

REMARKS ON THE PAPER¹
"M. KOLIBIAR, ON A CONSTRUCTION OF SEMIGROUPS"

Attila Nagy

Department of Algebra
 Budapest University of Technology and Economics
 P.O. Box 91
 1521 Budapest
 Hungary

Abstract

In his paper "On a construction of semigroups", M. Kolibiar gives a construction for a semigroup T (beginning from a semigroup S) which is said to be derived from the semigroup S by a θ -construction. He asserted that every semigroup T can be derived from the factor semigroup $T/\theta(T)$ by a θ -construction, where $\theta(T)$ is the congruence on T defined by: $(a, b) \in \theta(T)$ if and only if $xa = xb$ for all $x \in T$. Unfortunately, the paper contains some incorrect part. In our present paper we give a revision of the paper.

A semigroup S is called a left reductive semigroup ([1], [4]) if, for every $a, b \in S$, whenever $xa = xb$ holds for all $x \in S$ then $a = b$. It is known that, for an arbitrary semigroup S , the relation $\theta(S)$ defined by $(a, b) \in \theta(S)$ for some $a, b \in S$ if and only if $xa = xb$ for all $x \in S$ is a congruence on S . In [3], the author examined this congruence. He defined a sequence θ_n ($n = 1, 2, \dots$) of congruences on an arbitrary semigroup S as follows: $\theta_1 = \theta(S)$, and if θ_n is given, θ_{n+1} is the congruence relation on S , induced by the congruence relation $\theta(S)/\theta_n$ ([2]). In Lemma 2 of [3] it is shown that $(a, b) \in \theta_n$ for some $a, b \in S$ if and only if $xa = xb$ for all $x \in S^n$.

In Theorem 1 of [3], it is asserted that $\theta^* = \cup_{n=1}^{\infty} \theta_n$ is the least element in the set of all congruence relations θ on S such that S/θ is a left reductive semigroup. The proof of this theorem is not correct. The author asserts that, from the result $(xa, xb) \in \theta^*$ for all $x \in S$, it follows that $txa = txb$ for all $x \in S$ and all $t \in S^k$ for some $k \in \mathbb{N}$ (that is $(xa, xb) \in \theta_k$ for all $x \in S$). This is not correct, because $(xa, xb) \in \theta^*$ for all $x \in S$ means that, for every $x \in S$, there is a positive n_x such that $(xa, xb) \in \theta_{n_x}$. From this it does not follow that there is a positive integer k such that $(xa, xb) \in \theta_k$ for all $x \in S$.

Theorem 2 of [3] is also incorrect. In part (a) of Theorem 2, the following construction is given: For a semigroup S and each $x \in S$, let $T_x \neq \emptyset$ be a set such that $T_x \cap T_y = \emptyset$ if $x \neq y$. Let a mapping $f_x^y : T_x \mapsto T_y$ be given for all couples $x, y \in S$ such that $y = xu$ (in the paper $y = ux$, but it is a missprint) for some $u \in S$. Suppose further $f_y^z \circ f_x^y = f_x^z$. Given $a, b \in \cup_{x \in S} T_x = T$, set $a \circ b = f_x^{xy}(a)$, where $a \in T_x, b \in T_y$. Then $(T; \circ)$ is a semigroup (this semigroup

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is said to be derived from S by a θ -construction), and each set T_x is contained in a θ -class of T . If S is left reductive then the θ -classes of T are exactly the sets T_x ($x \in S$). The assertions of this part of the theorem is correct. In part (b) of Theorem 2, it is asserted that, for an arbitrary semigroup T , if S denotes the factor semigroup $T/\theta(T)$ and $T_{[x]\theta}$ denotes the $\theta(T)$ -class $[x]\theta$ of T containing x , then $f_{[x]\theta}^{[xy]\theta} : a \mapsto ab$ ($a \in [x]\theta, b \in [y]\theta$) is a mapping on $[x]\theta$ to $[xy]\theta$ and, for all $c, d \in T$, $cd = f_{[c]\theta}^{[cd]\theta}(c)$. This last equation means that $cd = c \circ d$ for all $c, d \in S$. The next example shows that the assertion $cd = f_{[c]\theta}^{[cd]\theta}(c)$ for all $c, d \in T$ is not correct.

