Forum of Mathematics, Sigma (2023), **Vol. 11**:e61 1–29
doi:10.1017/fms.2023.39

RESEARCH ARTICLE

Quantum systems at the brink: existence of bound states, critical potentials, and dimensionality

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Received: 01 April 2022; **Revised:** 23 February 2023; **Accepted:** 06 April 2023

2020 Mathematics Subject Classification: *Primary* – 81Q05; *Secondary* – 35Q40, 81V45

Abstract
One of the crucial properties of a quantum system is the existence of bound states. While the existence of eigenvalues below zero, that is, below the essential spectrum, is well understood, the situation of zero energy bound states at the edge of the essential spectrum is far less understood. We present complementary sharp criteria for the existence and nonexistence of zero energy ground states. Our criteria give a straightforward explanation for the folklore that there is a spectral phase transition with critical dimension four, concerning the existence versus nonexistence of zero energy ground states.

Contents

1. Introduction

The existence of bound states plays a crucial role for the properties of quantum systems. Of special importance is the ground state, that is, the eigenfunction corresponding to the lowest eigenvalue of the Hamiltonian describing the system. In this paper, we consider a Schrödinger operator of the form

$$H = -\Delta + V \tag{1.1}$$

on $L^2(\mathbb{R}^d)$, where $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a real-valued potential, such that the operator H is a well-defined self-adjoint realization of the formal differential operator $-\Delta + V$ which is bounded from below. Moreover, we need that eigenfunctions of H are continuous. The precise conditions are given in Assumption 1.1 below.

We are particularly interested in the special case when the ground state energy of the Schrödinger operator H is at the threshold of the essential spectrum. By shifting the potential by a constant, one can assume that the essential spectrum of H starts at zero. One also often assumes that the potential V decays to zero at infinity, such that the essential spectrum $\sigma_{\mathrm{ess}}(H) = [0, \infty)$ (see the discussion just before Theorem 1.7). Under these conditions, the zero-energy level is at the edge of two regions with very distinct behavior: the point and the continuous spectrum. It is well-known that positive eigenvalues embedded in the continuum appear only due to a special combination of oscillations and slow decay of the potential. This goes back to [75], see also [16, 21, 30, 68] and the references therein. Criteria for the absence of positive eigenvalues were developed in [1, 22, 30, 36, 43, 68].

Thanks to the min-max theorem, see, for example [64], and the Birman–Schwinger principle [11, 66], the existence and nonexistence of eigenvalues below zero is well understood. Birman's monumental work provides not only a quadratic form approach for the study of the discrete spectrum of Schrödinger operators below zero, it is also the first paper where the spectral stability/instability of the point zero, that is, the lower edge of the essential spectrum, is investigated. It is well-known in physics, see [44, Problems 1 and 2 in Chapter 45], that one- and two-dimensional Schrödinger operators have weakly coupled bound states, that is, discrete spectrum below zero appears for arbitrarily weak attractive potentials V. On the other hand, Schrödinger operators in dimension $d \geq 3$ do not have negative discrete spectrum below zero for weak potentials V. This follows immediately from Hardy's inequality or from the famous Cwikel–Lieb–Rozenblum bound [13, 47, 65] (see also [20, 28]). In fact, in dimensions $d \geq 3$, the Iorio–O'Carroll theorem shows that Schrödinger operators with a weak potentials V are unitarily equivalent to the free Schrödinger operator (see [64, Theorem XIII.27]).

The emergence of weakly coupled bound states is due to virtual levels, that is, zero energy "eigenfunctions" of the unperturbed Schrödinger operator which have finite kinetic energy but are not square integrable. The necessary tools, homogeneous Sobolev spaces, for studying such eigenfunctions were already developed in [11, Section 2, in particular, Section 2.4]. It was rigorously shown in [69] that discrete spectrum appears below zero for arbitrary weak attractive potentials for one- and two-dimensional Schrödinger operators. This was extended to general second order elliptic operators in [56, 57] (see [61] for a review). In [25], a purely variational approach is developed, which gives necessary and sufficient conditions for the existence, respectively nonexistence, of weakly coupled bound states for a large class of generalized Schrödinger operators.

The question whether a zero energy eigenfunction is normalizable or not is of great importance for the time decay of solutions of the time-dependent Schrödinger equation (see, e.g. [31, 80]).

While the discrete spectrum below zero is well-understood, the question whether zero is actually a threshold eigenvalue, that is, an eigenvalue at *the edge of the continuum* is a very difficult problem, in general. Early results on existence or nonexistence of zero-energy eigenvalues go back to [1, 31, 32, 33, 37, 41, 42, 48, 52, 62, 63, 70, 80].

In [31], the authors studied the behavior of resonances and eigenstates at the zero-energy threshold in $d = 3$. Furthermore, based on a remark by the referee, the authors of [31] note that using properties of Riesz potentials, one can show that zero energy resonances cannot exist in dimensions $d > 4$, they become true zero energy eigenvalues (see [32]). The case $d = 4$ was considered in [33].

For radial potentials $V \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, it was shown in [80] that zero energy bound states cannot exist in dimensions $d \leq 4$. For slowly decaying *negative* potentials which, amongst other conditions, obey $V(x) \sim -c|x|^{-\gamma}$ for some $c > 0$ and $0 < \gamma < 2$ in the limit $|x| \to \infty$, the nonexistence of zero energy eigenstates was shown in [17, 19], while it was noted in [12] that a long-range Coulomb part can create zero energy eigenstates (see also [51, 79]). An analysis of eigenstates and resonances at the threshold for the case of certain nonlocal operators appeared in [35].

A condition for nonintegrability of zero-energy ground states of Schrödinger operators in three dimensions was given in [48, Lemma 7.18], by reducing it to an effective one-dimensional problem with the help of spherical averaging. This work inspired the results of [10], where it was shown that for Schrödinger operators on $L^2(\mathbb{R}^3)$ with spherical symmetric potentials $V \in L^p(\mathbb{R}^3)$ with $p > 3/2$, whose positive part satisfies $V_+(x) \leq 3/(4|x|^2)$ for $|x|$ large enough, zero is not an eigenvalue corresponding to a positive square integrable ground state eigenfunction. This extends to potentials with $V_+ \leq |x|^{-2}\Big(3/4 + \ln^{-1}(|x|)\Big)$ near infinity in $\mathbb{R}^3$, the constants $3/4$ and 1 are optimal. For similar results, see [24], which reproved a slightly weaker nonexistence result compared to [10] and additionally showed that if $V(x) \geq C|x|^{-2}$ for some constant $C > 3/4$ and $|x|$ large enough, then zero is an eigenvalue for critical potentials. Thus, a repulsive part can stabilize zero energy bound states of quantum systems (see Definition 1.5 for the precise notion of critical potentials). Informally, a potential V is critical, if the Schrödinger operator $H = -\Delta + V$ is bounded from below by zero, but for any nontrivial perturbation $W \geq 0$, the operator $H - W$ is not bounded from below by zero anymore. In this case, one also says that H has a virtual level (at zero). The study of critical potentials and, moreover, general critical operators, has been put to a high art (see the review [61] and the references therein).

While the paper [10] considers only continuous potentials on $\mathbb{R}^3$, they note that the condition $V \in L^p(\mathbb{R}^3)$ with $p > 3/2$ is enough to guarantee continuity of ground states, due to a Harnack inequality for positive eigenfunctions, which is all they need. We also note that compactly supported zero-energy eigenfunctions were constructed in [38, 43] for potential $V \in L^p(\mathbb{R}^d)$ with $p < d/2$ and compact support. For these potentials, a Harnack inequality for the ground state cannot hold.

In this paper, we extend all previous results, in particular, the ones of [10] and [24], by proving a *family of sharp criteria* for the existence and nonexistence of zero energy ground states at *the edge of the essential spectrum* for Schrödinger operators in *arbitrary* dimensions. In particular, our results apply to Schrödinger operators with a so-called virtual level at zero energy, and they explain when such a virtual level is a true ground state, that is, square integrable, or when it is not a ground state, that is, not square integrable.

Our results, summarized in Theorems 1.3 and 1.7 below, clearly explain why increasing the dimension makes it easier for a virtual level to be a true ground state. In particular, our work gives a straightforward explanation for the folklore wisdom that *dimension 4 is critical*. We also would like to emphasize that while dimension four shares some similarity with the case of lower dimensions, *second order terms* from our criteria are needed to settle the case of four-dimensional Schrödinger operators.

Our main assumption on the potential V are given by

Assumption 1.1. *The potential V is in the local Kato-class $K_{d,\mathrm{loc}}(\mathbb{R}^d)$ and the negative part $V_- = \sup(-V, 0)$ is relatively form small w.r.t. $-\Delta + V_+$, that is, there exist $0 \leq a < 1$ and $b \geq 0$, such that*

$$\langle \psi, V_- \psi \rangle = \|\sqrt{V_-}\psi\|^2 \leq a(\|\nabla\psi\|^2 + \|\sqrt{V_+}\psi\|^2) + b\|\psi\|^2 \tag{1.2}$$

for all $\psi \in H^1(\mathbb{R}^d) \cap \mathcal{D}(\sqrt{V_+})$. Here, $\mathcal{D}(\sqrt{V_+})$ is the domain of the multiplication operator $\sqrt{V_+}$ on $L^2(\mathbb{R}^d)$, also called the form domain of V_+ and often written as $\mathcal{Q}(V_+)$.

Note that what we call *relatively form small* is usually called *relatively form-bounded with relative bound $a < 1$*. We will call a potential W infinitesimally form bounded (w.r.t. $-\Delta + V_+$) if, for all $a > 0$, there exist $b \geq 0$, such that the positive and negative parts $W_\pm$ satisfy (1.2) (with V_- replaced by $W_\pm$).

Remark 1.2. The (local) Kato-class $K_{d,\mathrm{loc}}(\mathbb{R}^d) \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$, whose definition is recalled below, see (1.14), contains most, if not all physically relevant potentials. This assumption is only made to guarantee that all weak local eigenfunctions of H are continuous (see [3, 67, 71]).

One could relax the assumption that $V \in K_{d,\mathrm{loc}}(\mathbb{R}^d)$ to $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, if some other condition guaranteed that weak local eigenfunctions of H are continuous. In fact, it would be sufficient to have that eigenfunctions are locally bounded and that a ground state of H is bounded away from zero on compact sets. As will become clear from the proofs, we can allow for severe local singularities. It is enough to assume that V is in the local Kato-class outside some compact set $K \subset \mathbb{R}^d$.

https://doi.org/10.1017/fms.2023.39 Published online by Cambridge University Press

If $V_\pm \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and (1.2) holds, the Kato-Lions-Lax-Milgram-Nelson (KLMN) theorem shows that there exists a unique self-adjoint operator H, informally given by the differential operator $-\Delta + V$, such that its quadratic form, which with a slight abuse of notation, we write as

$$\langle \psi, H\psi \rangle := \langle \nabla\psi, \nabla\psi \rangle + \langle \sqrt{V_+}\psi, \sqrt{V_+}\psi \rangle - \langle \sqrt{V_-}\psi, \sqrt{V_-}\psi \rangle \tag{1.3}$$

is well-defined for $\psi \in \mathcal{Q}(H) := H^1(\mathbb{R}^d) \cap \mathcal{Q}(V_+)$. Moreover, it is closed and bounded from below on the quadratic form domain $\mathcal{Q}(H)$ (see also the discussion at the beginning of the next section).

To formulate our main results, we recall the definition of the iterated logarithms $\ln_n$ defined, for natural numbers $n \in \mathbb{N}$, by $\ln_1(r) := \ln(r)$ for $r > 0$ and inductively for $r > e_n$ by $\ln_{n+1}(r) := \ln(\ln_n(r))$. Here, $e_0 = 0$ and $e_n = e^{e_{n-1}}$ for $n \in \mathbb{N}$. Our first main result can be summarized as follows

Theorem 1.3 (Absence of a zero energy ground state). *Assume that the potential V satisfies Assumption 1.1 and $\sigma(H) = [0, \infty)$. If for some $m \in \mathbb{N}_0$ and $R > e_m$*

$$V(x) \le \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2} \sum_{j=1}^{m} \prod_{k=1}^{j} \ln_k^{-1}(|x|) \tag{1.4}$$

for all $|x| \ge R$, then zero is not a ground state eigenvalue of the Schrödinger operator H.

As usual, the empty product is 1 and the empty sum equals 0. The condition $R > e_m$ guarantees that the second term in the right-hand side of (1.4), involving (iterated) logarithms, is positive. So the only sign changing term in the right-hand side of (1.4) is the first summand, which is positive for $1 \le d \le 3$, zero for $d = 4$, and negative for $d \ge 5$.

Remark 1.4. Note that the right-hand side of (1.4) has a long-range positive tail proportional to $|x|^{-2}$ in dimensions $d \le 3$, respectively $|x|^{-2} \ln^{-1}(|x|)$ in dimension $d = 4$. So compactly supported potentials V, or even short-range potentials for which $V(x) = O(|x|^{-2-\delta})$ for some $\delta > 0$ near infinity, cannot create zero energy eigenstates of Schrödinger operators in dimension $d \le 4$. This is well-known folklore for compactly supported potentials, see, for example [80], but seems less well-known for short-range potentials.

More importantly, Theorem 1.3 gives a sufficient criterion for the absence of zero energy ground states at the edge of the essential spectrum for Schrödinger operators in any dimension $d \ge 1$ and Theorem 1.7 below shows the sharpness of condition (1.4) on the potential V for the absence of such embedded ground states.

Our second main result shows that critical potentials create zero energy ground states if they are not too small at infinity. We call a potential $W \ge 0$ nontrivial, if it is strictly positive on a set of positive Lebesgue measure.

Definition 1.5 (Critical and subcritical potentials). A potential V is *critical* if the Schrödinger operator H has spectrum $\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0, \infty)$ and for all nontrivial compactly supported potentials $W \ge 0$, which are infinitesimally form bounded with respect to $-\Delta + V_+$, the family of operators $H_\lambda = H - \lambda W$ has essential spectrum $\sigma_{\mathrm{ess}}(H_\lambda) = [0, \infty)$ and a negative energy bound state for all $\lambda > 0$. The potential V is called *subcritical* if the Schrödinger operator H has spectrum $\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0, \infty)$ and there exists a nontrivial potential $W \ge 0$, which is infinitesimally form bounded with respect to $-\Delta + V_+$, such that $H - \lambda W \ge 0$ for some $\lambda > 0$.

Remark 1.6. Alternatively, one often says that $-\Delta + V$ has a virtual level (at zero) if the potential is critical. The notion of critical and subcritical potentials has been extended to a much broader class of second order partial differential operators (see the review [61]).

