

Hyperbolic Anderson model with time-independent rough noise: Gaussian fluctuations

Raluca M. Balan^{*†} Wangjun Yuan^{‡§}

May 8, 2023

Abstract

In this article, we study the hyperbolic Anderson model in dimension 1, driven by a time-independent rough noise, i.e. the noise associated with the fractional Brownian motion of Hurst index $H \in (1/4, 1/2)$. We prove that, with appropriate normalization and centering, the spatial integral of the solution converges in distribution to the standard normal distribution, and we estimate the speed of this convergence in the total variation distance. We also prove the corresponding functional limit result. Our method is based on a version of the second-order Gaussian Poincaré inequality developed recently in [27], and relies on delicate moment estimates for the increments of the first and second Malliavin derivatives of the solution. These estimates are obtained using a connection with the wave equation with delta initial velocity, a method which is different than the one used in [27] for the parabolic Anderson model.

Mathematics Subject Classifications (2020): Primary 60H15; Secondary 60H07, 60G15, 60F05

Keywords: hyperbolic Anderson model, rough noise, Malliavin calculus, Stein’s method for normal approximations

^{*}Corresponding author. University of Ottawa, Department of Mathematics and Statistics, STEM Building, 150 Louis-Pasteur Private, Ottawa, Ontario, K1N 6N5, Canada. E-mail: rbalan@uottawa.ca.

[†]Research supported by a grant from Natural Sciences and Engineering Research Council of Canada.

[‡]University of Luxembourg, Department of Mathematics, Maison du Nombre 6, Avenue de la Fonte L-4364 Esch-sur-Alzette. Luxembourg. E-mail: ywangjun@connect.hku.hk.

[§]The author gratefully acknowledges the financial support of ERC Consolidator Grant 815703 ”STAMFORD: Statistical Methods for High Dimensional Diffusions”

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1 Introduction

The study of stochastic partial differential equations (SPDEs) using the random field approach originates in Walsh’ lecture notes [32], which introduced the general framework, focusing mostly on equations driven by space-time Gaussian white noise in dimension 1. In the seminal article [13], Dalang extended the martingale measure method of Walsh to equations driven by spatially homogeneous Gaussian noise (white in time), and introduced powerful techniques for analyzing these equations. Since then, this area has been growing at an accelerated pace. One of the tools that has been used extensively is Malliavin calculus. This tool is especially useful when the noise is colored in time (or time-independent), and Itô calculus techniques cannot be used, due to a lack of martingale structures. We refer the reader to [7, 10, 8, 9, 14, 16, 19, 20, 17, 18, 24, 31] for a small sample of relevant contributions related to SPDEs with various types of Gaussian noise.

In [21], a new line of investigations has been opened up in this area, focusing on the asymptotic behaviour of the spatial integral of the solution of the stochastic heat equation with space-time Gaussian white noise, as the size of the integration region becomes large. The main result of [21] states that, with suitable normalization and centering, this integral converges to the standard normal distribution, and gives the speed of this convergence in the total variation distance. This result, called the “Quantitative Central Limit Theorem”

(QCLT) is obtained by combining Malliavin calculus with Stein's method for normal approximations. Similar results and extensions have been obtained in the subsequent papers [22, 28, 27, 6] for the solution of the stochastic heat equation with colored noise in space/time (or with time-independent noise), respectively in [12, 15, 4, 5] for the solution of the stochastic wave equation. In both cases, the noise enters the equation multiplied by a Lipschitz function $\sigma(u)$ of the solution. The most difficult case is when the noise is *rough in space*, i.e. it behaves in space like the fractional Brownian motion (fBm) with Hurst index $H < 1/2$. This case has been studied in [27] for the *parabolic Anderson model* (pAm), the stochastic heat equation with a linear term $\sigma(u) = u$ multiplying the noise, using technical arguments that rely heavily on properties of the heat kernel.

The goal of the present article is to present the first study of this problem for the wave equation with rough noise in space. We will assume that the noise is time-independent, and we postpone the treatment of the time-dependent noise for future work. More precisely, in this article we consider the *hyperbolic Anderson model* (hAm) in dimension 1:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + \sqrt{\theta}u(t, x)\dot{W}(x), & t > 0, x \in \mathbb{R}, \\ u(0, x) = 1, \quad \frac{\partial u}{\partial t}(0, x) = 0. \end{cases} \quad (1)$$

Often, we are interested in the case $\theta = 1$. The reason we introduce a parameter $\theta > 0$ is the following. Our main result, the QCLT for the spatial average of the solution u_θ of (1), is obtained by applying a version of the second-order Gaussian Poincaré inequality (Proposition 2.4 of [27]). For this, we need to estimate the *fourth* moment of the increments of the Malliavin derivative Du_θ and of the rectangular increments of the second Malliavin derivative D^2u_θ . The fact that we include a parameter $\theta > 0$ in equation (1) allows us to compare the p -th moment (for $p > 2$) of the solution u_θ , of its Malliavian derivatives, or of their increments, with the *second* moment of the similar quantities corresponding to the solution $u_{(p-1)\theta}$, which are then treated using their chaos expansions. This comparison between moments plays a crucial role in the present article, and is derived using a hypercontractivity property for general SPDEs, which is of independent interest and is included in Appendix A.

The noise W is time-independent and fractional in space with Hurst index $H \in (\frac{1}{4}, \frac{1}{2})$, being given by a zero-mean Gaussian process $\{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R})\}$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with covariance

$$\mathbb{E}[W(\varphi)W(\psi)] = c_H \int_{\mathbb{R}} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}|\xi|^{1-2H}d\xi =: \langle \varphi, \psi \rangle_{\mathcal{P}_0}.$$

Here $\mathcal{D}(\mathbb{R})$ is the space of C^∞ functions with compact support in \mathbb{R} , $\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\varphi(x)dx$ is the Fourier transform of φ , and

$$c_H = \frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi}.$$

By approximation, the noise can be extended to an isonormal Gaussian process $\{W(\varphi); \varphi \in \mathcal{P}_0\}$, as defined in Malliavin calculus, where \mathcal{P}_0 is the Hilbert space defined as completion

of $\mathcal{D}(\mathbb{R})$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{P}_0}$. In particular, $1_{[0,x]} \in \mathcal{P}_0$ for all $x \in \mathbb{R}$, and the process $\{W(x) = W(1_{[0,x]}); x \in \mathbb{R}\}$ is a fBm of index H .

We say that a process $u_\theta = \{u_\theta(t, x); t \geq 0, x \in \mathbb{R}\}$ is a (mild Skorohod) **solution** to equation (1) if it satisfies the following integral equation:

$$u_\theta(t, x) = 1 + \sqrt{\theta} \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) u_\theta(s, y) W(\delta y) ds$$

where the $W(\delta y)$ corresponds to the divergence operator δ (or Skorohod integral) defined in Section 2.2 below, and G_t is the fundamental solution of the wave equation on $\mathbb{R}_+ \times \mathbb{R}$:

$$G_t(x) := \frac{1}{2} 1_{\{|x| < t\}}, \quad t > 0, x \in \mathbb{R}. \quad (2)$$

We let $G_t(x) = 0$ when $t \leq 0$. Note that the Fourier transform of G_t is given by:

$$\mathcal{F}G_t(\xi) = \int_{\mathbb{R}} e^{-i\xi x} G_t(x) dx = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0. \quad (3)$$

We are interested on the asymptotic behaviour as $R \rightarrow \infty$ of the spatial average:

$$F_{R,\theta}(t) = \int_{-R}^R (u_\theta(t, x) - 1) dx.$$

More precisely, letting $\sigma_{R,\theta}^2(t) = \text{Var}(F_{R,\theta}(t))$, we would like to show that

$$\frac{F_{R,\theta}(t)}{\sigma_{R,\theta}(t)} \xrightarrow{d} Z \sim N(0, 1) \quad \text{as } R \rightarrow \infty,$$

and to give an estimate for the speed of this convergence in the total variation distance:

$$d_{\text{TV}}(X, Y) = \sup_{B \in \mathcal{B}(\mathbb{R})} |\mathbb{P}(X \in B) - \mathbb{P}(Y \in B)|.$$

The following theorems are the main results of this article.

Theorem 1.1 (Limiting Covariance). *For any $\theta > 0$, $t > 0$ and $s > 0$*

$$\lim_{R \rightarrow \infty} \frac{\mathbb{E}[F_{R,\theta}(t) F_{R,\theta}(s)]}{R} = K_\theta(t, s), \quad (4)$$

where $K_\theta(t, s)$ is finite and is given by (65) below. Moreover, for any $\theta > 0$ and $t > 0$,

$$\lim_{R \rightarrow \infty} \frac{\sigma_{R,\theta}^2(t)}{R} = K_\theta(t, t) > 0. \quad (5)$$

Theorem 1.2 (QCLT). *For any $\theta > 0$ and $t > 0$,*

$$d_{\text{TV}} \left(\frac{F_{R,\theta}(t)}{\sigma_{R,\theta}(t)}, Z \right) \leq C_{t,H,\theta} R^{-1/2},$$

where $C_{t,H,\theta} > 0$ is a constant depending on (t, H, θ) and $Z \sim N(0, 1)$.

Theorem 1.3 (FCLT). *For any $\theta > 0$, the process $\{R^{-1/2}F_{R,\theta}(t); t \geq 0\}$ has a continuous modification which converges in distribution in $C([0, \infty))$ as $R \rightarrow \infty$, to a zero-mean Gaussian process $\{\mathcal{G}_\theta(t); t \geq 0\}$ with covariance*

$$\mathbb{E}[\mathcal{G}_\theta(t)\mathcal{G}_\theta(s)] = K_\theta(t, s),$$

where $C([0, \infty))$ is equipped with the topology of uniform convergence on compact sets.

For the proof of Theorem 1.1 we use the Wiener chaos decomposition of $F_{R,\theta}$, by observing that the projection on the first chaos space does not contribute to the limit, while the projections on the other chaos spaces give rise to a convergent series. A similar phenomena has been observed in [26] for (pAm) with rough noise in space, that is colored in time. This is in contrast with the QCLT for equations with “regular” noise in space (studied in [28, 27, 5]) for which only the projection on the first chaos space contributes to the limit. In both cases, the QCLT is *non-chaotic*, in the sense described on page 3 of [15], which means that not all the projections on the chaos spaces contribute to the limit.

Theorem 1.2 follows by applying a version of the second-order Gaussian Poincaré inequality (Proposition 2.4 of [27]) for the time-independent noise. This leads to a study of the increments of Du_θ , as well as the rectangular increments of D^2u_θ . Analyzing these increments requires significant effort. For this task, we use a different method than in [27] for (pAm). In addition to the moment comparison technique mentioned above, we develop a connection with the solution v_θ of (hAm) with delta initial velocity. This introduces a new problem, which leads us to examine the increments of u_θ and v_θ . A key role in this analysis is played by some translation-invariance properties of these processes. In addition, we exploit the fact that the increments of v_θ are closely related to those of the solution V_θ of (hAm) driven by a Gaussian noise \mathfrak{X} , which is *white in time* and fractional in space with the same index $H \in (\frac{1}{4}, \frac{1}{2})$. The covariance of \mathfrak{X} is given by (33) below.

For the proof of Theorem 1.3, as usually, we need to show two things : (i) tightness; and (ii) finite-dimensional convergence. (i) follows by Kolmogorov-Centsov theorem, while (ii) follows using the same method as in [5], based on a variance estimation that is proved using the same arguments as for the QCLT.

This article is organized as follows. In Section 2, we include the background and some preliminary results from Malliavin calculus, and we prove the existence of the solution u_θ . In Section 3, we examine some properties of v_θ and V_θ . In Section 4, we derive some estimates for the increments of Du_θ and the rectangular increments of D^2u_θ . The increments of the processes u_θ and v_θ are studied in Section 5. Finally, in Section 6, we present the proofs of Theorems 1.1, 1.2 and 1.3. The moment comparison result for a general SPDE is included in Appendix A. In Appendix B, we study two stochastic Volterra equations driven by \mathfrak{X} , which are of the same form as the equations satisfied by the increments of V_θ , and allow us to derive some properties of these increments. Appendix C contains some elementary inequalities which are used in the sequel.

2 Preliminaries

In this section, we include some preliminary results related to Malliavin calculus, and we prove the existence and uniqueness of the solution to (1). We let $\|\cdot\|_p$ be the $L^p(\Omega)$ -norm.

2.1 Multiple Wiener integrals

We start by introduce some basic ingredients of Malliavin calculus, following [25].

We denote by J_n the projection from $L^2(\Omega)$ to $\mathcal{P}_{0,n}$, where $\mathcal{P}_{0,n}$ is the n -th Wiener chaos space, defined at the closed linear span in $L^2(\Omega)$ of $\{H_n(W(\varphi); \varphi \in \mathcal{P}_0)\}$, with $H_n(x)$ being the n -th Hermite polynomial. Let $I_n : \mathcal{P}_0^{\otimes n} \rightarrow \mathcal{P}_{0,n}$ be the multiple Wiener integral of order n with respect to W , where $\mathcal{P}_0^{\otimes n}$ is the n -th tensor product of \mathcal{P}_0 . Since I_n is surjective, for any $F \in L^2(\Omega)$, $J_n(F) = I_n(f_n)$ for some $f_n \in \mathcal{P}_0^{\otimes n}$.

The Wiener chaos spaces $\mathcal{P}_{0,n}$ are orthogonal in $L^2(\Omega)$. More precisely, for any $f \in \mathcal{P}_0^{\otimes n}$ and $g \in \mathcal{P}_0^{\otimes m}$,

$$E[I_n(f)I_m(g)] = \begin{cases} n! \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{P}_0^{\otimes n}} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

where \tilde{f} is the symmetrization of f in all n variables:

$$\tilde{f}(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\rho \in S_n} f(x_{\rho(1)}, \dots, x_{\rho(n)}),$$

and S_n is the set of all permutations of $\{1, \dots, n\}$. Note that $I_n(\tilde{f}) = I_n(f)$ and

$$\|\tilde{f}\|_{\mathcal{P}_0^{\otimes n}} \leq \|f\|_{\mathcal{P}_0^{\otimes n}}. \quad (6)$$

Every random variable $F \in L^2(\Omega)$ which is measurable with respect to W has the *Wiener chaos expansion*:

$$F = E(F) + \sum_{n \geq 1} J_n(F) = E(F) + \sum_{n \geq 1} I_n(f_n), \quad (7)$$

where the series converges in $L^2(\Omega)$. Moreover,

$$E|F|^2 = \sum_{n \geq 0} E|I_n(f_n)|^2 = \sum_{n \geq 0} n! \|\tilde{f}_n\|_{\mathcal{P}_0^{\otimes n}}^2,$$

where I_0 is the identity map on \mathbb{R} and $f_0 = E(F)$. Occasionally, we denote $J_0(F) = \mathbb{E}(F)$.

In general, $I_{p+q}(f \otimes g) \neq I_p(f)I_q(g)$. The exact expression of $I_{p+q}(f \otimes g)$ is given by the *product formula*, which will not be used in the present paper. Instead, we use the following result.

Lemma 2.1. *Let $p, q \in \mathbb{N}$.*

(a) *For $f \in \mathcal{P}_0^{\otimes p}$, $g \in \mathcal{P}_0^{\otimes q}$, we have*

$$\|I_{p+q}(f \otimes g)\|_2^2 \leq \frac{(p+q)!}{p!q!} \|I_p(f)\|_2^2 \|I_q(g)\|_2^2.$$

(b) *For $f_1 \in \mathcal{P}_0^{\otimes p_1}$, $f_2 \in \mathcal{P}_0^{\otimes p_2}$ and $f_3 \in \mathcal{P}_0^{\otimes p_3}$, we have*

$$\|I_{p_1+p_2+p_3}(f_1 \otimes f_2 \otimes f_3)\|_2^2 \leq \frac{(p_1+p_2+p_3)!}{p_1!p_2!p_3!} \|I_{p_1}(f_1)\|_2^2 \|I_{p_2}(f_2)\|_2^2 \|I_{p_3}(f_3)\|_2^2.$$

Proof. (a) Since $f \otimes g$ and $\tilde{f} \otimes \tilde{g}$ have the same symmetrization, $I_{p+q}(\tilde{f} \otimes \tilde{g}) = I_{p+q}(f \otimes g)$. Hence,

$$\begin{aligned}
\|I_{p+q}(f \otimes g)\|_2^2 &= \|I_{p+q}(\tilde{f} \otimes \tilde{g})\|_2^2 = (p+q)! \left\| \widetilde{\tilde{f} \otimes \tilde{g}} \right\|_{\mathcal{P}_0^{\otimes(p+q)}}^2 \\
&\leq (p+q)! \left\| \tilde{f} \otimes \tilde{g} \right\|_{\mathcal{P}_0^{\otimes(p+q)}}^2 = (p+q)! \int_{\mathbb{R}^{p+q}} \left| \mathcal{F}(\tilde{f} \otimes \tilde{g})(\xi_1, \dots, \xi_{p+q}) \right|^2 \prod_{j=1}^{p+q} |\xi_j|^{1-2H} d\xi_j \\
&= (p+q)! \int_{\mathbb{R}^{p+q}} \left| \mathcal{F}\tilde{f}(\xi_1, \dots, \xi_p) \right|^2 \left| \mathcal{F}\tilde{g}(\xi_{p+1}, \dots, \xi_{p+q}) \right|^2 \prod_{j=1}^{p+q} |\xi_j|^{1-2H} d\xi_j \\
&= \frac{(p+q)!}{p!q!} \|\tilde{f}\|_{\mathcal{P}_0^{\otimes p}}^2 \|\tilde{g}\|_{\mathcal{P}_0^{\otimes q}}^2 = \frac{(p+q)!}{p!q!} \|I_p(f)\|_2^2 \|I_q(g)\|_2^2
\end{aligned}$$

(b) We use (a) twice, the first time for $p = p_1$, $q = p_2 + p_3$, $f = f_1$, $g = f_2 \otimes f_3$, and the second time for $p = p_2$, $q = p_3$, $f = f_2$, $g = f_3$. \square

Next, we extend Lemma 2.1 to processes indexed by elements in a measure space.

Lemma 2.2. *Let (E, \mathcal{E}, μ) be a measure space. For each $\theta > 0$ and $r \in E$, let $F_\theta(r), G_\theta(r)$ be random variables in $L^2(\Omega)$ with the Wiener chaos expansions $F_\theta(r) = \sum_{p \geq 0} \theta^{p/2} I_p(f_p(\cdot, r))$ and $G_\theta(r) = \sum_{q \geq 0} \theta^{q/2} I_q(g_q(\cdot, r))$. Then*

$$\sum_{p \geq 0} \sum_{q \geq 0} \theta^{p+q} \left\| I_{p+q} \left(\int_E f_p(\cdot, r) \otimes g_q(\cdot, r) \mu(dr) \right) \right\|_2^2 \leq \left(\int_E \|F_{2\theta}(r)\|_2 \|G_{2\theta}(r)\|_2 \mu(dr) \right)^2.$$

Moreover, if for each $\theta > 0$ and $r \in E$, $H_\theta(r)$ is a random variable in $L^2(\Omega)$ with the Wiener chaos expansion $H_\theta(r) = \sum_{k \geq 0} \theta^{k/2} I_k(h_k(\cdot, r))$, then

$$\begin{aligned}
&\sum_{p \geq 0} \sum_{q \geq 0} \sum_{k \geq 0} \theta^{p+q+k} \left\| I_{p+q+k} \left(\int_E f_p(\cdot, r) \otimes g_q(\cdot, r) \otimes h_k(\cdot, r) \mu(dr) \right) \right\|_2^2 \\
&\leq \left(\int_E \|F_{3\theta}(r)\|_2 \|G_{3\theta}(r)\|_2 \|H_{3\theta}(r)\|_2 \mu(dr) \right)^2.
\end{aligned}$$

Proof. We use the same idea as in the proof of Lemma 2.1.(a). More precisely,

$$\begin{aligned}
&\left\| I_{p+q} \left(\int_E f_p(\cdot, r) \otimes g_q(\cdot, r) \mu(dr) \right) \right\|_2^2 = \left\| I_{p+q} \left(\int_E \tilde{f}_p(\cdot, r) \otimes \tilde{g}_q(\cdot, r) \mu(dr) \right) \right\|_2^2 \\
&\leq (p+q)! \left\| \int_E \tilde{f}_p(\cdot, r) \otimes \tilde{g}_q(\cdot, r) \mu(dr) \right\|_{\mathcal{P}_0^{\otimes(p+q)}}^2 \\
&= (p+q)! \int_{\mathbb{R}^{p+q}} \left| \int_E \mathcal{F}(\tilde{f}_p(\cdot, r) \otimes \tilde{g}_q(\cdot, r))(\xi_1, \dots, \xi_{p+q}) dr \right|^2 \prod_{j=1}^{p+q} |\xi_j|^{1-2H} d\xi_j \\
&= (p+q)! \int_{\mathbb{R}^{p+q}} \int_{E^2} \mathcal{F}\tilde{f}_p(\cdot, r_1)(\xi_1, \dots, \xi_p) \mathcal{F}\tilde{g}_q(\cdot, r_2)(\xi_{p+1}, \dots, \xi_{p+q})
\end{aligned}$$

$$\begin{aligned}
& \times \overline{\mathcal{F}\tilde{f}_p(\cdot, r_2)(\xi_1, \dots, \xi_p)\mathcal{F}\tilde{g}_q(\cdot, r_2)(\xi_{p+1}, \dots, \xi_{p+q})\mu(dr_1)\mu(dr_2)} \prod_{j=1}^{p+q} |\xi_j|^{1-2H} d\xi_j \\
& = (p+q)! \int_{E^2} \left\langle \tilde{f}_p(\cdot, r_1), \tilde{f}_p(\cdot, r_2) \right\rangle_{\mathcal{P}_0^{\otimes p}} \left\langle \tilde{g}_q(\cdot, r_1), \tilde{g}_q(\cdot, r_2) \right\rangle_{\mathcal{P}_0^{\otimes q}} \mu(dr_1)\mu(dr_2) \\
& = \frac{(p+q)!}{p!q!} \int_{E^2} \mathbb{E} [I_p(f_p(\cdot, r_1)) I_p(f_p(\cdot, r_2))] \mathbb{E} [I_q(g_q(\cdot, r_1)) I_q(g_q(\cdot, r_2))] \mu(dr_1)\mu(dr_2).
\end{aligned}$$

Noting that $\frac{(p+q)!}{p!q!} \leq 2^{p+q}$, we obtain:

$$\begin{aligned}
& \sum_{p \geq 1} \sum_{q \geq 1} \theta^{p+q} \left\| I_{p+q} \left(\int_E f_p(\cdot, r) \otimes g_q(\cdot, r) \mu(dr) \right) \right\|_2^2 \\
& \leq \int_{E^2} \sum_{p \geq 1} (2\theta)^p \mathbb{E} [I_p(f_p(\cdot, r_1)) I_p(f_p(\cdot, r_2))] \sum_{q \geq 1} (2\theta)^q \mathbb{E} [I_q(g_q(\cdot, r_1)) I_q(g_q(\cdot, r_2))] dr_1 dr_2 \\
& = \int_{E^2} \mathbb{E} [F_{2\theta}(r_1) F_{2\theta}(r_2)] \mathbb{E} [G_{2\theta}(r_1) G_{2\theta}(r_2)] \mu(dr_1)\mu(dr_2) \leq \left(\int_E \|F_{2\theta}(r)\|_2 \|G_{2\theta}(r)\|_2 \mu(dr) \right)^2,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality for the last inequality.

The second estimate is proved in a similar way, using the inequality $\frac{(p+q+k)!}{p!q!k!} \leq 3^{p+q+k}$. \square

2.2 The operators D and δ

In this section, we recall briefly the definitions of the Malliavin derivative D and the divergence operator δ . We refer the reader to [25] for more details.

Let \mathcal{S} be the class of “smooth” random variables, i.e variables of the form

$$F = f(W(\varphi_1), \dots, W(\varphi_n)), \quad (8)$$

where $f \in C_b^\infty(\mathbb{R}^n)$, $\varphi_i \in \mathcal{P}_0$, $n \geq 1$, and $C_b^\infty(\mathbb{R}^n)$ is the class of bounded C^∞ -functions on \mathbb{R}^n , whose partial derivatives of all orders are bounded. The *Malliavin derivative* of a random variable F of the form (8) is the \mathcal{P}_0 -valued random variable given by:

$$DF := \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(\varphi_1), \dots, W(\varphi_n)) \varphi_i.$$

We endow \mathcal{S} with the norm $\|F\|_{\mathbb{D}^{1,2}} := (E|F|^2)^{1/2} + (E\|DF\|_{\mathcal{P}_0}^2)^{1/2}$. The operator D can be extended to the space $\mathbb{D}^{1,2}$, the completion of \mathcal{S} with respect to $\|\cdot\|_{\mathbb{D}^{1,2}}$.

For any integer $k \geq 2$, we let D^k be the Malliavin derivative of order k , whose domain is denoted by $\mathbb{D}^{k,2}$.

If F is a random variable in $L^2(\Omega)$ with the Wiener chaos expansion $F = \sum_{n \geq 0} I_n(f_n) = \sum_{n \geq 0} J_n(F)$ for some *symmetric* functions $f_n \in \mathcal{P}_0^{\otimes n}$, then

$$D_x F = \sum_{n \geq 1} n I_{n-1}(f_n(\cdot, x)). \quad (9)$$

In particular, $D_x J_n(F) = nI_{n-1}(f_n(\cdot, x)) = J_{n-1}(D_x F)$ for any $n \geq 1$, and $D_x J_0(F) = 0$. In general,

$$D^k J_n(F) = \begin{cases} J_{n-k}(D^k F) & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases} \quad (10)$$

The *divergence operator* δ is the adjoint of the operator D . The domain of δ , denoted by $\text{Dom } \delta$, is the set of $u \in L^2(\Omega; \mathcal{P}_0)$ such that

$$|E\langle DF, u \rangle_{\mathcal{H}}| \leq c(E|F|^2)^{1/2}, \quad \forall F \in \mathbb{D}^{1,2},$$

where c is a constant depending on u . If $u \in \text{Dom } \delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$E(F\delta(u)) = E\langle DF, u \rangle_{\mathcal{P}_0}, \quad \forall F \in \mathbb{D}^{1,2}. \quad (11)$$

In particular, $E(\delta(u)) = 0$. If $u \in \text{Dom } \delta$, we use the notation

$$\delta(u) = \int_{\mathbb{R}} u(x)W(\delta x),$$

and we say that $\delta(u)$ is the *Skorohod integral* of u with respect to W .

