Universality in low-dimensional three-body systems

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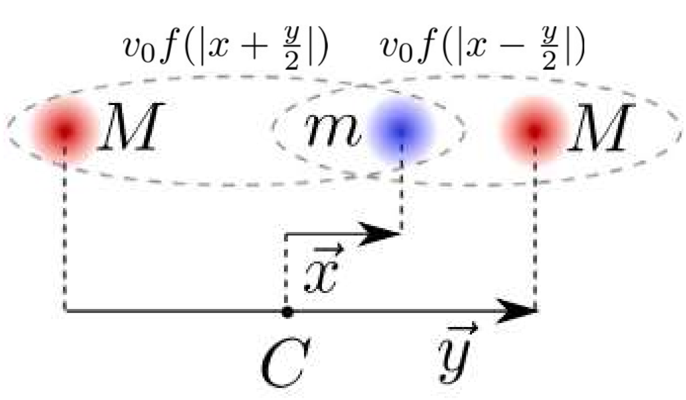
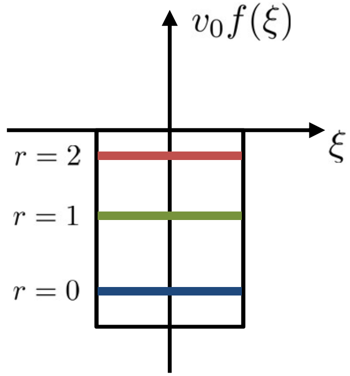
Outline

- ❑ 1D three-body system
- ❑ The ground-state resonance: universality for energy and wave-functions
- ❑ The excited-state resonance: universality for energy and wave-functions
- ❑ Universality in 2D three-body system
- ❑ Summary and open questions

1D three-body problem



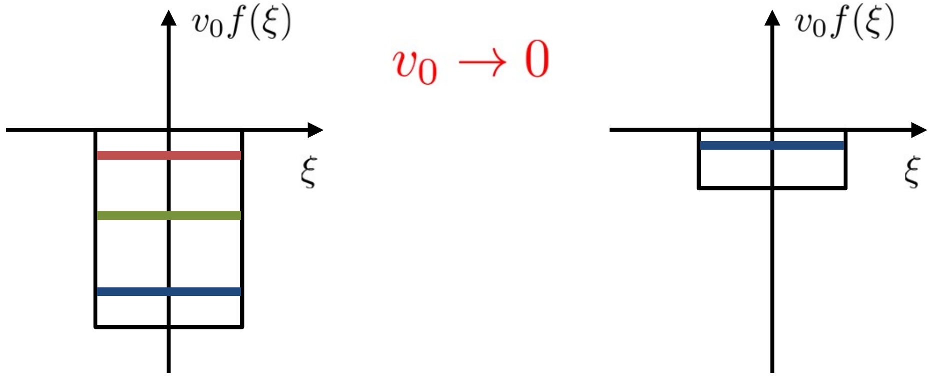


$$\left[-\frac{\alpha_x}{2}\frac{\partial^2}{\partial x^2}-\frac{\alpha_y}{2}\frac{\partial^2}{\partial y^2}+v_0 f(x+\tfrac{y}{2})+v_0 f(x-\tfrac{y}{2})\right]\psi_n=\mathcal{E}_n^{(3)}\psi_n$$

$$\alpha_x=\frac{m+2M}{2(m+M)}\quad \alpha_y=\frac{2m}{m+M}$$

Question: is there *universality*, $f(\xi)$-independence, in the system?

The limit $v_0 \to 0$: $f(\xi) = \delta(\xi)$



Common wisdom:

$$\mathcal{E}_0^{(2)} \to 0$$

$$f(\xi) \simeq \delta(\xi)$$

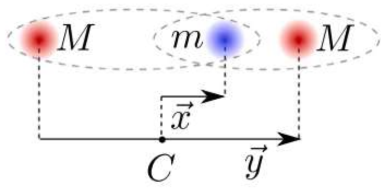
Universal constants:

$$\epsilon_n^\star = \frac{\mathcal{E}_n^{(3)}}{|\mathcal{E}_0^{(2)}|} = \epsilon_n^\star\left(\frac{M}{m}\right)$$