The operator $H_\lambda = H - \lambda W$ in Definition 1.5 is well-defined with quadratic form methods for all λ (see Remark 2.1). In order to guarantee that $\sigma_{\mathrm{ess}}(H_\lambda) = [0, \infty)$, some decay of the potential V is required. A well-known sufficient criterion for this is that V is relatively form compact with respect to

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the kinetic energy $P^2 = -\Delta$ (see [74]). This also implies that V is infinitesimally form bounded, that is, relatively form small with relative bound zero, w.r.t. $P^2 = -\Delta$, which excludes Hardy type potentials. A *much less restrictive criterion* for $\sigma_{\mathrm{ess}}(H) = [0, \infty)$ only assumes that V *vanishes asymptotically with respect to the kinetic energy*. More precisely, if

$$|\langle \varphi, V\varphi\rangle| \le a_n \|\nabla\varphi\|^2 + b_n\|\varphi\|^2 \tag{1.5}$$

for all $\varphi \in H^1(\mathbb{R}^d)$ with support $\operatorname{supp}(\varphi) \subset \{|x| \ge R_n\}$ for some sequences $0 \le a_n, b_n \to 0$ and $R_n \to \infty$ as $n \to \infty$, then $\sigma_{\mathrm{ess}}(H) = [0, \infty)$ (see [4, 34]).

This criterion is clearly in line with the physical heuristic that *only the asymptotic behavior of the potential near infinity* determines the essential spectrum, and it allows for strongly singular potentials which are not infinitesimally form bounded. It also shows that $\sigma_{\mathrm{ess}}(H_\lambda) = \sigma_{\mathrm{ess}}(H) = [0, \infty)$ for all $\lambda > 0$ when W has compact support and is infinitesimally form bounded w.r.t. $-\Delta$ and V is form small w.r.t. $-\Delta$ and satisfies (1.5).

Theorem 1.7 (Existence of a zero energy ground state for critical potentials). *Assume that the potential V satisfies Assumption 1.1 and that it is critical. If for some $m \in \mathbb{N}_0$, $\epsilon > 0$, and $R > e_m$*

$$V(x) \ge \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2}\sum_{j=1}^{m}\prod_{k=1}^{j} \ln_k^{-1}(|x|) + \frac{\epsilon}{|x|^2}\prod_{k=1}^{m}\ln_k^{-1}(|x|) \tag{1.6}$$

for all $|x| \ge R$, then zero is an eigenvalue of H.

Remark 1.8. Clearly, the right-hand sides of (1.4) and (1.6) are, for each fixed $m \in \mathbb{N}$, complementary. Thus, our criteria for existence and nonexistence of zero energy ground states at the edge of the essential spectrum are sharp! Considering the simplest case $m = 0$, we have

$$V(x) \le \frac{d(4-d)}{4|x|^2} \tag{1.7}$$

for the absence and

$$V(x) \ge \frac{d(4-d)+\epsilon}{4|x|^2} \tag{1.8}$$

for the existence with $\epsilon > 0$ and all $|x|$ large enough. For $d = 3$, this recovers the results proved in [24] for the special case of three dimensions.

Using the higher-order corrections from Equations (1.4) and (1.6), we obtain a sharp distinction between existence and nonexistence in the case of a critical potential. For example, the cases $m = 1, 2$ show that if

$$V(x) \le \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2\ln(|x|)} \quad \text{or} \tag{1.9}$$

$$V(x) \le \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2\ln(|x|)} + \frac{1}{|x|^2\ln(|x|)\ln_2(|x|)} \tag{1.10}$$

for all large enough $|x|$, then zero will not be a ground state eigenvalue. Conversely, for critical potentials, the bound

$$V(x) \ge \frac{d(4-d)}{4|x|^2} + \frac{1+\epsilon}{|x|^2\ln(|x|)} \quad \text{or} \tag{1.11}$$

$$V(x) \ge \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2\ln(|x|)} + \frac{1+\epsilon}{|x|^2\ln(|x|)\ln_2(|x|)} \tag{1.12}$$

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for all large enough $|x|$ and some $\epsilon > 0$ implies that zero is a ground state eigenvalue. Using $d = 3$ in (1.9) recovers the nonexistence result of [10]. The $d = 3$ case in (1.11) provides a complementary existence result, which was missing in [10].

More importantly, our results provide, to arbitrary order, a whole family of complementary sharp criteria which are not restricted to three dimensions and our proofs are considerably simpler than the approaches based on delicate estimates for Green's functions.

Remark 1.9. In Appendix A, we construct a family of potentials $V_{\alpha,d}$ on $\mathbb{R}^d$ for $\alpha \in \mathbb{R}$ and $d \in \mathbb{N}$, such that the Schrödinger operator $H_{\alpha,d} = -\Delta + V_{\alpha,d}$ has spectrum $\sigma(H_{\alpha,d}) = [0, \infty)$. Moreover, $V_{\alpha,d}$ is subcritical for $\alpha < 0$ and critical for $\alpha \geq 0$. The Schrödinger operator $H_{\alpha,d}$ has a zero energy resonance for $0 \leq \alpha \leq 1$ and a zero energy bound state for $\alpha > 1$ in any dimension.

Remark 1.10. The bounds on the potential in Theorems 1.3 and 1.7 are similar in spirit to logarithmic corrections to the Hardy inequality. For $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus \{|x| \leq e_m\})$ and $m \in \mathbb{N}_0$, one has

$$\langle \nabla\psi, \nabla\psi\rangle \geq \Big\langle \psi, \Big(\frac{(d-2)^2}{4|x|^2} + \frac{1}{4|x|^2}\sum_{j=1}^{m}\prod_{k=1}^{j} \ln_k^{-2}(|x|)\Big)\psi\Big\rangle, \tag{1.13}$$

see [54], which also discusses conditions on the potential, such that $-\Delta + V$ has infinitely many, respectively finitely many, negative eigenvalues. Bounds on the number of negative eigenvalues are given in [49, 50]. Certain logarithmic refinements of Hardy's inequality have been used to study the existence of resonances of Schrödinger operators and the Efimov effect in low dimensions (see [9]). Similar logarithmic corrections were also derived for L^p Hardy inequalities in [5].

Remark 1.11. Theorems 1.3 and 1.7 show a *spectral phase transition* concerning the existence of zero energy ground states for Schrödinger operators with critical dimension $d = 4$: The sign of the leading order term in (1.4) and (1.6) strongly depends on the dimension d being positive if $d \leq 3$, zero in dimension $d = 4$, and negative if $d \geq 5$.

The four-dimensional case is *critical,* since the leading order term vanishes in dimension $d = 4$. The next order correction, the term with $m = 1$ from Theorems 1.3 and 1.7, becomes dominant. This shows that zero energy bound states of Schrödinger operators $-\Delta + V$ do not exist in dimension $d = 4$, unless the potential has a positive tail, which is larger than $|x|^2 \ln^{-1}(|x|)$ near infinity. So a slightly faster decay of the potential near infinity is allowed in dimension $d = 4$ compared to dimensions $d \leq 3$.

Since the new leading order term for $d = 4$ is also positive, the four-dimensional case is similar to the case of lower dimensions. In particular, in dimension $d \leq 4$, only potentials with a "long-range" positive tail can create zero energy ground states (see also Remark 1.4). While in dimension $d \geq 5$, critical potentials which are nonpositive or short-range have zero energy ground states (see also the discussion in Section 2 of [27]).

Other indications that dimension $d = 4$ is critical for spectral properties of Schrödinger operators appeared in the literature in various contexts, that is, the so-called localization of binding for Schrödinger operators (see [39, 58, 59]). In particular, the reason why the famous Efimov effect exists for a system of three particles interacting with short-range pair potentials in $\mathbb{R}^3$ is related to the fact that the two particle subsystems, which are equivalent to effective one particle Schrödinger operators in $\mathbb{R}^3$, can only have zero energy virtual levels, or resonances, but no zero energy ground state [77, 78] (see also [53, 72, 73]). In dimension $d \geq 5$, virtual levels of Schrödinger operators with short-range potential become true zero energy ground states (see, e.g., Theorem 1.7). The arguments of [76] can then be used to prove that no Efimov effect can exist for three particles in $\mathbb{R}^d$ when $d \geq 5$. In [6] it was shown that the Efimov effect for three particles also ceases to exist in dimension $d = 4$. Systems of N particles with $N \geq 4$ were studied in [8, 9], the existence and decay of zero energy bound states for multiparticle systems with short-range interactions was also studied in [7]. The reason that the Efimov effect depends highly on the absence of zero energy ground states is one of the motivations for our work.

Another motivation is as follows. Assume that the potential V is infinitesimally form bounded w.r.t. $-\Delta$ and has compact support. Then $\sigma_{\text{ess}}(-\Delta + \beta V) = [0, \infty)$ for all $\beta \geq 0$, see [4, 74], and a simple

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application of the min–max principle shows that as soon as negative eigenvalues of $-\Delta + \beta V$ exist, they are decreasing in $\beta > 0$ (see [64, Proposition after Theorem XIII.2, page 79]). Let $\beta_0 > 0$ be the value of the coupling constant when the ground energy of $-\Delta + \beta V$ hits zero. The asymptotic behavior of this ground state energy in $\beta - \beta_0 > 0$ depends not only on the dimension, but also strongly on the existence of the ground state at $\beta = \beta_0$ (see [40]). Theorem 1.3 shows that for short-range potentials, $-\Delta + \beta_0 V$ has no zero energy ground state when $d \leq 4$, and Theorem 1.7 shows that it has a zero energy ground state in dimension $d \geq 5$.

Lastly, recall that the Kato-class K_d is given by all real-valued potentials V, such that in dimension $d \geq 2$

$$\lim_{\alpha \downarrow 0} \sup_{|x| \in \mathbb{R}^d} \int_{|x-y| \leq \alpha} g_d(x-y)|V(y)|dy = 0, \tag{1.14}$$

where

$$g_d(x) := \begin{cases} |x|^{2-d} & \text{if } d \geq 3 \\ |\ln(|x|)| & \text{if } d = 2 \end{cases}. \tag{1.15}$$

The Kato class in one dimension is given by $K_1 := L^1_{\text{loc,unif}}(\mathbb{R})$, the space of uniformly locally integrable functions on $\mathbb{R}$. We say that the potential V is in the local Kato-class $K_{d,\text{loc}}$ if $V\mathbb{1}_K \in K_d$ for all compact sets $K \subset \mathbb{R}^d$. It is clear that $K_d \subset L^1_{\text{loc,unif}}(\mathbb{R}^d)$ and $K_{d,\text{loc}} \subset L^1_{\text{loc}}(\mathbb{R}^d)$. Moreover, it is well-known that any potential $V \in K_d$ is infinitesimally form small with respect to $-\Delta$ (see [14]).

Thus, if $V = V_+ - V_-$ with $V_\pm \geq 0$, $V_+ \in K_{d,\text{loc}}$, and $V_- \in K_d$, then all of the claims of Assumption 1.1 hold. This class of potentials is large enough to include most, if not all, physically relevant potentials, except maybe for some highly oscillatory potentials.

The structure of our paper is as follows: In Section 2, we present all the necessary technical tools to precisely formulate our main results. Theorem 1.3 is proven in Section 3. The proof is by contradiction, assuming that a zero energy ground state exists, and then deriving a lower bound which shows that it cannot be square integrable. To construct such a lower bound, one only needs to know that a ground state, if it exists, can be chosen to be positive and that it is locally bounded away from zero. It is well-known that ground states of a Schrödinger operator H in $L^2(\mathbb{R}^d)$ are unique, up to global phase, and can be chosen to be strictly positive as soon as they exist (see [18, 23] or [64, Section XIII.12]). Thus, if one knows that the ground state is bounded away from zero, one can relax the condition on V to $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ and V_- satisfies (1.2). The assumption that V is in the local Kato-class is only needed to guarantee that eigenfunctions of H are continuous (see [3, 71] and also [67]). This continuity then guarantees that the positive ground state is bounded away from zero on compact sets.

The proof of Theorem 1.7 is given in Section 4. The main tool is an upper bound for the spacial decay of ground states of the approximating Schrödinger operators H_λ, see Definition 1.5, which is *uniform* in $\lambda > 0$.

Since it will be necessary to have a positive ground state for the nonexistence proof, we cannot prove the absence of ground state under symmetry constraints which destroy the positivity of ground states, such as fermionic particle statistics. However, the existence proof still works under symmetry restrictions (see Remark 4.10).

In Appendix A, we construct an explicit example of a family of potentials which exhibits all possible different scenarios.

2. Definitions and preparations

Assume that $V_\pm \in L^1_{\text{loc}}(\mathbb{R}^d)$ and (1.2) holds for V_-. The KLMN theorem [64, 74] then shows that there exists a unique self-adjoint operator H corresponding to a quadratic form

$$\langle \psi, H\psi \rangle := \langle \nabla\psi, \nabla\psi \rangle + \langle \sqrt{V_+}\psi, \sqrt{V_+}\psi \rangle - \langle \sqrt{V_-}\psi, \sqrt{V_-}\psi \rangle \tag{2.1}$$

with the usual slight abuse of notation. Here, $\psi \in \mathcal{Q}(H) := H^1(\mathbb{R}^d) \cap \mathcal{Q}(V_+)$, the form domain of H, where $H^1(\mathbb{R}^d)$ is the usual L^2-based Sobolev space of functions $\psi \in L^2(\mathbb{R}^d)$ whose weak (distributional) gradient $\nabla\psi \in L^2(\mathbb{R}^d)$ and

$$\mathcal{Q}(V_+) := \mathcal{D}(\sqrt{V_+}) = \left\{\psi \in L^2(\mathbb{R}^d) : \sqrt{V_+}\psi \in L^2(\mathbb{R}^d)\right\} \tag{2.2}$$

is the quadratic form domain of the multiplication operator V_+.

Since $\sqrt{V_+} \in L^2_{\mathrm{loc}}$, we clearly have $\mathcal{C}_0^\infty(\mathbb{R}^d) \subset \mathcal{Q}(H)$. Note that $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is a form core, that is, dense in $H^1(\mathbb{R}^d) \cap \mathcal{D}(\sqrt{V_+})$ with respect to the norm

$$\|\psi\|_1 := (\|\psi\|_{H^1}^2 + \|\sqrt{V_+}\psi\|^2)^{1/2} \tag{2.3}$$

(see [14, 45]). In addition, Friedrich's extension theorem, see, for example [74, Theorem 2.13], implies that the operator H and its domain $\mathcal{D}(H)$ are given by

$$\begin{aligned} \mathcal{D}(H) &= \left\{\psi \in H^1(\mathbb{R}^d) \cap \mathcal{Q}(V_+) : (-\Delta + V)_{\mathrm{distr}}\, \psi \in L^2(\mathbb{R}^d)\right\} \\ H\psi &= (-\Delta + V)_{\mathrm{distr}}\, \psi \end{aligned}, \tag{2.4}$$

where $(-\Delta + V)_{\mathrm{distr}}\psi$ is in the sense of distributions when acting on $\psi \in L^2(\mathbb{R}^d)$.

