THE DEPTH OF SUBGROUPS OF SUZUKI GROUPS

L. HÉTHELYI, E. HORVÁTH, AND F. PETÉNYI

Abstract. We determine the combinatorial depth of certain subgroups of simple Suzuki groups $Sz(q)$, among others, the depth of their maximal subgroups. We apply these results to determine the ordinary depth of these subgroups.

1. The combinatorial depth and ordinary depth

In this paper we shall study the depth of subgroups of Suzuki groups. The notion of combinatorial depth was defined in [1]. The related concept of ordinary depth has its origins in von-Neumann algebras, see [9].

First we remind the reader to some of the basic definitions, results and notations on combinatorial depth, see [1].

Let G be a finite group, H a subgroup of G. To define the combinatorial depth of H in G, we need some facts about bisets.

Let J, K, L be finite groups, let X be a (J, K)-biset, let Y be a (K, L)-biset. Then $X \times Y$ will be a (J, L)-biset. The group K also acts on $X \times Y$ by $k \cdot (x, y) := (xk^{-1}, ky)$ for $x \in X, y \in Y, k \in K$. The set of K-orbits of this action is denoted by $X \times_K Y$. This set also inherits a (J, L)-biset structure. Let $\Theta_1(H, G)$ be the (G, G)-biset G, using left and right multiplication of G on itself. Let $\Theta_{i+1}(H, G) := \Theta_i(H, G) \times_H G$ for $i \geq 1$. We denote by $\Theta'_i(H, G)$ the set $\Theta_i(H, G)$ considered as a (H, H)-biset. We denote by $\Theta^l_i(H, G)$ and $\Theta^r_i(H, G)$, the set $\Theta_i(H, G)$ considered as a (H, G)-biset or (G, H)-biset, respectively. Furthermore, $\Theta'_0(H, G) := H$, as a (H, H)-biset. The subgroup H is said to be of combinatorial depth $2i$ in G if there exist natural numbers a_1, a_2 such that $\Theta^r_{i+1}(H, G)$ is a direct summand of $a_1 \cdot \Theta^r_i(H, G)$ for some $i \geq 1$ and Θ^l_{i+1} is a direct summand of $a_2 \cdot \Theta^l_i(H, G)$, respectively. Moreover H has combinatorial depth $2i+1$ if ${\Theta'}_{i+1}(H, G)$ is a direct summand of $a \cdot {\Theta'}_i(H, G)$ for some natural number a. The minimal combinatorial depth $d_c(H, G)$ is the smallest positive integer i such that H has combinatorial depth i in G. This number is well defined.

Let us denote by $H^x := x^{-1}Hx$ and $H_{x_1,\ldots,x_n} := H \cap H^{x_1} \cap \ldots \cap H^{x_n}$, for $x_1, \ldots, x_n \in G$. Let $\mathcal{U}_i := \mathcal{U}_i(H, G) := \{H_{x_1,\ldots x_i} : x_1, \ldots x_i \in G\}$ and $\mathcal{U}_\infty := \mathcal{U}_\infty(H, G) := \cup_{i \geq 0} \mathcal{U}_i$, where $\mathcal{U}_0 := \{H\}$. In [1] the following characterization of combinatorial depth $d_c(H, G)$ is proved:

Theorem 1.1. [1, Thm 3.9] *Let H be a subgroup of the finite group G and let $i \geq 1$. Then:*

(i) $d_c(H, G) \leq 2i \leftrightarrow \mathcal{U}_{i-1} = \mathcal{U}_i \leftrightarrow \mathcal{U}_{i-1} = \mathcal{U}_\infty \leftrightarrow$ *for any* $x_1, \ldots, x_i \in G$, *there exist* $y_1, \ldots, y_{i-1} \in G$ *with* $H_{x_1,\ldots,x_i} = H_{y_1,\ldots,y_{i-1}}$.

(ii) *Let* $i > 1$*. Then* $d_c(H,G) \leq 2i-1 \leftrightarrow$ *for any* $x_1, \dots x_i \in G$ *there exist* $y_1, \dots y_{i-1} \in G$ *with* $H_{x_1,\dots x_i} = H_{y_1,\dots,y_{i-1}}$ *and* $x_1hx_1^{-1} = y_1hy_1^{-1}$ *for all* $h \in H_{x_1,\dots x_i}$.

