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ON SOME STIELTJES INTEGRAL INEQUALITIES

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In the theory of the differential and integral equations the famous Bellman-Gronwall inequality, its generalizations and similar inequalities have several applications.

Usually these inequalities are of the following form: Under suitable assumptions from

$x(t) \leqq F(t, \int_a^t g(t,s,x(s))\, ds)$ we come to the inequality

$x(t) \leqq F_1(t, \int_a^t h(t,s)ds)$.

Recently in some papers we find the problem whether these inequalities are valid also for Stieltjes integrals (see e.g. P.G. Das - R.R. Sharma, Some Stieltjes integral inequalities, Journal of Math. Anal. and Appl. 73(1980), 423-433 and its references). In these papers special inequalities are investigated.

Naturally it arises the question to find a method which can be applied on several known inequalities to extend them to the Stieltjes integral case.

We tried the method sketched briefly in the following (here we use Riemann - Stieltjes integral):

Suppose that $x(t) \leq F(t, \int_a^t g(t,s,x(s))d\phi(s))$, where $\phi:[a,b] \to R$ is monotone. Assume that we have a sequence of continuously differentiable monotone functions $\{\phi_n\}$ with the property: $\phi_n(x) \to \phi(x)$, for $x \in [a,b]$, $n = 1,2,\ldots$.

Hence we obtain

$$x(t) \leq F(t, \int_a^t g(t,s,x(s))d\phi(s)) = F(t, \int_a^t g(t,s,x(s))d\phi_n(s) +$$

$$+ \int_a^t g(t,s,x(s))d(\phi(s) - \phi_n(s))) \leq F(t, \int_a^t g(t,s,x(s))\phi'_n(s)ds +$$

$$+ \varepsilon_n(t)).$$

If $\varepsilon_n(t) \leq 0$ and F is an increasing function in its second variable then

$$x(t) \leq F(t, \int_a^t g(t,s,x(s))\ \phi'_n(s)ds$$

hence

$$x(t) \leq F_1(t, \int_a^t h_1(t,s,\phi'_n(s))ds).$$

If the right hand side may be written in the form

$$F_1(t, \int_a^t h_2(t,s)\phi'_n(s)ds) \quad \text{or} \quad F_1(t, \int_a^t h_2(t,s)d\phi_n(s)),$$

then applying the Helly-Bray theorem we have

$$x(t) \leqq F_1(t, \int_a^t h_1(t,s)d\phi(s)).$$

$\varepsilon_n(t) \leqq 0$ is not true in general. It is valid e.g. if $g(u,v,w) \geqq 0$ and $\phi - \phi_n$ is a monotone decreasing function, $n = 1,2,\ldots,$. (But using Fubini's theorem we obtain that a singular monotone function can't be a limit of continuously differentiable monotone functions with $\phi'_n(x) \geqq \phi'_{n+1}(x)$.)

The wanted sequence $\{\phi_n\}$ may be constructed from polynomials too. We use S.W. Young's theorem (Bull. Amer. Math. Soc. (73(1967), 642-643.).

Suppose that n is a positive integer, $x_{i-1} < x_i$ and $y_{i-1} < y_i$, $i=1,\ldots,n$, then there exists a polynomial P such that $P(x_i) = y_i$, $i=0,1,\ldots,n$, and P is monotone in each of the intervals $[x_{i-1},x_i]$, $i=1,\ldots,n$.

By the aid of this theorem we sketch our construction. For the sake of simplicity suppose that ϕ is a continuous and strictly monotone increasing function.

Divide the interval $[a,b]$ into equal parts:

$a = x_o < x_1 < \ldots < x_{2^n} = b$. Put $y_k = \phi(x_{k-1})$ if $k=1,2,\ldots,2^n$ and choose $y_{on} < \phi(x_o)$. Then $y_{on} < y_1 < \ldots < y_{2^n}$. From the above mentioned theorem we come to a polynomial ϕ_n with $\phi_n(x_k) = y_k$, $k = 1,2,\ldots,2^n$, $\phi_n(x_o) = y_{on}$ and as ϕ_n increases on every interval $[x_{i-1},x_i]$, it is increasing on $[a,b]$.

Constructing these polynomials on $n = 1,2,\dots,k,\dots$ it is clear that $\phi_n(x) < \phi(x)$ and $\phi_n(x) \to \phi(x)$ if $n \to \infty$ and $y_{on} \to \phi(a)$. (With a slight modification of the construction we can obtain a sequence of polynomials having the property $\phi_n(x) \leqq \phi_{n+1}(x)$ too.)