Remark 2.1. If $V, W \in L^1_{\mathrm{loc}}$ and V_- is form small and $|W|$ is form bounded with respect to $-\Delta + V_+$, that is, (1.2) holds for V_- for some $0 \le a_1 < 1$, $b \ge 0$ and it also holds with V_- replaced by $|W|$ for some $a_2, b_2 \ge 0$, then

$$\|\sqrt{V_-}\psi\|^2 + \lambda\|\sqrt{|W|}\psi\|^2 \le (a_1 + \lambda a_2)\big(\|\nabla\psi\|^2 + \|\sqrt{V_+}\psi\|^2\big) + (b_1 + \lambda b_2)\|\psi\|^2 \tag{2.5}$$

for all $\psi \in H^1 \cap \mathcal{Q}(V_+)$. So for any $0 < \lambda_0 < (1 - a_1)/a_2$, we can construct the family of Schrödinger operators H_λ as the unique self-adjoint operator given by the quadratic forms

$$\langle\psi, H_\lambda\psi\rangle := \langle\nabla\psi, \nabla\psi\rangle + \langle\psi, V_+\psi\rangle - \langle\psi, V_-\psi\rangle - \lambda\langle\psi, W\psi\rangle, \tag{2.6}$$

with quadratic form domain $\mathcal{Q}(H_\lambda) = H^1(\mathbb{R}^d) \cap \mathcal{Q}(V_+) = \mathcal{Q}(H)$ for all $0 \le \lambda \le \lambda_0$. For $\lambda = 0$, one recovers H. If W is infinitesimally form bounded w.r.t. $-\Delta + V_+$, then $\lambda_0 = \infty$.

One can relax the conditions on V to hold only on a connected, open set $U \subset \mathbb{R}^d$, which contains infinity. In this case, one assumes $V_+ \in L^1_{\mathrm{loc}}(U)$, and (1.2) holds for all $\psi \in H^1_0(U) \cap \mathcal{Q}^U(V_+)$, where $H^1_0(U)$ is the usual Sobolev space with Dirichlet boundary conditions on the boundary ∂U and $\mathcal{Q}^U(V_+) = \{\psi \in L^2(U) : \sqrt{V_+}\psi \in L^2(U)\}$. In this case, H is the Schrödinger operator (with Dirichlet boundary conditions) defined by the quadratic form (2.1) which is restricted to $\psi \in \mathcal{Q}^U(H) = H^1_0(U) \cap \mathcal{Q}^U(V_+)$. Again, it is well known that $\mathcal{C}_0^\infty(U)$ is dense in $\mathcal{Q}^U(H)$ w.r.t. the norm given in (2.3). The same holds for H_δ and any $\delta > 0$ small enough.

Now assume that the real-valued potential $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, that its negative part V_- is form small w.r.t. $-\Delta + V_+$, that is, (1.2) holds, and let H be the associated Schrödinger operator defined by quadratic form methods as above. For an open set $U \subset \mathbb{R}^d$, we consider weak (local) eigenfunctions of H at energy E, that is, (weak local) solutions of the Schrödinger equation

$$H\psi = E\psi \quad \text{in } U. \tag{2.7}$$

We are mainly interested in the case that $E = 0$.

With a slight abuse of notation, we denote by $\langle\varphi, H\psi\rangle$ the sesquilinear form given by

$$\langle\varphi, H\psi\rangle := \langle\nabla\varphi, \nabla\psi\rangle + \langle\varphi, V\psi\rangle = \int (\overline{\nabla\varphi} \cdot \nabla\psi + \overline{\varphi}V\psi)\, dx \tag{2.8}$$

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whenever the right-hand side makes sense. This is the case, if $\varphi, \psi \in \mathcal{Q}(H) = H^1(\mathbb{R}^d) \cap \mathcal{Q}(V_+)$ but also if $\varphi \in \mathcal{Q}_c^U(H)$ and $\psi \in \mathcal{Q}_{\rm loc}^U(H)$, where for some open set $U \subset \mathbb{R}^d$, the local quadratic form domain

$$\mathcal{Q}_{\rm loc}^U(H) := \left\{\psi \in L^2_{\rm loc}(U) : \chi\psi \in \mathcal{Q}(H) \text{ for all } \chi \in \mathcal{C}_0^\infty(U)\right\} \tag{2.9}$$

is the vector space of functions which are locally (in U) in the quadratic form domain of H and

$$\mathcal{Q}_c^U(H) := \left\{\psi \in \mathcal{Q}(H) : \operatorname{supp}(\psi) \subset U \text{ is compact }\right\} \tag{2.10}$$

is the set of functions in $\mathcal{Q}(H)$ with compact support inside U. If $\varphi \in \mathcal{Q}_c^U(H)$ and $\psi \in \mathcal{Q}_{\rm loc}^U(H)$, then the integral on the right-hand side of (2.8) can be restricted to the set U. Clearly, $\mathcal{Q}_{\rm loc}^U(H) = \{\psi \in H^1_{\rm loc}(U) : \sqrt{V_+}\psi \in L^2_{\rm loc}(U)\} = H^1_{\rm loc}(U) \cap \mathcal{Q}_{\rm loc}^U(V_+)$.

Similarly, one can define the local domain of H, relative to some open set $U \subset \mathbb{R}^d$, by

$$\mathcal{D}_{\rm loc}^U(H) := \left\{\psi \in L^2_{\rm loc}(U) : \chi\psi \in \mathcal{D}(H) \text{ for all } \chi \in \mathcal{C}_0^\infty(U)\right\}. \tag{2.11}$$

Remark 2.2. Note that the definitions of $\mathcal{Q}_{\rm loc}^U(H)$ and $\mathcal{D}_{\rm loc}^U(H)$ are consistent in the sense that for any $\chi \in \mathcal{C}_0^\infty(U)$, one has $\chi\psi \in \mathcal{Q}(H)$ (even $\chi\varphi \in \mathcal{Q}_c^U(H)$) for any $\psi \in \mathcal{Q}(H)$ and $\chi\psi \in \mathcal{D}(H)$ for any $\psi \in \mathcal{D}(H)$. This is clear when, $\psi \in \mathcal{Q}_{\rm loc}^U(H) = H^1_{\rm loc}(U) \cap \mathcal{Q}_{\rm loc}^U(V_+)$ since for $\chi \in \mathcal{C}_0^\infty(U)$, we have $\chi\psi \in H^1(\mathbb{R}^d)$ for any $\psi \in H^1_{\rm loc}(U)$ and $\chi\psi \in \mathcal{Q}(V_+)$ for any $\psi \in \mathcal{Q}_{\rm loc}^U(V_+)$. In addition, if $\psi \in \mathcal{D}(H)$, then

$$(-\Delta + V)_{\rm distr}\, \chi\psi = \chi(-\Delta + V)_{\rm distr}\, \psi - 2\nabla\chi\nabla\psi - (\Delta\chi)\psi \in L^2(\mathbb{R}^d),$$

so $\chi\psi \in \mathcal{D}(H)$. Moreover, with $\mathcal{C}^\infty(U)$, the infinitely differentiable functions on U, it is easy to see that

$$\mathcal{C}^\infty(U) \subset \mathcal{Q}_{\rm loc}^U(H), \tag{2.12}$$

since $\mathcal{C}_0^\infty(U) \subset \mathcal{Q}^U(H)$. However, the inclusion $\mathcal{C}^\infty(U) \subset \mathcal{D}_{\rm loc}^U(H)$ is wrong in general, since the construction of the Schrödinger operator H with the help of quadratic forms allows for rather singular potentials V.

Thus, we define weak solutions, supersolutions, and subsolutions of (2.7) in the following quadratic form sense.

Definition 2.3. a) u is a (weak) eigenfunction of the Schrödinger operator H with energy E if $u \in \mathcal{Q}(H)$ and

$$\langle\varphi, (H - E)u\rangle = 0 \tag{2.13}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^d)$.

b) u is a (weak) local eigenfunction of the Schrödinger operator H with energy E in $U \subset \mathbb{R}^d$ if $u \in \mathcal{Q}_{\rm loc}^U(H)$ and

$$\langle\varphi, (H - E)u\rangle = 0 \tag{2.14}$$

for every $\varphi \in C_0^\infty(U)$.

c) u is a supersolution of the Schrödinger operator H with energy E in $U \subset \mathbb{R}^d$ if $u \in \mathcal{Q}_{\rm loc}^U(H)$ and

$$\langle\varphi, (H - E)u\rangle \geq 0 \tag{2.15}$$

for every nonnegative $\varphi \in C_0^\infty(U)$.

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d) u is a subsolution of the Schrödinger operator H with energy E in $U \subset \mathbb{R}^d$ if $u \in \mathcal{Q}^U_{\mathrm{loc}}(H)$ and

$$\langle \varphi, (H - E)u \rangle \le 0 \tag{2.16}$$

for every nonnegative $\varphi \in C_0^\infty(U)$.

Remark 2.4. Using the density of $\mathcal{C}_0^\infty$ in $\mathcal{Q}(H)$, it is easy to see that once (2.13) holds for all $\varphi \in \mathcal{C}_0^\infty$, it holds for all $\varphi \in \mathcal{Q}(H)$. Similarly, (2.14) holds for all $\varphi \in \mathcal{Q}^U_c(H)$, and (2.15), respectively (2.16), hold for all $0 \le \varphi \in \mathcal{Q}^U_c(H)$.

One should note that one does not have to distinguish between weak eigenfunctions and eigenfunctions, and similarly for local eigenfunctions.

Lemma 2.5. *Every weak eigenfunction $u \in \mathcal{Q}(H)$ of H is in $\mathcal{D}(H)$ given by (2.4). Similarly, if $u \in \mathcal{Q}^U_{\mathrm{loc}}(H)$ is a weak local eigenfunction of H in an open domain $U \subset \mathbb{R}^d$, then u is locally in the domain of H, that is, $u \in \mathcal{D}^U_{\mathrm{loc}}(H)$ given by (2.11).*

Proof. This is probably a standard argument for weak eigenfunctions but not standard for weak local eigenfunctions. Let $f \in L^2(\mathbb{R}^d)$ and ψ be a weak solution of the equation $H\psi = f$, that is,

$$\langle \varphi, H\psi \rangle = \langle \varphi, f \rangle \tag{2.17}$$

for all $\varphi \in \mathcal{Q}(H)$. Recall that $\langle \varphi, H\psi \rangle$ denotes the quadratic form given by (2.8). Then for any $\lambda \in \mathbb{R}$, we have

$$\langle \varphi, (H + \lambda)\psi \rangle = \langle \varphi, \lambda\psi + f \rangle \tag{2.18}$$

for all $\varphi \in \mathcal{Q}(H)$. Since H is bounded from below, $-\lambda$ will be in the resolvent set of H for all large λ. So for all large enough λ, we can choose $\varphi = (H + \lambda)^{-1}\xi$, with $\xi \in L^2(\mathbb{R}^d)$ in (2.18) to get

$$\langle \xi, \psi \rangle = \langle (H + \lambda)^{-1}\xi, (H + \lambda)\psi \rangle = \langle (H + \lambda)^{-1}\xi, \lambda\psi + f \rangle = \langle \xi, (H + \lambda)^{-1}(\lambda\psi + f) \rangle. \tag{2.19}$$

This holds for all $\xi \in L^2(\mathbb{R}^d)$, so

$$\psi = (H + \lambda)^{-1}(\lambda\psi + f) \in \mathcal{D}(H), \tag{2.20}$$

since $\psi, f \in L^2(\mathbb{R}^d)$ and the resolvent $(H + \lambda)^{-1}$ maps $L^2(\mathbb{R}^d)$ onto $\mathcal{D}(H)$.

Note that if ψ is a weak eigenfunction of H, at energy E, then we can use $f = E\psi$. Thus, weak eigenfunctions are eigenfunctions in the domain of H.

Now assume that $f \in L^2_{\mathrm{loc}}(U)$ and $\psi \in \mathcal{Q}^U_{\mathrm{loc}}(H)$ is a weak local solution of

$$\langle \varphi, H\psi \rangle = \langle \varphi, f \rangle \tag{2.21}$$

for all $\varphi \in \mathcal{Q}^U_c(H)$. Take any $\chi \in \mathcal{C}_0^\infty(U)$. Replacing φ by $\overline{\chi}\varphi$ in (2.21), one sees that

$$\langle \overline{\chi}\varphi, H\psi \rangle = \langle \varphi, \chi f \rangle \tag{2.22}$$

for all $\varphi \in \mathcal{Q}(H)$. Using that χ, $\nabla\chi$, and $\Delta\chi$ have compact supports, a straightforward calculation shows

$$\langle \nabla(\overline{\chi}\varphi), \nabla\psi \rangle = \langle \nabla\varphi, \nabla(\chi\psi) \rangle + \langle \varphi, (\Delta\chi + 2\nabla\chi\nabla)\psi \rangle.$$

Using this and the definition (2.1) of the quadratic form in (2.22) yield

$$\langle \varphi, H\chi\psi \rangle = \langle \varphi, \chi f - (\Delta\chi + 2\nabla\chi\nabla)\psi \rangle \tag{2.23}$$

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for all $\varphi \in \mathcal{Q}(H)$. Adding again $\langle \varphi, \lambda\chi\psi\rangle$ on both sides and choosing $\varphi = (H+\lambda)^{-1}\xi$ with λ large enough, one sees that

$$\langle \xi, \chi\psi\rangle = \langle \xi, (H+\lambda)^{-1}\big(\chi(\lambda\psi + f) - (\Delta\chi)\psi + 2\nabla\chi\nabla\psi\big)\rangle \tag{2.24}$$

for all $\xi \in L^2(\mathbb{R}^d)$. Hence

$$\chi\psi = (H+\lambda)^{-1}\big(\chi(\lambda\psi + f) - (\Delta\chi)\psi + 2\nabla\chi\nabla\psi\big) \in \mathcal{D}(H), \tag{2.25}$$

for any $\chi \in \mathcal{C}_0^\infty(U)$. Thus, $\psi \in \mathcal{D}^U_{\mathrm{loc}}(H)$. Again, choosing $f = E\psi$ shows that any weak local eigenfunctions of H at energy E is locally in the domain of H. □

Finally, let us note that the definition of critical potential and virtual levels is rather natural. It is easy to see that any potential which creates a zero energy ground state is critical.

Lemma 2.6. *Assume that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and that V_- is form small and $0 \le W$ is infinitesimally form small w.r.t. $-\Delta + V_+$. Furthermore, let H and $H_\lambda = H - \lambda W$, $0 < \lambda \le \lambda_0$, be the associated Schrödinger operators (see Remark 2.1). Assume also that $\sigma(H) = \sigma_{\mathrm{ess}}(H_\lambda) = [0,\infty)$ and that H has a zero energy ground state. Then the potential V is critical.*

Proof. This is probably well-known. We provide the short proof for the convenience of the reader. Let ψ be a zero energy normalized ground state of H. Then for any small enough $\lambda > 0$

$$\langle \psi, H_\lambda\psi\rangle = \langle \psi, H\psi\rangle - \lambda\langle\psi, W\psi\rangle = -\lambda\langle\psi, W\psi\rangle < 0$$

since $|\psi|^2 > 0$ almost everywhere. Thus, as soon as $\sigma_{\mathrm{ess}}(H_\lambda) = [0,\infty)$, the min–max principle shows that H_λ has eigenvalues below zero. □

The converse to Lemma 2.6 does not hold (see the example from Appendix A).

