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REMARKS ON CLOSURE OPERATIONS

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§.1. INTRODUCTION

The relational datamodel was defined by E.F.Codd [3]. Many papers have appeared since that dealing with the combinational characterization problems of functional dependencies.

The main purpose of this paper is to investigate the connection of closure operations with the minimal keys and antikeys.

§.2. DEFINITIONS

In this section, we present some necessary definitions.

Definition 2.1. Let $X = \{1,\dots,n\}$. The function $F: 2^x \to 2^x$ is called a closure operation if for every $A,B \subseteq X$

(i) $A \subseteq F(A)$ (extensive)

(ii) $A \subseteq B \Rightarrow F(A) \subseteq F(B)$ (monotone)

(iii) $F(F(A)) = F(A)$ (idempotent)

Let M be an $m x n$ matrix and X be the set of its columns. Let $F_M(A)$, $A \subseteq X$, be a function such that $F_M(A)$ contains the ith column of M iff any two rows identical in columns belonging to A are also equal in the ith column.

It is clear that $F_M(A)$ is a closure operation.

Definition 2.2. Let F be a closure operation. We say that M represents the closure operation F if $F = F_M$.
It is known [1] that any closure operation is representable by an appropriate matrix M.

Definition 2.3. Let F be a closure operation and $A \subseteq X$. A is a key of F if $F(A) = X$.

Definition 2.4. Let F be a closure operation. We define

$$K_F = \{A : F(A)=X, (\forall B \subseteq A)(F(B)=F(A) \Rightarrow B=A)\}$$

That is: K_F is a set of minimal keys. We say that an $m \times n$ matrix M represents the family K iff $K = K_{F_M}$.
It is easy to see that the family of keys of a closure operation create a Sperner-system.
We denote $\Delta(K) = \min\{m : K = K_{F_M} : M \text{ is an } m \times n \text{ matrix}\}$.
where K is a Sperner-system over X.

§.3. THE PROPERTIES OF THE CLOSURE OPERATIONS

It is easy to prove that if F is a closure operation and $A_i \subseteq X$ $(1 \leq i \leq m)$, then $F(\bigcup_1^m A_i) = F(\bigcup_1^m F(A_i))$ and $F(\bigcap_1^m F(A_i)) = \bigcap_1^m F(A_i)$.

Definition 3.1. Let F be a closure operation. We say that $A (A \subseteq X)$ is a maximal element of F iff for all $B (B \supseteq A) : F(B) = F(A)$ implies $B = A$.

Denote by $M(F)$ the set of the maximal elements of F i.e.

$$M(F) = \{A: (\forall B \subseteq A)(F(B)=F(A) \Rightarrow B = A)\}$$

Theorem 3.2. Let F be a closure operation. Then

$$M(F) = \{A: (\forall C)(A \subseteq F(C) \Rightarrow C \not\subset A)\}$$

Proof. Assume the $A \in M(F)$, but $\exists C$ such that $A \subseteq F(C)$ and $C \subset A$. We have $F(A) \subseteq F(F(C)) = F(C)$ by (ii) and (iii). $C \subset A$ implies $F(C) \subseteq F(A)$, so $F(A) = F(C)$ holds. Consequently there exists $C \subset A$ such that $F(A) = F(C)$. This conteadicts to the assumption $A \in M(F)$.

Now, assume that

$$\forall C\ (A \subseteq F(C) \Rightarrow C/A) \quad (*)$$

but $A \notin M(F)$. This means that there is a set B such that $B \subset A$ and $F(B) = F(A)$. (i) implies $A \subseteq F(A) = F(B)$. Consequently, there is B such that $A \subseteq F(B)$ and $B \subset A$. This contradicts the fact that A satisfies $(*)$. The theorem is proved.

