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A Momentum Subspace-based Model Order Reduction for Finite Element Models in Nonlinear Dynamic Analyses

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- Introduction
- Theoretical formulation
- Adaptation for dynamics
- Results and Discussions
- Conclusions and Outlook

Introduction

- Structural nonlinearities is of interest in several engineering fields.
- Full finite element (FE) solution can be computationally expensive.
- Reduced order models (ROM) provide an efficient solution to such problems.
- The momentum subspace ROM, discussed here, is an adaptation of the Koiter-Newton reduction technique (K. Liang et al., 2013).
- Extended to dynamics (Sinha et al., 2020) with focus on panel structures.
- Current studies on cantilevers.

Theoretical Formulation

- The equilibrium equations (statics) are expanded up to the third order in Taylor series.

$$f(\boldsymbol{u}) = \mathcal{L}(\boldsymbol{u}) + Q(\boldsymbol{u},\boldsymbol{u}) + C(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}) = \boldsymbol{f}_{\mathbf{ext}} = \mathbf{F}\boldsymbol{\phi} \qquad (1)$$

- The equilibrium displacement $\boldsymbol{u}$ is parametrised by generalized displacements ξ.

$$\mathbf{u}(\xi) = \mathbf{u}_{\boldsymbol{\alpha}}\xi_\alpha + \mathbf{u}_{\boldsymbol{\alpha\beta}}\xi_\alpha\xi_\beta \qquad (2)$$

- In the reduced subspace, a similar assumption is made for the equilibrium equation.

$$\bar{\mathcal{L}}(\boldsymbol{\xi}) + \bar{Q}(\boldsymbol{\xi},\boldsymbol{\xi}) + \bar{C}(\boldsymbol{\xi},\boldsymbol{\xi},\boldsymbol{\xi}) = \phi \qquad (3)$$

- Work equivalence to fix the parametrisation.

$$(\mathbf{F}\boldsymbol{\phi})' \, . \, \delta\mathbf{u} = \boldsymbol{\phi}' \, . \, \delta\xi \qquad (4)$$

Theoretical Formulation

- By regrouping the coefficients of ξ, a set of ROM equations are obtained.

$$\begin{bmatrix} \mathbf{L} & -\mathbf{F} \\ -\mathbf{F} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_\alpha \\ \bar{\mathbf{L}}_\alpha \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{E}_\alpha \end{Bmatrix}$$

$$\mathbf{u}(\xi) = \mathbf{u}_\alpha \xi_\alpha + \mathbf{u}_{\alpha\beta} \xi_\alpha \xi_\beta$$

$$\begin{bmatrix} \mathbf{L} & -\mathbf{F} \\ -\mathbf{F} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\alpha\beta} \\ \bar{\mathbf{Q}}_{\alpha\beta} \end{Bmatrix} = \begin{Bmatrix} -\mathrm{Q}(\mathbf{u}_\alpha, \mathbf{u}_\beta) \\ \mathbf{0} \end{Bmatrix}$$

$$\bar{\boldsymbol{C}}_{\alpha\beta\gamma\delta} = \boldsymbol{C}(\boldsymbol{u}_\alpha, \boldsymbol{u}_\beta, \boldsymbol{u}_\gamma, \boldsymbol{u}_\delta) - \frac{2}{3}\left[\boldsymbol{u}^t{}_{\alpha\beta}\, \boldsymbol{L}(\boldsymbol{u}_{\delta\gamma}) + \boldsymbol{u}^t{}_{\beta\gamma}\, \boldsymbol{L}(\boldsymbol{u}_{\delta\alpha}) + \boldsymbol{u}^t{}_{\gamma\alpha}\, \boldsymbol{L}(\boldsymbol{u}_{\delta\beta})\right]$$

- The stiffness tensors are obtained as higher order derivatives of strain energy.

Adaptations for dynamics

- Full FE equations described by

$$\boldsymbol{M}\ddot{\mathbf{u}} + \boldsymbol{D}\dot{\mathbf{u}} + \boldsymbol{L}\mathbf{u} + \boldsymbol{Q}\mathbf{uu} + \boldsymbol{C}\mathbf{uuu} = \mathbf{F}(t)$$

- 1st order differential equations in order to perform parametric continuation (AUTO, Doedel, 2007)

- Hamiltonian formulation to derive the equations of motion.

$$H(\boldsymbol{u},\boldsymbol{p}) = T(\boldsymbol{u},\boldsymbol{p}) + V(\boldsymbol{u})$$

- Conservative system - damping and external force excluded initially.

- An assumption is made for the momentum:

$$\mathbf{p} = \mathbf{P}\pi, \quad \mathbf{P} = \mathbf{M\Phi}$$

where P is the basis matrix, π is a vector of amplitudes for the momentum vectors.

Adaptations for dynamics

- Potential energy in the reduced subspace:

$$\bar{V} = \frac{1}{2}\bar{L}_{\alpha\beta}\xi_\alpha\xi_\beta + \frac{1}{3}\bar{Q}_{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma + \frac{1}{4}\bar{C}_{\alpha\beta\gamma\delta}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta$$

- Kinetic energy in the reduced subspace:

$$\bar{T} = \frac{1}{2}\,\pi'\,\underbrace{(\boldsymbol{\Phi}'\mathbf{M}\,\boldsymbol{\Phi})}_{\bar{\mathbf{M}}^{-1}}\,\pi$$

- For a conservative system,

$$\dot{\boldsymbol{\xi}} = \frac{\partial\bar{H}}{\partial\boldsymbol{\pi}} = \bar{\mathbf{M}}^{-1}\boldsymbol{\pi}$$

$$\dot{\boldsymbol{\pi}} = \frac{\partial\bar{H}}{\partial\boldsymbol{\xi}} = -\{\bar{\boldsymbol{L}}\boldsymbol{\xi} + \bar{\boldsymbol{Q}}\boldsymbol{\xi}\boldsymbol{\xi} + \bar{\boldsymbol{C}}\boldsymbol{\xi}\boldsymbol{\xi}\boldsymbol{\xi}\}$$

Adaptations for dynamics

- Potential energy in the reduced subspace:

$$\bar{V} = \frac{1}{2}\bar{L}_{\alpha\beta}\xi_\alpha\xi_\beta + \frac{1}{3}\bar{Q}_{\alpha\beta\gamma}\xi_\alpha\xi_\beta\xi_\gamma + \frac{1}{4}\bar{C}_{\alpha\beta\gamma\delta}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta$$

