# Multi-Dimensional Parameter Estimation of Heavy-Tailed Moving Averages 

Mathias Mørck Ljungdahl<br>Department of Mathematics, Aarhus University

Mark Podolskij<br>Department of Mathematics, University of Luxembourg


#### Abstract

In this paper we present a parametric estimation method for certain multi-parameter heavy-tailed Lévy-driven moving averages. The theory relies on recent multivariate central limit theorems obtained via Malliavin calculus on Poisson spaces. Our minimal contrast approach is related to previous papers, which propose to use the marginal empirical characteristic function to estimate the one-dimensional parameter of the kernel function and the stability index of the driving Lévy motion. We extend their work to allow for a multi-parametric framework that in particular includes the important examples of the linear fractional stable motion, the stable Ornstein-Uhlenbeck process, certain CARMA $(2,1)$ models and Ornstein-Uhlenbeck processes with a periodic component among other models. We present both the consistency and the associated central limit theorem of the minimal contrast estimator. Furthermore, we demonstrate numerical analysis to uncover the finite sample performance of our method.


Keywords: Heavy tails, Lévy processes, limit theorems, low frequency, parametric estimation

## 1 Introduction

Steadily through the last decades estimation procedures for various classes of continuous time moving averages and related processes have been proposed, see, e.g. $[2,11,16]$ for estimation of the parameters in the linear fractional stable motion model and [9, 10] for the more general class of self-similar processes among many others. The bedrock of these techniques are of course the underlying limit theory for various functionals of the processes at hand. One such seminal paper is [18], which gives conditions for bounded functionals of a large class of moving averages and was later extended in [19] to certain unbounded functions. In a similar framework [5] gives an almost complete picture of the 'law of large numbers' for the classical case of the power variation functionals. The article [4] extends the functionals

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from power variation to a large class of statistically interesting functionals and for a class of symmetric $\beta$-stable moving averages. This paper also provides an almost complete picture of the corresponding weak limit theorems, at least in the setting of Appell rank $>1$ (such as is the case for power variation and the (real part) of the characteristic function).

Previous estimation methods suggested in $[14,15,16]$ relied on functionals of the onedimensional marginal law of the process and specific properties of the process at hand. Since the marginal distribution of the considered models have been symmetric $\beta$-stable, only the scale and the stability parameters can be estimated via such statistics. In particular, they are typically not sufficient to estimate kernel functions that depend on a multi-dimensional parameter, which excludes many interesting models. Indeed, this discrepancy is observed in [14], where the characteristic function of the one-dimensional law is not sufficient and instead the authors have to rely on a combination with other statistics to ensure estimation of all parameters.

The aim of this paper is to construct estimators of the kernel function and the stability index in the general setting of a multi-dimensional parameter space. Instead of relying on existing theory $[4,5,18]$, which only accounts for the marginal law of the underlying model, we shall use the framework from the recent paper [3], which is tailor-made for the study of Gaussian fluctuations of functionals of multiple heavy-tailed moving averages, to estimate the multi-dimensional parameter. While our approach is similar in spirit to the univariate framework of $[14,15]$ from the theoretical viewpoint, there are some important differences. First of all, the parameter identifiability and non-degeneracy condition (see Condition (A)(4) below) are not trivial in the multi-dimensional setting and we demonstrate the corresponding techniques for various examples. Secondly, the theoretical results of our paper shed light on parameter estimation for a large class of models; in particular, we will present statistical inference for a certain $\operatorname{CARMA}(2,1)$ model, which is novel in the literature. Finally, we remark that while Theorem 2.7 is a multivariate extension of [15, Proposition 1], its proof is somewhat more complex than in the univariate setting.

Let us now define the class of moving average processes for which the underlying limit theory applies. Let $L=\left(L_{t}\right)_{t \in \mathbb{R}}$ be a standard symmetric $\beta$-stable Lévy process and consider the model

$$
\begin{equation*}
X_{t}=\int_{-\infty}^{t} g(t-s) \mathrm{d} L_{s}, \quad t \in \mathbb{R} \tag{1.1}
\end{equation*}
$$

for some measurable $g: \mathbb{R} \rightarrow \mathbb{R}$. Necessary and sufficient conditions for the integral to exist are given in [20] and we mention that in our setting a sufficient condition is $\int_{\mathbb{R}}|g(s)|^{\beta} \mathrm{d} s<\infty$. The kernel function $g$ is assumed to have a power behaviour around 0 and at infinity. More specifically, we shall assume the existence of a constant $K>0$ together with powers $\alpha>0$ and $\kappa \in \mathbb{R}$ for which it holds

$$
\begin{equation*}
|g(x)| \leq K\left(x^{\kappa} \mathbb{1}_{[0,1)}(x)+x^{-\alpha} \mathbb{1}_{[1, \infty)}(x)\right) \quad \text { for all } x \in \mathbb{R} \tag{1.2}
\end{equation*}
$$

We are interested in (scaled) partial sums of multivariate functionals of the vectors (( $X_{s+1}$, $\left.\left.\ldots, X_{s+m}\right)\right)_{s \geq 0}$ :

$$
\begin{equation*}
V_{n}(X ; f)=\frac{1}{\sqrt{n}} \sum_{s=0}^{n-m}\left(f\left(X_{s+1}, \ldots, X_{s+m}\right)-\mathbb{E}\left[f\left(X_{1}, \ldots, X_{m}\right)\right]\right) \tag{1.3}
\end{equation*}
$$

where $f: \mathbb{R}^{m} \rightarrow \mathbb{R}^{d}$ is a suitable Borel function. Adhering to [3, Remark 2.4(iii)] the following result holds. Below $C_{b}^{2}\left(\mathbb{R}^{m}, \mathbb{R}^{d}\right)$ denotes the space of twice differentiable functions $f: \mathbb{R}^{m} \rightarrow \mathbb{R}^{d}$ such that $f$ and all of its first and second order derivatives are bounded and continuous.

Theorem 1.1 ([3, Theorem 2.3]). Let $\left(X_{t}\right)_{t \in \mathbb{R}}$ be a moving average as in (1.1) with kernel function $g$ satisfying (1.2). Assume that $\alpha \beta>2$ and $\kappa>-1 / \beta$. Let $f=\left(f_{1}, \ldots, f_{d}\right) \in$ $C_{b}^{2}\left(\mathbb{R}^{m}, \mathbb{R}^{d}\right)$ and consider the statistic $V_{n}(X ; f)$ introduced at (1.3). Then as $n \rightarrow \infty$

$$
\begin{equation*}
\Sigma_{n}^{i, j}:=\operatorname{Cov}\left(V_{n}(X ; f)\right) \rightarrow \Sigma^{i, j}:=\sum_{s \in \mathbb{Z}} \operatorname{Cov}\left(f_{i}\left(X_{s+1}, \ldots, X_{s+m}\right), f_{j}\left(X_{1}, \ldots, X_{m}\right)\right) \tag{1.4}
\end{equation*}
$$

for any $1 \leq i, j \leq d$. Moreover, $V_{n}(X ; f) \xrightarrow{\mathcal{L}} \mathcal{N}_{d}(0, \Sigma)$ as $n \rightarrow \infty$.

The paper [3] additionally provides Berry-Esseen type bounds for an appropriate distance between probability laws on $\mathbb{R}^{d}$, but Theorem 1.1 is sufficient for our statistical analysis. We remark that the limit theory for bounded $f$ in the case of $m=1$ and general $d \in \mathbb{N}$ is handled in [19], but it is actually the reverse situation, i.e. $m \in \mathbb{N}$ and $d=1$, that is the most needed. Specifically, $f$ will be the empirical characteristic function of the joint distribution $\left(X_{s+1}, \ldots, X_{s+m}\right)$, which then grants us the ability to estimate parameters which are not determined by the one-dimensional distribution of $X_{1}$, see Examples 2.2-2.6 below.

The paper is organized as follows. In Section 2 we introduce the parametric model, numerous assumptions and the main theoretical results of the paper, which show the strong consistency and the asymptotic normality of the minimal contrast estimator. Section 3 is devoted to a numerical analysis of the finite sample performance of our estimator. Finally, all proofs are collected in Appendix A.

## 2 The setting and main results

### 2.1 The model and assumptions

In the following we will consider a Lévy-driven moving average $X=\left(X_{t}\right)_{t \in \mathbb{R}}$ given by

$$
\begin{equation*}
X_{t}=\int_{\mathbb{R}} g_{\beta, \theta}(t-s) \mathrm{d} L_{s}, \quad t \in \mathbb{R} \tag{2.1}
\end{equation*}
$$

where $L$ is a symmetric $\beta$-stable Lévy process with unit scale and $\beta \in \Upsilon$ for some open subset $\Upsilon \subseteq(0,2)$, and $\left\{g_{\beta, \theta} \mid \beta \in \Upsilon, \theta \in \Theta\right\}$ is a measurable family of functions parametrized by an open subset $\Upsilon \times \Theta \subseteq(0,2) \times \mathbb{R}^{d}$ for some $d \geq 1$. For ease of notation we shall often denote the joint parameter with $\xi=(\beta, \theta)$ and the open subset by $\Xi=\Upsilon \times \Theta$.

The main goal of this section is to extend the theory of [15] from a one-dimensional parameter space, i.e. $d=1$, to a general multi-dimensional theory. Such multi-dimensional parameter spaces include important examples of the linear fractional stable motion, the stable Ornstein-Uhlenbeck process, certain CARMA $(2,1)$ models, and Ornstein-Uhlenbeck processes with a periodic component among others. One of the main difficulties in extending from $d=1$ to $d \in \mathbb{N}$ is that, quite naturally, the parameters $(\beta, \theta)$ should be identifiable

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from the (theoretical) statistic, which in the case of [15] is the one-dimensional characteristic function:

$$
\phi_{\beta, \theta}(u)=\mathbb{E}\left[\mathrm{e}^{\mathrm{i} u X_{1}}\right]=\exp \left(-\left\|u g_{\beta, \theta}\right\|_{\beta}^{\beta}\right)
$$

This identification can very well be an unreasonable assumption if $d>1$, see Example 2.2. But if we instead consider the characteristic function of the joint distribution $\left(X_{1}, \ldots, X_{m}\right)$,

$$
\begin{equation*}
\varphi_{\beta, \theta}^{m}\left(u_{1}, \ldots, u_{m}\right)=\mathbb{E}\left[\mathrm{e}^{\mathrm{i} \sum_{k=1}^{m} u_{k} X_{k}}\right]=\exp \left(-\left\|\sum_{k=1}^{m} u_{k} g_{\beta, \theta}(\cdot+k)\right\|_{\beta}^{\beta}\right) \tag{2.2}
\end{equation*}
$$

such an identification may be possible. Let us discuss this in more details. The underlying stability index $\beta$ is always identifiable from (2.2), since the stability index of a stable random variable is unique. The problem is then reduced to whether the parametrization of the kernel $\theta \mapsto g_{\beta, \theta}$ specifies the distribution of $X$ uniquely. The question now becomes a matter of uniqueness for the spectral representation of moving averages, which has been studied in, e.g. [21]. Translating the question to the characteristic functions of the finite dimensional distributions, $\left(X_{1}, \ldots, X_{m}\right), m \in \mathbb{N}$, we ask whether the $\beta$-norm of linear combinations of translations of the kernel specifies $g_{\beta, \theta}$ uniquely. This is known as Kanter's theorem in the literature and first appeared in [12], but for exposition sake let us repeat it here. Suppose $\beta \in(0, \infty)$ is not an even integer and let $g, h \in \mathcal{L}^{\beta}(\mathbb{R})$. Then Kanter's theorem states that if for all $n \in \mathbb{N}$ and $u_{1}, t_{1}, \ldots, u_{n}, t_{n} \in \mathbb{R}$ it holds that

$$
\left\|\sum_{i=1}^{n} u_{i} g\left(\cdot+t_{i}\right)\right\|_{\beta}^{\beta}=\left\|\sum_{i=1}^{n} u_{i} h\left(\cdot+t_{i}\right)\right\|_{\beta}^{\beta},
$$

then there exists an $\epsilon \in\{ \pm 1\}$ and a $\tau \in \mathbb{R}$ such that $g=\epsilon h(\cdot+\tau)$ almost everywhere. Kanter's theorem then implies that the distribution of $X$ is the same under $\theta$ and $\theta^{\prime}$ if and only if there exists $\epsilon \in\{ \pm 1\}$ and $\tau \in \mathbb{R}$ such that

$$
\epsilon g_{\beta, \theta}(\cdot+\tau)=g_{\beta, \theta^{\prime}} \quad \text { almost everywhere. }
$$

For many concrete examples of the kernel family $\left\{g_{\xi} \mid \xi \in \Xi\right\}$ it is often straightforward to check that such an identity only occurs if $\epsilon=1, \tau=0$ and $\theta=\theta^{\prime}$.

Due to the preceding discussion it is reasonable to make the following assumptions on the family of kernels and we note that similar identification requirements are often explicitly or implicitly required in the literature. An important remark is that our theory allows for a general $m \in \mathbb{N}$ instead of only $m \in\{1,2\}$, where the statistics in the case $m=2$ are often autocorrelations. We denote by $\partial_{\xi}^{i, j} f_{\xi}$ the partial derivative of $f$ with respect to the $i$ th and the $j$ th parameters $\xi_{i}$ and $\xi_{j}$ evaluated at $\xi \in \Xi$. Note in particular that $\partial_{\xi}^{1}$ is the derivative with respect to $\beta$ and $\partial_{\xi}^{i}$ for $i \geq 2$ is the derivative with respect to the $\theta_{i}$-coordinate according to the notational convention given below (2.1).

Assumption (A). There exists an $m \in \mathbb{N}$ such that:
(1) $0<\left\|g_{\beta, \theta}\right\|_{\beta}<\infty$ for all $(\beta, \theta) \in \Upsilon \times \Theta$.
(2) The map $\theta \mapsto \varphi_{\beta, \theta}^{m}$ given in (2.2) is injective.
(3) The function $(\beta, \theta) \mapsto\left\|\sum_{i=1}^{m} u_{i} g_{\beta, \theta}(\cdot+i)\right\|_{\beta}^{\beta}$ is $C^{2}(\Upsilon \times \Theta)$ for each $u_{1}, \ldots, u_{m} \in \mathbb{R}$.
(4) $u \mapsto \partial_{\xi}^{1} \varphi_{\xi}^{m}(u), \partial_{\xi}^{2} \varphi_{\xi}^{m}(u), \ldots, \partial_{\xi}^{d+1} \varphi_{\xi}^{m}(u)$ are linearly independent continuous functions.