Example 1 Let $T = \{e, a, u, v, 0\}$ be a semigroup defined by Table 1.

\cdot	e	a	u	v	0
e	e	a	0	0	0
a	a	e	0	0	0
u	u	v	0	0	0
v	v	u	0	0	0
0	0	0	0	0	0

Table 1:

Let θ denote the congruence $\theta(T)$. It is easy to see that the factor semigroup $S = T/\theta$ is a left reductive semigroup. We note that S is a semigroup which can be obtained from a two-element group $\{[e]\theta, [a]\theta\}$ by adjunction the zero $[0]\theta$.

Consider the semigroup $(T; \circ)$ which is derived from T/θ by a θ -construction (see (b) of Theorem 2 of [3]). As $[0]\theta = [0]\theta[0]\theta$, $[0]\theta = [0]\theta[e]\theta$, $[0]\theta = [0]\theta[a]\theta$, we have three possibility to define a mapping $f_{[0]\theta}^{[0]\theta}$ from $[0]\theta$ into $[0]\theta$.

In case $f_{[0]\theta}^{[0]\theta} x \mapsto x \cdot 0 = 0$ for all $x \in [0]\theta$, we have

$$u \circ e = f_{[u]\theta}^{[u \cdot e]\theta}(u) = f_{[0]\theta}^{[0]\theta}(u) = 0 \neq u = u \cdot e.$$

In case $f_{[0]\theta}^{[0]\theta} x \mapsto x \cdot e$ for all $x \in [0]\theta$, we get

$$u \circ a = f_{[u]\theta}^{[u \cdot a]\theta}(u) = f_{[0]\theta}^{[0]\theta}(u) = u \cdot e = u \neq v = u \cdot a.$$

In case $f_{[0]\theta}^{[0]\theta} x \mapsto x \cdot a$ for all $x \in [0]\theta$, we have

$$u \circ e = f_{[u]\theta}^{[u \cdot e]\theta}(u) = f_{[0]\theta}^{[0]\theta}(u) = u \cdot a = v \neq u = u \cdot e.$$

Under any possible choosing of the mapping $f_{[0]\theta}^{[0]\theta}$ we have a contradiction. Thus T can not be derived from T/θ by a θ -construction.

In the next we formulate a new and correct version of Theorem 2 of [3].

Theorem 2 (a) Let S be a semigroup. For each $x \in S$, associate a set $T_x \neq \emptyset$. Assume that $T_x \cap T_y = \emptyset$ for all $x \neq y$. Assume that, for each triple $(x, y, xy) \in S \times S \times S$ is associated a mapping $f_{(x,y,xy)}$ of T_x into T_{xy} acting on the left. Suppose further $f_{(xy,z,xyz)} \circ f_{(x,y,xy)} = f_{(x,yz,xyz)}$ for all triples $(x, y, z) \in S \times S \times S$. Given $a, b \in \cup_{x \in S} T_x = T$, set $a \circ b = f_{(x,y,xy)}(a)$, where $a \in T_x$ and $b \in T_y$. Then $(T; \circ)$ is a semigroup, and each set T_x is contained in a $\theta(T)$ -class of T . If S is left reductive then the $\theta(T)$ -classes of the semigroup T are exactly the sets T_x ($x \in S$).

(b) Let $(T; \cdot)$ be a semigroup. Let θ denote the congruence $\theta(T)$, and let $S = T/\theta$. For an element $[x]\theta \in S$, let $T_{[x]\theta}$ be the θ -class $[x]\theta$ of T . For arbitrary triple $([x]\theta, [y]\theta, [x \cdot y]\theta) \in S \times S \times S$, let $f_{([x]\theta, [y]\theta, [x \cdot y]\theta)} : a \mapsto a \cdot b$ ($a \in [x]\theta$), where b is an arbitrary element of $[y]\theta$. (We note that $a \cdot b = a \cdot [b]\theta$ for all $a, b \in T$. We also note that we consider all of the mappings $f_{([x]\theta, [z]\theta, [x \cdot z]\theta)}$, where $[z]\theta \in S$ satisfies $[x]\theta[y]\theta = [x]\theta[z]\theta$.) For all $c, d \in T$, let $c \circ d = f_{([c]\theta, [d]\theta, [c \cdot d]\theta)}(c)$. Then $(T; \circ)$ is a semigroup which is isomorphic to the semigroup $(T; \cdot)$. Consequently, every semigroup $(T; \cdot)$ is isomorphic to a semigroup derived from the semigroup $S = T/\theta$ using the construction in part (a) of the theorem.