	Atomic mixture (M/m)		
	^{87}Rb–^{40}K (2.2)	^{87}Rb–^{7}Li (12.4)	^{133}Cs–^{6}Li (22.2)
$\epsilon_0^\star$	−2.1966	−2.5963	−2.7515
$\epsilon_1^\star$	−1.0520	−1.4818	−1.6904
$\epsilon_2^\star$		−1.1970	−1.3604
$\epsilon_3^\star$		−1.0377	−1.1479
$\epsilon_4^\star$		−1.0002	−1.0525
$\epsilon_5^\star$			−1.0040

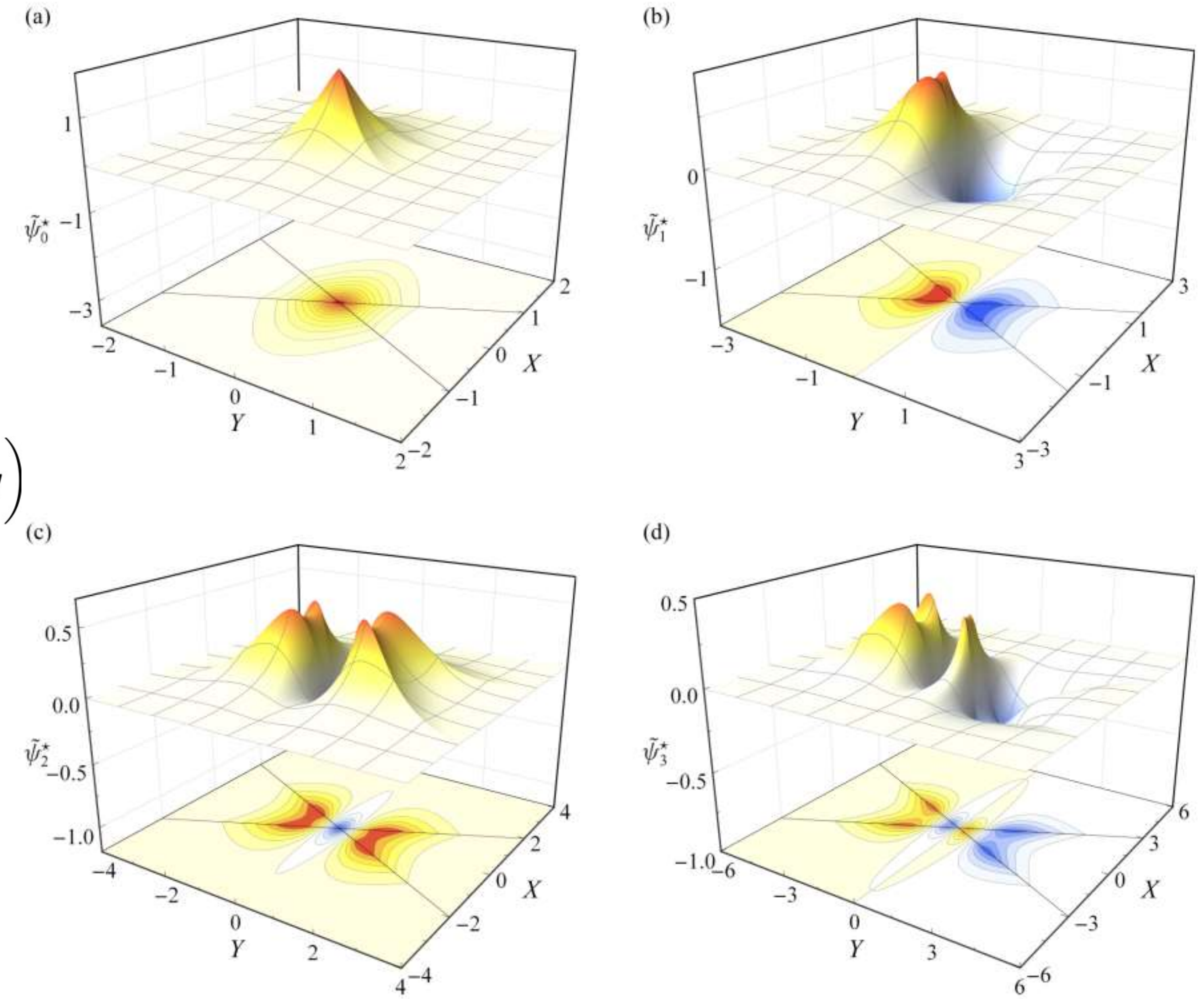
*L Happ et al., Phys. Rev. A **100**, 012709 (2019)*

The limit $v_0 \to 0$: $f(\xi) = \delta(\xi)$



Universal wave functions:

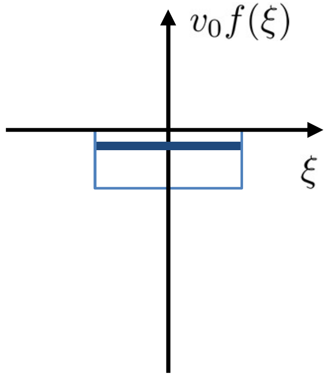
$$\psi_n(x, y) = \psi_n^\star\left(\sqrt{2|\mathcal{E}_0^{(2)}|}\,x, \sqrt{2|\mathcal{E}_0^{(2)}|}\,y\right)$$



L Happ et al., Phys. Rev. A ***100****, 012709 (2019)*

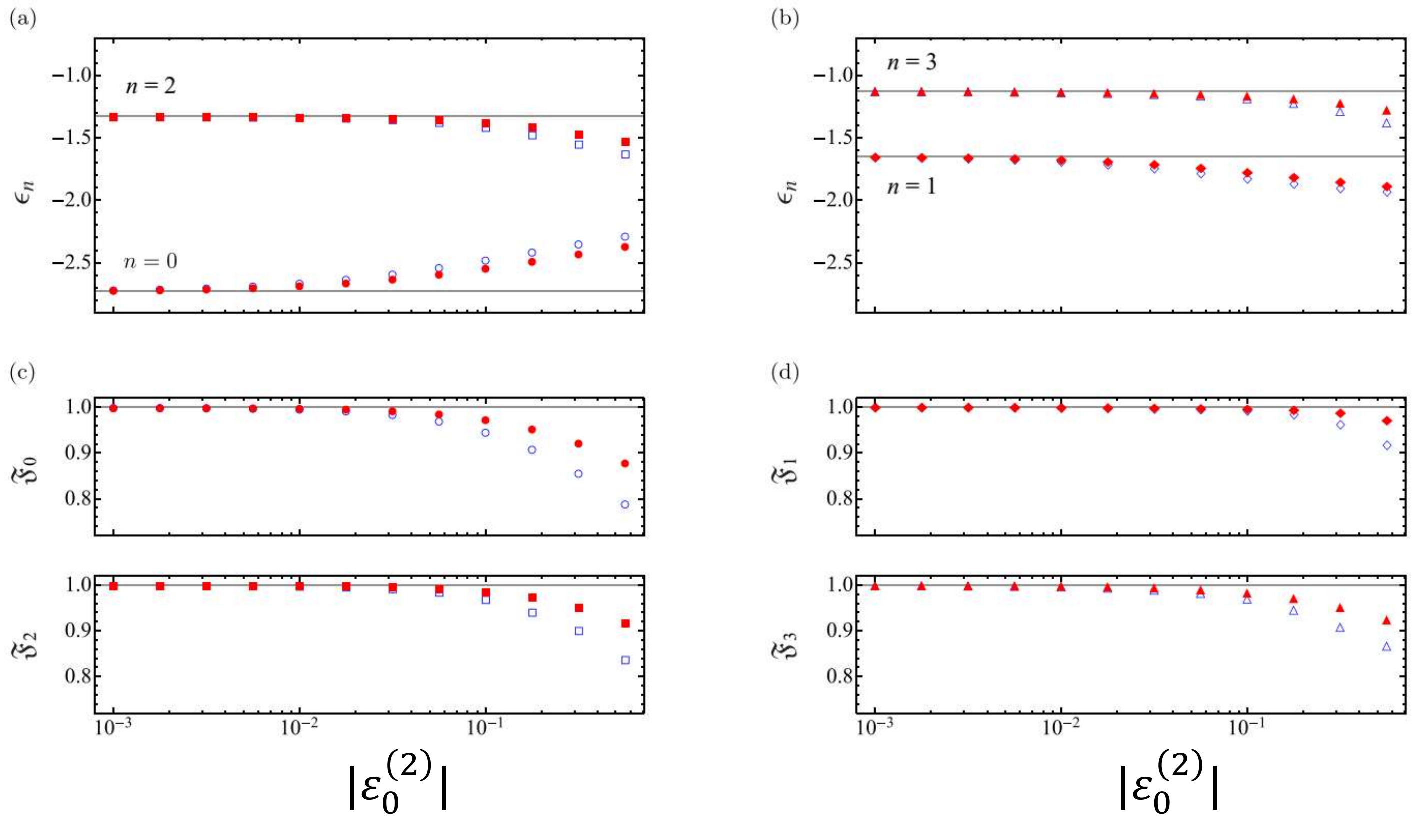
General potential $f(\xi)$

$$\epsilon_n = \frac{\varepsilon_n^{(3)}}{|\varepsilon_0^{(2)}|} \qquad \mathfrak{F}_n = (\langle \psi_n | \psi_n^\star \rangle)^2$$