- Kinetic energy in the reduced subspace:

$$\bar{T} = \frac{1}{2}\pi' \underbrace{(\boldsymbol{\Phi}'\mathbf{M}\,\boldsymbol{\Phi})}_{\bar{\mathbf{M}}^{-1}} \pi$$

- For a non-conservative system,

$$\dot{\boldsymbol{\xi}} = \frac{\partial \bar{H}}{\partial \boldsymbol{\pi}} = \bar{\mathbf{M}}^{-1}\boldsymbol{\pi}$$

$$\dot{\boldsymbol{\pi}} = -\frac{\partial \bar{H}}{\partial \boldsymbol{\xi}} = -\{\bar{\mathbf{L}}\boldsymbol{\xi} + \bar{\mathbf{Q}}\boldsymbol{\xi}\boldsymbol{\xi} + \bar{\mathbf{C}}\boldsymbol{\xi}\boldsymbol{\xi}\boldsymbol{\xi}\} - \bar{\mathbf{D}}\bar{\mathbf{M}}^{-1}\boldsymbol{\pi} + \bar{\boldsymbol{\phi}}(t)$$

Rayleigh damping

Quadratic damping model

Discussion of results

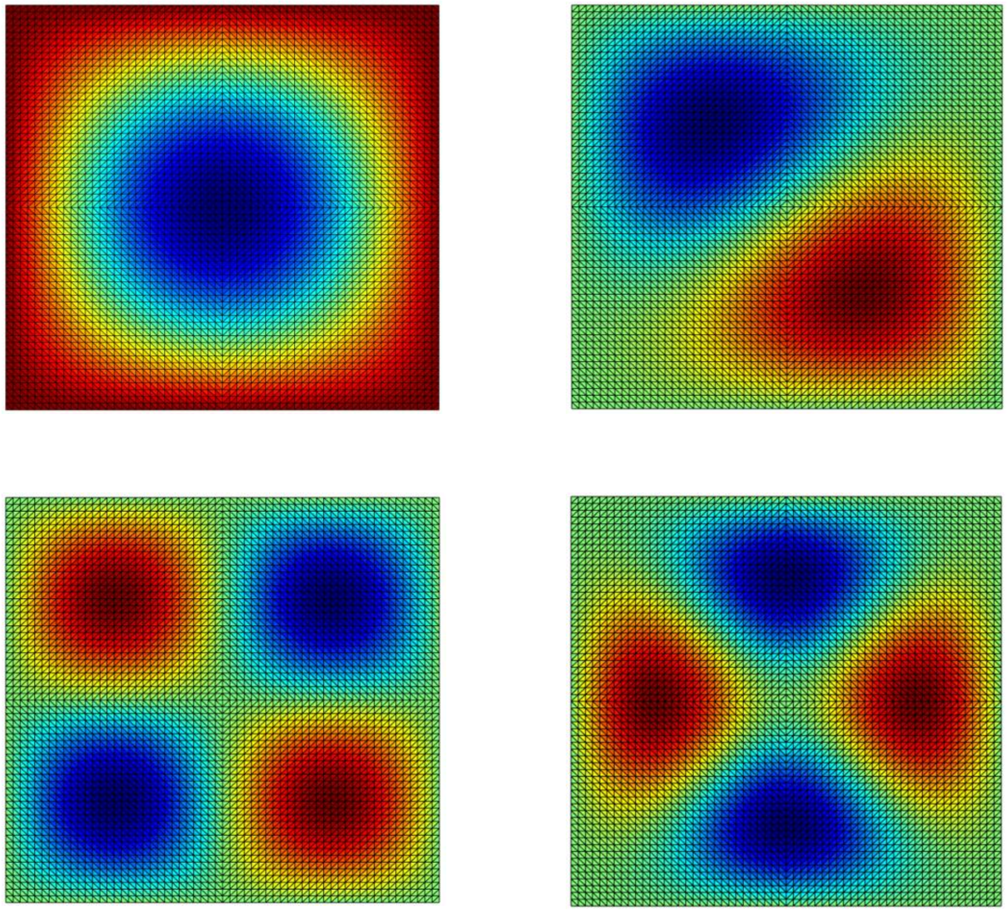
- Test case description

1. Test case 1 : Simply supported square plate (M. Amabili, 2004).
2. Test case 2 : Stiffened plate with free boundary conditions (Sinha et al, 2020).
3. Test case 3 : Ongoing studies, cantilever beam.

Discussion of results

Test case 1: Rectangular plate, simply supported (M. Amabili, 2004)

Pre-processing: Linear modes analysis



Analysis parameters:

l = b = $0.3\ \mathrm{m}$

$\mathrm{t} = 0.001\ \mathrm{m}$

Damping ratio $\zeta = 0.065$

Applied force $f_{ext} = 1.74\ \mathrm{N}$ (centre of the plate)

Mesh size = 60 x 60

Material parameters:

E = $70\ \mathrm{GPa}$

$\rho = 2778\,\mathrm{kg/m^3}$

Discussion of results

Test case 1: Rectangular plate, simply supported (M. Amabili, 2004)