Let $M_1(F) = \{A: A \quad F(A) \text{ and } (\forall B \subseteq A)(F(B)=F(A) \Rightarrow A=B)\}$

Denoting by M_n the extremum of $/M_1(F)/$ it can be proved that $\lim\limits_{n\to\infty} \frac{M_n}{2^n} = 1$, see [2].

We denote by N_n the extremum of $/M(F)/$. It is clear that $M_n \leq N_m \leq 2^n$, hence $\lim\limits_{n\to\infty} \frac{N_n}{2^n} = 1$.

Definition 3.3. Let F be a closure operation over X, we call the image $F(A)$ of A as a nontrivial one if $A \subset F(A)$.

Let $P(F) = \{F(A): A \subset F(A)\}$ and denote by P_n the extremum of $/P(F)/$.

Theorem 3.4. $P_n = 2^{n-1}$.

Proof. Let $T(F) = \{A : A \subset F(A)\}$. It is clear that $/T(F)/ \geq /P(F)/$ (*). On the other hand (iii) implies $F(F(A))=F(A)$, so $P(F) \cap T(F)=\emptyset$ holds. Consequently, $/P(F)/+/T(F)/ \leq 2^n$, we obtain $2/P(F)/ \leq 2^n$ by (*). Hence $/P(F)/ \leq 2^{n-1}$ Take $b \in X$ and let $F(A)=A \cup \{b\}$ for every $A \subseteq X$. It is easily seen that F is a closure operation and $/P(F)/=2^{n-1}$. The theorem is proved.

§.4. THE CONNECTION BETWEEN THE MINIMAL KEYS AND ANTIKEYS

Let K be a Sperner-system. We define the set of the antikeys of K, denoted by K^{-1}, as follows:

$$K^{-1} = \{A \subseteq X : (B \in K \Rightarrow B \not\subseteq A) \text{ and } (A \quad C) \Rightarrow (\exists B \in K)(B \subseteq C)\}.$$

That is: the antikeys of K are the subsets of X not containing the elements of K and which are maximal for this property. Ot is clear that K^{-1} is a Sperner-system.

Remark 4.1. In [1,4], it has been proved that if K is an arbitrary Sperner-system then there exists a closure operation $F(F')$ for which $K = K_F$ $(K=K_F^{-1})$.

The antikeys play important roles for the evaluation of $\Delta(K)$ as well as for the construction of a concrete matrix representing a family K or for finding minimal keys.

The algorithm for finding the set of antikeys:

Let $K = \{B_1,\dots,B_m\}$ be the Sperner-system over X we have to contruct K^{-1}. For every $q = 1,\dots,n$, we construct $K_q = \{B_1,\dots,B_q\}^{-1}$ by induction.

Step 1: Construct K_1 in the following way:

$$K_1 = \{B_1\}^{-1} = \{X \setminus \{C\}: C \in B_1\}$$

Step $q+1$: Construct K_{q+1} in the following way:
By the inductive hypothesis we have constructed $K_q = \{B_1, \ldots, B_q\}^{-1}$. Suppose that $X_1, \ldots, X_p$ are the elements containing B_{q+1} of K_q. So

$$K_q = \{X_1, \ldots, X_p\} \cup \{A \in K_q : B_{q+1} \not\subseteq A\}.$$

Denote $\{A \in K_q (B_{q+1} \not\subseteq A\}$ by F_q. For all i $(i=1,\ldots,p)$ we construct the antikeys of $\{B_{q+1}\}$ on X_i in the analogous way of as in step 1, which are the maximal subsets of X_i not containing B_{q+1}. Denote them by $A^i_1, \ldots, A^i_{R_i}$ $(i=1,\ldots,p)$.