(iii) $d_c(H,G) = 1 \leftrightarrow$ *for any* $x \in G$ *there exists a* $y \in H$ *with* $xhx^{-1} = yhy^{-1}$ *for all* $h \in H \leftrightarrow G = HC_G(H)$.

In [1] the *depth of a partially ordered set* X is introduced, as the length of a largest chain in X, and δ is defined as the depth of $\mathcal{U}_\infty(H,G)$. The symbol δ_* denotes the smallest positive integer k such that $Core_G(H)$ can be written as the intersection of k conjugates of H, and δ^* denotes the smallest positive integer k such that the intersection of any k distinct conjugates of H is equal to $Core_G(H)$.

Theorem 1.2. [1, Thm 3.11] *Let* G *be a finite group,* H *a subgroup of* G *and* $K := Core_G(H)$*. Then:*
(a) $2\delta_* - 1 \leq d_c(H,G) \leq 2\delta$
(b) $\delta_* \leq \delta \leq \delta^* \leq |G : N_G(H)|$
(c) *If* $\delta_* = \delta$ *and* $K \leq Z(G)$ *then* $d_c(H,G) = 2\delta - 1$.

Now let us remind the reader to the notion of *ordinary depth* of a subgroup H in the finite group G. We say that the *depth of the group algebra inclusion* $\mathbb{C}H \subseteq \mathbb{C}G$ is $2n$ if $\mathbb{C}G \otimes_{\mathbb{C}H} \cdots \otimes_{\mathbb{C}H} \mathbb{C}G$ ($n+1$-times $\mathbb{C}G$) is isomorphic to a direct summand of $\oplus_{i=1}^{a} \mathbb{C}G \otimes_{\mathbb{C}H} \cdots \otimes_{\mathbb{C}H} \mathbb{C}G$ (n times $\mathbb{C}G$) as $\mathbb{C}G - \mathbb{C}H$-bimodules for some positive integer a.

Furthermore $\mathbb{C}H$ is said to have depth $2n+1$ in $\mathbb{C}G$ if the same assertion holds for $\mathbb{C}H - \mathbb{C}H$-bimodules. Finally $\mathbb{C}H$ has depth 1 in $\mathbb{C}G$ if $\mathbb{C}G$ is isomorphis to a direct summand of $\oplus_{i=1}^{a} \mathbb{C}H$ as $\mathbb{C}H - \mathbb{C}H$ bimodules. The minimal depth of group algebra inclusion $\mathbb{C}H \subseteq \mathbb{C}G$ is called (minimal) ordinary depth of H in G, which we denote by $d(H,G)$. This is well defined.

This depth can be obtained from the so called inclusion matrix M. If $\chi_1, \dots, \chi_s$ are all irreducible characters of G and $\psi_1, \dots, \psi_r$ are all irreducible characters of H, then $m_{i,j} := (\psi_i^G, \chi_j)$. The "powers" of M are defined by $M^{2l} := M^{2l-1}M^T$ and $M^{2l+1} := M^{2l}M$ The odinary depth $d(H,G)$ can be obtained as the smallest integer n such that $M^{n+1} \leq aM^{n-1}$. This is well defined. The results on characters in [2] help to determine $d(H,G)$. Two irreducible characters $\alpha, \beta \in \mathrm{Irr}(H)$ are related, $\alpha \sim \beta$, if they are constituents of some χ_H, for $\chi \in \mathrm{Irr}(G)$. The distance $d(\alpha,\beta) = m$ is the smallest integer m such that there is a chain of irreducible characters of H such that $\alpha = \psi_0 \sim \psi_1 \ldots \sim \psi_m = \beta$. If there is no such chain then $d(\alpha,\beta) = -\infty$ and if $\alpha = \beta$ then the distance is zero. If X is the set of irreducible constituents of χ_H then $m(\chi) := \max_{\alpha \in \mathrm{Irr}(H)} \min_{\psi \in X} d(\alpha,\psi)$. The following results from [2] will be useful.

Theorem 1.3. [2, Thm 3.9, Thm 3.13]
(i) *Let* $m \geq 1$*. Then* H *has ordinary depth* $\leq 2m+1$ *in* G *if and only if the distance between two irreducible characters of* H *is at most* m*.*
(ii) *Let* $m \geq 2$*. Then* H *has ordinary depth* $\leq 2m$ *in* G *if and only if* $m(\chi) \leq m-1$ *for all* $\chi \in Irr(G)$*.*

Theorem 1.4. [2, Thm 6.9] *Suppose that* H *is a subgroup of a finite group* G *and* $N = Core_G(H)$ *is the intersection of* m *conjugates of* H*. Then* H *has ordinary depth* $\leq 2m$ *in* G*. If* $N \leq Z(G)$ *then* $d(H,G) \leq 2m-1$.