Let us give some remarks about the imposed conditions.
Remark 2.1.
(i) The assumption $(\mathrm{A})(1)$ is a necessary and sufficient condition for $X$ to be welldefined and non-degenerate. Moreover, (A)(1) makes it apparent why an explicit dependence on $\beta$ of the kernel $g_{\beta, \theta}$ could be useful. This case of dependence is also necessary for some processes such as increments of the linear fractional stable motion, see Example 2.3 below.
(ii) Condition $(\mathrm{A})(2)$ is necessary to ensure that the model (2.1) is parametrized properly. Note that the non-existence of an $m \in \mathbb{N}$ such that $(\mathrm{A})(2)$ holds would imply that the parameters could never be inferred from any finite data sample making the inference of $\theta$ impossible in practice. The identification of the parameters in a continuous time model from samples at equidistant time points is known in the literature as the aliasing problem.
(iii) Condition (A)(3) is a minimal requirement for our method of proof (see also [15, Assumption (A)]). In particular, it ensures existence of the derivatives in $(A)(4)$.
(iv) We note here that under assumption $(A)(1)$ condition $(A)(4)$ follows from linear independence of the smaller subset: $u \mapsto \partial_{\xi}^{2} \varphi_{\xi}^{m}(u), \ldots, \partial_{\xi}^{d+1} \varphi_{\xi}^{m}(u)$. Indeed, this follows from the particular form of the $\beta$-derivative as shown in Appendix A.2.

In order to use Theorem 1.1 we need to make additional assumptions on our kernel and for this we need to introduce some more notation. Consider a strictly positive weight function $w \in \mathcal{L}^{1}\left(\mathbb{R}_{+}^{m}\right)$ and define the weighted inner product and norms

$$
\langle g, h\rangle_{w}=\int_{\mathbb{R}_{+}^{m}} g(x) h(x) w(x) \mathrm{d} x \quad \text { and } \quad\|h\|_{w, p}^{p}=\int_{\mathbb{R}_{+}^{m}}|h(x)|^{p} w(x) \mathrm{d} x, \quad p \in\{1,2\}
$$

Let $\mathcal{L}_{w}^{p}\left(\mathbb{R}_{+}^{m}\right)$ denote the corresponding Banach $\mathcal{L}^{p}$-space of Borel functions.

## Assumption (B).

(1) Assume that for all $(\beta, \theta) \in \Upsilon \times \Theta$ there exist $\kappa \in \mathbb{R}$ and $\alpha>0$ such that $\kappa>-1 / \beta$, $\alpha \beta>2$ and (1.2) holds for $g_{\beta, \theta}$.
(2) The functions $u \mapsto\left|\partial_{\xi}^{i, k} \varphi_{\xi}(u)\right|,\left|\partial_{\xi}^{i} \varphi_{\xi}(u)\right|, i, k \in\{1, \ldots, d+1\}$, are locally dominated in $\mathcal{L}_{w}^{2}\left(\mathbb{R}_{+}^{m}\right)$. That is, there exists for all $\xi \in \Xi$ a neighbourhood $\Xi_{0} \ni \xi$ such that the supremum of these functions over $\xi \in \Xi_{0}$ are dominated by a function in $\mathcal{L}_{w}^{2}\left(\mathbb{R}_{+}^{m}\right)$.

Assumption (B)(1) is imposed to ensure that we may employ Theorem 1.1. While (B)(2) seems strict it is always satisfied in the one-dimensional case $m=1$ and we shall need it to ensure validity of the implicit function theorem in our setup.

We now demonstrate some examples, which satisfy Assumption (A) for $m \geq 2$ but not for $m=1$.

Example 2.2 (Stable Ornstein-Uhlenbeck process). Let $\left(X_{t}\right)_{t \in \mathbb{R}}$ denote the $\beta$-stable OrnsteinUhlenbeck process with parameter $\lambda>0$ and scale parameter $\sigma>0$. That is, $\left(X_{t}\right)_{t \in \mathbb{R}}$ is a stationary solution of the stochastic differential equation

$$
\mathrm{d} X_{t}=-\lambda X_{t} \mathrm{~d} t+\sigma \mathrm{d} L_{t}
$$

It has the representation (2.1) with kernel function $g_{\theta}(u)=\sigma \exp (-\lambda u) \mathbb{1}_{(0, \infty)}(u)$ and $\theta=(\sigma, \lambda) \in(0, \infty)^{2}$. It is clear that the one-dimensional characteristic function does not characterize the parameter $\theta$, hence Assumption (A)(2) is not satisfied for $m=1$. Consider therefore the case $m=2$. Here the characteristic function is uniquely determined by $\theta$ if the $\beta$-norms are. Indeed, using the binomial series one may deduce the following formula:

$$
\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}=\frac{\sigma^{\beta}}{\beta \lambda}\left[u_{2}^{\beta}(1-\exp (-\beta \lambda))+\left(u_{1}+u_{2} \exp (-\lambda)\right)^{\beta}\right], \quad u_{1}>u_{2} \geq 0
$$

It is then straightforward to check that these equations in $u_{1}>u_{2} \geq 0$ determine $\theta \in$ $(0, \infty)^{2}$ uniquely. Additionally, $(A)(4)$ can be checked in a manner similar to Example 2.4 below and we refer to Appendix A. 3 for the derivation of these statements.

There are a number of alternative estimation methods for a stable Ornstein-Uhlenbeck model. When the stability parameter $\beta$ is known, $\lambda$ can be estimated with convergence rate $(n / \log n)^{1 / \beta}$ as it has been shown in [24]. In the discrete-time setting of the $\operatorname{AR}(1)$ model with heavy-tailed i.i.d. noise, it is known that a Gaussian limit can be obtained, cf. [13], but this method again lacks joint estimation with the parameter $\beta$. In a similar framework the paper [1] investigates the asymptotic behaviour of the maximum likelihood estimator. In particular, their results imply that the parameters $\sigma$ and $\beta$ can be estimated with a $\sqrt{n}$-precision, while the drift parameter $\lambda$ has a faster convergence rate of $n^{1 / \beta}$.

Example 2.3 (Linear fractional stable motion). Let $\left(Y_{t}\right)_{t \in \mathbb{R}}$ be the linear fractional stable motion with self-similarity $H \in(0,1)$, stability index $\beta \in(0,2)$ and scale parameter $\sigma>0$. That is,

$$
Y_{t}=\int_{\mathbb{R}} \sigma\left[(t-s)_{+}^{H-1 / \beta}-(-s)_{+}^{H-1 / \beta}\right] \mathrm{d} L_{s}
$$

Consider the low frequency $k$ th order increment at rate $r(k, r \in \mathbb{N})$ defined as

$$
\Delta_{i, k}^{r} Y=\sum_{j=0}^{k}(-1)^{j}\binom{k}{j} Y_{i-r j}, \quad i \geq r k
$$

For example, for $k=2$ we have that

$$
\Delta_{i, 2}^{r} Y=Y_{i}-2 Y_{i-r}+Y_{i-2 r}, \quad i \geq 2 r
$$

Our example process $X$ will be the $k$ th increments at rate $r=1$, i.e. $X_{i}:=\Delta_{i, k}^{1} Y$. The corresponding kernel of the process $\left(X_{i}\right)$ for a general $k$ is given by

$$
g_{\beta, H, \sigma}(u):=\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}(u-j)_{+}^{H-1 / \beta}
$$

where $x_{+}=x \vee 0$ is the positive part and $x_{+}^{a}:=0$ for all $x \leq 0$. We note the asymptotic behaviour

$$
\frac{g_{\beta, H, \sigma}(u)}{K u^{H-1 / \beta-k}} \longrightarrow 1 \quad \text { as } u \rightarrow \infty
$$

for some constant $K>0$ depending on $\alpha, H$ and $k$. Hence the kernel $g_{\beta, H, \sigma}$ for $X$ always satisfies the assumption (1.2) with $\kappa:=H-1 / \beta$ and $\alpha:=k+1 / \beta-H$, since $H>0$ and $k \geq 1>H$.

In this case Assumption (B) for $g_{\beta, H, \sigma}$ can simply be translated into an assumption on the parameter space $\Upsilon \times \Theta$, e.g.

$$
\Upsilon \times \Theta=\left\{(\beta, H, \sigma) \mid 0<H<k-1 / \beta, \frac{1}{C}<\sigma<C\right\}
$$

for some arbitrary but finite constant $C>0$. It is well-known that $X$ has a version with continuous paths if and only if $H-1 / \beta>0$, so if we want to do inference in the continuous case we have the two parameter inequalities:

$$
\begin{equation*}
0<H-1 / \beta \quad \text { and } \quad H<k-1 / \beta \tag{2.3}
\end{equation*}
$$

We note that these inequalities never hold for $k=1$, but they are always satisfied for $k \geq 2$. Indeed, the first inequality implies that $H \in(1 / 2,1)$ and $\beta \in(1,2)$.

Now, the stability index $\beta$ and the scale parameter $\sigma>0$ are identifiable from the one-dimensional characteristic function, since these parameters are unique in a stable distribution. The $H$-self-similarity of the linear fractional stable motion $Y$ implies that

$$
\frac{\mathbb{E}\left[\left|\Delta_{2 k, k}^{2} Y\right|^{p}\right]}{\mathbb{E}\left[\left|\Delta_{k, k} Y\right|^{p}\right]}=2^{p H} \quad \text { for } p \in(-1,0)
$$

For $k=2$ the term $\Delta_{4,2}^{2} Y$ is a linear combination of $X_{2}=\Delta_{2,2}^{1} Y, X_{3}=\Delta_{3,2}^{1} Y$ and $X_{4}=\Delta_{4,2}^{1} Y$. Hence $H$ is identifiable from the characteristic function of the three-dimensional distribution ( $X_{2}, X_{3}, X_{4}$ ), in other words, $m=3$ in the case $k=2$.

Example 2.4 (OU-type model with a periodic component). The next example we consider is a periodic extension of the stable Ornstein-Uhlenbeck process from Example 2.2. Let $\theta=$ $\left(\theta_{1}, \theta_{2}\right) \in(0, \infty)^{2}$ and consider the kernel function:

$$
g_{\theta}(u)=\exp \left(-\theta_{1} u-\theta_{2} f(u)\right) \mathbb{1}_{(0, \infty)}(u), \quad u \in \mathbb{R},
$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is a bounded measurable function which is either non-negative or nonpositive and has period 1, i.e. $f(x+1)=f(x)$ for all $x$. If $f$ does not vanish except on Lebesgue null set, then $\theta \mapsto \varphi_{\beta, \theta}^{m}$ for $m=2$ is injective. If, in addition, $f$ is negative then Assumption (B)(2) is satisfied except possibly at $\beta=1$, and condition (A)(4) holds. We refer to Appendix A. 3 for the proof of these statements.

Example 2.5 (Modulated $O U$ ). Consider the process $X$ defined at (2.1) with kernel given by

$$
\begin{equation*}
g_{\theta}(s)=\theta_{1} s \exp \left(-\theta_{2} s\right) \mathbb{1}_{(0, \infty)}(s), \quad s \in \mathbb{R} \tag{2.4}
\end{equation*}
$$

Under the assumptions on the parameters $\theta \in(0, \infty)^{2}$ and $\beta \in(1,2)$ it is possible to prove that $\theta$ is not identifiable from $m=1$ while it is in the case $m=2$. Furthermore, condition (A)(4) is satisfied. We refer to Appendix A. 4 for the full exposition of these claims.

Example 2.6 (CARMA processes). Consider integers $p>q$. The $\operatorname{CARMA}(p, q)$ process $\left(Y_{t}\right)_{t \in \mathbb{R}}$ with parameters $a_{1}, \ldots, a_{p}, b_{0}, \ldots, b_{q-1} \in \mathbb{R}$ driven by $L$ is the solution to the stochastic differential equation

$$
\begin{equation*}
X_{t}=b^{\top} Y_{t} \quad \text { with } \quad \mathrm{d} Y_{t}-A Y_{t} \mathrm{~d} t=e \mathrm{~d} L_{t} \tag{2.5}
\end{equation*}
$$

where $e$ and $b$ are the $p$-dimensional column vectors given by

$$
e=(0, \ldots, 0,1)^{\top} \quad \text { and } \quad b=\left(b_{0}, \ldots, b_{p-1}\right)^{\top}
$$

where $b_{q}=1$ and $b_{i}=0$ for all $q<i<p$ and $A$ is the $p \times p$ matrix given by

$$
A=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 1 \\
-a_{p} & -a_{p-1} & a_{p-2} & \cdots & -a_{1}
\end{array}\right)
$$

CARMA $(p, q)$ processes fit within the framework of (2.1) since if the eigenvalues of $A$ have strictly negative real part, then a unique stationary solution of (2.5) exists and is given by

$$
X_{t}=\int_{\mathbb{R}} b^{\top} \mathrm{e}^{A(t-s)} e \mathbb{1}_{[0, \infty)}(t-s) \mathrm{d} L_{s}, \quad t \in \mathbb{R}
$$

see [7, Proposition 1]. In this example we discuss a specific three-dimensional sub-class of CARMA $(2,1)$ processes, which corresponds to the choice $\lambda:=-\sqrt{a_{2}}$ and $a_{1}=2 \sqrt{a_{2}}=-2 \lambda$. The parameter of interest becomes $\xi=\left(\beta, b_{0}, \lambda\right)$ and we further assume that $\beta \in(1,2)$ and $\theta:=b_{0}+\lambda>0$. In this setting the matrix $A$ is given by

$$
A=\left(\begin{array}{cc}
0 & 1 \\
-\lambda^{2} & 2 \lambda
\end{array}\right)
$$

and $\lambda<0$ is the only eigenvalue of $A$. We thus obtain the Jordan normal form

$$
A=S\left(\begin{array}{cc}
\lambda & 1 \\
0 & \lambda
\end{array}\right) S^{-1}, \quad S=\left(\begin{array}{cc}
1 & 0 \\
\lambda & 1
\end{array}\right), \quad S^{-1}=\left(\begin{array}{cc}
1 & 0 \\
-\lambda & 1
\end{array}\right)
$$

Using this representation elementary matrix algebra yields the identity

$$
g(s)=b^{\top} \exp (s A) e \mathbb{1}_{[0, \infty)}(s)=(1+\theta s) \exp (\lambda s) \mathbb{1}_{[0, \infty)}(s)
$$

In Appendix A. 5 we show that, under the condition $\beta \in(1,2)$, the parameters of the model are identifiable in the case $m=2$ and condition $(\mathrm{A})(4)$ holds.

