# Splitting the source term for the Einstein equation to classical and quantum parts

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We consider the special and general relativistic extensions of the action principle behind the Schrödinger equation distinguishing classical and quantum contributions. Postulating a particular quantum correction to the source term in the classical Einstein equation we identify the conformal content of the above action and obtain classical gravitation for massive particles, but with a cosmological term representing off-mass-shell contribution to the energy-momentum tensor. In this scenario the - on the Planck scale surprisingly small - cosmological constant stems from quantum bound states (gravonium) having a Bohr radius *a* as being  $\Lambda = 3/a^2$ .

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# I. INTRODUCTION

The enigma of the cosmological constant, in the modern view interpreted as dark energy, is mainly due to its surprisingly small, yet nonzero magnitude: should it have namely a quantum gravity origin (analogous to a symmetric broken phase with nonzero Higgs fields in the Standard Model), then the natural scale for a ground state (vacuum) energy density would be of  $M_P^4$  order, with  $M_P$  being the Planck mass. In the symmetric phase on the other hand it should be *exactly zero*. According to astronomical observations, however, the effect is about 120 decimal orders of magnitudes too small for the energy density (and still 30 orders too small in the linear energy scale) while it is definitely non zero [1–16].

In spite of this naturalness problem the dark energy is responsible for 68 - 72% of the evolution of the Universe observed presently in the standard cosmological models based on Friedmann's first calculation. A remaining 24 - 28% of the effect is called dark matter, about which more ideas have been already discussed in the literature. The classical naturalness problem has probably nothing to do with quantum gravity. Its appearance may be caused by quantum effects on the source term in the Hilbert-Einsteinian gravity theory.

In this paper we test a particular idea based on a conformal treatment of the Schrödinger equation: as if the quantum mechanical problem of obtaining wave functions, and the special relativistic field theory problem with scalar fields, could be splitted to a massive and a conformal part in line with a classical – quantum partition [17–20]. Considering in a relativistic setting (but without spin effects) the free Klein-Gordon action is inspected. Identifying the quantum part as belonging to a traceless relativistic energy-momentum tensor, we suggest to generalize its Bohm-Takabayashi form [21–26] and connect the remaining classical part to Einstein's gravity equation [27–29] in form of a dust matter source of massive point particles moving on Bohm trajectories. The resulting energy-momentum tensor from this procedure agrees with the proper handling by variation against the metric tensor for the conformal invariant part of the action. In this scenario the quantum nature of the scalar field reveals itself in deviations from the classical on-mass-shell relation  $P_{\mu}P^{\mu} = (mc)^2$ , and our suggested natural coupling to gravity makes a simple conformal transformation of the full Einstein tensor expedient. After this transformation the classical part (dust gravity) separates from quantum effects which among others include a cosmological term. This term represents negative pressure e.g. for stationary quantum bound states in a simple attractive  $-\alpha/r$  gravitational potential, with a mass of about 140 MeV for the pairwise composite object.

In this paper we first recall the Schrödinger equation with complex magnitude – phase variables, together with the underlying action principle. Then the Klein-Gordon quantum action is analyzed in the same way, aiming at the determination of the physically correct energy-momentum tensor. Here we emphasize the quantum-conformal (in the Bohm-like contribution traceless) construction possibility. Based on this we apply a naturally emerging conformal transformation to the Einstein equation with the generalized Bohm-Takabayashi energy-momentum tensor as matter source. Then one identifies the classical Einstein tensor, the quantization volume, the conformal symmetry of the quantum part and a cosmological term proportional to the off-mass-shell part in the flat space.

# II. NONRELATIVISTIC QUANTUM MECHANICS

Although this paper is ultimately written with the purpose of testing an idea about a non-conventional explanation for the origin of the classical cosmological constant, at this point we recapitulate a few conceptual issues in non-relativistic quantum mechanics because the suggested modification to the energy-momentum tensor, as the source term of the classical Einstein equation describing gravity, is motivated by a particular view on quantum binding energy effects. These effects will be re-interpreted in the next section in the framework of special relativistic quantum field theory as off-mass-shell contributions.

### A. Schrödinger equation with Madelung variables

The Madelung picture [30] of the Schrödinger equation has been criticized due to various reasons (see e.g. [31–33]). On the other hand it is a permanent source of inspiration equally in applied and fundamental quantum research. Jánossy and his coworkers stressed the fluid interpretation up to its limits [34–37]. Then Bialynicki-Birula and Mycielski suggested an additive nonlinear extension of the Schrödinger equation [38, 39], that was treated in detail with more traditional concepts by Weinberg [40]. Later Bialynicki-Birula researched the Weyl equation and also the quantum mechanics of massless particles with the help of the hydrodynamic form [41–43]. Kuz'menkov and Maksimov researched fermion systems providing a statistical background for the hydrodynamic view [44, 45]. The connection of vortices in quantum fluids and electromagnetism has been explored by Bialynicki-Birula and Bialynicka-Birula [46–48] and recently related predictions were confirmed by experiments in slow ionatom collisions [49, 50]. The first relativistic extension is due to Takabayashi [51]. Jackiw and coworkers proved that quantum field theories can be reformulated in a hydrodynamic form [52, 53].

