On optimal tests for circular reflective symmetry about an unknown central direction

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Jose Ameijeiras-Alonso, Christophe Ley, Arthur Pewsey & Thomas Verdebout

Abstract

Symmetry is one of the most fundamental of dividing hypotheses, its rejection, or not, heavily influencing subsequent modeling strategies. In this paper, the authors construct tests for circular reflective symmetry about an unknown central direction that are asymptotically valid within a semi-parametric class of distributions and maintain certain parametric local and asymptotic optimality properties. The asymptotic distributions of the test statistics under the null hypothesis and under local alternatives are established, and a pre-existing omnibus test is identified as a special case of the proposed construction. The finite-sample properties of the semi-parametric tests are compared with those of other testing approaches in a simulation experiment, and recommendations made regarding testing for reflective symmetry in practice. Analyses of data on the directions of cracks in hip replacements illustrate the proposed methodology.

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Author information

Authors and Affiliations

Statistics Section, Department of Mathematics, KU Leuven, Leuven, Belgium

Jose Ameijeiras-Alonso

Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Ghent, Belgium

Christophe Ley

Department of Mathematics, University of Extremadura, Cáceres, Spain

Arthur Pewsey

Mathematics Department, Université Libre de Bruxelles, Brussels, Belgium

Thomas Verdebout

Corresponding author

Correspondence to Jose Ameijeiras-Alonso.

Additional information

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Electronic supplementary material

Below is the link to the electronic supplementary material.

Supplementary material 1 (pdf 138 KB)

Appendices

Proof of Lemma 1

We show that $\hat{\Gamma}_{f_0,g_0;11} - \Gamma_{f_0,g_0;11} = o_{\mathrm{P}}(1)$ as $n \to \infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$. Showing that $\hat{\Gamma}_{g_0,p;12} - \Gamma_{g_0,p;12} = o_{\mathrm{P}}(1)$ as $n \to \infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ proceeds along the same lines. In this proof, we set $\mu^{(n)} := \mu + n^{-1/2}\tau_1^{(n)}$ for some bounded sequence $\tau_1^{(n)}$ as in Theorem 1. Because of the local discreteness of $\hat{\mu}^{(n)}$ (Assumption B), it is sufficient to show that

$$n^{-1}\sum_{i=1}^{n}\dot{\varphi}_{f_0}\left(\Theta_i - \mu^{(n)}\right) - \mathrm{E}_{g_0}\left[\dot{\varphi}_{f_0}\left(\Theta_i - \mu\right)\right] = o_{\mathrm{P}}(1)$$

as $n \to \infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$. The law of large numbers leads to

$$n^{-1}\sum_{i=1}^{n}\dot{\varphi}_{f_0}\left(\Theta_i - \mu\right) - \mathrm{E}_{g_0}\left[\dot{\varphi}_{f_0}\left(\Theta_i - \mu\right)\right] = o_{\mathrm{P}}(1)$$

as $n \to \infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ so that it only remains to show that

$$S_n := n^{-1} \sum_{i=1}^{n} \left(\dot{\varphi}_{f_0} \left(\Theta_i - \mu^{(n)} \right) - \dot{\varphi}_{f_0} \left(\Theta_i - \mu \right) \right)$$

is $o_{\mathrm{P}}(1)$ as $n \to \infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$. As the Θ_i are i.i.d., we find

$$\begin{aligned} \mathrm{E}[|S_n|] \leq & n^{-1} \sum_{i=1}^{n} \mathrm{E}[|\dot{\varphi}_{f_0} \left(\Theta_i - \mu^{(n)} \right) - \dot{\varphi}_{f_0} \left(\Theta_i - \mu \right)|] \\ = & \mathrm{E}[|\dot{\varphi}_{f_0} \left(\Theta_1 - \mu^{(n)} \right) - \dot{\varphi}_{f_0} \left(\Theta_1 - \mu \right)|]. \end{aligned}$$

