Analytic self-similar solutions of the Oberbeck-Boussinesq equations

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Abstract

In this article we will present pure two-dimensional analytic solutions for the coupled non-compressible Newtoniain Navier-Stokes — with Boussinesq approximation — and the heat conduction equation. The system was investigated from E.N. Lorenz half a century ago with Fourier series and pioneered the way to the paradigm of chaos. We present a novel analysis of the same system where the key idea is the two-dimensional generalization of the well-known self-similar Ansatz of Barenblatt which will be interpreted in a geometrical way. The results, the pressure, temperature and velocity fields are all analytic and can be expressed with the help of the error functions. The temperature field has a strongly damped oscillating behavior which is an interesting feature.

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I. INTRODUCTION

The investigation of the dynamics of viscous fluids has a long past. Enormous scientific literature is available from the last two centuries for fluid motion even without any kind of heat exchange. Thanks to new exotic materials like nanotubes, heat conduction in solid bulk phase (without any kind of material transport) is an other quickly growing independent research area as well. The combination of both processes are even more complex which lacks general existence theorems for unique solutions. The most simple way to couple these two phenomena together is the Boussinesq [1] approximation which is used in the field of buoyancy-driven flow (also known as natural convection). It states that density differences are sufficiently small to be neglected, except where they appear in terms multiplied by g, the acceleration due to gravity. The main idea of the Boussinesq approximation is that the difference in inertia is negligible but gravity is sufficiently strong to make the specific weight appreciably different between the two fluids. When the Boussinesq approximation is used than no sound wave can be described in the fluid, because sound waves move via density variation.

Boussinesq flows are quite common in nature (such as oceanic circulations, atmospheric fronts or katabatic winds), industry (fume cupboard ventilation or dense gas dispersion), and the built environment (like central heating, natural ventilation). The approximation is extremely accurate for such flows, and makes the mathematics and physics much simpler and transparent.

The advantage of the approximation arises because when investigation a flow of, say, warm and cold waters with densities ρ_1 and ρ_2 are considered, the difference $\Delta\rho = \rho_1 - \rho_2$ is negligible and one needs only a single density ρ. It can be shown with the help of dimensional analysis, under these circumstances, the only sensible way that acceleration due to gravity g should enter into the equations of motion is in the reduced gravity $g' = g(\rho_1 - \rho_2)$. The corresponding dimensionless numbers for such flows are the Richardson and Rayleigh numbers. The used mathematics is therefore much simpler because the density ratio (ρ_1/ρ_2 a dimensionless number) is exactly one and does not affect the features of the investigated flow system.

In the following we analyze the dynamics of a two-dimensional viscous fluid with additional heat conduction mechanism. Such systems were first investigated by Boussinesq [1] and Oberbeck [2] in the nineteenth century. Oberbeck used a finite series expansion. He developed a model to study the heat convection in fluids taking into account the flow of the fluid as a result of temperature difference. He applied the model to the normal atmosphere.

More than half a century later Saltzman [3] tried to solve the same model with the help of Fourier series. At the same time Lorenz [4] analyzed the solutions with computers and published the plot of a strange attractor which was a pioneering results and the advent of the studies of chaotic dynamical systems. The literature of chaotic dynamics is enormous but a modern basic introduction can be found in [5].

Later till to the first beginning years of the millennium [4] Lorenz analyzed the final first order chaotic ordinary differential equation(ODE) system with different numerical methods. This ODE system becomes an emblematic object of chaotic systems and attracts much interest till today [6].

On the other side critical studies came to light which go beyond the simplest truncated Fourier series. Curry for example gives a transparent proof that the finite dimensional approximations have bounded solutions [7]. Musielak *et al* [8] in three papers analyzed large number of truncated systems with different kinds and found chaotic and periodic solutions as well. The messages of these studies will be shortly mentioned later.