Similarly as in the case of combinatorial depth, $d(H,G) \leq 2$ if and only if H is normal in G. The ordinary depth does not change if we replace the complex number field by any field of characteristic zero. By [1, Remark 4.5] the ordinary depth $d(H,G)$ is bounded from above by the combinatorial depth $d_c(H,G)$.

For papers computing combinatorial or ordinary depth in some other groups, see [5],[6][7], where $PSL(2,q)$ and S_n are considered.

2. The Suzuki groups

Let q always denote an odd power of 2, say $q = 2^{2m+1}$, where $m > 0$. The Suzuki groups can be defined in various ways. They are twisted simple groups of Lie type ${}^2B_2(q)$, see e.g. [3, p. 43]. (In fact $Sz(2) = {}^2B_2(2)$ is also defined, but it is not simple. It is a Frobenius group of order 20.)

As permutation groups, the Suzuki groups belong to the class of Zassenhaus groups, i.e. doubly transitive permutation groups without any regular normal subgroups, where any non-identity element has at most two fixed points, and there are elements with exactly two fixed points. According to a well-known theorem of Feit, see [11, Thm. 6.1 in Ch. XI], the degree of a Zassenhaus group is always $q_1 + 1$, where q_1 is a prime power. For even q_1, there are two series of such groups, $PSL(2, 2^m)$ and $Sz(2^{2m+1})$. Suzuki proved that these are all the Zassenhaus groups for even q_1. See e. g. [13, 14].

The Suzuki groups can also be defined as subgroups of $GL(4,q)$. See e.g. [11, Ch XI. p. 182].

Let π be the unique automorphism of the field $K = GF(q)$ with $x\pi^2 = x^2$ for every $x \in K$.

Then $Sz(q) := \langle S(a,b), M(\lambda), T | a,b \in GF(q), \lambda \in GF(q)^\times \rangle$, where

$$S(a,b) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & a\pi & 1 & 0 \\ a^2(a\pi) + ab + b\pi & a(a\pi) + b & a & 1 \end{pmatrix}$$

$$M(\lambda) := \begin{pmatrix} \lambda^{1+2^m} & 0 & 0 & 0 \\ 0 & \lambda^{2^m} & 0 & 0 \\ 0 & 0 & \lambda^{-2^m} & 0 \\ 0 & 0 & 0 & \lambda^{-1-2^m} \end{pmatrix}$$

and

$$T := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The $q^2 + 1$ points $\{p_\infty, p(x,y) | x, y \in K\}$, where $p(x,y) = [xy + (x\pi)x^2 + y\pi, y, x, 1]$, form an ovoid $\mathcal{O}$, in the projective space $P(3,q)$. It means that no 3 points of $\mathcal{O}$ are on one line. The group $Sz(q)$ is the group of all projective collineations on $P(3,q)$ which leave invariant the set of $q^2 + 1$ points of this ovoid. This action is faithful and $Sz(q)$ as a permutation group on these points is a Zassenhaus group of order $(q^2 + 1)q^2(q - 1)$.

Let $G = Sz(q)$. Elements which only fix the point p_∞ constitute the nontrivial elements of a Sylow 2-subgroup F of G. The pointwise stabilizer of the set $\{p(0,0), p_\infty\}$ is a cyclic subgroup H of order $q-1$, which normalizes F.

Thus the stabilizer of p_∞ is $FH = N_G(F)$ which is a Frobenius group.

The normalizer $N_G(H)$ is a dihedral group of order $2(q-1)$.

We note that the numbers $q^2+1, q^2, q-1$ are pairwise relatively prime.

For odd prime p the Sylow p-subgroups of G are cyclic. $Sz(q)$ is a CN-group, i.e. every nontrivial element has a nilpotent centralizer.

For the proofs of the above statements, see e.g. [11, Ch. XI p. 182–194].

The Sylow 2-subgroup F of G is a Suzuki 2-group. This means that it is a non-Abelian 2-group, having more than one involution, and having a solvable group of automorphisms which permutes the set of involutions of F transitively.

See [10, p. 299] for details. It is a class 2 group of order q^2 and exponent 4. Moreover its center $Z(F)$ is of order q. The involutions in F together with the identity element constitute $Z(F)$. The subgroup H acts sharply 1-transitively on the involutions of F. The centralizer in G of every nontrivial element of F is a subgroup of F.