Our proofs of Theorems 1.3 and 1.7 rely on the so-called subharmonic comparison lemma, which has already seen wide use in the study of the asymptotic decay of eigenfunctions of Schrödinger operators (see, e.g. [15, 26]). We use the version of [2, Theorem 2.7] since it allows for a quadratic form version which needs only minimal regularity assumptions.

Theorem 2.7 (Agmon's version of the comparison principle). *Let w be a positive supersolution of the Schrödinger operator H at energy E in a neighborhood of infinity $U_R := \{x \in \mathbb{R}^d : |x| > R\}$. Let v be a subsolution of H at energy E in U_R. Suppose that*

$$\liminf_{N\to\infty}\left(\frac{1}{N^2}\int_{N\le|x|\le\alpha N} |v|^2 \mathrm{d}x\right) = 0 \tag{2.26}$$

for some $\alpha > 1$. If for some $\delta > 0$ and $0 \le C < \infty$, one has

$$v(x) \le Cw(x) \ \textit{on the annulus}\ R < |x| \le R + \delta, \tag{2.27}$$

then

$$v(x) \le Cw(x) \ \textit{for all}\ x \in U_R. \tag{2.28}$$

Remark 2.8. We note that the condition (2.26) is trivially satisfied as soon as $v \in L^2(\mathbb{R}^d)$, but it also allows for subsolutions v which are not square integrable at infinity. This is crucial for the proof of our nonexistence result. A slight extension of Agmon's comparison principle, which allows to relax the continuity assumptions and works for domains U which are not necessarily neighborhoods of infinity, is derived in [27].

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Remark 2.9. Agmon also assumes that the supersolution w and the subsolution v are continuous in $\overline{U_R} = \{|x| \geq R\}$ in [2]. However, the form of Theorem 2.7 is what is really proven in [2] (see also [27]). The additional assumption that the supersolution $w > 0$ and the subsolution v are continuous in $\{|x| \geq R\}$ are only made in [2] to guarantee that (2.27) holds with constant $C = c_2/c_1$, where $c_1 := \inf_{R\leq|x|\leq R+\delta} w(x) > 0$ and $c_2 := \sup_{R\leq|x|\leq R+\delta} |v(x)| < \infty$, by continuity, for arbitrary $\delta > 0$.

Before we give the proofs of Theorems 1.3 and 1.7, we sketch the simple proof of the $m = 0$ version of Theorem 1.3 using the comparison theorem:

For $\gamma > 0$, set $\psi_\gamma(x) = |x|^{-\gamma}$, so $\psi_\gamma \in \mathcal{C}^\infty(U_R)$ and all $R > 0$. A short calculation shows $\Delta\psi_\gamma(x) = \gamma(\gamma + 2 - d)|x|^{-\gamma-2}$ for $x \neq 0$. Hence, with

$$W_\gamma(x) = \frac{\Delta\psi_\gamma(x)}{\psi_\gamma(x)} = \gamma(\gamma + 2 - d)|x|^{-2}, \tag{2.29}$$

one sees that $(-\Delta + W_\gamma(x))\psi_\gamma(x) = 0$ for $x \neq 0$. Since $\psi_\gamma \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$, integration by parts shows that ψ_γ is a weak local eigenfunction of $-\Delta + W_\gamma$ in the sense of Definition 2.3 in the open sets U_R for any $R > 0$.

Moreover, Remark 2.2 shows that $\psi_\gamma \in \mathcal{Q}^{U_R}_{\mathrm{loc}}(H)$ for any $R > 0$ and any Schrödinger operator H with potential $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for which V_- is form small w.r.t. $-\Delta + V_+$. Thus, if

$$V(x) \leq W_\gamma(x) \tag{2.30}$$

for some $\gamma > 0$ and all large enough $|x| > R$, then

$$\langle \varphi, H\psi_\gamma \rangle \leq \langle \varphi, (-\Delta + W_\gamma)\psi_\gamma \rangle = 0 \tag{2.31}$$

for all $0 \leq \varphi \in \mathcal{C}^\infty_0(U_R)$, that is, ψ_γ is a zero energy subsolution of H. One easily checks that

a) $\psi_\gamma \notin L^2(U_R)$ if and only if $0 < \gamma \leq d/2$.
b) ψ_γ satisfies (2.26) if and only if $\gamma > (d-2)/2$.
c) $0 < \gamma \mapsto W_\gamma$ is increasing if and only if $\gamma > (d-2)/2$.

The last part shows that one should choose γ as large as possible in order to guarantee that (2.31) holds.

Now assume that $H \geq 0$ has zero as an eigenvalue with corresponding unique ground state ψ, which can be chosen to be positive [18, 23]. Since V is locally in the Kato class, one also knows that ψ is continuous, [71]. If

$$V(x) \leq \frac{d(4-d)}{4|x|^2} = W_{d/2}(x), \tag{2.32}$$

then the above discussion shows that $\psi_{d/2}$ is a zero energy subsolution of H which is not square integrable at infinity but for which (2.26) holds. Using $c^1_R = \inf_{R\leq|x|\leq R+1} \psi(x) > 0$, $c^2_R = \sup_{R\leq|x|\leq R+1} \psi_{d/2}(x)$, and $C = c^2_R/c^1_R$ ensures $\psi_{d/2}(x) \leq C\psi(x)$ for all $R \leq |x| \leq R+1$. Then Theorem 2.7 shows that

$$\psi_{d/2}(x) \leq C\psi(x) \tag{2.33}$$

for all $|x| > R$, in particular, $\psi \notin L^2(\mathbb{R}^d)$, hence, ψ is not an eigenfunction. Thus, zero is not an eigenvalue of the Schrödinger operator H, which proves the $m = 0$ version of Theorem 1.3.

Remark 2.10. Note that for $\gamma > d/2$, the function ψ_γ is in $L^2(U_R)$ for any $R > 0$, so $\gamma = d/2$ is the largest possible choice in order to get the nonexistence result. The higher-order condition for nonexistence uses logarithmic corrections to $\psi_{d/2}$.

Remark 2.11. Choosing $\gamma = (d-2)/2$ yields the Hardy potential $W_{(d-2)/2}(x) = -\frac{(d-2)^2}{4|x|^2}$. It is well known that a Schrödinger operator with a Hardy potential is nonnegative. It is curious that for the absence of zero energy eigenfunctions, the choice $\gamma = d/2$ becomes relevant.

For the existence result, we want to reverse the roles of the eigenfunction ψ and ψ_γ. If

$$V(x) \geq \frac{d(4-d)+\varepsilon}{4|x|^2} \tag{2.34}$$

for all $|x| > R$ and some $\varepsilon > 0$, then ψ_γ is a zero energy supersolution of H in U_R, where $\gamma > d/2$ is the unique solution of $\gamma(\gamma+2-d) = (d(4-d)+\varepsilon)/4$. Arguing as above, one sees that any positive zero energy ground state ψ of H satisfies the upper bound

$$\psi(x) \leq C\psi_\gamma(x) \tag{2.35}$$

for all $|x| > R$, hence, it is square integrable at infinity since $\psi_\gamma \in L^2(U_R)$ as soon as $\gamma > d/2$. Of course, this is a circular reasoning, since we need the existence of a square integrable bound state, or at least the existence of a local zero energy bound state which satisfies (2.26). The rigorous argument uses the fact that $H \geq 0$ is assumed to have a virtual level at zero. From Definition 1.5, we see that for any nontrivial infinitesimally form bounded potential $W \geq 0$, this implies that the operators $H_\lambda = H - \lambda W$ have negative energy ground states for arbitrary small $\lambda > 0$. These ground states will converge to a zero energy ground state of H in the limit $\lambda \to 0$ (see Section 4).

3. Proof of the nonexistence result

Recall the iterated logarithms $\ln_n$ defined by $\ln_1(r) := \ln(r)$ for $r > 0$ and, for $r > e_n$, inductively by $\ln_{n+1}(r) := \ln(\ln_n(r))$ when $n \in \mathbb{N}$. Here, $e_0 = 0$ and $e_n = e^{e_{n-1}}$ for $n \in \mathbb{N}$.

A convenient sequence of functions at the edge of L^2-integrability near infinity is given by

$$\psi_{\ell,m}(x) := |x|^{-d/2} \prod_{j=1}^{m} \ln_j^{-1/2}(|x|) \quad \text{for } |x| > e_m. \tag{3.1}$$

As usual, the empty product is one, so $\psi_{\ell,0}(x) = |x|^{-d/2} = \psi_{d/2}(x)$. We note that the condition $|x| > e_m$ guarantees the positivity of $\ln_j(x)$ for $1 \leq j \leq m$.

We still have $\psi_{\ell,m} \in \mathcal{C}^\infty(\{|x| > e_m\})$, in particular, $\psi_{\ell,m} \in \mathcal{Q}_{\text{loc}}^{U_R}(H)$, for $R \geq e_m$ and any Schrödinger operator constructed via quadratic form methods as in Section 2. In order to mimic the proof sketched at the end of Section 2, we need to know the potential W_m for which $(-\Delta + W_m)\psi_{\ell,m} = 0$ in $U_{e_m} = \{|x| > e_m\}$. This is a bit more complicated than the previous calculation for ψ_γ.

Lemma 3.1. *For any $m \in \mathbb{N}_0$, we have $(-\Delta + W_m)\psi_{\ell,m} = 0$ in $U_R = \{x \in \mathbb{R}^d : |x| > R\}$, for all large enough $R \geq e_m$, where*

$$\begin{aligned} W_m(x) :={}& \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2}\sum_{j=1}^{m}\prod_{k=1}^{j}\ln_k^{-1}(|x|) + \frac{1}{4|x|^2}\left(\sum_{j=1}^{m}\prod_{k=1}^{j}\ln_k^{-1}(|x|)\right)^2 \\ &+ \frac{1}{2|x|^2}\sum_{j=1}^{m}\sum_{l=1}^{j}\prod_{s=1}^{l}\prod_{t=1}^{j}\ln_s^{-1}(|x|)\ln_t^{-1}(|x|) \end{aligned} \tag{3.2}$$

is well-defined for $|x| > e_m$.

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Proof. Clearly, if $W_m = \frac{\Delta\psi_{\ell,m}}{\psi_{\ell,m}}$, then $(-\Delta + W_m)\psi_{\ell,m} = 0$ in U_R, for all large enough $R > 0$. For any radial function depending only on the radius $r = |x|$, we have

$$\Delta\psi(x) = \partial_r^2\psi(x) + \frac{d-1}{|x|}\partial_r\psi(x). \tag{3.3}$$

By a straightforward but slightly tedious calculation, one sees that

$$\begin{aligned}
\partial_r\psi_{\ell,m}(x) &= -\psi_{\ell,m}(x)\left(\frac{d}{2|x|} + \frac{1}{2|x|}\sum_{j=1}^{m}\prod_{k=1}^{j}\ln_k^{-1}(|x|)\right),\\
\partial_r^2\psi_{\ell,m}(x) &= \psi_{\ell,m}(x)\left(\frac{d}{2|x|} + \frac{1}{2|x|}\sum_{j=1}^{m}\prod_{k=1}^{j}\ln_k^{-1}(|x|)\right)^2 + \psi_{\ell,m}(x)\frac{d}{2|x|^2}\\
&\quad + \psi_{\ell,m}(x)\left(\frac{1}{2|x|^2}\sum_{j=1}^{m}\prod_{k=1}^{j}\ln_k^{-1}(|x|) + \frac{1}{2|x|^2}\sum_{j=1}^{m}\sum_{l=1}^{j}\prod_{s=1}^{l}\prod_{t=1}^{j}\ln_s^{-1}(|x|)\ln_t^{-1}(|x|)\right),
\end{aligned}$$

where we used

$$\partial_r \ln_1(r) = \frac{1}{r} \quad\text{and}\quad \partial_r \ln_j(r) = \frac{1}{\ln_{j-1}(r)}\frac{1}{\ln_{j-2}(r)}\cdots\frac{1}{\ln_1(r)}\frac{1}{r}.$$

Thus

$$\begin{aligned}
W_m(x) = \frac{\Delta\psi_{\ell,m}(x)}{\psi_{\ell,m}(x)} &= \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2}\sum_{j=1}^{m}\prod_{k=1}^{j}\ln_k^{-1}(|x|) + \frac{1}{4|x|^2}\left(\sum_{j=1}^{m}\prod_{k=1}^{j}\ln_k^{-1}(|x|)\right)^2\\
&\quad + \frac{1}{2|x|^2}\sum_{j=1}^{m}\sum_{l=1}^{j}\prod_{s=1}^{l}\prod_{t=1}^{j}\ln_s^{-1}(|x|)\ln_t^{-1}(|x|),
\end{aligned}$$

and we have $(-\Delta + W_m)\psi_{\ell,m} = 0$ in U_R as long as $R > 0$ is large enough, so that the iterated logarithms are well-defined. □

Remark 3.2. Alternatively, one can compute $W_m = \frac{\Delta\psi_{\ell,m}}{\psi_{\ell,m}}$ inductively. For radial functions f, g, that is, with the usual abuse of notation $f(x) = f(r)$ and $g(x) = g(r)$ for $r = |x|$, we have

$$\Delta(gf) = f\Delta g + 2\partial_r f\partial_r g + (\Delta f)g. \tag{3.4}$$

Using (3.4) and $\psi_{\ell,m+1}(x) = \psi_{\ell,m}(x)\ln_{m+1}^{-\frac{1}{2}}(|x|)$, we obtain

$$W_{m+1}(x) = \frac{\Delta\psi_{\ell,m+1}(x)}{\psi_{\ell,m+1}(x)} = W_m + \frac{\Delta\ln_{m+1}^{-\frac{1}{2}}(|x|)}{\ln_{m+1}^{-\frac{1}{2}}(|x|)} + 2\frac{\partial_r\psi_{\ell,m}(x)}{\psi_{\ell,m}(x)}\frac{\partial_r\ln_{m+1}^{-\frac{1}{2}}(|x|)}{\ln_{m+1}^{-\frac{1}{2}}(|x|)}.$$

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A straightforward calculation yields

$$\frac{\Delta \ln_{m+1}^{-\frac{1}{2}}(|x|)}{\ln_{m+1}^{-\frac{1}{2}}(|x|)} = \frac{1}{4|x|^2}\prod_{k=1}^{m+1}\ln_k^{-2}(|x|) + \frac{2-d}{2|x|^2}\prod_{k=1}^{m+1}\ln_k^{-1}(|x|)$$
$$+ \frac{1}{2|x|^2}\sum_{j=1}^{m+1}\prod_{s=1}^{m+1}\prod_{t=1}^{j}\ln_s^{-1}(|x|)\ln_t^{-1}(|x|),$$

$$2\frac{\partial_r\psi_{\ell,m}(x)}{\psi_{\ell,m}(x)}\frac{\partial_r \ln_{m+1}^{-\frac{1}{2}}(|x|)}{\ln_{m+1}^{-\frac{1}{2}}(|x|)} = \left(\frac{d}{|x|} + \frac{1}{|x|}\sum_{j=1}^{m}\prod_{k=1}^{j}\ln_k^{-1}(|x|)\right)\left(\frac{1}{2|x|}\prod_{s=1}^{m+1}\ln_s^{-1}(|x|)\right),$$

$$W_{m+1}(x) = W_m(x) + \frac{3}{4|x|^2}\prod_{k=1}^{m+1}\ln_k^{-2}(|x|) + \frac{1}{|x|^2}\prod_{k=1}^{m+1}\ln_k^{-1}(|x|)$$
$$+ \frac{1}{|x|^2}\sum_{j=1}^{m}\prod_{s=1}^{m+1}\prod_{t=1}^{j}\ln_s^{-1}(|x|)\ln_t^{-1}(|x|).$$

Since $W_0(x) = \frac{d(4-d)}{4|x|^2}$, this yields W_m via induction.