In our study we apply a completely different investigation approach, namely the two-dimensional generalization of the self-similar Ansatz which is well-known for one dimension from more than half a century [9–11]. This generalized Ansatz was successfully applied to the three dimensional compressible and non-compressible Navier-Stokes equations [12, 13] from us in the last years. We investigated one dimensional Euler equations with heat conduction as well [14] which can be understood as the precursor of the recent study.

To our knowledge this kind of investigation method was not yet applied to the Oberbeck-Boussinesq (OB) system. In the next section we outline our theoretical investigation together with the results. The paper ends with a short summary.

II. THEORY AND RESULTS

We consider the original partial differential equation(PDE) system of Saltzman [3] to describe heat conduction in a two dimensional viscous incompressible fluid. In Cartesian coordinates and Eulerian description these equations have the following form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} - eGT_1 - \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) = 0,$$

$$\frac{\partial T_1}{\partial t} + u\frac{\partial T_1}{\partial x} + w\frac{\partial T_1}{\partial z} - \kappa\left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial z^2}\right) = 0,$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \qquad (1)$$

where u, w, denote respectively the x and z velocity coordinates, T_1 is the temperature difference relative to the average ($T_1 = T - T_{av}$) and P is the scaled pressure over the density . The free physical parameters are ν, e, G, κ kinematic viscosity, coefficient of volume expansion, acceleration of gravitation and coefficient of thermal diffusivity. (To avoid further misunderstanding we use G for gravitation acceleration and g which is reserved for a self-similar solution.) The first two equations are the Navier-Stokes equations, the third one is the heat conduction equation and the last one is the continuity equation all are for two spatial dimensions. The Boussinesq approximation means the way how the heat conduction is coupled to the second NS equation. Chandrasekhar [15] presented a wide-ranging discussion of the physics and mathematics of Rayleigh-Benard convection along with many historical references.

Every two dimensional flow problem can be reformulated with the help of the stream function Ψ via $u = \Psi_y$ and $v = -\Psi_x$ which automatically fulfills the continuity equation. The subscripts mean partial derivations. After introducing dimensionless quantities the system of (1) is reduced to the next two PDEs

$$(\Psi_{xx} + \Psi_{yy})_t + \Psi_x(\Psi_{xxz} + \Psi_{yyz}) - \Psi_z(\Psi_{xxx} + \Psi_{zzx}) -$$

$$\sigma(\theta_x - \Psi_{xxxx} - \Psi_{zzzz} - 2\Psi_{xxzz}) = 0,$$

$$\theta_t + \Psi_x\theta_z - \Psi_z\theta_x - R\Psi_x - (\theta_{xx} + \theta_{zz}) = 0, \qquad (2)$$

where Θ is the scaled temperature, $\sigma = \nu/\kappa$ is the Prandtl Number and $R = \frac{GeH^3\Delta T_0}{\kappa\nu}$ is the Rayleigh number and H is the height of the fluid. A detailed derivation of (2) can be found in [3].

All the mentioned studies in the introduction, investigated these two PDEs with the help of some truncated Fourier series, different kind of truncations are available which result different ordinary differential equation(ODE) systems. The derivation of the final non-linear ODE system from the PDE system can be found in the original papers [3, 4]. Bergé *et al.* [16] contains a slightly different development of the Lorenz model equatios, and in addition, provides more details on how the dynamics evolve as the reduced Rayleigh number changes. The book of Sparrow [17] gives a detailed treatment of the Lorenz model and its behavior as well. Hilborn [18] presents the idea of the derivation in a transparent and easy way. Therefore, we do not mention this derivation in our manuscript.

Some truncations violates energy conservation [6] and some not. Roy and Musiliak [8] in his exhausting three papers present various energy-conserving truncations. Some of them contain horizontal modes, some of them contain vertical modes and some of them both kind of modes in the truncations. All these models show different features some of them are chaotic and some of them - in well-defined parameter regimes - show periodic orbits in the projections of the phase space. This is a true indication of the complex nature of the original flow problem. It is also clear that the Fourier expansion method which is a two hundred year old routine tool for linear PDEs fails for a relevant non-linear PDE system.