### 2.2 Parametric estimation via minimal contrast approach

We note first that the discrete time process $\left(X_{t}\right)_{t \in \mathbb{Z}}$ is ergodic according to [8], and so is the sequence

$$
Y_{i}=f\left(X_{i+1}, \ldots, X_{i+m}\right), \quad i \in \mathbb{Z}
$$

for any measurable function $f$. Hence, we obtain by Birkhoff's ergodic theorem the strong consistency (of the real part) of the joint empirical characteristic function:

$$
\begin{equation*}
\varphi_{n}\left(u_{1}, \ldots, u_{m}\right)=\frac{1}{n} \sum_{i=0}^{n-m} \cos \left(\sum_{k=1}^{m} u_{k} X_{i+k}\right) \xrightarrow{\text { a.s. }} \mathbb{E}\left[\cos \left(\sum_{k=0}^{m-1} u_{k} X_{1+k}\right)\right]=\varphi_{\xi}^{m}\left(u_{1}, \ldots, u_{m}\right), \tag{2.6}
\end{equation*}
$$

where $\xi=(\beta, \theta) \in \Xi$ denotes the unknown parameter of the model. To reduce cumbersome notation we drop the dependence on $m$ in the characteristic function and simply write $\varphi_{\xi}$ from now on. For a weight function $w$ introduced in the previous section, we denote by $F: \mathcal{L}_{w}^{2}\left(\mathbb{R}_{+}^{m}\right) \times \Xi \rightarrow \mathbb{R}$ the map

$$
F(\psi, \xi)=\left\|\psi-\varphi_{\xi}\right\|_{w, 2}^{2}
$$

The minimal contrast estimator $\xi_{n}$ of $\xi$ is then defined as

$$
\begin{equation*}
\xi_{n} \in \underset{\xi \in \Xi}{\operatorname{argmin}} F\left(\varphi_{n}, \xi\right)=\underset{\xi \in \Xi}{\operatorname{argmin}} \int_{\mathbb{R}_{+}^{m}}\left(\varphi_{n}(u)-\varphi_{\xi}(u)\right)^{2} w(u) \mathrm{d} u, \tag{2.7}
\end{equation*}
$$

and we remark that $\xi_{n}$ can be chosen universally measurable by [23, Theorem 2.17(d)]. To obtain the asymptotic normality of the minimal contrast estimator $\xi_{n}$ we will show a central limit theorem for the statistic $\sqrt{n}\left(\varphi_{n}\left(u_{1}, \ldots, u_{m}\right)-\varphi_{\xi}\left(u_{1}, \ldots, u_{m}\right)\right)$ using Theorem 1.1 and then apply a functional version of the implicit function theorem. For this purpose we introduce a centred Gaussian field $\left(G_{u}\right)_{u \in \mathbb{R}_{+}^{m}}$ whose covariance kernel is defined as

$$
\begin{equation*}
\operatorname{Cov}\left(G_{u}, G_{v}\right)=\sum_{l \in \mathbb{Z}} \operatorname{Cov}\left(\cos \left(\left\langle u, Z_{0}\right\rangle_{\mathbb{R}^{m}}\right), \cos \left(\left\langle v, Z_{l}\right\rangle_{\mathbb{R}^{m}}\right)\right), \tag{2.8}
\end{equation*}
$$

where $Z_{k}=\left(X_{1+k}, \ldots, X_{m+k}\right)$. The main theoretical result of the paper is the strong consistency and asymptotic normality of the minimal contrast estimator $\xi_{n}$.

Theorem 2.7. Let $\left(\xi_{n}\right)$ be the minimal contrast estimator at (2.1) associated with the true parameter $\xi_{0}=\left(\beta_{0}, \theta_{0}\right)$. Suppose that Assumptions $(A)$ and ( $B$ ) hold for the underlying family of kernels $\left(g_{\xi}\right)_{\xi \in \Xi}$. Assume that the weight function $w$ is continuous and $\int_{\mathbb{R}_{+}^{m}}\|u\|_{\mathbb{R}^{m}}^{2} w(u) \mathrm{d} u<\infty$.
(i) $\xi_{n} \rightarrow \xi_{0}$ almost surely as $n \rightarrow \infty$.
(ii) The convergence as $n \rightarrow \infty$

$$
\sqrt{n}\left(\xi_{n}-\xi_{0}\right) \xrightarrow{\mathcal{L}}\left(\nabla_{\xi}^{2} F\left(\varphi_{\xi_{0}}, \xi_{0}\right)\right)^{-1}\left(\left\langle\partial_{\xi}^{i} \varphi_{\xi_{0}}, G\right\rangle_{w}\right)_{i=1, \ldots, d+1}
$$

holds, where $G=\left(G_{u}\right)_{u \in \mathbb{R}_{+}^{m}}$ is a continuous zero-mean Gaussian random field with covariance kernel defined by (2.8). In particular, the above limit is a normally distributed $(d+1)$-dimensional random vector.

Direct computation shows that the Hessian $\nabla_{\xi}^{2} F\left(\varphi_{\xi_{0}}, \xi_{0}\right)$ is a Gramian matrix with respect to the first order derivatives $\left(\partial_{\xi}^{i} \varphi_{\xi}\right)_{i=1, \ldots, d+1}$ and the inner product $\langle\cdot, \cdot\rangle_{w}$ and therefore Assumption (A)(4) ensures that the Hessian is invertible. In principle, the normal limit in Theorem 2.7 is explicit up to the knowledge of the parameter $\xi_{0}$, but due to the complex covariance kernel of the process $G$ it is hard to apply the central limit theorem to obtain
confidence regions. Instead one may use a parametric bootstrap approach as it has been suggested in [14, Section 4.2].

We remark that the convergence rate is $\sqrt{n}$ for all parameters. Due to the non-Markovian structure of the general model (2.1) it is a non-trivial task to assess the optimality of this rate. As we have discussed in Example 2.2 the rate $\sqrt{n}$ can be suboptimal in the particular case of the drift parameter in an Ornstein-Uhlenbeck model.

Remark 2.8 (Extension to general Lévy drivers). If we drop the requirement for estimation of $\beta$ we can consider a larger class of Lévy drivers. Indeed, according to [3] the statement of Theorem 1.1 still holds for a symmetric Lévy process $L$, which admits a Lévy density $\nu$ such that

$$
\nu(x) \leq C|x|^{-1-\beta} \quad \text { for all } x \neq 0
$$

In this case the characteristic function takes on a more complicated form. Indeed, by [20, Theorem 2.7] it holds that

$$
\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\left\langle u,\left(X_{1}, \ldots, X_{m}\right)\right\rangle_{\mathbb{R}^{m}}}\right]=\exp \left(\int_{\mathbb{R}} \int_{\mathbb{R}}\left[\cos \left(\left\langle u, x\left(g_{\xi}(z+i)\right)_{i=0, \ldots, m-1}\right\rangle_{\mathbb{R}^{m}}\right)-1\right] \nu(\mathrm{d} x) \mathrm{d} z\right)
$$

In principle, the asymptotic theory of Theorem 2.7 can be extended to this more general setting. However, the proof of the asymptotic normality relies on the existence of a continuous modification of the random field $\left(G_{u}\right)_{u \in \mathbb{R}_{+}^{m}}$ and the behaviour of $\mathbb{E}\left[G_{u}^{2}\right]$ in $u \in \mathbb{R}_{+}^{m}$ (cf. Appendix A.1.1), which requires a different treatment compared to the $\beta$-stable case.

## 3 A simulation study

In this section we will demonstrate the finite sample performance of our estimator for three examples, which are supposed to highlight different aspects of the minimal contrast approach. First, we will consider the linear fractional stable motion (cf. Example 2.3) and use $m=3$ to estimate the three-dimensional parameter of the model. The second model is the Ornstein-Uhlenbeck process considered in Example 2.2. We will examine the performance for the full model and also for a known scale parameter $\sigma$ to assess the improvement of the estimation procedure. In the latter setting both $m=1$ and $m=2$ are used to estimate the drift $\lambda$ and the stability index $\beta$, and the aim of the numerical simulation is to test how the choice of higher index $m$ affects the performance of the estimator. The third example is the generalized modulated OU-process, which has not been shown to satisfy the main assumptions of the paper. We will use $m=2$ to estimate the three-dimensional parametric model and test how our method works in this framework.

Since the weight function $w$ depends on $m$ implicitly via its domain we need a function, which is reasonably compatible between different dimensions and we consider therefore throughout this study the $m$-dimensional Gaussian density with zero mean and a scaled unit covariance matrix $\nu^{2} I_{m}$ :

$$
\begin{equation*}
w_{\nu}(u)=\left(2 \pi \nu^{2}\right)^{-m / 2} \exp \left(-\frac{\|u\|_{\mathbb{R}^{m}}^{2}}{2 \nu^{2}}\right), \quad u \in \mathbb{R}^{m}, \quad \nu>0 \tag{3.1}
\end{equation*}
$$

The choice of $\nu$ varies between the three example processes and it is a subject for future research to automatically determine an optimal weight. For the computation of the weighted
integral in (2.7) we use Gauss-Laguerre quadrature which is a weighted sum of function values and the number of weights will also vary depending on the process.

We note additionally that the minimization involved in computing the minimal contrast estimator at (2.7) has to be done numerically and for this we use the method of [17], which requires picking a starting point which naturally will depend on the example kernel at hand. Lastly, we remark that the $\beta$-norm of the kernel function is generally not known explicitly, hence the theoretical characteristic function is approximated as well.

All tables in this section are based on at least 200 Monte Carlo repetitions.

### 3.1 Linear fractional stable motion

Recall from the discussion in Example 2.3 that it is prudent to take higher order increments, and we fix throughout $k=2$. Moreover, to properly identify the parameters we consider the characteristic function of the three-dimensional joint distribution, hence $m=3$. Next we consider throughout the weight function at (3.1) with standard deviation $\nu=10$ and the weighted integral is approximated with $12^{3}=1728$ number of weights. The starting point for the minimization algorithm is $(\beta, H, \sigma)=(1.5,0.5,2)$.

The estimator is tested in the continuous case, so only parameter combinations resulting in the equality $H-1 / \beta>0$ are considered. Table 1 reports the bias and standard deviation in the case of $n=1000$ for different parameter combinations, while Table 2 explores the case $n=10000$. We observe a rather good performance of all estimators with superior results in the setting $n=10000$ as expected from our theoretical statements. We note that the estimator of the scale parameter $\sigma$ performs the best, which is in line with earlier findings of [16].

Table 1: Absolute value of bias ( $|\operatorname{Bias}|$ ) and standard deviation (Std) for $n=1000$ and $\sigma=0.3$ for the linear fractional stable motion.

|  |  | $\mid$ Bias $\mid$ |  |  |  |  | Std |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $H$ | $\beta$ | $\widehat{\beta}_{n}$ | $\widehat{H}_{n}$ | $\widehat{\sigma}_{n}$ |  | $\widehat{\beta}_{n}$ | $\widehat{H}_{n}$ | $\widehat{\sigma}_{n}$ |  |
| 0.6 | 1.8 | 0.0176 | 0.0478 | 0.0362 |  | 0.1950 | 0.2730 | 0.0705 |  |
| 0.7 | 1.6 | 0.0705 | 0.1710 | 0.0834 |  | 0.2409 | 0.3581 | 0.0947 |  |
|  | 1.8 | 0.0106 | 0.0041 | 0.0120 |  | 0.1766 | 0.2094 | 0.0429 |  |
| 0.8 | 1.4 | 0.0862 | 0.2444 | 0.0862 |  | 0.2348 | 0.3457 | 0.1044 |  |
|  | 1.6 | 0.0250 | 0.0597 | 0.0270 |  | 0.1783 | 0.2466 | 0.0541 |  |
|  | 1.8 | 0.0120 | 0.0060 | 0.0044 |  | 0.1452 | 0.1578 | 0.0287 |  |

## 3 A simulation study

Table 2: Absolute value of bias (|Bias|) and standard deviation (Std) for $n=10000$ and $\sigma=0.3$ for the linear fractional stable motion.

|  |  | $\mid$ Bias $\mid$ |  |  |  |  | Std |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $H$ | $\beta$ | $\widehat{\beta}_{n}$ | $\widehat{H}_{n}$ | $\widehat{\sigma}_{n}$ |  | $\widehat{\beta}_{n}$ | $\widehat{H}_{n}$ | $\widehat{\sigma}_{n}$ |  |
| 0.6 | 1.8 | 0.0133 | 0.0456 | 0.0254 |  | 0.1272 | 0.2007 | 0.0532 |  |
| 0.7 | 1.6 | 0.0238 | 0.0818 | 0.0331 |  | 0.1005 | 0.2147 | 0.0685 |  |
|  | 1.8 | 0.0060 | 0.0147 | 0.0066 |  | 0.0869 | 0.1153 | 0.0173 |  |
| 0.8 | 1.4 | 0.0347 | 0.1536 | 0.0504 |  | 0.1095 | 0.2546 | 0.0865 |  |
|  | 1.6 | 0.0078 | 0.0053 | 0.0008 |  | 0.0665 | 0.0843 | 0.0085 |  |
|  | 1.8 | 0.0032 | 0.0020 | 0.0009 |  | 0.0597 | 0.0732 | 0.0067 |  |

### 3.2 Ornstein-Uhlenbeck

In this subsection we consider the Ornstein-Uhlenbeck kernel from Example 2.2. We start with a two-parameter submodel, where $\sigma=1$ is fixed. In this case Assumption (A) is satisfied for both $m=2$ and $m=1$, and we will compare the performance for each of these dimensions. Akin to Section 3.3 we pick $20^{m}, m=1,2$, number of weights in the integral approximation with weight function chosen as in (3.1) with $\nu=1$. The starting point for the minimization algorithm is throughout $(\beta, \lambda)=(1.5,0.5)$.

Tables 3 and 4 demonstrate the simulation results for $m=1$ and $m=2$, respectively. We observe a rather convincing performance for both estimators in all settings, but the choice $m=1$ clearly outperforms the setting $m=2$. We conjecture that it has a theoretical background, i.e. the asymptotic variances in Theorem 2.7(ii) are smaller for $m=1$, and a numerical background. Indeed, the minimization algorithm has a worse performance for higher values of $m$. For this reason it is advisable to use the minimal $m$, which identifies the parameters of the model.

We now consider the full Ornstein-Uhlenbeck model from Example 2.2 with parameters $\beta \in(0,2), \lambda>0$ and a non-fixed scale $\sigma>0$. For comparison with the case of fixed scale we consider as starting point $(\beta, \lambda, \sigma)=(1.5,0.5,1.1)$ for the minimization algorithm. To avoid innumerable parameter combinations we consider only $\beta \in\{1.4,1.6\}, \lambda \in\{0.25,0.75\}$ and $\sigma \in\{0.9,1\}$. Lastly, we have $\nu=1$ as in the previous simulation.

Comparing Table 5 to Table 4 we see that a fixed, known $\sigma$ significantly increases the performance of the estimator especially when the starting point for, e.g. $\sigma$ is further away. Nevertheless, the estimation results in Table 5 are still quite reliable for most parameter settings.