Comparison to full FE solution in time domain simulation

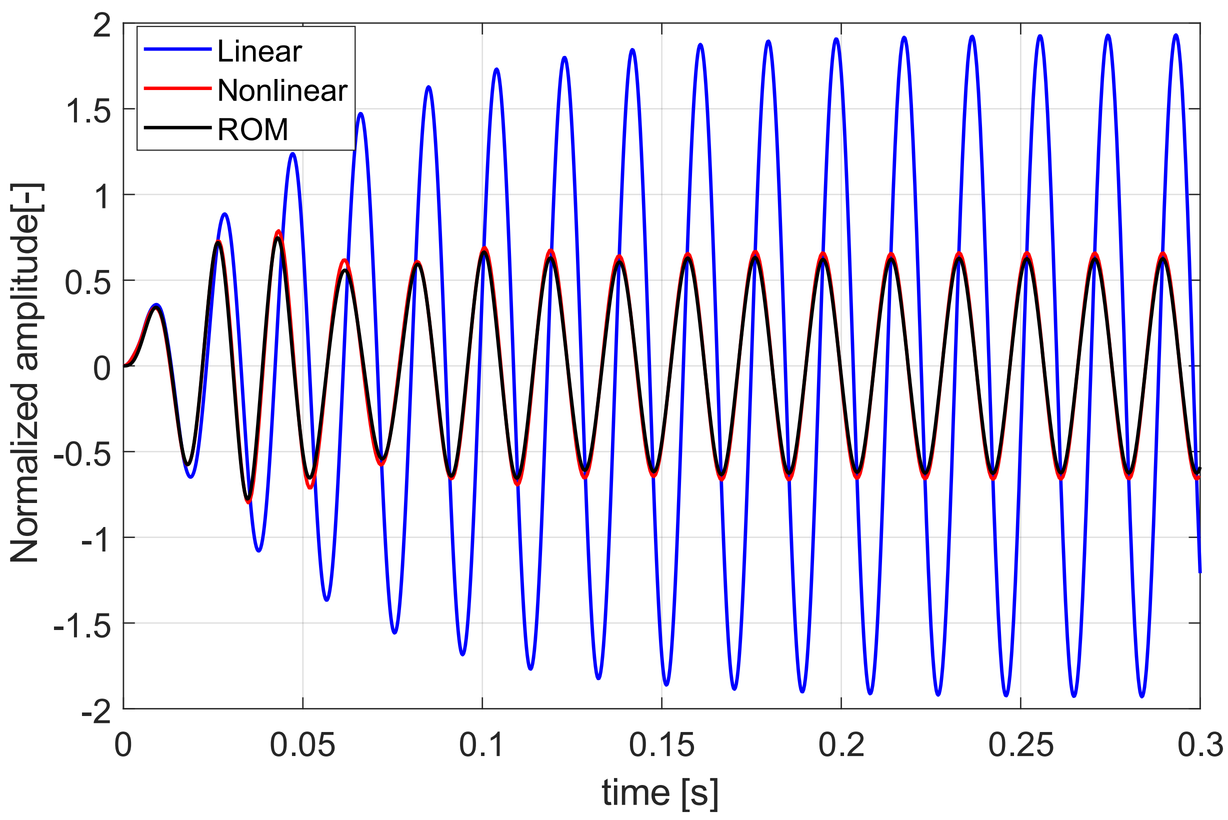
Excitation at ~ 0.997 . ω_1

Full FE solution time = 274.6 sec

ROM solution time = 3.9 sec

ROM pre-processing = 8.4 sec
includes formulation of ROM parameters, stiffness tensors and modal eigenvalue analysis

Total code run-time = 12.3 sec



Discussion of results

Test case 1: Rectangular plate, simply supported (M. Amabili, 2004)

Analysis using the software AUTO (Doedel, 2007)

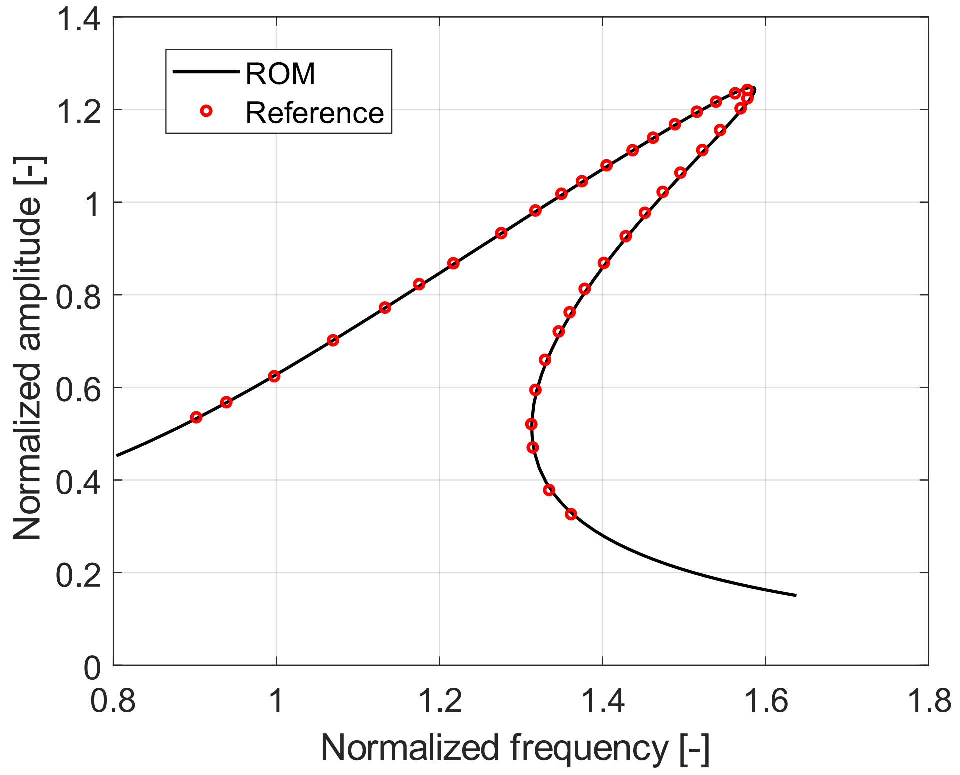
Linear modal analysis (pre-processing) = 0.65 sec

ROM formulation time = 0.99 sec

AUTO analysis = 5.59 sec (1560 data points along the solution curve)