2.3 OU semigroup and hypercontractivity

In this section, we review the definition and some properties of the Ornstein-Uhlenbeck semigroup.

The *Ornstein-Uhlenbeck (OU) semigroup* is the family $(T_t)_{t \geq 0}$ of contraction operators on $L^2(\Omega)$ defined by:

$$T_t(F) = \sum_{n \geq 0} e^{-nt} J_n(F).$$

Using (10), we derive that

$$D^k(T_t(F)) = \sum_{n \geq 0} e^{-nt} D^k(J_n(F)) = \sum_{n \geq k} e^{-nt} J_{n-k}(D^k F) = e^{-kt} T_t(D^k F). \quad (12)$$

The OU semigroup has a *hypercontractivity property*: for any $t > 0$ and $F \in L^p(\Omega)$,

$$\|T_t F\|_{q(t)} \leq \|F\|_p, \quad \text{where } q(t) = e^{2t}(p-1) + 1 > p > 1. \quad (13)$$

An important consequence of this property is that for any $1 < p < q < \infty$, the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on the same chaos space $\mathcal{P}_{0,n}$. More precisely,

$$\|F\|_q \leq \left(\frac{q-1}{p-1}\right)^{n/2} \|F\|_p, \quad \text{for any } F \in \mathcal{P}_{0,n} \text{ and } 1 < p < q. \quad (14)$$

The generator of the Ornstein-Uhlenbeck semigroup is given by:

$$LF = \sum_{n \geq 1} n J_n(F). \quad (15)$$

Its domain $\text{Dom } L$ consists of random variables $F \in L^2(\Omega)$ for which the series in (15) converges in $L^2(\Omega)$. Note that $F \in \text{Dom } L$ if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom } \delta$, and in this case, $LF = -\delta(DF)$.

The *pseudo-inverse* of L is the operator L^{-1} defined by

$$L^{-1}F = \sum_{n \geq 1} \frac{1}{n} J_n(F).$$

For any $F \in \mathbb{D}^{1,2}$ with $\mathbb{E}(F) = 0$, the process $u = -DL^{-1}F$ belongs to $\text{Dom } \delta$ and

$$F = \delta(-DL^{-1}F). \quad (16)$$

2.4 Existence of solution

In this section, we show that equation (1) has a unique solution.

It is known that, if it exists, the solution to equation (1) has the Wiener chaos expansion:

$$u_\theta(t, x) = 1 + \sum_{n \geq 1} \theta^{n/2} I_n(f_n(\cdot, x; t)), \quad (17)$$

with kernel $f_n(\cdot, x; t)$ given by:

$$f_n(x_1, \dots, x_n, x; t) = \int_{T_n(t)} f_n(t_1, x_1, \dots, t_n, x_n, t, x) dt, \quad (18)$$

where $T_n(t) = \{0 < t_1 < \dots < t_n < t\}$ and

$$f_n(t_1, x_1, \dots, t_n, x_n, t, x) = G_{t-t_n}(x - x_n) \dots G_{t_2-t_1}(x_2 - x_1) 1_{\{0 < t_1 < \dots < t_n < t\}}. \quad (19)$$

is the kernel which appears in the chaos decomposition of the solution of (hAm) with time dependent noise (studied in [4]). In this case,

$$\mathbb{E}|u_\theta(t, x)|^2 = 1 + \sum_{n \geq 1} n! \theta^n \|\tilde{f}_n(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}}^2. \quad (20)$$

In fact, the solution exists if and only if the series (17) converges in $L^2(\Omega)$, i.e. the series (20) converges. In this case, the solution is unique.

The next result gives the existence of the solution and provides an upper bound for its moments. For this result, we will use the fact that for any $a > 0$, there exist some constants $c_1 > 0$ and $c_2 > 0$ depending on a such that

$$\sum_{n \geq 0} \frac{x^n}{(n!)^a} \leq c_1 \exp(c_2 x^{1/a}) \quad \text{for any } x > 0. \quad (21)$$

In addition, we will need the value of the following integral: for any $t > 0$,

$$\int_{\mathbb{R}} |\mathcal{F}G_t(\xi)|^2 |\xi|^\alpha d\xi = \int_{\mathbb{R}} \frac{\sin^2(t|\xi|)}{|\xi|^2} |\xi|^\alpha d\xi = c_\alpha t^{1-\alpha}. \quad (22)$$

The integral converges if and only if $\alpha \in (-1, 1)$; see for instance, Lemma 3.1 of [3].

Theorem 2.3. For any $H \in (\frac{1}{4}, \frac{1}{2})$, equation (1) has a unique (Skorohod) solution. Moreover, for any $\theta > 0$, $p \geq 2$, $t > 0$ and $x \in \mathbb{R}$,

$$\|u_\theta(t, x)\|_p \leq c_1 \exp\left(c_2 \theta^{\frac{1}{2H+1}} p^{\frac{1}{2H+1}} t^{\frac{2H+2}{2H+1}}\right), \quad (23)$$

where $c_1 > 0$ and $c_2 > 0$ are constants that only depend on H . Consequently, $C_{\theta,p,T,u} := \sup_{(t,x) \in [0,T] \times \mathbb{R}} \|u_\theta(t, x)\|_p < \infty$.

Proof. Note that $\mathcal{F}f_n(t_1, \cdot, \dots, t_n, \cdot, t, x)(\boldsymbol{\xi}) = e^{-i(\xi_1 + \dots + \xi_n)x} \prod_{j=1}^n \mathcal{F}G_{t_{j+1}-t_j}(\xi_1 + \dots + \xi_j)$, where $t_{n+1} = t$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\mathcal{F}f_n(\cdot, x; t)(\boldsymbol{\xi})|^2 &= \left| \int_{T_n(t)} \prod_{j=1}^n \mathcal{F}G_{t_{j+1}-t_j}(\xi_1 + \dots + \xi_j) dt \right|^2 \\ &\leq \frac{t^n}{n!} \int_{T_n(t)} \prod_{j=1}^n |\mathcal{F}G_{t_{j+1}-t_j}(\xi_1 + \dots + \xi_j)|^2 dt, \end{aligned}$$

where $\mathbf{t} = (t_1, \dots, t_n)$. By Fubini's theorem,

$$\begin{aligned} \|f_n(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}}^2 &= c_H^n \int_{\mathbb{R}^n} |\mathcal{F}f_n(\cdot, x; t)(\boldsymbol{\xi})|^2 \prod_{j=1}^n |\xi_j|^{1-2H} d\boldsymbol{\xi} \\ &\leq c_H^n \frac{t^n}{n!} \int_{T_n(t)} \int_{\mathbb{R}^n} \prod_{j=1}^n |\mathcal{F}G_{t_{j+1}-t_j}(\xi_1 + \dots + \xi_j)|^2 \prod_{j=1}^n |\xi_j|^{1-2H} d\boldsymbol{\xi} dt \\ &= c_H^n \frac{t^n}{n!} \int_{T_n(t)} \int_{\mathbb{R}^n} \prod_{j=1}^n |\mathcal{F}G_{t_{j+1}-t_j}(\eta_j)|^2 |\eta_1|^{1-2H} \prod_{j=2}^n |\eta_j - \eta_{j-1}|^{1-2H} d\boldsymbol{\eta} dt, \end{aligned}$$

where for the last line, we used the change of variables $\eta_j = \xi_1 + \dots + \xi_j$ for $j = 1, \dots, n$, and we denoted $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$. We now use the inequality $|a+b|^{1-2H} \leq |a|^{1-2H} + |b|^{1-2H}$, followed by the identity:

$$x_1 \prod_{j=2}^n (x_j + x_{j-1}) = \sum_{a \in A_n} x_j^{\alpha_j},$$

where A_n is a set of multi-indices $a = (a_1, \dots, a_n)$ with $\text{card}(A_n) = 2^{n-1}$ such that $a_1 \in \{1, 2\}$, $a_n \in \{0, 1\}$, $a_2, \dots, a_{n-1} \in \{0, 1, 2\}$ and $\sum_{j=1}^n a_j = n$. The exact description of the set A_n is given in Lemma 2.2 of [2]. We obtain that:

$$|\eta_1|^{1-2H} \prod_{j=2}^n |\eta_j - \eta_{j-1}|^{1-2H} \leq \sum_{a \in A_n} \prod_{j=1}^n |\eta_j|^{(1-2H)a_j} = \sum_{\alpha \in D_n} \prod_{j=1}^n |\eta_j|^{\alpha_j}, \quad (24)$$

where D_n is the set of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j = (1 - 2H)a_j$ for all $j = 1, \dots, n$ and $a = (a_1, \dots, a_n) \in A_n$. Therefore,

$$\|f_n(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}}^2 \leq c_H^n \frac{t^n}{n!} \sum_{\alpha \in D_n} \int_{T_n(t)} \prod_{j=1}^n \left(\int_{\mathbb{R}} |\mathcal{F}G_{t_{j+1}-t_j}(\eta_j)|^2 |\eta_j|^{\alpha_j} d\eta_j \right) dt.$$

The inner integral above is calculated using (22). In our case, $\alpha_j = 2(1 - 2H) < 1$, since $H > 1/4$. Using the rough bound (6), we infer that:

$$\|\tilde{f}_n(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}}^2 \leq \|f_n(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}}^2 \leq C^n \frac{t^n}{n!} \sum_{\alpha \in D_n} \int_{T_n(t)} \prod_{j=1}^n (t_{j+1} - t_j)^{1-\alpha_j} dt \leq C^n \frac{t^{(2H+2)n}}{(n!)^{2H+2}}, \quad (25)$$

where $C > 0$ is a constant that depends on H which is different in each of its appearances. Using Minkowski inequality, hypercontractivity property (14), and (25), we obtain:

$$\begin{aligned} \|u_\theta(t, x)\|_p &\leq \sum_{n \geq 0} \theta^{n/2} (p-1)^{n/2} \|I_n(f_n(\cdot, x; t))\|_2 = \sum_{n \geq 0} \theta^{n/2} (p-1)^{n/2} (n!)^{1/2} \|\tilde{f}_n(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}} \\ &\leq \sum_{n \geq 0} \theta^{n/2} (p-1)^{n/2} C^{n/2} \frac{t^{(H+1)n}}{(n!)^{H+1/2}}. \end{aligned}$$

Relation (23) now follows using (21). \square

3 Wave equation with delta initial condition

In this section, we study the (hAm) on $[r, \infty)$ with zero initial condition initial velocity given by the measure δ_z . The relation between the solution of this model and the Malliavin derivative Du_θ plays an important role in the present article.

For any $\theta > 0$, $r > 0$ and $z \in \mathbb{R}$ fixed, we consider the following equation:

$$\begin{cases} \frac{\partial^2 v}{\partial t^2}(t, x) = \frac{\partial^2 v}{\partial x^2}(t, x) + \sqrt{\theta} v(t, x) \dot{W}(x), & t > r, x \in \mathbb{R} \\ v(r, \cdot) = 0, \quad \frac{\partial v}{\partial t}(r, \cdot) = \delta_z. \end{cases} \quad (26)$$

We say that $v_\theta^{(r,z)}$ is a (mild Skorohod) *solution* of (26) if it satisfies the following equation:

$$v_\theta^{(r,z)}(t, x) = G_{t-r}(x - z) + \sqrt{\theta} \int_r^t \int_{\mathbb{R}} G_{t-s}(x - y) v_\theta^{(r,z)}(s, y) W(\delta y) ds, \quad t \geq r. \quad (27)$$

It can be proved that if a solution $v_\theta^{(r,z)}$ exists, then it is unique and it has the chaos expansion:

$$v_\theta^{(r,z)}(t, x) = G_{t-r}(x - z) + \sum_{n \geq 1} \theta^{n/2} I_n(g_n(\cdot, z, x; r, t)), \quad (28)$$

where

$$g_n(x_1, \dots, x_n, z, x; r, t) = \int_{r < t_1 < \dots < t_n < t} g_n(t_1, x_1, \dots, t_n, x_n, r, z, t, x) dt_1 \dots dt_n, \quad (29)$$

and

$$g_n(t_1, x_1, \dots, t_n, x_n, r, z, t, x) = G_{t-t_n}(x - x_n) \dots G_{t_1-r}(x_1 - z)$$

We denote $g_0(z, x; r, t) = g_0(r, z, t, x) = G_{t-r}(x-z)$. The necessary and sufficient condition for the existence of the solution $v_\theta^{(r,z)}$ is that the series in (28) converges in $L^2(\Omega)$, i.e.

$$\sum_{n \geq 1} \theta^n n! \|\tilde{g}_n(\cdot, z, x; r, t)\|_{\mathcal{P}_0^{\otimes n}}^2 < \infty, \quad (30)$$

where $\tilde{g}_n(\cdot, z, x; r, t)$ is the symmetrization of $g_n(\cdot, z, x; r, t)$.

Due to the special form (2) of G , $v_\theta^{(r,z)}(t, x)$ satisfies the following identity:

$$v_\theta^{(r,z)}(t, x) = 2G_{t-r}(x-z)v_\theta^{(r,z)}(t, x). \quad (31)$$

The existence of the solution v_θ and some of its properties follow from a connection with the (hAm) model with another Gaussian noise (called \mathfrak{X}), which is *white in time* and fractional in space (with the same Hurst index H as the noise W):

$$\begin{cases} \frac{\partial^2 V}{\partial t^2}(t, x) = \frac{\partial^2 V}{\partial x^2}(t, x) + \sqrt{\theta}V(t, x)\dot{\mathfrak{X}}(t, x), & t > r, x \in \mathbb{R} \\ V(r, \cdot) = 0, \quad \frac{\partial V}{\partial t}(r, \cdot) = \delta_z. \end{cases} \quad (32)$$

More precisely, $\{\mathfrak{X}(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ is a zero-mean Gaussian process with covariance

$$\mathbb{E}[\mathfrak{X}(\varphi)\mathfrak{X}(\psi)] = c_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t, \cdot) \overline{\mathcal{F}\psi(t, \cdot)}(\xi) |\xi|^{1-2H} d\xi dt =: \langle \varphi, \psi \rangle_{\mathcal{H}_0}, \quad (33)$$

which can be extended to an isonormal Gaussian process $\mathfrak{X} = \{\mathfrak{X}(\varphi); \varphi \in \mathcal{H}_0\}$, where \mathcal{H}_0 is the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$.

We say that $V_\theta^{(r,z)}$ is a (*mild Skorohod*) solution of (32) if it satisfies:

$$V_\theta^{(r,z)}(t, x) = G_{t-r}(x-z) + \sqrt{\theta} \int_r^t \int_{\mathbb{R}} G_{t-s}(x-y) V_\theta^{(r,z)}(s, y) \mathfrak{X}(ds, dy), \quad t \geq r. \quad (34)$$

Using a similar method to the proof of Theorem 3.8 of [3], we show in Appendix B that equation (32) has a unique solution $V_\theta^{(r,z)}$, and its moments are uniformly bounded:

$$K_{\theta, T, p} := \sup_{0 \leq r < t \leq T} \sup_{x, z \in \mathbb{R}} \|V_\theta^{(r,z)}(t, x)\|_p < \infty. \quad (35)$$

See Example (B.2). Moreover, the solution $V_\theta^{(r,z)}$ has the following chaos expansion

$$V_\theta^{(r,z)}(t, x) = G_{t-r}(x-z) + \sum_{n \geq 1} \theta^{n/2} I_n^{\mathfrak{X}}(g_n(\cdot, r, z, t, x)), \quad (36)$$

where $I_n^{\mathfrak{X}}$ is the n th multiple integral with respect to \mathfrak{X} . Since the solution $V_\theta^{(r,z)}$ exists, the series (36) converges in $L^2(\Omega)$. i.e.

$$\sum_{n \geq 1} n! \theta^n \|\tilde{g}_n(\cdot, r, z, t, x)\|_{\mathcal{H}_0^{\otimes n}}^2 < \infty, \quad (37)$$

where $\tilde{g}_n(\cdot, r, z, t, x)$ is the symmetrization of $g_n(\cdot, r, z, t, x)$.

We now establish the existence of the solution $v_\theta^{(r,z)}$ and we show that its moments are also uniformly bounded.

Theorem 3.1. For any $r > 0$, $z \in \mathbb{R}$, equation (26) has a unique solution $v_\theta^{(r,z)}$. Moreover, for any $\theta > 0$, $p \geq 2$ and $T > 0$, we have:

$$C_{\theta,p,T,v} := \sup_{0 \leq r \leq t \leq T} \sup_{z,x \in \mathbb{R}} \|v_\theta^{(r,z)}(t,x)\|_p < \infty. \quad (38)$$

Proof. We denote $\boldsymbol{\xi} = (\xi_1 \dots d\xi_n)$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \|g_n(\cdot, z, x; r, t)\|_{\mathcal{P}_0^{\otimes n}}^2 \\ &= c_H^n \int_{\mathbb{R}^n} \left| \int_{r < t_1 < \dots < t_n < t} \mathcal{F}g_n(t_1, \cdot, \dots, t_n, \cdot, r, z, t, x)(\boldsymbol{\xi}) dt \right|^2 \prod_{j=1}^n |\xi_j|^{1-2H} d\xi \\ &\leq c_H^n \frac{(t-r)^n}{n!} \int_{\mathbb{R}^n} \int_{r < t_1 < \dots < t_n < t} |\mathcal{F}g_n(t_1, \cdot, \dots, t_n, \cdot, r, z, t, x)(\boldsymbol{\xi})|^2 dt \prod_{j=1}^n |\xi_j|^{1-2H} d\xi \\ &= \frac{(t-r)^n}{n!} \|g_n(\cdot, r, z, t, x)\|_{\mathcal{H}_0^{\otimes n}}^2. \end{aligned}$$

Since $g_n(t_1, x_1, \dots, t_n, x_n, r, z, t, x)$ contains the indicator of the set $t_1 < \dots < t_n$,

$$\|g_n(\cdot, r, z, t, x)\|_{\mathcal{H}_0^{\otimes n}}^2 = n! \|\tilde{g}_n(\cdot, r, z, t, x)\|_{\mathcal{H}_0^{\otimes n}}^2.$$

We obtain that:

$$\|\tilde{g}_n(\cdot, z, x; r, t)\|_{\mathcal{P}_0^{\otimes n}}^2 \leq \|g_n(\cdot, z, x; r, t)\|_{\mathcal{P}_0^{\otimes n}}^2 \leq (t-r)^n \|\tilde{g}_n(\cdot, r, z, t, x)\|_{\mathcal{H}_0^{\otimes n}}^2,$$

The converges of the series (30) follows from (37). This proves the existence of the solution $v_\theta^{(r,z)}$. Moreover,

$$\begin{aligned} \mathbb{E}|v_\theta^{(r,z)}(t,x)|^2 &= \sum_{n \geq 0} \theta^n n! \|\tilde{g}_n(\cdot, z, x; r, t)\|_{\mathcal{P}_0^{\otimes n}}^2 \\ &\leq \sum_{n \geq 0} \theta^n n! (t-r)^n \|\tilde{g}_n(\cdot, r, z, t, x)\|_{\mathcal{H}_0^{\otimes n}}^2 = \mathbb{E}|V_{\theta(t-r)}^{(r,z)}(t,x)|^2. \end{aligned} \quad (39)$$

To treat the higher moments, we use Lê's hypercontractivity principle. Note that this principle was originally developed in [23] for the heat equation, but it is in fact valid for a larger class of SPDEs (see Theorem B.1 of [1]). From this principle, combined with (39), we deduce that

$$\|v_\theta^{(r,z)}(t,x)\|_p \leq \|v_{(p-1)\theta}^{(r,z)}(t,x)\|_2 \leq \|V_{(p-1)\theta(t-r)}^{(r,z)}(t,x)\|_2 \leq \|V_{(p-1)\theta T}^{(r,z)}(t,x)\|_2.$$

Relation (38) follows from (35). \square

We conclude this section with a uniform bound for some integrals of v_θ , which will be used in the proof of Theorem 1.2.

Lemma 3.2. For any $\theta > 0$, $p \geq 2$, $q > 0$ and $t > 0$,

$$\sup_{r \in [0,t]} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}} \|v_\theta^{(r,z)}(t,x)\|_p^q dx \leq 2t C_{\theta,p,t,v}^q \quad \text{and} \quad \sup_{r \in [0,t]} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \|v_\theta^{(r,z)}(t,x)\|_p^q dz \leq 2t C_{\theta,p,t,v}^q,$$

where $C_{\theta,p,t,v}$ is the constant given by relation (38).

Proof. Using identity (31) and relation (38), we have:

$$\begin{aligned} \int_{\mathbb{R}} \|v_{\theta}^{(r,z)}(t, x)\|_p^q dx &= 2^q \int_{\mathbb{R}} G_{t-r}^q(x-z) \|v_{\theta}^{(r,z)}(t, x)\|_p dx \\ &\leq C_{\theta,t,p,v}^q 2^q \int_{\mathbb{R}} G_{t-r}^q(x-z) dx = 2C_{\theta,t,p,v}^q (t-r). \end{aligned}$$

The other statement is proved similarly. \square

4 Estimates for the Mallivian derivatives

In this section, we give some estimates for $Du_{\theta}(t, x)$ and $D^2u_{\theta}(t, x)$ and their increments, which are obtained using a connection with v_{θ} . If we denote by U_{θ} the solution of (hAm) with white noise in time \mathfrak{X} , it can be shown that $D_{r,z}U_{\theta}(t, x) = U_{\theta}(r, z)V_{\theta}^{(r,z)}(t, x)$. This relation does not hold for the time-independent noise. Luckily, with the help of Lemmas 2.2 and 2.1, we can still develop a connection between $Du_{\theta}(t, x)$ and v_{θ} .

We start by observing that for any fixed $z \in \mathbb{R}$, $D_z u_{\theta}(t, x)$ has the chaos expansion:

$$D_z u_{\theta}(t, x) = \sum_{n \geq 1} \theta^{n/2} n I_{n-1}(\tilde{f}_n(\cdot, z, x; t)). \quad (40)$$

See (9). For any $z \in \mathbb{R}^d$ fixed, we have the decomposition:

$$\tilde{f}_n(\cdot, z, x; t) = \frac{1}{n} \sum_{j=1}^n h_j^{(n)}(\cdot, z, x; t), \quad (41)$$

where $h_j^{(n)}(\cdot, z, x; t)$ is the symmetrization of the function $f_j^{(n)}(\cdot, z, x; t)$ given by:

$$\begin{aligned} f_j^{(n)}(x_1, \dots, x_{n-1}, z, x; t) &= f_n(x_1, \dots, x_{j-1}, z, x_j, \dots, x_{n-1}, x; t) \\ &= \int_{\{0 < t_1 < \dots < t_{j-1} < r < t_j < \dots < t_{n-1} < t\}} G_{t-t_{n-1}}(x-x_{n-1}) \dots G_{t_j-r}(x_j-z) G_{r-t_{j-1}}(z-x_{j-1}) \dots \\ &\quad G_{t_2-t_1}(x_2-x_1) dt_1 \dots dt_{n-1} dr. \end{aligned}$$

If the series (40) converges in $L^2(\Omega)$, then $u_{\theta}(t, x) \in \mathbb{D}^{1,2}$ and $D.u_{\theta}(t, x)$ is a function in \mathfrak{z} . In this case,

$$D_z u_{\theta}(t, x) = \sum_{n \geq 1} \theta^{n/2} \sum_{j=1}^n I_{n-1}(f_j^{(n)}(\cdot, z, x; t)). \quad (42)$$

For the proof of the QCLT in Section 6.2 below, we will need some estimates for the increments of the Mallivian derivatives of u_{θ} . We include these estimates in the next two theorems. Parts a) of these theorems give some estimates which are not needed in the present paper. We include these estimates for the sake of comparison with similar results that exist in the literature, e.g. Theorem 1.3 of [4] for the colored noise in time, respectively Theorems 3.1-3.2 of [5] for the time-independent noise.

Theorem 4.1. a) For any $\theta > 0$, $p \geq 2$, $0 \leq t \leq T$ and $x, z \in \mathbb{R}$,

$$\|D_z u_\theta(t, x)\|_p \leq C \int_0^t G_{t-r}(x - z) dr, \quad (43)$$

where $C = 2\sqrt{\theta}C_{4(p-1)\theta, 2, T, u}C_{4(p-1)\theta, 2, T, v}$, and $C_{\theta, p, T, u}, C_{\theta, p, T, v}$ are the constants given by Theorems 2.3 and 3.1.

b) For any $\theta > 0$, $p \geq 2$, $0 \leq t \leq T$ and $x, z, z' \in \mathbb{R}$,

$$\|D_z u_\theta(t, x) - D_{z+z'} u_\theta(t, x)\|_p \leq \sqrt{2\theta} \int_0^t I_1(z, z', x; r, t) dr,$$

where

$$\begin{aligned} I_1(z, z', x; r, t) &= \|u_\eta(r, z) - u_\eta(r, z + z')\|_2 \|v_\eta^{(r, z)}(t, x)\|_2 + \\ &\quad \|u_\eta(r, z + z')\|_2 \|v_\eta^{(r, z)}(t, x) - v_\eta^{(r, z+z')}(t, x)\|_2 \end{aligned}$$

and $\eta = 4(p-1)\theta$.