Therefore, we apply another investigation method which is common for non-linear PDEs. At first we introduce the two dimensional generalization of the self-similar Ansatz

$$v(x,t) = t^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) := t^{-\alpha} f(\eta) \tag{3}$$

where $v(x,t)$ can be an arbitrary variable of a PDE and t means time and x means spatial dependence. The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude $v(x,t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution for $t > 0$. The most powerful result of this Ansatz is the fundamental or Gaussian solution of the Fourier heat conduction equation (or for Fick's diffusion equation) with $\alpha = \beta = 1/2$. These solutions are exhibited on Figure 1. for time-points $t_1 < t_2$. This transformation is based on the assumption that a self-similar solution exists, i.e., every physical parameter preserves its shape during the expansion. Self-similar solutions usually describe the asymptotic behavior of an unbounded or a far-field problem; the time t and the space coordinate x appear only in the combination of $f(x/t^{\beta})$. It means that the existence of self-similar variables implies the lack of characteristic lengths and times. These solutions are usually not unique and do not take into account the initial stage of the physical expansion process. It is also transparent from (3) that to avoid singularity at $t = 0$ the following transformation $\tilde{t} = t + t_0$ is valid.

There is a reasonable generalization of (3) in the form of $v(x,t) = h(t) \cdot f[x/g(t)]$, where $h(t), g(t)$ are continuous functions. The choice of $h(t) = g(t) = \sqrt{t_0 - t}$ is called the blow-up solution, which means that the solution becomes infinity after a well-defined finite time duration.

These kind of solutions describe the intermediate asymptotic of a problem: they hold when the precise initial conditions are no longer important, but before the system has reached its final steady state. For some systems it can be shown that the self-similar solution fulfills the source type (Dirac delta) initial condition. They are much simpler than the full solutions and so easier to understand and study in different regions of parameter space. A final reason for studying them is that they are solutions of a system of ODEs and hence do not suffer the extra inherent numerical problems of the full PDEs. In some cases self-similar solutions helps to understand diffusion-like properties or the existence of compact supports of the solution.

Let's introduce the two dimensional generalization of the self-similar Ansatz (3) which might have the general form of

$$v(x,z,t) = t^{-\alpha} f\left(\frac{F(x,z)}{t^{\beta}}\right) \tag{4}$$

where $F(x,z)$ could be understood as an implicit parametrization of a one-dimensional space curve with continuous first and second derivatives. In our former studies [12, 13] we explain in heavy details that for the Navier-Stokes type of non-linearity unfortunately only the $F(x,z) = x + z + c$ function is valid in Cartesian coordinates which is a straight line. It basically comes from the symmetry properties of the left-hand of the NS equation. Only this function fulfills the following relation $u_x = u_z$. (Other locally orthogonal coordinate systems e.q. spherical are not investigated yet.)

We may investigate both dynamical systems, the original hydrodynamical (1) or the other one (2) which is valid for the stream functions.

Similar to the former studies [3, 4] try to solve the PDEs for the dimensionless stream and temperature functions in the form of

$$\Psi = t^{-\alpha} f(\eta), \; \theta = t^{-\epsilon} h(\eta), \; \eta = \frac{x+z}{t^{\beta}}. \tag{5}$$

Unfortunately, after some algebra it becomes clear that the constraints which should fix the values of the exponents become contradictory, therefore no unambiguous ODE can be