Difference from reference solution (Amabili, 2004) = 0.43 %

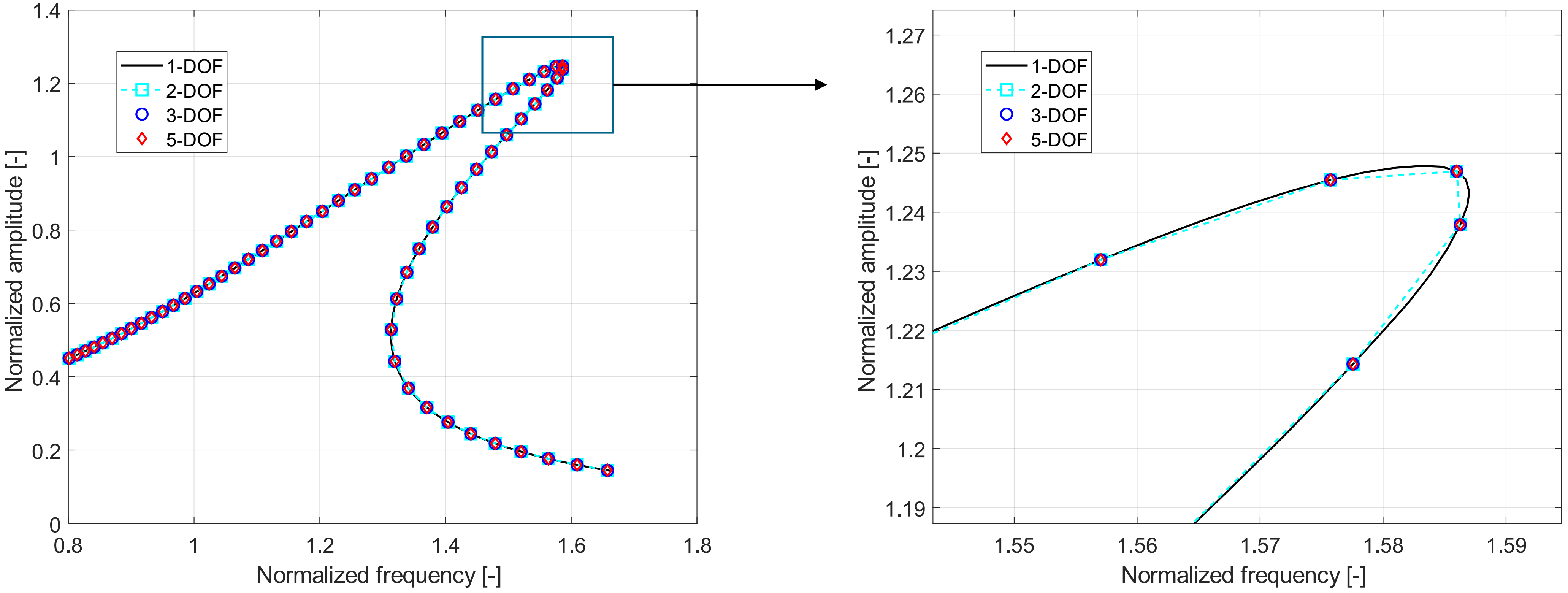
1-DOF model



Discussion of results

Test case 1: Rectangular plate, simply supported (M. Amabili, 2004)

Convergence study - increase the number of modes in the reduction subspace.



Discussion of results

Test case 2: Stiffened plate with free boundary conditions (Sinha et al, 2020).

Analysis parameters:

$\mathrm{l} = 0.5\ \mathrm{m},\ \ \mathrm{b} = 0.4\ \mathrm{m}\ ,\ \mathrm{t} = 0.002\ \mathrm{m}$

$\mathrm{ls} = 0.4\ \mathrm{m},\ \ \mathrm{bs} = 0.008\ \mathrm{m}\ ,\ \mathrm{ts} = 0.005\ \mathrm{m}\ (\mathrm{stiffener})$

Damping ratio $\zeta = 0.0012$ (initial guess)

Applied force $f_{ext} = 0.2 - 1$ N

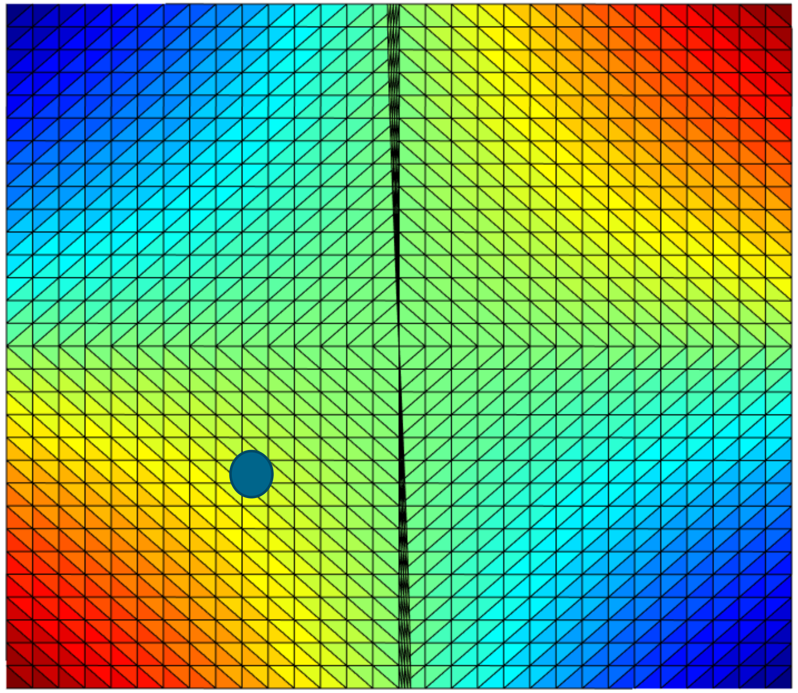
Excitation frequency = 32 – 40 Hz (sweep)

Material parameters:

E = 70 GPa

$\rho = 2660\ \mathrm{kg/m^3}$

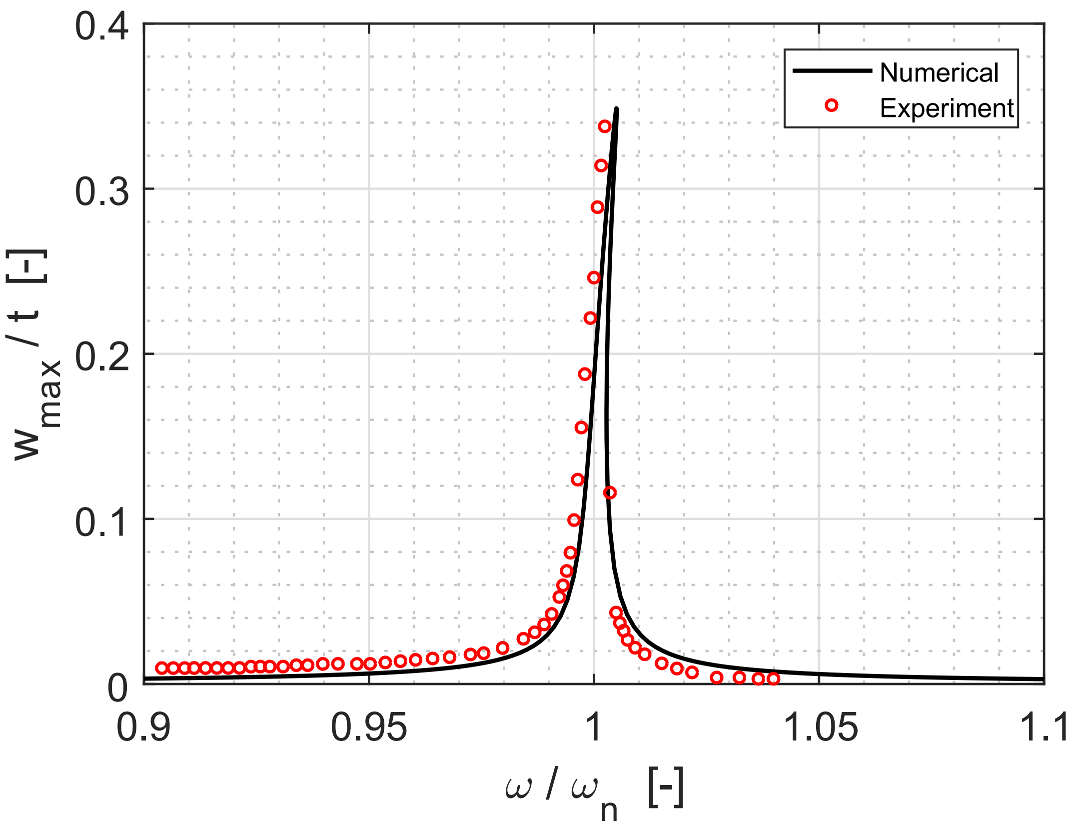
1st elastic mode at 36.78 Hz



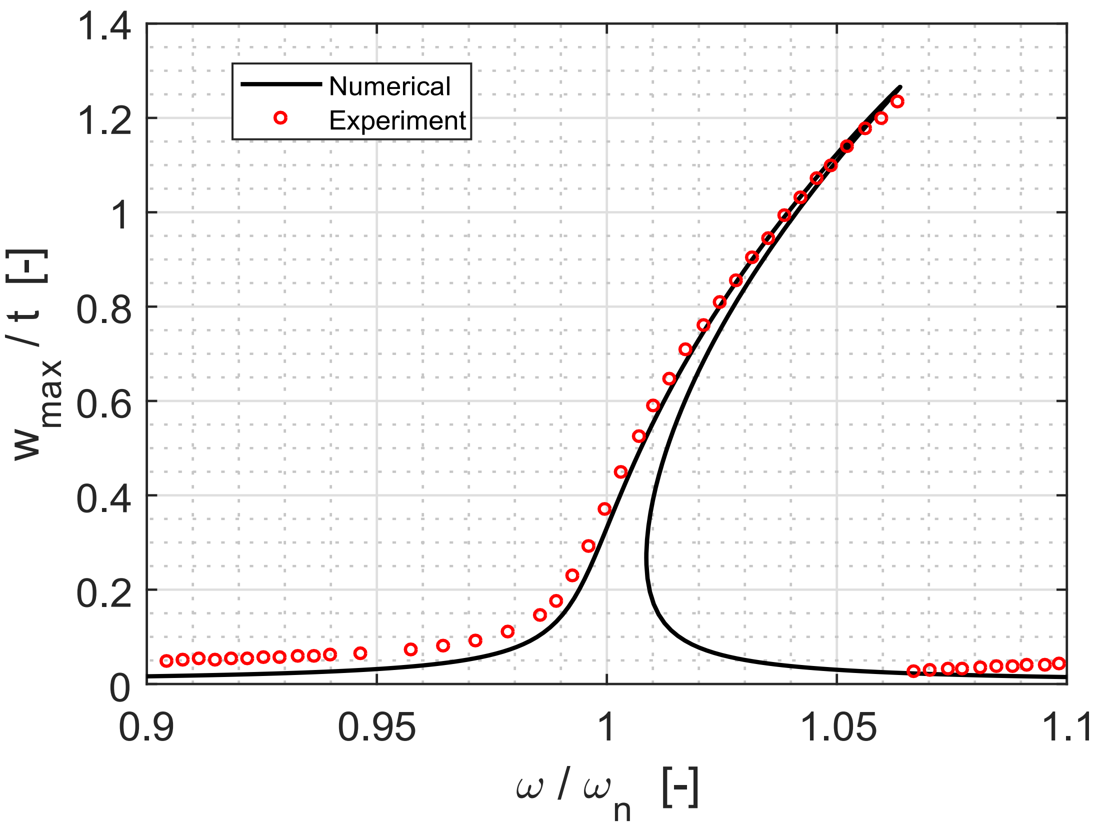
Excitation at (x, y) = (0.2, 0.16 m)

Discussion of results

Test case 2: Stiffened plate with free boundary conditions (Sinha et al, 2020).



0.2 N



1.2 N

Discussion of results

Test case 3: Ongoing studies, cantilever beam (Pany and Rao, 2002)

1-cosine profile

Analysis parameters:

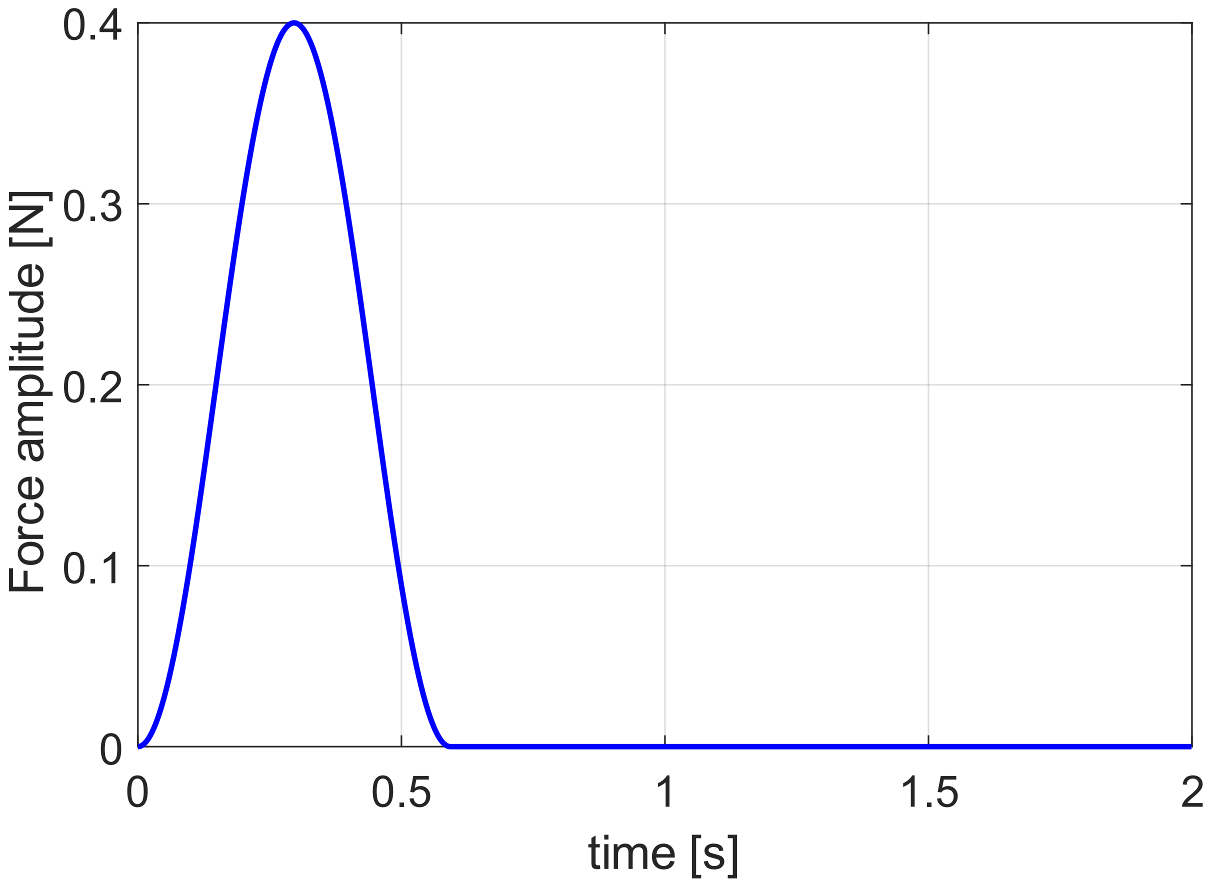
l = 0.693166 m

$t = 0.001$ m

Damping ratio $\zeta = 0.0467$

Applied force $f_{ext} = 0.2$ N

Excitation frequency = 10.6 rad/s



Material parameters:

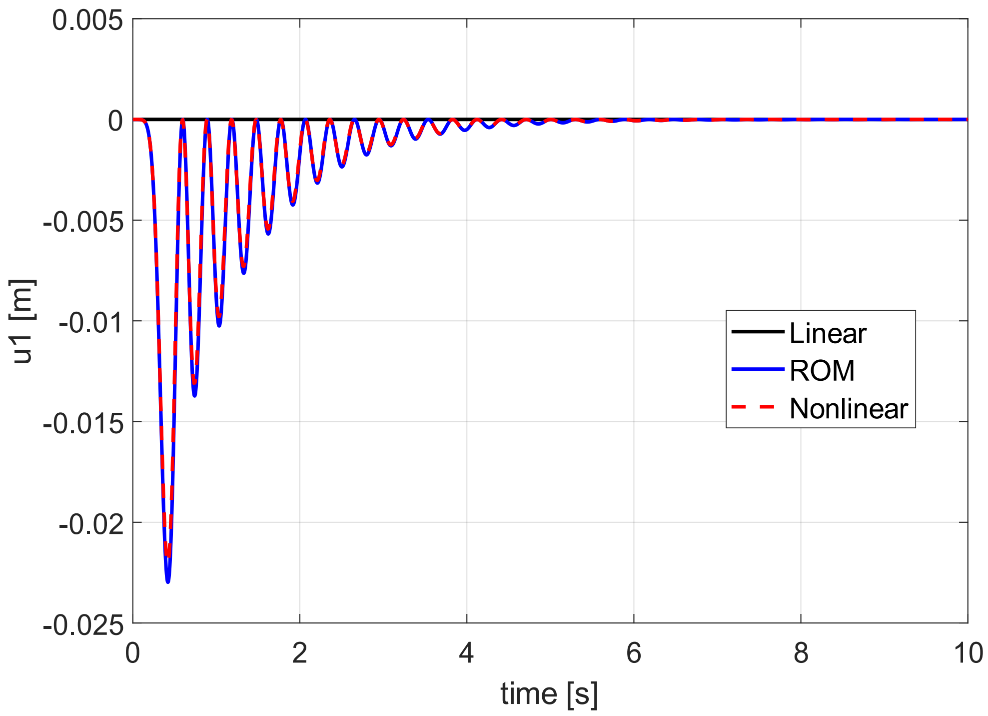
E = 200 GPa

$\rho = 7800\ \mathrm{kg/m^3}$

Discussion of results

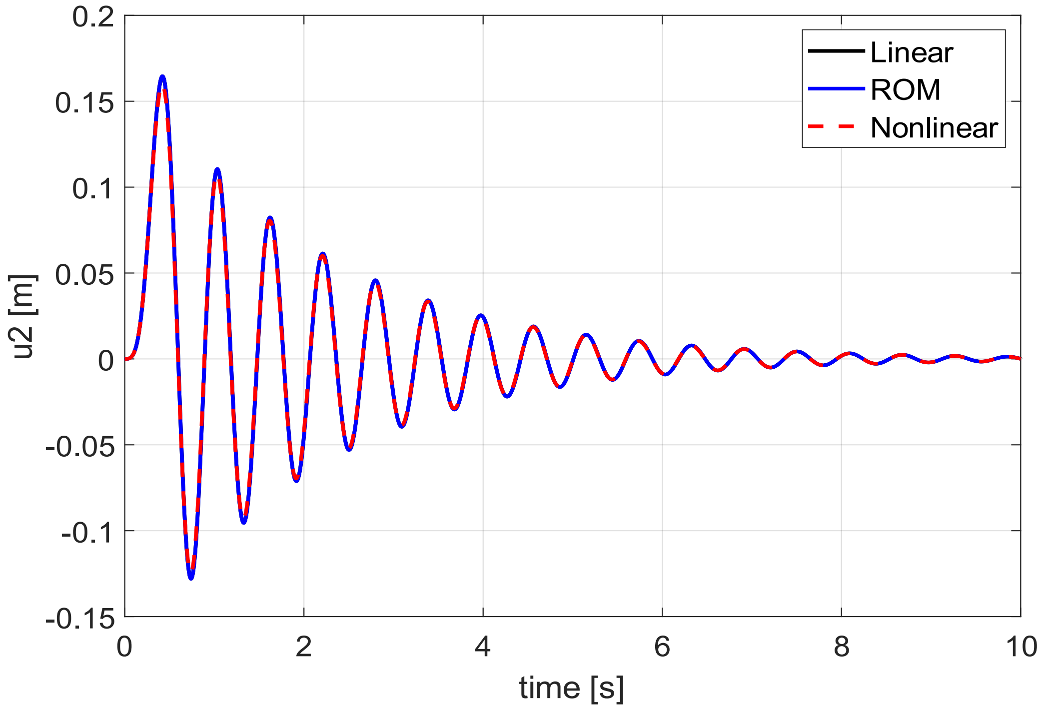
Test case 3: Ongoing studies, cantilever beam (Pany and Rao, 2002)