With this enumeration we want to express that in our opinion different interpretations may be mind provoking and hence useful [54]. This is also valid for the Bohm potential based approach [21, 22, 55, 56], that we consider closely related to the hydrodynamic one [57]. It leads to important observations, too, e.g. recently in the understanding of quantum tunneling [58].

Here we would like to explore an important aspect of the use of the magnitude and phase of the complex wave function field,  $\varphi$ . Being interested in a splitting of the fundamental quantum mechanical equation into a "classical" and a "quantum" part namely, the representation

$$\varphi = R \, e^{\frac{i}{\hbar}\alpha} \tag{1}$$

is of genuine use. Here  $\alpha$  plays the role of the classical action for the corresponding classical dynamics and the canonical classical momentum and energy are derived accordingly as

$$E = -\frac{\partial \alpha}{\partial t}, \qquad P = \nabla \alpha. \tag{2}$$

The Schrödinger equation in its well-known form,

$$-\frac{\hbar^2}{2m}\nabla^2\varphi + V(x)\varphi = i\hbar\frac{\partial}{\partial t}\varphi,\tag{3}$$

then can be rewritten in terms of the classical momentum, energy and the quantum factor R by observing the following derivatives:

$$\frac{\partial}{\partial t}\varphi = \left(\frac{1}{R}\frac{\partial R}{\partial t} - \frac{i}{\hbar}E\right)\varphi, \qquad \nabla\varphi = \left(\frac{1}{R}\nabla R + \frac{i}{\hbar}P\right)\varphi. \tag{4}$$

The Laplacian becomes

$$\nabla^2 \varphi = \left[ \nabla \left( \frac{\nabla R}{R} + \frac{i}{\hbar} P \right) + \left( \frac{\nabla R}{R} + \frac{i}{\hbar} P \right)^2 \right] \varphi.$$
(5)

Now the Schrödinger equation (3) is separated into its real and imaginary parts as follows: The real part connects the classical energy and momentum according to the classical formula,  $E = P^2/2m$ , and reveals a quantum correction, called the Bohm potential [21–23, 55]:

$$E = V - \frac{\hbar^2}{2m} \left[ \nabla \frac{\nabla R}{R} + \left( \frac{\nabla R}{R} \right)^2 - \frac{P^2}{\hbar^2} \right]$$
(6)

The interpretation of this energy expression (6) as a sum of a classical energy and a quantum modification,

$$E = \left(\frac{P^2}{2m} + V\right) - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R},\tag{7}$$

reveals a position-dependent quantum correction to the classical energy, E. The imaginary part on the other hand leads to a first order time-evolution constraint equation,

$$\frac{i\hbar}{R}\frac{\partial R}{\partial t} = -\frac{\hbar^2}{2m}\frac{i}{\hbar}\left[\nabla P + \frac{2}{R}P\cdot\nabla R\right].$$
(8)

Simplifying this imaginary part leads to an analogue of the mass density continuity equation,

$$m\frac{\partial R^2}{\partial t} + \nabla \left(R^2 P\right) = 0. \tag{9}$$

Upon introducing the velocity field via P = mv and the local fluid density  $\rho = R^2 = |\varphi|^2$ , this relation was interpreted as a continuity equation for the mass current carried by a "Madelung fluid"

$$\frac{\partial \rho}{\partial t} + \nabla \left( \rho \, v \right) = 0. \tag{10}$$

Similarly equation (7) is a "Bernoulli equation" of the corresponding rotation free momentum balance of a special Korteweg fluid [57, 59] or an off mass shell relation [60]. From now on we follow the classical – quantum separation hint by using the variables R and  $\alpha$ . At the end we shall realize that exactly this splitting makes it possible to identify a conformal part in the quantum dynamics of the massive particles having a traceless contribution to the energy-momentum tensor.

# B. Schrödinger equation from action principle

According to Schrödinger's original article about his equation the following Action Principle can be formulated: instead of fulfilling the classical Hamilton-Jacobi equation [61], it is violated so that its space-time integral weighted by  $|\varphi|^2$  achieves a variational extremum. The use of this form of the weighting factor may be argued for by noting that only this leads to a linear Euler-Lagrange variational equation. The Quantum Action Principle behind the Schrödinger equation is given by [17–19]

$$\mathfrak{S} = \int \left(\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V\right) |\varphi|^2 \, d^3x \, dt. \tag{11}$$

It has been interpreted via a "Boltzmannian" eikonal ansatz:  $S = \frac{\hbar}{i} \ln \varphi$ . Using this ansatz leads to the following complex bilinear form of the quantum action:

$$\mathfrak{S} = \int \left[\frac{\hbar}{i}\varphi^*\frac{\partial\varphi}{\partial t} + \frac{\hbar^2}{2m}\nabla\varphi^*\cdot\nabla\varphi + V\varphi^*\varphi\right]d^3x\,dt\tag{12}$$

Finally variation against  $\varphi^*$  delivers the well-known Schrödinger equation, linear in the complex wave function  $\varphi$ :

$$\frac{\delta\mathfrak{S}}{\delta\varphi^*} = \frac{\hbar}{i}\frac{\partial\varphi}{\partial t} - \frac{\hbar^2}{2m}\nabla^2\varphi + V\varphi = 0 \tag{13}$$

It is straightforward to check by variation against  $\varphi$  followed by a complex conjugation that the eikonal coefficient,  $\hbar/i$ , has to be pure imaginary.