Proof. a) Using notations (18) and (29), we can write

$$f_j^{(n)}(\cdot, z, x; t) = \int_0^t f_{j-1}(\cdot, z; r) \otimes g_{n-j}(\cdot, z, x; r, t) dr. \quad (44)$$

By stochastic Fubini's theorem, we can commute the dr integral with the multiple Wiener integral I_{n-1} . We obtain:

$$I_{n-1}(f_j^{(n)}(\cdot, z, x; t)) = \int_0^t I_{n-1}(f_{j-1}(\cdot, z; r) \otimes g_{n-j}(\cdot, z, x; r, t)) dr. \quad (45)$$

By the definition (29) and the special form $2G_t(x) = \mathbf{1}_{|x| \leq t} = 4G_t^2(x)$, we have the identity $g_{n-j}(\cdot, z, x; r, t) = 2G_{t-r}(x - z)g_{n-j}(\cdot, z, x; r, t)$. Hence, by (45) and Cauchy-Schwarz inequality, as well as the Lemma 2.1, we have

$$\begin{aligned} \left\| I_{n-1}(f_j^{(n)}(\cdot, z, x; t)) \right\|_2^2 &= \mathbb{E} \left[\left(\int_0^t 2G_{t-r}(x - z) I_{n-1}(f_{j-1}(\cdot, z; r) \otimes g_{n-j}(\cdot, z, x; r, t)) dr \right)^2 \right] \\ &\leq \int_0^t 2G_{t-r}(x - z) dr \int_0^t \|I_{n-1}(f_{j-1}(\cdot, z; r) \otimes g_{n-j}(\cdot, z, x; r, t))\|_2^2 dr \\ &\leq \binom{n-1}{j-1} \int_0^t 2G_{t-r}(x - z) dr \int_0^t \|I_{j-1}(f_{j-1}(\cdot, z; r))\|_2^2 \|I_{n-j}(g_{n-j}(\cdot, z, x; r, t))\|_2^2 dr. \end{aligned}$$

Thus, by (42), orthogonality and Cauchy-Schwarz inequality as well as the inequalities $(\sum_{j=1}^n a_j)^2 \leq n \sum_{j=1}^n a_j^2$, $n \leq 2^{n-1}$ and $\binom{n-1}{j-1} \leq 2^{n-1}$, we have

$$\|D_z u_\theta(t, x)\|_2^2 = \sum_{n \geq 1} \theta^n \left\| \sum_{j=1}^n I_{n-1}(f_j^{(n)}(\cdot, z, x; t)) \right\|_2^2 \leq \sum_{n \geq 1} \theta^n n \sum_{j=1}^n \left\| I_{n-1}(f_j^{(n)}(\cdot, z, x; t)) \right\|_2^2$$

$$\begin{aligned}
&\leq \int_0^t 2G_{t-r}(x-z)dr \sum_{n \geq 1} 4^{n-1}\theta^n \sum_{j=1}^n \int_0^t \|I_{j-1}(f_{j-1}(\cdot, z; r))\|_2^2 \|I_{n-j}(g_{n-j}(\cdot, z, x; r, t))\|_2^2 dr \\
&= \theta \int_0^t 2G_{t-r}(x-z)dr \\
&\int_0^t \sum_{j \geq 1} (4\theta)^{j-1} \|I_{j-1}(f_{j-1}(\cdot, z; r))\|_2^2 \left(\sum_{n \geq j} (4\theta)^{n-j} \|I_{n-j}(g_{n-j}(\cdot, z, x; r, t))\|_2^2 \right) dr \\
&= \theta \int_0^t 2G_{t-r}(x-z)dr \int_0^t \|u_{4\theta}(r, z)\|_2^2 \|v_{4\theta}^{(r,z)}(t, x)\|_2^2 dr.
\end{aligned}$$

Using identity (31), and Theorems 2.3 and 3.1, we have

$$\begin{aligned}
\|D_z u_\theta(t, x)\|_2^2 &\leq 4\theta \int_0^t G_{t-r}(x-z)dr \int_0^t \|u_{4\theta}(r, z)\|_2^2 \|v_{4\theta}^{(r,z)}(t, x)\|_2^2 G_{t-r}(x-z)dr \\
&\leq 4\theta C_{4\theta, 2, T, u}^2 C_{4\theta, 2, T, v}^2 \left(\int_0^t G_{t-r}(x-z)dr \right)^2.
\end{aligned}$$

By Lemma A.1, we have

$$\|D_z u_\theta(t, x)\|_p \leq \frac{1}{\sqrt{p-1}} \|D_z u_{(p-1)\theta}(t, x)\|_2 \leq 2\sqrt{\theta} C_{4(p-1)\theta, 2, t, u} C_{4(p-1)\theta, 2, t, v} \int_0^t G_{t-r}(x-z)dr.$$

b) By Lemma A.1, (42), orthogonality, Cauchy-Schwarz inequality, the inequality $n \leq 2^{n-1}$, Lemma 2.2, we have:

$$\begin{aligned}
\|D_z u_\theta(t, x) - D_{z+z'} u_\theta(t, x)\|_p^2 &\leq \frac{1}{p-1} \|D_z u_{(p-1)\theta}(t, x) - D_{z+z'} u_{(p-1)\theta}(t, x)\|_2^2 \\
&= \frac{1}{p-1} \sum_{n \geq 1} ((p-1)\theta)^n \left\| \sum_{j=1}^n I_{n-1} \left(f_j^{(n)}(\cdot, z, x; t) - f_j^{(n)}(\cdot, z+z', x; t) \right) \right\|_2^2 \\
&\leq \frac{1}{p-1} \sum_{n \geq 1} ((p-1)\theta)^n n \sum_{j=1}^n \left\| I_{n-1} \left(f_j^{(n)}(\cdot, z, x; t) - f_j^{(n)}(\cdot, z+z', x; t) \right) \right\|_2^2 \\
&\leq \theta \sum_{n \geq 1} (2(p-1)\theta)^{n-1} \sum_{j=1}^n \left\| I_{n-1} \left(f_j^{(n)}(\cdot, z, x; t) - f_j^{(n)}(\cdot, z+z', x; t) \right) \right\|_2^2.
\end{aligned}$$

Let $\tau = 2(p-1)\theta$. Using (44) for expressing $f_j^{(n)}(\cdot, z, x; t)$ and $f_j^{(n)}(\cdot, z+z', x; t)$, we have:

$$\|D_z u_\theta(t, x) - D_{z+z'} u_\theta(t, x)\|_p^2 \leq 2\theta(T_1 + T_2),$$

where

$$T_1 = \sum_{n \geq 1} \tau^{n-1} \sum_{j=1}^n \left\| I_{n-1} \left(\int_0^t (f_{j-1}(\cdot, z; r) - f_{j-1}(\cdot, z+z'; r)) \otimes g_{n-j}(\cdot, z, x; r, t) dr \right) \right\|_2^2$$

$$T_2 = \sum_{n \geq 1} \tau^{n-1} \sum_{j=1}^n \left\| I_{n-1} \left(\int_0^t f_{j-1}(\cdot, z + z'; r) \otimes (g_{n-j}(\cdot, z, x; r, t) - g_{n-j}(\cdot, z + z', x; r, t)) dr \right) \right\|_2^2.$$

We now use the fact that for any $\theta > 0$ and for any functions $(h_n)_{n \geq 0}$ and $(h'_n)_{n \geq 0}$ for which the integrals below are well-defined, by Lemma 2.2, we have:

$$\begin{aligned} & \sum_{n \geq 1} \theta^{n-1} \sum_{j=1}^n \left\| I_{n-1} \left(\int_0^t h_{j-1}(\cdot, r) \otimes h'_{n-j}(\cdot, r) dr \right) \right\|_2^2 \\ &= \sum_{p \geq 0} \sum_{q \geq 0} \theta^{p+q} \left\| I_{p+q} \left(\int_0^t h_p(\cdot, r) \otimes h'_q(\cdot, r) dr \right) \right\|_2^2 \\ &\leq \left(\int_0^t \left\| \sum_{p \geq 0} (2\theta)^{p/2} I_p(h_p(\cdot, r)) \right\|_2 \left\| \sum_{q \geq 0} (2\theta)^{q/2} I_q(h'_q(\cdot, r)) \right\|_2 dr \right)^2. \end{aligned}$$

The conclusion follows using the chaos decomposition of $u_\theta(t, x)$ and $v_\theta^{(r,z)}(t, x)$. \square

We now examine the second Malliavin derivative. For any fixed $w, z \in \mathbb{R}$, we have the following chaos expansion

$$D_{w,z}^2 u_\theta(t, x) = \sum_{n \geq 2} n(n-1) \theta^{n/2} I_{n-2}(\tilde{f}_n(\cdot, w, z, x; t)). \quad (46)$$

Note that

$$\tilde{f}_n(\cdot, w, z, x; t) = \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^n h_{ij}^{(n)}(\cdot, w, z, x; t), \quad (47)$$

where $h_{ij}^{(n)}(\cdot, w, z, x; t)$ is the symmetrization of the function $f_{ij}^{(n)}(\cdot, w, z, x; t)$ defined as follows. If $i < j$,

$$\begin{aligned} & f_{ij}^{(n)}(x_1, \dots, x_{n-2}, w, z, x; t) \\ &= f_n(x_1, \dots, x_{i-1}, w, x_i, \dots, x_{j-2}, z, x_{j-1}, \dots, x_{n-2}, x; t) \\ &= \int_{\{0 < t_1 < \dots < t_{i-1} < s < t_i < \dots < t_{j-2} < r < t_{j-1} < \dots < t_{n-2} < t\}} G_{t-t_{n-2}}(x - x_{n-2}) \dots G_{t_{j-1}-r}(x_{j-1} - z) \\ & G_{r-t_{j-2}}(z - x_{j-2}) \dots G_{t_i-s}(x_i - w) G_{s-t_{i-1}}(w - x_{i-1}) \dots G_{t_2-t_1}(x_2 - x_1) dt_1 \dots dt_{n-2} dr ds \\ &= \int_{0 < s < r < t} dr ds f_{i-1}(x_1, \dots, x_{i-1}, w; s) g_{j-i-1}(x_i, \dots, x_{j-2}, w, z; s, r) g_{n-j}(x_{j-1}, \dots, x_{n-2}, z, x; r, t). \end{aligned}$$

If $j < i$,

$$\begin{aligned} & f_{ij}^{(n)}(x_1, \dots, x_{n-2}, w, z, x; t) \\ &= \int_{0 < s < r < t} dr ds f_{j-1}(x_1, \dots, x_{j-1}, z; s) g_{i-j-1}(x_j, \dots, x_{i-2}, z, w; s, r) g_{n-i}(x_{i-1}, \dots, x_{n-2}, w, x; r, t) \end{aligned}$$

In both cases, w is on position i and z is on position j . Consequently,

$$D_{w,z}^2 u_\theta(t, x) = \sum_{n \geq 2} \theta^{n/2} \sum_{i,j=1, i \neq j}^n I_{n-2}(f_{ij}^{(n)}(\cdot, w, z, x; t)). \quad (48)$$

Theorem 4.2. a) For any $\theta > 0$, $p \geq 2$, $0 \leq t \leq T$ and $x, z, w \in \mathbb{R}$,

$$\|D_{w,z}^2 u_\theta(t, x)\|_p \leq C \tilde{f}_2(w, z, t, x) \quad (49)$$

where $C = 4\theta C_{6\theta(p-1), 2, T, u} C_{6\theta(p-1), 2, T, v}^2$.

b) For any $\theta > 0$, $p \geq 2$, $0 \leq t \leq T$ and $x, z, z', w, w' \in \mathbb{R}$,

$$\begin{aligned} & \left\| D_{z,y}^2 u_\theta(t, x) - D_{z,y+y'}^2 u_\theta(t, x) - D_{z+z',y}^2 u_\theta(t, x) + D_{z+z',y+y'}^2 u_\theta(t, x) \right\|_p \\ & \leq 4\theta \int_{0 < s < r < t} (I_2(z, z', y, y', x; s, r, t) + I_2(y, y', z, z', x; s, r, t)) ds dr, \end{aligned} \quad (50)$$

where $I_2(z, z', y, y', x; s, r, t) = \sum_{k=1}^4 I_{2,k}(z, z', y, y', x; s, r, t)$ and

$$\begin{aligned} I_{2,1}(z, z', y, y', x; s, r, t) &= \|u_\eta(s, z) - u_\eta(s, z + z')\|_2 \|v_\eta^{(s,z)}(r, y)\|_2 \left\| v_\eta^{(r,y)}(t, x) - v_\eta^{(r,y+y')}(t, x) \right\|_2 \\ I_{2,2}(z, z', y, y', x; s, r, t) &= \|u_\eta(s, z + z')\|_2 \left\| v_\eta^{(s,z)}(r, y) - v_\eta^{(s,z+z')}(r, y) \right\|_2 \left\| v_\eta^{(r,y)}(t, x) - v_\eta^{(r,y+y')}(t, x) \right\|_2 \\ I_{2,3}(z, z', y, y', x; s, r, t) &= \|u_\eta(s, z) - u_\eta(s, z + z')\|_2 \|v_\eta^{(s,z)}(r, y) - v_\eta^{(s,z)}(r, y + y')\|_2 \left\| v_\eta^{(r,y+y')}(t, x) \right\|_2 \\ I_{2,4}(z, z', y, y', x; s, r, t) &= \|u_\eta(s, z + z')\|_2 \\ & \left\| v_\eta^{(s,z)}(r, y) - v_\eta^{(s,z)}(r, y + y') - v_\eta^{(s,z+z')}(r, y) + v_\eta^{(s,z+z')}(r, y + y') \right\|_2 \left\| v_\eta^{(r,y+y')}(t, x) \right\|_2. \end{aligned}$$

and $\eta = 6(p-1)\theta$.

Proof. a) The proof is similar to the proof of Theorem 4.1.a). We omit the details.

b) By Lemma A.1, (48), orthogonality, and the inequality $(\sum_{k=1}^N a_k)^2 \leq N \sum_{k=1}^N a_k^2$,

$$\begin{aligned} & \left\| D_{z,y}^2 u_\theta(t, x) - D_{z,y+y'}^2 u_\theta(t, x) - D_{z+z',y}^2 u_\theta(t, x) + D_{z+z',y+y'}^2 u_\theta(t, x) \right\|_p^2 \\ & \leq \frac{1}{(p-1)^2} \left\| D_{z,y}^2 u_{(p-1)\theta}(t, x) - D_{z,y+y'}^2 u_{(p-1)\theta}(t, x) - D_{z+z',y}^2 u_{(p-1)\theta}(t, x) + D_{z+z',y+y'}^2 u_{(p-1)\theta}(t, x) \right\|_2^2 \\ & = \frac{1}{(p-1)^2} \sum_{n \geq 2} ((p-1)\theta)^n \left\| \sum_{i,j=1, i \neq j}^n I_{n-2} \left(f_{ij}^{(n)}(\cdot, z, y, x; t) - f_{ij}^{(n)}(\cdot, z, y + y', x; t) \right. \right. \\ & \quad \left. \left. - f_{ij}^{(n)}(\cdot, z + z', y, x; t) + f_{ij}^{(n)}(\cdot, z + z', y + y', x; t) \right) \right\|_2^2 \\ & \leq \frac{1}{(p-1)^2} \sum_{n \geq 2} ((p-1)\theta)^n n(n-1) \sum_{i,j=1, i \neq j}^n \left\| I_{n-2} \left(f_{ij}^{(n)}(\cdot, z, y, x; t) - f_{ij}^{(n)}(\cdot, z, y + y', x; t) \right. \right. \\ & \quad \left. \left. - f_{ij}^{(n)}(\cdot, z + z', y, x; t) + f_{ij}^{(n)}(\cdot, z + z', y + y', x; t) \right) \right\|_2^2 =: \frac{1}{(p-1)^2} (S_1 + S_2). \end{aligned} \quad (51)$$

where S_1, S_2 denote the sums corresponding to $i < j$, respectively $j < i$.

We first treat the sum S_1 . When $i < j$, we can write

$$f_{ij}^{(n)}(\cdot, z, y, x; t) = \int_{0 < s < r < t} f_{i-1}(\cdot, z; s) \otimes g_{j-i-1}(\cdot, z, y; s, r) \otimes g_{n-j}(\cdot, y, x; r, t) ds dr.$$

Besides, we have the following decomposition (we pair the first two terms and the last two terms in the first step, and then pair the first and the third term in the second step):

$$\begin{aligned}
& a_z \otimes b_{z,y} \otimes c_y - a_z \otimes b_{z,y+y'} \otimes c_{y+y'} - a_{z+z'} \otimes b_{z+z',y} \otimes c_y + a_{z+z'} \otimes b_{z+z',y+y'} \otimes c_{y+y'} \\
= & a_z \otimes b_{z,y} \otimes (c_y - c_{y+y'}) + a_z \otimes (b_{z,y} - b_{z,y+y'}) \otimes c_{y+y'} \\
& - a_{z+z'} \otimes b_{z+z',y} \otimes (c_y - c_{y+y'}) - a_{z+z'} \otimes (b_{z+z',y} - b_{z+z',y+y'}) \otimes c_{y+y'} \\
= & (a_z - a_{z+z'}) \otimes b_{z,y} \otimes (c_y - c_{y+y'}) + a_{z+z'} \otimes (b_{z,y} - b_{z+z',y}) \otimes (c_y - c_{y+y'}) \\
& + (a_z - a_{z+z'}) \otimes (b_{z,y} - b_{z,y+y'}) \otimes c_{y+y'} + a_{z+z'} \otimes (b_{z,y} - b_{z,y+y'} - b_{z+z',y} + b_{z+z',y+y'}) \otimes c_{y+y'}.
\end{aligned}$$

Hence, for any $1 \leq i < j \leq n$,

$$\begin{aligned}
& f_{ij}^{(n)}(\cdot, z, y, x; t) - f_{ij}^{(n)}(\cdot, z, y + y', x; t) - f_{ij}^{(n)}(\cdot, z + z', y, x; t) + f_{ij}^{(n)}(\cdot, z + z', y + y', x; t) \\
& = \sum_{k=1}^4 \int_{0 < s < r < t} A_{ij,k}^{(n)}(s, r) ds dr,
\end{aligned}$$

where

$$\begin{aligned}
A_{ij,1}^{(n)}(s, r) & := (f_{i-1}(\cdot, z; s) - f_{i-1}(\cdot, z + z'; s)) \otimes g_{j-i-1}(\cdot, z, y; s, r) \\
& \quad \otimes (g_{n-j}(\cdot, y, x; r, t) - g_{n-j}(\cdot, y + y', x; r, t)) \\
A_{ij,2}^{(n)}(s, r) & := f_{i-1}(\cdot, z + z'; s) \otimes (g_{j-i-1}(\cdot, z, y; s, r) - g_{j-i-1}(\cdot, z + z', y; s, r)) \\
& \quad \otimes (g_{n-j}(\cdot, y, x; r, t) - g_{n-j}(\cdot, y + y', x; r, t)) \\
A_{ij,3}^{(n)}(s, r) & := (f_{i-1}(\cdot, z; s) - f_{i-1}(\cdot, z + z'; s)) \otimes (g_{j-i-1}(\cdot, z, y; s, r) - g_{j-i-1}(\cdot, z, y + y'; s, r)) \\
& \quad \otimes g_{n-j}(\cdot, y + y', x; r, t) \\
A_{ij,4}^{(n)}(s, r) & := f_{i-1}(\cdot, z + z'; s) \otimes (g_{j-i-1}(\cdot, z, y; s, r) - g_{j-i-1}(\cdot, z, y + y'; s, r) - g_{j-i-1}(\cdot, z + z', y; s, r) \\
& \quad + g_{j-i-1}(\cdot, z + z', y + y'; s, r)) \otimes g_{n-j}(\cdot, y + y', x; r, t)
\end{aligned}$$

Using inequality $n(n-1) \leq 2^n$ and the inequality $(\sum_{k=1}^4 a_k)^4 \leq 4 \sum_{k=1}^4 a_k^2$ we have

$$\begin{aligned}
S_1 & \leq \sum_{n \geq 2} \sum_{1 \leq i < j \leq n} (2(p-1)\theta)^n \left\| I_{n-2} \left(f_{ij}^{(n)}(\cdot, z, y, x; t) - f_{ij}^{(n)}(\cdot, z, y + y', x; t) - f_{ij}^{(n)}(\cdot, z + z', y, x; t) \right. \right. \\
& \quad \left. \left. + f_{ij}^{(n)}(\cdot, z + z', y + y', x; t) \right) \right\|_2^2 \\
& \leq 16(p-1)^2 \theta^2 \sum_{k=1}^4 \sum_{n \geq 2} (2(p-1)\theta)^{n-2} \sum_{1 \leq i < j \leq n} \left\| I_{n-2} \left(\int_{0 < s < r < t} A_{ij,k}^{(n)}(s, r) dr dr \right) \right\|_2^2.
\end{aligned}$$

We now use the fact that for any $\theta > 0$ and for any functions h_n, h'_n, h''_n for which the integrals below are well-defined, by Lemma 2.2, we have:

$$\sum_{n \geq 2} \theta^{n-2} \sum_{i,j=1, i < j}^n \left\| I_{n-2} \left(\int_{0 < s < r < t} h_{j-1}(\cdot, s) \otimes h'_{j-i-1}(\cdot, s, r) \otimes h''_{n-j}(\cdot, r) ds dr \right) \right\|_2^2$$

$$\begin{aligned}
&= \sum_{p \geq 0} \sum_{q \geq 0} \sum_{k \geq 0} \theta^{p+q+k} \left\| I_{p+q+k} \left(\int_{0 < s < r < t} h_p(\cdot, s) \otimes h'_q(\cdot, s, r) \otimes h''_k(\cdot, r) ds dr \right) \right\|_2^2 \\
&\leq \left(\int_{0 < s < r < t} \left\| \sum_{p \geq 0} (3\theta)^{p/2} I_p(h_p(\cdot, s)) \right\|_2 \left\| \sum_{q \geq 0} (3\theta)^{q/2} I_q(h'_q(\cdot, s, r)) \right\|_2 \left\| \sum_{k \geq 0} (3\theta)^{k/2} I_k(h''_k(\cdot, r)) \right\|_2 ds dr \right)^2.
\end{aligned}$$

Using the chaos decompositions of $u_\theta(t, x)$ and $v_\theta^{(r,z)}(t, x)$, we infer that:

$$S_1 \leq 16(p-1)^2 \theta^2 \sum_{k=1}^4 \left(\int_{0 < s < r < t} I_{2,k}(z, z', y, y', x; s, r, t) ds dr \right)^2.$$

Using the inequality $\sum_{i=1}^4 a_i^2 \leq (\sum_{i=1}^4 a_i)^2$ for $a_i > 0$, we obtain:

$$S_1 \leq \left(4(p-1)\theta \int_{0 < s < r < t} I_2(z, z', y, y', x; s, r, t) ds dr \right)^2. \quad (52)$$

A similar formula holds for the sum S_2 (which corresponds to the case $j < i$), which is obtained by swapping (y, y', i) and (z, z', j) :

$$S_2 \leq \left(4(p-1)\theta \int_{0 < s < r < t} I_2(y, y', z, z', x_3; s, r, t) ds dr \right)^2. \quad (53)$$

Relation (71) follows from (51), (52) and (53). \square

5 Increments of u_θ and v_θ

In this section, study some integrals involving the increments of the solutions u_θ and v_θ , which will be used in the proof of the QCLT. A key role is played by some stationarity properties of these processes.

The first result examines the increments of u_θ .

Lemma 5.1. *For any $\theta > 0$, $p \geq 2$ and $T > 0$,*

$$C'_{u,p,H,T,\theta} := \sup_{(t,x) \in [0,T] \times \mathbb{R}} \int_{\mathbb{R}} \|u_\theta(t, x) - u_\theta(t, x+h)\|_p^2 |h|^{2H-2} dh < \infty.$$

Proof. By Theorem 2.3, since $H < 1/2$, we have:

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} \int_{|h| > 1} \|u_\theta(t, x) - u_\theta(t, x+h)\|_p^2 |h|^{2H-2} dh \leq 2C_{\theta,p,T,u} \int_{|h| > 1} |h|^{2H-2} dh < \infty.$$

It remains to treat the integral over the set $|h| \leq 1$.

We use the chaos expansion $u_\theta(t, x) - u_\theta(t, x+h) = \sum_{n \geq 1} \theta^{n/2} I_n(f_{n,h}(\cdot, x; t))$ where $f_{n,h}(\cdot, x; t) = f_n(\cdot, x; t) - f_n(\cdot, x+h; t)$. By hypercontractivity property (14),

$$\|u(t, x) - u(t, x+h)\|_p \leq \sum_{n \geq 1} (p-1)^{n/2} \theta^{n/2} \|I_n(f_{n,h}(\cdot, x; t))\|_2 = \sum_{n \geq 1} (p-1)^{n/2} \theta^{n/2} [J_{n,h}(t)]^{1/2} \quad (54)$$

where $J_{n,h}(t) = n! \|\tilde{f}_{n,h}(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}}^2$. Note that

$$f_{n,h}(x_1, \dots, x_n, x; t) = \int_{T_n(t)} f_{n,h}(t_1, x_1, \dots, t_n, x_n, t, x) dt,$$

where $f_{n,h}(\cdot, t, x) = f_n(\cdot, t, x) - f_n(\cdot, t, x + h)$ and $f_n(\cdot, t, x)$ is given by (19). Using the rough bound (6), followed by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} J_{n,h}(t) &\leq n! \|f_{n,h}(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}}^2 \\ &= c_H^n n! \int_{\mathbb{R}^n} \left| \int_{T_n(t)} \mathcal{F} f_{n,h}(t_1, \cdot, \dots, t_n, \cdot, t, x)(\xi_1, \dots, \xi_n) dt \right|^2 \prod_{j=1}^n |\xi_j|^{1-2H} d\xi \\ &\leq c_H^n t^n \int_{T_n(t)} \int_{\mathbb{R}^n} |\mathcal{F} f_{n,h}(t_1, \cdot, \dots, t_n, \cdot, t, x)(\xi_1, \dots, \xi_n)|^2 \prod_{j=1}^n |\xi_j|^{1-2H} d\xi dt \\ &\leq c_H^n t^n \int_{T_n(t)} \int_{\mathbb{R}^n} |1 - e^{-i(\xi_1 + \dots + \xi_n)h}|^2 \prod_{j=1}^n |\mathcal{F} G_{t_{j+1}-t_j}(\xi_1 + \dots + \xi_j)|^2 \prod_{j=1}^n |\xi_j|^{1-2H} d\xi dt \\ &= c_H^n t^n \int_{T_n(t)} \int_{\mathbb{R}^n} |1 - e^{-i\eta_n h}|^2 \prod_{j=1}^n |\mathcal{F} G_{t_{j+1}-t_j}(\eta_j)|^2 \prod_{j=1}^n |\eta_j - \eta_{j-1}|^{1-2H} d\eta dt, \end{aligned}$$

where we change the variable $\xi_1 + \dots + \xi_j = \eta_j$ with the convention $t_{n+1} = t$ and $\eta_0 = 0$.