In-plane (IP) displacement



3.2 % (of length) maximum IP deflection

Out-of-plane (OOP) displacement



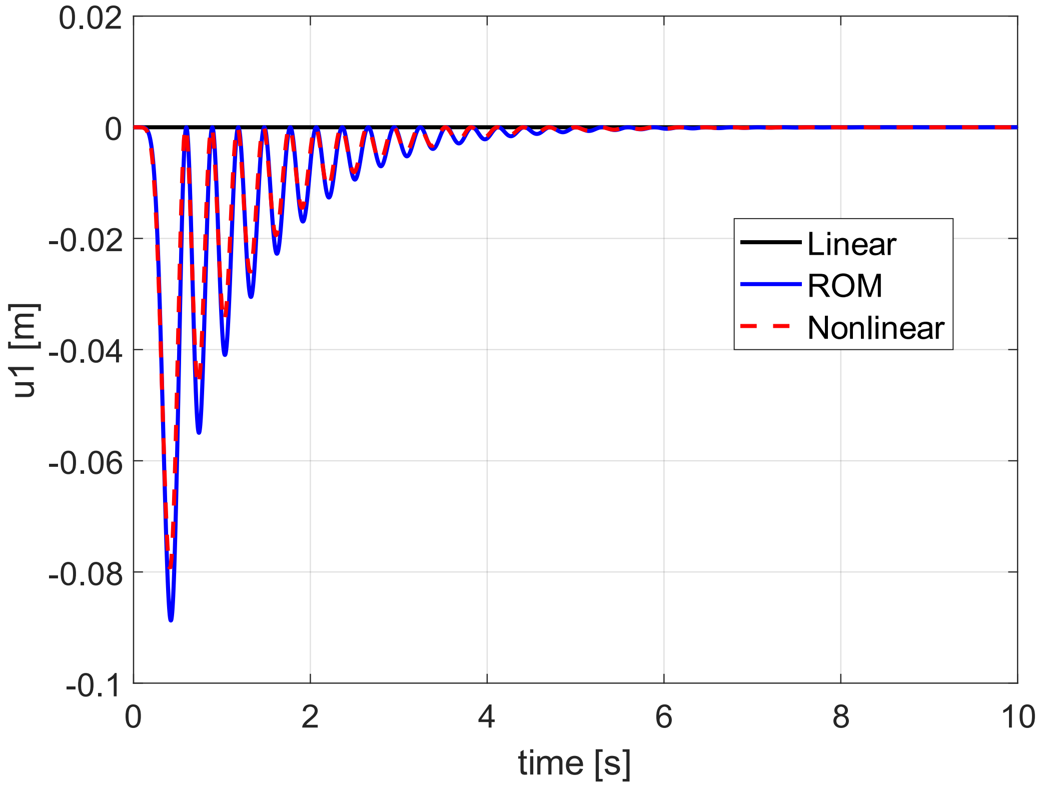
23.5 % (of length) maximum OOP deflection

Simulation time (4-DOF ROM) = 2.4 sec, simulation time (full Nonlinear) = 223 sec

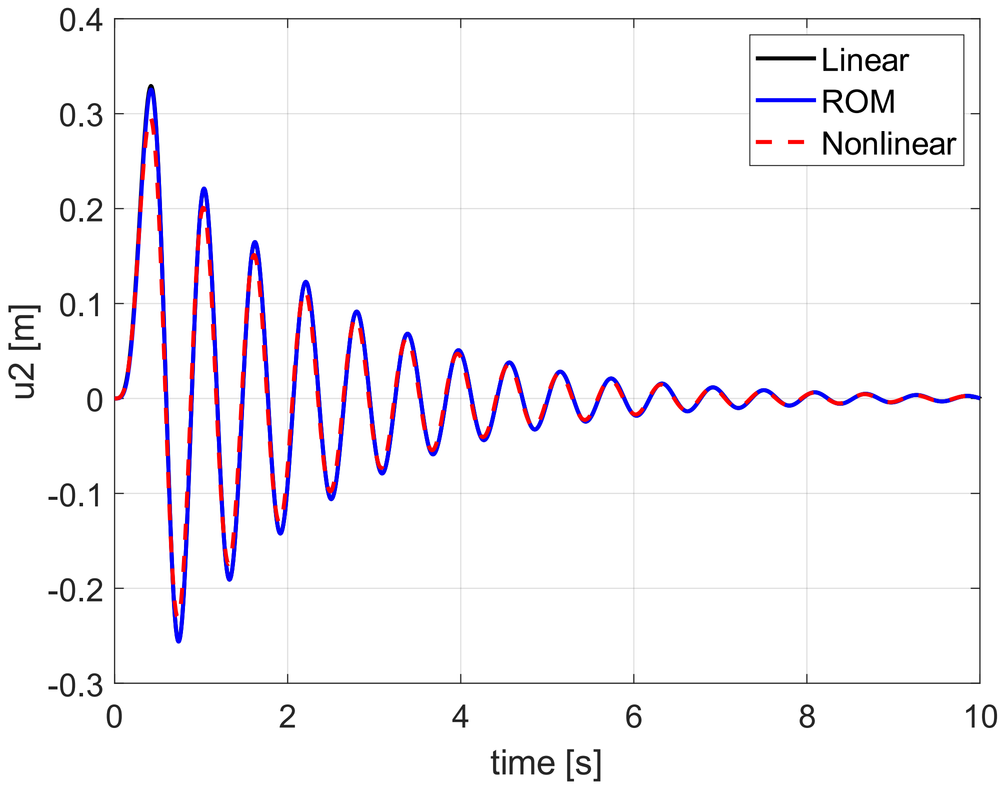
Discussion of results

Test case 3: Ongoing studies, cantilever beam (Pany and Rao, 2002)

Double the force amplitude



11.5 % (of length) maximum IP deflection



42.7 % (of length) maximum OOP deflection

Outlook and Conclusions

- ROM works well for various boundary conditions.
- Experiments show us a need of nonlinear damping model.
- Limited region of ROM validity.
- Ongoing studies aim to extend the limit of validity for larger deflections, specially in cantilevers.
- Intended application towards large scale model reduction of generic FE models.