Now we come to the

Proof of Theorem 1.3. Using Lemma 3.1 and $\psi_{\ell,m} \in \mathcal{Q}_{\mathrm{loc}}^{U_R}(H)$, one sees that

$$\begin{aligned}\langle\varphi, H\psi_{\ell,m}\rangle &= \langle\nabla\varphi, \nabla\psi_{\ell,m}\rangle + \langle\varphi, V\psi_{\ell,m}\rangle \le \langle\nabla\varphi, \nabla\psi_{\ell,m}\rangle + \langle\varphi, W_m\psi_{\ell,m}\rangle \\ &= \langle\varphi, (-\Delta + W_m)\psi_{\ell,m}\rangle = 0\end{aligned} \tag{3.5}$$

for all $0 \le \varphi \in \mathcal{C}_0^\infty(U_R)$ as soon as $V \le W_m$ in U_R. So $\psi_{\ell,m}$ is a zero energy subsolution of H in U_R as soon as $V \le W_m$ in U_R. Since V is in the local Kato class, we know from [3, 67, 71] that any eigenfunction of H is continuous. Moreover, it is well-known that the ground state eigenfunction can be chosen to be strictly positive [18, 23, 64].

So if $\psi > 0$ is a zero energy ground state of H, then, with $c_R^1 = \inf_{R\le|x|\le R+1}\psi(x) > 0$ and $c_R^2 = \sup_{R\le|x|\le R+1}\psi_{\ell,m}(x) < \infty$, we can set $C := c_R^2/c_R^1$ to see that $\psi_{\ell,m}(x) \le C\psi(x)$ for $R \le |x| \le R+1$. Moreover, ψ being a zero energy solution is also a zero energy supersolution, so Theorem 2.7 shows that

$$\psi_{\ell,m}(x) \le C\psi(x) \tag{3.6}$$

for all $|x| > R$. In particular, ψ cannot be square integrable as soon as $V \le W_m$ on U_R for some large enough $R > 0$, since $\psi_{\ell,m}$ is positive and not in $L^2(U_R)$.

Finally, setting $V_m(x) = \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2}\sum_{j=1}^m\prod_{k=1}^j \ln_k^{-1}(|x|)$, we note that $W_m(x) \ge V_m(x)$ for all large enough $|x|$. This proves Theorem 1.3. □

Remark 3.3. Of course, the proof of Theorem 1.3 given above shows that if

$$V(x) \le W_m(x) \qquad \text{for all } |x| > R \tag{3.7}$$

for large enough $R > 0$ and some $m \in \mathbb{N}_0$, then the Schrödinger operator H cannot have any zero energy ground state.

4. Proof of the existence result

For the existence result, we need to modify our comparison functions $\psi_{\ell,m}$ to make them barely square integrable near infinity. Given an arbitrary $\varepsilon > 0$, we set

$$\psi_{u,m,\epsilon}(x) := \psi_{\ell,m}(x) \ln_m^{-\varepsilon/2}(|x|), \tag{4.1}$$

where $\psi_{\ell,m}$ is defined in (3.1). For each $m \in \mathbb{N}_0$ and $\varepsilon > 0$, we have $\psi_{u,m,\epsilon} \in \mathcal{C}^\infty(\{|x| > e_m\})$, and it is not hard to see that for any $\varepsilon > 0$ and any large enough $R > 0$, the function $\psi_{u,m,\epsilon}$ is barely in $L^2(U_R)$. The potential $Y_{m,\epsilon}$ for which $(-\Delta + Y_{m,\epsilon})\psi_{u,m,\epsilon} = 0$ in $U_R = \{x \in \mathbb{R}^d : |x| > R\}$ is given by

Lemma 4.1. *For any $m \in \mathbb{N}_0$, we have $(-\Delta + Y_{m,\epsilon})\psi_{u,m,\epsilon} = 0$ in $U_{e_m} = \{x \in \mathbb{R}^d : |x| > e_m\}$, where the potential $Y_{m,\epsilon}$ is given by*

$$\begin{aligned} Y_{m,\epsilon}(x) = W_m(x) &+ \frac{\epsilon^2}{4|x|^2} \prod_{k=1}^{n} \ln_k^{-2}(|x|) + \frac{\epsilon}{|x|^2} \prod_{k=1}^{n} \ln_k^{-1}(|x|) \\ &+ \frac{\epsilon}{|x|^2} \sum_{j=1}^{n} \prod_{k=1}^{n} \prod_{m=1}^{j} \ln_k^{-1}(|x|) \ln_m^{-1}(|x|), \end{aligned} \tag{4.2}$$

with W_m given in (3.2).

Proof. As in the proof of Lemma 3.1, we have to calculate $Y_{m,\epsilon} := \frac{\Delta \psi_{u,m,\epsilon}}{\psi_{u,m,\epsilon}}$. We use (3.4) to see that

$$\begin{aligned} Y_{m,\epsilon}(x) &= \frac{\Delta \psi_{\ell,m}(x)}{\psi_{\ell,m}(x)} + \frac{\Delta \ln_{m+1}^{-\frac{\epsilon}{2}}(|x|)}{\ln_{m+1}^{-\frac{\epsilon}{2}}(|x|)} + 2\frac{\partial_r \ln_{m+1}^{-\frac{\epsilon}{2}}(|x|)}{\ln_{m+1}^{-\frac{\epsilon}{2}}(|x|)} \frac{\partial_r \psi_{\ell,m}(|x|)}{\psi_{\ell,m}(|x|)} \\ &= W_m(x) + \frac{\epsilon^2}{4|x|^2} \prod_{k=1}^{m} \ln_k^{-2}(|x|) + \frac{\epsilon}{|x|^2} \prod_{k=1}^{m} \ln_k^{-1}(|x|) \\ &\quad + \frac{\epsilon}{|x|^2} \sum_{j=1}^{m} \prod_{s=1}^{m} \prod_{t=1}^{j} \ln_s^{-1}(|x|) \ln_t^{-1}(|x|), \end{aligned}$$

which is (4.2). $\square$

We want to show that ground states of Schrödinger operators H with critical potentials V exist using suitable eigenfunctions of H_λ. For this, the following is convenient.

Lemma 4.2. *Assume that the potential V satisfies Assumption 1.1 and W is a nonnegative potential which is form bounded w.r.t. $-\Delta + V_+$. Let $(H_\lambda)_{\lambda \geq 0}$ be the family of Schrödinger operators constructed in Remark 2.1 (for small enough $\lambda > 0$). Moreover, assume that there exists a sequence $0 < \lambda_n \to 0$ as $n \to \infty$, such that the operators $H_n = H_{\lambda_n}$ have eigenvalues $E_n = E_{\lambda_n}$ with corresponding normalized weak eigenfunctions $\psi_n = \psi_{\lambda_n}$. If*

a) *the sequence of eigenvalues $(E_n)_n$ of H_n is bounded from above and*
b) *the sequence ψ_n is a Cauchy sequence in L^2,*

then the sequence ψ_n is Cauchy w.r.t. the quadratic form norm $\|\cdot\|_1$ given in (2.3), hence, its limit $\psi = \lim_{n\to\infty} \psi_n \in \mathcal{Q}(H)$. Moreover, $E = \lim_{n\to\infty} E_n$ exists and ψ is a normalized weak eigenfunction of H with eigenvalue E.

Remark 4.3. Lemma 2.5 shows even that $\psi \in \mathcal{D}(H)$. Moreover, we do not need that V is in the local Kato-class, only that V_- is relatively form small w.r.t. $-\Delta + V_+$.

Remark 4.4. We apply Lemma 4.2 when λ_n converges monotonically to zero and E_n is a ground state of H_n, in which case, one can simplify the proof. For example, if E_n are ground state energies of H_n, then since as quadratic forms $H_\lambda \le H_{\lambda'}$ for all $0 < \lambda' \le \lambda \le \lambda_0$, the limit $\lim_{n\to\infty} E_n$ exists, by monotonicity. However, the result of Lemma 4.2 is needed when one considers not only ground states, but also excited states which hit the bottom of the essential spectrum.

Remark 4.5. Clearly, any eigenvalue of H_n is bounded from below uniformly in $n \in \mathbb{N}$ since $H_n \ge H_{\lambda_{\max}}$ with $\lambda_{\max} = \max_n \lambda_n$ as quadratic forms for all $n \in \mathbb{N}$. In particular, all the eigenvalues E_n are bounded uniformly in $n \in \mathbb{N}$ once they are bounded from above.

Moreover, if the essential spectrum of H is not empty and E_n is an eigenvalue of H_n below the essential spectrum of H_n, then E_n is bounded from above, since as quadratic forms $H_n \le H$ and by Persson's theorem [46, 55]

$$\begin{aligned} \inf\sigma_{\mathrm{ess}}(H_n) &= \lim_{R\to\infty} \inf\left\{\langle\varphi, H_n\varphi\rangle : \varphi \in \mathcal{Q}(H), \|\varphi\| = 1, \mathrm{supp}(\varphi) \subset \{|x| > R\}\right\} \\ &\le \lim_{R\to\infty} \inf\left\{\langle\varphi, H\varphi\rangle : \varphi \in \mathcal{Q}(H), \|\varphi\| = 1, \mathrm{supp}(\varphi) \subset \{|x| > R\}\right\} = \inf\sigma_{\mathrm{ess}}(H). \end{aligned} \tag{4.3}$$

Thus

$$-\infty < \inf\sigma(H_{\lambda_{\max}}) \le \inf\sigma(H_n) \le E_n \le \inf\sigma_{\mathrm{ess}}(H_n) \le \inf\sigma_{\mathrm{ess}}(H)$$

for all $n \in \mathbb{N}$, which shows that $\sup_n |E_n| < \infty$ as soon as $\sigma_{\mathrm{ess}}(H)$ is not empty.

Proof of Lemma 4.2. Let ψ_n be a normalized sequence of eigenfunctions of H_n with eigenvalue E_n, which is also a Cauchy sequence in L^2. In particular,

$$\langle\psi_n, H_n\psi_n\rangle = E_n\langle\psi_n, \psi_n\rangle = E_n. \tag{4.4}$$

Let $0 < a_1 < 1$ and $b_1 \ge 0$ be, such that (1.2) holds and $a_2, b_2 \ge 0$, such that (1.2) with V_- replaced by W holds. Then

$$\begin{aligned} \langle\psi_n, H_n\psi_n\rangle &= \|\nabla\psi_n\|^2 + \|\sqrt{V_+}\psi_n\|^2 - \|\sqrt{V_-}\psi_n\|^2 - \lambda_n\|\sqrt{W}\psi_n\|^2 \\ &\ge \|\nabla\psi_n\|^2 + \|\sqrt{V_+}\psi_n\|^2 - (a_1 + \lambda_n a_2)\|\nabla\psi_n\|^2 - (b_1 + \lambda_n b_2)\|\psi_n\|^2 \\ &\ge (1 - a_1 - \lambda_n a_2)\|\psi_n\|_1^2 - (b_1 + \lambda_n b_2), \end{aligned}$$

since ψ_n is normalized. We also used the quadratic form norm $\|\cdot\|_1$ given by (2.3). Using (4.4), this implies

$$(1 - a_1 - \lambda_n a_2)\|\psi_n\|_1^2 \le b_1 + \lambda_n b_2 + E_n, \tag{4.5}$$

which shows that we have $\limsup_{n\to\infty} \|\psi_n\|_1 < \infty$, since $a_1 < 1$, $\lambda_n \to 0$ for $n \to \infty$, and E_n is bounded from above uniformly in $n \in \mathbb{N}$. Thus, both the Sobolev norm $\|\psi_n\|_{H^1}^2 = \|\psi_n\|^2 + \|\nabla\psi_n\|^2$ and $\|\sqrt{V_+}\psi_n\|$ are bounded in n.

Now consider

$$\begin{aligned} \langle\varphi, H(\psi_n - \psi_m)\rangle &= \langle\varphi, H_n\psi_n\rangle + \lambda_n\langle\varphi, W\psi_n\rangle - \langle\varphi, H_m\psi_m\rangle - \lambda_m\langle\varphi, W\psi_m\rangle \\ &= E_n\langle\varphi, \psi_n\rangle + \lambda_n\langle\varphi, W\psi_n\rangle - E_m\langle\varphi, \psi_m\rangle - \lambda_m\langle\varphi, W\psi_m\rangle \end{aligned} \tag{4.6}$$

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for $\varphi \in \mathcal{Q}(H)$. The choice $\varphi = \psi_n - \psi_m$ and the Cauchy–Schwarz inequality yields

$$\begin{aligned}
&\langle \psi_n - \psi_m, H(\psi_n - \psi_m)\rangle \\
&\quad \le |E_n|\|\varphi\|\|\psi_n\| + \lambda_n\|\sqrt{W}\varphi\|\|\sqrt{W}\psi_n\| + |E_m|\|\varphi\|\|\psi_m\| + \lambda_m\|\sqrt{W}\varphi\|\|\sqrt{W}\psi_m\| \\
&\quad \le \big(|E_n|\|\psi_n\| + |E_m|\|\psi_m\|\big)\|\varphi\| + \frac{\lambda_n}{2}\big(\|\sqrt{W}\varphi\|^2 + \|\sqrt{W}\psi_n\|^2\big) \\
&\qquad + \frac{\lambda_m}{2}\big(\|\sqrt{W}\varphi\|^2 + \|\sqrt{W}\psi_m\|^2\big) \\
&\quad \le \big(|E_n|\|\psi_n\| + |E_m|\|\psi_m\|\big)\|\varphi\| + \frac{\lambda_n+\lambda_m}{2}\big(a_2\|\nabla\varphi\|^2 + b_2\|\varphi\|^2\big) \\
&\qquad + \frac{\lambda_n}{2}\big(a_2\|\nabla\psi_n\|^2 + b_2\|\psi_n\|^2\big) + \frac{\lambda_m}{2}\big(a_2\|\nabla\psi_m\|^2 + b_2\|\psi_m\|^2\big).
\end{aligned} \tag{4.7}$$

On the other hand,

$$\begin{aligned}
\langle \psi_n - \psi_m, H(\psi_n - \psi_m)\rangle &= \|\nabla(\psi_n-\psi_m)\|^2 + \|\sqrt{V_+}(\psi_n-\psi_m)\|^2 - \|\sqrt{V_-}(\psi_n-\psi_m)\|^2 \\
&\ge (1-a_1)\|\nabla(\psi_n-\psi_m)\|^2 + \|\sqrt{V_+}(\psi_n-\psi_m)\|^2 - b_1\|\psi_n-\psi_m\|^2.
\end{aligned}$$