The product above is estimated as usually, using inequality (24). Then, using identity (22), we obtain:

$$\begin{aligned} J_{n,h}(t) &\leq c_H^n t^n \sum_{\alpha \in D_n} \int_{T_n(t)} \int_{\mathbb{R}^n} |1 - e^{-i\eta_n h}|^2 \prod_{j=1}^n \int_{\mathbb{R}} |\mathcal{F} G_{t_{j+1}-t_j}(\eta_j)| \prod_{j=1}^n |\eta_j|^{\alpha_j} d\eta dt \\ &\leq c^n t^n \sum_{\alpha \in D_n} \int_0^t \left(\int_{\mathbb{R}} |1 - e^{-i\eta_n h}|^2 |\mathcal{F} G_{t-t_n}(\eta_n)|^2 |\eta_n|^{\alpha_n} \eta_n \right) \\ &\quad \left(\int_{T_{n-1}(t_n)} \prod_{j=1}^{n-1} (t_{j+1} - t_j)^{1-\alpha_j} dt_1 \dots dt_{n-1} \right) dt_n \\ &\leq c_{t,H} \frac{c^n t^n}{(n!)^{2H+1}} \int_0^t \int_{\mathbb{R}} |1 - e^{-i\eta_n h}|^2 (1 + |\eta_n|^{1-2H}) \frac{\sin^2((t-t_n)|\eta_n|)}{|\eta_n|^2} t_n^{(2H+1)n-2} d\eta_n dt_n, \end{aligned}$$

where $c_{t,H} = t^{1-2H} + 1$ and $c > 0$ is a constant depending on H (which may be different from line to line). Using the bound $\sin^2((t-t_n)|\eta_n|) \leq 1$, we obtain:

$$\begin{aligned} J_{n,h}(t) &\leq c_{t,H} \frac{c^n t^n}{(n!)^{2H+1}} t^{(2H+1)n-1} \int_{\mathbb{R}} |1 - e^{-i\eta h}|^2 (1 + |\eta|^{1-2H}) \frac{1}{|\eta|^2} d\eta \\ &\leq c_{t,H} \frac{c^n}{(n!)^{2H+1}} t^{(2H+2)n-1} (|h| + |h|^{2H}). \end{aligned}$$

The estimate above is now inserted in (54). We obtain:

$$\begin{aligned} \|u(t, x) - u(t, x + h)\|_p &\leq c_{t,H}^{1/2} \sum_{n \geq 1} (p-1)^{n/2} \theta^{n/2} \frac{c^{n/2}}{(n!)^{H+1/2}} t^{(H+1)n-1/2} (|h|^{1/2} + |h|^H) \\ &=: C_{t,p,\theta,H} (|h|^{1/2} + |h|^H), \end{aligned}$$

where $C_{t,p,\theta,H} > 0$ is a constant which depends in t, p, θ, H and is increasing in t . Hence,

$$\int_{|h| \leq 1} \|u(t, x) - u(t, x + h)\|_p^2 |h|^{2H-2} dh \leq C_{t,p,\theta,H}^2 \int_{|h| \leq 1} (|h| + |h|^{2H}) |h|^{2H-2} dh.$$

The last integral is finite due to the condition $H > 1/4$. \square

The following lemma gives some translation invariance properties of u_θ and v_θ .

Lemma 5.2. *For any $\theta > 0$, $0 \leq r \leq t$, $x, z, h \in \mathbb{R}$, we have:*

$$\begin{aligned} u_\theta(t, x) - u_\theta(t, x + h) &\stackrel{d}{=} u_\theta(t, 0) - u_\theta(t, h) \\ v_\theta^{(r,z)}(t, x) - v_\theta^{(r,z+h)}(t, x) &\stackrel{d}{=} v_\theta^{(r,0)}(t, x - z) - v_\theta^{(r,h)}(t, x - z) \\ v_\theta^{(r,z)}(t, x) - v_\theta^{(r,z)}(t, x + h) &\stackrel{d}{=} v_\theta^{(r,0)}(t, x - z) - v_\theta^{(r,0)}(t, x + h - z) \end{aligned}$$

Moreover, for the rectangular difference, we have:

$$\begin{aligned} v_\theta^{(r,z)}(t, x) - v_\theta^{(r,z+h)}(t, x) - v_\theta^{(r,z)}(t, x + k) + v_\theta^{(r,z+h)}(t, x + k) &\stackrel{d}{=} \\ v_\theta^{(r,0)}(t, x - z) - v_\theta^{(r,h)}(t, x - z) - v_\theta^{(r,0)}(t, x + k - z) + v_\theta^{(r,h)}(t, x + k - z). &\quad (55) \end{aligned}$$

Proof. The lemma follows using the chaos expansions of these differences and the fact that the noise W is spatially homogeneous, i.e. $W \stackrel{d}{=} W^{(x)}$ for any $x \in \mathbb{R}$, where $W^{(x)}(\varphi) = W(\varphi(\cdot - x))$. \square

Remark 5.3. For any $r \in [0, t]$, $h \in \mathbb{R}$, the following integrals do not depend on $x, z \in \mathbb{R}$:

$$\int_{\mathbb{R}} \left\| v_\theta^{(r,z)}(t, x') - v_\theta^{(r,z+h)}(t, x') \right\|_p^q dx' = \int_{\mathbb{R}} \left\| v_\theta^{(r,z')}(t, x) - v_\theta^{(r,z'+h)}(t, x) \right\|_p^q dz' = I_{r,t,h}^{(\theta,p,q)}, \quad (56)$$

$$\int_{\mathbb{R}} \left\| v_\theta^{(r,z)}(t, x') - v_\theta^{(r,z)}(t, x' + h) \right\|_p^q dx' = \int_{\mathbb{R}} \left\| v_\theta^{(r,z')}(t, x) - v_\theta^{(r,z')}(t, x + h) \right\|_p^q dz' = J_{r,t,h}^{(\theta,p,q)},$$

where

$$I_{r,t,h}^{(\theta,p,q)} = \int_{\mathbb{R}} \left\| v_\theta^{(r,0)}(t, x) - v_\theta^{(r,h)}(t, x) \right\|_p^q dx, \quad J_{r,t,h}^{(\theta,p,q)} = \int_{\mathbb{R}} \left\| v_\theta^{(r,0)}(t, x) - v_\theta^{(r,0)}(t, x + h) \right\|_p^q dx.$$

To see this, we apply Lemma 5.2, followed by the changes of variables $x'' = x' - z$ (for the integrals on the left), respectively $z'' = x - z'$ (for the integrals on the right). Moreover, the same argument shows that the two integrals below also do not depend on $x, z \in \mathbb{R}$:

$$\begin{aligned} &\int_{\mathbb{R}} \left\| v_\theta^{(r,z)}(t, x') - v_\theta^{(r,z+h)}(t, x') - v_\theta^{(r,z)}(t, x' + k) + v_\theta^{(r,z+h)}(t, x' + k) \right\|_p^q dx' \\ &= \int_{\mathbb{R}} \left\| v_\theta^{(r,z')}(t, x) - v_\theta^{(r,z'+h)}(t, x) - v_\theta^{(r,z')}(t, x + k) + v_\theta^{(r,z'+h)}(t, x + k) \right\|_p^q dz'. \end{aligned}$$

The next result examines the increments of v_θ . Its proof uses the connection between v_θ and V_θ , and the fact that similar inequalities hold for V_θ .

Lemma 5.4. *For any $\theta > 0$, $p \geq 2$ and $t > 0$,*

$$\begin{aligned}
a) \quad & \sup_{r \in [0, t]} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^2} \left\| v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z+h)}(t, x) \right\|_p^2 |h|^{2H-2} dx dh \leq C_{v, p, H, t, \theta}, \\
b) \quad & \sup_{r \in [0, t]} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}^2} \left\| v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z+h)}(t, x) \right\|_p^2 |h|^{2H-2} dz dh \leq C_{v, p, H, t, \theta}, \\
c) \quad & \sup_{r \in [0, t]} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^2} \left\| v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z)}(t, x+h) \right\|_p^2 |h|^{2H-2} dx dh \leq C_{v, p, H, t, \theta}, \\
d) \quad & \sup_{r \in [0, t]} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}^2} \left\| v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z)}(t, x+h) \right\|_p^2 |h|^{2H-2} dz dh \leq C_{v, p, H, t, \theta}, \\
e) \quad & \sup_{r \in [0, t]} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^3} \left\| v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z+h)}(t, x) - v_\theta^{(r, z)}(t, x+k) + v_\theta^{(r, z+h)}(t, x+k) \right\|_p^2 \\
& \quad |h|^{2H-2} |k|^{2H-2} dx dh dk \leq C_{v, p, H, t, \theta}, \\
f) \quad & \sup_{r \in [0, t]} \sup_{x \in \mathbb{R}} \int_{\mathbb{R}^3} \left\| v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z+h)}(t, x) - v_\theta^{(r, z)}(t, x+k) + v_\theta^{(r, z+h)}(t, x+k) \right\|_p^2 \\
& \quad |h|^{2H-2} |k|^{2H-2} dz dh dk \leq C_{v, p, H, t, \theta},
\end{aligned}$$

where $C_{v, p, H, t, \theta} > 0$ is a constant that depends on (p, H, t, θ) and is increasing in t .

Proof. Due to Remark 5.3 and (55), we only have to prove a), c) and e).

a) We use the following fact: for any $x, z \in \mathbb{R}$,

$$\left\| v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z+h)}(t, x) \right\|_p^2 \leq \left\| V_{(p-1)\theta t}^{(r, z)}(t, x) - V_{(p-1)\theta t}^{(r, z+h)}(t, x) \right\|_2^2. \quad (57)$$

This can be shown as in the proof of Theorem 3.1: we first prove it for $p = 2$ (replacing $g_n(\cdot, z, x; r, t)$ by $g_n(\cdot, z, x; r, t) - g_n(\cdot, z+h, x; r, t)$), and then we use the fact that $\|v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z+h)}(t, x)\|_p \leq \|v_{(p-1)\theta}^{(r, z)}(t, x) - v_{(p-1)\theta}^{(r, z+h)}(t, x)\|_2$, due to Lê's hypercontractivity principle (Theorem B.1 of [1]), noting that $v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z+h)}(t, x)$ satisfies an equation of the form given in Appendix B.1 of [1].

Then, the inequality in part a) follows from relation (98) (Example B.5), with the constant $C_{v, p, H, t, \theta} = C_{V, 2, H, t, (p-1)\theta t}$, where $C_{V, p, H, t, \theta}$ is given by (100).

c) For any $x, z \in \mathbb{R}$,

$$\begin{aligned}
\left\| v_\theta^{(r, z)}(t, x) - v_\theta^{(r, z)}(t, x+h) \right\|_p^2 & \leq \left\| v_{(p-1)\theta}^{(r, z)}(t, x) - v_{(p-1)\theta}^{(r, z)}(t, x+h) \right\|_2^2 \\
& \leq \left\| V_{(p-1)\theta t}^{(r, z)}(t, x) - V_{(p-1)\theta t}^{(r, z)}(t, x+h) \right\|_2^2,
\end{aligned}$$

where the first inequality is due to Lemma A.1 and the second inequality is proved similarly to (57). Therefore, it suffices to show that the integral

$$\int_{\mathbb{R}^2} \left\| V_{\theta t}^{(r, z)}(t, x) - V_{\theta t}^{(r, z)}(t, x+h) \right\|_2^2 |h|^{2H-2} dx dh$$

is uniformly bounded, for all $r \in [0, t]$ and $z \in \mathbb{R}$. To treat this integral, we cannot use the method of Example B.5, since $V_\theta^{(r,z)}(t, x) - V_\theta^{(r,z)}(t, x+h)$ does not satisfy an integral equation similar to the one given in Theorem B.4. So we need to proceed differently. The idea is to move the increment from the x variable to the z variable, and then use the bound given by (98) for the resulting integral. From the chaos expansion, we have:

$$\begin{aligned} \left\| V_{\theta t}^{(r,z)}(t, x) - V_{\theta t}^{(r,z)}(t, x+h) \right\|_2^2 &= \left| G_{t-r}(x-z) - G_{t-r}(x+h-z) \right|^2 + \\ &\quad \sum_{n \geq 1} (\theta t)^n n! \left\| \tilde{g}_n(\cdot, r, z, t, x) - \tilde{g}_n(\cdot, r, z, t, x+h) \right\|_{\mathcal{H}^{\otimes n}}^2. \end{aligned}$$

By direct calculation, it can be shown that

$$\left\| \tilde{g}_n(\cdot, r, z, t, x) - \tilde{g}_n(\cdot, r, z, t, x+h) \right\|_{\mathcal{H}^{\otimes n}}^2 = \left\| \tilde{g}_n(\cdot, 0, x, t-r, z) - \tilde{g}_n(\cdot, 0, x+h, t-r, z) \right\|_{\mathcal{H}^{\otimes n}}^2,$$

and hence

$$\left\| V_{\theta t}^{(r,z)}(t, x) - V_{\theta t}^{(r,z)}(t, x+h) \right\|_2^2 = \left\| V_{\theta t}^{(0,x)}(t-r, z) - V_{\theta t}^{(0,x+h)}(t-r, z) \right\|_2^2.$$

We conclude that for any $z, x \in \mathbb{R}$,

$$\begin{aligned} &\int_{\mathbb{R}^2} \left\| V_{\theta t}^{(r,z)}(t, x') - V_{\theta t}^{(r,z)}(t, x'+h) \right\|_2^2 |h|^{2H-2} dx' dh \\ &= \int_{\mathbb{R}^2} \left\| V_{\theta t}^{(0,x')} (t-r, z) - V_{\theta t}^{(0,x'+h)} (t-r, z) \right\|_2^2 |h|^{2H-2} dx' dh \\ &= \int_{\mathbb{R}^2} \left\| V_{\theta t}^{(0,x)} (t-r, z') - V_{\theta t}^{(0,x+h)} (t-r, z') \right\|_2^2 |h|^{2H-2} dz' dh, \end{aligned}$$

where the last equality is proved exactly as (56), using some translation invariance properties of V_θ (similar to those given in Lemma 5.2 for v_θ). Finally, due to relation (98), the last integral above is bounded by the constant $C_{V,2,H,t,\theta t}$.

e) For fixed $r > 0$, $z \in \mathbb{R}$ and $h \in \mathbb{R}$, the process $U_\theta^{(r,z,h)}(t, x) := v_\theta^{(r,z)}(t, x) - v_\theta^{(r,z+h)}(t, x)$ satisfies the following equation: for any $t \geq r$ and $x \in \mathbb{R}$,

$$U_\theta^{(r,z,h)}(t, x) = G_{t-r}(x-z) - G_{t-r}(x-z-h) + \int_r^t \int_{\mathbb{R}} G_{t-s}(x-y) U_\theta^{(r,z,h)}(s, y) W(\delta y) ds.$$

Therefore, by applying Lemma A.1, we infer that

$$\left\| U_\theta^{(r,z,h)}(t, x) - U_\theta^{(r,z,h)}(t, x+k) \right\|_p \leq \left\| U_{(p-1)\theta}^{(r,z,h)}(t, x) - U_{(p-1)\theta}^{(r,z,h)}(t, x+k) \right\|_2,$$

which reduces the problem to the case $p = 2$. Next, we use the fact that for any $x, z \in \mathbb{R}$,

$$\begin{aligned} &\left\| v_\theta^{(r,z)}(t, x) - v_\theta^{(r,z+h)}(t, x) - v_\theta^{(r,z)}(t, x+k) + v_\theta^{(r,z+h)}(t, x+k) \right\|_2^2 \leq \\ &\quad \left\| V_{\theta t}^{(r,z)}(t, x) - V_{\theta t}^{(r,z+h)}(t, x) - V_{\theta t}^{(r,z)}(t, x+k) + V_{\theta t}^{(r,z+h)}(t, x+k) \right\|_2^2, \end{aligned}$$

which is proved similarly to (57). The conclusion follows from relation (99). \square

The next result is derived using Lemma 5.4.

Lemma 5.5. *For any $\theta > 0$, $p \geq 2$ and $t > 0$,*

$$\begin{aligned}
a) \quad & \sup_{r \in [0, t]} \int_{\mathbb{R}} \sup_{z \in \mathbb{R}} \left(\int_{\mathbb{R}} \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z+h)}(t, x) \right\|_p dx \right)^2 |h|^{2H-2} dh \leq C'_{v, p, H, t, \theta}, \\
b) \quad & \sup_{r \in [0, t]} \int_{\mathbb{R}} \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z+h)}(t, x) \right\|_p dz \right)^2 |h|^{2H-2} dh \leq C'_{v, p, H, t, \theta}, \\
c) \quad & \sup_{r \in [0, t]} \int_{\mathbb{R}} \sup_{z \in \mathbb{R}} \left(\int_{\mathbb{R}} \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z)}(t, x+h) \right\|_p dx \right)^2 |h|^{2H-2} dh \leq C'_{v, p, H, t, \theta}, \\
d) \quad & \sup_{r \in [0, t]} \int_{\mathbb{R}} \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z)}(t, x+h) \right\|_p dz \right)^2 |h|^{2H-2} dh \leq C'_{v, p, H, t, \theta}, \\
e) \quad & \sup_{r \in [0, t]} \int_{\mathbb{R}^2} \sup_{z \in \mathbb{R}} \left(\int_{\mathbb{R}} \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z+h)}(t, x) - v_{\theta}^{(r, z)}(t, x+k) + v_{\theta}^{(r, z+h)}(t, x+k) \right\|_p dx \right)^2 \\
& \quad \times |h|^{2H-2} |k|^{2H-2} dh dk \leq C'_{v, p, H, t, \theta} \\
f) \quad & \sup_{r \in [0, t]} \int_{\mathbb{R}^2} \sup_{x \in \mathbb{R}} \left(\int_{\mathbb{R}} \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z+h)}(t, x) - v_{\theta}^{(r, z)}(t, x+k) + v_{\theta}^{(r, z+h)}(t, x+k) \right\|_p dz \right)^2 \\
& \quad \times |h|^{2H-2} |k|^{2H-2} dh dk \leq C'_{v, p, H, t, \theta},
\end{aligned}$$

where $C'_{v, p, H, t, \theta} > 0$ is a constant that depends on (p, H, t, θ) and is increasing in t .

Proof. By Remark 5.3, it suffices to prove a), c) and e).

a) We denote by I the integral appearing in this inequality. Then $I = I_1 + I_2$, where I_1 and I_2 are the integrals on $|h| > 1$, respectively $|h| \leq 1$. By triangle inequality and Lemma 3.2, we have

$$\int_{\mathbb{R}} \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z+h)}(t, x) \right\|_p dx \leq \int_{\mathbb{R}} \left\| v_{\theta}^{(r, z)}(t, x) \right\|_p dx + \int_{\mathbb{R}} \left\| v_{\theta}^{(r, z+h)}(t, x) \right\|_p dx \leq 4tC_{\theta, p, t, v},$$

and hence $I_1 \leq (4t^2C_{\theta, p, t, v})^2 \int_{|h| > 1} |h|^{2H-2} dh < \infty$.

It remains to treat I_2 . For this, we insert artificially the G function, using identity (31). Then, by triangle inequality,

$$\begin{aligned}
\left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z+h)}(t, x) \right\|_p & \leq 2G_{t-r}(x-z) \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z+h)}(t, x) \right\|_p \\
& \quad + 2|G_{t-r}(x-z) - G_{t-r}(x-z-h)| \left\| v_{\theta}^{(r, z+h)}(t, x) \right\|_p.
\end{aligned}$$

Hence $I_2 \leq 8(I_{2,1} + I_{2,2})$, where

$$\begin{aligned}
I_{2,1} & = \int_{|h| \leq 1} \left(\int_{\mathbb{R}} G_{t-r}(x-z) \left\| v_{\theta}^{(r, z)}(t, x) - v_{\theta}^{(r, z+h)}(t, x) \right\|_p dx \right)^2 |h|^{2H-2} dh \\
I_{2,2} & = \int_{|h| \leq 1} \left(\int_{\mathbb{R}} |G_{t-r}(x-z) - G_{t-r}(x-z-h)| \left\| v_{\theta}^{(r, z+h)}(t, x) \right\|_p dx \right)^2 |h|^{2H-2} dh.
\end{aligned}$$

We treat $I_{2,1}$. The the presence of G allows us to apply the Cauchy-Schwarz inequality to bring the square inside the integral. Using the fact that $\int_{\mathbb{R}} G_{t-r}^2(x-z)dx = (t-r)/2$, we obtain:

$$\left(\int_{\mathbb{R}} G_{t-r}(x-z) \left\| v_{\theta}^{(r,z)}(t,x) - v_{\theta}^{(r,z+h)}(t,x) \right\|_p dx \right)^2 \leq t \int_{\mathbb{R}} \left\| v_{\theta}^{(r,z)}(t,x) - v_{\theta}^{(r,z+h)}(t,x) \right\|_p^2 dx.$$

The desired bound for $I_{2,1}$ follows now from Lemma 5.4.a).

For $I_{2,2}$, we use Theorem 3.1, and $\int_{\mathbb{R}} |G_{t-r}(x-z) - G_{t-r}(x-z-h)|dx \leq |h|$. Hence,

$$\begin{aligned} I_{2,2} &\leq C_{\theta,p,t,v}^2 \int_{|h|\leq 1} \left(\int_{\mathbb{R}} |G_{t-r}(x-z) - G_{t-r}(x-z-h)|dx \right)^2 |h|^{2H-2} dh \\ &\leq C_{\theta,p,t,v}^2 \int_{|h|\leq 1} |h|^{2H} dh < \infty. \end{aligned}$$

c) We use the same argument as in part a), using Lemma 5.4.c).

e) We denote by I' the integral appearing in this inequality. Inserting artificially the term $G_{t-r}(x-z)$ (using identity (31)), we obtain:

$$\begin{aligned} &v_{\theta}^{(r,z)}(t,x) - v_{\theta}^{(r,z+h)}(t,x) - v_{\theta}^{(r,z)}(t,x+k) + v_{\theta}^{(r,z+h)}(t,x+k) = \\ &2G_{t-r}(x-z)v_{\theta}^{(r,z)}(t,x) - 2G_{t-r}(x-z-h)v_{\theta}^{(r,z+h)}(t,x) - \\ &2G_{t-r}(x+k-z)v_{\theta}^{(r,z)}(t,x+k) + 2G_{t-r}(x+k-z-h)v_{\theta}^{(r,z+h)}(t,x+k). \end{aligned}$$

We now use the following identity:

$$\begin{aligned} a_1b_1 - a_2b_2 - a_3b_3 + a_4b_4 &= a_1(b_1 - b_2) + (a_1 - a_2)b_2 - a_3(b_3 - b_4) - (a_3 - a_4)b_4 \\ &= a_1(b_1 - b_2 - b_3 + b_4) + (a_1 - a_3)(b_3 - b_4) + (a_1 - a_2 - a_3 + a_4)b_2 + (a_3 - a_4)(b_2 - b_4). \end{aligned}$$

Using this identity together with triangle inequality, we have

$$\begin{aligned} &\left\| v_{\theta}^{(r,z)}(t,x) - v_{\theta}^{(r,z+h)}(t,x) - v_{\theta}^{(r,z)}(t,x+k) + v_{\theta}^{(r,z+h)}(t,x+k) \right\|_p \\ &\leq 2G_{t-r}(x-z) \left\| v_{\theta}^{(r,z)}(t,x) - v_{\theta}^{(r,z+h)}(t,x) - v_{\theta}^{(r,z)}(t,x+k) + v_{\theta}^{(r,z+h)}(t,x+k) \right\|_p \\ &+ |2G_{t-r}(x-z) - 2G_{t-r}(x+k-z)| \left\| v_{\theta}^{(r,z)}(t,x+k) - v_{\theta}^{(r,z+h)}(t,x+k) \right\|_p \\ &+ |2G_{t-r}(x-z) - 2G_{t-r}(x-z-h) - 2G_{t-r}(x+k-z) + 2G_{t-r}(x+k-z-h)| \left\| v_{\theta}^{(r,z+h)}(t,x) \right\|_p \\ &+ |2G_{t-r}(x+k-z) - 2G_{t-r}(x+k-z-h)| \left\| v_{\theta}^{(r,z+h)}(t,x) - v_{\theta}^{(r,z+h)}(t,x+k) \right\|_p \\ &:= \sum_{j=1}^4 F_j(x,h,k). \end{aligned}$$

Using the inequality $(\sum_{j=1}^4 e_j)^2 \leq 4 \sum_{j=1}^4 e_j^2$, we have

$$I' \leq 4 \sum_{j=1}^4 \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} F_j(x,h,k) dx \right)^2 |h|^{2H-2} |k|^{2H-2} dh dk =: 4 \sum_{j=1}^4 I'_j.$$

Since F_j is a product of two terms (which involve G , respectively v_θ), we bound $(\int_{\mathbb{R}} F_j(x, h, k) dx)^2$ using the Cauchy-Schwarz inequality. For I'_1 , we use Lemma 5.4.e) and $\int_{\mathbb{R}} G_{t-r}^2(x-r) dx = (t-r)/2$. For I'_2 , we use Lemma 5.4.a), and

$$\int_{\mathbb{R}^2} |G_{t-r}(x-z) - G_{t-r}(x+k-z)|^2 |k|^{2H-2} dx dk = c_H (t-r)^{2H}, \quad (58)$$

where $c_H > 0$ is a constant depending on H . For I'_3 , we use Lemma 3.2 and

$$\int_{\mathbb{R}^3} |G_{t-r}(x-z) - G_{t-r}(x-z-h) - G_{t-r}(x+k-z) + G_{t-r}(x+k-z-h)|^2 |h|^{2H-2} |k|^{2H-2} dh dk dx = c'_H (t-r)^{4H-1}, \quad (59)$$

where $c'_H > 0$ is a constant that depends on H . For I'_4 , we use Lemma 5.4.c) and (58). \square

Remark 5.6. As stated in Remark 5.3, the integral dx which appears in part a) of Lemma 5.5 does not depend on z , so instead of taking the supremum over $z \in \mathbb{R}$, we could have evaluated this integral at one particular point z_0 , for instance $z_0 = 0$. We chose to present the result in this way, since this will be the form that we will use in Section 6.2 below, for the proof of the QCLT. The same comment applies to parts b)-f).