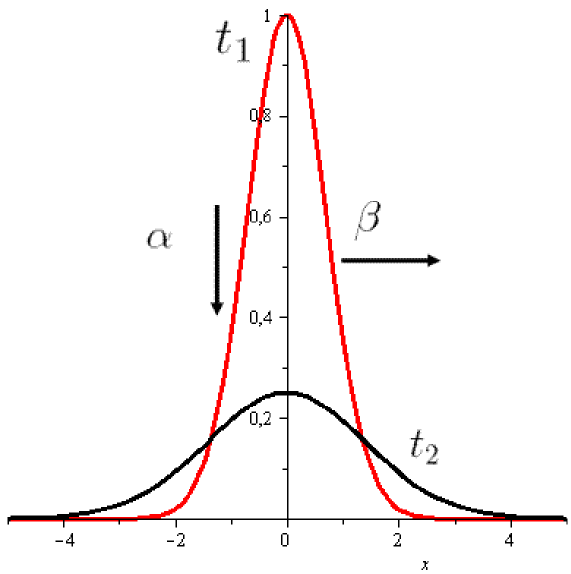

FIG. 1: A self-similar solution of Eq. (3) for $t_1 < t_2$. The presented curves are Gaussians for regular heat conduction.

derived. This means that the PDE of the stream function and the dimensionless temperature do not have self-similar solutions. In other words these functions have no such a diffusive property which could be investigated with the self-similar Ansats, which is a very instructive example of the applicability of the trial function of (5). Our experience shows that, most of the investigated PDEs have a self-similar ODE system and this is a remarkable exception.

Now investigate the original hydrodynamical system with the next Ansatz

$$u(\eta) = t^{-\alpha} f(\eta), \;\; w(\eta) = t^{-\delta} g(\eta), \;\; P(\eta) = t^{-\epsilon} h(\eta), \;\; T_1(\eta) = t^{-\omega} l(\eta), \tag{6}$$

where the new variable is $\eta = (x+z)/t^{\beta}$. All the five exponents $\alpha, \beta, \delta, \epsilon, \omega$ are real numbers. (Solutions with integer exponents are the self-similar solutions of the first kind and sometimes can be obtained from dimensional considerations.) The f, g, h, l objects are called the shape functions of the corresponding dynamical variables.

After some algebraic manipulations the following constrains are fixed among the self-similarity exponents : $\alpha = \delta = \beta = 1/2$, $\epsilon = 1$ and $\omega = 3/2$ which are called the universality relations. At this point it is worth to mention that now all the exponents have a fix numerical value which simplifies the structure of the solutions. There is no free exponential parameter in the original dynamical system, like an exponent in the equation of state. As an example we mention one of our former study where the compressible NS equation was investigated

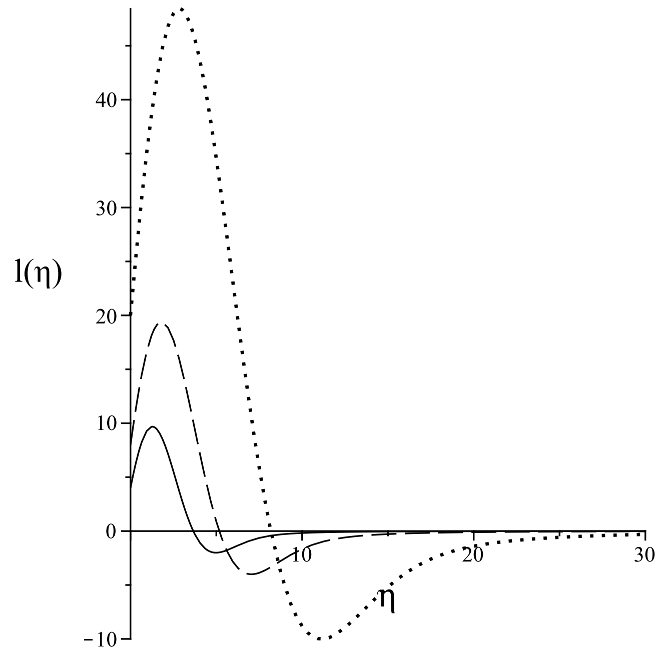

FIG. 2: Different shape functions of the temperature Eq. (9) as a function of η for different thermal diffusivity. The integration constants are $c_1 = c_2 = 1$ the same for all the three curves. The solid the dashed and the dotted lines are for $\kappa = 1, 2, 5$, respectively.