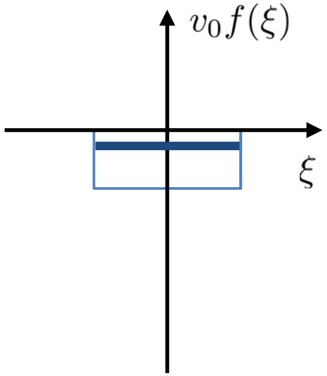
$f_G(\xi) = e^{-\xi^2}$

$f_L(\xi) = \frac{1}{(1+\xi^2)^3}$

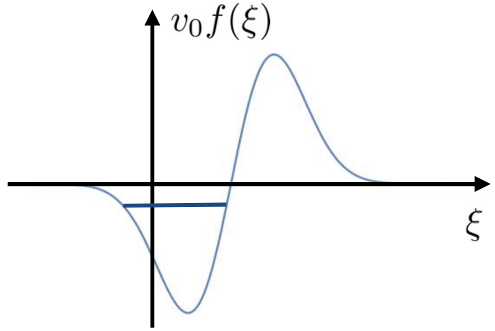


L Happ et al., Phys. Rev. A **100**, 012709 (2019)

The limit $v_0 \to 0$: general $f(\xi)$



$$F(p) \equiv \int e^{-ip\xi} f(\xi) d\xi$$



$$\int v_0 f(\xi) d\xi < 0$$

$$\int v_0 f(\xi) d\xi = 0$$

$$\mathcal{E}_0^{(2)}(v_0 \to 0) = -v_0^2 \, [F(0)]^2$$

$$\mathcal{E}_0^{(2)}(v_0 \to 0) = -v_0^4 \left[\int \frac{dp}{\pi} \frac{|F(p)|^2}{p^2} \right]^2$$

B. Simon, Ann. Phys. ***97****, 279 (1976)*

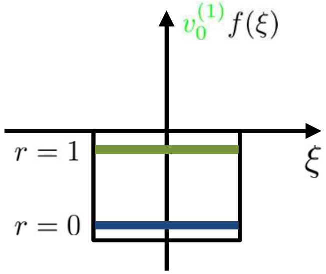
Universality: $(v_0 \to 0)$

$$\mathcal{E}_n^{(3)} = -\epsilon_n^\star |\mathcal{E}_0^{(2)}|$$

$$\psi_n(x, y) = \psi_n^\star \left(\sqrt{2|\mathcal{E}_0^{(2)}|}\, x, \sqrt{2|\mathcal{E}_0^{(2)}|}\, y \right)$$

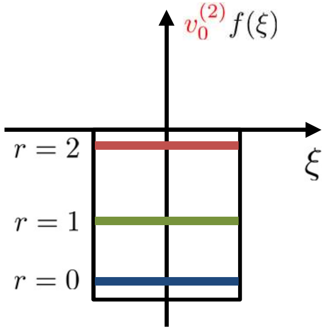
L Happ and M Efremov, J. Phys. B ***54****, 21LT01 (2021)*

1D three-body problem: excited states



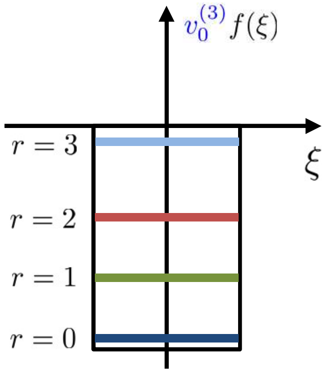
$$\mathcal{E}^{(2)}_{r=1}(v_0^{(1)} \to v_1) \to 0$$