6 Proofs of the main results

In this section, we include the proofs of Theorems 1.1, 1.2 and 1.3.

6.1 Limiting covariance

In this section, we give the proof of Theorem 1.1. Let

$$\rho_{t,s,\theta}(x-y) := \mathbb{E} [(u_\theta(t,x) - 1)(u_\theta(s,y) - 1)] = \sum_{n \geq 1} \frac{1}{n!} \theta^n \alpha_n(x-y; t, s), \quad (60)$$

where

$$\begin{aligned} \alpha_n(x-y; t, s) &= (n!)^2 \langle \tilde{f}_n(\cdot, x; t), \tilde{f}_n(\cdot, y; s) \rangle_{\mathcal{P}_0^{\otimes n}} = (n!)^2 \langle f_n(\cdot, x; t), \tilde{f}_n(\cdot, y; s) \rangle_{\mathcal{P}_0^{\otimes n}} \\ &= n! \sum_{\sigma \in S_n} c_H^n \int_{\mathbb{R}^n} \mathcal{F} f_n(\cdot, x; t)(\xi_1, \dots, \xi_n) \overline{\mathcal{F} f_n(\cdot, y; s)(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)})} \prod_{j=1}^n |\xi_j|^{1-2H} d\xi \\ &= n! \sum_{\sigma \in S_n} c_H^n \int_{\mathbb{R}^n} e^{-i \sum_{j=1}^n \xi_j (x-y)} \prod_{j=1}^n |\xi_j|^{1-2H} d\xi_1 \dots d\xi_n \int_{T_n(t)} dt \int_{T_n(s)} ds \\ &\quad \times \prod_{j=1}^n \mathcal{F} G_{t_{j+1}-t_j}(\xi_1 + \dots + \xi_j) \prod_{j=1}^n \mathcal{F} G_{s_{j+1}-s_j}(\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}). \end{aligned} \quad (61)$$

Here we use the convention $t_{n+1} = t$ and $s_{n+1} = s$. This shows that $\alpha_n(x-y; t, s)$ and $\rho_{t,s}(x-y)$ depend on x and y only through the difference $x-y$. In particular, $\{u(t, x); x \in \mathbb{R}\}$ is a (wide-sense) stationary process with covariance function $\rho_{t,s,\theta}$.

By Fubini theorem,

$$\begin{aligned} \frac{1}{R} \mathbb{E}[F_{R,\theta}(t)F_{R,\theta}(s)] &= \frac{1}{R} \int_{-R}^R \int_{-R}^R \rho_{t,s,\theta}(x-y) dx dy = \sum_{n \geq 1} \theta^n \frac{1}{Rn!} \int_{-R}^R \int_{-R}^R \alpha_n(x-y; t, s) dx dy \\ &=: \sum_{n \geq 1} \theta^n Q_{n,R}(t, s), \end{aligned} \quad (62)$$

The application of Fubini theorem is justified by (25), using that fact that:

$$\alpha_n(x-y; t, s) \leq (n!)^2 \|f_n(\cdot, x; t)\|_{\mathcal{P}_0^{\otimes n}} \|\tilde{f}_n(\cdot, y; s)\|_{\mathcal{P}_0^{\otimes n}} \leq c^n \frac{t^{(2H+2)n}}{(n!)^{2H}}.$$

We now show that the first term in series (62) does not contribute to the limit.

Lemma 6.1. *For any $t, s > 0$,*

$$Q_{1,R}(t, s) = \frac{1}{R} \int_{-R}^R \int_{-R}^R \alpha_1(x-y; t, s) dx dy \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Proof. By Fubini theorem,

$$\begin{aligned} \alpha_1(x-y; t, s) &= c_H \int_{\mathbb{R}} \left(\int_0^t e^{-i\xi x} \frac{\sin((t-t_1)|\xi|)}{|\xi|} dt_1 \right) \left(\int_0^s e^{i\xi y} \frac{\sin((s-s_1)|\xi|)}{|\xi|} ds_1 \right) \\ &= c_H \int_0^t \int_0^s \int_{\mathbb{R}} e^{-i\xi(x-y)} \frac{\sin((t-t_1)|\xi|)}{|\xi|} \frac{\sin((s-s_1)|\xi|)}{|\xi|} d\xi ds_1 dt_1. \end{aligned}$$

The application of this theorem is justified using the inequality:

$$\frac{|\sin((t-t_1)|\xi|)|}{|\xi|} \frac{|\sin((s-s_1)|\xi|)}{|\xi|} \leq (t-t_1)(s-s_1) \mathbf{1}_{\{|\xi| \leq 1\}} + \frac{1}{|\xi|^2} \mathbf{1}_{\{|\xi| > 1\}}. \quad (63)$$

Another application of Fubini theorem, justified also by (63), shows that:

$$\begin{aligned} &\int_{-R}^R \int_{-R}^R \alpha_1(x-y; t, s) dx dy \\ &= c_H \int_0^t \int_0^s \int_{\mathbb{R}} \frac{4 \sin^2(R|\xi|)}{|\xi|^2} \frac{\sin((t-t_1)|\xi|)}{|\xi|} \frac{\sin((s-s_1)|\xi|)}{|\xi|} |\xi|^{1-2H} d\xi ds_1 dt_1, \end{aligned}$$

where we used the fact that:

$$\left(\int_{-R}^R \int_{-R}^R e^{-i\xi(x-y)} dx dy \right) = \left| \int_{-R}^R e^{-i\xi x} dx \right|^2 = \frac{4 \sin^2(R|\xi|)}{|\xi|^2} := 4\pi R \ell_R(\xi),$$

We denote

$$I_1(t_1, s_1) := \int_{\mathbb{R}} \frac{\sin^2(R|\xi|)}{|\xi|^2} \frac{\sin((t-t_1)|\xi|)}{|\xi|} \frac{\sin((s-s_1)|\xi|)}{|\xi|} |\xi|^{1-2H} d\xi = I_1(t_1, s_1) + I_2(t_1, s_1),$$

where $I_1(t_1, s_1)$ and $I_2(t_1, s_1)$ correspond to the integration regions $\{|\xi| \leq \varepsilon\}$, respectively $\{|\xi| > \varepsilon\}$. To estimate $I_2(t_1, s_1)$, we bound all three sin functions by 1, obtaining

$$|I_2(t_1, s_1)| \leq \int_{|\xi| > \varepsilon} |\xi|^{-3-2H} d\xi =: C_{\varepsilon, H}.$$

To bound $I_1(t_1, s_1)$, we choose $\varepsilon > 0$ such that $(t+s)\varepsilon < \frac{\pi}{2}$, and hence $\sin((t-t_1)|\xi|) \geq 0$ and $\sin((s-s_1)|\xi|) \geq 0$ for any $t_1 \in [0, t]$, $s_1 \in [0, s]$ and ξ with $|\xi| \leq \varepsilon$. Using the inequality $|\sin(x)| \leq |x|$ for the last two sin functions, followed by the inequality $|\sin x| \leq |x|^\theta$ with $\theta \in [0, 1]$ for the last sin function, we obtain that:

$$\begin{aligned} I_1(t_1, s_1) &\leq (t-t_1)(s-s_1) \int_{|\xi| \leq \varepsilon} \frac{\sin^2(R|\xi|)}{|\xi|^2} |\xi|^{1-2H} d\xi \\ &\leq (t-t_1)(s-s_1) R^{2\theta} \int_{|\xi| \leq \varepsilon} |\xi|^{2\theta-2H-1} d\xi =: (t-t_1)(s-s_1) R^{2\theta} C_{\varepsilon, \theta, H}, \end{aligned}$$

provided that $\theta > H$. Summarizing, for any $t_1 \in [0, t]$, $s_1 \in [0, s]$ and $\theta \in (H, 1]$,

$$|I(t_1, s_1)| \leq (t-t_1)(s-s_1) R^{2\theta} C_{\varepsilon, \theta, H} + C_{\varepsilon, H}.$$

Therefore,

$$\left| \int_{-R}^R \int_{-R}^R \alpha_1(x-y; t, s) dx dy \right| \leq 4c_H \int_0^t \int_0^s |I(t_1, s_1)| ds_1 dt_1 \leq C(t^2 s^2 R^{2\theta} + ts).$$

Then $|Q_{1,R}(t, s)| \leq C(t^2 s^2 R^{2\theta-1} + ts R^{-1}) \rightarrow 0$ as $R \rightarrow \infty$, if we choose $\theta \in (H, \frac{1}{2})$. \square

Next, we study the terms corresponding to $n \geq 2$. We use the fact that the function

$$\ell_R(\xi) = \frac{\sin^2(R|\xi|)}{\pi R |\xi|^2}, \quad \xi \in \mathbb{R}$$

is an approximation of the identity, when $R \rightarrow \infty$ (see Lemma 2.1 of [28]).

Lemma 6.2. *For any $t > 0$ and $s > 0$,*

$$K_\theta(t, s) := \lim_{R \rightarrow \infty} \sum_{n \geq 2} \theta^n Q_{n,R}(t, s) \quad \text{exists and is finite.}$$

Proof. We use expression (61) for $\alpha_n(x-y; t, s)$, and we integrate $dx dy$ on $[-R, R]^2$. Using Fubini's theorem, we obtain the following expression for $Q_{n,R}(t, s)$:

$$\begin{aligned} Q_{n,R}(t, s) &= \frac{1}{R} \frac{1}{n!} \int_{-R}^R \int_{-R}^R \alpha_n(x-y; t, s) dx dy \\ &= 4\pi c_H^n \sum_{\sigma \in S_n} \int_{T_n(t)} \int_{T_n(s)} \int_{\mathbb{R}^n} \ell_R(\xi_1 + \dots + \xi_n) \prod_{j=1}^n |\xi_j|^{1-2H} \\ &\quad \times \frac{\sin((t-t_n)|\xi_1 + \dots + \xi_n|)}{|\xi_1 + \dots + \xi_n|} \frac{\sin((s-s_n)|\xi_1 + \dots + \xi_n|)}{|\xi_1 + \dots + \xi_n|} \end{aligned}$$

$$\begin{aligned}
& \times \prod_{j=1}^{n-1} \frac{\sin((t_{j+1} - t_j)|\xi_1 + \dots + \xi_j|)}{|\xi_1 + \dots + \xi_j|} \prod_{j=1}^{n-1} \frac{\sin((s_{j+1} - s_j)|\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}|)}{|\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}|} d\xi ds dt \\
& := 4\pi c_H^n \sum_{\sigma \in S_n} \int_{T_n(t)} \int_{T_n(s)} Q_{n,R}(\mathbf{t}, \mathbf{s}, \sigma) ds dt
\end{aligned} \tag{64}$$

To evaluate $Q_{n,R}(\mathbf{t}, \mathbf{s}, \sigma)$, we use the change the variables $\xi_n \rightarrow \eta_n = \xi_1 + \dots + \xi_n$. For any fixed permutation σ , we denote $j_0 = \sigma^{-1}(n)$, so that $\sigma(j_0) = n$. Then for all $j < j_0$, the sum $\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}$ does not involve the variable ξ_n , and remains the same when we perform the change of variables. For $j \geq j_0$, the sum $\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}$ contains ξ_n and will be replaced after the change of variables by

$$\sum_{k=1}^{j_0-1} \xi_{\sigma(k)} + \left(\eta_n - \sum_{j=1}^{n-1} \xi_j\right) + \sum_{k=j_0+1}^j \xi_{\sigma(k)} = \eta_n - \xi_{\sigma(j+1)} - \dots - \xi_{\sigma(n)},$$

where $\sum_{\emptyset} = 0$. Using the convention $\prod_{\emptyset} = 1$, we obtain the expression:

$$\begin{aligned}
Q_{n,R}(\mathbf{t}, \mathbf{s}, \sigma) &= \int_{\mathbb{R}^n} \ell_R(\eta_n) \frac{\sin((t - t_n)|\eta_n|)}{|\eta_n|} \frac{\sin((s - s_n)|\eta_n|)}{|\eta_n|} \\
&\times \prod_{j=1}^{n-1} \frac{\sin((t_{j+1} - t_j)|\xi_1 + \dots + \xi_j|)}{|\xi_1 + \dots + \xi_j|} \prod_{j < \sigma^{-1}(n)} \frac{\sin((s_{j+1} - s_j)|\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}|)}{|\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}|} \\
&\times \prod_{j \geq \sigma^{-1}(n)} \frac{\sin((s_{j+1} - s_j)|\eta_n - \xi_{\sigma(j+1)} - \dots - \xi_{\sigma(n)}|)}{|\eta_n - \xi_{\sigma(j+1)} - \dots - \xi_{\sigma(n)}|} \\
&\times \prod_{j=1}^{n-1} |\xi_j|^{1-2H} |\eta_n - \xi_1 - \dots - \xi_{n-1}|^{1-2H} d\xi_1 \dots d\xi_{n-1} d\eta_n := \left(\ell_R * g_{\mathbf{t},\mathbf{s},\sigma}^{(n)}\right)(0),
\end{aligned}$$

where the function $g_{\mathbf{t},\mathbf{s},\sigma}^{(n)}$ is given by:

$$\begin{aligned}
g_{\mathbf{t},\mathbf{s},\sigma}^{(n)}(x) &:= \frac{\sin((t - t_n)|x|)}{|x|} \frac{\sin((s - s_n)|x|)}{|x|} \\
&\times \int_{\mathbb{R}^{n-1}} \prod_{j=1}^{n-1} \frac{\sin((t_{j+1} - t_j)|\xi_1 + \dots + \xi_j|)}{|\xi_1 + \dots + \xi_j|} \prod_{j < \sigma^{-1}(n)} \frac{\sin((s_{j+1} - s_j)|\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}|)}{|\xi_{\sigma(1)} + \dots + \xi_{\sigma(j)}|} \\
&\times \prod_{j \geq \sigma^{-1}(n)} \frac{\sin((s_{j+1} - s_j)|x - \xi_{\sigma(j+1)} - \dots - \xi_{\sigma(n)}|)}{|x - \xi_{\sigma(j+1)} - \dots - \xi_{\sigma(n)}|} \\
&\times \prod_{j=1}^{n-1} |\xi_j|^{1-2H} |x - \xi_1 - \dots - \xi_{n-1}|^{1-2H} d\xi_1 \dots d\xi_{n-1}.
\end{aligned}$$

To define $g_{\mathbf{t},\mathbf{s},\sigma}(0)$, we use the convention $\frac{\sin x}{x} \Big|_{x=0} = 1$.

Since ℓ_R is an approximation of the identity as $R \rightarrow \infty$, by the Dominated Convergence Theorem, we obtain that

$$\sum_{n \geq 2} \theta^n Q_{n,R}(t, s) = 4\pi \sum_{n \geq 2} \theta^n c_H^n \int_{T_n(t)} \int_{T_n(s)} \sum_{\sigma \in S_n} (\ell_R * g_{\mathbf{t},\mathbf{s},\sigma})(0) ds dt$$

$$\rightarrow 4\pi \sum_{n \geq 2} \theta^n c_H^n \int_{T_n(t)} \int_{T_n(s)} \sum_{\sigma \in S_n} g_{\mathbf{t}, \mathbf{s}, \sigma}(0) ds dt =: K_\theta(t, s), \quad \text{as } R \rightarrow \infty. \quad (65)$$

It remains to justify the application of the Dominated Convergence Theorem. For this, we will prove that there exists a function $H_n(\mathbf{t}, \mathbf{s})$ such that

$$\left| \sum_{\sigma \in S_n} (\ell_R * g_{\mathbf{t}, \mathbf{s}, \sigma})(0) \right| \leq H_n(\mathbf{t}, \mathbf{s})$$

for any $n \geq 2$, $R \geq 1$, $\mathbf{t} \in T_n(t)$, $\mathbf{s} \in T_n(s)$, and

$$\sum_{n \geq 2} \theta^n c_H^n \int_{T_n(t)} \int_{T_n(s)} H_n(\mathbf{t}, \mathbf{s}) dt ds < \infty. \quad (66)$$

In particular, this will imply that the limit $K_\theta(t, s)$ is finite.

We will use the following inequality: for any functions h_1, h_2 on \mathbb{R}^n and for any symmetric measure μ_n on \mathbb{R}^n ,

$$\left| \sum_{\sigma \in S_n} \int_{\mathbb{R}^n} h_1(\xi_1, \dots, \xi_n) h_2(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) \mu_n(d\xi_1, \dots, d\xi_n) \right| \leq \frac{n!}{2} \left\{ \int_{\mathbb{R}^n} |h_1(\xi_1, \dots, \xi_n)|^2 \mu_n(d\xi_1, \dots, d\xi_n) + \int_{\mathbb{R}^n} |h_2(\xi_1, \dots, \xi_n)|^2 \mu_n(d\xi_1, \dots, d\xi_n) \right\},$$

which can be proved applying the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$. It follows that

$$\begin{aligned} \left| \sum_{\sigma \in S_n} (\ell_R * g_{\mathbf{t}, \mathbf{s}, \sigma})(0) \right| &\leq \frac{n!}{2} \left\{ \int_{\mathbb{R}^n} \ell_R(\xi_1 + \dots + \xi_n) \prod_{j=1}^n \frac{\sin^2((t_{j+1} - t_j)|\xi_1 + \dots + \xi_j|)}{|\xi_1 + \dots + \xi_j|^2} \prod_{j=1}^n |\xi_j|^{1-2H} d\xi \right. \\ &\quad \left. + \int_{\mathbb{R}^n} \ell_R(\xi_1 + \dots + \xi_n) \prod_{j=1}^n \frac{\sin^2((s_{j+1} - s_j)|\xi_1 + \dots + \xi_j|)}{|\xi_1 + \dots + \xi_j|^2} \prod_{j=1}^n |\xi_j|^{1-2H} d\xi \right\} \\ &=: \frac{n!}{2} (\mathcal{I}_{n,R}(\mathbf{t}) + \mathcal{I}_{n,R}(\mathbf{s})). \end{aligned}$$

We only evaluate the first integral, the second one being similar. We use the change of variables $\eta_j = \xi_1 + \dots + \xi_j$ for $j = 1, \dots, n$ (with $\eta_0 = 0$), followed by inequality (24):

$$\begin{aligned} \mathcal{I}_{n,R}(\mathbf{t}) &\leq \sum_{\alpha \in D_n} \prod_{j=1}^{n-1} \left(\int_{\mathbb{R}} \frac{\sin^2((t_{j+1} - t_j)|\eta_j|)}{|\eta_j|^2} |\eta_j|^{\alpha_j} d\eta_j \right) \left(\int_{\mathbb{R}} \frac{\sin^2((t - t_n)|\eta_n|)}{|\eta_n|^2} \ell_R(\eta_n) |\eta_n|^{\alpha_n} d\eta_n \right) \\ &\leq c^{n-1} \sum_{\alpha \in D_n} \prod_{j=1}^{n-1} (t_{j+1} - t_j)^{1-\alpha_j} (t - t_n)^2 \frac{1}{\pi R} \int_{\mathbb{R}} \frac{\sin^2(R|\eta_n|)}{|\eta_n|^2} |\eta_n|^{\alpha_n} d\eta_n \\ &= c^n \sum_{\alpha \in D_n} \prod_{j=1}^{n-1} (t_{j+1} - t_j)^{1-\alpha_j} (t - t_n)^2 R^{-\alpha_n} \leq c^n \sum_{\alpha \in D_n} t^{1+\alpha_n} \prod_{j=1}^n (t_{j+1} - t_j)^{1-\alpha_j} \end{aligned}$$

Therefore, assuming that $s \leq t$, we obtain:

$$\left| \sum_{\sigma \in S_n} (\ell_R * g_{\mathbf{t}, \mathbf{s}, \sigma})(0) \right| \leq n! c^n \sum_{\alpha \in D_n} t^{1+\alpha_n} \left(\prod_{j=1}^n (t_{j+1} - t_j)^{1-\alpha_j} + \prod_{j=1}^n (s_{j+1} - s_j)^{1-\alpha_j} \right) =: H_n(\mathbf{t}, \mathbf{s}).$$

To see that (66) holds, we note that

$$\begin{aligned} \int_{T_n(t)} \int_{T_n(s)} H_n(\mathbf{t}, \mathbf{s}) d\mathbf{s} d\mathbf{t} &\leq 2c^n \sum_{\alpha \in D_n} t^{1+\alpha_n} \int_{T_n(t)} \prod_{j=1}^n (t_{j+1} - t_j)^{1-\alpha_j} d\mathbf{t} \\ &\leq 2c^n \sum_{\alpha \in D_n} t^{1+\alpha_n} \frac{t^{(2H+1)n}}{\Gamma((2H+1)n+1)}. \end{aligned}$$

□

Finally, we prove that when $s = t$, the limit $K_\theta(t, t)$ is non-zero. For this, we use the following lemma.

Lemma 6.3. *If h is a measurable function on $(\mathbb{R}^d)^n$ and μ_n is a symmetric measure on $(\mathbb{R}^d)^n$, then*

$$\begin{aligned} \sum_{\sigma \in S_n} \int_{(\mathbb{R}^d)^n} h(\xi_1, \dots, \xi_n) h(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}) \mu_n(d\xi_1, \dots, d\xi_n) \\ = \frac{1}{n!} \int_{(\mathbb{R}^d)^n} |\tilde{h}(\xi_1, \dots, \xi_n)|^2 \mu_n(d\eta_1, \dots, d\eta_n). \end{aligned} \quad (67)$$

Proof. We denote by I the left-hand side of (67). For any $\rho \in S_n$, let $h_\rho(\xi_1, \dots, \xi_n) = h(\xi_{\rho(1)}, \dots, \xi_{\rho(n)})$. Then for any $\rho \in S_n$, $I = n! \int \tilde{h} h d\mu_n = n! \int h_\rho \tilde{h} d\mu_n$, where the second equality is due to the symmetry of μ_n . Taking the sum over all $\rho \in S_n$, we obtain $n! I = n! \int \sum_{\rho \in S_n} h_\rho \tilde{h} d\mu_n = (n!)^2 \int \tilde{h}^2 d\mu_n$. □

Lemma 6.4. *For any $t > 0$, $K_\theta(t, t) > 0$.*

Proof. We will first prove that $Q_{n,R}(t, t) \geq 0$ for all $n \geq 1$ and $R > 0$. For this, we use (64). Applying Lemma 6.3 to the measure

$$\mu_n(d\xi_1, \dots, d\xi_n) = \ell_R(\xi_1 + \dots + \xi_n) \prod_{j=1}^n |\xi_j|^{1-2H} d\xi_1 \dots d\xi_n$$

and the function

$$h(\xi_1, \dots, \xi_n) = \int_{T_n(t)} \frac{\sin((t - t_n)|\xi_1 + \dots + \xi_n|)}{|\xi_1 + \dots + \xi_n|} \prod_{j=1}^{n-1} \frac{\sin((t_{j+1} - t_j)|\xi_1 + \dots + \xi_j|)}{|\xi_1 + \dots + \xi_j|} d\mathbf{t},$$

we obtain:

$$Q_{n,R}(t, t) = 4\pi c_H^n n! \int_{\mathbb{R}^n} |\tilde{h}(\xi_1, \dots, \xi_n)|^2 \mu_n(d\xi_1, \dots, d\xi_n) \geq 0.$$

This implies that $Q_n(t, t) := \lim_{R \rightarrow \infty} Q_{n,R}(t, t) \geq 0$ for all $n \geq 1$, and hence $K_\theta(t, t) = \sum_{n \geq 2} \theta^n Q_n(t, t) \geq \theta^2 Q_2(t, t)$. Therefore, it is enough to prove that $Q_2(t, t) > 0$.

