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SOME APPLICATIONS OF QUADRATIC LYAPUNOV FUNCTIONS

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1. INTRODUCTION

Consider the linear differential equation:

$$\dot{x} = A(t)x \tag{1}$$

where $A(t)$ is an $n \times n$ continuous matrix function. The object of this study is the stability of the trivial solution.

For the estimate of the solutions, W.A. Coppel gave the formula:

$$e^{-\int_{t_o}^{t} \mu(-A(\tau))d\tau} \le \frac{|x(t)|}{|x(t_o)|} \le e^{\int_{t_o}^{t} \mu(A(\tau)d\tau},$$

where

$$\mu(A) = \overline{\lim_{h \to +o}} \frac{|I - hA| - 1}{h}.$$

<u>*Note:*</u>

$$\mu(A) = \sup_{|x|=1} Re\, x^* VAx \quad \text{if}$$

$$|x| = \sqrt{x^* Vx}$$

where

V is positive definite Hermetian and

$x^* = \overline{x}^T$ the conjugate transposed of x.

Example 1 Let be

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

In this case, of course, $\lim_{t\to\infty} |x(t)| = 0$ which can be prooved with Coppel's estimate too because $\mu(A) = -1$.

Example 2 Let be

$$A = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}, \text{ then}$$

$\lim_{t\to\infty} |x(t)| = 0$ nevertheless $\mu(A) > 0$.

Coppel's estimate is, generally, not good enough. This is the problem I am going to treat.

2. ONE SOLUTION OF THE PROBLEM

I have found that transforming the variable x the estimate can be improved.

Suppose the system (1) is asymptotically stable.

Let be $V(t)$ a continuously differentiable, positive definite, Hermitian matrix function and $v(t,x)$ a Lyapunov function such that

$$v(t,x) = x^* V(t)x ,$$

$$\alpha^2 |x|^2 \leq v(t,x) \leq \beta^2 |x|^2 ,$$

$$\dot{v}_{(1)}(t,x) \leq -\gamma^2 |x|^2 .$$

Now we choose a continuously differentiable regular matrix function such that

$$V = W^* W.$$

Note: $|W^{-1}(t)|$ is bounded.

The transformation

$$y = W(t) x$$

carries (1) into a new system:

$$\dot{y} = C(t) y$$

and applying Coppel's formula for this case:

$$|W(t)^{-1}| |W(t_o)^{-1}|^{-1} e^{-\int_{t_o}^{t} \mu(-C(\tau)) d\tau} \leq \frac{|x(t)|}{|x(t_o)|} \leq |W(t_o)| |W^{-1}(t)| \cdot e^{\int_{t_o}^{t} \mu(C(\tau)) d\tau} , \qquad (2)$$

where

$$\mu(C(t)) \leq -\frac{1}{2} \frac{\gamma^2}{\beta^2}$$

if in the definition of μ we use the Euclidean norm. (Inversely: from a proper transformation matrix $W(t)$ we can get a positive definite quadratic Lyapunov function $v(t,x) = x^* W^* Wx$, the derivative of which alonge the solution of (1) is negative definite.)

3. FURTHER IMPROVEMENT OF THE METHOD

The transformation matrix can be restricted to a part of the unterval $[t_o, \infty]$ and a countable set of transformation matrices can be chosen in the following way:

$$t_o < t_1 < t_2 < \dots < t_i \quad \dots ,$$

$$I_i = [t_{i-1}, t_i] \quad ,$$

$$\Delta t_i = t_i - t_{i-1} \quad ,$$

$$y = W_i(t)\, x \quad \text{if} \quad t \in I_i \ ,$$

where $W_i(t)$ is continuously differentiable, regular in I_i.

This kind of transformation has proved to be more efficient than applying only one continuously differentiable, regular matrix function $W(t)$ in the whole interval $[t_o, \infty)$.

Example 3

Let be $$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & -p(t) \end{bmatrix} , \qquad t \geq 0 ,$$

$$p(t) \geq 0 \quad ; \quad p(t) \quad \text{is continuous.}$$

For *lower* estimate Coppel's formula is very good.

$$-\mu(-A(t)) = -p(t).$$

If $\int_{t_o}^{\infty} p(t)dt < \infty$ then $\lim_{t\to\infty} |x(t)| > 0$.

For *upper* estimate the formula is very rough:

$$\mu(A(t)) = 0$$

and so the only fact what can be stated is that
$|x(t)| \leq |x(t_o)|$ for $t \geq t_o$.

With proper transformations (see below) I could prove the following

Theorem:

Let be $0 < \delta_i < 1$ and $0 < \varepsilon_i$ some constants such that

$0 \leq p(t) \leq \frac{1}{\delta_i}$ if $t \in I_i$, and $\delta_i \leq p(t) \leq \frac{1}{\delta_i}$ if $t \in I_{io}$

where
$$I_{io} = I_i \setminus \bigcup_{j=1}^{k_i} I_{ij}$$

$$I_{ij} \cap I_{i\ell} = \emptyset \quad \text{if} \quad j \neq \ell$$

$$I_{ij} = [\tau_{ij_1} ; \tau_{ij_2}] \subset I_i$$

then

$$\Delta t_{io} > 4 + \frac{4\varepsilon_i}{\delta_i}$$

implies

$$\frac{|x(t_i)|}{|x(t_{i-1})|} < e^{-\varepsilon_i} ,$$

where

$$\Delta t_{io} = \Delta t_i - 3 \sum_{j=1}^{k_i} \Delta\tau_{ij} > 0$$

and

$$t_i - t_{i-1} > \Delta\tau_{ij} = \tau_{ij_2} - \tau_{ij_1} \geq 0.$$

Corollary

The asymptotical stability holds if

$$\frac{1}{t} \leq p(t) \leq t \quad \text{for} \quad 1 \leq t_o \leq t.$$

For proving the theorem I applied the following matrices:

$$W_i = D_i M_i$$

where

$$D_i = \begin{bmatrix} \sqrt{\lambda_{1i}} & 0 \\ 0 & \sqrt{\lambda_{2i}} \end{bmatrix} ,$$

$$M_i = \frac{1}{\sqrt{1+q_i^2}} \begin{bmatrix} q_i & 1 \\ -1 & q_i \end{bmatrix} ,$$

$$q_i = \frac{2}{\delta_i} - \lambda_{1i} ,$$

$$\lambda_{1i} = \frac{\delta_i}{2} + \frac{2}{\delta_i} + \sqrt{1 + \frac{\delta_i^2}{4}} ,$$

$$\lambda_{2i} = \frac{\delta_i}{2} + \frac{2}{\delta_i} - \sqrt{1 + \frac{\delta_i}{4}} .$$

With these:

$$|W_i \| W_i^{-1}| = - q_i \quad \text{and}$$

$$\mu(W_i \; A(t) \; W_i^{-1}) < - \frac{\delta_i}{4} \quad \text{if}$$

$$\delta_i \leq p(t) \leq \frac{1}{\delta_i} \quad \text{or}$$

$$\mu(W_i A(t) W_i^{-1}) \leq \frac{\delta_i}{2} \quad \text{if}$$

$$0 \leq p(t) < \delta_i .$$

Applying (2) (considering that in this case $C(t) = W_i A(t) \; W_i^{-1}$) we have the proof of the theorem.

For proving the corollary we need only to choose I_i in the following way:

$$t_i > \frac{t_{i-1}+4}{1 - 4\varepsilon}$$

where $0 < \varepsilon < 1/4$ is arbitrary and $\varepsilon_i = \varepsilon$ for every i.