Now we re-investigate this quantum action with magnitude-phase variables in order to see the effect of the quantum – classical splitting considered in the previous subsection. Indeed the action also splits into quantum and classical parts,

$$\mathfrak{S} = \int \left[\frac{\hbar^2}{2m} \left(\nabla R\right)^2 + R^2 \left(\frac{\partial \alpha}{\partial t} + V + \frac{(\nabla \alpha)^2}{2m}\right)\right] d^3x \, dt \tag{14}$$

The characteristic Lagrangian structure contained in this Quantum Action Principle can be summarized as follows:

$$\mathcal{L} = \hbar^2 (\text{quantum kinetic}) + R^2 (\text{classical Hamilton} - \text{Jacobi equation})$$

Finally we make some remarks about the relation between the pure classical action,  $\alpha$  and the complex action variable in the eikonal form, S (more commonly used in derivations). In fact one realizes that  $S = \alpha - i\hbar \ln R$ , i.e. the real part of S is the classical  $\alpha$ . Certainly for R = 1 the classical dynamics is recovered. In the quantum propagation of massive objects, however,  $\alpha$  and its derivatives, E and P, are not constants, their evolution couples to that of R(x, t) exactly via the Schrödinger equation. This fact typically reflects deviations from the classical momentum and energy, and - as we shall see - also from the on-mass-shell dispersion relation.

# III. SPECIAL RELATIVISTIC QUANTUM MECHANICS

In this section we present the quantum – classical splitting in the above terms for the special relativistic free Klein-Gordon theory. Although, in fact, the probability density interpretation is no more available in this case, a conserved current and the corresponding continuity equation is easily derived. Restricing to a single particle with mass m, this continuity reflects a content similar to the one in the previous section. The corresponding four-velocity is, however, either not normalized to one, and therefore is not a physical velocity, or its quantum, "off-mass-shell" contribution has to be splitted away from the classical part.

### A. Klein-Gordon Lagrangian

Disregarding the spin of the electron, the Schrödinger equation can be viewed as the non-relativistic approximation to the Klein-Gordon equation – in analogy to the non-relativistic approximation to the relativistic Hamilton-Jacobi equation based on the energy-momentum dispersion relation of the mass point m. Although the Klein-Gordon equation does not describe the quantum energy levels of the H-atom precisely, for the study of a quantum – classical splitting it is more suitable due to its simplicity. The quantum action is based on the Lagrange density

$$\mathcal{L} = -\frac{1}{2}\partial_{\mu}\psi^{*}\partial^{\mu}\psi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^{2}\psi^{*}\psi, \qquad (15)$$

containing a complex  $\psi(x, t)$  field. The action is a Lorentz-invariant integral,

$$\mathfrak{S} = \int \mathcal{L} \, d^4 x,\tag{16}$$

with  $dx^4 = (cdt, d\vec{r})$ . We use physical units in which  $[\mathcal{L}] = \text{energy density/c} = [mc/L^3]$  and the Lorentz form diag(1, -1, -1, -1). In order to keep the relation to the non-relativistic wave function description on the one hand and to the classical relativistic mass point action (Maupertuis action) on the other hand, we include some further factors. The complex

scalar field related to the wave function is written as

$$\psi = \frac{\hbar}{\sqrt{mc}} R e^{\frac{i}{\hbar}\alpha}.$$
(17)

Here  $\alpha$  is the (real) classical action, as used in the previous section. The physical units of R can be obtained from the mass term in the Lagrange density:

$$\left(\frac{mc}{\hbar}\right)^2 \psi^* \psi = mcR^2 \tag{18}$$

is part of  $\mathcal{L}$ , so it follows that  $R^2$  is a number density. Comparing this with the Maupertuis action for a classical mass point:

$$-\frac{1}{2}\int \left(\int mc^2 R^2 d^3 x\right) dt = -\int mc^2 d\tau \tag{19}$$

one would consider  $\int R^2 d^3x = 2$ . This step is of course not compulsory, one may re-interpret the scalar Klein-Gordon field as in the quantum field theory, representing an undetermined number of particles or more generally an undetermined value of mass, M. By doing so one reinterprets the Maupertuis action in eq.(19) as  $-\int Mc^2 d\tau$  rendering M = m to be a very particular choice. In this case  $\int R^2 d^3x = 2M/m$ . In the quantum interpretation here nothing excludes negative M-s, giving rise to Dirac's problem on describing holes in negative energy continua. For our purpose this debate is irrelevant; since we are only seeking a motivation for the optimal quantum – classical splitting for a given positive mass object.