Let $K_{q+1} = F_q \cup \{A^i_T : A^i_T \not\subset A,\ if\ A \in F_q,\ 1 \leq T \leq R_i,\ 1 \leq i \leq p\}$

Theorem 4.2. $K_m = K^{-1}$

Proof. We prove the theorem by induction. The fact $K_1 = \{B_1\}^{-1}$ is obvious. Now we have to prove $K_{q+1} = \{B_1, \ldots, B_{q+1}\}^{-1}$ using the induktive hypothesis $K_q = \{B_1, \ldots, B_q\}^{-1}$. We have to prove:

a) If $A \in K_{q+1}$ then A is the subset of X not containing B_T $(T=1,\ldots,q+1)$ and being maximal for this property.

b) Every $A \subseteq X$ not containing elements B_T $(T=1,\ldots,q+1)$ and being maximal for this property is a element of K_{q+1}. The proof for (a): Let $A \in K_{q+1}$. If $A \in F_q$ then A doesn't contain any one in $B_1, \ldots, B_q$ and A is maximal for this property and at the same time $B_{q+1} \not\subseteq A$. Consequently, A is a maximal subset of X not containing B_T $(T=1,\ldots,q+1)$.

Let $A \in K_{q+1} \setminus F_q$. It is clear that there is A^i_T $(1 \leq i \leq p$ and $1 \leq T \leq R_i)$ such that $A = A^i_T$. Our construction shows that $B_l \not\subseteq A^i_T$ $(l=1,\ldots,q+1)$. Because A^i_T is an antikey of $\{B_{q+1}\}$ for X_i, then $A^i_T = X_i \setminus \{b\}$ for some $b \in B_{q+1}$. Now it is obvious that $A^i_T \cup \{b\} \supseteq B_{q+1}$. If $a \in X \setminus X_i$ then, by inductive hypothesis, for

$A_T^i \cup \{a\}$ $\{b\} = X_i \cup \{a\}$ there is B_l $(l=1,\ldots,q)$ such that $B_l \subseteq A_T^i \cup \{a\} \cup \{b\}$. X_i doesn't contain $B_1,\ldots,B_q$ by $X_i \in K_q$. Hence $a \in B_l$. If $(B_l \setminus a) \subseteq A_T^i$ then $A_T^i \cup \{a\} \supseteq B_l$. For every B_l $(1 \leq l \leq q)$ such that $B_l \subseteq X_i \cup \{a\}$ and $B_l \not\subseteq A_T^i$ we have $b \in B_l$.
Hence $(B_l \setminus \{a,b\}) \subseteq A_T^i$. Consequently, there is $A_1 \in F_q$ such that A_T^i A_1. This contradicts $A \in K_{q+1} \setminus F_q$. So there exists B_l $(1 \leq l \leq q)$ such that A_T^i $\{a\} \supseteq B_l$.
The proof for (b): Suppose that A is the maximal subset of X not containing B_T $(1 \leq T \leq q+1)$. By inductive hypothesis, there is $Y \in K_q$ such that $A \subseteq Y$.
The first case: If $B_{q+1} \not\subseteq Y$ then Y doesn't contain $B_1,\ldots,B_{q+1}$. Because A is the maximal subset of X not containing B_T $(1 \leq T \leq q+1)$, then $A=Y$. $B_{q+1} \subseteq Y$ implies $A \in F_q$. Hence $A \in K_{q+1}$.
The second case: If $B_{q+1} \subseteq Y$ then $Y = X_i$ for some i in $\{1,\ldots,p\}$ and $A \subseteq A_T^i$ for some T' in $\{1,\ldots,R_i\}$. If there exists $A_1 \in F_q$ such that $A_T^i \subset A_1$, then A_1 doesn't contain $B_1,\ldots,B_{q+1}$. Hence A A_1. This contradicts the definition of A. Hence $A_T^i \in K_{q+1}$. It is clear that A_T^i doesn't contain $B_1,\ldots,B_{q+1}$. By the definition of A we obtain $A=A_T^i$. The theorem is proved.

It can be seen that K and K^{-1} are determined uniquely by each other. Because of this fact, the determination of K^{-1} based on the algorithm doesn't depend on the order of sequence $\{B_1,\ldots,B_m\}$.