To show this, we will use an alternative expression of $Q_n(t, t)$. We denote $H(\xi_1, \dots, \xi_n) = |\tilde{h}(\xi_1, \dots, \xi_n)|^2 \prod_{j=1}^n |\xi_j|^{1-2H}$, and we use the change of variables $\xi_n \mapsto \eta_n = \xi_1 + \dots + \xi_n$:

$$\begin{aligned} Q_{n,R}(t, t) &= 4\pi c_H^n n! \int_{\mathbb{R}^n} H(\xi_1, \dots, \xi_n) \ell_R(\xi_1 + \dots + \xi_n) d\xi_1 \dots d\xi_n \\ &= 4\pi c_H^n n! \int_{\mathbb{R}^n} H(-\xi_1, \dots, -\xi_n) \ell_R(\xi_1 + \dots + \xi_n) d\xi_1 \dots d\xi_n \\ &= 4\pi c_H^n n! \int_{\mathbb{R}^n} H(-\xi_1, \dots, -\xi_{n-1}, \xi_1 + \dots + \xi_{n-1} - \eta_n) \ell_R(\eta_n) d\xi_1 \dots d\xi_{n-1} d\eta_n \\ &= 4\pi c_H^n n! \int_{\mathbb{R}^{n-1}} \left(H(-\xi_1, \dots, -\xi_{n-1}, \cdot) * \ell_R \right) (\xi_1 + \dots + \xi_{n-1}) d\xi_1 \dots d\xi_{n-1}. \end{aligned}$$

Taking $R \rightarrow \infty$, we obtain:

$$Q_n(t, t) = 4\pi c_H^n n! \int_{\mathbb{R}^{n-1}} H(-\xi_1, \dots, -\xi_{n-1}, \xi_1 + \dots + \xi_{n-1}) d\xi_1 \dots d\xi_{n-1}.$$

We consider now the case $n = 2$. In this case,

$$\tilde{h}(\xi_1, \xi_2) = \frac{1}{2} \int_{T_2(t)} \frac{\sin((t - t_2)|\xi_1 + \xi_2|)}{|\xi_1 + \xi_2|} \left(\frac{\sin((t_2 - t_1)|\xi_1|)}{|\xi_1|} + \frac{\sin((t_2 - t_1)|\xi_2|)}{|\xi_2|} \right) dt_1 dt_2$$

and

$$H(\xi_1, \xi_2) = \frac{1}{4} \left(\int_{T_2(t)} \frac{\sin((t - t_2)|\xi_1 + \xi_2|)}{|\xi_1 + \xi_2|} \left(\frac{\sin((t_2 - t_1)|\xi_1|)}{|\xi_1|} + \frac{\sin((t_2 - t_1)|\xi_2|)}{|\xi_2|} \right) dt_1 dt_2 \right)^2 |\xi_1|^{1-2H} |\xi_2|^{1-2H}.$$

Using the convention $\frac{\sin x}{x}|_{x=0} = 1$, by direct calculation, we have:

$$H(-\xi_1, \xi_1) = \frac{1}{4} \left(\int_{T_2(t)} \frac{\sin((t_2 - t_1)|\xi_1|)}{|\xi_1|} dt_1 dt_2 \right)^2 |\xi_1|^{2(1-2H)} = \left(t - \frac{\sin(t|\xi_1|)}{|\xi_1|} \right)^2 |\xi_1|^{-4H-2}.$$

Since $H(-\xi_1, \xi_1) > 0$ for almost all $\xi_1 \in \mathbb{R}$, $Q_2(t, t) = \int_{\mathbb{R}} H(-\xi_1, \xi_1) d\xi_1 > 0$. \square

6.2 Quantitative CLT

In this section, we prove Theorem 1.2. We divide the proof into several steps. To simplify the notation, we drop the dependence on θ in this part, and we write $F_R(t)$ and $\sigma_R(t)$ instead of $F_{R,\theta}(t)$ and $\sigma_{R,\theta}(t)$.

Step 1. By applying a version of Proposition 2.4 of [27] for the time-independent noise, to $F = F_R(t)$, we get:

$$d_{TV} \left(\frac{F_R(t)}{\sigma_R(t)}, Z \right) \leq \frac{2\sqrt{3}}{\sigma_R^2(t)} \sqrt{C_H^3 \mathcal{A}^*},$$

where $Z \sim N(0, 1)$ and

$$\begin{aligned} \mathcal{A}^* = & \int_{\mathbb{R}^6} \|D_z F_R(t) - D_{z'} F_R(t)\|_4 \|D_w F_R(t) - D_{w'} F_R(t)\|_4 \\ & \|D_{z,y}^2 F_R(t) - D_{z,y'}^2 F_R(t) - D_{z',y}^2 F_R(t) + D_{z',y'}^2 F_R(t)\|_4 \\ & \|D_{w,y}^2 F_R(t) - D_{w,y'}^2 F_R(t) - D_{w',y}^2 F_R(t) + D_{w',y'}^2 F_R(t)\|_4 \\ & |y - y'|^{2H-2} |z - z'|^{2H-2} |w - w'|^{2H-2} dy dy' dz dz' dw dw'. \end{aligned}$$

By applying Minkowski's inequality to the four norms and change of variables $y' \rightarrow y + y'$, $z' \rightarrow z + z'$, $w' \rightarrow w + w'$, we have

$$\begin{aligned} \mathcal{A}^* \leq \mathcal{A} = & \int_{[-R,R]^4} \int_{\mathbb{R}^6} \|D_z u_\theta(t, x_1) - D_{z+z'} u_\theta(t, x_1)\|_4 \|D_w u_\theta(t, x_2) - D_{w+w'} u_\theta(t, x_2)\|_4 \\ & \|D_{z,y}^2 u_\theta(t, x_3) - D_{z,y+y'}^2 u_\theta(t, x_3) - D_{z+z',y}^2 u_\theta(t, x_3) + D_{z+z',y+y'}^2 u_\theta(t, x_3)\|_4 \\ & \|D_{w,y}^2 u_\theta(t, x_4) - D_{w,y+y'}^2 u_\theta(t, x_4) - D_{w+w',y}^2 u_\theta(t, x_4) + D_{w+w',y+y'}^2 u_\theta(t, x_4)\|_4 \\ & |y'|^{2H-2} |z'|^{2H-2} |w'|^{2H-2} dy dy' dz dz' dw dw' dx_1 dx_2 dx_3 dx_4. \quad (68) \end{aligned}$$

Since $\sigma_R^2(t) \sim C_t R$ as $R \rightarrow \infty$, it is enough to prove that $\mathcal{A} \leq C_t R$.

For the first two norms, we use Theorem 4.1.b) with $p = 4$ (and hence $\eta = 12\theta$). We obtain:

$$\|D_z u_\theta(t, x_1) - D_{z+z'} u_\theta(t, x_1)\|_4 \leq \sqrt{2\theta} \int_0^t I_1(z, z', x_1; r, t) dr \quad (69)$$

$$\|D_w u_\theta(t, x_2) - D_{w+w'} u_\theta(t, x_2)\|_4 \leq \sqrt{2\theta} \int_0^t I_1(w, w', x_2; r, t) dr. \quad (70)$$

For the last two norms, we use Theorem 4.2.b) with $p = 4$ (and hence $\eta = 18\theta$). We obtain:

$$\begin{aligned} & \|D_{z,y}^2 u_\theta(t, x_3) - D_{z,y+y'}^2 u_\theta(t, x_3) - D_{z+z',y}^2 u_\theta(t, x_3) + D_{z+z',y+y'}^2 u_\theta(t, x_3)\|_4 \\ & \leq 4\theta \int_{0 < s < r < t} (I_2(z, z', y, y', x_3; s, r, t) + I_2(y, y', z, z', x_3; s, r, t)) ds dr \quad (71) \end{aligned}$$

$$\begin{aligned} & \|D_{w,y}^2 u_\theta(t, x_4) - D_{w,y+y'}^2 u_\theta(t, x_4) - D_{w+w',y}^2 u_\theta(t, x_4) + D_{w+w',y+y'}^2 u_\theta(t, x_4)\|_4 \\ & \leq 4\theta \int_{0 < s < r < t} (I_2(w, w', y, y', x_4; s, r, t) + I_2(y, y', w, w', x_4; s, r, t)) ds dr. \quad (72) \end{aligned}$$

Hence, substituting the identities (69), (70), (71) and (72) to (68), we have

$$\mathcal{A} \leq (4\theta)^3 \int_{[-R,R]^4} dx_1 dx_2 dx_3 dx_4 \int_{\mathbb{R}^3} dy dz dw \int_{\mathbb{R}^3} |y'|^{2H-2} |z'|^{2H-2} |w'|^{2H-2} dy' dz' dw'$$

$$\begin{aligned}
& \int_0^t dr_1 \int_0^t dr_2 \int_{0 < s_3 < r_3 < t} ds_3 dr_3 \int_{0 < s_4 < r_4 < t} ds_4 dr_4 I_1(z, z', x_1; r_1, t) I_1(w, w', x_2; r_2, t) \\
& \times (I_2(z, z', y, y', x_3; s_3, r_3, t) + I_2(y, y', z, z', x_3; s_3, r_3, t)) \\
& \times (I_2(w, w', y, y', x_4; s_4, r_4, t) + I_2(y, y', w, w', x_4; s_4, r_4, t)). \tag{73}
\end{aligned}$$

Step 2. In this step, we evaluate the spatial integral in (73). We have

$$\begin{aligned}
& \int_{[-R, R]^4} dx_1 dx_2 dx_3 dx_4 \int_{\mathbb{R}^3} dy dz dw \int_{\mathbb{R}^3} |y'|^{2H-2} |z'|^{2H-2} |w'|^{2H-2} dy' dz' dw' \\
& I_1(z, z', x_1; r_1, t) I_1(w, w', x_2; r_2, t) (I_2(z, z', y, y', x_3; s_3, r_3, t) + I_2(y, y', z, z', x_3; s_3, r_3, t)) \\
& \times (I_2(w, w', y, y', x_4; s_4, r_4, t) + I_2(y, y', w, w', x_4; s_4, r_4, t)) \\
& \leq \int_{[-R, R]} dx_4 \int_{\mathbb{R}^3} |y'|^{2H-2} |z'|^{2H-2} |w'|^{2H-2} dy' dz' dw' \\
& \sup_{z \in \mathbb{R}} \int_{\mathbb{R}} I_1(z, z', x_1; r_1, t) dx_1 \sup_{w \in \mathbb{R}} \int_{\mathbb{R}} I_1(w, w', x_2; r_2, t) dx_2 \\
& \times \sup_{y \in \mathbb{R}} \int_{\mathbb{R}^2} (I_2(z, z', y, y', x_3; s_3, r_3, t) + I_2(y, y', z, z', x_3; s_3, r_3, t)) dz dx_3 \\
& \times \int_{\mathbb{R}^2} (I_2(w, w', y, y', x_4; s_4, r_4, t) + I_2(y, y', w, w', x_4; s_4, r_4, t)) dw dy \\
& \leq \int_{[-R, R]} dx_4 \left(\int_{\mathbb{R}^2} \left(\sup_{z \in \mathbb{R}} \int_{\mathbb{R}} I_1(z, z', x_1; r_1, t) dx_1 \sup_{w \in \mathbb{R}} \int_{\mathbb{R}} I_1(w, w', x_2; r_2, t) dx_2 \right)^2 |z'|^{2H-2} |w'|^{2H-2} dz' dw' \right)^{1/2} \\
& \left(\int_{\mathbb{R}^2} \left(\sup_{y \in \mathbb{R}} \int_{\mathbb{R}^2} (I_2(z, z', y, y', x_3; s_3, r_3, t) + I_2(y, y', z, z', x_3; s_3, r_3, t)) dz dx_3 \right)^2 |y'|^{2H-2} |z'|^{2H-2} dy' dz' \right)^{1/2} \\
& \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} (I_2(w, w', y, y', x_4; s_4, r_4, t) + I_2(y, y', w, w', x_4; s_4, r_4, t)) dw dy \right)^2 |y'|^{2H-2} |w'|^{2H-2} dy' dw' \right)^{1/2}. \tag{74}
\end{aligned}$$

where for the last inequality, we use Lemma 6.5 for the integral $\iiint dy' dz' dw'$

Next, we need to compute the three integrals $dz' dw'$, $dy' dz'$, $dy' dw'$ one by one. We start with the integral $dz' dw'$. By Theorem 2.3, Lemma 3.2 and Lemma 5.2, for any $r \in [0, t]$

$$\begin{aligned}
\int_{\mathbb{R}} I_1(z, z', x_1; r, t) dx_1 &= \int_{\mathbb{R}} \|u_{12\theta}(r, z) - u_{12\theta}(r, z + z')\|_2 \left\| v_{12\theta}^{(r, z)}(t, x_1) \right\|_2 dx_1 \\
&+ \int_{\mathbb{R}} \|u_{12\theta}(r, z + z')\|_2 \left\| v_{12\theta}^{(r, z)}(t, x_1) - v_{12\theta}^{(r, z+z')}(t, x_1) \right\|_2 dx_1 \\
&\leq 2t C_{12\theta, 2, t, v} \|u_{12\theta}(r, 0) - u_{12\theta}(r, z')\|_2 \\
&+ C_{12\theta, 2, t, u} \int_{\mathbb{R}} \left\| v_{12\theta}^{(r, z)}(t, x_1) - v_{12\theta}^{(r, z+z')}(t, x_1) \right\|_2 dx_1.
\end{aligned}$$

By Remark 5.3, the integral on the right-hand side does not depend on z . Hence, by Lemma 5.1 and Lemma 5.5, we have

$$\begin{aligned}
& \int_{\mathbb{R}} \left(\sup_{z \in \mathbb{R}} \int_{\mathbb{R}} I_1(z, z', x_1; r, t) dx_1 \right)^2 |z'|^{2H-2} dz' \\
& \leq 8t^2 C_{12\theta, 2, t, v}^2 \int_{\mathbb{R}} \|u_{12\theta}(r, 0) - u_{12\theta}(r, z')\|_2^2 |z'|^{2H-2} dz' \\
& \quad + 2C_{12\theta, 2, t, u}^2 \int_{\mathbb{R}} \left(\sup_{z \in \mathbb{R}} \int_{\mathbb{R}} \left\| v_{12\theta}^{(r, z)}(t, x_1) - v_{12\theta}^{(r, z+z')}(t, x_1) \right\|_2 dx_1 \right)^2 |z'|^{2H-2} dz' \\
& \leq 8t^2 C_{12\theta, 2, t, v}^2 C'_{u, 2, H, t, 12\theta} + 2C_{12\theta, 2, t, u}^2 C'_{v, 2, H, t, 12\theta}.
\end{aligned}$$

Similarly, for any $r \in [0, t]$, we have

$$\int_{\mathbb{R}} \left(\sup_{w \in \mathbb{R}} \int_{\mathbb{R}} I_1(w, w', x_2; r, t) dx_2 \right)^2 |w'|^{2H-2} dw' \leq 8t^2 C_{12\theta, 2, t, v}^2 C'_{u, 2, H, t, 12\theta} + 2C_{12\theta, 2, t, u}^2 C'_{v, 2, H, t, 12\theta}.$$

Therefore, for any $r_1, r_2 \in [0, t]$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^2} \left(\sup_{z \in \mathbb{R}} \int_{\mathbb{R}} I_1(z, z', x_1; r_1, t) dx_1 \sup_{w \in \mathbb{R}} \int_{\mathbb{R}} I_1(w, w', x_2; r_2, t) dx_2 \right)^2 |z'|^{2H-2} |w'|^{2H-2} dz' dw' \\
& \leq (8t^2 C_{12\theta, 2, t, v}^2 C'_{u, 2, H, t, 12\theta} + 2C_{12\theta, 2, t, u}^2 C'_{v, 2, H, t, 12\theta})^2.
\end{aligned} \tag{75}$$

Secondly, we deal with the integral $dy'dz'$, which consist of two terms. For the first term, by Theorem 2.3, Lemma 3.2 and Lemma 5.2, for any $0 \leq s < r \leq t$,

$$\begin{aligned}
& \int_{\mathbb{R}^2} I_2(z, z', y, y', x_3; s, r, t) dz dx_3 \\
& \leq 2t C_{18\theta, 2, t, v} \|u_{18\theta}(s, 0) - u_{18\theta}(s, z')\|_2 \int_{\mathbb{R}} \left\| v_{18\theta}^{(r, y)}(t, x_3) - v_{18\theta}^{(r, y+y')}(t, x_3) \right\|_2 dx_3 \\
& \quad + C_{18\theta, 2, t, u} \int_{\mathbb{R}} \left\| v_{18\theta}^{(s, z)}(r, y) - v_{18\theta}^{(s, z+z')}(r, y) \right\|_2 dz \int_{\mathbb{R}} \left\| v_{18\theta}^{(r, y)}(t, x_3) - v_{18\theta}^{(r, y+y')}(t, x_3) \right\|_2 dx_3 \\
& \quad + 2t C_{18\theta, 2, t, v} \|u_{18\theta}(s, 0) - u_{18\theta}(s, z')\|_2 \int_{\mathbb{R}} \left\| v_{18\theta}^{(s, z)}(r, y) - v_{18\theta}^{(s, z)}(r, y+y') \right\|_2 dz \\
& \quad + 2t C_{18\theta, 2, t, v} C_{18\theta, 2, t, u} \int_{\mathbb{R}} \left\| v_{18\theta}^{(s, z)}(r, y) - v_{18\theta}^{(s, z)}(r, y+y') - v_{18\theta}^{(s, z+z')}(r, y) + v_{18\theta}^{(s, z+z')}(r, y+y') \right\|_2 dz.
\end{aligned} \tag{76}$$

For the second term, we integrate dx_3 first and then dz . We have

$$\begin{aligned}
& \int_{\mathbb{R}^2} I_2(y, y', z, z', x_3; s, r, t) dz dx_3 \\
& \leq 2t C_{18\theta, 2, t, v} \|u_{18\theta}(s, 0) - u_{18\theta}(s, y')\|_2 \sup_{z \in \mathbb{R}} \int_{\mathbb{R}} \left\| v_{18\theta}^{(r, z)}(t, x_3) - v_{18\theta}^{(r, z+z')}(t, x_3) \right\|_2 dx_3 \\
& \quad + C_{18\theta, 2, t, u} \int_{\mathbb{R}} \left\| v_{18\theta}^{(s, y)}(r, z) - v_{18\theta}^{(s, y+y')}(r, z) \right\|_2 dz \sup_{z \in \mathbb{R}} \int_{\mathbb{R}} \left\| v_{18\theta}^{(r, z)}(t, x_3) - v_{18\theta}^{(r, z+z')}(t, x_3) \right\|_2 dx_3
\end{aligned}$$

$$\begin{aligned}
& + 2tC_{18\theta,2,t,v} \|u_{18\theta}(s, 0) - u_{18\theta}(s, y')\|_2 \int_{\mathbb{R}} \left\| v_{18\theta}^{(s,y)}(r, z) - v_{18\theta}^{(s,y)}(r, z + z') \right\|_2 dz \\
& + 2tC_{18\theta,2,t,v} C_{18\theta,2,t,u} \int_{\mathbb{R}} \left\| v_{18\theta}^{(s,y)}(r, z) - v_{18\theta}^{(s,y)}(r, z + z') - v_{18\theta}^{(s,y+y')}(r, z) + v_{18\theta}^{(s,y+y')}(r, z + z') \right\|_2 dz.
\end{aligned} \tag{77}$$

One can see from the Remark 5.3 that the integrals on the right-hand side of (76) and (77) do not depend on y . Hence, by Lemma 5.1 and Lemma 5.5 together with the inequality $(\sum_{i=1}^4 a_i)^2 \leq 4 \sum_{i=1}^4 a_i^2$, we have

$$\begin{aligned}
& \max \left\{ \int_{\mathbb{R}^2} \left(\sup_{y \in \mathbb{R}} \int_{\mathbb{R}^2} I_2(z, z', y, y', x_3; s_3, r_3, t) dz dx_3 \right)^2 |y'|^{2H-2} |z'|^{2H-2} dy' dz', \right. \\
& \left. \int_{\mathbb{R}^2} \left(\sup_{y \in \mathbb{R}} \int_{\mathbb{R}^2} I_2(y, y', z, z', x_3; s_3, r_3, t) dz dx_3 \right)^2 |y'|^{2H-2} |z'|^{2H-2} dy' dz' \right\} \\
& \leq 8(2tC_{18\theta,2,t,v})^2 C'_{u,2,H,t,18\theta} C'_{v,2,H,t,18\theta} + 4C_{18\theta,2,t,u}^2 (C'_{v,2,H,t,18\theta})^2 + 4(2tC_{18\theta,2,t,v} C_{18\theta,2,t,u})^2 C'_{v,2,H,t,18\theta}.
\end{aligned}$$

Therefore, using the inequality $(a_1 + a_2)^2 \leq 2a_1^2 + 2a_2^2$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^2} \left(\sup_{y \in \mathbb{R}} \int_{\mathbb{R}^2} (I_2(z, z', y, y', x_3; s_3, r_3, t) + I_2(y, y', z, z', x_3; s_3, r_3, t)) dz dx_3 \right)^2 |y'|^{2H-2} |z'|^{2H-2} dy' dz' \\
& \leq 32(2tC_{18\theta,2,t,v})^2 C'_{u,2,H,t,18\theta} C'_{v,2,H,t,18\theta} + 16C_{18\theta,2,t,u}^2 (C'_{v,2,H,t,18\theta})^2 + 16(2tC_{18\theta,2,t,v} C_{18\theta,2,t,u})^2 C'_{v,2,H,t,18\theta}.
\end{aligned} \tag{78}$$

Thirdly, we deal with the integral $dy'dw'$. By change of variables $(w, w') \leftrightarrow (y, y')$, one has

$$\begin{aligned}
& \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} I_2(w, w', y, y', x_4; s_4, r_4, t) dw dy \right)^2 |y'|^{2H-2} |w'|^{2H-2} dy' dw' \\
& = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} I_2(y, y', w, w', x_4; s_4, r_4, t) dw dy \right)^2 |y'|^{2H-2} |w'|^{2H-2} dy' dw'.
\end{aligned}$$

Thus, it is enough to compute the first integral. Let $0 \leq s < r \leq t$ be arbitrary. By Theorem 2.3, Lemma 3.2 and Lemma 5.2, we integrate dw first and then dy to obtain

$$\begin{aligned}
& \int_{\mathbb{R}^2} I_2(w, w', y, y', x_4; s, r, t) dw dy \\
& \leq 2tC_{18\theta,2,t,v} \|u_{18\theta}(s, 0) - u_{18\theta}(s, w')\|_2 \int_{\mathbb{R}} \left\| v_{18\theta}^{(r,y)}(t, x_4) - v_{18\theta}^{(r,y+y')}(t, x_4) \right\|_2 dy \\
& + C_{18\theta,2,t,u} \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} \left\| v_{18\theta}^{(s,w)}(r, y) - v_{18\theta}^{(s,w+w')}(r, y) \right\|_2 dw \int_{\mathbb{R}} \left\| v_{18\theta}^{(r,y)}(t, x_4) - v_{18\theta}^{(r,y+y')}(t, x_4) \right\|_2 dy \\
& + 2tC_{18\theta,2,t,v} \|u_{18\theta}(s, 0) - u_{18\theta}(s, w')\|_2 \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} \left\| v_{18\theta}^{(s,w)}(r, y) - v_{18\theta}^{(s,w)}(r, y + y') \right\|_2 dw \\
& + 2tC_{18\theta,2,t,v} C_{18\theta,2,t,u} \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} \left\| v_{18\theta}^{(s,w)}(r, y) - v_{18\theta}^{(s,w)}(r, y + y') - v_{18\theta}^{(s,w+w')}(r, y) + v_{18\theta}^{(s,w+w')}(r, y + y') \right\|_2 dw.
\end{aligned}$$

Thus, using Lemma 5.1 and Lemma 5.5 with the inequality $(\sum_{i=1}^4 a_i)^2 \leq 4 \sum_{i=1}^4 a_i^2$, we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} I_2(w, w', y, y', x_4; s_4, r_4, t) dw dy \right)^2 |y'|^{2H-2} |w'|^{2H-2} dy' dw' \\ & \leq 8 (2tC_{18\theta,2,t,v})^2 C'_{u,2,H,t,18\theta} C'_{v,2,H,t,18\theta} + 4C_{18\theta,2,t,u}^2 (C'_{v,2,H,t,18\theta})^2 + 4 (2tC_{18\theta,2,t,v} C_{18\theta,2,t,u})^2 C'_{v,2,H,t,18\theta}. \end{aligned} \quad (79)$$

Therefore, using the inequality $(a_1 + a_2)^2 \leq 2a_1^2 + 2a_2^2$, we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} (I_2(w, w', y, y', x_4; s_4, r_4, t) + I_2(y, y', w, w', x_4; s_4, r_4, t)) dw dy \right)^2 |y'|^{2H-2} |w'|^{2H-2} dy' dw' \\ & \leq 32 (2tC_{18\theta,2,t,v})^2 C'_{u,2,H,t,18\theta} C'_{v,2,H,t,18\theta} + 16C_{18\theta,2,t,u}^2 (C'_{v,2,H,t,18\theta})^2 + 16 (2tC_{18\theta,2,t,v} C_{18\theta,2,t,u})^2 C'_{v,2,H,t,18\theta}. \end{aligned} \quad (80)$$

Lastly, substituting (75), (78) and (80) to (74), we have the following estimate for spatial integral:

$$\begin{aligned} & \int_{[-R,R]^4} dx_1 dx_2 dx_3 dx_4 \int_{\mathbb{R}^3} dy dz dw \int_{\mathbb{R}^3} |y'|^{2H-2} |z'|^{2H-2} |w'|^{2H-2} dy' dz' dw' \\ & I_1(z, z', x_1; r_1, t) I_1(w, w', x_2; r_2, t) (I_2(z, z', y, y', x_3; s_3, r_3, t) + I_2(y, y', z, z', x_3; s_3, r_3, t)) \\ & \times (I_2(w, w', y, y', x_4; s_4, r_4, t) + I_2(y, y', w, w', x_4; s_4, r_4, t)) \\ & \leq \int_{[-R,R]} dx_4 (8t^2 C_{12\theta,2,t,v}^2 C'_{u,2,H,t,12\theta} + 2C_{12\theta,2,t,u}^2 C'_{v,2,H,t,12\theta}) \\ & \times (32 (2tC_{18\theta,2,t,v})^2 C'_{u,2,H,t,18\theta} C'_{v,2,H,t,18\theta} + 16C_{18\theta,2,t,u}^2 (C'_{v,2,H,t,18\theta})^2 + 16 (2tC_{18\theta,2,t,v} C_{18\theta,2,t,u})^2 C'_{v,2,H,t,18\theta}). \end{aligned}$$

Therefore, coming back to (73), we obtain $\mathcal{A} \leq C(\theta, t, H)R$, where $C(\theta, t, H)$ is a positive constant only depends on θ, t, H and is increasing in t . This concludes the proof.