We consider now the derivatives of the complex field,  $\psi$  in Lorentz-covariant notation. The first derivative of  $\psi$  is given by

$$\partial_{\mu}\psi = \left(\frac{\partial_{\mu}R}{R} + \frac{i}{\hbar}\partial_{\mu}\alpha\right)\psi.$$
(20)

The derivative of the classical action is again a classical four-momentum and a four-velocity field also can be introduced analogous to the non-relativistic treatment:

$$P_{\mu} = \partial_{\mu}\alpha, \qquad u_{\mu} = P_{\mu}/(mc). \tag{21}$$

# B. Action principle with Madelung variables

The special relativitic quantum action of the free massive particle can again be splitted into a classical and a quantum part by using the magnitude-phase variables. As a functional of the fields R(x) and  $\alpha(x)$  it reads as

$$\mathfrak{S} = \frac{\hbar^2}{2mc} \int \left[ \partial_\mu R \,\partial^\mu R \,+\, \frac{R^2}{\hbar^2} \left( \partial_\mu \alpha \,\partial^\mu \alpha - (mc)^2 \right) \right] \,d^4x. \tag{22}$$

Rewriting this expression, it is transformed into  $\hbar^2$  times quantum kinetic plus  $R^2$  times classical part:

$$\mathfrak{S} = \int \left[\frac{\hbar^2}{2mc}\partial_\mu R\,\partial^\mu R \,+\, \frac{R^2}{2mc}\left(P_\mu\,P^\mu - (mc)^2\right)\right]\,d^4x \tag{23}$$

Now the classical part is the relativistic energy-momentum mass-shell expression, which is classically zero, but in the quantum mechanics in general it differs from zero, unless R is a constant. As it is well-known the Klein-Gordon action possesses a U(1) phase symmetry of the  $\psi(x)$  field. The corresponding U(1) Noether current is given by

$$-J^{\mu} = \frac{i}{2\hbar} \left( \psi \,\partial^{\mu} \psi^{*} - \psi^{*} \partial^{\mu} \psi \right) = \frac{1}{mc} R^{2} P^{\mu} = R^{2} u^{\mu} \tag{24}$$

constituting a number density 4-current  $R^2 u^{\mu} = \rho u^{\mu}$  based on the fluid picture. It is interesting to realize that the variation of the quantum action  $\mathfrak{S}$  with respect to the classical action (phase)  $\alpha$  results in the conservation of this current:

$$\frac{\delta\mathfrak{S}}{\delta\alpha} = -\partial_{\mu}\left(\frac{1}{mc}R^{2}\partial^{\mu}\alpha\right) = \partial_{\mu}J^{\mu} = 0.$$
<sup>(25)</sup>

For our present seek for the quantum - classical splitting of the content of quantum physics it is, however, more important to

study the other Euler-Lagrange equation of motion, the one obtained by variation against R. It delivers

$$\frac{\delta\mathfrak{S}}{\delta R} = -\frac{\hbar^2}{mc}\Box R + \frac{R}{mc}\left(P_{\mu}P^{\mu} - (mc)^2\right) = 0.$$
<sup>(26)</sup>

Here  $\partial_{\mu}^{\mu} = \Box$ . This equation constitutes an off-mass shell dispersion relation for the classical 4-momentum

$$P_{\mu}P^{\mu} - (mc)^2 = \hbar^2 \frac{\Box R}{R}.$$
(27)

Either one interprets this as quantum effects causing the free scalar field be off-mass shell even without any further interaction, or one speculates that perhaps the underlying space-time metric recieves corrections if R(x) is not a constant. In the latter case we consider a metric view:

$$g_{\mu\nu}u^{\mu}u^{\nu} = 1 + \left(\frac{\hbar}{mc}\right)^2 \frac{\Box R}{R}.$$
(28)

This is a Compton wavelength scaled, locally Lorentzian spacetime metric. Although this observation does not enforces a quantum origin of the space-time metric itself, and therefore our suggestion for the classical – quantum splitting in general has no intersection with quantum gravity theories, this metric view calls the attention to the fact that a certain handling of the quantum nature of the source term alone may modify the Einstein equation. This will be the basis of our starting point in section IV.

### C. Klein-Gordon Energy-Momentum tensor

In order to execute the above outlined program, one has to investigate the source term of gravity, the energy-momentum tensor, more closely. So, before turning to the Einstein equation, we turn to the calculation of the Klein-Gordon energy-momentum tensor. First we review the textbook derivation [62], the one using  $\psi$  and  $\psi^*$ . We note already here that in the context of general relativity the energy-momentum tensor is obtained from the variation of the action against the metric tensor, not like below. However, the result of our final choice on fixing the freedom of adding a total divergence to the Lagrange density and likewise a divergenceless contribution to the energy-momentum tensor, will be in accord to the classical definition. Without considering general relativity, as a first step, the canonically conjugated complex "momentum" field is obtained,

$$\Pi_{\mu} = \frac{\delta \mathcal{L}}{\delta \partial^{\mu} \psi} = \frac{1}{2} \partial_{\mu} \psi^{*}, \qquad (29)$$

and then according to the familiar Legendre-transformation-like definition the following energy-momentum tensor is presented:

$$T_{\mu\nu} = \Pi_{\mu}\partial_{\nu}\psi + \Pi_{\mu}^{*}\partial_{\nu}\psi^{*} - g_{\mu\nu}\mathcal{L}.$$
(30)