Plugging this lower bound into (4.7) and using that ψ_n is normalized, we arrive at

$$\begin{aligned}
&\Big(1 - a_1 - \frac{\lambda_n+\lambda_m}{2}a_2\Big)\|\nabla(\psi_n-\psi_m)\|^2 + \|\sqrt{V_+}(\psi_n-\psi_m)\|^2 \\
&\qquad \le \Big(|E_n| + |E_m| + \frac{\lambda_n+\lambda_m}{2}b_2\Big)\|\psi_n-\psi_m\| + b_1\|\psi_n-\psi_m\|^2 \\
&\qquad\quad + \frac{\lambda_n}{2}\big(a_2\|\nabla\psi_n\|^2 + b_2\big) + \frac{\lambda_m}{2}\big(a_2\|\nabla\psi_m\|^2 + b_2\big).
\end{aligned} \tag{4.8}$$

By assumption and Remark 4.5, the sequence of eigenvalues E_n is bounded and, because of (4.5), we also have that $\|\nabla\psi_n\|$ is bounded uniformly in $n \in \mathbb{N}$. Since $\lambda_n \to 0$ and $\|\psi_n - \psi_m\| \to 0$ as $n, m \to \infty$, (4.8) implies

$$\limsup_{n,m\to\infty}\Big((1-a_1)\|\nabla(\psi_n-\psi_m)\|^2 + \|\sqrt{V_+}(\psi_n-\psi_m)\|^2\Big) \le 0.$$

That is, the sequence of normalized weak eigenfunctions ψ_n of H_n is Cauchy in $\mathcal{Q}(H)$ with respect to the form norm $\|\cdot\|_1$ as soon as it is Cauchy in L^2 and the sequence of eigenvalues $(E_n)_{n\in\mathbb{N}}$ is bounded. In particular, the limit $\psi = \lim_{n\to\infty}\psi_n$ exists in $\mathcal{Q}(H)$. Thus, $\|\nabla\psi\| = \lim_{n\to\infty}\|\nabla\psi_n\|$, $\|\sqrt{V_+}\psi\| = \lim_{n\to\infty}\|\sqrt{V_+}\psi_n\|$, and, since $W \ge 0$ is form bounded w.r.t. $-\Delta + V_+$ also $\|\sqrt{W}\psi\| = \lim_{n\to\infty}\|\sqrt{W}\psi_n\|$. Hence, $\sup_n\|\sqrt{W}\psi_n\| < \infty$.

Now assume additionally that E_n converges to some E as $n \to \infty$. In this case, using that ψ_n converges to ψ in $\mathcal{Q}(H)$, we get

$$\begin{aligned}
\langle\varphi, H\psi\rangle &= \lim_{n\to\infty}\langle\varphi, H\psi_n\rangle = \lim_{n\to\infty}\big(\langle\varphi, H_n\psi_n\rangle + \lambda_n\langle\sqrt{W}\varphi, \sqrt{W}\psi_n\rangle\big) \\
&= \lim_{n\to\infty}\big(E_n\langle\varphi,\psi_n\rangle + \lambda_n\langle\sqrt{W}\varphi,\sqrt{W}\psi_n\rangle\big) = E\langle\varphi,\psi\rangle
\end{aligned} \tag{4.9}$$

for all $\varphi \in \mathcal{Q}(H)$ since $\lambda_n \to 0$ and $\sup_n|\langle\sqrt{W}\varphi,\sqrt{W}\psi_n\rangle| \le \|\sqrt{W}\varphi\|\sup_n\|\sqrt{W}\psi_n\| < \infty$. Thus, we proved that the limit $\psi = \lim_{n\to\infty}\psi_n \in \mathcal{Q}(H)$ exists, $\|\psi\| = 1$, and ψ is a weak eigenfunction of H with eigenvalue $E = \lim_{n\to\infty}E_n$ under the *additional assumption* that the limit $E = \lim_{n\to\infty}E_n$ exists.

Finally, it is easy to see that the sequence of eigenvalues E_n must converge. Assume that E_n does not converge as $n \to \infty$. Since E_n is bounded in $n \in \mathbb{N}$, there exist two different limit points $E_1 \ne E_2$ of E_n

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corresponding to two subsequences $E_{\sigma_1(n)} \to E_1$ and $E_{\sigma_2(n)} \to E_2$, where $\sigma_1, \sigma_2 : \mathbb{N} \to \mathbb{N}$ are strictly increasing functions.

Clearly, $\psi = \lim_{n\to\infty} \psi_{\sigma_1(n)} = \lim_{n\to\infty} \psi_{\sigma_2(n)}$ since ψ_n converges to ψ in $\mathcal{Q}(H)$. So (4.9) shows that ψ is a weak eigenfunction of H corresponding to the two different eigenvalues E_1 and E_2, which is impossible. Hence, the eigenvalues E_n converges. This finishes the proof of Lemma 4.2. □

Lemma 4.6. *Assume that the potentials V and W satisfy Assumption 1.1, except that the relative bound of W does not have to be less than one. Let $(H_\lambda)_{0\le\lambda\le\lambda_0}$ be the family of perturbed Schrödinger operators constructed in Remark 2.1 for some small enough $0 < \lambda_0$. Moreover, assume that for some sequence $0 < \lambda_n \le \lambda_0$, the operators $H_n = H_{\lambda_n}$ have eigenvalues E_n with corresponding weak eigenfunctions ψ_n.*

If $\|\psi_n\| = 1$ for all $n \in \mathbb{N}$ and $\sup_n E_n < \infty$, then the weak eigenfunctions ψ_n are pointwise locally bounded uniformly in $n \in \mathbb{N}$, that is,

$$\sup_{n\in\mathbb{N}} \sup_{x\in S} |\psi_n(x)| < \infty \tag{4.10}$$

for any bounded set $S \subset \mathbb{R}^d$.

Remark 4.7. Since eigenfunctions are continuous if the potential is locally in the Kato-class, $\psi_n(x)$ makes sense for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

Proof. Note that ψ_n is a zero energy weak eigenfunction of the Schrödinger operator $\widetilde{H}_n$ with potential $\widetilde{V}_n$ given by $\widetilde{V}_n = V - \lambda_n W - E_n$. If V and W are in the local Kato class, so is $\widetilde{V}_n$. Hence, for any $x \in \mathbb{R}^d$ the subsolution estimate

$$|\psi_n(x)| \le C_{x,n} \int_{|x-y|<1} |\psi_n(y)|\, dy \tag{4.11}$$

holds (see [71, Theorem C.1.2] and also [3, 67]). Moreover, it is shown in [71] that the constants $C_{x,n}$ depend only on

$$\|\mathbb{1}_{B_1(x)}(\widetilde{V}_n)_-\|_{K^d}$$

with $(\widetilde{V}_n)_-$ being the negative part of $\widetilde{V}_n$ and the Kato norm $\|\cdot\|_{K^d}$ given by

$$\|V\|_{K^d} := \sup_{x\in\mathbb{R}^d} \int_{|x-y|\le 1} \widetilde{g}_d(x-y)|V(y)|dy \tag{4.12}$$

with $\widetilde{g}_d = g_d$ when $d \ge 3$, $\widetilde{g}_2 = 1 + g_2$, and $\widetilde{g}_1 = 1$, where g_d is defined in (1.15). Adding 1 to g_2 is necessary since $g_2(x) = 0$ when $|x| = 1$.

For any set $S \subset \mathbb{R}^d$ and any potential V, we have

$$\sup_{x\in S} \|\mathbb{1}_{B_1(x)} V\|_{K^d} \le \|\sup_{x\in S} \mathbb{1}_{B_1(x)} V\|_{K^d} = \|\mathbb{1}_{S_1} V\|_{K^d}, \tag{4.13}$$

where $S_1 = \{y \in \mathbb{R}^d : \operatorname{dist}(y, S) < 1\}$.

Now let $S \subset \mathbb{R}^d$ be bounded. Then S_1 is bounded and, since the Kato norm of a constant function is finite and $(\widetilde{V}_n)_- = (V - \lambda_n W - E_n)_- \le V_- + \lambda_n W_+ + (E_n)_+$, we have

$$\sup_n \|\mathbb{1}_{S_1}(\widetilde{V}_n)_-\|_{K^d} \le \big(\|\mathbb{1}_{S_1} V_-\|_{K^d} + \sup_n \lambda_n \|\mathbb{1}_{S_1} W_+\|_{K^d} + \sup_n (E_n)_+ \|1\|_{K^d}\big) < \infty$$

for any bounded set S, using that $\sup_n E_n < \infty$ and $\sup_n \lambda_n < \infty$, by assumption, and $\|\mathbb{1}_{S_1} V_-\|_{K^d} < \infty$ and $\|\mathbb{1}_{S_1} W_+\|_{K^d} < \infty$, since S_1 is bounded and V and W are locally in the Kato class.

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Thus, for any bounded set $S \subset \mathbb{R}^d$, there exists a constant $C < \infty$, such that

$$|\psi_n(x)| \leq C \int_{|x-y|<1} |\psi_n(y)|\, dy \tag{4.14}$$

for all $x \in S$ and $n \in \mathbb{N}$. Using the normalization $\|\psi_n\| = 1$, we have

$$\int_{|x-y|<1} |\psi_n(y)|\, dy \leq |B_1^d|^{1/2}\|\psi_n\| = |B_1^d|^{1/2} \tag{4.15}$$

for all $x \in S$ and $n \in \mathbb{N}$. Hence, (4.10) follows immediately from (4.14). □

The last result which we need is

Lemma 4.8. *Assume that $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and V_- is form small w.r.t. $-\Delta + V_+$ and ψ is a real-valued weak eigenfunction of H at energy E. Then $|\psi|$ is a subsolution of H at energy E.*

Proof. If ψ is a real-valued eigenfunction of H at energy E, then it is also a subsolution, hence, [2, Lemma 2.9] shows that its positive part $\psi_+ = \sup(\psi, 0)$ is a subsolution. The same argument applied to $-\psi$, which is also a weak solution, shows that its negative part $\psi_- = \sup(-\psi, 0)$ is a subsolution. Hence, $|\psi| = \psi_+ + \psi_-$ is a subsolution of H at energy E. □

Remark 4.9. It is well-known that for the type of Schrödinger operator H we consider here, the eigenfunctions can be chosen to be real-valued. Since H is self-adjoint, all eigenvalues are real. Moreover, H commutes with complex conjugation, so for any complex-valued eigenfunction ψ of H, also the real and imaginary parts $\mathrm{Re}(\psi) = \frac{1}{2}(\psi + \overline{\psi})$ and $\mathrm{Im}(\psi) = \frac{1}{2i}(\psi - \overline{\psi})$ are eigenfunction of H at energy E. This is not true anymore if one considers Schrödinger operators with magnetic fields, since they do not commute with complex conjugation, in general.

Now we are ready to give the

Proof of Theorem 1.7. By assumption, the potential V is critical. Thus, $\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0, \infty)$. Moreover, for any nontrivial potential $W \geq 0$ which is infinitesimally form small w.r.t. $-\Delta + V_+$ and has compact support, the Schrödinger operators $H_\lambda = H - \lambda W$, constructed in Remark 2.1, have nontrivial discrete spectrum below zero. That is, $\sigma_{\mathrm{ess}}(H_\lambda) = [0, \infty)$, and there exist eigenvalues $E_\lambda < 0$ of H_λ with associated normalized weak eigenfunctions ψ_λ for all $\lambda > 0$. We take any sequence $(\lambda_n)_{n\in\mathbb{N}}$ which is monotonically decreasing to zero and abbreviate $H_n = H_{\lambda_n}$, $E_n = E_{\lambda_n}$, and $\psi_n = \psi_{\lambda_n}$.

Recall that we also assume that the potential V satisfies the lower bound

$$V(x) \geq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2}\sum_{j=1}^{m}\prod_{k=1}^{j} \ln_k^{-1}(|x|) + \frac{2\epsilon}{|x|^2}\prod_{k=1}^{m} \ln_k^{-1}(|x|)$$

for $|x| > R$, some $m \in \mathbb{N}_0$, $\epsilon > 0$, and all large enough $R > 0$. We replaced ϵ by 2ϵ in (1.6). Increasing R, if necessary, it is easy to see that this implies

$$V(x) \geq Y_{m,\epsilon}(x) \text{ for all } |x| \geq R, \tag{4.16}$$

where the family of comparison functions $Y_{m,\epsilon}$ is defined in (4.2).

Since W has compact support, we can also assume that R is so large that its support $\mathrm{supp}(W) \subset B_R(x)$. Thus, with $U_R = \{|x| > R\}$, we have $W\varphi = 0$ for all $\varphi \in \mathcal{C}_0^\infty(U_R)$. Lemma 4.1 and (4.16) imply

$$\begin{aligned} &\langle \varphi, (H_n - E_n)\psi_{u,m,\epsilon}\rangle = \langle \varphi, (H - E_n)\psi_{u,m,\epsilon}\rangle \\ &\quad = \langle \varphi, (-\Delta + Y_{m,\epsilon} - E_n)\psi_{u,m,\epsilon}\rangle + \langle \varphi, (V - Y_{m,\epsilon})\psi_{u,m,\epsilon}\rangle \geq -E_n\langle \varphi, \psi_{u,m,\epsilon}\rangle \geq 0 \end{aligned} \tag{4.17}$$

for all $0 \leq \varphi \in \mathcal{C}_0^\infty(U_R)$. Here, $\psi_{u,m,\epsilon} > 0$ is defined in (4.1), and we used that $E_n \leq 0$.