In the argument above, we used the following generalized version of the Cauchy-Schwarz inequality.

Lemma 6.5. *Let $n_1, n_2, n_3 \in \mathbb{N}$, and let $f \in L^2(\mathbb{R}^{n_1+n_2}), g \in L^2(\mathbb{R}^{n_2+n_3}), h \in L^2(\mathbb{R}^{n_3+n_1})$. Then the following inequality holds.*

$$\begin{aligned} & \int f(x_1, x_2) g(x_2, x_3) h(x_3, x_1) dx_1 dx_2 dx_3 \\ & \leq \left(\int f^2(x_1, x_2) dx_1 dx_2 \right)^{1/2} \left(\int g^2(x_2, x_3) dx_2 dx_3 \right)^{1/2} \left(\int h^2(x_3, x_1) dx_3 dx_1 \right)^{1/2}. \end{aligned}$$

Proof. Applying Cauchy-Schwarz inequality for the integral $dx_1 dx_2$ first, and then Cauchy-Schwarz inequality to dx_3 , we have

$$\int f(x_1, x_2) g(x_2, x_3) h(x_3, x_1) dx_1 dx_2 dx_3$$

$$\begin{aligned}
&\leq \left(\int f^2(x_1, x_2) dx_1 dx_2 \right)^{1/2} \left(\int dx_1 dx_2 \left(\int g(x_2, x_3) h(x_3, x_1) dx_3 \right)^2 \right)^{1/2} \\
&\leq \left(\int f^2(x_1, x_2) dx_1 dx_2 \right)^{1/2} \left(\int dx_1 dx_2 \left(\int g^2(x_2, x_3) dx_3 \right) \left(\int h^2(x_3, x_1) dx_3 \right) \right)^{1/2} \\
&= \left(\int f^2(x_1, x_2) dx_1 dx_2 \right)^{1/2} \left(\int g^2(x_2, x_3) dx_2 dx_3 \right)^{1/2} \left(\int h^2(x_3, x_1) dx_3 dx_1 \right)^{1/2}.
\end{aligned}$$

□

6.3 Functional CLT

In this section, we give the proof of Theorem 1.3.

Step 1. (tightness) For this, we show that for any $p \geq 2$, $0 \leq s < t \leq T$ and $R \geq 1$,

$$\|F_R(t) - F_R(s)\|_p \leq CR^{1/2}(t - s), \quad (81)$$

where $C = C_{p, \theta, T} > 0$ is a constant that depends on (p, θ, T) . By Kolmogorov's continuity theorem, it will follow that the process $F_R = \{F_R(t)\}_{t \in [0, T]}$ has a continuous modification (which we denote also by F_R). Moreover, by Kolmogorov-Censtov Theorem (Theorem 12.3 of [11]), the family $\{F_R\}_{R > 1}$ is tight in $C([0, T])$.

To prove (81), we proceed as in Section 4.3 of [5]. Using the chaos expansion, we have: $F_R(t) - F_R(s) = \sum_{n \geq 1} \theta^{n/2} I_n(g_{n,R}(\cdot; t, s))$, where

$$g_{n,R}(x_1, \dots, x_n; t, s) = \int_{B_R} (f_n(x_1, \dots, x_n, x; t) - f_n(x_1, \dots, x_n, x; s)) dx.$$

By hypercontractivity, for any $p \geq 2$,

$$\|F_R(t) - F_R(s)\|_p \leq \sum_{n \geq 1} \theta^{n/2} (p-1)^{n/2} (n!)^{1/2} \|\tilde{g}_{n,R}(\cdot; t, s)\|_{\mathcal{P}_0^{\otimes n}}. \quad (82)$$

Using the same argument as for (54)-(55) of [5], followed by the change of variables $\eta_j = \xi_1 + \dots + \xi_j$ and inequality (24), we obtain:

$$\begin{aligned}
&n! \|\tilde{g}_{n,R}(\cdot; t, s)\|_{\mathcal{P}_0^{\otimes n}}^2 \leq (t-s)^2 t^n c_H^n \\
&\times \int_{T_n(t)} \int_{\mathbb{R}^n} |\mathcal{F}\mathbf{1}_{[-R, R]}(\xi_1 + \dots + \xi_n)|^2 \left| \prod_{j=1}^{n-1} \mathcal{F}G_{t_{j+1}-t_j}(\xi_1 + \dots + \xi_j) \right|^2 \prod_{j=1}^n |\xi_j|^{1-2H} d\xi dt \\
&\leq c_H^n (t-s)^2 t^n \sum_{\alpha \in D_n} \int_{T_n(t)} \prod_{j=1}^{n-1} \left(\int_{\mathbb{R}} |\mathcal{F}G_{t_{j+1}-t_j}(\eta_j)|^2 |\eta_j|^{\alpha_j} d\eta_j \right) \left(\int_{\mathbb{R}} |\mathcal{F}\mathbf{1}_{[-R, R]}(\eta_n)|^2 |\eta_n|^{\alpha_n} d\eta_n \right) dt.
\end{aligned}$$

Note that $\mathcal{F}\mathbf{1}_{[-R, R]}(\eta) = \frac{2 \sin(\eta R)}{\eta}$. Using Lemma C.2 and the fact that $\alpha_n \in \{0, 1 - 2H\}$, we obtain:

$$\int_{\mathbb{R}} |\mathcal{F}\mathbf{1}_{[-R, R]}(\eta_n)|^2 |\eta_n|^{\alpha_n} d\eta_n = \int_{\mathbb{R}} \left| \frac{2 \sin(\eta_n R)}{\eta_n} \right|^2 |\eta_n|^{\alpha_n} d\eta_n = 4R^{1-\alpha_n} \int_{\mathbb{R}} \frac{\sin^2 \eta_n}{|\eta_n|^2} |\eta_n|^{\alpha_n}$$

$$\leq 8R^{1-\alpha_n} \left(\frac{1}{1-\alpha_n} + \frac{1}{1+\alpha_n} \right) \leq 8R \left(\frac{1}{H} + 1 \right).$$

Similarly, for any $j = 1, \dots, n-1$, using Lemma C.2, we have:

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{F}G_{t_{j+1}-t_j}(\eta_j)|^2 |\eta_j|^{\alpha_j} d\eta_j &= \int_{\mathbb{R}} \frac{\sin^2((t_{j+1}-t_j)|\eta_j|)}{|\eta_j|^2} |\eta_j|^{\alpha_j} d\eta_j \leq \frac{2}{1-\alpha_j} + \frac{2(t_{j+1}-t_j)^2}{1+\alpha_j} \\ &\leq 2 \left(\frac{1}{4H-1} + 1 \right) (1+t^2), \end{aligned}$$

since $\alpha_j \in \{0, 1-2H, 2(1-2H)\}$ and $H \in (\frac{1}{4}, \frac{1}{2})$. Hence,

$$n! \|\tilde{g}_{n,R}(\cdot; t, s)\|_{\mathcal{P}_0^{\otimes n}}^2 \leq (t-s)^2 t^{2n} C_H^m (1+t^2)^{n-1} \frac{1}{n!} R, \quad (83)$$

where $C_H > 0$ is a constant that depends on H . Relation (81) follows from (82), (83) and Stirling's formula.

Step 2. (finite-dimensional convergence) Let $Q_R(t) = R^{-1/2} F_R(t)$. We have to show that for any $m \geq 1$, $0 \leq t_1 < \dots < t_m \leq T$,

$$(Q_R(t_1), \dots, Q_R(t_m)) \xrightarrow{d} (\mathcal{G}_1(t_1), \dots, \mathcal{G}_m(t_m))$$

Using the same argument as in the proof of Theorem 1.3.(iii) (Step 2) of [5], it is enough to prove that for any $i, j = 1, \dots, m$,

$$\text{Var}(\langle DF_R(t_i), -DL^{-1}F_R(t_j) \rangle_{\mathcal{P}_0}) \leq CR.$$

To estimate this variance, we use Proposition B.1 of [6]. It remains to show that

$$\begin{aligned} &\int_{[-R,R]^4} \int_{\mathbb{R}^6} \|D_z u_\theta(t_j, x_1) - D_{z+z'} u_\theta(t_j, x_1)\|_4 \|D_w u_\theta(t_j, x_2) - D_{w+w'} u_\theta(t_j, x_2)\|_4 \\ &\|D_{z,y}^2 u_\theta(t_i, x_3) - D_{z,y+y'}^2 u_\theta(t_i, x_3) - D_{z+z',y}^2 u_\theta(t_i, x_3) + D_{z+z',y+y'}^2 u_\theta(t_i, x_3)\|_4 \\ &\|D_{w,y}^2 u_\theta(t_i, x_4) - D_{w,y+y'}^2 u_\theta(t_i, x_4) - D_{w+w',y}^2 u_\theta(t_i, x_4) + D_{w+w',y+y'}^2 u_\theta(t_i, x_4)\|_4 \\ &|y'|^{2H-2} |z'|^{2H-2} |w'|^{2H-2} dy dy' dz dz' dw dw' dx_1 dx_2 dx_3 dx_4 \leq CR. \end{aligned}$$

This can be proved using the same argument as for $\mathcal{A}^* \leq CR$ in Section 6.2. We omit the details.

A Moment comparison using hypercontractivity

In this section, we provide a moment comparison result for the solution of a general SPDE, in the spirit of Theorem B.1 of [1].

Let $W = \{W(\varphi); \varphi \in \mathcal{H}\}$ be an isonormal Gaussian process, associated to a Hilbert space \mathcal{H} . Assume that either one of the following conditions hold:

(i) \mathcal{H} consists of functions (or distributions) on $\mathbb{R}_+ \times \mathbb{R}^d$, i.e. W is *time-dependent*; and

(ii) \mathcal{H} consists of functions (or distributions) on \mathbb{R}^d , i.e. W is *time-independent*.

Let \mathcal{L} be a second-order pseudo-differential operator of $\mathbb{R}_+ \times \mathbb{R}^d$ and u_θ be the solution of the SPDE:

$$\mathcal{L}u(t, x) = \sqrt{\theta}u(t, x)\dot{W}, \quad t > 0, x \in \mathbb{R}^d \quad (84)$$

with (deterministic) initial condition. By definition, the (*mild Skorohod*) solution to (84) is an adapted square-integrable process $u_\theta = \{u_\theta(t, x); t > 0, x \in \mathbb{R}^d\}$ which satisfies

$$u_\theta(t, x) = w(t, x) + \sqrt{\theta} \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)u_\theta(s, y)W(\delta s, \delta y),$$

if the noise W is time-dependent, respectively

$$u_\theta(t, x) = w(t, x) + \sqrt{\theta} \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)u_\theta(s, y)W(\delta y)ds,$$

if the noise W is time-independent, provided that these integrals are well-defined. Here $W(\delta s, \delta y)$ (respectively $W(\delta y)$) denotes the Skorohod integral with respect to W , G is the fundamental solution of \mathcal{L} on $\mathbb{R}_+ \times \mathbb{R}^d$, and w is the solution of the deterministic equation $\mathcal{L}u = 0$ on $\mathbb{R}_+ \times \mathbb{R}^d$, with the same initial condition as (84).

Lemma A.1. *Suppose that for any $\theta > 0$, equation (84) has a unique solution U_θ and $\mathbb{E}|U_\theta(t, x)|^p < \infty$ for any $t > 0, x \in \mathbb{R}^d$ and $p > 1$.*

(a) *For any $\theta > 0, q > p > 1, t > 0, x \in \mathbb{R}^d$ and $h \in \mathbb{R}^d$, we have:*

$$\left\| U_{\frac{p-1}{q-1}\theta}(t, x+h) - U_{\frac{p-1}{q-1}\theta}(t, x) \right\|_q \leq \|U_\theta(t, x+h) - U_\theta(t, x)\|_p.$$

In particular, for any $\theta > 0, p > 2, t > 0, x \in \mathbb{R}^d$ and $h \in \mathbb{R}^d$, we have:

$$\|U_\theta(t, x+h) - U_\theta(t, x)\|_p \leq \|U_{(p-1)\theta}(t, x+h) - U_{(p-1)\theta}(t, x)\|_2.$$

(b) *Assume that for any $\theta > 0, t > 0$ and $x \in \mathbb{R}^d$, $U_\theta(t, x)$ is Malliavin differentiable of order k , its Malliavin derivative is a function in \mathcal{H} , and $\mathbb{E}|D_z U_\theta(t, x)|^p < \infty$ for any z and $p > 1$. Then, for any $q > p > 1, t > 0, x \in \mathbb{R}^d, l \in \mathbb{N}$, for any real numbers $\{a^{(j)} : 1 \leq j \leq l\}$ and for any $\{z_i^{(j)} : 1 \leq i \leq k, 1 \leq j \leq l\}$ (chosen in $\mathbb{R}_+ \times \mathbb{R}^d$ if the noise is time-dependent, or in \mathbb{R}^d if the noise is time-independent), we have:*

$$\left\| \sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_{\frac{p-1}{q-1}\theta}(t, x) \right\|_q \leq \left(\frac{p-1}{q-1} \right)^{k/2} \left\| \sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_\theta(t, x) \right\|_p.$$

In particular, for any $\theta > 0, p > 2, t > 0$ and $x \in \mathbb{R}^d$, we have:

$$\left\| \sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_\theta(t, x) \right\|_p \leq \left(\frac{1}{p-1} \right)^{k/2} \left\| \sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_{(p-1)\theta}(t, x) \right\|_2.$$

Proof. From the proof of Theorem B.1 of [1], for any $\tau > 0$, $t > 0$ and $x \in \mathbb{R}^d$, we have:

$$T_\tau(U_\theta(t, x)) = U_{e^{-2\tau\theta}}(t, x) \quad \text{a.s.} \quad (85)$$

where $(T_t)_{t \geq 0}$ is the Ornstein-Uhlenbeck semigroup (see relation (B.6) of [1]).

(a) Using (85) and the fact that T_τ is a linear operator, we have:

$$T_\tau(U_\theta(t, x+h) - U_\theta(t, x)) = U_{e^{-2\tau\theta}}(t, x+h) - U_{e^{-2\tau\theta}}(t, x).$$

Hence, using the hypercontractivity property (13) of the OU semigroup, we obtain:

$$\begin{aligned} \|U_{e^{-2\tau\theta}}(t, x+h) - U_{e^{-2\tau\theta}}(t, x)\|_{q(\tau)} &= \|T_\tau(U_\theta(t, x+h) - U_\theta(t, x))\|_{q(\tau)} \\ &\leq \|U_\theta(t, x+h) - U_\theta(t, x)\|_p, \end{aligned}$$

where $q(\tau) = e^{2\tau}(p-1) + 1$. We choose τ such that $q(\tau) = q$, which means that $e^{2\tau} = \frac{q-1}{p-1}$.

(b) By (85) and 12,

$$D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_{e^{-2\tau\theta}}(t, x) = D_{z_1^{(j)}, \dots, z_k^{(j)}}^k (T_\tau(U_\theta(t, x))) = e^{-k\tau} T_\tau(D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_\theta(t, x)).$$

Since T_τ is a linear operator,

$$\sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_{e^{-2\tau\theta}}(t, x) = e^{-k\tau} T_\tau \left(\sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_\theta(t, x) \right).$$

Using the hypercontractivity property (13) of the OU semigroup, we obtain:

$$\begin{aligned} \left\| \sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_{e^{-2\tau\theta}}(t, x) \right\|_{q(\tau)} &= e^{-k\tau} \left\| T_\tau \left(\sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_\theta(t, x) \right) \right\|_{q(\tau)} \\ &\leq e^{-k\tau} \left\| \sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_\theta(t, x) \right\|_p, \end{aligned}$$

where $q(\tau) = e^{2\tau}(p-1) + 1$. We choose τ such that $q(\tau) = q$, which means that $e^{2\tau} = \frac{q-1}{p-1}$. \square

Remark A.2. Using the same argument as above, one can prove the following extension of Lemma A.1. Consider a family $\mathcal{L}^{(1)}, \dots, \mathcal{L}^{(l)}$ of operators, and let $U_\theta^{(j)}$ be the unique solution of equation (84) with operator \mathcal{L} replaced by $\mathcal{L}^{(j)}$, and some initial data $w^{(j)}$. Then

$$\left\| \sum_{j=1}^l a_j U_{\frac{p-1}{q-1}\theta_j}^{(j)}(t, x_j) \right\|_q \leq \left\| \sum_{j=1}^l a_j U_{\theta_j}^{(j)}(t, x_j) \right\|_p,$$

$$\left\| \sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_{\frac{p-1}{q-1}\theta}^{(j)}(t, x_j) \right\|_q \leq \left(\frac{p-1}{q-1} \right)^{k/2} \left\| \sum_{j=1}^l a^{(j)} D_{z_1^{(j)}, \dots, z_k^{(j)}}^k U_\theta^{(j)}(t, x_j) \right\|_p.$$

B Stochastic Volterra equations with white noise in time

In this section, we study two models involving parametric families of stochastic Volterra equations driven by the Gaussian noise \mathfrak{X} with covariance (33), with $H \in (\frac{1}{4}, \frac{1}{2})$. This study will allow us to develop some properties of the solution V_θ of the (hAm) model (32) with noise \mathfrak{X} and delta initial velocity, which are needed in the sequel.

The stochastic heat and wave equations with noise \mathfrak{X} and affine function $\sigma(u) = au + b$ multiplying the noise were studied in [3]. Several facts from [3] will be needed here. For instance, from Theorem 2.9 of [3] and Minkowski's inequality, we have: for any $p \geq 2$,

$$\left\| \int_0^T \int_{\mathbb{R}} S(t, x) \mathfrak{X}(dt, dx) \right\|_p^2 \leq C_{p,H} \int_0^T \int_{\mathbb{R}^2} \|S(t, x) - S(t, y)\|_p^2 |x - y|^{2H-2} dx dy dt, \quad (86)$$

where $C_{p,H} > 0$ is a constant which depends on p and H .

Let $T > 0$ be arbitrary. For each $t \in [0, T]$, let Γ_t be a deterministic non-negative function on \mathbb{R} . Consider the following functions:

$$\begin{aligned} J_1(t) &:= \int_{\mathbb{R}^2} |\Gamma_t(x) - \Gamma_t(x+h)|^2 |h|^{2H-2} dx dh \\ J_2(t) &:= \int_0^t J_3(s) J_4(t-s) ds \\ J_3(t) &:= \int_{\mathbb{R}^3} |(\Gamma_t(x+h) - \Gamma_t(x)) - (\Gamma_t(x+h+k) - \Gamma_t(x+k))|^2 |h|^{2H-2} |k|^{2H-2} dh dk dx \\ J_4(t) &:= \int_{\mathbb{R}} \Gamma_t^2(x) dx. \end{aligned}$$

We impose the following assumption:

Assumption A. $J_i(t) < \infty$ for any $t \in [0, T]$ and $i = 1, 2, 3, 4$, and

$$Q_i(T) := \int_0^T J_i(t) dt < \infty \quad \text{for } i = 1, 2.$$

Assumption A holds when $\Gamma_t(x) = G_t(x)$ is the fundamental solution of the wave equation (given by (2)), or when $\Gamma_t(x) = g_t(x) := (2\pi t)^{-1/2} \exp(-\frac{x^2}{2t})$ is the fundamental solution of the heat equation: by Lemmas 3.1 and 3.3 of [3],

$$\begin{aligned} J_1(t) &= C \int_{\mathbb{R}} |\mathcal{F}\Gamma_t(\xi)|^2 |\xi|^{1-2H} d\xi = \begin{cases} Ct^{2H} & \text{for wave equation} \\ Ct^{H-1} & \text{for heat equation} \end{cases} \\ J_3(t) &= C \int_{\mathbb{R}} |\mathcal{F}\Gamma_t(\xi)|^2 |\xi|^{2(1-2H)} d\xi = \begin{cases} Ct^{4H-1} & \text{for wave equation} \\ Ct^{2H-3/2} & \text{for heat equation} \end{cases} \\ J_4(t) &= \begin{cases} Ct & \text{for wave equation} \\ Ct^{-1/2} & \text{for heat equation} \end{cases} \quad \text{and} \quad J_2(t) = \begin{cases} Ct^{4H+1} & \text{for wave equation} \\ Ct^{2H-1} & \text{for heat equation} \end{cases} \end{aligned}$$

Here C is a constant that depends on H and may be different in each of its appearances.

Let $p \geq 2$ be arbitrary. For any $r \in [0, T]$, let \mathcal{X}_r be the space of predictable processes $\{X(t, x); t \in [r, T], x \in \mathbb{R}\}$ such that $\|X\|_{\mathcal{X}_r} := \|X\|_{\mathcal{X}_{r,1}} + \|X\|_{\mathcal{X}_{r,2}} < \infty$, where

$$\begin{aligned} \|X\|_{\mathcal{X}_{r,1}} &= \sup_{(t,x) \in [r,T] \times \mathbb{R}} \|X(t, x)\|_p \\ \|X\|_{\mathcal{X}_{r,2}} &= \sup_{(t,x) \in [r,T] \times \mathbb{R}} \left(\int_r^t \int_{\mathbb{R}^d} \Gamma_{t-s}^2(x-y) \|X(s, y) - X(s, y+h)\|_p^2 |h|^{2H-2} dh dy ds \right)^{1/2}. \end{aligned}$$

Theorem B.1. *Suppose Assumption A holds. Let Λ be an arbitrary set.*

a) *For any $r \in [0, T]$ and $\alpha \in \Lambda$, the family of stochastic Volterra equations:*

$$X(t, x) = X_0^{(r,\alpha)}(t, x) + \int_r^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) X(s, y) \mathfrak{X}(ds, dy), \quad (87)$$

with $t \in [r, T]$ and $x \in \mathbb{R}$, has a unique solution $X^{(r,\alpha)}$ in \mathcal{X}_r , provided that $X_0^{(r,\alpha)} \in \mathcal{X}_r$.

b) *If in addition,*

$$\sup_{r \in [0, T]} \sup_{\alpha \in \Lambda} \|X_0^{(r,\alpha)}\|_{\mathcal{X}_r} < \infty, \quad (88)$$

then

$$\sup_{r \in [0, T]} \sup_{\alpha \in \Lambda} \|X^{(r,\alpha)}\|_{\mathcal{X}_r} < \infty. \quad (89)$$

Proof. a) For any $n \geq 0$, we define the Picard iterations:

$$X_{n+1}(t, x) = X_0^{(r,\alpha)}(t, x) + \int_r^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) X_n(s, y) \mathfrak{X}(ds, dy), \quad (90)$$

for all $t \in [r, T]$ and $x \in \mathbb{R}$. Note that $X_n = X_n^{(r,\alpha)}$ depends on (r, α) . To simplify the writing, we omit writing the upper index (r, α) in the first part of the argument.

