,\|w_{s}\|_{\ell^{q}}^{q}\,, \end{aligned}
$$

because  $\sum_{j=1}^{m} |J^j| = h$  and  $m \leq h-1$  for  $J \neq *$ . We conclude applying [\(4.36\)](#page-17-9) and [\(4.48\)](#page-19-9).  $\Box$ 

**C.4. Proof of Proposition [4.23.](#page-19-8)** Let us set for short  $p := \frac{q}{q-1}$  $\frac{q}{q-1}$  (so that  $\frac{1}{p} + \frac{1}{q} = 1$ ). We are going to use a key functional inequality from [\[CSZ23,](#page-35-1) Lemma 6.8], in the improved version from  $[LZ21+, eq. (3.21)$  in the proof of Proposition 3.3]:

<span id="page-33-0"></span>
$$
\sum_{\mathbf{z}\in(\mathbb{Z}^2)^h, \mathbf{x}\in(\mathbb{Z}^2)^h} \frac{f(\mathbf{z})g(\mathbf{x})}{(1+|\mathbf{x}-\mathbf{z}|^2)^{h-1}} \leq C_1 p q \|f\|_{\ell^p} \|g\|_{\ell^q} \quad \text{where} \quad C_1 := 2^{2h} (1+\pi)^h. \tag{C.17}
$$

(The value of  $C_1$  is extracted from [\[LZ21+,](#page-36-1) proof of Proposition 3.3] where  $C_1 \leq 2^{3h+1}$  ( $\frac{c}{2}$ )  $\frac{c}{2}$ <sup>h-1</sup>pq with  $c \leq 1 + \pi$  from [\[LZ21+,](#page-36-1) proof of Lemma A.1], hence  $C_1 \leq 2^{2h+2} (1 + \pi)^{h-1}$ .)

We show below the following bound on  $\hat{Q}_L^{*,*}(\mathbf{z}, \mathbf{x}) = \sum_{n=1}^L \prod_{i=1}^h q_n(x^i - z^i)$ :

<span id="page-33-1"></span>
$$
\hat{Q}_{L}^{*,*}(\mathbf{z}, \mathbf{x}) \leq \frac{C_2 e^{-\frac{|\mathbf{x} - \mathbf{z}|^2}{16cL}}}{(1 + |\mathbf{x} - \mathbf{z}|^2)^{h-1}} \quad \text{where} \quad C_2 := h! (200 \, \text{c}^2)^h. \tag{C.18}
$$

Recalling [\(4.28\)](#page-16-6), since  $\hat{\mathbf{Q}}_L^{I,J}(\mathbf{z}, \mathbf{x}) = \hat{\mathbf{Q}}_L^{*,*}(\mathbf{z}, \mathbf{x}) \mathbb{1}_{\{\mathbf{z} \sim I, \mathbf{x} \sim J\}}$ , see [\(4.7\)](#page-12-0)-[\(4.13\)](#page-14-6), we obtain

$$
\big(\mathcal{W}_t\,\widehat{\mathsf{Q}}^{I,J}_{L}\textstyle \frac{1}{\mathcal{W}_t}\big)(\mathbf{z},\mathbf{x})\leqslant \frac{C_2\,\mathbbm{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}}{\big(1+|\mathbf{x}-\mathbf{z}|^2\big)^{h-1}}\prod_{i=1}^h\mathrm{e}^{t|z^i-x^i|-\frac{|z^i-x^i|^2}{16c\,L}}\leqslant \frac{C_2\,\mathrm{e}^{8\mathrm{c}ht^2L}\,\mathbbm{1}_{\{\mathbf{z}\sim I,\mathbf{x}\sim J\}}}{\big(1+|\mathbf{x}-\mathbf{z}|^2\big)^{h-1}}\,,
$$

because  $\max_{a \in \mathbb{R}} \{ta - \frac{a^2}{16c}\}\$  $\frac{a^2}{16cL}$ } = 8c *t*<sup>2</sup>*L*. Applying [\(C.17\)](#page-33-0), get [\(4.52\)](#page-19-6) since 800(1 +  $\pi$ ) ≤ 4000. We next prove [\(4.53\)](#page-20-4). Let *I*, *J* be pairs, say  $I = \{(a, b), (c): c \neq a, c \neq b\}$  and  $J =$ 