[13] with a free parameter which described different materials.

These universality relations dictate the corresponding coupled ODE system which has the following form of

$$
\begin{aligned}
-\frac{f}{2} - \frac{f'\eta}{2} + ff' + gf' + h' - 2\nu f'' &= 0, \\
-\frac{g}{2} - \frac{g'\eta}{2} + fg' + gg' + h' - eGl - 2\nu g'' &= 0, \\
-\frac{3l}{2} - \frac{l'\eta}{2} + fl' + gl' - 2\kappa l'' &= 0, \\
f' + g' &= 0.
\end{aligned} \tag{7}
$$

From the last (continuity) equation we automatically get the $f + g = c$ and $f'' + g'' = 0$ conditions which are necessary in the following.

Going through a straightforward derivation the next single ODE for the shape function of the temperature distribution can be separated

$$2\kappa l'' + \frac{l'\eta}{2} + \frac{3l}{2} = 0. \tag{8}$$

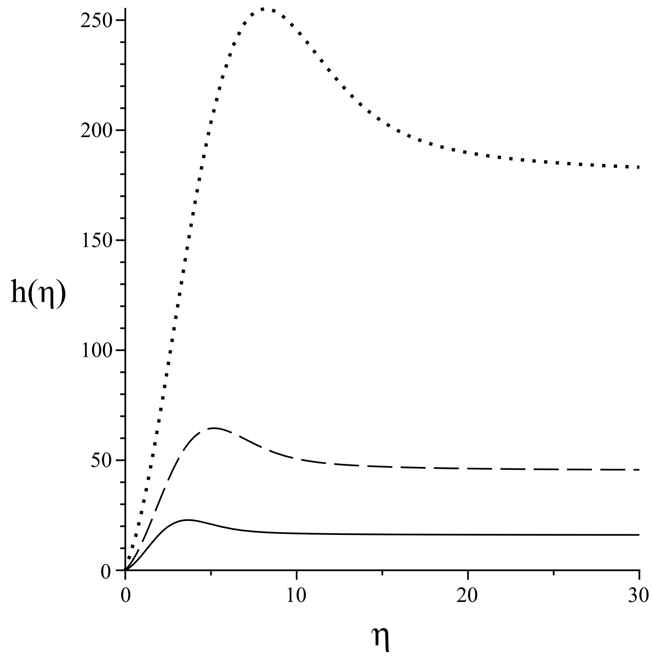

FIG. 3: Different shape functions of the pressure Eq. (12) as a function of η for different thermal diffusivity. The integration constants are taken $c_1 = c_2 = 1$ for all the three curves. We fixed the value of $eG = 1$ as well. The solid, the dashed and the dotted lines are for $\kappa = 1, 2, 5$ numerical values, respectively.

The solution is

$$l = c_1 \left[4erfi\left(\frac{\sqrt{2}\eta}{4\sqrt{\kappa}}\right)\sqrt{2\pi}\left(\kappa - \frac{\eta^2}{4}\right) e^{-\frac{\eta^2}{8\kappa}} + 4\sqrt{\kappa}\eta \right] + c_2 e^{-\frac{\eta^2}{8\kappa}}(4\kappa - \eta^2) \tag{9}$$