Table 3: Absolute value of bias (|Bias|) and standard deviation (Std) for $m=1$ and $n \in\left\{10^{3}, 10^{4}\right\}$.

| $n=1000$ |  | $\mid$ Bias $\mid$ |  |  | Std |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\beta$ | $\lambda$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ |  | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ |
| 1.2 | 0.25 | 0.0185 | 0.0030 |  | 0.1072 | 0.0544 |
|  | 0.75 | 0.0144 | 0.0061 |  | 0.0627 | 0.0683 |
|  | 1 | 0.0107 | 0.0018 |  | 0.0573 | 0.0755 |
|  | 1.25 | 0.0084 | 0.0062 |  | 0.0531 | 0.0856 |
|  | 1.5 | 0.0135 | 0.0058 |  | 0.0561 | 0.0901 |
|  | 2 | 0.0044 | 0.0028 |  | 0.0543 | 0.1282 |
|  | 2.5 | 0.0122 | 0.0150 |  | 0.0583 | 0.1530 |
| 1.4 | 0.25 | 0.0097 | 0.0079 |  | 0.1214 | 0.0530 |
|  | 0.75 | 0.0047 | 0.0029 |  | 0.0661 | 0.0669 |
|  | 1 | 0.0036 | 0.0093 |  | 0.0593 | 0.0646 |
|  | 1.25 | 0.0042 | 0.0018 |  | 0.0572 | 0.0757 |
|  | 1.5 | 0.0138 | 0.0023 |  | 0.0550 | 0.0826 |
|  | 2 | 0.0091 | 0.0039 |  | 0.0595 | 0.1072 |
|  | 2.5 | 0.0060 | 0.0072 |  | 0.0608 | 0.1507 |
| 1.6 | 0.25 | 0.0099 | 0.0012 |  | 0.1143 | 0.0513 |
|  | 0.75 | 0.0071 | 0.0066 |  | 0.0604 | 0.0604 |
|  | 1 | 0.0076 | 0.0016 |  | 0.0590 | 0.0669 |
|  | 1.25 | 0.0116 | 0.0042 |  | 0.0533 | 0.0759 |
|  | 1.5 | 0.0020 | 0.0039 |  | 0.0563 | 0.0781 |
|  | 2 | 0.0101 | 0.0074 |  | 0.0540 | 0.1021 |
|  | 2.5 | 0.0144 | 0.0061 |  | 0.0567 | 0.1283 |
| 1.8 | 0.25 | 0.0106 | 0.0004 |  | 0.1013 | 0.0417 |
|  | 0.75 | 0.0111 | 0.0007 |  | 0.0586 | 0.0597 |
|  | 1 | 0.0021 | 0.0007 |  | 0.0529 | 0.0649 |
|  | 1.25 | 0.0088 | 0.0043 |  | 0.0453 | 0.0764 |
|  | 1.5 | 0.0092 | 0.0136 |  | 0.0494 | 0.0825 |
|  | 2 | 0.0084 | 0.0025 |  | 0.0481 | 0.1015 |
|  | 2.5 | 0.0144 | 0.0045 |  | 0.0446 | 0.1273 |


| $n=10000$ |  | \|Bias| |  | Std |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ | $\lambda$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ |
| 1.2 | 0.25 | 0.0016 | 0.0004 | 0.0321 | 0.0174 |
|  | 0.75 | 0.0039 | 0.0028 | 0.0187 | 0.0199 |
|  | 1 | 0.0008 | 0.0000 | 0.0197 | 0.0244 |
|  | 1.25 | 0.0000 | 0.0028 | 0.0189 | 0.0265 |
|  | 1.5 | 0.0004 | 0.0033 | 0.0164 | 0.0323 |
|  | 2 | 0.0003 | 0.0030 | 0.0161 | 0.0365 |
|  | 2.5 | 0.0016 | 0.0090 | 0.0179 | 0.0466 |
| 1.4 | 0.25 | 0.0035 | 0.0014 | 0.0377 | 0.0172 |
|  | 0.75 | 0.0017 | 0.0010 | 0.0202 | 0.0194 |
|  | 1 | 0.0017 | 0.0022 | 0.0196 | 0.0241 |
|  | 1.25 | 0.0016 | 0.0026 | 0.0174 | 0.0274 |
|  | 1.5 | 0.0000 | 0.0092 | 0.0177 | 0.0281 |
|  | 2 | 0.0016 | 0.0069 | 0.0166 | 0.0362 |
|  | 2.5 | 0.0014 | 0.0155 | 0.0194 | 0.0404 |
| 1.6 | 0.25 | 0.0079 | 0.0019 | 0.0439 | 0.0169 |
|  | 0.75 | 0.0022 | 0.0014 | 0.0184 | 0.0170 |
|  | 1 | 0.0008 | 0.0020 | 0.0182 | 0.0212 |
|  | 1.25 | 0.0015 | 0.0032 | 0.0183 | 0.0239 |
|  | 1.5 | 0.0003 | 0.0056 | 0.0169 | 0.0271 |
|  | 2 | 0.0005 | 0.0133 | 0.0166 | 0.0325 |
|  | 2.5 | 0.0020 | 0.0192 | 0.0178 | 0.0401 |
| 1.8 | 0.25 | 0.0015 | 0.0011 | 0.0392 | 0.0152 |
|  | 0.75 | 0.0013 | 0.0010 | 0.0187 | 0.0176 |
|  | 1 | 0.0019 | 0.0051 | 0.0162 | 0.0190 |
|  | 1.25 | 0.0020 | 0.0067 | 0.0159 | 0.0232 |
|  | 1.5 | 0.0012 | 0.0113 | 0.0151 | 0.0263 |
|  | 2 | 0.0032 | 0.0159 | 0.0146 | 0.0298 |
|  | 2.5 | 0.0000 | 0.0259 | 0.0143 | 0.0411 |

Table 4: Absolute value of bias (|Bias|) and standard deviation (Std) for $m=2$ and $n \in\left\{10^{3}, 10^{4}\right\}$.

| $n=1000$ |  | \|Bias ${ }^{\text {\| }}$ |  | Std |  | $n=10000$ |  | \|Bias| |  | Std |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\beta$ | $\lambda$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\beta$ | $\lambda$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ |
| 1.2 | 0.25 | 0.3988 | 0.1113 | 0.1044 | 0.0623 | 1.2 | 0.25 | 0.3922 | 0.1142 | 0.0464 | 0.0098 |
|  | 0.75 | 0.0666 | 0.0363 | 0.2178 | 0.1932 |  | 0.75 | 0.0005 | 0.0013 | 0.0391 | 0.0480 |
|  | 1 | 0.0150 | 0.0081 | 0.0928 | 0.1339 |  | 1 | 0.0019 | 0.0012 | 0.0260 | 0.0421 |
|  | 1.25 | 0.0125 | 0.0104 | 0.0681 | 0.1226 |  | 1.25 | 0.0007 | 0.0024 | 0.0220 | 0.0377 |
|  | 1.5 | 0.0054 | 0.0063 | 0.0626 | 0.1181 |  | 1.5 | 0.0003 | 0.0030 | 0.0218 | 0.0428 |
|  | 2 | 0.0073 | 0.0090 | 0.0646 | 0.1472 |  | 2 | 0.0005 | 0.0050 | 0.0195 | 0.0454 |
|  | 2.5 | 0.0613 | 0.1477 | 0.0636 | 0.0627 |  | 2.5 | 0.0387 | 0.1186 | 0.0204 | 0.0024 |
| 1.4 | 0.25 | 0.2028 | 0.0531 | 0.1407 | 0.1061 | 1.4 | 0.25 | 0.1916 | 0.0599 | 0.0731 | 0.0402 |
|  | 0.75 | 0.0484 | 0.0204 | 0.1793 | 0.1621 |  | 0.75 | 0.0024 | 0.0015 | 0.0439 | 0.0500 |
|  | 1 | 0.0063 | 0.0063 | 0.0848 | 0.1165 |  | 1 | 0.0019 | 0.0024 | 0.0257 | 0.0363 |
|  | 1.25 | 0.0124 | 0.0067 | 0.0714 | 0.1096 |  | 1.25 | 0.0009 | 0.0002 | 0.0235 | 0.0361 |
|  | 1.5 | 0.0025 | 0.0067 | 0.0721 | 0.1204 |  | 1.5 | 0.0012 | 0.0004 | 0.0211 | 0.0381 |
|  | 2 | 0.0080 | 0.0203 | 0.0572 | 0.1269 |  | 2 | 0.0027 | 0.0020 | 0.0227 | 0.0397 |
|  | 2.5 | 0.0593 | 0.1395 | 0.0734 | 0.0482 |  | 2.5 | 0.0505 | 0.1184 | 0.0243 | 0.0006 |
| 1.6 | 0.25 | 0.1120 | 0.1078 | 0.3009 | 0.2139 | 1.6 | 0.25 | 0.0051 | 0.0138 | 0.1794 | 0.1028 |
|  | 0.75 | 0.0481 | 0.0210 | 0.1669 | 0.1602 |  | 0.75 | 0.0084 | 0.0061 | 0.0451 | 0.0479 |
|  | 1 | 0.0165 | 0.0159 | 0.0909 | 0.1164 |  | 1 | 0.0002 | 0.0023 | 0.0253 | 0.0324 |
|  | 1.25 | 0.0072 | 0.0017 | 0.0666 | 0.1039 |  | 1.25 | 0.0003 | 0.0047 | 0.0206 | 0.0305 |
|  | 1.5 | 0.0012 | 0.0078 | 0.0667 | 0.0990 |  | 1.5 | 0.0003 | 0.0040 | 0.0200 | 0.0334 |
|  | 2 | 0.0037 | 0.0133 | 0.0688 | 0.1171 |  | 2 | 0.0015 | 0.0060 | 0.0200 | 0.0389 |
|  | 2.5 | 0.0873 | 0.1364 | 0.0850 | 0.0431 |  | 2.5 | 0.0604 | 0.1185 | 0.0287 | 0.0015 |
| 1.8 | 0.25 | 0.2478 | 0.1751 | 0.3584 | 0.2232 | 1.8 | 0.25 | 0.2109 | 0.1160 | 0.2539 | 0.1351 |
|  | 0.75 | 0.0194 | 0.0015 | 0.1182 | 0.1253 |  | 0.75 | 0.0016 | 0.0023 | 0.0389 | 0.0395 |
|  | 1 | 0.0112 | 0.0007 | 0.0755 | 0.1010 |  | 1 | 0.0001 | 0.0025 | 0.0243 | 0.0316 |
|  | 1.25 | 0.0098 | 0.0083 | 0.0587 | 0.0881 |  | 1.25 | 0.0004 | 0.0036 | 0.0178 | 0.0266 |
|  | 1.5 | 0.0150 | 0.0020 | 0.0540 | 0.0973 |  | 1.5 | 0.0001 | 0.0042 | 0.0173 | 0.0280 |
|  | 2 | 0.0187 | 0.0121 | 0.0632 | 0.1106 |  | 2 | 0.0012 | 0.0092 | 0.0181 | 0.0371 |
|  | 2.5 | 0.0948 | 0.1346 | 0.0802 | 0.0502 |  | 2.5 | 0.0801 | 0.1184 | 0.0343 | 0.0010 |

Table 5: Absolute value of bias (|Bias|) and standard deviation (Std) for $m=2$ and $n=10000$.

|  |  |  | $\mid$ Bias $\mid$ |  |  |  |  | Std |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: |
|  | $\lambda$ | $\sigma$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\sigma}_{n}$ |  | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\sigma}_{n}$ |  |
| 1.4 | 0.25 | 0.9 | 0.0095 | 0.1946 | 0.4041 |  | 0.1971 | 0.2708 | 0.3998 |  |
| 1.4 | 0.25 | 1 | 0.0606 | 0.0736 | 0.2801 |  | 0.0101 | 0.0157 | 0.0079 |  |
| 1.4 | 0.75 | 0.9 | 0.0013 | 0.0512 | 0.0442 |  | 0.0305 | 0.0506 | 0.0411 |  |
| 1.4 | 0.75 | 1 | 0.0043 | 0.0410 | 0.0444 |  | 0.0451 | 0.0637 | 0.0444 |  |
| 1.6 | 0.25 | 0.9 | 0.0527 | 0.2136 | 0.3167 |  | 0.2701 | 0.1852 | 0.1998 |  |
| 1.6 | 0.25 | 1 | 0.1569 | 0.2084 | 0.2401 |  | 0.2380 | 0.2242 | 0.2904 |  |
| 1.6 | 0.75 | 0.9 | 0.0011 | 0.0027 | 0.0045 |  | 0.0300 | 0.0384 | 0.0379 |  |
| 1.6 | 0.75 | 1 | 0.0026 | 0.0480 | 0.0415 |  | 0.0428 | 0.0309 | 0.0205 |  |

3 A simulation study

### 3.3 Generalized modulated OU process

The generalized modulated OU process is defined via equation (2.1) with kernel function

$$
g_{\theta}(s)=s^{\sigma} \exp (-\lambda s) \mathbb{1}_{(0, \infty)}(s), \quad s \in \mathbb{R},
$$

where $\theta=(\sigma, \lambda) \in(0, \infty)^{2}$. This class of kernels has not been shown to satisfy the main assumption of the paper, but it is easily seen that $m=1$ is not enough to identify the parameters in $\theta$. We take $m=2$ and set the number of weights to 20 , hence the weighted integral approximation is based on $20^{2}=400$ nodes. Moreover, the weight function is as in (3.1) with $\nu=0.1$. Lastly, we pick as starting point for the minimization algorithm $(\beta, \lambda, \sigma)=(1.5,1,1)$.

Tables 6 and 7 report the finite sample performance of the estimators for $n=10000$, and $\sigma=0.5$ and $\sigma=2$, respectively. We observe a good performance of the estimator $\widehat{\beta}_{n}$ and a very unsatisfactory performance of the estimator $\widehat{\sigma}_{n}$. We conjecture that the reason for the suboptimal performance lies in the choice of the weight function $w$, which may have opposite effects on different parameters of the model, as well as in the minimization algorithm, since it has a tendency to get stuck in local minima.

Table 6: Absolute value of bias ( $|\operatorname{Bias}|$ ) and standard deviation for $n=10000$ and $\sigma=0.5$ for the generalized modulated OU kernel.

|  |  | $\mid$ Bias $\mid$ |  |  |  |  | Std |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: | :---: | :---: |
| $\beta$ | $\lambda$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\sigma}_{n}$ |  | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\sigma}_{n}$ |  |
| 1.8 | 0.5 | 0.0111 | 0.1585 | 0.5982 |  | 0.0460 | 0.0444 | 0.1353 |  |
|  | 0.75 | 0.0196 | 0.0925 | 0.5620 |  | 0.0542 | 0.0494 | 0.1718 |  |
|  | 1.25 | 0.0147 | 0.0064 | 0.0671 |  | 0.0813 | 0.1152 | 0.0946 |  |
|  | 1.5 | 0.0029 | 0.0361 | 0.0969 |  | 0.0856 | 0.1006 | 0.1728 |  |
| 1.2 | 0.5 | 0.0062 | 0.1881 | 0.6967 |  | 0.0349 | 0.0732 | 0.2415 |  |
|  | 0.75 | 0.0044 | 0.1787 | 0.8088 |  | 0.0440 | 0.0443 | 0.0486 |  |
|  | 1.25 | 0.0103 | 0.0089 | 0.6124 |  | 0.0468 | 0.0594 | 0.1307 |  |
|  | 1.5 | 0.0110 | 0.0886 | 0.5869 |  | 0.0519 | 0.0951 | 0.2115 |  |

Table 7: Absolute value of bias (|Bias $\mid$ ) and standard deviation (Std) for $n=10000$ and $\sigma=2$ for the generalized modulated OU kernel.

|  |  | $\mid$ Bias $\mid$ |  |  |  |  | Std |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: | :---: | :---: |
| $\beta$ | $\lambda$ | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\sigma}_{n}$ |  | $\widehat{\beta}_{n}$ | $\widehat{\lambda}_{n}$ | $\widehat{\sigma}_{n}$ |  |
| 1.8 | 0.5 | 0.0076 | 0.0314 | 0.1458 |  | 0.1730 | 0.2052 | 0.7289 |  |
|  | 0.75 | 0.0028 | 0.2089 | 0.6515 |  | 0.0309 | 0.0241 | 0.0729 |  |
|  | 1.25 | 0.0314 | 0.2521 | 1.3273 |  | 0.0727 | 0.0641 | 0.1244 |  |
|  | 1.5 | 0.0626 | 0.0066 | 1.3147 |  | 0.0889 | 0.1085 | 0.1725 |  |
| 1.2 | 0.5 | 0.0165 | 0.0220 | 0.1531 |  | 0.2724 | 0.1923 | 0.6673 |  |
|  | 0.75 | 0.0011 | 0.2065 | 0.6793 |  | 0.0335 | 0.0521 | 0.1611 |  |
|  | 1.25 | 0.0037 | 0.2068 | 0.7685 |  | 0.0474 | 0.0454 | 0.0362 |  |
|  | 1.5 | 0.0019 | 0.1720 | 1.0176 |  | 0.0635 | 0.0995 | 0.1928 |  |

## A Proofs

In this section $C>0$ denotes a generic constant, which may change from line to line. Recall moreover the shorthand $\xi=(\beta, \theta)$ for the joint parameters.