 $\{\{\tilde{a}, \tilde{b}\}, \{c\}: c \neq \tilde{a}, c \neq \tilde{b}\}.$  For  $\mathbf{z} \sim I$  and  $\mathbf{x} \sim J$  we have  $z^a = z^b$ , hence

$$
\frac{1}{\mathcal{V}_s^I(\mathbf{x})} \leqslant e^{s|x^a - x^b|} \leqslant e^{s\{|x^a - z^a| + |z^a - z^b| + |z^b - x^b|\}} = e^{s|x^a - z^a|} e^{s|z^b - x^b|},
$$

and similarly  $\frac{1}{\mathcal{V}_s^J(\mathbf{z})} \leqslant e^{s|x^{\tilde{a}}-z^{\tilde{a}}|} e^{s|z^{\tilde{b}}-x^{\tilde{b}}|}$ . Arguing as above, we obtain [\(4.53\)](#page-20-4):

$$
\begin{split} \left(\frac{\mathcal{W}_t}{\mathcal{V}_s^J}\,\widehat{\mathsf{Q}}_L^{I,J}\frac{1}{\mathcal{W}_t\,\mathcal{V}_s^I}\right)({\bf z},{\bf x}) &\leq \frac{C_2\,\mathbbm{1}_{\{{\bf z}\sim I, {\bf x}\sim J\}}}{\left(1+|{\bf x}-{\bf z}|^2\right)^{h-1}}\prod_{i=1}^h{\rm e}^{(t+2s)|z^i-x^i|-\frac{1}{16c\,L}|z^i-x^i|^2}\\ &\leqslant \frac{C_2\,{\rm e}^{8\mathrm{ch}(t+2s)^2L}\,\mathbbm{1}_{\{{\bf z}\sim I, {\bf x}\sim J\}}}{\left(1+|{\bf x}-{\bf z}|^2\right)^{h-1}}\,. \end{split}
$$

Let us prove [\(C.18\)](#page-33-1). By the bound  $q_n(x) \leq \frac{6c}{n} e^{-\frac{|x|^2}{8cn}}$  $\frac{8c_n}{c}$  proved in Lemma [B.1](#page-29-1) we obtain

$$
\mathsf{Q}_n^{*,*}(\mathbf{z}, \mathbf{x}) = \prod_{i=1}^h q_n (x^i - z^i) \leqslant \frac{(6c)^h}{n^h} e^{-\frac{|\mathbf{x} - \mathbf{z}|^2}{8cn}},
$$

hence for  $\mathbf{x} = \mathbf{z}$  we get  $\widehat{Q}_L^{*,*}(\mathbf{x}, \mathbf{x}) = \sum_{n=1}^L Q_n^{*,*}(\mathbf{z}, \mathbf{x}) \leq (6c)^h \sum_{n=1}^\infty \frac{1}{n^2}$  $\frac{1}{n^2} = (6c)^h \frac{\pi^2}{6} \leq 2 (6c)^h$ which is compatible with [\(C.18\)](#page-33-1). We next assume that  $\mathbf{x} \neq \mathbf{z}$ : note that for  $A = \frac{|\mathbf{x} - \mathbf{z}|^2}{8c} > 0$ 

$$
\sum_{n=1}^{L} \frac{e^{-\frac{A}{n}}}{n^h} \leq \frac{e^{-\frac{A}{2L}}}{A^{h-1}} \left\{ \frac{1}{A} \sum_{n=1}^{\infty} \varphi(\frac{n}{A}) \right\} \quad \text{where} \quad \varphi(t) := \frac{e^{-\frac{1}{2t}}}{t^h}.
$$

Since  $\varphi(\cdot)$  is unimodal, we can bound  $\frac{1}{A} \sum_{n=1}^{\infty} \varphi\left(\frac{n}{A}\right)$  $\left(\frac{n}{A}\right) \leqslant \int_0^\infty \varphi(t) \, \mathrm{d}t + \frac{1}{A}$  $\frac{1}{A}$ || $\varphi$ ||<sub>∞</sub> and note that  $\int_0^\infty \varphi(t) = 2^{h-1} \int_0^\infty s^{h-2} e^{-s} ds = 2^{h-1} \frac{A}{(h-2)!}$  while  $\|\varphi\|_\infty = (2h)^h e^{-h} \leq 2^h h!/\sqrt{2\pi h} \leq$ 

1  $\frac{1}{2}2^h h!$ , therefore for  $A \geq 1$  we get  $\frac{1}{A} \sum_{n=1}^{\infty} \varphi\left(\frac{n}{A}\right)$  $\left(\frac{n}{A}\right) \leq 2^h h!$ . Overall, recalling [\(4.13\)](#page-14-6), we have for  $\mathbf{x} \neq \mathbf{z}$ 