3. Subgroups of Suzuki groups

We collect some of the basic facts about the subgroups of Suzuki groups. For these and their proofs, see [12, Thm. 4.12] and [11, Thm 3.10 in Ch. XI].

Theorem 3.1 (Suzuki)**.** *Let $G = Sz(q)$, where $q = 2^{2m+1}$, for some positive integer m. Then G has the following subgroups:*

1. *the Hall subgroup $N_G(F) = FH$, which is a Frobenius group of order $q^2(q-1)$.*
2. *the dihedral group $B_0 = N_G(H)$ of order $2(q-1)$.*
3. *the cyclic Hall subgroups A_1, A_2 of orders $q+2r+1, q-2r+1$, respectively, where $r = 2^m$ and $|A_1||A_2| = q^2+1$.*
4. *the Frobenius subgroups $B_1 = N_G(A_1), B_2 = N_G(A_2)$ of orders $4|A_1|, 4|A_2|$, respectively.*
5. *the subgroups of form $Sz(s)$, where s is an odd power of 2, $s \geq 8$, and $q = s^n$ for some positive integer n. Moreover, for every odd 2-power s, where $s^n = q$ for some positive integer n, there exists a subgroup isomorphic to $Sz(s)$.*
6. *Subgroups (and their conjugates) of the above groups.*

Theorem 3.2. *Let $q = 2^{2m+1}$, $m > 0$, $r = 2^m$ and $G = Sz(q)$.*

a) *Let $i \in \{1,2\}$ and let $u_i \in A_i$, $u_i \neq 1$. Then $C_G(u_i) = A_i$. If $B_i = N_G(A_i)$ then $B_i = \langle A_i, t_i\rangle$, where t_i is an element of order 4, and $u^{t_i} = u^q$, for all $u \in A_i$. Moreover, $N_G(A_i)$ is a Frobenius group with kernel A_i.*

b) *Let F, H, A_1, A_2 as in Theorem 3.1. Then the conjugates of F, H, A_1, A_2 form a partition of G. In particular F, H, A_1, A_2, their conjugates and the conjugates of their characteristic subgroups are TI sets in G.*

4. The combinatorial depth of subgroups of $Sz(q)$

The main result of this paper is the following:

Theorem 4.1. a) *Let us list the representatives of conjugacy classes of the maximal subgroups of the Suzuki group $G = Sz(q)$. By Theorem 3.1 they are the following: $N_G(F)$, $B_0 = N_G(H)$, $B_1 = N_G(A_1)$, $B_2 = N_G(A_2)$ and $Sz(s)$ for*

maximal s such that $s^t = q$ for some positive integer $t > 1$. The combinatorial depths of these subgroups are:
$d_c(N_G(F), G) = 5$, $d_c(B_0, G) = 4$, $d_c(B_1, G) = 4$, $d_c(B_2, G) = 4$, *and* $d_c(Sz(s), G) = 4$.

b) *The following often used subgroups and their conjugates have combinatorial depth* 3*:*
   - *F or characteristic subgroups of F*
   - *subgroups of H*
   - *subgroups of A_1*
   - *subgroups of A_2*
   - *subgroups of order* 2
   - *cyclic subgroups of order* 4
   - *$S_1 \in Syl_2(Sz(s))$ for some Suzuki subgroup $Sz(s)$.*

c) *Some* 2*-subgroups (and their conjugates) have the following combinatorial depths:*
   - *the Klein four subgroups K_4 have $d_c(K_4, G) = 4$*
   - *$L \leq Z(F)$ of order 2^{f-1}, where $|Z(F)| = 2^f$ have $d_c(L, G) = 2f - 2$.*

d) *$d_c(Sz(s), G) = 4$ for any s such that $s^t = q$ for some positive integer $t > 1$.*

e) *Viewing G as a twisted group of Lie type, some subgroups of G are of special importance. Using the notations of* [3] *they are $B^1 = N_G(F)$, $H^1 = H$, $N^1 = N_G(H)$, $U^1 = F$. Their depths can be read off from the above lists.*

We will prove this theorem in a series of propositions and lemmas.

Note that by [1, Thm. 3.12 (b),(c)] every TI subgroup of a group G has combinatorial depth at most 3 and has combinatorial depth at most 2 if and only if it is normal.