It is easy to see that the construction goes also if ϕ is monotone nondecreasing and continuous from the left. (One can construct such sequence $\{\phi_n\}$ in another way too.)

As an application let us have an inequality of Deo (1971): If the functions x,a,k are defined on $J = [\alpha,\beta], k(t) \geqq 0$, the function $g : I \to R$ is monotone nondecreasing positive subadditive and submultiplicative, $x(J) \subset I$; the function h is defined on an interval Δ, $0 \in \Delta$, $h(\Delta) \subset I$, and h is monotone nondecreasing.

Suppose further that

$$x(t) \leqq a(t) + b(t)h\left(\int_\alpha^t k(s)g(x(s))\,ds\right) \quad \text{for} \quad t \in J \quad \text{then}$$

$$x(t) \leqq a(t)+b(t)h\left\{G^{-1}\left[\int_\alpha^t k(s)g(b(s))ds \; + \right.\right.$$

$$\left.\left. + \; G\left(\int_\alpha^t k(s)g(a(s))ds\right)\right]\right\}, \quad \text{where} \quad G(u) = \int_{u_0}^{u} \frac{ds}{g(s)}, \quad u \in \Delta.$$

By the above sketched method this inequality is valid for Stieltjes integrals too with function ϕ which is monotone nondecreasing and continuous from the left:

$$x(t) \leqq a(t) + b(t)h\left(\int_\alpha^t k(s)g(x(s))d\,\phi(s)\right) = a(t) \; +$$

$$+ \; b(t)h\left(\int_\alpha^t k(s)g(x(s))d(\phi(s)-\phi_n(s)) + \int_\alpha^t k(s)g(x(s))d\phi_n(s)\right) \leqq$$

$$\leqq a(t) + b(t)h\left(\int_\alpha^t k(s)g(x(s))d\,(\phi(s) - \phi_n(s))\right) +$$

$$+ b(t)h\left(\int_\alpha^t k(s)g(x(s))d\phi_n(s)\right) \leqq a(t) + A +$$

$$+ b(t)h\left(\int_\alpha^t k(s)g(x(s))\,\phi'_n(s)ds\right),$$

$A \geqq 0$ fixed number, $n \geqq N(A)$.

Therefore

$$x(t) \leqq a(t) + A + b(t)h\,\{G^{-1}[\int_\alpha^t k(s)g(b(s))d\,\phi(s) +$$

$$+ G(\int_\alpha^t k(s)g(a(s) + A)d\,\phi(s))]\}.$$

Since this is true for every $A \geqq 0$, we obtain

$$x(t) \leqq a(t) + b(t)h\,\{G^{-1}[\int_\alpha^t k(s)g(b(s))d\,\phi(s) +$$

$$+ G(\int_\alpha^t k(s)g(a(s))d\phi(s))]\}.$$

There are problems if the function ϕ'_n appears repeatedly in the obtained inequality. For example the following inequality was published by Gamidov in 1969.

If a, v_i, φ_i are continuous positive functions on $[a,b]$ and

$$x(t) \leqq a(t) + \sum_{i=1}^{k} v_i(t) \int_{\alpha}^{t} \varphi_i(s)x(s)ds \,,$$ then

$$x(t) \leqq a(t) + v(t) \int_{\alpha}^{t} \sum_{i=1}^{k} \varphi_i(s)a(s) \, exp(\int_{s}^{t} \varphi_i(u)v(u)du)ds,$$

where $v(t) = \sup_{1 \leqq i \leqq k} v_i(t)$.

In this case applying our method we come to the inequality:

$$x(t) \leqq a(t) + v(t) \int_{\alpha}^{t} \sum_{i=1}^{k} \varphi_i(s)a(s) \, exp(\int_{s}^{t} \sum_{i=1}^{k} \varphi_i(u)d\phi_n(u))d\phi_n(s) \,,$$

and the problem is whether the right-hand side tends to

$$a(t) + v(t) \int_{\alpha}^{t} \sum_{i=1}^{k} \varphi_i(s)a(s) \, exp(\int_{s}^{t} \sum_{i=1}^{k} \varphi_i(u)d\phi(u))d\phi(s)$$

or not.

In this and in similar cases the Lebesgue-Stieltjes or more general integral may be useful but this question will be treated in another note.