## A. 1 Proof of Theorem 2.7

This section is devoted to the proof of Theorem 2.7, which is divided into three steps. In Appendix A.1.1 we analyse the smoothness properties of the limiting Gaussian field $\left(G_{u}\right)_{u \in \mathbb{R}_{+}^{m}}$. Appendix A.1.2 presents a general weak convergence statement for integrals of stochastic processes. Finally, Appendix A.1.3 demonstrates proofs of the convergence results in Theorem 2.7.

## A.1.1 The limiting Gaussian field

To characterize the covariance of the asymptotic Gaussian field $\left(G_{u}\right)_{u \in \mathbb{R}_{+}^{m}}$ we define a dependence measure between two $m$-dimensional stable vectors $Y=\left(\int h_{1} \mathrm{~d} L, \ldots, \int h_{m} \mathrm{~d} L\right)$ and $Z=\left(\int g_{1} \mathrm{~d} L, \ldots, \int g_{m} \mathrm{~d} L\right)$ :

$$
U_{Y, Z}(u, v):=\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle(u, v),(Y, Z)\rangle_{\mathbb{R}^{2 m}}}\right]-\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle u, Y\rangle_{\mathbb{R}^{m}}}\right] \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle v, Z\rangle_{\mathbb{R}^{m}}}\right], \quad u, v \in \mathbb{R}^{m}
$$

This is a straightforward multivariate extension of the measure defined in [19]. We now apply Theorem 1.1 in conjunction with the smooth and bounded functions

$$
f_{u}(x)=\cos \left(\langle u, x\rangle_{\mathbb{R}^{m}}\right), \quad u, x \in \mathbb{R}^{m},
$$

such that we obtain the finite dimensional convergence of the processes:

$$
\begin{equation*}
\sqrt{n}\left(\varphi_{n}(u)-\varphi_{\xi}(u)\right)_{u \in \mathbb{R}_{+}^{m}} \xrightarrow[n \rightarrow \infty]{\mathrm{fdi}}\left(G_{u}\right)_{u \in \mathbb{R}_{+}^{m}} . \tag{A.1}
\end{equation*}
$$

Let $Z_{0}=\left(X_{1}, \ldots, X_{m}\right)$ and $Z_{\ell}=\left(X_{1+\ell}, \ldots, X_{m+\ell}\right)$, then the covariance function $R$ : $\mathbb{R}^{m} \times \mathbb{R}^{m} \rightarrow \mathbb{R}$ of $G$ is, cf. (1.4), given by

$$
R(u, v)=\sum_{\ell \in \mathbb{Z}} r_{\ell}(u, v),
$$

where for $\ell \in \mathbb{Z}$

$$
r_{\ell}(u, v)=\operatorname{Cov}\left(\cos \left(\left\langle u, Z_{0}\right\rangle\right), \cos \left(\left\langle v, Z_{\ell}\right\rangle\right)\right), \quad u, v \in \mathbb{R}^{m} .
$$

We will now prove that there exists a version of $G$, which is locally Hölder continuous up to any order less than $\beta / 4$. By Kolmogorov's criteria and Gaussianity it is enough to prove that for any $T>0$ there exists a constant $C_{T} \geq 0$ such that

$$
\begin{equation*}
\mathbb{E}\left[\left(G_{u}-G_{v}\right)^{2}\right] \leq C_{T}\|u-v\|^{\beta / 2} \quad \text { for all } u, v \in[0, T]^{m}, \tag{A.2}
\end{equation*}
$$

where $\|u-v\|=\sum_{i=1}^{m}\left|u_{i}-v_{i}\right|$ denotes the $\ell_{1}$-norm throughout the rest of this paper. To prove (A.2) note the decomposition

$$
\mathbb{E}\left[\left(G_{u}-G_{v}\right)^{2}\right]=R(u, u)-R(u, v)+R(v, v)-R(u, v) .
$$

Hence by symmetry it suffices to consider the term

$$
R(u, u)-R(u, v)=\sum_{\ell \in \mathbb{Z}}\left(r_{\ell}(u, u)-r_{\ell}(u, v)\right) .
$$

The main difficulty lies in establishing a bound on $r_{\ell}(u, u)-r(u, v)$ which is both $\frac{\beta}{2}$-Hölder in $(u, v)$ and summable in $\ell$. Using the standard identity $\cos (x)=\left(\mathrm{e}^{\mathrm{i} x}+\mathrm{e}^{-\mathrm{i} x}\right) / 2$ and the symmetry of $L_{1}$ we deduce the identity

$$
2\left(r_{\ell}(u, u)-r_{\ell}(u, v)\right)=\left[U_{Z_{0}, Z_{\ell}}(u,-u)-U_{Z_{0}, Z_{\ell}}(u,-v)\right]+\left[U_{Z_{0}, Z_{\ell}}(u, u)-U_{Z_{0}, Z_{\ell}}(u, v)\right] .
$$

The two terms in the square brackets are treated very similarly so we consider only the first one. Before diving into the tedious calculations we recall the following inequalities for $x, y \in \mathbb{R}$ :

$$
\begin{align*}
\left|\mathrm{e}^{-x}-\mathrm{e}^{-y}\right| \leq|x-y| & & \text { if } x, y \geq 0  \tag{A.3}\\
|x+y|^{\beta} \leq|x|^{\beta}+|y|^{\beta} & & \text { for } \beta \in(0,1]  \tag{A.4}\\
\left||x|^{\beta}-|y|^{\beta}\right| \leq|x-y|^{\beta} & & \text { for } \beta \in(0,1]  \tag{A.5}\\
\left||x+y|^{\beta}-|x|^{\beta}-|y|^{\beta}\right| \leq|x y|^{\beta / 2} & & \text { for } \beta \in(0,2) . \tag{A.6}
\end{align*}
$$

Define additionally the two quantities

$$
\rho_{i}=\int_{\mathbb{R}}\left|g_{\xi}(x) g_{\xi}(x+i)\right|^{\beta / 2} \mathrm{~d} x \quad \text { and } \quad \mu_{i}=\int_{-m}^{\infty}\left|g_{\xi}(x+i)\right|^{\beta} \mathrm{d} x, \quad i \in \mathbb{Z} .
$$

We shall need the following lemma.
Lemma A.1. There exists a constant $C>0$ such that for any $i \in \mathbb{N}$
(i) $\rho_{i} \leq C i^{-\alpha \beta / 2}$.
(ii) If $i>m$ then $\mu_{i} \leq C(i-m)^{1-\alpha \beta}$.

Proof. (i) follows as in [6, Lemma 4.1]. For (ii) note if $k>m$ then $x+k>1$ for any $x>-m$, so according to assumption (1.2)

$$
\mu_{i} \leq C \int_{-m}^{\infty}(x+k)^{-\alpha \beta}=C(k-m)^{1-\alpha \beta},
$$

where we used that $\alpha \beta>2$.

Using the expression for the characteristic function of a symmetric $\beta$-stable random variable we decompose as follows

$$
\begin{aligned}
& U_{Z_{0}, Z_{\ell}}(u,-u)-U_{Z_{0}, Z_{\ell}}(u,-v) \\
& =\exp \left(-\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta}^{\beta}\right)-\exp \left(-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right) \\
& -\left[\exp \left(-\left\|\sum_{i=1}^{m} u_{i} g(i-\cdot)-v_{i} g(i+\ell-\cdot)\right\|_{\beta}^{\beta}\right)\right. \\
& \left.-\exp \left(-\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}-\left\|\sum_{i=1}^{m} v_{i} g_{\xi}(i+\ell-\cdot)\right\|_{\beta}^{\beta}\right)\right] \\
& =\exp \left(2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right) \\
& \times\left[\exp \left(-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right)-\exp \left(-\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}-\left\|\sum_{i=1}^{m} v_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right)\right] \\
& \times\left[\exp \left(-\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta}^{\beta}\right)-\exp \left(-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right)\right] \\
& +\exp \left(-\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}-\left\|\sum_{i=1}^{m} v_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right) \\
& \times\left[\exp \left(-\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta}^{\beta}+2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right)\right. \\
& \left.-\exp \left(-\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)-v_{i} g_{\xi}(i+\ell-\cdot)\right\|_{\beta}^{\beta}+\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}+\left\|\sum_{i=1}^{m} v_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right)\right] \\
& =r_{\ell}^{1}(u, v)+r_{\ell}^{2}(u, v) .
\end{aligned}
$$

For the first term, $r_{\ell}^{1}$, we notice that the exponential term in front is bounded in $u \in[0, T]^{m}$ (and of course in $\ell \in \mathbb{Z}$ as well), hence by (A.3)

$$
\begin{aligned}
r_{\ell}^{1}(u, v) \leq & C_{T}\left|\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}-\left\|\sum_{i=1}^{m} v_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right| \\
& \times\left|\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta}^{\beta}-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right|
\end{aligned}
$$

The first absolute value term will give the Hölder continuity of order $\beta / 2$ and the second will ensure summability in $\ell$. For the first term we may bound as follows in the case $\beta \in(0,1]$ using (A.5) and (A.4)

$$
\begin{aligned}
\left|\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}-\left\|\sum_{i=1}^{m} v_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right| & \leq \int_{\mathbb{R}}\left(\sum_{i=1}^{m}\left|u_{i}-v_{i}\right|\left|g_{\xi}(i-x)\right|\right)^{\beta} \mathrm{d} x \\
& \leq\|u-v\|^{\beta} \sum_{i=1}^{m} \int_{\mathbb{R}}\left|g_{\xi}(i-x)\right|^{\beta} \mathrm{d} x \\
& \leq C_{T}\|u-v\|^{\beta / 2}
\end{aligned}
$$

If instead $\beta>1$, then the map is $u \mapsto\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}$ is continuously differentiable, hence by the mean value theorem it is Hölder continuous of any order less than or equal to 1 , and since $\beta \in(0,2)$ Hölder continuity of order $\beta / 2$ then holds. For the second absolute value term it follows by (A.6) and (A.4)

$$
\begin{aligned}
& \left|\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta}^{\beta}-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right| \\
& \quad=\mid\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta}^{\beta}-\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta} \\
& \quad-\left\|-\sum_{i=1}^{m} u_{i} g_{\xi}(i+\ell-\cdot)\right\|_{\beta}^{\beta} \mid \\
& \quad \leq 2\left\|\left(\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right)\left(\sum_{k=1}^{m} u_{k} g_{\xi}(k+\ell-\cdot)\right)\right\|_{\beta / 2}^{\beta / 2} \\
& \quad \leq 2 T^{\beta} \sum_{i, k=1}^{m}\left\|g_{\xi}(i-\cdot) g_{\xi}(k+\ell-\cdot)\right\|_{\beta / 2}^{\beta / 2} \\
& \quad=2 T^{\beta} \sum_{i, k=1}^{m} \rho_{\ell+k-i},
\end{aligned}
$$

which is summable in $\ell$ by Lemma A. 1 and the assumption $\alpha \beta>2$. We now turn our attention to the more complicated second term $r_{\ell}^{2}(u, v)$. Utilizing (A.3) we have that

$$
\begin{aligned}
r_{\ell}^{2}(u, v) \leq & \mid\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta}^{\beta}-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta} \\
& +\left\|\sum_{i=1}^{m} v_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}+\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta} \\
& -\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)-v_{i} g_{\xi}(i+\ell-\cdot)\right\|_{\beta}^{\beta} \mid \\
= & \mid \int_{-m}^{\infty}\left[\left|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(x+i)-g_{\xi}(i+\ell+x)\right)\right|^{\beta}\right. \\
& \left.-\left|\sum_{i=1}^{m} u_{i} g_{\xi}(i+x)-v_{i} g_{\xi}(i+\ell+x)\right|^{\beta}\right] \\
& +\left[\left|\sum_{i=1}^{m} v_{i} g_{\xi}(i+\ell+x)\right|^{\beta}-\left|\sum_{i=1}^{m} u_{i} g_{\xi}(i+\ell+x)\right|^{\beta}\right] \mathrm{d} x \mid \\
\leq & \left.\int_{-m}^{\infty}| | \sum_{i=1}^{m} u_{i}\left(g_{\xi}(x+i)-g_{\xi}(i+\ell+x)\right)\right|^{\beta} \\
& -\left|\sum_{i=1}^{m} u_{i} g_{\xi}(i+x)-v_{i} g_{\xi}(i+\ell+x)\right|^{\beta} \mid \mathrm{d} x \\
& +\left.\int_{-m}^{\infty}| | \sum_{i=1}^{m} v_{i} g_{\xi}(i+\ell+x)\right|^{\beta}-\left|\sum_{i=1}^{m} u_{i} g_{\xi}(i+\ell+x)\right|^{\beta} \mid \mathrm{d} x \\
= & r_{\ell}^{2,1}(u, v)+r_{\ell}^{2,2}(u, v) .
\end{aligned}
$$