EXAMPLE: *Let* $X = \{1,2,3,4,5,6\}$ *and*
$K = \{(1,2),(2,3,4),(2,4,5),(4,6)\}$

According to the above algorithm we have:

$K_1 = \{(1,3,4,5,6),(2,3,4,5,6)\}$; $K_2 = \{(1,3,4,5,6),(2,3,5,6)(2,4,5,6)\}$

$K_3 = \{(1,3,4,5,6),(2,3,5,6),(2,4,6)\}$; $K_4 = \{(2,3,5,6)(1,3,4,5)(1,3,5,6),(2,4)\}$

$K^{-1} = K_4$.

We consider the following matrix:

The attributes: 1 2 3 4 5 6

$$M = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 3 & 0 & 0 \\ 4 & 0 & 4 & 0 & 4 & 4 \end{matrix}$$

M represents K, see [4].

Now we describe the "reverse" algorithm:
For given Sperner-system considered as the set of antikeys, we construct it's origin.

The following definition is necessary for us.
Let F be a closure operation over X. Denote:

$$Z(F) = \{A : F(A)=A\} \quad and \quad Y(F) = \{A \subset X : F(A)=A \quad and \ \nexists B \in Z(F)\setminus\{X\} : A \subset B\}$$

The elements of $Z(F)$ are called closure sets. It is clear that $Y(F)$ is the family of maximal closure sets.

Now we prove the following lemma:

Lemma 4.3.: A is an antikey if and only if A is the maximal closure set. That is: $K_F^{-1} = Y(F)$.

Proof. Let A is an antikey and suppose that $A \subset F(A)$. Hence $F(F(A))=F(A)=X$. Consequently A is a key. This contradicts to $\forall B \in K_F : B \not\subseteq A$. If there is A' such that $A \subset A'$ and $A' \in Z(F)\setminus\{X\}$, then A' is a key. This contradicts to $A' \subset X$.

On the other side if A is a maximal closure set but there exists $B (B \in K_F)$ such that $B \subseteq A$, then $F(A)=X$. This contradicts to $A \subset X$. If $A \subset D$ $(D \subseteq X)$ then it is clear that $F(D)=X$ (because A is the maximal closure set). Consequently A is anantikey.

The lemma is proved.

An algorithm finding a minimal key:
Let H be the Sperner-system, $B \in H$ and $a \in X \setminus B$. Suppose that $B = \{b_1, \ldots, b_m\}$. Let $G = \{B_T \in H : a \notin B_T\}$ and $T_o = B \cup \{a\}$

$$T_{q+1} = \begin{cases} T_q \setminus \{b_{q+1}\} & \text{if } \forall B_i \in H \setminus G : T_q \setminus \{b_{q+1}\} \not\subseteq B_i \\ T_q & \text{otherwise} \end{cases}$$

Theorem 4.4. If H is a set of antikeys, then $\{T_o, \ldots, T_m\}$ are the keys and T_m is a minimal key.

Proof. By the remark 4.1. there is a closure operation F such that $H = K_F^{-1}$. We prove the theorem by the induction.
It is obvious that T_o is a key. If T_q is the key and $T_{q+1} = T_q$, then T_{q+1} is a key. If $T_{q+1} = T_q \setminus \{b_{q+1}\}$ and $F(T_{q+1}) \neq X$, then by lemma 4.3 there is $B_T \in H$ such that $F(T_{q+1}) \subseteq B_T$. Hence $T_{q+1} \subseteq B_T$. This constradicts to $\forall B_T \in H : T_{q+1} \not\subseteq B_T$. Consequently, T_{q+1} is a key.