Thank you for your attention !

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Extra

- Orthogonality conditions derived from constraint equations:

$$\mathrm{f}'_{\alpha}\, u_{\beta} \;=\; \delta_{\alpha\beta}$$
$$\mathrm{f}'_{\alpha}\, u_{\beta\gamma} = 0$$

- Deriving the reduced force from conditions of work equivalence:

$$(\mathbf{F}\boldsymbol{\phi})' \,.\, \delta\mathbf{u} = \boldsymbol{\phi}' \,.\, \delta\xi$$

Substitute for u:

$$\mathbf{u}(\xi) = \mathbf{u}_{\boldsymbol{\alpha}}\xi_{\alpha} + \mathbf{u}_{\boldsymbol{\alpha\beta}}\xi_{\alpha}\xi_{\beta}$$

With use of the orthogonality constraints we get, $\boldsymbol{\phi} = \mathbf{f}_{\mathbf{ext}}\ \mathbf{u}_{\boldsymbol{\alpha}}$

Extra

- Dissipation energy

$$E_d = \frac{1}{2}\ \dot{\boldsymbol{u}}'\ \mathbf{D}\ \dot{\boldsymbol{u}} = \frac{1}{2}\ (\boldsymbol{M}^{-1}\mathbf{P}\boldsymbol{\pi})'\ \mathrm{D}\ (\boldsymbol{M}^{-1}\mathbf{P}\boldsymbol{\pi})$$

$$E_d = \frac{1}{2}\ \dot{\boldsymbol{\xi}}\ (\bar{\boldsymbol{M}}\ \mathbf{P}'\boldsymbol{M}^{-1}\ \mathbf{D}\ \boldsymbol{M}^{-1}\mathbf{P}\ \bar{\boldsymbol{M}})\ \dot{\boldsymbol{\xi}}$$

- von Karman Strain, beam element

$$\epsilon = u_{,x} + \frac{1}{2}(u_{,x}^2 + w_{,x}^2)$$

$$\chi = w_{,xx}$$

Discussion of results

Test case 3: Ongoing studies, cantilever beam (Pany and Rao, 2002)

Analysis parameters:

l = 0.693166 m

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Damping ratio $\zeta = 0.0467$

Applied force $f_{ext} = 0.2$ N

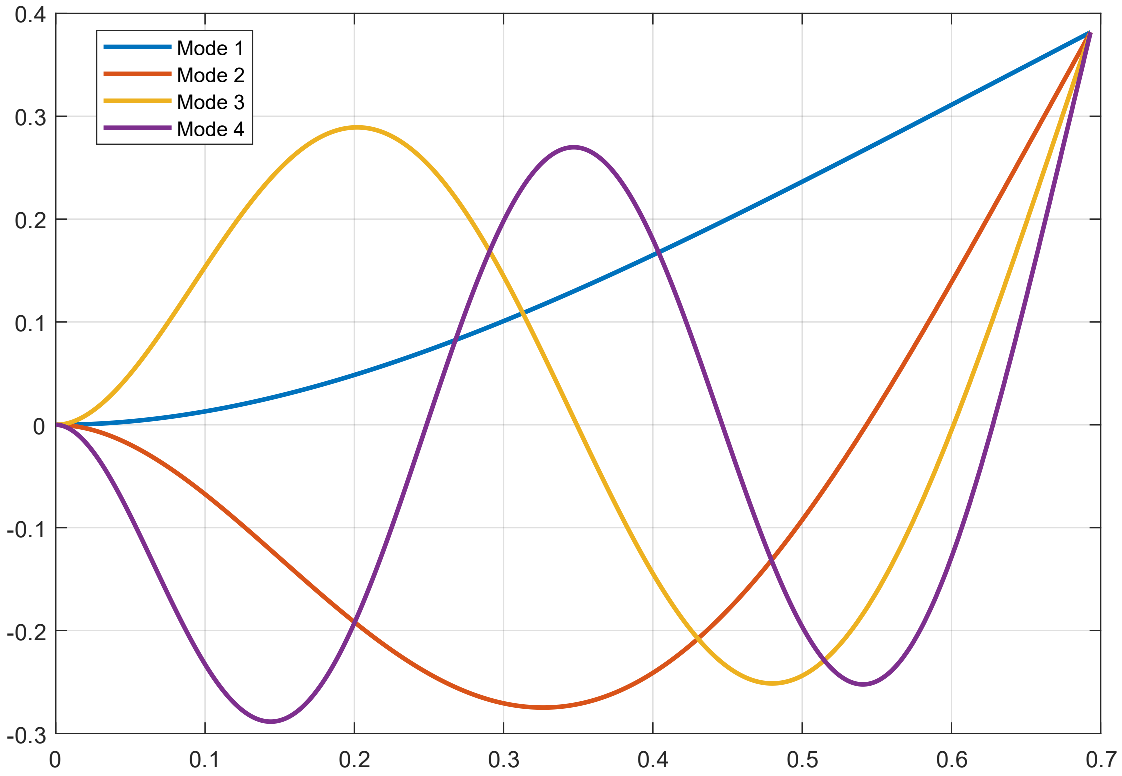
Excitation frequency = 10.6 rad/s

Material parameters:

E = 200 GPa

$\rho = 7800\ \mathrm{kg/m^3}$

Pre-processing: Linear modes analysis



Discussion of results

Test case 3: Ongoing studies, cantilever beam (Pany and Rao, 2002)

Convergence analysis

Number of modes	u_1 [m]	u_2 [m]	Simulation time [sec]
1	-0.0222	0.1627	0.83
2	-0.0229	0.1643	1.65
3	-0.0229	0.1645	2.26
4	-0.0230	0.1645	2.40
5	-0.0230	0.1646	2.74
8	-0.0230	0.1646	4.68
10	-0.0230	0.1646	6.58