3-body energies
(Faddeev equations): $\mathcal{E}^{(3)}_{n,1}$



$$\mathcal{E}^{(2)}_{r=2}(v_0^{(2)} \to v_2) \to 0$$

3-body energies
(Faddeev equations): $\mathcal{E}^{(3)}_{n,2}$



$$\mathcal{E}^{(2)}_{r=3}(v_0^{(3)} \to v_3) \to 0$$

3-body energies
(Faddeev equations): $\mathcal{E}^{(3)}_{n,3}$

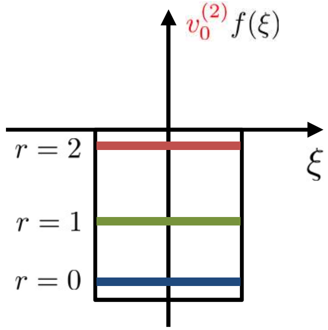
Universality:

$$\frac{\mathcal{E}^{(3)}_{n,r}}{|\mathcal{E}^{(2)}_{r}|} = -\epsilon^{\star}_{n}$$

as for $f^{\star}(\xi) = \delta(\xi)$

L Happ, M Zimmermann and M Efremov, J. Phys. B ***55****, 015301 (2022)*

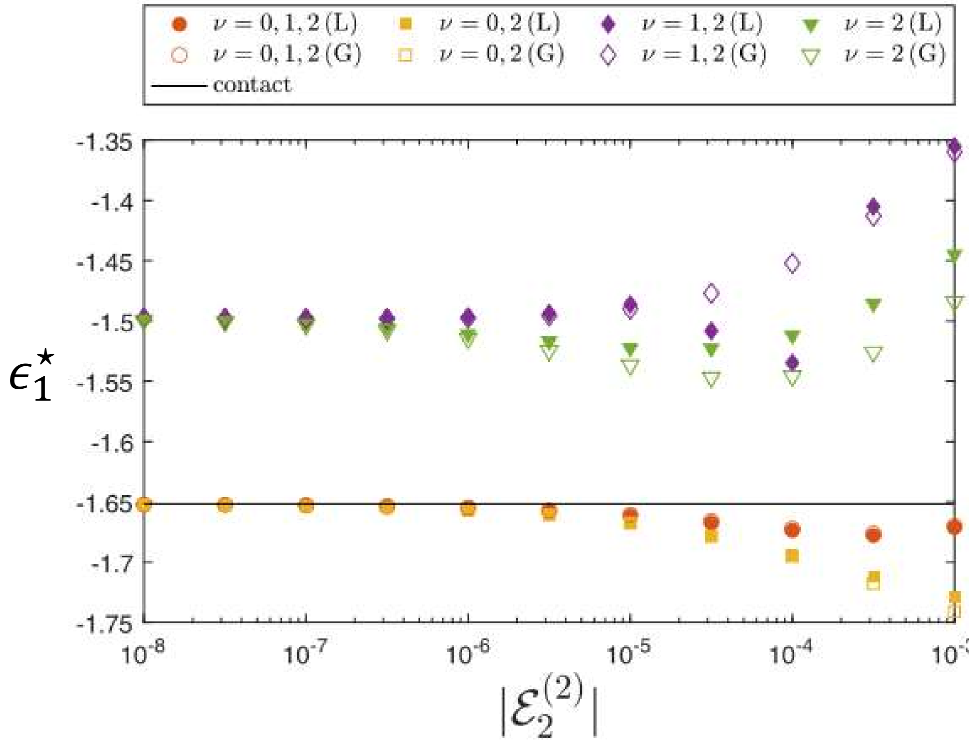
Role of the deeply-bound two-body states