$$
\widehat{\mathsf{Q}}_{L}^{*,*}(\mathbf{z},\mathbf{x}) \leqslant \sum_{n=1}^{L} \mathsf{Q}_{n}^{*,*}(\mathbf{z},\mathbf{x}) \leqslant \frac{(48\,\mathrm{c}^2)^h \,\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{16\,\mathrm{c}\,L}}}{|\mathbf{x}-\mathbf{z}|^{2(h-1)}} \, 2^h \, h! \leqslant \frac{h! \, (200\,\mathrm{c}^2)^h \,\mathrm{e}^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{16\,\mathrm{c}\,L}}}{(1+|\mathbf{x}-\mathbf{z}|^2)^{h-1}},
$$

where we last bounded  $|\mathbf{x} - \mathbf{z}|^2 \ge \frac{1}{2}(1 + |\mathbf{x} - \mathbf{z}|^2)$  for  $\mathbf{x} \ne \mathbf{z}$ . We have proved [\(C.18\)](#page-33-1).  $\square$ 

#### **C.5. Proof of Proposition [4.24.](#page-20-6)** Let us define  $p := \frac{q}{q-1}$  $\frac{q}{q-1}$  so that  $\frac{1}{p} + \frac{1}{q} = 1$ . Since

$$
\|A\|_{\ell^q\to\ell^q}:=\sup_{f,g\colon \|f\|_{\ell^p}\leqslant 1,\, \|g\|_{\ell^q}\leqslant 1}\ \sum_{\mathbf{z},\mathbf{x}\in(\mathbb{Z}^2)_I^h}f(\mathbf{z})\,A(\mathbf{z},\mathbf{x})\,g(\mathbf{x})\,,
$$

we can bound  $\sum_{\mathbf{z},\mathbf{x}} f(\mathbf{z}) |\hat{\mathbf{U}}|^I(\mathbf{z},\mathbf{x}) g(\mathbf{x}) \le (\sum_{\mathbf{z},\mathbf{x}} f(\mathbf{z})^p |\hat{\mathbf{U}}|^I(\mathbf{z},\mathbf{x}))^{1/p} (\sum_{\mathbf{z},\mathbf{x}} |\hat{\mathbf{U}}|^I(\mathbf{z},\mathbf{x}) g(\mathbf{x})^q)^{1/q}$ by Cauchy-Schwarz, hence we obtain

<span id="page-34-3"></span>
$$
\|\mathbf{A}\|_{\ell^{q}\to\ell^{q}}\leqslant\max\left\{\sup_{\mathbf{z}\in(\mathbb{Z}^{2})_{I}^{h}}\sum_{\mathbf{x}\in(\mathbb{Z}^{2})_{I}^{h}}\mathsf{A}(\mathbf{z},\mathbf{x}),\sup_{\mathbf{x}\in(\mathbb{Z}^{2})_{I}^{h}}\sum_{\mathbf{z}\in(\mathbb{Z}^{2})_{I}^{h}}\mathsf{A}(\mathbf{z},\mathbf{x})\right\}.
$$
 (C.19)

We will prove [\(4.55\)](#page-20-3) and [\(4.56\)](#page-20-5) exploiting this bound.

We recall that  $U_{n,\beta}(x)$  is defined in [\(4.11\)](#page-13-5) and we define

<span id="page-34-0"></span>
$$
U_{n,\beta} := \sum_{x \in \mathbb{Z}^2} U_{n,\beta}(x) = \sum_{k=1}^{\infty} (\sigma_{\beta}^2)^k \sum_{0 = :n_0 < n_1 < \dots < n_k := n} \prod_{i=1}^k q_{2(n_i - n_{i-1})}(0) \,. \tag{C.20}
$$

When we sum  $U_{n,\beta}$  for  $n = 1, \ldots, L$ , if we enlarge the sum range in [\(C.20\)](#page-34-0) by letting each increment  $m_i := n_i - n_{i-1}$  vary freely in  $\{1, \ldots, M\}$ , recalling  $(4.54)$  we obtain

<span id="page-34-4"></span>
$$
\sum_{n=1}^{L} e^{-\lambda n} U_{n,\beta} \leq \sum_{k=1}^{\infty} (\sigma_{\beta}^{2})^{k} \bigg( \sum_{m=1}^{L} e^{-\lambda m} q_{2m}(0) \bigg)^{k} = \sum_{k=1}^{\infty} (\sigma_{\beta}^{2} R_{L}^{(\lambda)})^{k} = \frac{\sigma_{\beta}^{2} R_{L}^{(\lambda)}}{1 - \sigma_{\beta}^{2} R_{L}^{(\lambda)}}.
$$
 (C.21)

We next estimate the exponential spatial moments of  $U_{n,\beta}(x)$ . Pluggin the second bound from [\(4.32\)](#page-17-8) into [\(4.11\)](#page-13-5), writing  $x = (x^1, x^2)$  and  $x^a = \sum_{i=1}^k (x_i^a - x_{i-1}^a)$ , we obtain

$$
\forall a = 1, 2: \qquad \sum_{x \in \mathbb{Z}^2} e^{tx^a} U_{n,\beta}(x) \leqslant e^{c \frac{t^2}{2} n} U_{n,\beta}.
$$