We deal first with the second term $r_{\ell}^{2,2}$. First, if $\beta \in(0,1]$, then by (A.5) and (A.4)

$$
r_{\ell}^{2,2}(u, v) \leq \int_{-m}^{\infty}\left|\sum_{i=1}^{m}\left(u_{i}-v_{i}\right) g_{\xi}(i+\ell+x)\right|^{\beta} \mathrm{d} x \leq\|u-v\|^{\beta} \sum_{i=1}^{m} \mu_{i+\ell}
$$

and by Lemma A.1(ii) we obtain a bound which is summable in $\ell>m$. If instead $\beta \in(1,2)$ the map

$$
h(u)=\int_{-m}^{\infty}\left|\sum_{i=1}^{m} u_{i} g_{\xi}(i+\ell+x)\right|^{\beta} \mathrm{d} x, \quad u \in \mathbb{R}^{m}
$$

is continuously differentiable and the absolute value of the derivative is bounded as follows for any $u \in[0, T]^{m}$ and $\ell>m$ :

$$
\begin{aligned}
\left|\frac{\partial}{\partial u_{k}} h(u)\right| & \leq \int_{-m}^{\infty}\left|\sum_{i=1}^{m} u_{i} g_{\xi}(i+\ell+x)\right|^{\beta-1}\left|g_{\xi}(k+\ell+x)\right| \mathrm{d} x \\
& \leq T^{\beta-1} \sum_{i=1}^{m} \int_{-m}^{\infty}\left|g_{\xi}(i+\ell+x)\right|^{\beta-1}\left|g_{\xi}(k+\ell+x)\right| \mathrm{d} x \\
& \leq C T^{\beta-1} m(\ell-m)^{1-\alpha \beta}
\end{aligned}
$$

where we have argued as in Lemma A.1(ii) in the last inequality. Hence, in the case $\beta \in(1,2)$ we obtain by the mean value theorem

$$
r_{\ell}^{2,2}(u, v) \leq \sup _{z \in[0, T]^{m}}\|\nabla h(z)\|\|u-v\| \leq C_{T}(\ell-m)^{1-\alpha \beta}\|u-v\|^{\beta}
$$

and as $\alpha \beta>2$ we have obtained a bound summable in $\ell$.
It remains to consider the term $r_{\ell}^{2,1}$. Here it follows from the inequality $\left||x|^{\beta}-|y|^{\beta}\right| \leq$ $\left|x^{2}-y^{2}\right|^{\beta / 2}$ and the triangle inequality that the integrand is bounded by

$$
\begin{aligned}
\left|\mid \sum_{i=1}^{m}\right. & \left.u_{i}\left(g_{\xi}(i+x)-g_{\xi}(i+\ell+x)\right)\right|^{\beta}-\left|\sum_{i=1}^{m} u_{i} g_{\xi}(i+x)-v_{i} g_{\xi}(i+\ell+x)\right|^{\beta} \mid \\
\leq & \mid \sum_{i, k=1}^{m} u_{i} u_{k}\left(g_{\xi}(i+x)-g_{\xi}(i+\ell+x)\right)\left(g_{\xi}(k+x)-g_{\xi}(k+\ell+x)\right) \\
& \quad-\left.\left(u_{i} g_{\xi}(i+x)-v_{i} g_{\xi}(i+\ell+x)\right)\left(u_{k} g_{\xi}(k+x)-v_{k} g_{\xi}(k+\ell+x)\right)\right|^{\beta / 2} \\
= & \mid \sum_{i, k=1}^{m}\left[\left(u_{i} u_{k}-v_{i} v_{k}\right) g_{\xi}(i+\ell+x) g_{\xi}(k+\ell+x)\right. \\
& +u_{i}\left(v_{k}-u_{k}\right) g_{\xi}(i+x) g_{\xi}(k+\ell+x) \\
& \left.+u_{k}\left(v_{i}-u_{i}\right) g_{\xi}(i+\ell+x) g_{\theta, \beta}(k+x)\right]\left.\right|^{\beta / 2} \\
\leq & C_{T}\|u-v\|^{\beta / 2} \sum_{i, k=1}^{m}\left[\left|g_{\xi}(i+\ell+x) g_{\xi}(k+\ell+x)\right|^{\beta / 2}+\left|g_{\xi}(i+x) g_{\xi}(k+\ell+x)\right|^{\beta / 2}\right]
\end{aligned}
$$

Hence, we obtain with arguments as in Lemma A.1(ii) that

$$
r_{\ell}^{2,1}(u, v) \leq C_{T}\|u-v\|^{\beta / 2}\left((\ell-m)^{1-\alpha \beta}+\sum_{i, k=1}^{m} \rho_{\ell+k-i}\right)
$$

which is summable in $\ell$ as $\alpha \beta>2$.
Lastly, we shall prove that $\left(G_{u}\right)_{u \in \mathbb{R}_{+}^{m}}$ has paths in $\mathcal{L}_{w}^{1}\left(\mathbb{R}_{+}^{m}\right)$ almost surely, such that $\int_{\mathbb{R}_{+}^{m}} G_{u} w(u) \mathrm{d} u$ is well-defined. A sufficient criteria for this is $\int_{\mathbb{R}_{+}^{m}} \operatorname{Var}\left[G_{u}\right]^{1 / 2} w(u) \mathrm{d} u<\infty$, since $G$ is centred. For this we need to study $r_{\ell}(u, u)$ again. Recall that

$$
r_{\ell}(u, u)=U_{Z_{0}, Z_{\ell}}(u,-u)+U_{Z_{0}, Z_{\ell}}(u, u) .
$$

As both terms are treated almost identically it suffices to consider the first one. Here it follows from the inequality $\left|\mathrm{e}^{x}-1\right| \leq \mathrm{e}^{|x|}|x|, x \in \mathbb{R}$, and (A.6), that

$$
\begin{aligned}
& \left|U_{Z_{0}, Z_{\ell}}(u,-u)\right| \\
& \quad=\left|\exp \left(-\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta}^{\beta}\right)-\exp \left(-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right)\right| \\
& \quad \leq \exp \left(-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right) \\
& \quad \times\left|\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-x)\right)\right\|_{\beta}^{\beta}-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right| \\
& \quad \times \exp \left(\left|\left\|\sum_{i=1}^{m} u_{i}\left(g_{\xi}(i-\cdot)-g_{\xi}(i+\ell-x)\right)\right\|_{\beta}^{\beta}-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}\right|\right) \\
& \leq \\
& \quad \exp \left(-2\left\|\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right\|_{\beta}^{\beta}+2\left\|\left(\sum_{i=1}^{m} u_{i} g_{\xi}(i-\cdot)\right)\left(\sum_{i=1}^{m} u_{i} g_{\xi}(i+\ell-\cdot)\right)\right\|_{\beta / 2}^{\beta / 2}\right) \\
& \quad \times\|u\|^{\beta} \sum_{i, k=1}^{m} \rho_{\ell+k-i} \\
& \leq\|u\|^{\beta} \sum_{i, k=1}^{m} \rho_{\ell+k-i},
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality in the last line. Summing over $\ell$ yields an element in $\mathcal{L}_{w}^{1}\left(\mathbb{R}_{+}^{m}\right)$ by the assumption on the weight function $w$.

## A.1.2 Convergence of integral functionals

In Appendix A.1.1 we saw that the empirical characteristic functions suitably scaled and centred converge to a Gaussian process in finite dimensional sense. We wish to extend this convergence to integrals of our processes. For this we need to extend [15, Lemma 1] to a multivariate case. For $x \in \mathbb{R}$ let $\lfloor x\rfloor$ denote the largest integer $l$ such that $l \leq x$ and for a vector $u=\left(u_{1}, \ldots, u_{m}\right) \in \mathbb{R}^{m}$ we let $\lfloor u\rfloor=\left(\left\lfloor u_{1}\right\rfloor, \ldots,\left\lfloor u_{m}\right\rfloor\right)$.
Lemma A.2. Let $\left(Y_{u}^{n}\right)_{u \in \mathbb{R}_{+}^{m}}$ and $\left(Y_{u}\right)_{u \in \mathbb{R}_{+}^{m}}$ be continuous random fields with $Y^{n} \xrightarrow{\text { fidi }} Y$. Assume that $\int_{\mathbb{R}_{+}^{m}} \mathbb{E}\left[\left|Y_{u}^{n}\right|\right] \mathrm{d} u<\infty$ and $\int_{\mathbb{R}_{+}^{m}} \mathbb{E}\left[\left|Y_{u}\right|\right] \mathrm{d} u<\infty$, and set for $k, \ell, n \in \mathbb{N}$

$$
X_{n, k, \ell}=\int_{[0, \ell]^{m}} Y_{[u k\rfloor / k}^{n} \mathrm{~d} u \quad \text { and } \quad X_{n, \ell}=\int_{[0, \ell]^{m}} Y_{u}^{n} \mathrm{~d} u .
$$

Suppose that

$$
\lim _{\ell \rightarrow \infty} \limsup _{n \rightarrow \infty} \int_{\mathbb{R}_{+}^{m-1-i}} \int_{\ell}^{\infty} \int_{\mathbb{R}_{+}^{i}} \mathbb{E}\left[\left|Y_{u}^{n}\right|\right] \mathrm{d} u=0, \quad \lim _{k \rightarrow \infty} \limsup _{n \rightarrow \infty} \mathbb{P}\left(\left|X_{n, k, \ell}-X_{n, \ell}\right|>\varepsilon\right)=0
$$

where the first convergence holds for all $i \in\{0, \ldots, m-1\}$ and the latter for all $\varepsilon, \ell>0$. Then convergence in distribution holds:

$$
\int_{\mathbb{R}_{+}^{m}} Y_{u}^{n} \mathrm{~d} u \xrightarrow{\mathcal{L}} \int_{\mathbb{R}_{+}^{m}} Y_{u} \mathrm{~d} u \quad \text { as } n \rightarrow \infty
$$

Proof. Observe for each $\ell>0$ the decomposition

$$
\int_{\mathbb{R}_{+}^{m}} Y_{u}^{n} \mathrm{~d} u=X_{n, k, \ell}+\left(X_{n, \ell}-X_{n, k, \ell}\right)+\sum_{i=0}^{m-1} \int_{\mathbb{R}_{+}^{m-1-i}} \int_{\ell}^{\infty} \int_{[0, \ell]^{i}} Y_{u}^{n} \mathrm{~d} u
$$

Conclude now as in [15, Lemma 1].

## A.1.3 Convergence of the estimator

First, $\xi_{n} \xrightarrow{\text { a.s. }} \xi_{0}$ follows by standard arguments which in particular uses Assumption (A), see, e.g. [15], where one uses

$$
\begin{equation*}
\left\|\varphi_{n}-\varphi_{\xi_{0}}\right\|_{w} \xrightarrow{\text { a.s. }} 0 \quad \text { as } n \rightarrow \infty \tag{A.7}
\end{equation*}
$$

which is a consequence of Lebesgue's dominated convergence theorem. Indeed, denote by $\lambda$ the Lebesgue measure on $\mathbb{R}^{m}$. For (A.7) it is by dominated convergence enough to prove that there exists a $\mathbb{P}$-null set $N$ such that for all $\omega \in \Omega \backslash N$

$$
\varphi_{n}(u, \omega) \rightarrow \varphi_{\xi_{0}}(u) \quad \text { for } \lambda \text {-almost all } u \in \mathbb{R}_{+}^{m}
$$

To see this set $A=\left\{(\omega, u) \mid \varphi_{n}(u, \omega) \nrightarrow \varphi_{\xi_{0}}(u)\right\}$ and note that by Tonelli's theorem:

$$
\begin{aligned}
\int_{\Omega} \lambda\left(\left\{u \mid \varphi_{n}(u, \omega) \nrightarrow \varphi_{\xi_{0}}(u)\right\}\right) \mathbb{P}(\mathrm{d} \omega) & =\int_{\Omega} \int_{\mathbb{R}^{m}} \mathbb{1}_{A}(\omega, u) \lambda(\mathrm{d} u) \mathbb{P}(\mathrm{d} \omega) \\
& =\int_{\mathbb{R}^{m}} \mathbb{P}\left(\left\{\omega \mid \varphi_{n}(u, \omega) \nrightarrow \varphi_{\xi_{0}}(u)\right\}\right) \lambda(\mathrm{d} u)=0
\end{aligned}
$$

where the last equality follows from Birkhoff's ergodic theorem which states that for each $u \in \mathbb{R}^{m}$, then

$$
\varphi_{n}(u) \rightarrow \varphi_{\xi_{0}}(u) \quad \mathbb{P} \text {-almost surely as } n \rightarrow \infty
$$

To derive the central limit theorem for the estimator, we consider instead the requirement

$$
\nabla_{\xi} F(\varphi, \xi)=0 \quad \varphi \in \mathcal{L}_{w}^{2}\left(\mathbb{R}_{+}^{m}\right), \quad \xi \in \Xi
$$

which is satisfied at $\left(\varphi_{\xi_{0}}, \xi_{0}\right)$. The problem may now be viewed from a implicit functional point of view. To this end we recall the implicit function theorem on general Banach spaces. Consider a Fréchet differentiable map $g: U_{1} \times U_{2} \rightarrow B_{3}$ where $U_{1}$ and $U_{2}$ are open subsets of the Banach spaces $B_{1}$ and $B_{2}$, respectively, and $B_{3}$ is an additional Banach space. Let $D_{h_{i}}^{i} g\left(p_{1}, p_{2}\right), i \in\{1,2\}$, denote the partial derivatives at the point $\left(p_{1}, p_{2}\right) \in U_{1} \times U_{2}$ in the direction $h_{i} \in B_{i}$. If $\left(p_{1}^{0}, p_{2}^{0}\right) \in U_{1} \times U_{2}$ is a point such that $g\left(p_{1}^{0}, p_{2}^{0}\right)=0$ and the map $h \mapsto D_{h}^{2} g\left(p_{1}^{0}, p_{2}^{0}\right): B_{2} \rightarrow B_{3}$ is a continuous and invertible function, then there exists open subsets $V_{1} \subseteq U_{1}$ and $V_{2} \subseteq U_{2}$ such that $\left(p_{1}^{0}, p_{2}^{0}\right) \in V_{1} \times V_{2}$ and a Fréchet differentiable and bijective (implicit) function $\Phi: V_{1} \rightarrow V_{2}$ such that

$$
g\left(p_{1}, p_{2}\right)=0 \quad \Longleftrightarrow \quad \Phi\left(p_{1}\right)=p_{2}
$$

In addition, the derivative is given by

$$
\begin{equation*}
D_{h} \Phi(p)=-\left(D^{2} g(p, \Phi(p))\right)^{-1}\left(D_{h}^{1} g(p, \Phi(p))\right), \quad h \in B_{1}, \quad p \in V_{1} \tag{A.8}
\end{equation*}
$$

As might be apparent we shall consider the specific setup of $g=\nabla_{\xi} F, B_{1}=U_{1}=\mathcal{L}_{w}^{2}\left(\mathbb{R}_{+}^{m}\right)$, $U_{2}=\Xi \subseteq B_{2}=\mathbb{R}^{d+1}$. We note that Assumption (B)(2) ensures the existence and continuity of the first and second order derivatives of $F$. Moreover, Assumption (A)(4) yields the invertibility of the Hessian $\nabla_{\xi}^{2} F\left(\varphi_{\xi_{0}}, \xi_{0}\right)$.