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So for fixed $m \in \mathbb{N}$, large enough $R > 0$, and small enough $\epsilon > 0$, the function $\psi_{u,m,\epsilon}$ is a supersolution of H_n at energy E_n in U_R for all $n \in \mathbb{N}$. Moreover, since $\|\psi_n\| = 1$, we have

$$c_R^1 := \sup_{n\in\mathbb{N}} \sup_{R\le|x|\le R+1} |\psi_n(x)| < \infty$$

by Lemma 4.6. Since $\psi_{u,m,\epsilon} > 0$ is continuous away from zero, we also have

$$c_R^2 = \inf_{R\le|x|\le R+1} \psi_{u,m,\epsilon}(x) > 0$$

and using $C_R = c_R^1/c_R^2$, one gets $|\psi_n(x)| \le C_R\psi_{u,m,\epsilon}(x)$, hence, also

$$\widetilde{\psi}_n(x) := |\mathrm{Re}(\psi_n(x))| + |\mathrm{Im}(\psi_n(x))| \le \sqrt{2}|\psi_n(x)| \le \sqrt{2}C_R\psi_{u,m,\epsilon}(x) \tag{4.18}$$

for all $R \le |x| \le R+1$ and all $n \in \mathbb{N}$. Clearly, $|\psi_n| \le \widetilde{\psi}_n$. Since $\widetilde{\psi}_n$ is a nonnegative subsolution of H_n at energy E_n by Lemma 4.8 and Remark 4.9, we can use $w = \psi_{u,m,\epsilon}$ and $v = \widetilde{\psi}_n$ in Theorem 2.7 to see that

$$|\psi_n(x)| \le \widetilde{\psi}_n(x) \le \sqrt{2}C_R\psi_{u,m,\epsilon}(x) \quad \text{for all } |x| \ge R \tag{4.19}$$

uniformly in $n \in \mathbb{N}$. Since $\psi_{u,m,\epsilon}$ is square integrable at infinity for any fixed $m \in \mathbb{N}$ and $\epsilon > 0$, the bound (4.19) yields tightness in x-space, that is,

$$\lim_{R\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|>R} |\psi_n(x)|^2 \mathrm{d}x = 0. \tag{4.20}$$

From (4.5), one gets $\sup_{n\in\mathbb{N}} \|\psi_n\|_{H^1} < \infty$. In particular, we have

$$\lim_{L\to\infty} \sup_{n\in\mathbb{N}} \int_{|\eta|>L} |\widehat{\psi}_n(\eta)|^2 \mathrm{d}\eta = 0, \tag{4.21}$$

which is tightness in momentum space. Here, $\widehat{\psi}_n$ is the Fourier transform of ψ_n.

Moreover, since ψ_n is bounded in $H^1(\mathbb{R}^d)$, there exists a subsequence which converges weakly in H^1 and L^2. By a slight abuse of notation, we also write ψ_n for this subsequence. Let $\psi \in L^2(\mathbb{R}^d)$ be the weak limit of ψ_n. Tightness and weak convergence then imply that ψ_n converges to ψ in L^2 (see, e.g. [29, Appendix A]). Hence, $\|\psi\| = \lim_{n\to\infty} \|\psi_n\| = 1$.

Lemma 4.2 shows that $E = \lim_{n\to\infty} E_n \le 0$ exists and that ψ is a normalized weak eigenfunction of H with eigenvalue E. Clearly, $E = 0$ since $\sigma(H) = [0, \infty)$. So zero is the ground state eigenvalue of H, which is at the edge of the essential spectrum of H. This finishes the proof of Theorem 1.7. □

Remark 4.10. Note that we could have simplified some parts of the proof by using that ground states can be chosen to be strictly positive. We intentionally avoided the use of strict positivity of ground state eigenfunctions. This allows to use Theorem 1.7 also for systems with symmetry restrictions, or for the existence of higher eigenstates with energies above the ground state energy, provided one suitably modifies the assumption of a virtual level for such systems. These modifications are straightforward.

A. An example in search of a theorem

It is well-known that the zero potential is critical in dimensions one and two (see [69] and also [44, Problems 1 and 2 in Chapter 45]). This phenomenon can be explained by the nonintegrability of $\eta \mapsto |\eta|^{-2}$ near $\eta = 0$ in $\mathbb{R}^d$ (see [25]). The Iorio-O'Carroll theorem [64, Theorem XII.27] shows that shallow potential wells cannot create ground states in dimension $d \ge 3$ and that the corresponding Schrödinger operators are even unitarily equivalent to the free Laplacian. Of course, in order to construct zero energy resonances or zero energy ground states, one can take any Schrödinger operator H which

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has essential spectrum $[0,\infty)$ and finitely many negative eigenvalues. Adding a suitable local positive perturbation then moves the ground state energy to zero, creating a zero energy resonance, or zero energy ground state, depending, for example, on which a priori bound from Theorem 1.3 or Theorem 1.7 holds. Specific examples of critical potentials in dimension one and two which are different from the zero potential seem to be rare. In the following, we construct a family of potentials $V_{\alpha,d}$ in any dimension which are critical for $\alpha \geq 0$, having a zero energy resonance when $0 \leq \alpha \leq 1$ and a zero energy ground state when $\alpha > 1$, and which are not critical when $\alpha < 0$. To the best of our knowledge, our example is new.

Remark A.1. There are different definitions for a zero energy resonance available in the literature. One often calls ψ a zero energy resonance if it is a local positive eigenfunction of a Schrödinger operator H which is not square integrable on $\mathbb{R}^d$, but its gradient $\nabla\psi$ is square integrable. We will follow this convention, except that we also allow that the L^2-norm of $\nabla\psi$ is logarithmically divergent at infinity.

For $\alpha \in \mathbb{R}$ and $d \in \mathbb{N}$, define the potential $V_{\alpha,d}$ on $\mathbb{R}^d$ by

$$V_{\alpha,d}(x) := \frac{4\alpha^2 - (d-2)^2}{4\big(1+|x|^2\big)} + \frac{1-(\alpha+d/2)^2}{\big(1+|x|^2\big)^2}. \tag{A.1}$$

Clearly, $V_{\alpha,d}$ is bounded, continuous, and goes to zero at infinity. In particular, for all $d \geq 1$ and $\alpha \in \mathbb{R}$, the potentials $V_{\alpha,d}$ are both infinitesimally operator bounded and infinitesimally form bounded, w.r.t. $-\Delta$ and in the Kato-class K_d. Therefore, the Schrödinger operator $H_{\alpha,d} = -\Delta + V_{\alpha,d}$ is a well-defined self-adjoint operator on the domain $H^2(\mathbb{R}^d)$ with form domain $H^1(\mathbb{R}^d)$.

The key to understanding why the potentials $V_{\alpha,d}$ are critical for all $\alpha \geq 0$ and $d \geq 1$, not critical for $\alpha < 0$, and switch from having zero energy resonances to having zero energy ground states at $\alpha = 1$ is

Lemma A.2 (Ground state representation of $H_{\alpha,d}$). *Let $\alpha \in \mathbb{R}$, $d \geq 1$, and define*

$$\psi_{\alpha,d}(x) = (1+|x|^2)^{(2-d)/4-\alpha/2} \tag{A.2}$$

for $x \in \mathbb{R}^d$ and the measure

$$\mu_{\alpha,d}(B) = \int_B \psi_{\alpha,d}^2\,dx \tag{A.3}$$

on the Borel sets B in $\mathbb{R}^d$. Then the map $U_{\alpha,d} : L^2(\mathbb{R}^d, d\mu_{\alpha,d}) \to L^2(\mathbb{R}^d)$ given by

$$(U_{\alpha,d}\varphi) = \psi_{\alpha,d}\,\varphi \tag{A.4}$$

is unitary with

$$U_{\alpha,d}^{-1}(H^1(\mathbb{R}^d)) = \{\varphi \in L^2(\mathbb{R}^d, d\mu_{\alpha,d}) : \nabla\varphi \in L^2(\mathbb{R}^d, d\mu_{\alpha,d})\}. \tag{A.5}$$

Moreover, $U_{\alpha,d}H_{\alpha,d}U_{\alpha,d}^{-1} = -\Delta$ in the sense that for all $\psi \in H^1(\mathbb{R}^d)$, the form domain of $H_{\alpha,d}$,

$$\langle\psi, H_{\alpha,d}\psi\rangle = \langle\nabla\psi,\nabla\psi\rangle - \langle\psi, V_{\alpha,d}\psi\rangle = \int_{\mathbb{R}^d} |\nabla\varphi|^2\,\psi_{\alpha,d}^2\,dx, \tag{A.6}$$

where $\varphi = U_{\alpha,d}^{-1}\psi$.

Remark A.3. Lemma A.2 shows that the Schrödinger operator $H_{\alpha,d}$ is equivalent to the nonnegative Dirichlet form $q(\varphi) = \langle\nabla\varphi,\nabla\varphi\rangle_{L^2(\mathbb{R}^d, d\mu_{\alpha,d})}$ on the weighted L^2-space with measure $d\mu_{\alpha,d} = (1+|x|^2)^{-(d-2)/2-\alpha}dx$. Note that this measure is finite if and only if $\alpha > 1$.

We give the proof of the lemma at the end of the Appendix.

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Theorem A.4. *Let $d \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $H_{\alpha,d} = -\Delta + V_{\alpha,d}$ be the self-adjoint Schrödinger operator with potential $V_{\alpha,d}$ given by (A.1). Then*

a) $\sigma(H_{\alpha,d}) = \sigma_{\mathrm{ess}}(H_{\alpha,d}) = [0,\infty)$.
b) *For all $\alpha \geq 0$, the potential $V_{\alpha,d}$ is critical, that is, the Schrödinger operator $H_{\alpha,d}$ has a virtual level.*
c) *Zero is not an eigenvalue of $H_{\alpha,d}$ when $0 \leq \alpha \leq 1$. For $\alpha > 1$, zero is an eigenvalue. The local zero energy ground state, when $0 \leq \alpha \leq 1$, respectively ground state, when $\alpha > 1$, is given by (A.2).*
d) *For $\alpha < 0$, the potential $V_{\alpha,d}$ is subcritical, hence zero is neither an eigenvalue nor a virtual level.*

Remark A.5. Using the early result of Kato, [36], see also [1, 68], the operator $H_{\alpha,d}$ has no strictly positive embedded eigenvalues. Since the potential $V_{\alpha,d}$ is short-range, the spectrum of $H_{\alpha,d}$ is even purely absolutely continuous inside $(0,\infty)$ (see [14, Theorem 5.10]).

Proof. Using standard methods, [74], one sees that $\sigma_{\mathrm{ess}}(H_{\alpha,d}) = \sigma_{\mathrm{ess}}(-\Delta) = [0,\infty)$ since $V_{\alpha,d}$ is bounded and goes to zero at infinity. Moreover, the ground state representation (A.6) implies $\sigma(H_{\alpha,d}) \subset [0,\infty)$. Hence, $\sigma(H_{a,d}) = \sigma_{\mathrm{ess}}(H_{a,d}) = [0,\infty)$. This proves claim a).

Given $\varphi \in L^2(\mathbb{R}^d, d\mu_{\alpha,d})$, let $\psi := U_{\alpha,d}\varphi$. Taking $\varphi = 1$ gives $\psi = \psi_{\alpha,d} > 0$ which is in $H^1(\mathbb{R}^d)$ if and only if $\alpha > 1$. In this case, (A.6) and standard arguments show that $\psi_{\alpha,d}$ is the ground state of $H_{\alpha,d}$ corresponding to the eigenvalue zero. This proves the second claim in c). Lemma 2.6 also shows that the potential $V_{\alpha,d}$ is critical when $\alpha > 1$.

In addition, note that the right-hand side of (A.6) is strictly positive unless φ is constant. When $0 \leq \alpha \leq 1$, $\psi_{\alpha,d}$ is not square integrable anymore. Nevertheless, one can bilinearize the ground state representation (A.6) to see that

$$\langle \psi_1, H_{\alpha,d}\psi_2\rangle = \langle \nabla\varphi_1, \psi_{\alpha,d}^2 \nabla\varphi_2\rangle,$$

where $\varphi_j = \psi_{\alpha,d}^{-1}\psi_j$, for $j = 1,2$. Since for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ also $\varphi = \psi_{\alpha,d}^{-1}\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, we have

$$\langle \psi, H_{\alpha,d}\psi_{\alpha,d}\rangle = \langle \nabla\varphi, \psi_{\alpha,d}^2 \nabla 1\rangle = 0, \tag{A.7}$$

so $\psi_{\alpha,d}$ is a local ground state of $H_{\alpha,d}$ when $0 \leq \alpha \leq 1$.

To show that zero is a virtual level when $0 < \alpha \leq 1$, we take any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\varphi(t) = 1$ for $|t| \leq 1$, $\varphi(t) = 0$ for $|t| \geq 2$, and define

$$\varphi_R(x) = \varphi(|x|/R)$$

for $R > 0$. Then $|\nabla\varphi_R(x)| = R^{-1}|\varphi'|(|x|/R)$. Using $\psi_R = \psi_{\alpha,d}\,\varphi_R$, we get

$$\begin{aligned}\langle \psi_R, H_{\alpha,d}\psi_R\rangle &= R^{-2}\int |\varphi'(|x|/R)|^2 (1+|x|^2)^{(2-d)/2-\alpha}\,dx \\ &\lesssim R^{-2}\int_R^{2R} (1+r^2)^{(2-d)/2-\alpha}\,r^{d-1}dr \sim R^{-2}\int_R^{2R}(1+r^2)^{-\alpha}\,r\,dr \lesssim R^{-2\alpha} \to 0\end{aligned}$$

for $R \to \infty$ and $\alpha > 0$. Now let $W \geq 0$ have compact support, be infinitesimally form bounded w.r.t. $-\Delta$, and $W > 0$ on a set of positive Lebesgue measure. Since $\psi_R(x) \to (1+|x|^2)^{(2-d)/4-\alpha/2}$ as $R \to \infty$ uniformly on compact sets, we have

$$\lim_{R\to\infty}\langle \psi_R, (H_{\alpha,d} - \lambda W)\psi_R\rangle = -\lambda \int W(x)(1+|x|^2)^{(2-d)/2-\alpha}\,dx < 0$$

for all $\lambda > 0$. Thus, $\langle \psi_R, (H_{\alpha,d} - \lambda W)\psi_R\rangle < 0$ for all large enough $R > 0$. Since $\sigma_{\mathrm{ess}}(H_{\alpha,d} - \lambda W) = [0,\infty)$, the Rayleigh–Ritz principle shows that $H_{\alpha,d} - \lambda W$ has a negative eigenvalue for any $\lambda > 0$.

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Thus, the potential $V_{\alpha,d}$ is critical. Clearly, zero cannot be an eigenvalue because $\psi_{\alpha,d}$ is not L^2 for $\alpha \le 1$ (see (A.7)). To see that $V_{0,d}$ is critical, one needs to modify the ansatz function. Let $\delta > 0$, and set

$$\varphi_\delta(x) := \begin{cases} 1 & \text{if } |x| \le 1 \\ (1 - \delta \ln |x|)_+ & \text{if } |x| > 1 \end{cases}$$

and $\psi_\delta = U_{\alpha,d}\varphi_\delta$. A straightforward calculation shows

$$\begin{aligned} \langle \psi_\delta, H_{0,d}\psi_\delta \rangle &= \delta^2 \int_{1 \le |x| \le e^{1/\delta}} (1 + |x|^2)^{(2-d)/2} |x|^{-2}\, dx \\ &\lesssim \delta^2 \int_1^{e^{1/\delta}} (1 + r^2)^{(2-d)/2}\, r^{d-3} dr \sim \delta^2 \int_1^{e^{1/\delta}} (1 + r^2)^{-1}\, r dr \\ &= \frac{\delta^2}{2} \ln(1 + e^{1/\delta}) \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0. \end{aligned}$$

Thus, $\lim_{\delta \to 0} \langle \psi_\delta, (H_{0,d} - \lambda W)\psi_\delta \rangle = -\lambda \int W(x)(1 + |x|^2)^{(2-d)/2}\, dx < 0$. As before, this shows that $V_{0,d}$ is critical. Moreover, even though $\psi_{\alpha,d} \notin L^2(\mathbb{R}^d)$ when $0 \le \alpha \le 1$, its gradient $\nabla\psi_{\alpha,d}$ is in $L^2(\mathbb{R}^d)$ when $0 < \alpha \le 1$ and the L^2-norm of $\nabla\psi_{0,d}$ is only logarithmically divergent. Hence, $\psi_{\alpha,d}$ is a zero energy resonance for $H_{\alpha,d}$ when $0 \le \alpha \le 1$. This finishes the proofs of claims b), c).