Now suppose that A is a proper subset of T_m. If $a \notin A$, then clearly $F(A) \neq X$. If $a \in A$, then there is $b_q \in B$ such that $b_q \in T_m \setminus A$ $(1 \leq q)$. By the given algorithm there is $B_T \in H \setminus G$ such that $T_{q-1}\{b_q\} \subseteq B_T$. We obtain $A \subseteq T_m \setminus \{b_q\} \subseteq T_{q-1} \setminus \{b_q\} \subseteq B_T$ by $T_m \subseteq T_q$ $(0 \leq q \leq m-1)$. Hence $F(A) \neq X$. Consequently, T_m is a minimal key. The theorem is proved.

Remark 4.5:

- It is best to choose B such that $/B/$ is minimal.
- If there is B such that $\forall B_T \in H$ and $B_T \neq B : B \cap B_T = \emptyset$ then $a \cup b$ is a minimal key $(\forall b \in B)$
- If $X \setminus \bigcup_{B_T \in H} B_T \neq \emptyset$ then $a \in X \setminus \bigcup_{B_T \in H} B_T$ is a minimal key.
- Let $Y = \bigcup_{B_T \in H} B_T$ $(B_T \neq B)$. If $B \setminus Y \neq \emptyset$ then it is best to choose

$T_o = B \cap Y \cup \{a\} \cup \{b\}$ $(b \in B \setminus Y)$.

Remark 4.6: Let H be an arbitrary Sperner-system and $A \subset X$. We can give an algorithm (which is analogous to the above one) to decide whether A is or isn't a key. If A is the key, then this algorithm find one A' such that $A' \subseteq A$ and A' is a minimal key.

Basing on theorem 4.4. We can find the minimal keys in concrete cases.

In the paper [4] the equalitysets of the relation are defined: Let R be a relation and h_i, $h_T \in R$. Denote

$$E(h_i, h_T) = \{a \in X : h_i(a) = h_T(a)\} \qquad (i \neq T)$$

Remark 4.7. Let R be a relation over X. $R = \{h_1, \ldots, h_m\}$. Let $E_{iT} = \{a \in X : h_i(a) = h_T(a)\}$ where $1 \leq i \leq m$, $1 \leq T \leq m$ and $i \neq T$. Denote $M = \{E_{iT}$: there isn't $E_{s\tau}$ such that $E_{iT} \subset E_{s\tau}\}$ practically, it is possible that there are many E_{iT} which equal to each other. We choose one E_{iT} from M. According to Lemma 4.3 it can be seen that M is the set of antikeys. Basing on the theorem 4.4. and the Remark 4.7 we find the minimal keys.

EXAMPLE. Let $X = \{1,2,3,4,5,6\}$ and

R be the relation:

0	1	0	0	1	0
1	0	1	0	0	1
2	0	0	1	2	2
0	1	2	2	0	3
3	2	1	0	0	0

It can be seen that $M = \{(1,2),(3,4,5),(4,6)\}$, where $E_{14} = \{1,2\}$, $E_{15} = \{4,6\}$ and $E_{25} = \{3,4,5\}$.

By the Theorem 4.4 and the Remark 4.5 it is clear that: $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{1,6\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{2,6\}$ are the minimal keys. We use the algorithm (Theorem 4.4) with $T_o = \{3,4,6\}$ and $T_o = \{4,5,6\}$. It can be seen that $\{3,6\}$ and $\{5,6\}$ are the minimal keys.

Let K be an arbitrary Sperner-system. The following theorem has been proved in [2].

Theorem 4.8. ([2]). $\binom{\Delta(K)}{2} \geq |K^{-1}| \geq \Delta(K)-1.$

Denote by $\binom{X}{k}$ the family of all k-element subsets of X. Let

$F_k(n) = max\ \{\Delta(K) : K \subseteq \binom{X}{k},\ |X| = n\}$

Theorem 4.9 ([5]).

$$F_k(n) \geq \sqrt{2}\ \binom{2k-2}{k-1}^{\frac{1}{2}\left[\frac{n}{2k-1}\right]}$$