Remark 4.2. It follows that every nontrivial subgroup L of $Sz(q)$ for $q \geq 8$ which is a TI set has combinatorial depth 3, $\delta_*(L) = \delta(L) = \delta^*(L) = 2$. In particular, by Theorem 3.2 this holds for F, H, A_1, A_2 and for the conjugates of their nontrivial characteristic subgroups.

From now on let $G = Sz(q)$, $G_1 = Sz(s)$ a Suzuki subgroup of G.

Proposition 4.3. *Let $N = N_G(F)$. The combinatorial depth $d_c(N, G)$ of the Hall-subgroup N is* 5*, $\delta_*(N) = \delta(N) = \delta^*(N) = 3$.*

Proof. The subgroup N is a one-point stabilizer in the Zassenhaus group G. Let N^{x_1} and N^{x_2} be two conjugates of N, which are different from each other and from N. Then $N \cap N^{x_1} \cap N^{x_2}$ stabilizes 3 points, hence it is the identity. However, if y_1 does not normalize N then $N \cap N^{y_1}$ is a two point stabilizer in G, which is nontrivial. Thus by Theorem 1.2 (b), $\delta_*(N) = \delta(N) = \delta^*(N) = 3$. Since $Core_G(N) = 1 \leq Z(G)$, it follows from Theorem 1.2 (c) that $d_c(N, G) = 2\delta(N) - 1 = 5$. □

Lemma 4.4. *Let $1 \leq m_1, m_2 \leq f$ be natural numbers. Then two subgroups H_1, H_2 of $Z(F)$ of orders 2^{m_1} and 2^{m_2} respectively have intersection of order at least $2^{m_1+m_2-f}$, where $|Z(F)| = q = 2^f$ $(f = 2m + 1)$.*

Proof. This follows from the dimension formula for vector spaces, (see Lemma 3.2 in [4]). □

Proposition 4.5. *Let $L \leq Z(F)$ be a subgroup of order 2^{f-1}, where $|Z(F)| = 2^f$. Then $d_c(L, G)) = 2f - 2$.*

Proof. Our argument is similar to that of [4, Satz 3.3]. It is easy to see that the depth δ of $\mathcal{U}_\infty := \mathcal{U}_\infty(L,G)$ is at most f. Since $N_G(F)$ acts transitively on the involutions of F, and these generate $Z(F)$, we have that $Core_{N_G(F)}(L) = 1$. By Lemma 4.4 we need at least f conjugates of L by elements of $N_G(F)$, i.e. $L_1,\dots,L_f$ so that their intersection should be 1. Hence $\delta(L)$ is also f, and this is the same as $\delta_*(L)$, where conjugation is considered by elements of $N_G(F)$. By Theorem 1.2 (c), $d_c(L, N_G(F)) = 2f-1$. It means among other things that $L\cap L^{x_1}\cap \dots L^{x_f} = L\cap L^{y_1}\cap \dots L^{y_{f-1}}$ for suitable elements y_i, where $i = 1,\dots,f-1$ and $\mathcal{U}_{f-2} \neq \mathcal{U}_{f-1} = \mathcal{U}_f$. We want to show that in G the following holds: $\mathcal{U}_{f-3} \neq \mathcal{U}_{f-2} = \mathcal{U}_{f-1}$. Let us consider $L_{x_1,\dots,x_{f-1}} = L\cap L^{x_1}\cap \dots L^{x_{f-1}}$. We have to find elements $y_1,\dots y_{f-2}$ in G such that $L_{y_1,\dots y_{f-2}} = L_{x_1,\dots x_{f-1}}$. If $L_{x_1,\dots,x_{f-1}} = 1$ then y_1 can be chosen outside $N_G(F)$, since F is TI, thus if $y_1 \in G\backslash N_G(F)$ then $L\cap L^{y_1} = 1$. If the intersection $L_{x_1,\dots,x_{f-1}}$ is of order 2 then if we intersect one by one, in some step we do not get a smaller subgroup. So one of the L^{x_i} can be cancelled. Further cancellation is impossible by Lemma 4.4, so the intersection cannot belong to $\mathcal{U}_{f-3}$. However, there are subgroups $L, L^{x_1}\dots, L^{x_{f-1}}$ such that their intersection is of order 2, since the depth of $\mathcal{U}_\infty(L,G)$ is f. The depth $d_c(L,G)$ cannot be $2f-3$, for if we consider the situation when $L_{x_1,\dots,x_{f-1}}$ is of order 2 and $x_1 = 1$, then by Theorem 1.1 (ii) y_1 has to centralize an involution in the intersection, thus $y_1 \in F$ must hold. Hence it centralizes L. But it is impossible since by Lemma 4.4, a subgroup of order 2 cannot be the intersection of less than $f-1$ subgroups. Thus $d_c(L,G) = 2f-2$. □

The next result is about some other subgroups of F.