where c_1, c_2 are free integration constants. The erfi means the imaginary error function defined via the integral $2/\sqrt{\pi}\int_0^x exp(x^2)dx$ for more details see [19]. It is interesting, that the temperature distribution is separated from the other three dynamical variables an does not depend on the viscosity coefficients as well. We may say, that among the solution obtained from the self-similar Ansazt the temperature has the highest priority and this quantity defines the pressure and the velocity field. That is a remarkable feature. In a former study, where the one-dimensional Euler system was investigated with heat conduction [14] we found the opposite property, the density and the velocity field were much simpler than the temperature field. Figure 2 presents different shape functions of the temperature for different thermal diffusivity values. The first message is clear, the larger the thermal diffusivity the larger the shape function of the temperature distribution. A detailed analysis of Eq. (9) shows that for any reasonable κ and c values the main property of the function is not changing - has one global maximum and minimum with a strong decay for large ηs. A second remarkable feature is the single oscillation which is not a typical behavior for self-similar solutions. We investigated numerous non-linear PDE systems till today [12–14] some of them are even not hydrodynamical [20] and never found such a property. This analysis clearly shows that at least the temperature distribution in this physical system has a single-period anharmonic oscillation. For a fixed time value and a well-chosen z the difference of values of η where $l(\eta)$ yields a minima and a maxima correspondsto that Δx at which the temperature (and density) fluctuation may start the Behnard convection. To go a step further we may calculate the Fourier transform of the shape function, $l(\eta)$ Eq. (9) to study the spectral distribution. (An analytic expression for the Fourier transform is available, which we skip now.) The first term (which is proportion to c_1) becomes a complex function, however the general overall shape remains the same, a single-period anharmonic oscillation with a global minimum and maximum like on Figure 2. Of course, the zero transition of the function depends on the value of κ. The second term of the Fourier transformed function which is proportional to c_2 remains a Gaussian which is not interesting.

For completeness we give the full two dimensional temperature field as follows

$$T_1(x,z,t) = c_1 \; t^{-3/2} \left[4erfi\left(\frac{x+z}{4(\kappa t)^{1/2}}\right)\sqrt{2\pi}\left(\kappa - \frac{(x+z)^2}{4t}\right) e^{-\frac{(x+z)^2}{8\kappa t}} + \frac{4\sqrt{\kappa}(x+z)}{t^{1/2}}\right] + c_2 \; t^{-3/2} e^{-\frac{(x+z)^2}{8\kappa t}} \left(4\kappa - \frac{(x+z)^2}{t}\right). \quad (10)$$

The shape function of the pressure field can be obtained from the temperature shape function via the following equation

$$h' = \frac{eGl}{2} \quad (11)$$

with a similar solution to (9)

$$h = c_1 \left[2\kappa\sqrt{2\pi} eG \cdot erfi\left(\frac{\sqrt{2}\eta}{4\sqrt{\kappa}}\right) \eta e^{-\frac{\eta^2}{8\kappa}}\right] + c_2 2eG\kappa\eta e^{-\frac{\eta^2}{8\kappa}} + c_3, \quad (12)$$

this can be understood that the derivative of the pressure is proportional to the temperature. With the known numerical value of the exponent $\epsilon = 1$ the scaled pressure field can be expressed as well $P(x,z,t) = t^{-1}h([x+z]/t^{-1/2})$. Note, the difference between the ω and the ϵ exponents, which are responsible for the different asymptotic decays. The temperature field has a stronger damping for large η than the pressure field. (It is worth to mention that for the three dimensional NS equation, without any heat exchange the decay exponent of the pressure term is also different to the velocity field [12].)

At last the ODE for the shape function of the velocity component z reads

$$4\nu g'' + g'\eta + g + eGl = 0 \tag{13}$$

which directly depends on the temperature on $l(\eta)$ and all the physical parameters ν, e, G, κ, of course. In contrast to the pressure and temperature field there is no closed solutions available for a general parameter set. The formal, most general solution is

$$g = \tilde{c}_2 e^{-\frac{\eta^2}{8\nu}} + e^{-\frac{\eta^2}{8\nu}} \left\{ \int \frac{1}{4\nu} \left[\left(\tilde{c}_1 - 4eG\kappa c_2 \eta e^{-\frac{\eta^2}{8\kappa}} - 4eG\kappa\sqrt{2\pi} c_1 erfi\left(\frac{\sqrt{2}\eta}{4\sqrt{\kappa}}\right) \eta e^{-\frac{\eta^2}{8\kappa}} \right) e^{\frac{\eta^2}{8\nu}} \right] d\eta \right\} \tag{14}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are the recent integration constants. Note, that the integral can be analytically evaluated if and only if $\nu = \kappa$ which is a great restriction to the physical system. We skip this solution now. The other way is to fix $c_1 = 0$ and let κ and μ free. The solution has the next form of