We define the function $f_{2k-1}: N \to N$ (N-the set of natural numbers) in following way

$$f_{2k-1}(n) = \begin{cases} \binom{2k-1}{k-1}^{\frac{n}{2k-1}} & if\ n \equiv 0 \quad (mod\ (2k-1)) \\ \binom{2k-1}{k-1}^{\left[\frac{n}{2k-1}\right]-1} \times \binom{2k-1+p}{k-1} & if\ n \equiv p \quad (mod\ (2k-1))\ and\ 1 \leq p \leq k-1 \\ \binom{2k-1}{k-1}^{\left[\frac{n}{2k-1}\right]} \times \binom{p}{k-1} & if\ n \equiv p \quad (mod\ (2k-1))\ and\ k \leq p \leq 2k-2 \end{cases}$$

and

$$f_{2k-2}(n) = \begin{cases} \binom{2k-2}{k-1}^{\frac{n}{2n-2}} & if\ n \equiv 0 \quad (mod\ (2k-2)) \\ \binom{2k-2}{k-1}^{\left[\frac{n}{2n-1}\right]-1} \times \binom{2k-2+p}{k-1} & if\ n \equiv p \quad (mod\ (2k-2))\ and\ 1 \leq p \leq k-1 \\ \binom{2k-2}{k-1}^{\left[\frac{n}{2n-1}\right]} \times \binom{p}{k-1} & if\ n \equiv p \quad (mod\ (2k-2))\ and\ k \leq p \leq 2k-3 \end{cases}$$

It is clear that $2k-1$ and $2k-2 \leq n$

Take a partition $X = X_1 \cup \ldots \cup X_m \cup W$, where $m = \left[\frac{n}{2k-1}\right]$ and $|X_i| = 2k-1\ (1 \leq i \leq m)$. Let

$$K = \{B : |B| = k,\ B \subseteq X_i,\ \forall_i\} \quad \text{if} \quad |W| = 0$$

$K = \{B: |B|=k,\ B \subseteq X_i\ (1 \le i \le m-1) \text{ and } B \subseteq X_m \cup W\}$ if $1 \le |W| \le k-1$

$K = \{B: |B|=k,\ B \subseteq X_i\ (1 \le i \le m) \text{ and } B \subseteq W\}$ if $k \le |W| \le 2k-2$

It is clear that $K^{-1} = \{A: |A \cap X_i| = k-1,\ \forall_i\}$ if $|W| = 0$.

$K^{-1} = \{A: |A \cap X_i| = k-1\ (1 \le i \le m-1) \text{ and } |A \cap (X_m \cup W)| = k-1\}$ if $1 \le |W| \le k-1$

$K^{-1} = \{A: |A \cap X_i| = k-1\ (1 \le i \le m) \text{ and } |A \cap W| = k-1\}$ if $k \le |W| \le 2k-2$

It can be seen that $f_{2k-1}(n) = |K^{-1}|$

By the analogous way we take a partition

$X = X_1 \cup \ldots \cup X_m \cup W$, where $m = [\frac{n}{2k-2}]$ and $|X_i| = 2k-2$

Let $K = \{B: |B|=k,\ B \subseteq X_i,\ \forall_i\}$ if $|W|=0$

$K = \{B: |B|=k,\ B \subseteq X_i\ (1 \le i \le m-1) \text{ and } B \subseteq X_m \cup W\}$ if $1 \le |W| \le k-1$

$K = \{B: |B|=k,\ B \subseteq X_i\ (1 \le i \le m) \text{ and } B \subseteq W\}$ if $k \le |W| \le 2k-3$

It is clear that $f_{2k-2}(n) = |K^{-1}|$ and $f_{2k-2}(n) \ge \binom{2k-2}{k-1}^{[\frac{n}{2k-2}]}$

Theorem 4.10. Let $X = \{1, \ldots, n\}$.