Since $\dot{\varphi}_{f_0}$ is continuous on a compact support, it is bounded. The result then follows by applying Lebesgue's dominated convergence theorem. □

Proof of Lemma 2

To prove (i), we start by showing that $\Delta^{(n)\star}_{f_0,j}(\hat{\mu}^{(n)}) - \Delta^{(n)\star}_{f_0,j}(\mu) = o_P(1)$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n \to \infty$. First note that, due to Assumption B, we have under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n \to \infty$ that

$$\begin{aligned} n^{-1/2} \sum_{i=1}^{n} \varphi_{f_0}(\Theta_i - \hat{\mu}^{(n)}) = & \Delta^{(n)}_{f_0}(\mu) - \mathrm{E}_{g_0}[\dot{\varphi}_{f_0}(\Theta_i - \mu)] n^{1/2}(\hat{\mu}^{(n)} - \mu) + o_{\mathrm{P}}(1) \\ = & \Delta^{(n)}_{f_0}(\mu) - \mathit{\Gamma}_{f_0,g_0;11} n^{1/2}(\hat{\mu}^{(n)} - \mu) + o_{\mathrm{P}}(1). \end{aligned}$$

(12)

Therefore, using (4) with (12), it follows that

$$\begin{aligned} \Delta^{(n)\star}_{f_0,j}(\hat{\mu}^{(n)}) = & \Delta^{(n)}_{j}(\mu) - \eta \Delta^{(n)}_{f_0}(\mu) - (\mathit{\Gamma}_{g_0,j;12} - \eta \mathit{\Gamma}_{f_0,g_0;11}) n^{1/2}(\hat{\mu}^{(n)} - \mu) + o_{\mathrm{P}}(1) \\ = & \Delta^{(n)\star}_{f_0,j}(\mu) + o_{\mathrm{P}}(1) \end{aligned}$$

under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n \to \infty$ since $\mathit{\Gamma}_{g_0,j;12} - \eta \mathit{\Gamma}_{f_0,g_0;11} = 0$. It remains to show that

$$\hat{\Delta}^{(n)\star}_{f_0,j}(\hat{\mu}^{(n)}) - \Delta^{(n)\star}_{f_0,j}(\hat{\mu}^{(n)}) = -\left(\frac{\hat{\mathit{\Gamma}}_{g_0,j;12}}{\hat{\mathit{\Gamma}}_{f_0,g_0;11}} - \frac{\mathit{\Gamma}_{g_0,j;12}}{\mathit{\Gamma}_{f_0,g_0;11}}\right)\Delta^{(n)}_{f_0}\left(\hat{\mu}^{(n)}\right) = o_{\mathrm{P}}(1)$$

(13)

under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n \to \infty$. To prove (13), first note that (12) and the central limit theorem (CLT) imply that $\Delta^{(n)}_{f_0}\left(\hat{\mu}^{(n)}\right)$ is $O_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n \to \infty$. Therefore, we only need to show that

$$\frac{\hat{\mathit{\Gamma}}_{g_0,j;12}}{\hat{\mathit{\Gamma}}_{f_0,g_0;11}} - \frac{\mathit{\Gamma}_{g_0,j;12}}{\mathit{\Gamma}_{f_0,g_0;11}} = o_{\mathrm{P}}(1)$$

(14)

as $n \to \infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n \to \infty$. Since

$$\frac{\hat{\mathit{\Gamma}}_{g_0,j;12}}{\hat{\mathit{\Gamma}}_{f_0,g_0;11}} - \frac{\mathit{\Gamma}_{g_0,j;12}}{\mathit{\Gamma}_{f_0,g_0;11}} = \frac{\hat{\mathit{\Gamma}}_{g_0,j;12} - \mathit{\Gamma}_{g_0,j;12}}{\hat{\mathit{\Gamma}}_{f_0,g_0;11}} - \frac{\mathit{\Gamma}_{g_0,j;12}\left(\hat{\mathit{\Gamma}}_{f_0,g_0;11} - \mathit{\Gamma}_{f_0,g_0;11}\right)}{\hat{\mathit{\Gamma}}_{f_0,g_0;11}\mathit{\Gamma}_{f_0,g_0;11}},$$

the result follows directly from Lemma 1.