This can be rewritten in terms of R and  $\alpha$  as follows:

$$T_{\mu\nu} = mcR^2 w_{\mu\nu} + \frac{\hbar^2}{mc} U_{\mu\nu},$$
  

$$w_{\mu\nu} = u_{\mu}u_{\nu} - \frac{1}{2}g_{\mu\nu}(u_{\alpha}u^{\alpha} - 1),$$
  

$$U_{\mu\nu} = \partial_{\mu}R \partial_{\nu}R - \frac{1}{2}g_{\mu\nu}\partial_{\alpha}R \partial^{\alpha}R.$$
(31)

Here we note that the term proportional to  $(u_{\alpha}u^{\alpha}-1)$  is also of quantum nature, in the order of  $\hbar^2$ . The only classical contribution to  $T_{\mu\nu}$  is therefore  $mcR^2u_{\mu}u_{\nu}$ , that of the dust consisting of point-particles with mass m moving on Bohm trajectories according to the velocity field  $u_{\mu}(x)$ . Replacing back the off-mass-shell relation (28) into this expression leads to:

$$T_{\mu\nu} = mcR^2 u_{\mu}u_{\nu} + \frac{\hbar^2}{2mc} \left( 2\partial_{\mu}R \,\partial_{\nu}R - g_{\mu\nu} \left( \partial_{\alpha}R\partial^{\alpha}R + R\Box R \right) \right). \tag{32}$$

Here the  $\mathcal{O}(\hbar^2)$  part is the quantum contribution, the rest is classical dust. There are, however, other derivations of the energymomentum tensor with a formally different result [63–65]. In the next subsection we explore the differences.

## D. Generalized Bohm-Takabayashi Energy-Momentum Tensor

Although the Bohm-Takabayashi energy-momentum tensor [25, 26] was originally derived in the Madelung fluid picture, its validity is independent of the fluid interpretation. To begin with one takes the derivative of the off-mass-shell equation (28) and multiplies it by  $R^2/2$ :

$$\frac{R^2}{2} \partial_\mu \left[ u_\nu u^\nu - 1 - \frac{\hbar^2}{(mc)^2} \frac{\Box R}{R} \right] = 0.$$
(33)

Introducing now the Compton wavelength  $L_C = \hbar/mc$  and expanding the derivative of  $u_{\nu}u^{\nu}$  we obtain

$$R^2 u^{\nu} \partial_{\mu} u_{\nu} - \frac{1}{2} L_C^2 R^2 \partial_{\mu} \left(\frac{\Box R}{R}\right) = 0.$$
(34)

One utilizes also the following identity (the Madelung fluid is irrotational)

$$\partial_{\mu}u_{\nu} = \frac{1}{mc}\partial_{\mu}\partial_{\nu}\alpha = \frac{1}{mc}\partial_{\nu}\partial_{\mu}\alpha = \partial_{\nu}u_{\mu}.$$
(35)

Therefore

$$R^2 u^{\nu} \partial_{\mu} u_{\nu} = R^2 u^{\nu} \partial_{\nu} u_{\mu} = \partial_{\nu} (R^2 u^{\nu} u_{\mu}) - u_{\mu} \partial_{\nu} (R^2 u^{\nu})$$
(36)

and due to continuity (eq.9) the last term vanishes. By these manipulations we obtain

$$\partial_{\nu} \left( R^2 u^{\nu} u_{\mu} \right) = \frac{1}{2} L_C^2 R^2 \partial_{\mu} \left( \frac{\Box R}{R} \right).$$
(37)

Further use of the Leibniz rule in this formula leads to

$$R^{2}\partial_{\mu}\left(\frac{\Box R}{R}\right) = R\Box\partial_{\mu}R - \partial_{\mu}R\,\Box R = \partial^{\nu}\left(R\partial_{\nu}\partial_{\mu}R - \partial_{\nu}R\,\partial_{\mu}R\right). \tag{38}$$

This form already reveals a vanishing divergence of the Bohm-Takabayashi tensor

$$\mathcal{T}_{\mu\nu} = mcR^2 u_{\mu}u_{\nu} - \frac{\hbar^2}{2mc} \left(R\partial_{\mu}\partial_{\nu}R - \partial_{\mu}R\,\partial_{\nu}R\right). \tag{39}$$

Obviously this expression differs from the Klein-Gordon one (32) by

$$\Delta_{\mu\nu} = T_{\mu\nu} - \mathcal{T}_{\mu\nu} = \frac{\hbar^2}{2mc} \left( \partial_\mu R \partial_\nu R + R \partial_\mu \partial_\nu R - g_{\mu\nu} (\partial_\alpha R \partial^\alpha R + R \Box R) \right). \tag{40}$$