The recurrence relation (90) also holds for $n = -1$, letting $X_{-1}(t, x) = 0$. For any $n \geq 0$ and $t \in [r, T]$, we define

$$\begin{aligned} V_n(t) &= \sup_{x \in \mathbb{R}} \|X_n(t, x) - X_{n-1}(t, x)\|_p^2 \\ W_n(t) &= \sup_{x \in \mathbb{R}} \int_r^t \int_{\mathbb{R}^d} \Gamma_{t-s}^2(x-y) \\ &\quad \|(X_n(s, y) - X_{n-1}(s, y)) - (X_n(s, y+h) - X_{n-1}(s, y+h))\|_p^2 |h|^{2H-2} dh dy ds. \end{aligned}$$

As in the proof of Theorem 3.8 of [3], for any $t \in [r, T]$ and $n \geq 0$, we have:

$$\begin{aligned} V_{n+1}(t) &\leq 2C_{p,H} \left(\int_r^t V_n(s) J_1(t-s) ds + W_n(t) \right) \\ W_{n+1}(t) &\leq 2C_{p,H} \left(\int_r^t V_n(s) J_2(t-s) ds + \int_r^t W_n(s) J_1(t-s) ds \right), \end{aligned}$$

where $C_{p,H}$ is the constant from (86). Letting $f_n(t) = V_n(t) + W_n(t)$, we have: for $n \geq 1$,

$$f_{n+1}(t) \leq 2C_{p,H}(C_{p,H} + 1) \int_r^t (f_n(s) + f_{n-1}(s))ds, \quad \text{for all } t \in [r, T].$$

Note that this is precisely the recurrence relation appearing in Lemma 3.10 of [3]. To apply this lemma, we need to show that f_0 and f_1 are uniformly bounded. For $n = 0$,

$$\sup_{t \in [r, T]} \sqrt{f_0(t)} \leq \sup_{t \in [r, T]} \sqrt{V_0(t)} + \sup_{t \in [r, T]} \sqrt{W_0(t)} = \|X_0^{(r,\alpha)}\|_{\mathcal{X}_r} =: M_0$$

For $n = 1$, using the recurrence relations above, we have:

$$\begin{aligned} V_1(t) &\leq 2C_{p,H} \left(\int_r^t V_0(s)J_1(t-s)ds + W_0(t) \right) \leq 2C_{p,H} \|X_0^{(r,\alpha)}\|_{\mathcal{X}_r}^2 (Q_1(t-r) + 1) \\ W_1(t) &\leq 2C_{p,H} \left(\int_r^t V_0(s)J_2(t-s)ds + \int_r^t W_0(s)J_1(t-s)ds \right) \\ &\leq 2C_{p,H} \|X_0^{(r,\alpha)}\|_{\mathcal{X}_r}^2 (Q_1(t-r) + Q_2(t-r)), \end{aligned}$$

and hence

$$\sup_{t \in [r, T]} f_1(t) \leq 2C_{p,H} \|X_0^{(r,\alpha)}\|_{\mathcal{X}_r}^2 (2Q_1(T-r) + Q_2(T-r) + 1) =: M_1.$$

Let $M = M_0 + M_1$. Applying now Lemma 3.10 of [3], we infer that there exists a sequence $(a_n)_{n \geq 0}$ of positive numbers with the property $\sum_{n \geq 0} a_n^{1/p} < \infty$ for any $p > 1$, such that

$$\sup_{t \in [r, T]} f_n(t) \leq Ma_n \quad \text{for all } n \geq 0.$$

It follows that $\|X_n - X_{n-1}\|_{\mathcal{X}_r} = \sup_{t \in [r, T]} \sqrt{V_n(t)} + \sup_{t \in [r, T]} \sqrt{W_n(t)} \leq 2\sqrt{Ma_n}$. Hence $(X_n)_{n \geq 0}$ is a Cauchy sequence in \mathcal{X}_r . Its limit X is the unique solution of (87) in \mathcal{X}_r .

b) We now prove the last statement regarding the uniform bound in (r, α) . We include the upper indices (r, α) in this part. We have:

$$\|X_n^{(r,\alpha)}\|_{\mathcal{X}_r} \leq \|X_0^{(r,\alpha)}\|_{\mathcal{X}_r} + \sum_{k=1}^n \|X_k^{(r,\alpha)} - X_{k-1}^{(r,\alpha)}\|_{\mathcal{X}_r} \leq \|X_0^{(r,\alpha)}\|_{\mathcal{X}_r} + 2(M^{(r,\alpha)})^{1/2} \sum_{n \geq 1} a_n^{1/2}.$$

Relation (89) follows letting $n \rightarrow \infty$, using the fact that $M^{(r,\alpha)}$ is uniformly bounded for all $r \in [0, T]$ and $\alpha \in \Lambda$, due to condition (88). \square

Example B.2. As an application of Theorem B.1, we consider the (hAm) model (32) with noise \mathfrak{X} and Dirac delta initial velocity. For any $r \in [0, T]$ and $z \in \mathbb{R}$ fixed, the solution $V_\theta^{(r,z)}$ satisfies the integral equation (34). This is precisely the stochastic Volterra equation (87) with initial condition $X_0^{(r,z)}(t, x) = G_{t-r}(x - z)$ and $\Gamma_t = \sqrt{\theta}G_t$. In this application of Theorem B.1, $\alpha = z$ and $\Lambda = \mathbb{R}$. Condition (88) clearly holds since:

$$\|X_0^{(r,z)}\|_{\mathcal{X}_{r,1}}^2 = \sup_{(t,x) \in [r, T] \times \mathbb{R}} G_{t-r}^2(x - z) \leq \frac{1}{4}$$

$$\begin{aligned} \|X_0^{(r,z)}\|_{\mathcal{X}_{r,2}}^2 &= \theta \sup_{(t,x) \in [r,T] \times \mathbb{R}} \int_r^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) |G_{s-r}(y-z) - G_{s-r}(y+h-z)|^2 |h|^{2H-2} dh dy ds \\ &\leq \frac{1}{4} \theta C_{H,1} \sup_{t \in [r,T]} \int_r^t (s-r)^{2H} ds = \theta C'_{H,1} (T-r)^{2H+1}. \end{aligned}$$

We conclude that

$$\sup_{r \in [0,T]} \sup_{z \in \mathbb{R}} \|V_\theta^{(r,z)}\|_{\mathcal{X}_r} < \infty. \quad (91)$$

Remark B.3. Theorem B.1 cannot be applied to the (pAm) model with Dirac delta initial condition, since $\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}} g_t(x) = \infty$, where $g_t(x) = (2\pi t)^{-1/2} e^{-x^2/t}$ is the heat kernel. For any $x \neq 0$, the function $t \mapsto g_t(x)$ attains its maximum at $t = x^2$, and if $|x| \leq T$, this maximum value is $c|x|^{-1}$ for some constant $c > 0$.

We study now a second parametric family of stochastic Volterra equations, which will be useful for treating the increments of V_θ . We introduce the following assumption.

Assumption B. For any $i = 1, 3, 4$, $J_i(t) < \infty$ for any $t \in [0, T]$ and

$$Q_i(T) := \int_0^T J_i(t) dt < \infty.$$

Let $p \geq 2$ be arbitrary. For any $r \in [0, T]$, let \mathcal{Y}_r be the set of all processes $\{Y(t, x, z, h); t \in [r, T], (x, z, h) \in \mathbb{R}^3\}$ such that the map $(\omega, t, x, z, h) \mapsto Y(\omega, t, x, z, h)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^2)$ -measurable and $\|Y\|_{\mathcal{Y}_r} := \|Y\|_{\mathcal{Y}_{r,1}} + \|Y\|_{\mathcal{Y}_{r,2}} < \infty$, where

$$\begin{aligned} \|Y\|_{\mathcal{Y}_{r,1}} &= \sup_{t \in [r,T]} \sup_{z \in \mathbb{R}} \left(\int_{\mathbb{R}^2} \|Y(t, x, z, h)\|_p^2 |h|^{2H-2} dx dh \right)^{1/2} \\ \|Y\|_{\mathcal{Y}_{r,2}} &= \sup_{t \in [r,T]} \sup_{z \in \mathbb{R}} \left(\int_{\mathbb{R}^3} \|Y(t, x, z, h) - Y(t, x+k, z, h)\|_p^2 |h|^{2H-2} |k|^{2H-2} dx dh dk \right)^{1/2}. \end{aligned}$$

Here \mathcal{P} is the predictable σ field on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$.

Theorem B.4. *Suppose Assumption B holds.*

a) *For any $r \in [0, T]$, the family of stochastic Volterra equations:*

$$X(t, x, z, h) = X_0^{(r)}(t, x, z, h) + \int_r^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) X(s, y, z, h) \mathfrak{X}(ds, dy), \quad (92)$$

with $t \in [r, T]$ and $(x, z, h) \in \mathbb{R}^3$, has a unique solution $X^{(r)}$ in \mathcal{Y}_r , provided that $X_0^{(r)} \in \mathcal{Y}_r$.

b) *If in addition,*

$$M := \sup_{r \in [0,T]} \|X_0^{(r)}\|_{\mathcal{Y}_r} < \infty,$$

then

$$\sup_{r \in [0,T]} \|X^{(r)}\|_{\mathcal{Y}_r} \leq C_{T,H,p} M, \quad (93)$$

where $C_{T,H,p} > 0$ is a constant that depends on (T, H, p) .

Proof. a) We fix $r \in [0, T]$. To illustrate the main ideas, assume first that a solution $X^{(r)}$ exists. Denote $X^{(r,z,h)}(t, x) := X^{(r)}(t, x, z, h)$.

We use the notation $a \lesssim b$ if $a \leq Cb$ and $C > 0$ is a constant that depends on p and H . By the BDG inequality (86) and triangular inequality,

$$\begin{aligned} \|X^{(r,z,h)}(t, x)\|_p^2 &\lesssim |X_0^{(r,z,h)}(t, x)|^2 + \\ &\int_r^t \int_{\mathbb{R}^2} \|\Gamma_{t-s}(x-y)X^{(r,z,h)}(s, y) - \Gamma_{t-s}(x-y-k)X^{(r,z,h)}(s, y+k)\|_p^2 |k|^{2H-2} dk dy ds \\ &\lesssim |X_0^{(r,z,h)}(t, x)|^2 + \int_r^t \int_{\mathbb{R}} (\Gamma_{t-s}(x-y) - \Gamma_{t-s}(x-y-k))^2 \|X^{(r,z,h)}(s, y)\|_p^2 |k|^{2H-2} dk dy ds \\ &\quad + \int_r^t \int_{\mathbb{R}} \Gamma_{t-s}^2(x-y-k) \|X^{(r,z,h)}(s, y) - X^{(r,z,h)}(s, y+k)\|_p^2 |k|^{2H-2} dk dy ds. \end{aligned}$$

We multiply by $|h|^{2H-2}$ and we integrate $dx dh$ on \mathbb{R}^2 . For the second term, we use

$$\int_{\mathbb{R}^2} (\Gamma_{t-s}(x-y) - \Gamma_{t-s}(x-y-k))^2 |k|^{2H-2} dx dk = J_1(t-s),$$

and for the third term, we use $\int_{\mathbb{R}} \Gamma_{t-s}^2(x-y-k) dx = J_4(t-s)$. We obtain:

$$\begin{aligned} \int_{\mathbb{R}^2} \|X^{(r,z,h)}(t, x)\|_p^2 |h|^{2H-2} dx dh &\lesssim \int_{\mathbb{R}^2} |X_0^{(r,z,h)}(t, x)|^2 |h|^{2H-2} dx dh \\ &+ \int_r^t J_1(t-s) \left(\int_{\mathbb{R}^2} \|X^{(r,z,h)}(s, y)\|_p^2 |h|^{2H-2} dy dh \right) ds \\ &+ \int_r^t J_4(t-s) \left(\int_{\mathbb{R}^3} \|X^{(r,z,h)}(s, y) - X^{(r,z,h)}(s, y+k)\|_p^2 |h|^{2H-2} |k|^{2H-2} dy dh dk \right) ds. \end{aligned}$$

Taking the supremum over $z \in \mathbb{R}$, we obtain that for any $t \in [r, T]$,

$$\alpha^{(r)}(t) \lesssim \|X_0^{(r)}\|_{\mathcal{Y}_{r,1}}^2 + \int_r^t (J_1(t-s)\alpha^{(r)}(s) + J_4(t-s)\beta^{(r)}(s)) ds, \quad (94)$$

where

$$\begin{aligned} \alpha^{(r)}(t) &= \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^2} \|X^{(r,z,h)}(t, x)\|_p^2 |h|^{2H-2} dx dh \\ \beta^{(r)}(t) &= \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^3} \|X^{(r,z,h)}(t, x) - X^{(r,z,h)}(t, x+k)\|_p^2 |h|^{2H-2} |k|^{2H-2} dx dh dk. \end{aligned}$$

To make this argument work, relation (94) should be paired with a similar inequality for $\beta^{(r)}(t)$. We show below how to obtain this. Again, by BDG inequality (86) and triangle inequality,

$$\begin{aligned} \|X^{(r,z,h)}(t, x) - X^{(r,z,h)}(t, x+k)\|_p^2 &\lesssim |X_0^{(r,z,h)}(t, x) - X_0^{(r,z,h)}(t, x+k)|^2 + \\ &\int_r^t \int_{\mathbb{R}^2} \|(\Gamma_{t-s}(x-y) - \Gamma_{t-s}(x+k-y))X^{(r,z,h)}(s, y) - \end{aligned}$$

$$\begin{aligned}
& \left(\Gamma_{t-s}(x-y-w) - \Gamma_{t-s}(x+k-y-w) \right) X^{(r,z,h)}(s, y+w) \Big|_p^2 |w|^{2H-2} dw dy ds \\
& \lesssim |X_0^{(r,z,h)}(t, x) - X_0^{(r,z,h)}(t, x+k)|^2 + \int_r^t \int_{\mathbb{R}^2} \left| \Gamma_{t-s}(x-y) - \Gamma_{t-s}(x+k-y) \right| - \\
& \quad \left(\Gamma_{t-s}(x-y-w) - \Gamma_{t-s}(x+k-y-w) \right) \Big|_p^2 \|X^{(r,z,h)}(t, x)\|_p^2 |w|^{2H-2} dw dy ds + \\
& \int_r^t \int_{\mathbb{R}^2} \left| \Gamma_{t-s}(x-y-w) - \Gamma_{t-s}(x+k-y-w) \right|^2 \|X^{(r,z,h)}(s, y) - X^{(r,z,h)}(s, y+w)\|_p^2 \\
& \quad |w|^{2H-2} dw dy ds.
\end{aligned}$$

We multiply by $|h|^{2H-2}|k|^{2H-2}$ and we integrate $dx dh dk$ on \mathbb{R}^3 . For the second term, we use

$$\begin{aligned}
& \int_{\mathbb{R}^3} \left| \left(\Gamma_{t-s}(x-y) - \Gamma_{t-s}(x+k-y) \right) - \left(\Gamma_{t-s}(x-y-w) - \Gamma_{t-s}(x+k-y-w) \right) \right|^2 \\
& \quad |k|^{2H-2} |w|^{2H-2} dw dx dk = J_3(t-s),
\end{aligned}$$

and for the third one, $\int_{\mathbb{R}^2} \left| \Gamma_{t-s}(x-y-w) - \Gamma_{t-s}(x+k-y-w) \right|^2 |k|^{2H-2} dx dk = J_1(t-s)$. We obtain:

$$\begin{aligned}
& \int_{\mathbb{R}^3} \|X^{(r,z,h)}(t, x) - X^{(r,z,h)}(t, x+k)\|_p^2 |h|^{2H-2} |k|^{2H-2} dx dh dk \lesssim \\
& \int_{\mathbb{R}^3} |X_0^{(r,z,h)}(t, x) - X_0^{(r,z,h)}(t, x+k)|^2 |h|^{2H-2} |k|^{2H-2} dx dh dk + \\
& \int_r^t J_3(t-s) \left(\int_{\mathbb{R}^2} \|X^{(r,z,h)}(t, x)\|_p^2 |h|^{2H-2} dy dh \right) ds + \\
& \int_r^t J_1(t-s) \left(\int_{\mathbb{R}^3} \|X^{(r,z,h)}(s, y) - X^{(r,z,h)}(s, y+w)\|_p^2 |h|^{2H-2} |w|^{2H-2} dy dw dh \right) ds.
\end{aligned}$$

Taking the supremum over $z \in \mathbb{R}$, we obtain that for any $t \in [r, T]$,

$$\beta^{(r)}(t) \lesssim \|X_0^{(r)}\|_{\mathcal{Y}_{r,2}}^2 + \int_r^t \left(J_3(t-s) \alpha^{(r)}(s) + J_1(t-s) \beta^{(r)}(s) \right) ds. \quad (95)$$

We now prove the existence of solution of equation (92), for fixed $r \in [0, T]$. For $n \geq 0$, we define the Picard iterations:

$$X_{n+1}(t, x, z, h) = X_0^{(r)}(t, x, z, h) + \int_r^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) X_n(s, y, z, h) \mathfrak{X}(ds, dy),$$

for all $t \in [r, T]$ and $x, z, h \in \mathbb{R}$. Denote $X_n^{(r,z,h)}(t, x) := X_n^{(r)}(t, x, z, h)$. Letting $X_{-1}^{(r,z,h)} = 0$, we see that the following recurrence relation holds for any $n \geq 0$:

$$(X_{n+1}^{(r,z,h)} - X_n^{(r,z,h)})(t, x) = \int_r^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) (X_n^{(r,z,h)} - X_{n-1}^{(r,z,h)})(s, y) \mathfrak{X}(ds, dy).$$

For any $n \geq 0$ and $t \in [r, T]$, we define

$$\alpha_n^{(r)}(t) = \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^2} \|(X_n^{(r,z,h)} - X_{n-1}^{(r,z,h)})(t, x)\|_p^2 |h|^{2H-2} dx dh$$

$$\beta_n^{(r)}(t) = \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^3} \left\| (X_n^{(r,z,h)} - X_{n-1}^{(r,z,h)})(t, x) - (X_n^{(r,z,h)} - X_{n-1}^{(r,z,h)})(t, x+k) \right\|_p^2 |h|^{2H-2} |k|^{2H-2} dx dh dk.$$

Similarly to (94) and (95), we obtain that for any $n \geq 0$ and $t \in [r, T]$,

$$\begin{aligned} \alpha_{n+1}^{(r)}(t) &\leq C_{p,H} \int_r^t (J_1(t-s)\alpha_n^{(r)}(s) + J_4(t-s)\beta_n^{(r)}(s)) ds \\ \beta_{n+1}^{(r)}(t) &\leq C_{p,H} \int_r^t (J_3(t-s)\alpha_n^{(r)}(s) + J_1(t-s)\beta_n^{(r)}(s)) ds, \end{aligned}$$

where $C_{p,H}$ is a constant depending on p and H . Denote $f_n^{(r)} = \alpha_n^{(r)} + \beta_n^{(r)}$ and $J = J_1 + J_3 + J_4$. Then, for any $n \geq 0$ and $t \in [r, T]$,

$$f_{n+1}^{(r)}(t) \leq C_{p,H} \int_r^t f_n^{(r)}(s) J(t-s) ds.$$

For the initial term, we have:

$$M_0^{(r)} := \sup_{t \in [r, T]} (\alpha_0^{(r)}(t) + \beta_0^{(r)}(t)) \leq \|X_0^{(r)}\|_{\mathcal{Y}_r}^2. \quad (96)$$

By Assumption A, $Q(T) := C_{p,H} \int_0^T J(t) dt < \infty$. By Lemma 15 of [13] (an extension of Gronwall Lemma), there exists a sequence $(a_n)_{n \geq 0}$ of non-negative real numbers with the property $\sum_{n \geq 1} a_n^{1/q} < \infty$ for any $q \geq 1$, such that

$$f_n^{(r)}(t) \leq M_0^{(r)} a_n \quad \text{for all } t \in [r, T].$$

More precisely,

$$a_n = a_n(p, H, T) = Q(T)^n P(S_n \leq T),$$

where $S_n = \sum_{i=1}^n X_i$ and $(X_i)_{i \geq 1}$ are i.i.d. random variables with values in $[0, T]$ of law $J(\cdot) / \int_0^T J(t) dt$. It follows that:

$$\|X_n^{(r)} - X_{n-1}^{(r)}\|_{\mathcal{Y}_r} = \sup_{t \in [r, T]} (\alpha_n^{(r)}(t))^{1/2} + \sup_{t \in [r, T]} (\beta_n^{(r)}(t))^{1/2} \leq 2(M_0^{(r)})^{1/2} a_n^{1/2}. \quad (97)$$

From this, we deduce that $\sum_{n \geq 0} \|X_n^{(r)} - X_{n-1}^{(r)}\|_{\mathcal{Y}_r} < \infty$. Therefore, $(X_n^{(r)})_{n \geq 0}$ is a Cauchy sequence in \mathcal{Y}_r . Its limit $X^{(r)}$ is a solution of equation (92). Uniqueness follows by standard methods.

b) From (96) and (97), we obtain that $\|X_n^{(r)} - X_{n-1}^{(r)}\|_{\mathcal{Y}_r} \leq 2\|X_0^{(r)}\|_{\mathcal{Y}_r} a_n^{1/2}$. Hence,

$$\|X_n^{(r)}\|_{\mathcal{Y}_r} \leq \|X_0^{(r)}\|_{\mathcal{Y}_r} + \sum_{k=1}^n \|X_k^{(r)} - X_{k-1}^{(r)}\|_{\mathcal{Y}_r} \leq \left(1 + 2 \sum_{k=1}^n a_k^{1/2}\right) \|X_0^{(r)}\|_{\mathcal{Y}_r}.$$

Letting $n \rightarrow \infty$, we deduce that $\|X^{(r)}\|_{\mathcal{Y}_r} \leq \left(1 + 2 \sum_{n \geq 1} a_n^{1/2}\right) \|X_0^{(r)}\|_{\mathcal{Y}_r}$ for any $r \in [0, T]$.

Relation (93) follows with

$$C_{p,H,T} = 1 + 2 \sum_{n \geq 1} a_n^{1/2}$$

□

Example B.5. As an application of Theorem B.4 we consider the (hAm) model (32) with noise \mathfrak{X} and Dirac delta initial velocity. For any $r \in [0, T]$ and $z \in \mathbb{R}$ fixed, the solution $V_\theta^{(r,z)}$ satisfies the integral equation (34). Then

$$X^{(r)}(t, x, z, h) = X^{(r,z,h)}(t, x) = V_\theta^{(r,z)}(t, x) - V_\theta^{(r,z+h)}(t, x)$$

is the unique solution of the stochastic Volterra equation (92) with $\Gamma_t := \sqrt{\theta}G_t$ and initial condition:

$$X_0^{(r)}(t, x, z, h) = G_{t-r}(x - z) - G_{t-r}(x - z - h).$$

In this case, the functions J_1, J_3, J_4 are replaced, respectively, by

$$\bar{J}_1(t) = \theta C_{H,1} t^{2H}, \quad \bar{J}_3(t) = \theta C_{H,3} t^{4H-1}, \quad \bar{J}_4(t) = \theta C_{H,4} t,$$

and the constant a_n is replaced by

$$\begin{aligned} \bar{a}_n &= \bar{a}_n(p, H, T, \theta) = \bar{Q}(T)^n P(S_n \leq T) \\ &= C_{p,H}^n \theta^n (C'_{H,1} T^{2H+1} + C'_{H,3} T^{4H} + C'_{H,4} T^2)^n P(S_n \leq T), \end{aligned}$$

where $\bar{Q}(T) := C_{p,H} \int_0^T \bar{J}(t) dt$ and $\bar{J} = \bar{J}_1 + \bar{J}_3 + \bar{J}_4$. Here $S_n = \sum_{i=1}^n X_i$ where $(X_i)_{i \geq 1}$ are i.i.d. random variables on $[0, T]$ of law $\bar{J}(\cdot) / \int_0^T \bar{J}(T)$ (which does not depend on θ). As for the bound for the initial condition, we have:

$$\begin{aligned} \|X_0^{(r)}\|_{\mathcal{Y}_{r,1}}^2 &= \sup_{t \in [r, T]} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^2} |G_{t-r}(x - z) - G_{t-r}(x - z - h)|^2 |h|^{2H-2} dx dh \\ &= C_{H,1} \sup_{t \in [r, T]} (t - r)^{2H} = C_{H,1} (T - r)^{2H} \\ \|X_0^{(r)}\|_{\mathcal{Y}_{r,2}}^2 &= \sup_{t \in [r, T]} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^3} |(G_{t-r}(x - z) - G_{t-r}(x - z - h)) - \\ &\quad (G_{t-r}(x + k - z) - G_{t-r}(x + k - z - h))|^2 |h|^{2H-2} |k|^{2H-2} dx dh dk \\ &= C_{H,3} \sup_{t \in [r, T]} (t - r)^{4H-1} = C_{H,3} (T - r)^{4H-1}, \end{aligned}$$

and hence

$$M := \sup_{r \in [0, T]} \|X_0^{(r)}\|_{\mathcal{Y}_r} = \sup_{r \in [0, T]} (C_{H,1}^{1/2} (T - r)^H + C_{H,3}^{1/2} (T - r)^{2H-1/2}) \leq C_H^{1/2} (T^H + T^{2H-1/2}),$$

where $C_H = \max(C_{H,1}, C_{H,3})$. By Theorem B.4, we infer that:

$$\sup_{0 \leq r \leq t \leq T} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^2} \|V_\theta^{(r,z)}(t, x) - V_\theta^{(r,z+h)}(t, x)\|_p^2 |h|^{2H-2} dx dh \leq C_{V,p,H,T,\theta} \quad (98)$$

$$\begin{aligned} \sup_{0 \leq r \leq t \leq T} \sup_{z \in \mathbb{R}} \int_{\mathbb{R}^3} \|V_\theta^{(r,z)}(t, x) - V_\theta^{(r,z+h)}(t, x) - V_\theta^{(r,z)}(t, x + k) + V_\theta^{(r,z+h)}(t, x + k)\|_p^2 \\ |h|^{2H-2} |k|^{2H-2} dx dh dk \leq C_{V,p,H,T,\theta}, \end{aligned} \quad (99)$$

where

$$C_{V,p,H,T,\theta} = 2C_H (T^{2H} + T^{4H-1}) \left(1 + 2 \sum_{n \geq 1} \bar{a}_n(p, H, T, \theta)^{1/2}\right)^2. \quad (100)$$

Remark B.6. Theorem B.4 also holds if we consider processes $\{Y(t, x, z, \mathbf{h})\}$ with multi-parameter $\mathbf{h} = (h_1, \dots, h_m) \in \mathbb{R}^m$ instead of h , and we replace the integral $|h|^{2H-2}dh$ on \mathbb{R} by the integral $\prod_{i=1}^m |h_i|^{2H-2}d\mathbf{h}$ on \mathbb{R}^m in the definitions of the norms $\|\cdot\|_{\mathcal{Y}_{r,1}}$ and $\|\cdot\|_{\mathcal{Y}_{r,2}}$. In addition, Theorem B.4 holds if we remove h from the parametrization, i.e. we consider processes $\{Y(t, x, z)\}$, and we drop the integral $|h|^{2H-2}dh$ from the definitions of the norms $\|\cdot\|_{\mathcal{Y}_{r,1}}$ and $\|\cdot\|_{\mathcal{Y}_{r,2}}$.

C Integral inequalities

In this section, we give some basic inequalities related to the function $\mathcal{F}G_t(\xi) = \frac{\sin(t|\xi|)}{|\xi|}$.

Lemma C.1. *Let $\varphi \in L^1(\mathbb{R})$. For $\beta \in (0, 2)$ and $t, s \in \mathbb{R}$, we have*

$$\int_{\mathbb{R}} \frac{|\sin(t|x|)| |\sin(s|x|)|}{|x|^2} |\varphi(x)| |x|^\beta dx \leq (1 + |ts|) \|\varphi\|_{L^1(\mathbb{R})}.$$

Proof. We denote by I_1, I_2 the integrals over the regions $|x| \leq 1$, respectively $|x| > 1$. In these regions, we use the inequalities $|\sin(x)| \leq |x|$, respectively $|\sin(x)| \leq 1$. Then,

$$\begin{aligned} I_1 &\leq |ts| \int_{|x| \leq 1} |\varphi(x)| |x|^\beta dx \leq |ts| \int_{|x| \leq 1} |\varphi(x)| dx \\ I_2 &\leq \int_{|x| > 1} |\varphi(x)| |x|^{\beta-2} dx \leq \int_{|x| > 1} |\varphi(x)| dx. \end{aligned}$$

□

The next result is obtained by a similar method.

Lemma C.2. *For $\beta \in (0, 1)$ and $t, s \in \mathbb{R}$, we have*

$$\int_{\mathbb{R}} \frac{|\sin(t|x|)| |\sin(s|x|)|}{|x|^2} |x|^\beta dx \leq \frac{2}{1-\beta} + \frac{2|ts|}{1+\beta}.$$

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