If $n \equiv 0,\ (\text{mod } (2k-2)(2k-1))$ then $f_{2k-1}(n) > f_{2k-2}(n)$

If we fix k, then $\lim\limits_{n\to\infty} \dfrac{f_{2k-2}(n)}{f_{2k-2}(n)} = \infty$

Proof. If $k=2$ then it is easy to prove that $\forall_n: f_3(n) \ge f_2(n)$.
If $n=6$ or $n \ge 8$ then $f_3(n) > f_2(n)$.

$$\text{Let}\quad F = \frac{\binom{2k-1}{k-1}^{\frac{n}{2k-1}}}{\binom{2k-2}{k-1}^{\frac{n}{2k-2}}} = \frac{\left(\frac{2k-1}{k}\right)^{\frac{n}{2k-1}}}{\binom{2k-2}{k-1}^{\frac{n}{(2k-2)(2k-1)}}}$$

It is known that $n! = \sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n e^{\frac{\theta n}{12n}}$, where $0 < \theta_n < 1$.

So

$$F \geq \frac{\left(\frac{2k-1}{k}\right)^{\frac{n}{2k-1}}}{\left(\frac{e^{\frac{\theta n}{12(2k-2)}}}{\sqrt{\pi(k-1)}}\right)^{\frac{n}{(2k-2)(2k-1)}} \times 2^{\frac{n}{2k-1}}} \geq \frac{\left(1-\frac{1}{2k}\right)^{\frac{n}{2k-1}}}{\left(\frac{e^{\frac{1}{24(k-1)}}}{\sqrt{\pi(k-1)}}\right)^{\frac{n}{(2k-2)(2k-2)}}} = E$$

$$\ln E = \frac{n}{2k-1}\left(\ln\left(1-\frac{1}{2k}\right)+\frac{1}{2k-2}\left(\frac{1}{2}\ln(\pi(k-1))-\frac{1}{24(k-2)}\right)\right) = T$$

$$T \geq \frac{n}{2k-1}\left(\frac{1}{2k-2}\left(\frac{1}{2}\ln(\pi(k-1))-\frac{1}{24(k-1)}\right)-\frac{1}{2k-1}\right) \text{ by } \left|\ln\left(1-\frac{1}{2k}\right)\right| \leq \frac{1}{2k-1}$$

It is clear that if $k=3$ then $\frac{1}{2k-2}\left(\frac{1}{2}\ln(\pi(k-1))-\frac{1}{24(k-1)}\right)-\frac{1}{2k-1}>0$ and for every $k \geq 4$: $\frac{1}{2}\ln(\pi(k-1))-\frac{1}{24(k-1)} > 1$. Hence $\frac{1}{2k-2}\left(\frac{1}{2}\ln(\pi(k-1))-\frac{1}{24(k-1)}\right)-\frac{1}{2k-1} > 0$. Consequently, if $n \equiv 0 \pmod{(2k-2)(2k-1)}$ then $f_{2k-1}(n) > f_{2k-2}(n)$.

Now let n be an arbitrary natural number and k fixed. It can be seen that there exists a number $M>0$ such that

$$\frac{\binom{2k-1+p}{k-1}}{\binom{2k-1}{k-1}^{1+\frac{p}{2k-1}}} < M, \quad \frac{\binom{p}{k-1}}{\binom{2k-1}{k-1}^{\frac{p}{2k-1}}} < M, \quad \frac{\binom{2k-2+p}{k-1}}{\binom{2k-2}{k-1}^{1+\frac{p}{2k-2}}} < M,$$

$$\frac{\binom{p}{k-1}}{\binom{2k-2}{k-1}^{\frac{p}{2k-2}}} < M.$$

It can be seen that $\ln E \underset{n\to\infty}{\to} \infty$. Hence $F \underset{n\to\infty}{\to} \infty$.

Consequently: $\frac{f_{2k-1}(n)}{f_{2k-2}(n)} \underset{n\to\infty}{\to} \infty$ (It is easily seen that $k=2$ is also true)

The theorem is proved.