This difference does not spoil the energy-momentum conservation, because it has a vanishing divergence. We note that

$$(g_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu})\frac{R^2}{2} = g_{\mu\nu}(\partial_{\alpha}R\partial^{\alpha}R + R\Box R) - \partial_{\mu}R\partial_{\nu}R - R\partial_{\mu}\partial_{\nu}R.$$
(41)

Using this identity one realizes that the difference between the familiar Klein-Gordon and the Bohm-Takabayashi tensor,

$$\Delta_{\mu\nu} = \frac{\hbar^2}{4mc} \left( \partial_\mu \partial_\nu - g_{\mu\nu} \Box \right) R^2, \tag{42}$$

has a vanishing divergence [66]. This difference can also be written as a divergence of a three-index tensor,  $\Delta_{\mu\nu} = \partial^{\alpha} f_{\alpha\mu\nu}$  with

$$f_{\alpha\mu\nu} = \frac{\hbar^2 R}{2mc} \left( g_{\alpha\mu} \partial_{\nu} R - g_{\mu\nu} \partial_{\alpha} R \right).$$
(43)

We note that in general it is allowed to add a term to the energy-momentum tensor with vanishing divergence, such a term does not change the conservation. Energy, however, has a physical meaning. In fact, the continuous symmetries of the underlying action govern the correct expression. The proper energy-momentum tensor can be obtained by taking into account all continuous symmetries via their infinitesimal generators, according to a procedure described in [67, 68]. The difference  $\Delta_{\mu\nu}$  is related to

the realization of the conformal symmetry. The full energy-momentum tensor is the proper mixture of the above expressions. The general tensor contains a parameter  $\lambda$  multiplying  $\Delta_{\mu\nu}$  and added to the Bohm-Takabayashi tensor (39):

$$\mathfrak{T}_{\mu\nu} = mcR^2 u_{\mu}u_{\nu} + \mathfrak{U}_{\mu\nu} + \lambda\Delta_{\mu\nu}.$$
(44)

The conformal part can be identified by inspecting the trace of the energy-momentum tensor,

$$\mathfrak{T}^{\mu}_{\mu} = \left\{ 1 + L_C^2 \, \frac{1 - 3\lambda}{4} \Box \right\} (mcR^2). \tag{45}$$

For  $\lambda = 0$  one arrives at the original Bohm-Takabayashi tensor. For  $\lambda = 1$  the original Klein-Gordon case emerges. Finally, for  $\lambda = 1/3$  only classical dust contributes to the trace and all terms proportional to  $L_C^2 \propto \hbar^2$  – the quantum part of the energy-momentum tensor – are altogether traceless. This will be the basis of the classical - quantum splitting of the Einstein equation when considering classical gravity with quantum sources. In order to prepare this study in the next section, we express the generalized Bohm-Takabayashi energy-momentum tensor in scaling variables. Using  $R = e^{\sigma}/\sqrt{V}$ , where V is constant, one easily gets

$$\mathfrak{T}_{\mu\nu} = \frac{mc}{V} e^{2\sigma} u_{\mu} u_{\nu} + \frac{\hbar^2}{2mcV} e^{2\sigma} \mathfrak{W}_{\mu\nu},\tag{46}$$

with

$$\mathfrak{W}_{\mu\nu} = 2\lambda\partial_{\mu}\sigma\,\partial_{\nu}\sigma + (\lambda - 1)\partial_{\mu}\partial_{\nu}\sigma - \lambda\eta_{\mu\nu}\left(2\partial_{\alpha}\sigma\partial^{\alpha}\sigma + \Box\sigma\right).\tag{47}$$

This expression readily reminds us to a dilaton field  $\sigma(x)$  [69–74].

# IV. GENERAL RELATIVISTIC QUANTUM MECHANICS

In this section we turn to the theory of general relativity. In particular we suggest to utilize the generalized energy-momentum tensor source term in the Einstein equation by using the entire  $\mathfrak{T}_{\mu\nu}$  given in eq. (46). This supports the classical source term with quantum contributions in general space-time:

$$G^{curved}_{\mu\nu} = \frac{8\pi G}{c^3} T^{curved}_{\mu\nu}.$$
(48)

With the help of a conformal transformation we shall obtain an equivalent form of this equation as

$$G^{flat}_{\mu\nu} - \Lambda \eta_{\mu\nu} = \frac{8\pi G}{c^3} T^{remnant}_{\mu\nu}.$$
(49)

Here  $T_{\mu\nu}^{remnant}$  shall have an essentially reduced form relative to the generalized Bohm-Takabayashi energy-momentum tensor, cf. eq. (46), discussed in the previous section. We design a conformal factor,  $e^{2s}$ , to compensate the  $R^2 = e^{2\sigma}/V$  factor discussed previously. By doing so further effects arise, among others a cosmological constant like source term. Finally we estimate, that if gravity had caused a quantum binding, in what mass range the pairwise bound objects should fall in order to count quantitatively for the cosmological constant effect observed to day.