In this case

$$
\Phi\left(\varphi_{n}\right)=\xi_{n} \quad \text { and } \quad \Phi\left(\varphi_{\xi_{0}}\right)=\xi_{0}
$$

Hence, by Fréchet differentiability we find that

$$
\begin{aligned}
\sqrt{n}\left(\xi_{n}-\xi_{0}\right) & =\sqrt{n}\left(\Phi\left(\varphi_{\xi_{0}}+\left(\varphi_{n}-\varphi_{\xi_{0}}\right)\right)-\Phi\left(\varphi_{\xi_{0}}\right)\right) \\
& =D_{\sqrt{n}\left(\varphi_{n}-\varphi_{\xi_{0}}\right.} \Phi\left(\varphi_{\xi_{0}}\right)+\sqrt{n}\left\|\varphi_{n}-\varphi_{\xi_{0}}\right\|_{w, 2} R\left(\varphi_{n}-\varphi_{\xi_{0}}\right)
\end{aligned}
$$

where the remainder term satisfies that $R\left(\varphi_{n}-\varphi_{\xi_{0}}\right) \xrightarrow{\text { a.s. }} 0$ as $\left\|\varphi_{n}-\varphi_{\xi_{0}}\right\|_{w, 2} \xrightarrow{\text { a.s. }} 0$. Recalling the derivative at (A.8) and the representation $F(\varphi, \xi)=\left\langle\varphi-\varphi_{\xi}, \varphi-\varphi_{\xi}\right\rangle_{w}$, it suffices to prove that

$$
\begin{gathered}
\sqrt{n}\left\|\varphi_{n}-\varphi_{\xi_{0}}\right\|_{w, 2} \xrightarrow{\mathcal{L}}\|G\|_{w, 2} \\
\left(\left\langle\partial_{\xi}^{i} \varphi_{\xi_{0}}, \sqrt{n}\left(\varphi_{n}-\varphi_{\xi_{0}}\right)\right\rangle_{w}\right)_{i=1, \ldots, d+1} \xrightarrow{\mathcal{L}}\left(\left\langle\partial_{\xi}^{i} \varphi_{\xi_{0}}, G\right\rangle_{w}\right)_{i=1 \ldots, d+1} .
\end{gathered}
$$

We focus on the last convergence since they are shown similarly. For this we wish to use (a vector version of) Lemma A. 2 which requires the finite dimensional convergence of the vector-valued process

$$
Z_{u}^{n}=\left(\partial_{\xi}^{i} \varphi_{\xi_{0}}(u) w(u) \sqrt{n}\left(\varphi_{n}(u)-\varphi_{\xi_{0}}(u)\right)\right)_{i=1, \ldots, d+1}
$$

But since it is the same underlying process, $\left(\sqrt{n}\left(\varphi_{n}(u)-\varphi_{\xi_{0}}(u)\right)\right)_{u \in \mathbb{R}_{+}^{m}}$, this simply follows from the continuous mapping theorem in conjunction with the finite dimensional convergence observed at (A.1). A small generalization of Lemma A. 2 shows that is sufficient to provide suitable moment estimates for each individual coordinate, that is, estimates for

$$
Y_{u}^{n}:=\partial_{\xi}^{i} \varphi_{\xi_{0}}(u) w(u) \sqrt{n}\left(\varphi_{n}(u)-\varphi_{\xi_{0}}(u)\right)=: h(u) G_{u}^{n}, \quad u \in \mathbb{R}_{+}^{m}, \quad n \in \mathbb{N}
$$

where $i \in\{1, \ldots, d+1\}$ is fixed and $h$ and $G^{n}$ are defined respectively as

$$
h(u)=\partial_{\xi}^{i} \varphi_{\xi_{0}}(u) w(u) \quad \text { and } \quad G_{u}^{n}=\sqrt{n}\left(\varphi_{n}(u)-\varphi_{\xi_{0}}(u)\right)
$$

Note that $h$ is continuous by Assumption (A)(4) and since the weight function is continuous.
Using arguments as in [15, Section 4.2] and the variance estimates from Appendix A.1.1 we deduce that

$$
\mathbb{E}\left[\left|Y_{u}^{n}\right|^{2}\right] \leq\left(\partial_{\xi}^{i} \varphi_{\xi_{0}}(u) w(u)\right)^{2} \sum_{\ell \in \mathbb{Z}}\left|r_{\ell}(u, u)\right| \leq C\|u\|^{\beta}\left(\partial_{\xi}^{i} \varphi_{\xi_{0}}(u) w(u)\right)^{2}
$$

Taking the square root we obtain a bound in $\mathcal{L}^{1}\left(\mathbb{R}_{+}^{m}\right)$ of $\mathbb{E}\left[\left|Y_{u}^{n}\right|\right]$ by the Cauchy-Schwarz inequality used together with Assumption $(\mathrm{B})(2)$ and that $u \mapsto\|u\|$ is an element of $\mathcal{L}_{w}^{2}\left(\mathbb{R}_{+}^{m}\right)$. Hence the first condition of Lemma A. 2 is satisfied. The second condition is slightly more involved, but let a $\ell>0$ be given and consider any $u, v \in[0, \ell]^{m}$. Then

$$
\begin{aligned}
\mathbb{E}\left[\left|Y_{u}^{n}-Y_{v}^{n}\right|^{2}\right]^{1 / 2} & \leq|h(u)-h(v)| \operatorname{Var}\left[G_{u}^{n}\right]^{1 / 2}+|h(v)| \operatorname{Cov}\left(G_{u}^{n}, G_{v}^{n}\right)^{1 / 2} \\
& \leq C_{\ell}(|h(u)-h(v)|+\|u-v\|)
\end{aligned}
$$

which by Markov's inequality yields the second condition of Lemma A.2.

## A. 2 Proof of Remark 2.1(iv)

Let $\left\{g_{\xi} \mid \xi \in \Xi\right\}$ be a family of measurable functions such that conditions (1) and (3) from Assumption (A) holds. Then condition (A)(4) holds if and only if $u \mapsto \partial_{\xi}^{2} \varphi_{\xi}(u), \ldots, \partial_{\xi}^{d+1} \varphi_{\xi}(u)$ are linearly independent. The only if part is trivial so we consider solely the if statement. Suppose therefore that there exists constants $a_{1}, \ldots, a_{d+1} \in \mathbb{R}$ such that

$$
\begin{equation*}
a_{1} \partial_{\xi}^{1} \varphi_{\xi}(u)+\cdots+a_{d+1} \partial_{\xi}^{d+1} \varphi_{\xi}(u)=0 \quad \text { for all } u \in \mathbb{R}_{+}^{m} \tag{A.9}
\end{equation*}
$$

We note first that for any $i \in\{1, \ldots, d+1\}$ and $u_{1}>0$

$$
\begin{align*}
\partial_{\xi}^{i} \varphi_{\xi}\left(u_{1}, 0, \ldots, 0\right) & =\varphi_{\xi}\left(u_{1}, 0, \ldots, 0\right) \partial_{\xi}^{i}\left(u_{1}^{\beta}\left\|g_{\xi}\right\|_{\beta}^{\beta}\right) \\
& =\varphi_{\xi}\left(u_{1}, 0, \ldots, 0\right) \times \begin{cases}u_{1}^{\beta} \partial_{\xi}^{i}\left\|g_{\xi}\right\|_{\beta}^{\beta}, & \text { if } i \neq 1 \\
u_{1}^{\beta} \log \left(u_{1}\right)\left\|g_{\xi}\right\|_{\beta}^{\beta}+u_{1}^{\beta} \partial_{\xi}^{1}\left\|g_{\xi}\right\|_{\beta}^{\beta}, & \text { if } i=1\end{cases} \tag{A.10}
\end{align*}
$$

Since $\varphi_{\xi}(u) \neq 0$ for all $u \in \mathbb{R}_{+}^{m}$ it follows from (A.10) that

$$
\frac{\partial_{\xi}^{i} \varphi_{\xi}\left(u_{1}, 0, \ldots, 0\right)}{\varphi_{\xi}\left(u_{1}, 0, \ldots, 0\right) u_{1}^{\beta} \log \left(u_{1}\right)} \underset{u_{1} \rightarrow \infty}{ } \begin{cases}0, & \text { if } i \neq 1 \\ \left\|g_{\xi}\right\|_{\beta}^{\beta}, & \text { if } i=1\end{cases}
$$

This proves that $a_{1}=0$ using condition $(\mathrm{A})(1)$. Then (A.9) reduces to

$$
a_{2} \partial_{\xi}^{2} \varphi_{\xi}(u)+\cdots+a_{d+1} \partial_{\xi}^{d+1} \varphi_{\xi}(u)=0 \quad \text { for all } u \in \mathbb{R}_{+}^{m}
$$

and $a_{2}=\cdots=a_{d+1}=0$ follows by the assumption of linear independence of the subset.

## A. 3 Proof of statements in Example 2.4

Consider the kernel ${ }^{1} g_{\theta}(u)=\exp \left(-\theta_{1} u-\theta_{2} f(u)\right) \mathbb{1}_{(0, \infty)}(u)$ for $\theta=\left(\theta_{1}, \theta_{2}\right) \in(0, \infty)^{2}$ and where $f$ is a bounded measurable 1-periodic function which does not vanish except on a Lebesgue null set. Assume moreover that $f$ is either non-positive and or non-negative. It is straightforward to see that in this case the characteristic function of $X_{1}$ does not determine the parameter $\theta$ uniquely. Consider instead the joint characteristic function $\varphi_{\beta, \theta}\left(u_{1}, u_{2}\right)$ of $\left(X_{1}, X_{2}\right)$ for the moving average $X$ with kernel $g_{\theta}$, which is given by:

$$
\varphi_{\beta, \theta}\left(u_{1}, u_{2}\right)=\exp \left(-\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}\right), \quad u_{1}, u_{2} \geq 0
$$

If $\varphi_{\beta, \theta}=\varphi_{\beta, \tilde{\theta}}$ for $\theta, \tilde{\theta} \in(0, \infty)^{2}$, then the $\beta$-norms must be equal. Recalling the generalized binomial theorem

$$
(x+y)^{\beta}=\sum_{k=0}^{\infty}\binom{\beta}{k} x^{\beta-k} y^{k} \quad x>y \geq 0
$$

we may calculate these norms explicitly for $u_{1}>u_{2} \geq 0$ :

$$
\begin{aligned}
& \left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta} \\
& \quad=u_{2}^{\beta} \int_{0}^{1} \exp \left(-\theta_{1} x-\theta_{2} f(x)\right) \mathrm{d} x
\end{aligned}
$$

[^0]\[

$$
\begin{aligned}
& +\int_{0}^{\infty} \sum_{k=0}^{\infty}\binom{\beta}{k} u_{1}^{\beta-k} u_{2}^{k} \exp \left(-(\beta-k)\left(\theta_{1} x+\theta_{2} f(x)\right)-k\left(\theta_{1}(x+1)+\theta_{2} f(x+1)\right)\right) \mathrm{d} x \\
= & u_{2}^{\beta} \int_{0}^{1} \exp \left(-\theta_{1} x-\theta_{2} f(x)\right) \mathrm{d} x \\
& +\int_{0}^{\infty} \sum_{k=0}^{\infty}\binom{\beta}{k} u_{1}^{\beta-k} u_{2}^{k} \exp \left(-\beta\left(\theta_{1} x+\theta_{2} f(x)\right)-k \theta_{1}\right) \mathrm{d} x \\
= & u_{2}^{\beta} \int_{0}^{1} \exp \left(-\theta_{1} x-\theta_{2} f(x)\right) \mathrm{d} x+\left(u_{1}+u_{2} \exp \left(-\theta_{1}\right)\right)^{\beta} \int_{0}^{\infty} \exp \left(-\beta\left(\theta_{1} x+\theta_{2} f(x)\right)\right) \mathrm{d} x
\end{aligned}
$$
\]

where the last equality follows from the generalized binomial theorem since $u_{1}>u_{2} \geq$ $u_{2} \exp \left(-\theta_{1}\right)$. Hence if $\varphi_{\beta, \theta}=\varphi_{\beta, \tilde{\theta}}$ then for all $u_{1}>u_{2} \geq 0$

$$
1=\frac{u_{2}^{\beta} \int_{0}^{1} \exp \left(-\theta_{1} x-\theta_{2} f(x)\right) \mathrm{d} x+\left(u_{1}+u_{2} \exp \left(-\theta_{1}\right)\right)^{\beta} \int_{0}^{\infty} \exp \left(-\beta\left(\theta_{1} x+\theta_{2} f(x)\right)\right) \mathrm{d} x}{u_{2}^{\beta} \int_{0}^{1} \exp \left(-\tilde{\theta}_{1} x-\tilde{\theta}_{2} f(x)\right) \mathrm{d} x+\left(u_{1}+u_{2} \exp \left(-\tilde{\theta}_{1}\right)\right)^{\beta} \int_{0}^{\infty} \exp \left(-\beta\left(\tilde{\theta}_{1} x+\tilde{\theta}_{2} f(x)\right)\right) \mathrm{d} x} .
$$

Inserting $u_{1}=1>0=u_{2}$ yields the identity:

$$
K:=\int_{0}^{\infty} \exp \left(-\beta\left(\theta_{1} x+\theta_{2} f(x)\right)\right) \mathrm{d} x=\int_{0}^{\infty} \exp \left(-\beta\left(\tilde{\theta}_{1} x+\tilde{\theta}_{2} f(x)\right)\right) \mathrm{d} x
$$

hence it suffices to prove that $\theta_{1}=\tilde{\theta}_{1}$. Moreover, inserting the above identity in $\varphi_{\beta, \theta}=\varphi_{\beta, \tilde{\theta}}$ and differentiating with respect to $u_{1}$ gives that for all $u_{1}>u_{2}$ :

$$
\left(u_{1}+u_{2} \exp \left(-\theta_{1}\right)\right)^{\beta-1} K=\left(u_{1}+u_{2} \exp \left(-\tilde{\theta}_{1}\right)\right)^{\beta-1} K,
$$

which proves that $\theta_{1}=\tilde{\theta}_{1}$ if $\beta \neq 1$.
Let us additionally show that $u \mapsto \partial_{\xi}^{2} \varphi_{\xi}$ and $u \mapsto \partial_{\xi}^{3} \varphi_{\xi}$ are linearly independent if the 1 -periodic function is negative and bounded and $\beta \neq 1$. Indeed, by Remark 2.1(iv) this is equivalent to Assumption (A)(4). Due to their exponential form these derivatives are linearly independent if the following functions (note that we only have an explicit formula when $u_{1}>u_{2} \geq 0$ ) are linearly independent in $u_{1}>u_{2} \geq 0$ :

$$
\begin{aligned}
& \partial_{\xi}^{2}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}=-K_{\theta, 1} u_{2}^{\beta}-K_{\theta, 2}\left(u_{1}+u_{2}\right)^{\beta-1} u_{2}-K_{\theta, 3}\left(u_{1}+u_{2} \exp \left(-\theta_{1}\right)\right)^{\beta} \\
& \partial_{\xi}^{3}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}=K_{\theta, 4} u_{2}^{\beta}+K_{\theta, 5}\left(u_{1}+u_{2} \exp \left(-\theta_{1}\right)\right)^{\beta},
\end{aligned}
$$

where the constant $K_{\theta, 1}, \ldots, K_{\theta, 5}$ are strictly positive, indeed the only constants which are not in general positive are:

$$
\begin{aligned}
& K_{\theta, 5}=-\int_{0}^{\infty} \beta \theta_{2} f(x) \exp \left(-\beta\left(\theta_{1} x+\theta_{2} f(x)\right)\right) \mathrm{d} x \\
& K_{\theta, 4}=-\int_{0}^{1} f(x) \exp \left(-\theta_{1} x-\theta_{2} f(x)\right) \mathrm{d} x
\end{aligned}
$$

but they are by our assumption $f<0$. The main observation needed is that these functions are of different order in $u_{1}$ when $u_{2} \neq 0$ and that their constants are of opposite sign. Indeed, for $a, b \in \mathbb{R}$ we have that

$$
0=\left(a \partial_{\xi}^{2}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}+b \partial_{\xi}^{3}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}\right) / u_{1}^{\beta}
$$

$$
\xrightarrow[u_{1} \rightarrow \infty]{ }-a K_{\theta, 3}+b K_{\theta, 5}
$$

The constants $a K_{\theta, 3}$ and $b K_{\theta, 5}$ must then be same and we have the following major simplification:

$$
\begin{aligned}
0 & =a \partial_{\xi}^{2}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}+b \partial_{\xi}^{3}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta} \\
& =-\left(a K_{\theta, 1}-b K_{\theta, 4}\right) u_{2}^{\beta}-a K_{\theta, 2}\left(u_{1}+u_{2}\right)^{\beta-1} u_{2}
\end{aligned}
$$

If $\beta>1$ then this is clearly unbounded in $u_{1}$, hence $a=0$, and therefore $b=0$ as well since $K_{\theta, 4}>0$. If $\beta<1$ then differentiating with respect to $u_{1}$ yields the simple equation:

$$
0=a K_{\theta, 2}\left(u_{1}+u_{2}\right)^{\beta-2} u_{2} \quad \text { for all } u_{1}>u_{2} \geq 0
$$

which yields $a=0$ and therefore $b=0$ since again $K_{\theta, 4}>0$.