$$\mathcal{E}^{(2)}_{r=2}(v_0^{(2)} \to v_2) \to 0$$

$$f_G(\xi) = e^{-\xi^2}$$

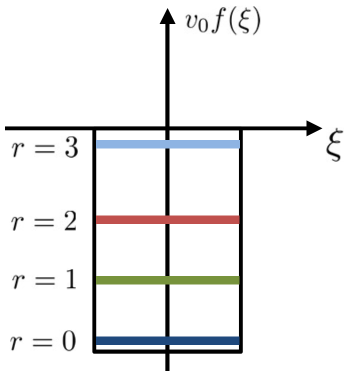
$$f_L(\xi) = \frac{1}{(1+\xi^2)^3}$$

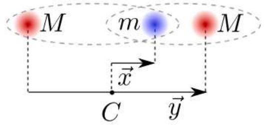


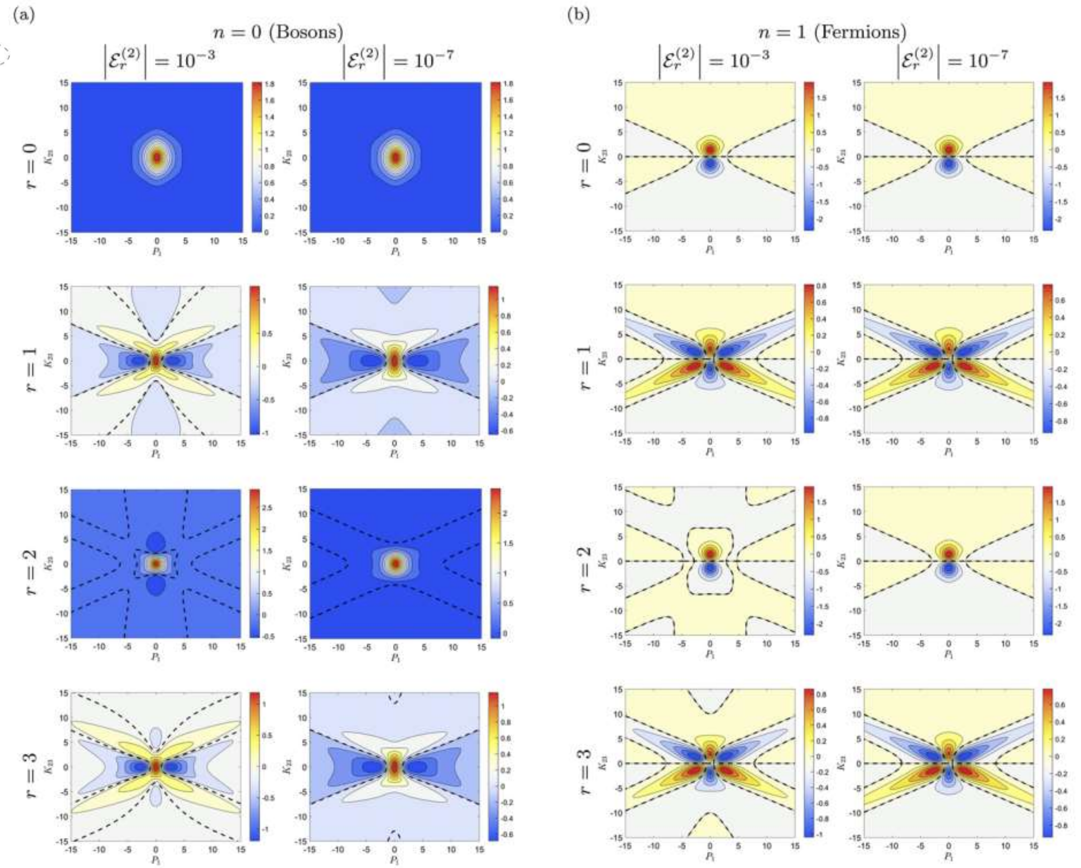
3D case:

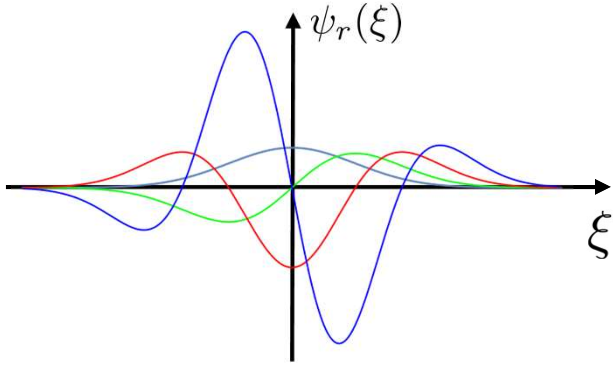
P. M. A. Mestrom, T. Secker, R. M. Kroeze, and S. J. J. M. F. Kokkelmans, Phys. Rev. A ***99****, 012702 (2019)*

1D three-body problem: excited states









Universality for even or odd *r* : $\mathcal{E}^{(3)}_{n,r} = -\epsilon^{\star}_{n} |\mathcal{E}^{(2)}_{r}|,\ \psi_{n,2l}(x,y) = \psi^{\star}_{n}\left(\sqrt{2|\mathcal{E}^{(2)}_{2l}|}\,x, \sqrt{2|\mathcal{E}^{(2)}_{2l}|}\,y\right)$