Proof. (a): Let $(x, y, z) \in S \times S \times S$ be an arbitrary triple and let $a \in T_x$, $b \in T_y$, $c \in T_z$ be arbitrary elements. Then

$$\begin{aligned} & a \circ (b \circ c) = \\ & = a \circ f_{(y,z,yz)}(b) = f_{(x,yz,xyz)}(a) = (f_{(xy,z,xyz)} \circ f_{(x,y,xy)})(a) = (f_{(x,y,xy)}(a)) \circ c = \\ & = (a \circ b) \circ c. \end{aligned}$$

Thus $(T; \circ)$ is a semigroup. Let $x \in S$ be arbitrary. We show that T_x is in a $\theta(T)$ -class of T . If $a, b \in T_x$ are arbitrary elements then, for every $y \in S$ and for all $t \in T_y$,

$$t \circ a = f_{(y,x,yx)}(t) = t \circ b$$

which implies that $(a, b) \in \theta(T)$.

Assume that S is left reductive. Let T_x and T_y be sets such that they are in the same $\theta(T)$ -class of T . Let $a \in T_x$ and $b \in T_y$ be arbitrary elements. Then, for every $z \in S$ and $t \in T_z$, $t \circ a = t \circ b \in T_{zx} \cap T_{zy}$. Thus $zx = zy$ for every $z \in S$. As S is left reductive, we get $x = y$. Hence the $\theta(T)$ -classes of T are exactly the sets T_x ($x \in S$).

(b): It is clear that, for every $x, y \in T$, $f_{([x]\theta, [y]\theta, [x \cdot y]\theta)}$ maps $[x]\theta = T_{[x]\theta}$ into $T_{[x \cdot y]\theta} = [x \cdot y]\theta$. For every $(x, y, z) \in T \times T \times T$ and $a \in [x]\theta$,

$$\begin{aligned} & (f_{([x \cdot y]\theta, [z]\theta, [x \cdot y \cdot z]\theta)} \circ f_{([x]\theta, [y]\theta, [x \cdot y]\theta)})(a) = \\ & = (a \cdot y) \cdot z = a \cdot (y \cdot z) = f_{([x]\theta, [y \cdot z]\theta, [x \cdot y \cdot z]\theta)}(a). \end{aligned}$$

Thus $(T; \circ)$ is a semigroup by part (a) of the theorem.

For every $x, y \in T$ and every $a \in [x]\theta$, $b \in [y]\theta$,

$$a \circ b = f_{([x]\theta, [y]\theta, [x \cdot y]\theta)}(a) = a \cdot b.$$

Thus $(T; \cdot)$ is isomorphic to the semigroup $(T; \circ)$. □

Example 3 Let T be the semigroup defined in Example 1. Let θ denote the congruence $\theta(T)$. Then $S = T/\theta$ is a semigroup which can be obtained from a two-element group by adjunction of a zero. Let $e, a, 0$ denote the θ -classes $[e]\theta = \{a\}$, $[a]\theta = \{a\}$, $[0]\theta = \{0, u, v\}$ of T , respectively. Then the Cayley-table of S is

	e	a	0
e	e	a	0
a	a	e	0
0	0	0	0

Table 2:

Apply our construction when the beginning semigroup is S . The sets T_x ($x \in S$) are $T_e = \{e\}$, $T_a = \{a\}$, $T_0 = \{0, u, v\}$. The mappings (see (b) of Theorem 2) are: $f_{(e,e,e)} : e \mapsto e$, $f_{(e,a,a)} : e \mapsto a$, $f_{(e,0,0)} : e \mapsto 0$, $f_{(a,e,a)} : a \mapsto a$, $f_{(a,a,e)} : a \mapsto e$, $f_{(a,0,0)} : a \mapsto 0$, $f_{(0,e,0)} : x \mapsto x \cdot e$ for every $x \in [0]\theta$, $f_{(0,a,0)} : x \mapsto x \cdot a$ for every $x \in [0]\theta$, $f_{(0,0,0)} : x \mapsto 0$ for every $x \in [0]\theta$. It is easy to see that the semigroup T is isomorphic to the semigroup $(T; \circ)$ derived from S by applying our construction in Theorem 2.

We formulate the new version of Theorem 3 of [3].

Theorem 4 *Let S be a semigroup whose congruence relations satisfy the ascending chain condition. Then there are semigroups $S_0 = S, S_1, \dots, S_n$ such that S_n is left reductive, and each S_{i-1} can be derived from S_i ($i = 1, \dots, n$) by the construction defined in (a) of Theorem 2.*

Proof. Let S be a semigroup. For a congruence ϱ on S , consider the congruence ϱ^* defined by $(a, b) \in \varrho^*$ for some $a, b \in S$ if and only if $(xa, xb) \in \varrho$ for all $x \in S$. (see [5]). Let

$$\varrho^{(0)} \subseteq \varrho^{(1)} \subseteq \dots \subseteq \varrho^{(n)} \subseteq \dots$$

be a sequence of congruences on S defined by $\varrho^{(0)} = \varrho$ and, for every nonnegative integer i , let $\varrho^{(i+1)} = (\varrho^{(i)})^*$ (see [5]). By Lemma 2 of [3], $\iota_S^{(i)} = \theta_i$ for every positive integer i , where ι_S denotes the identity relation on S and θ_i are the congruences examined above. As the congruence relations on S satisfy the ascending chain condition, there is a (least) nonnegative integer n such that $\iota_S^{(n)} = \iota_S^{(n+1)} = \dots$. Let $S_i = S/\iota_S^{(i)}$ for every $i = 0, 1, \dots, n$. Thus $S = S_0$. By Theorem 1 of [5], $\iota_S^{(n)} = \theta_n$ is a left reductive congruence which means that the factor semigroup $S_n = S/\theta_n$ is left reductive. For every $i = 1, \dots, n$,

$$S_i = S/\iota_S^{(i)} \cong (S/\iota_S^{(i-1)})/(\iota_S^{(i)}/\iota_S^{(i-1)}) \cong S_{i-1}/(\iota_S^{(i)}/\iota_S^{(i-1)}) \cong S_{i-1}/((\iota_S^{(i-1)})^*/\iota_S^{(i-1)})$$

using also Theorem 5.6 of [2]. By Lemma 7 of [5],

$$((\iota_S^{(i-1)})^*/\iota_S^{(i-1)}) = (\iota_{S_{i-1}})^* = \theta(S_{i-1}).$$

Thus

$$S_i \cong S_{i-1}/\theta(S_{i-1})$$

for every $i = 1, \dots, n$. By (b) of Theorem 2, the semigroup S_{i-1} ($i = 1, \dots, n$) can be derived from S_i by the construction in (a) of Theorem 2. \square

An addendum: It may be that the semigroup $S_n = S/\theta_n$ in Theorem 4 has only one element, that is, θ_n is the universal relation ω_S on S . The next theorem is about this case.

Theorem 5 *For a semigroup S , $\theta_n = \omega_S$ for some positive integer n if and only if S is an ideal extension of a left zero semigroup by a nilpotent semigroup.*

Proof. Let ι_S denote the identity relation on a semigroup S . By Theorem 5 of [5], $\iota_S^{(n)} = \omega_S$ for some non-negative integer n if and only if S is an ideal extension of a left zero semigroup by a nilpotent semigroup. As $\theta(S) = \iota_S^{(1)}$, our assertion is a consequence of Theorem 5 of [5]. \square

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