$$g = \tilde{c}_1 e^{-\frac{\eta^2}{8\nu}} erf\left(\frac{\eta}{4}\sqrt{-\frac{2}{\nu}}\right) 1 + \tilde{c}_2 e^{-\frac{\eta^2}{8\nu}} - \frac{4eGc_2\kappa^2 e^{-\frac{\eta^2}{8\kappa}}}{\kappa - \nu}. \tag{15}$$

Note, that now the $\nu \neq \kappa$ condition is obtained. The $\tilde{c}_1$ and $\tilde{c}_2$ are the recent integration constants as above, it is interesting that if both of them are set to zero, the solution is still not trivial. For a physical system the kinematic viscosity $\nu > 0$ is always positive, therefore in the case of $\tilde{c}_1 \neq 0$ the solution becomes complex. Figure 4 shows the shape function of the z velocity component. It is clear that the real part is a Gaussian function and the complex part is a Gaussian distorted with an error function, which is an interesting final result. In the literature we can find system which shows similarities like the work of Ernst [21] who presented a study where a the asymptotic normalized velocity autocorrelation function calculated from the linearized Navier-Stokes equation has an error function shape.

III. SUMMARY

We investigated the classical OB equation which is the starting point of countless dynamical and chaotic systems. Instead of the usual Fourier truncation method we applied the two-dimensional generalization of the self-similar Ansatz and found a coupled non-linear

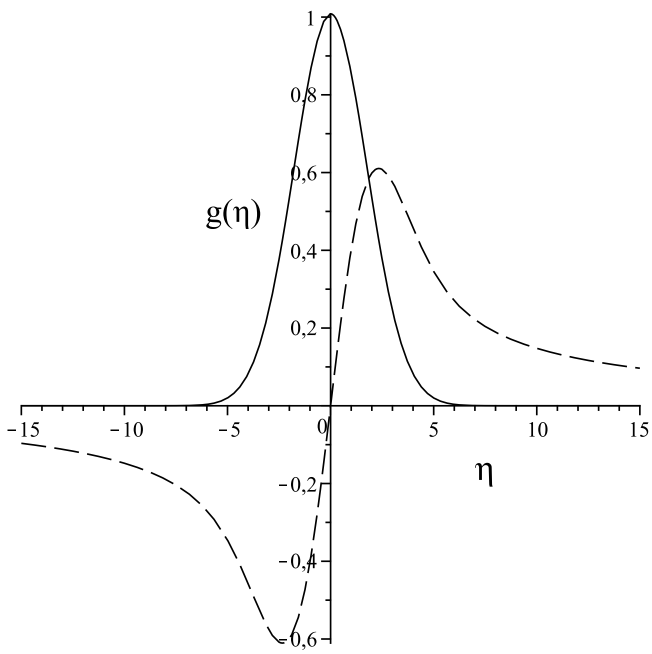

FIG. 4: The shape functions of the z velocity field $g(\eta)$ Eq. (15) as a function of η. The solid line is the real and the dotted is the complex part. All the integration constants are taken $\tilde{c}_1 = \tilde{c}_2 = c_2 = 1$. The physical constant $eG = 1$ as well. The $\kappa = 0.04$ and $\nu = 0.8$.

ODE system which can be solved with quadrature. The main result is that even these kind of solutions - build up from error functions - show some oscillating behavior. The resulting expressions show the decay of temperature and pressure fluctuations and velocity field in time. Due to our knowledge certain parts of the climate models are based on the OB equations therefore our results might be an interesting sign to climate experts.

IV. ACKNOWLEDGEMENT

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