Finally, we look at $V_{-\alpha,d}$ for $\alpha > 0$. A simple calculation shows

$$V_{-\alpha,d}(x) = V_{\alpha,d}(x) + 2\alpha d(1 + |x|^2)^{-2}.$$

Thus, with $W(x) = (1 + |x|^2)^{-2} > 0$ and $\lambda = 2\alpha d > 0$, we have

$$\langle \psi, (H_{-\alpha,d} - \lambda W)\psi \rangle = \langle \psi, H_{\alpha,d}\psi \rangle \ge 0$$

for all $\psi \in H^1(\mathbb{R}^d)$, since $\sigma(H_{\alpha,d}) = [0, \infty)$ by part a). Hence, $V_{-\alpha,d}$ is subcritical for $\alpha > 0$. □

Remark A.6. Our proof that $V_{\alpha,d}$ is critical for $\alpha \ge 0$ depends on the ground state representation (A.6). Instead of using this representation, one can also use the main result in [60] to show that $V_{\alpha,d}$ is critical for $\alpha \ge 0$.

The family of potential $V_{\alpha,d}$ has several interesting properties summarized in

Lemma A.7 (Properties of $V_{\alpha,d}$). *Let $\alpha \in \mathbb{R}$ and $d \in \mathbb{N}$. Then*

a) *In dimensions $d = 1, 2$, the potential $V_{\alpha,d}$ is nontrivial if $\alpha \ne |d - 2|/2$, and in dimension $d \ge 3$, it is nontrivial if $\alpha \ne (2 - d)/2$.*
b) *In dimension $d = 1$, we have $V_{\alpha,1} > 0$ for $\alpha \le -1/2$. If $-1/2 < \alpha < 1/2$, then $V_{\alpha,1} > 0$ near zero, and it has a negative tail, that is, $V_{\alpha,1}(x) < 0$ for large $|x|$. If $\alpha > 1/2$, then $V_{\alpha,1}$ is negative near zero, and it has a positive tail.*
c) *In dimension $d = 2$, we have $V_{\alpha,2} > 0$ for all $\alpha < 0$, that is, $V_{\alpha,2}$ is purely repulsive. For $\alpha > 0$, the potential $V_{\alpha,2}$ is negative near zero and has a positive tail.*
d) *In dimension $d \ge 3$, we have $V_{\alpha,d} > 0$ for $\alpha < (2 - d)/2$, that is, the potential is repulsive. For $(2 - d)/2 < \alpha \le (d - 2)/2$, we have $V_{\alpha,d} < 0$, that is, the potential is attractive. If $\alpha > (d - 2)/2$, then $V_{\alpha,d}$ is negative near zero and has a positive tail.*
e) *For $d = 1$, the potential $V_{\alpha,1}$ is integrable and*

$$\int_{-\infty}^{\infty} V_{\alpha,1}(x)\, dx = \frac{\pi}{2}(\alpha - 1/2)^2.$$

The integral is positive unless $\alpha = 1/2$.

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f) *For large enough R and all dimensions $d \geq 1$, the potentials $V_{\alpha,d}$ satisfy the bounds (1.7) for $0 \leq \alpha < 1$, respectively (1.9) for $\alpha = 1$, while they satisfy the complementary bound (1.8) for $0 < \epsilon < 4(\alpha^2 - 1)$ when $\alpha > 1$.*

Remark A.8. Claims b) and c) above are consistent with what is known about weakly coupled bound states in low dimensions. It is known that if $V \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} V\,dx \leq 0$, where V is supposed to be nontrivial when $\int V\,dx = 0$, then the operator $-\Delta + \lambda V$ always has a negative bound state, no matter how small the coupling parameter $\lambda > 0$ is, when $d = 1, 2$. See, for example, [69], where this is proved under some additional assumptions, or [25] for the full result. In particular, this implies that critical potentials in one and two dimensions have to change sign and, if they are integrable, then $\int_{\mathbb{R}^d} V\,dx > 0$ unless V is trivial.

In addition, claim d) is consistent with our nonexistence Theorem 1.3. Nonpositive potentials cannot have a zero energy ground state in dimensions $d \leq 4$. They need to have a strong enough positive tail in order to be able to have zero energy bound states. Claim f) together with the fact that the potential $V_{\alpha,d}$ supports zero energy ground states if and only if $\alpha > 1$, see Theorem A.4, is consistent with our Theorems 1.3 and 1.7.

It is illuminating to plot $V_{\alpha,d}(x)$ for $|x| = r$ to explicitly see the behavior of $V_{\alpha,d}$ for various values of the parameters α and d.

Proof. The first claim a) is easy to check. To prove the rest, let $a_{\alpha,d} = \alpha^2 - (d-2)^2/4$ and $b_{\alpha,d} = 1 - (\alpha + d/2)^2$. Then

$$4V_{\alpha,d}(0) = 4(a_{\alpha,d} + b_{\alpha,d}) = -2d(d - 2 + 2\alpha) > 0$$

if and only if $\alpha < (2-d)/2$. Moreover, unless $a_{\alpha,d} = 0$, the sign of $V_{\alpha,d}(x)$ for large $|x|$ is determined by the sign of $a_{\alpha,d}$. Since $a_{\alpha,d} > 0$ if and only if $|\alpha| > |d-2|/2$, it is straightforward to deduce the claims b), c), and d) from this.

Clearly, $V_{\alpha,1}$ is integrable. Using $\int_{-\infty}^{\infty}(1+x^2)^{-1}\,dx = \pi$ and $\int_{-\infty}^{\infty}(1+x^2)^{-2}\,dx = \pi/2$, claim e) follows from a simple calculation.

Since for large $|x|$ the second term in the definition of $V_{\alpha,d}$ is much smaller than the first, the last claim f) follows from a straightforward computation. $\square$

It remains to give the proof of the ground state representation.

Proof of Lemma A.2. Let $\gamma \in \mathbb{R}$, and set $\psi_\gamma(x) = (1 + |x|^2)^{-\gamma/2}$ for $x \in \mathbb{R}^d$, which is a regularized version of $|x|^{-\gamma}$ used at the end of Section 2. When ψ and φ are related by

$$\psi = \psi_\gamma\,\varphi, \tag{A.8}$$

then $\psi \in L^2(\mathbb{R}^d)$ is clearly equivalent to $\varphi \in L^2(\mathbb{R}^d, \psi_\gamma^2\,dx)$ and the corresponding norms are the same. So the map $U_\gamma : L^2(\mathbb{R}^d, \psi_\gamma^2\,dx) \to L^2(\mathbb{R}^d)$, $\varphi \mapsto \psi_\gamma\varphi$ preserves the corresponding norms. Its inverse is given by $U_\gamma^{-1}\psi = U_{-\gamma}\psi = \psi_\gamma^{-1}\psi$ and from this, one easily checks that U_γ is a unitary map from the weighted space $L^2(\mathbb{R}^d, \psi_\gamma^2\,dx)$ to $L^2(\mathbb{R}^d)$. This proves (A.4).

If $\psi \in L^2(\mathbb{R}^d)$ and $\varphi = U_\gamma^{-1}\psi \in L^2(\mathbb{R}^d, \psi_\gamma^2\,dx)$, then we have, in the sense of distributions,

$$\nabla\psi = \varphi\nabla\psi_\gamma + \psi_\gamma\nabla\varphi = -\gamma\psi_\gamma(1+|x|^2)^{-1}x\varphi + \psi_\gamma\nabla\varphi = -\gamma(1+|x|^2)^{-1}x\psi + \psi_\gamma\nabla\varphi \tag{A.9}$$

since $\psi_\gamma \in \mathcal{C}^\infty(\mathbb{R}^d)$. Clearly, $(1+|x|^2)^{-1}x$ is bounded on $\mathbb{R}^d$. Therefore, if $\varphi \in L^2(\mathbb{R}^d, \psi_\gamma^2\,dx)$ and $\nabla\varphi \in L^2(\mathbb{R}^d, \psi_\gamma^2\,dx)$, then (A.9) shows that $\nabla\psi \in L^2(\mathbb{R}^d)$. Hence, if φ and $\nabla\varphi$ are in $L^2(\mathbb{R}^d, \psi_\gamma^2\,dx)$, then $\psi = U_\gamma\varphi$ is in $H^1(\mathbb{R}^d)$.

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Conversely, if $\psi \in H^1(\mathbb{R}^d)$, then, as distributions, $\nabla\varphi = \gamma(1+|x|^2)^{\gamma/2-1}x\psi + (1+|x|^2)^{\gamma/2}\nabla\psi$, which shows that

$$\psi_\gamma \nabla\varphi = (1+|x|^2)^{-1}x\psi + \nabla\psi \in L^2(\mathbb{R}^d). \tag{A.10}$$

That is, if $\psi \in L^2(\mathbb{R}^d)$, then $\varphi = U_\gamma^{-1}\psi \in L^2(\mathbb{R}^d, \psi_\gamma^2\, dx)$ and if, in addition, $\nabla\psi \in L^2(\mathbb{R}^d)$, then (A.10) shows that $\nabla\varphi \in L^2(\mathbb{R}^d, \psi_\gamma^2\, dx)$. Altogether, this proves

$$U_\gamma^{-1}(H^1(\mathbb{R}^d)) = \left\{\varphi \in L^2(\mathbb{R}^d, \psi_\gamma^2\, dx) :\ \nabla\varphi \in L^2(\mathbb{R}^d, \psi_\gamma^2\, dx)\right\},$$

which is (A.5). Moreover, $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$ and since U_γ maps $\mathcal{C}_0^\infty(\mathbb{R}^d)$ into itself, it is also dense in $U_\gamma^{-1}(H^1(\mathbb{R}^d))$. So we only have to prove (A.6) for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.

Let $\gamma \in \mathbb{R}$ and $\psi = \psi_\gamma\varphi$ with $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Then, as already noticed before,

$$\nabla\psi(x) = -\gamma(1+|x|^2)^{-\gamma/2-1}x\varphi(x) + (1+|x|^2)^{-\gamma/2}\nabla\varphi(x),$$

hence

$$\begin{aligned}\langle\nabla\psi, \nabla\psi\rangle = \langle\nabla\varphi, (1+|x|^2)^{-\gamma}\nabla\varphi\rangle - 2\gamma\mathrm{Re}(\langle\nabla\varphi, (1+|x|^2)^{-\gamma-1}x\varphi\rangle)\\ + \gamma^2\langle\varphi, (1+|x|^2)^{-\gamma-2}|x|^2\varphi\rangle.\end{aligned} \tag{A.11}$$

An integration by parts shows

$$\begin{aligned}&\mathrm{Re}(\langle\nabla\varphi, (1+|x|^2)^{-\gamma-1}x\varphi\rangle) = -\mathrm{Re}(\langle\varphi, \nabla\cdot((1+|x|^2)^{-\gamma-1}x\varphi)\rangle)\\ &\quad = 2(\gamma+1)\langle\varphi, (1+|x|^2)^{-\gamma-2}|x|^2\varphi\rangle - d\langle\varphi, (1+|x|^2)^{-\gamma-1}\varphi\rangle - \mathrm{Re}(\langle\varphi, (1+|x|^2)^{-\gamma-1}x\nabla\varphi\rangle).\end{aligned}$$

Noticing that $\mathrm{Re}(\langle\varphi, (1+|x|^2)^{-\gamma-1}x\nabla\varphi\rangle) = \mathrm{Re}(\langle\nabla\varphi, (1+|x|^2)^{-\gamma-1}x\varphi\rangle)$, we get

$$2\gamma\mathrm{Re}(\langle\nabla\varphi, (1+|x|^2)^{-\gamma-1}x\varphi\rangle) = 2\gamma(\gamma+1)\langle\varphi, (1+|x|^2)^{-\gamma-2}|x|^2\varphi\rangle - d\gamma\langle\varphi, (1+|x|^2)^{-\gamma-1}\varphi\rangle,$$

and plugging this into (A.11), we arrive at

$$\begin{aligned}\langle\nabla\psi, \nabla\psi\rangle &= \langle\nabla\varphi, (1+|x|^2)^{-\gamma}\nabla\varphi\rangle - 2\gamma(\gamma+1)\langle\varphi, (1+|x|^2)^{-\gamma-2}|x|^2\varphi\rangle\\ &\quad + d\gamma\langle\varphi, (1+|x|^2)^{-\gamma-1}\varphi\rangle + \gamma^2\langle\varphi, (1+|x|^2)^{-\gamma-2}|x|^2\varphi\rangle\\ &= \langle\nabla\varphi, (1+|x|^2)^{-\gamma}\nabla\varphi\rangle - \gamma(\gamma+2-d)\langle\varphi, (1+|x|^2)^{-\gamma-1}\varphi\rangle\\ &\quad + \gamma(\gamma+2)\langle\varphi, (1+|x|^2)^{-\gamma-2}\varphi\rangle\\ &= \langle\nabla\varphi, (1+|x|^2)^{-\gamma}\nabla\varphi\rangle - \gamma(\gamma+2-d)\langle\psi, (1+|x|^2)^{-1}\psi\rangle\\ &\quad + \gamma(\gamma+2)\langle\psi, (1+|x|^2)^{-2}\psi\rangle.\end{aligned}$$

Choosing $\gamma = (d-2)/2 + \alpha$ finishes the proof of Lemma A.2. □

Remarks A.9. The proof of Lemma A.2 is clearly inspired by the proof of Hardy's inequality on $L^2(\mathbb{R}^d)$ for $d \geq 3$, where one considers $\psi(x) = |x|^{-\gamma/2}\varphi(x)$ for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus \{0\})$ and optimizes in $\gamma > 0$. One needs to restrict to $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d \setminus \{0\})$ due to the singularity of $|x|^{-\gamma/2}$ in zero. Since $\mathcal{C}_0^\infty(\mathbb{R}^d \setminus \{0\})$ is dense in $L^2(\mathbb{R}^d)$ only when $d \geq 3$, this leads to the well-known fact that Hardy's inequality only holds in dimensions $d \geq 3$.

Acknowledgements. It is a pleasure to thank the Centre International de Rencontres Mathématiques (CIRM), Luminy, for the Recherches en résidence (REB) (research in residence) program, where part of this work was done. Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173. This project also received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme

https://doi.org/10.1017/fms.2023.39 Published online by Cambridge University Press

(grant agreement MDFT No. 725528). Michal Jex also received financial support from the Ministry of Education, Youth and Sport of the Czech Republic under the Grant No. RVO 14000. Markus Lange was supported by NSERC of Canada and also acknowledges financial support from the ERC under the European Union Horizon 2020 research and innovation programme (ERC StG MaMBoQ, grant agreement No. 802901).

Competing Interest. The authors have no conflicts of interest to declare.

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