From this, by  $|x| \le |x^1| + |x^2|$ , Cauchy-Schwarz and  $e^{t|x^a|} \le e^{tx^a} + e^{-tx^a}$ , we deduce that

<span id="page-34-1"></span>
$$
\sum_{x \in \mathbb{Z}^2} e^{t|x|} U_{n,\beta}(x) \leqslant 2 e^{2ct^2 n} U_{n,\beta}.
$$
\n(C.22)

We now fix a partition  $I = \{I^1, ..., I^m\} \neq *$  and a *pair*  $J = \{\{a, b\}, \{c\} : c \neq a, b\}$ . Our goal is to prove  $(4.56)$ , which also yields  $(4.55)$  for  $s = 0$ . By  $(4.28)$  and  $(4.43)$  we have the following rough bound, for any  $a \in \{-1, +1\}$ :

<span id="page-34-2"></span>
$$
\frac{\mathcal{W}_t(\mathbf{z})\,\mathcal{V}_s^J(\mathbf{z})^a}{\mathcal{W}_t(\mathbf{x})\,\mathcal{V}_s^J(\mathbf{x})^a} \leq e^{2(t+s)|x^a - z^a|} \prod_{c \neq a,b} e^{(t+s)|x^c - z^c|}.
$$
\n(C.23)

We may order  $|I^1| \geq |I^2| \geq \ldots \geq |I^m|$ , so that  $|I^1| \geq 2$ . Given  $\mathbf{z}, \mathbf{x} \in (\mathbb{Z}^2)_I^h$ , denoting by  $x^{I^j}$  the common value of  $x^a$  for  $a \in I^j$ , by [\(4.7\)](#page-12-0) we can write

$$
Q_n^{I,I}(\mathbf{z}, \mathbf{x}) = q_n (x^{I^1} - z^{I^1})^{|I^1|} \prod_{j=2}^m q_n (x^{I^j} - z^{I^j})^{|I^j|} \leq q_n (x^{I^1} - z^{I^1})^2 \prod_{j=2}^m q_n (x^{I^j} - z^{I^j}),
$$

because  $q_n(\cdot) \leq 1$ . Since  $|\mathbb{E}[\xi_\beta^I]| \leq \sigma_\beta^2$  by assumption, from [\(4.8\)](#page-12-1) we can bound

$$
|\mathsf{U}|_{n,\beta}^{I}(\mathbf{z},\mathbf{x}) \leq U_{n,\beta}(x^{I^{1}} - z^{I^{1}}) \prod_{j=2}^{m} q_{n}(x^{I^{j}} - z^{I^{j}}),
$$

therefore by  $(C.22)$ ,  $(C.23)$  and the first bound in  $(4.33)$  we obtain

$$
\sum_{\mathbf{x}\in(\mathbb{Z}^2)_I^h} \left( |\mathsf{U}|_{n,\beta}^I(\mathbf{z}, \mathbf{x}) \frac{\mathcal{W}_t(\mathbf{z}) \mathcal{V}_s(\mathbf{z})^a}{\mathcal{W}_t(\mathbf{x}) \mathcal{V}_s(\mathbf{x})^a} \right) \leq 2^h e^{4hc (t+s)^2 n} U_{n,\beta},\tag{C.24}
$$

which yields, recalling [\(4.15\)](#page-14-8),

$$
\sup_{\mathbf{z}\in(\mathbb{Z}^2)_I^h} \sum_{\mathbf{x}\in(\mathbb{Z}^2)_I^h} |\widehat{\mathbf{U}}|_{L,\lambda,\beta}^J(\mathbf{z},\mathbf{x}) \frac{\mathcal{W}_t(\mathbf{z})\mathcal{V}_s(\mathbf{z})^a}{\mathcal{W}_t(\mathbf{x})\mathcal{V}_s(\mathbf{x})^a} \le 1 + 2^h e^{4h\mathbf{c}(t+s)^2 L} \sum_{n=1}^L e^{-\lambda n} U_{n,\beta}, \qquad (C.25)
$$

and the same holds exchanging **x** and **z** by symmetry (note that the bound [\(C.23\)](#page-34-2) is symmetric in  $\mathbf{x} \leftrightarrow \mathbf{z}$ ). Recalling [\(C.19\)](#page-34-3) and [\(C.21\)](#page-34-4), we obtain [\(4.56\)](#page-20-5) (hence [\(4.55\)](#page-20-3)).

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