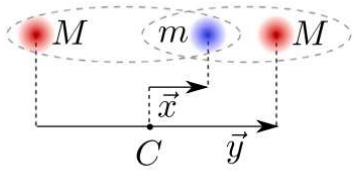
$$\psi_{n,2l+1}(x,y) = \psi_{n,2l+3}(x,y)$$

Summary and outlook

1D *M-m-M* system:

- If ***M-m subsystem*** has a shallow **ground** state:

$$\text{As}\quad \mathcal{E}_0^{(2)} \to 0: \quad \mathcal{E}_{n,0}^{(3)} = -\epsilon_n^\star |\mathcal{E}_0^{(2)}|,\ \psi_{n,0}(x,y) = \psi_n^\star\left(\sqrt{2|\mathcal{E}_0^{(2)}|}\,x, \sqrt{2|\mathcal{E}_0^{(2)}|}\,y\right)$$



- If ***M-m subsystem*** has a shallow **excited** state:

$$\text{As}\quad \mathcal{E}_{2l}^{(2)} \to 0: \quad \mathcal{E}_{n,2l}^{(3)} = -\epsilon_n^\star |\mathcal{E}_{2l}^{(2)}|,\ \psi_{n,2l}(x,y) = \psi_n^\star\left(\sqrt{2|\mathcal{E}_{2l}^{(2)}|}\,x, \sqrt{2|\mathcal{E}_{2l}^{(2)}|}\,y\right)$$

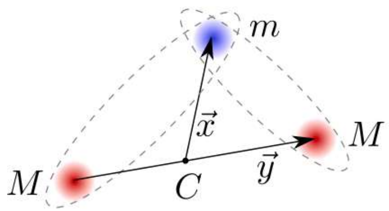
$$\text{As}\quad \mathcal{E}_{2l+1}^{(2)} \to 0: \quad \mathcal{E}_{n,2l+1}^{(3)} = -\epsilon_n^\star |\mathcal{E}_{2l+1}^{(2)}|,\ \psi_{n,2l+1}(x,y) = ?$$

$$f^\star(\xi) = \frac{d}{d\xi}\delta(\xi)?$$

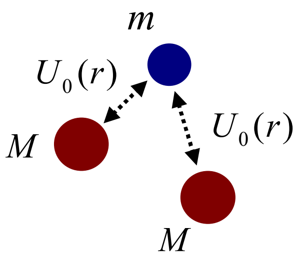
2D *M-m-M* system:

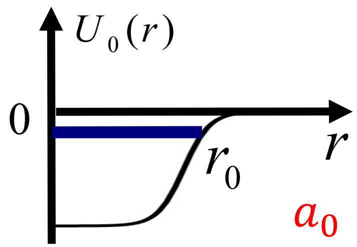
- If ***M-m subsystem*** has a shallow **ground** state: *universality for energies and wave-functions*

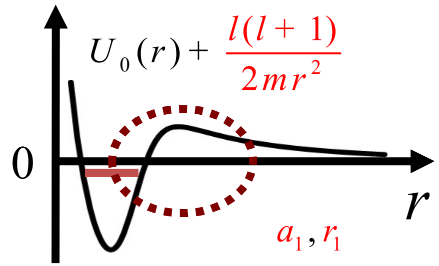
*J Thies, MT Hof, M Zimmermann and M Efremov, J. Comp. Science **64**, 101859 (2022)*



Summary and outlook







d	$l = 0$	$l = 1$
2	$E_n \sim \frac{1}{(n-\delta)^2}$; $N_b < \infty$ $V_{\text{eff}}(R) \sim -\frac{1}{mR}$; $r_0 \ll R \ll a_0^{(2)}$	1) $L = 0$: $E_n \sim ?$; $N_b < \infty$ 2) $L = \pm 1$: $E_n \sim \exp\left(-2\exp(2\pi \frac{m}{M} n + \theta)\right)$ $N_b = \infty$
3	$E_n \sim \exp(-\alpha n)$; $N_b = \infty$ $V_{\text{eff}}(R) \sim -\frac{1}{mR^2}$	$E_n \sim (n - n_*)^6$; $N_b < \infty$ $V_{\text{eff}}(R) \sim -\frac{1}{mR^3}$

*F F Belotti et al., J. Phys. B **46**, 055301 (2013)*

*S Moroz and Y Nishida, PRA **90**, 063631 (2014)*
*M Efremov et al., PRL **111**, 113201 (2013)*
*P M A Mestrom et al., PRA **103**, L051303 (2021)*

Many thanks for your attention

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