### A. Conformal transformation of the Einstein equation

Our starting point is the Einstein equation eq. (48) with the curved space-time version of the generalized Bohm-Takabayashi energy-momentum tensor as a source term. We introduce a conformal transformation that flattens out the metric of the Einstein equation, considering the curved geometry with a conformal metric tensor

$$g^{curved}_{\mu\nu} = e^{2s} \eta_{\mu\nu},\tag{50}$$

with  $\eta_{\mu\nu}$  being the Minkowski metric and s(x) a scalar function of the space-time coordinates. A conformal transformation of the energy-momentum tensor is given by [89]

$$T^{curved}_{\mu\nu} = e^{-2s} \mathfrak{T}_{\mu\nu} \tag{51}$$

and that of the Einstein tensor by [75]

$$G^{curved}_{\mu\nu} = G_{\mu\nu} + 2\partial^{\mu}s \,\partial_{\nu}s - 2\partial^{\mu}\partial_{\nu}s + \eta_{\mu\nu} \left(2\Box s + \partial_{\alpha}s\partial^{\alpha}s\right).$$
(52)

Quantities and partial derivatives on the right hand side refer to the flat metric  $\eta_{\mu\nu}$ . Substituting (46), (47), (51) and (52) into eq. (48) we arrive at

$$G_{\mu\nu} + 2\partial_{\mu}s \,\partial_{\nu}s - 2\partial_{\mu}\partial_{\nu}s + (2\Box s + \partial_{\alpha}s\partial^{\alpha}s) \,\eta_{\mu\nu} = \frac{8\pi G}{c^{3}}e^{-2s}\frac{\hbar^{2}}{2mcV}e^{2\sigma}\left[2\lambda\partial_{\mu}\sigma \,\partial_{\nu}\sigma + (\lambda - 1)\partial_{\mu}\partial_{\nu}\sigma - \lambda\eta_{\mu\nu}\left(2\partial_{\alpha}\sigma\partial^{\alpha}\sigma + \Box\sigma\right)\right].$$
(53)

It is obvious, that the  $s = \sigma$  choice represents the optimal reduction formula. The very same choice unifies several terms included in the curved Einstein and the generalized Bohm-Takabayashi tensors.

### B. Identifying the quantum effects

For the sake of deriving a most simple expression we may chose the trace parameter,  $\lambda$  and the wave function normalization volume V, so that the terms containing tensorial forms of partial derivatives compensate each other. This requires the equality of the coefficients of the terms  $\partial_{\mu} s \partial_{\nu} s$  and  $\partial_{\mu} \partial_{\nu} s$  on both sides of eq. (53):

$$2 = 2\lambda \frac{8\pi G}{c^3} \frac{\hbar^2}{2mcV},\tag{54}$$

$$-2 = (\lambda - 1) \frac{8\pi G}{c^3} \frac{\hbar^2}{2mcV}.$$
(55)

The solution of these requirements fixes  $\lambda = 1/3$ , the same value which leaves the classical part of the trace in the dust form  $\mathfrak{T}^{\mu}_{\mu} = mcR^2$ , cf. eq. (45). The optimal normalization volume becomes

$$V = \frac{4\pi}{3} L_S L_C^2,\tag{56}$$

upon using the Schwarzschild length by  $L_S = Gm/c^2$  and the Compton wavelength  $L_C = \hbar/mc$ . We note that V is the Planck volume scaled by  $M_p/m$ . The remaining terms in eq. (53) can be collected into an Einstein equation containing a cosmological term

$$G_{\mu\nu} - \Lambda \eta_{\mu\nu} = \frac{8\pi G}{c^3} \frac{m}{V} u_{\mu} u_{\nu}.$$
(57)

We note that  $8\pi Gm/(c^3V) = 8\pi L_S/V = 6/L_C^2$ .

Here we have identified a *cosmological term* proportional to the off-mass-shell effect on the Bohm trajectories:

$$\Lambda = -3\left(\Box\sigma + \partial_{\mu}\sigma\partial^{\mu}\sigma\right) = -3\frac{\Box R}{R}.$$
(58)

This result delivers a key to a new thinking about the cosmological constant. In this scenario quantum effects, in particular an attractive interaction, may lower the effective  $P_{\mu}P^{\mu}$  for a particle below the classical mass shell value, acting this way as an effective cosmological constant (cf. eq.27):

$$\Lambda = \frac{3}{\hbar^2} \left( (mc)^2 - P_\mu P^\mu \right). \tag{59}$$

Solutions of the quantization volume (56) and on the induced cosmological constant (58) might indicate a possible agreement with the scale-factor duality idea [73].

# C. Cosmological effect from quantum binding in gravitational $-\alpha/r$ potential

Based on this view we would like to make a new order of magnitude estimate for the source of a cosmological constant. A plane-wave solution of the Klein-Gordon equation leads to a zero cosmological term. Therefore we introduce a potential  $A^{\mu}$  in

$$\Lambda = 3 \, \frac{\nabla^2 R}{R} = 3 \, \left(\frac{1}{a^2} - \frac{2}{ar}\right) \tag{60}$$

with  $a = L_C/\alpha$  being the Bohr radius [76, 77]. Here the constant part must belong to the cosmological effect, while the -1/r like part to the potential energy term in the general non-relativistic Schrödinger equation. The relativistic treatment does not change the  $r \to \infty$  limit.