On the basis of theorem 4.10 and theorem 4.8 it is clear that

$$F_k(n) \geq \sqrt{2\, f_{2k-1}(n)} .$$

§.5. THE GENERAL FUNCTIONAL DEPENDENCY

In the paper [6] the concept of the general functional dependency is defined.

Let $X = \{1,\dots,n\}$, R be a relation over X.

$$h,h' \in R:\ t_i(h,h') = \begin{cases} 0 & \text{if } h(i) \neq h'(i) \\ 1 & \text{if } h(i) = h'(i) \end{cases}$$

Let $t(h,h') = (t_1(h,h'),\dots,t_n(h,h'))$

We say that (f,g) is a functional dependency iff f,g are the Boolean function of n variables.

Let $R \models (f,g) \iff \forall h,h' \in R: ft(h,h')=1 \Rightarrow gt(h,h')=1$

Denote by F the set of the functional dependencies, $B(f,g) = \{R : R \models (f,g)\}$, for $Y \subseteq F$ let $B(Y) = \bigcap_{(f,g)\in Y} B(f,g)$

Denote $Y \models (f,g)$ iff $B(Y) \subseteq B(f,g)$ and let $C(Y) = \{(f,g) \in F : Y \models (f,g)\}$.

We denote $f \leq f'$ iff $\forall t \in E_2^n : f(t)=1 \Rightarrow f'(t)=1$ and $Y (Y \subseteq F)$ is a closure set if $Y=C(Y)$.

Let Y be a closure set and

$MAX(Y) = \{(f',g') \in Y : g' = max(f),\ f'=min(g),\ (f,g) \in Y\}$

where $max(f) = \bigwedge_{(f,g)\in Y} g$ and $min(g) = \bigvee_{(f,g)\in Y} f$

Let $MIN(Y) = \{(f',g') \in Y : g'=min(f), f'=max(g),\ (f,g) \in Y\}$

where $min(f) = \bigvee_{(f,g)\in Y} g$ and $max(g) = \bigwedge_{(f,g)\in Y} f$

Theorem 5.1 ([6]). Let Y be a closure set. Then (f,g) is an element of Y if and only if there exists $(f',g')\in MAX(Y)$ such that $f\leq f'$ and $g'\leq g$.

Theorem 5.2. Let Y be a closure set. Then (f,g) is anelement of Y if and only if there exists $(f',g')\in MAX(Y)$ and $(f'',g'')\in MIN(Y)$ such that $f''\leq f\leq f'$ and $g'\leq g\leq g''$.

Proof. By the theorem 5.1. it is clear that we have only to prove: there is $(f'',g'')\in MIN(Y)$ such that $f''\leq f$ and $g\leq g''$.

Let $g''=min(f)$ and $f''=max(min(f))$. It is clear that $g\leq g''$ and we have $(f, min(f))\in Y$ by $Y\models(f, min(f))$.

Consequently, $max(min(f))\leq f$ by the definition of $MIN(Y)$. It is clear that $min(max(min(f)))\leq min(f)$. It can be seen that $min(f)\leq min(max(min(f)))$ (by $(f,g)\models(max(min(f)),min(f))$. Hence $min(f)=min(max(min(f)))$. We obtain $(max(min(f)),min(f))\in MIN(Y)$ by the definition of $MIN(Y)$. Hence $(f'',g'')\in MIN(Y)$ hold. The theorem is proved.

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ÖSSZEFOGLALÁS

MEGJEGYZÉSEK A LEZÁRÁSI OPERÁCIÓKHOZ

VU DUC THI

A dolgozatunkban a minimális kulcsok és antikulcsok és a lezárási operáció közötti kapcsolatot vizsgáljuk.

Р Е З Ю М Е

ЗАМЕЧАНИЯ ОБ ОПЕРАЦИЯХ ЗАМЫКАНИЯ

В настоящей работе изучается связь между минимальными ключами, антиключами и операциями замыкания.