## A. 4 Proof of statements in Example 2.5

Recall the moving average kernel from (2.4). First, we show that the one-dimensional characteristic function is not enough to identify $\theta=\left(\theta_{1}, \theta_{2}\right)$. Indeed, we see that for two parameters $\theta, \tilde{\theta} \in(0, \infty)^{2}$ equality of the one-dimensional characteristic functions gives

$$
\begin{align*}
\frac{\theta_{1}^{\beta} \Gamma(\beta+1)}{\left(\beta \theta_{2}\right)^{\beta+1}} & =\int_{0}^{\infty}\left(\theta_{1} s \exp \left(-\theta_{2} s\right)\right)^{\beta} \mathrm{d} s \\
& =\int_{0}^{\infty}\left(\tilde{\theta}_{1} s \exp \left(-\tilde{\theta}_{2} s\right)\right)^{\beta} \mathrm{d} s=\frac{\tilde{\theta}_{1}^{\beta} \Gamma(\beta+1)}{\left(\beta \tilde{\theta}_{2}\right)^{\beta+1}} \tag{A.11}
\end{align*}
$$

We claim that the two-dimensional characteristic function is enough to identify $\theta$. For this we recall the covariation between $X_{1}$ and $X_{0}$, cf. [22, Section 2.7], which is uniquely determined by the distribution of $\left(X_{1}, X_{0}\right)$ and hence by its joint characteristic function. If $\theta$ denotes the underlying parameter for the moving average $X$ and $\beta>1$, then the covariation is, cf. [22, Proposition 3.5.2],

$$
\begin{align*}
{\left[X_{1}, X_{0}\right]_{\beta} } & =\int_{\mathbb{R}} g_{\theta}(s+1) g_{\theta}(s)^{\beta-1} \mathrm{~d} s=\theta_{1}^{\beta} \int_{0}^{\infty}(s+1) \mathrm{e}^{-\theta_{2}(s+1)} s^{\beta-1} \mathrm{e}^{-(\beta-1) \theta_{2} s} \mathrm{~d} s \\
& =\theta_{1}^{\beta} \mathrm{e}^{-\theta_{2}}\left[\int_{0}^{\infty} s^{\beta} \mathrm{e}^{-\beta \theta_{2} s} \mathrm{~d} s+\int_{0}^{\infty} s^{\beta-1} \mathrm{e}^{-\beta \theta_{2} s} \mathrm{~d} s\right] \\
& =\theta_{1}^{\beta} \mathrm{e}^{-\theta_{2}}\left[\frac{\Gamma(\beta+1)}{\left(\beta \theta_{2}\right)^{\beta+1}}+\frac{\Gamma(\beta)}{\left(\beta \theta_{2}\right)^{\beta}}\right]  \tag{A.12}\\
& =\frac{\theta_{1}^{\beta} \Gamma(\beta+1)}{\left(\beta \theta_{2}\right)^{\beta+1}} \mathrm{e}^{-\theta_{2}}\left(1+\theta_{2}\right),
\end{align*}
$$

where we used the defining property: $\beta \Gamma(\beta)=\Gamma(\beta+1)$. Hence if $\theta$ and $\tilde{\theta}$ leads to the same distribution of $\left(X_{1}, X_{0}\right)$, then combining the identities (A.11) and (A.12) yields

$$
\left(1+\theta_{2}\right) \mathrm{e}^{-\theta_{2}}=\left(1+\tilde{\theta}_{2}\right) \mathrm{e}^{-\tilde{\theta}_{2}}
$$

It is straightforward to check that the function $x \mapsto(1+x) \mathrm{e}^{-x}$ is strictly decreasing on $(0, \infty)$, and therefore injective, which proves that $\theta_{2}=\tilde{\theta}_{2}$ and therefore $\theta_{1}=\tilde{\theta}_{1}$ as well, cf. (A.11).

Let us now check the condition (A)(4). According to Remark 2.1(iv) it suffices to prove linear independence of the functions $\partial_{\xi}^{2} \varphi_{\xi}$ and $\partial_{\xi}^{3} \varphi_{\xi}$. We obtain the identities

$$
\begin{aligned}
\partial_{\xi}^{2}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}= & \beta \theta_{1}^{\beta-1} \int_{\mathbb{R}}\left(u_{1} x \exp \left(-\theta_{2} x\right) \mathbb{1}_{(0, \infty)}(x)\right. \\
& \left.+u_{2}(x+1) \exp \left(-\theta_{2}(x+1)\right) \mathbb{1}_{(0, \infty)}(x+1)\right)^{\beta} \mathrm{d} x, \\
\partial_{\xi}^{3}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}= & -\beta \theta_{1}^{\beta} \int_{\mathbb{R}}\left(u_{1} x \exp \left(-\theta_{2} x\right) \mathbb{1}_{(0, \infty)}(x)\right. \\
& \left.+u_{2}(x+1) \exp \left(-\theta_{2}(x+1)\right) \mathbb{1}_{(0, \infty)}(x+1)\right)^{\beta-1} \\
& \times\left(u_{1} x^{2} \exp \left(-\theta_{2} x\right) \mathbb{1}_{(0, \infty)}(x)\right. \\
& \left.+u_{2}(x+1)^{2} \exp \left(-\theta_{2}(x+1)\right) \mathbb{1}_{(0, \infty)}(x+1)\right) \mathrm{d} x .
\end{aligned}
$$

Notice that it suffices to show linear independence of the functions

$$
f_{1}\left(u_{1}, u_{2}\right):=\frac{\partial_{\xi}^{2}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}}{\beta \theta_{1}^{\beta-1}}, \quad f_{2}\left(u_{1}, u_{2}\right):=\frac{\partial_{\xi}^{3}\left\|u_{1} g_{\theta}+u_{2} g_{\theta}(\cdot+1)\right\|_{\beta}^{\beta}}{\beta \theta_{1}^{\beta}} .
$$

Assume that there exist a constant $r$ such that $f_{1}\left(u_{1}, u_{2}\right)+r f_{2}\left(u_{1}, u_{2}\right)=0$ for any $u_{1}>$ $u_{2} \geq 0$. Next, setting $u_{2}=0$, we obtain the identities

$$
\begin{aligned}
& f_{1}\left(u_{1}, 0\right)=u_{1}^{\beta} \int_{0}^{\infty} x^{\beta} \exp \left(-\beta \theta_{2} x\right) \mathrm{d} x=u_{1}^{\beta}\left(\beta \theta_{2}\right)^{-\beta-1} \Gamma(\beta+1), \\
& f_{2}\left(u_{1}, 0\right)=-u_{1}^{\beta} \int_{0}^{\infty} x^{\beta+1} \exp \left(-\beta \theta_{2} x\right) \mathrm{d} x=-u_{1}^{\beta}\left(\beta \theta_{2}\right)^{-\beta-2} \Gamma(\beta+2) .
\end{aligned}
$$

Hence, it must hold that

$$
r=\frac{\beta \theta_{2}}{\beta+1} .
$$

In the next step we will show that $f_{1}\left(u_{1}, \exp \left(\theta_{2}\right)\right)+r f_{2}\left(u_{1}, \exp \left(\theta_{2}\right)\right) \rightarrow \infty$ as $u_{1} \rightarrow \infty$, which leads to the desired contradiction. Recall that $f_{1}\left(u_{1}, 0\right)+r f_{2}\left(u_{1}, 0\right)=0$ and hence we may instead consider $f_{1}\left(u_{1}, \exp \left(\theta_{2}\right)\right)-f_{1}\left(u_{1}, 0\right)+r\left(f_{2}\left(u_{1}, \exp \left(\theta_{2}\right)\right)-f_{2}\left(u_{1}, 0\right)\right)$. Applying the mean value theorem we conclude that

$$
f_{1}\left(u_{1}, \exp \left(\theta_{2}\right)\right)-f_{1}\left(u_{1}, 0\right)=\int_{0}^{1} x^{\beta} \exp \left(-\beta \theta_{2}(x-1)\right) \mathrm{d} x+u_{1}^{\beta-1} q_{1}+o\left(u_{1}^{\beta-1}\right),
$$

where

$$
q_{1}=\beta \int_{0}^{\infty} x^{\beta-1}(x+1) \exp \left(-\beta \theta_{2} x\right) \mathrm{d} x .
$$

Similarly, we deduce that

$$
f_{2}\left(u_{1}, \exp \left(\theta_{2}\right)\right)-f_{2}\left(u_{1}, 0\right)=-\int_{0}^{1} x^{\beta+1} \exp \left(-\beta \theta_{2}(x-1)\right) \mathrm{d} x+u_{1}^{\beta-1} q_{2}+o\left(u_{1}^{\beta-1}\right)
$$

with

$$
q_{2}=-\int_{0}^{\infty}\left(\beta x^{\beta+1}+(\beta+1) x^{\beta}+x^{\beta-1}\right) \exp \left(-\beta \theta_{2} x\right) \mathrm{d} x
$$

Since $u_{1}^{\beta-1} \rightarrow \infty$ as $u_{1} \rightarrow \infty$ because $\beta>1$, we only need to prove that $q_{1}+r q_{2} \neq 0$. We have that

$$
\begin{aligned}
& q_{1}=\beta\left(\left(\beta \theta_{2}\right)^{-\beta-1} \Gamma(\beta+1)+\left(\beta \theta_{2}\right)^{-\beta} \Gamma(\beta)\right) \\
& q_{2}=-\left(\beta\left(\beta \theta_{2}\right)^{-\beta-2} \Gamma(\beta+2)+(\beta+1)\left(\beta \theta_{2}\right)^{-\beta-1} \Gamma(\beta+1)+\left(\beta \theta_{2}\right)^{-\beta} \Gamma(\beta)\right)
\end{aligned}
$$

A straightforward calculation shows that

$$
q_{1}+r q_{2}=-r\left(\beta \theta_{2}\right)^{-\beta} \Gamma(\beta)<0
$$

Consequently, we have a contradiction and the functions $f_{1}$ and $f_{2}$ are linearly independent.

## A. 5 Proof of statements in Example 2.6

We consider a CARMA $(2,1)$ model of the form

$$
X_{t}=\int_{-\infty}^{t} b^{\top} \exp (A(t-s)) e d L_{s}, \quad t \in \mathbb{R}
$$

where $b=\left(b_{0}, 1\right)^{\top}, e=(0,1)^{\top}, L$ is a symmetric $\beta$-stable Lévy process with $\beta \in(1,2)$, and

$$
A=\left(\begin{array}{cc}
0 & 1 \\
-\lambda^{2} & 2 \lambda
\end{array}\right)
$$

with $\lambda<0$. We further assume that $\theta=b_{0}+\lambda>0$. Recall the definition of the incomplete gamma function:

$$
\Gamma(\beta ; x)=\int_{x}^{\infty} y^{\beta-1} \exp (-y) \mathrm{d} y, \quad \beta, x>0
$$

The following identity is due to partial integration: $\Gamma(\beta+1 ; x)=\beta \Gamma(\beta ; x)+x^{\beta} \exp (-x)$, or in other words

$$
\begin{equation*}
\Gamma(\beta ; x)=\beta^{-1}\left(\Gamma(\beta+1 ; x)-x^{\beta} \exp (-x)\right) \tag{A.13}
\end{equation*}
$$

The one-dimensional characteristic function of $X_{1}$ uniquely determines the term

$$
\begin{aligned}
\int_{\mathbb{R}}\left|g_{\xi}(x)\right|^{\beta} \mathrm{d} x & =\int_{0}^{\infty}(1+\theta x)^{\beta} \exp (\lambda \beta x) \mathrm{d} x=\left(\theta \exp \left(-\lambda \theta^{-1}\right)\right)^{\beta} \int_{\theta^{-1}}^{\infty} y^{\beta} \exp (\lambda \beta y) \mathrm{d} y \\
& =-\frac{1}{\lambda \beta}\left(-\frac{\theta \exp \left(-\lambda \theta^{-1}\right)}{\lambda \beta}\right)^{\beta} \Gamma\left(\beta+1 ;-\lambda \beta \theta^{-1}\right)=: c
\end{aligned}
$$

Now, we compute the covariation $\left[X_{1}, X_{0}\right]_{\beta}$ :

$$
\begin{aligned}
{\left[X_{1}, X_{0}\right]_{\beta} } & =\int_{\mathbb{R}} g_{\xi}(x+1) g_{\xi}(x)^{\beta-1} \mathrm{~d} x \\
& =\int_{0}^{\infty}(1+\theta(x+1)) \exp (\lambda(x+1))(1+\theta x)^{\beta-1} \exp (\lambda(\beta-1) x) \mathrm{d} x \\
& =-\frac{1}{\lambda \beta}\left(-\frac{\theta \exp \left(-\lambda \theta^{-1}\right)}{\lambda \beta}\right)^{\beta} \exp (\lambda)\left(\Gamma\left(\beta+1 ;-\lambda \beta \theta^{-1}\right)-\lambda \beta \Gamma\left(\beta ;-\lambda \beta \theta^{-1}\right)\right) \\
& =\exp (\lambda)\left(c(1-\lambda)-\beta^{-1}\right)
\end{aligned}
$$

where we used the formula (A.13). Since $c$ is uniquely determined, the quantity $\left[X_{1}, X_{0}\right]_{\beta}$ identifies the parameter $\lambda$ (note that $-c \lambda-\beta^{-1}>0$, and in particular this term is never equal to 0 ). Condition $(A)(4)$ is shown similarly to the previous example.

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[^0]:    ${ }^{1}$ Similar considerations can be done for the Ornstein-Uhlenbeck kernel, albeit easier and more explicit.