The quantum energy part is a spatial constant,  $3/a^2$  which may be in the correct order of magnitude.

In c = 1 units the gravitational Newtonian potential has the coupling constant  $\alpha = Gm^2/\hbar = (m/M_P)^2$  between two mass m objects. The corresponding Bohr radius amounts to  $a_B = \hbar/m\alpha = L_P(M_P/m)^3$ . Because in an equal mass "gravonium" the reduced mass is m/2, the radius we count with is  $a = 2a_B$ . This leads to the following estimate

$$\Lambda = \frac{3}{a^2} = \frac{3}{4L_P^2} \left(\frac{m}{M_P}\right)^6. \tag{61}$$

Using known values for  $M_P$ ,  $L_P$  and  $L_P^2 \Lambda = 2.56 \cdot 10^{-122}$  one arrives at  $m \approx 68$  MeV. This accounts to a total gravonium mass of  $2m \approx 138$  MeV.

### Summary

In summary we have explored the classical – quantum splitting of the Schrödinger equation by using the magnitude-phase representation of the complex wave function. By doing so not the Madelung fluid interpretation, but the partial conformal symmetry hidden in the relativistic Klein-Gordon Lagrangian, a simple relativistic generalization behind the Schrödinger Quantum Action, was in focus. Although the mass term breaks conformal invariance, in the limit of zero mass the rest of the theory should restore this. Accordingly the determination of the proper energy-momentum tensor has to take this symmetry into account.

Following the general mathematical recipe [66, 67, 78], we concluded that neither the naive expression - frequently found in textbooks - nor the Bohm-Takabayashi form of  $T_{\mu\nu}$  takes care of this symmetry. A conformal transformation of the Einstein tensor can be carried out which separates a classical fluid-like contribution of the free particle field to the classical gravity from quantum corrections in the energy-momentum tensor,  $T_{\mu\nu}$ , by assuming a simple (in the  $\hbar = 0$  limit vanishing) modification of the Einstein equation.

Moreover this quantum – classical splitting of the source term of the Einstein equation functions only if the Bohm potential part of  $T_{\mu\nu}$  is traceless ( $\lambda = 1/3$ ). Beyond this a cosmological term arises which was found to be proportional to the off-mass-shell measure of particles moving on Bohmian trajectories ( $\Lambda = -3\Box R/R$ ). As a small bonus the natural reference quantization volume belonging to the normalization of the total mass M represented by the scalar field, ( $R = |\varphi|$ ,  $\int R^2 d^3x = 2M/m$ ), becomes a Planck-scale based quantity ( $V = \frac{4\pi}{3}L_SL_C^2$ ). In fact we constructed a particular Jordan-Einstein frame change [79–81], which is optimally suited to simplify leading order quantum source effects.

Finally we investigated the induced cosmological term in case of quantum bound states in a simple, static  $-\alpha/r$  Newtonian gravitational potential of two mass m scalar objects. Due to eq.(61) the estimate for the total gravonium mass is 138 MeV. Since such objects - if they exist - are only superweakly bound (by gravity only), they cannot be mixed with ordinary matter.

As our first conclusive remark we note that recent approaches of quantum geometry recognize the connection to conformal transformation and Weyl geometry from various points of view. For example Carroll [82, 83] reviews many different works in this respect. On the other hand Koch summarizes and further elaborates some issues regarding reservations of nonstandard quantum interpretations [84, 85] (see also [86, 87] related specifically to the mentioned work of Wallstrom [33]). Our treatment is based on energetic considerations and focuses on the clear, formal, (universal) mathematical aspects, trying to avoid the traps of interpretational issues.

Our second remark corresponds to the trace anomaly of quantum field theories. At first sight our Klein-Gordon quantum mechanics has nothing to do with an effect that emerges in quantum fields in curved space-time [88]. The optimal choice  $\lambda = 1/3$ , removes the trace of the quantum part and leaves the classical part. Trace anomaly on the other hand usually occurs if quantum effects lead to nonvanishing  $T^{\mu}_{\mu}$  corrections to a classically traceless energy-momentum tensor. Our approach presented in this paper does not suit to the classification scheme of Flanagan [80]: our scalar field variable  $\sigma$  emerges as the quantum (non Hamilton-Jacobi) part of the action.

The essential difference lies in the separation of quantum parts. For the Schrödinger equation the Madelung variables (1) are

the classical real action  $\alpha$  and the  $\sigma = \ln R + \ln V/2$  characterizing the probability amplitude,

$$\psi = R e^{\frac{i}{\hbar}\alpha} = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}\alpha + \sigma} = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}(\alpha - i\hbar\sigma)}.$$

In a Feynman path integral formulation the transition probability is written as

$$\psi = \psi_0 e^{\frac{i}{\hbar}S} = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar}(Re(S) + iIm(S))}$$

where S is the action. Comparing the two expressions one realizes that a loop-expansion in the Feynman formalism requires a resummation in the Madelung variables and vice versa. The comparison could be more rewarding with the hydrodynamic version of quantum field theories [52, 53].

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