Turning to the proof of (ii), and working along the same lines as those at the end of the proof of Lemma 1, it is easily shown that

$$n^{-1}\sum_{i=1}^{n}\varphi^2_{f_0}(\Theta_i - \hat{\mu}^{(n)}) - \mathrm{E}_{g_0}[\varphi^2_{f_0}(\Theta_i - \mu)],$$
$$n^{-1}\sum_{i=1}^{n}\sin^2(j(\Theta_i - \hat{\mu}^{(n)})) - \mathrm{E}_{g_0}[\sin^2(j(\Theta_i - \mu))]$$

(15)

and

$$n^{-1}\sum_{i=1}^{n}\sin(j(\Theta_i-\hat{\mu}^{(n)}))\varphi_{f_0}(\Theta_i-\hat{\mu}^{(n)})-\mathrm{E}_{g_0}[\sin(j(\Theta_i-\mu))\varphi_{f_0}(\Theta_i-\mu)]$$

(16)

are $o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n\to\infty$. Using the law of large numbers, $C_{f_0,j}(\mu)$ converges to $\mathrm{E}_{g_0}\left[\left(\sin(j(\Theta_i-\mu))-\frac{\Gamma_{g_0,j;12}}{\Gamma_{f_0,g_0;11}}\varphi_{f_0}(\Theta_i-\mu)\right)^2\right]$, in probability, under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$. From this last result, it follows that $C_{f_0,j}(\hat{\mu}^{(n)})-C_{f_0,j}(\mu)=o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n\to\infty$. Therefore it remains to show that $\hat{C}_{f_0,j}(\hat{\mu}^{(n)})-C_{f_0,j}(\hat{\mu}^{(n)})$ is $o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n\to\infty$. We readily obtain that

$$\hat{C}_{f_0,j}\left(\hat{\mu}^{(n)}\right)-C_{f_0,j}\left(\hat{\mu}^{(n)}\right)=\left(\frac{\hat{\Gamma}^2_{g_0,j;12}}{\hat{\Gamma}^2_{f_0,g_0;11}}-\frac{\Gamma^2_{g_0,j;12}}{\Gamma^2_{f_0,g_0;11}}\right)n^{-1}\sum_{i=1}^{n}\varphi^2_{f_0}(\Theta_i-\hat{\mu}^{(n)})$$
$$-2\left(\frac{\hat{\Gamma}_{g_0,j;12}}{\hat{\Gamma}_{f_0,g_0;11}}-\frac{\Gamma_{g_0,j;12}}{\Gamma_{f_0,g_0;11}}\right)n^{-1}\sum_{i=1}^{n}\sin(j(\Theta_i-\hat{\mu}^{(n)}))\varphi_{f_0}(\Theta_i-\hat{\mu}^{(n)})$$

so (15) and (16) together with (14) and the continuous mapping theorem imply that $\hat{C}_{f_0,j}\left(\hat{\mu}^{(n)}\right)-C_{f_0,j}\left(\hat{\mu}^{(n)}\right)$ is $o_{\mathrm{P}}(1)$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ as $n\to\infty$. The result follows.

□

Proof of Theorem 2

Fix $g_0\in\mathcal{G}$ and $\mu\in[-\pi,\pi)$. Lemma 2 combined with Slutsky's lemma leads to

$$Q^{(n)}_{f_0,j}=\frac{\hat{\Delta}^{(n)\star}_{f_0,j}(\hat{\mu}^{(n)})}{\sqrt{\hat{C}_{f_0,j}(\hat{\mu}^{(n)})}}=\frac{\Delta^{(n)\star}_{f_0,j}(\mu)}{\sqrt{C_{f_0,j}(\mu)}}+o_P(1)=\frac{\Delta^{(n)\star}_{f_0,j}(\mu)}{V^{g_0}_{f_0}(j)}+o_P(1)$$

(17)

as $n\to\infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$. Part (i) then follows from the CLT.

Part (ii) is obtained via Le Cam's third lemma. First, it is necessary to calculate the joint distribution of $\hat{\Delta}^{(n)\star}_{f_0,j}(\hat{\mu}^{(n)})$ and $\log(d\mathrm{P}^{(n)}_{(\mu,n^{-1/2}\tau_2^{(n)})';g_0,k}/d\mathrm{P}^{(n)}_{(\mu,0)';g_0})$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$. We use Lemma 2 and the fact that

$$n^{-1/2}\sum_{i=1}^{n}\begin{pmatrix}\sin(j(\Theta_i-\mu))-\eta\varphi_{f_0}(\Theta_i-\mu)\\ \tau_2^{(n)}\sin(k(\Theta_i-\mu))\end{pmatrix}-\begin{pmatrix}0\\ \frac{1}{2}(\tau_2^{(n)})^2\Gamma_{g_0,k;22}\end{pmatrix}\xrightarrow{\mathcal{D}}$$
$$\mathcal{N}_2\left(\begin{pmatrix}0\\ -\frac{1}{2}(\tau_2)^2\Gamma_{g_0,k;22}\end{pmatrix},\begin{pmatrix}V^{g_0}_{f_0}(j) & \tau_2 C^{g_0}_{f_0}(j,k)\\ \tau_2 C^{g_0}_{f_0}(j,k) & (\tau_2)^2\Gamma_{g_0,k;22}\end{pmatrix}\right)$$

as $n\to\infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$, obtained using the multivariate CLT. Now, since $\mathrm{P}^{(n)}_{(\mu,0)';g_0}$ and $\mathrm{P}^{(n)}_{(\mu,n^{-1/2}\tau_2^{(n)})';g_0,k}$ are mutually contiguous, applying Le Cam's third lemma, we obtain that $\hat{\Delta}^{(n)\star}_{f_0,j}(\hat{\mu}^{(n)})\xrightarrow{\mathcal{D}}\mathcal{N}(\tau_2 C^{g_0}_{f_0}(j,k),V^{g_0}_{f_0}(j))$ under $\mathrm{P}^{(n)}_{(\mu,n^{-1/2}\tau_2^{(n)})';g_0,k}$, as $n\to\infty$.

We now turn to Part (iii). From (17), it is easily seen that

$$n^{-1/2}\sum_{i=1}^{n}\left[\sin(j(\Theta_i-\mu))-\frac{\Gamma_{f_0,j;12}}{\Gamma_{f_0,f_0;11}}\varphi_{f_0}(\Theta_i-\mu)\right]=\Delta^{(n)\star}_{f_0,j}(\hat{\mu}^{(n)})+o_{\mathrm{P}}(1),$$

(18)

as $n\to\infty$ under $\mathrm{P}^{(n)}_{(\mu,0)';f_0}$, and therefore under contiguous alternatives. Now, consider the following parametric testing problem

$$\begin{cases}\mathcal{H}_{0;f_0}:=\cup_{\mu\in[-\pi,\pi)}\mathrm{P}^{(n)}_{(\mu,0)';f_0},\\ \mathcal{H}_{1;f_0,j}:=\cup_{\lambda\neq0\in[-1,1]}\cup_{\mu\in[-\pi,\pi)}\mathrm{P}^{(n)}_{(\mu,\lambda)';f_0,j},\end{cases}$$

and consider the parametric test that rejects $\mathcal{H}_{0;f_0}$ when the absolute value of the parametric test statistic, that can be derived by taking as numerator the first part of the equality in (18), is greater than $z_{1-\alpha/2}$. It follows from Le Cam theory that this test is locally and asymptotically maximin for testing $\mathcal{H}_{0;f_0}$ against $\mathcal{H}_{1;f_0,j}$. The result in (iii) then follows from (18) and the optimality features of the parametric test for all $f_0\in\mathcal{G}$ satisfying Assumption A. ☐

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