Proposition 4.6. (i) *For every subgroup $F_1 \leq F$ of order 2 we have $d_c(F_1,G) = 3$.*
(ii) *We have $d_c(Z(F),G) = 3$.*
(iii) *Let K_4 be a subgroup of type $(2,2)$ in F. Then $d_c(K_4,G) = 4$.*
(iv) *Let C_4 be a cyclic subgroup of order 4 in F. Then $d_c(C_4,G) = 3$.*

Proof. (i) We note that F_1 is a TI set, so by Remark 4.2 we are done.
(ii) Follows from Remark 4.2.
(iii) First we note that $\delta_*(K_4) = 2$, since if we take an element x outside $N_G(F)$ then $K_4\cap K_4^x = 1$. According to Theorem 1.2 (a), $3 = 2\delta_*(K_4)-1 \leq d_c(K_4,G)$. We prove that the combinatorial depth of K_4 is at most 4. By Theorem 1.1, we have to show that for every $x_1, x_2 \in G$ there exists an element y_1 such that $K_4\cap K_4^{x_1}\cap K_4^{x_2} = K_4\cap K_4^{y_1}$ $(*)$. If the intersection on the left hand side of $(*)$ is 1 then we can choose any y_1 outside $N_G(F)$. We mention that $N_G(K_4) = C_G(K_4) = F$ since 3 does not divide the order of G. So if the intersection on the left hand side of $(*)$ is of order 4 then $x_1, x_2 \in F$, so we may choose $y_1 = x_1$. If the intersection on the left hand side of $(*)$ is of order 2 then $x_1, x_2 \in N_G(F)$. If $x_1 \notin F$ then we may choose $y_1 = x_1$. If the left hand side of $(*)$ is of order 2 and $x_2 \notin F$ then we may choose $y_1 = x_2$.
If the combinatorial depth would be 3 then by Theorem 1.1 (ii) for every $k \in K_{4\{x_1,x_2\}}$, $x_1kx_1^{-1} = y_1ky_1^{-1}$ should also hold. We mention that there exists always an intersection $K_4\cap K_4^{x_1}\cap K_4^{x_2}$ which is of order 2. Since if we take an element $x \in N_G(F)$ that takes an involution of K_4 to another, then $x \notin F$, so it cannot normalize K_4, but since $K_4\cap K_4^x \neq 1$, it can be only of order 2. Let us suppose now that the above intersection is of order 2. Let the elements of K_4 be $1, e_1, e_2, e_3$. Let us suppose that $x_1 \in F$ and $x_2 \notin F$, but

$x_2 \in N_G(F)$. Let us suppose that $e_1^{x_2} = e_2$, and the above intersection is $\langle e_2 \rangle$. Then the above intersection is $K_4 \cap K_4^{x_2}$. This is centralized by x_1. But if y_1 acts on the intersection as x_1 then it should also centralize this intersection. However, we know that the centralizer of every nontrivival element is in F, so $y_1 \in F$. Thus it centralizes K_4 and the intersections $K_4 \cap K_4^{x_1} \cap K_4^{x_2}$ and $K_4 \cap K_4^{y_1}$ cannot be equal. Thus $d_c(K_4, G) = 4$.

(iv) We first prove that the depth is at most 4. Let C be a cyclic subgroup of G of order 4. We may assume that $C \le F$. If C_{x_1,x_2} is of order 4 then let us take $y_1 = x_1$. If the order is 1 then let $y_1 \in G \backslash F$. If the order is 2 and C_{x_1} is of order 2 then let $y_1 = x_1$. If C_{x_1} is of order 4 and C_{x_2} is of order 2 then let $y_1 = x_2$. We have that C has depth at most 4. To prove that the depth is 3, we only have to deal with the case when the order of the intersection C_{x_1,x_2} is 2. If C_{x_1,x_2} is of order 2, C_{x_1} is of order 4 and C_{x_2} is of order 2 then x_2 centralizes the unique subgroup of C of order 2 and x_1 centralizes the whole C_{x_1,x_2}.

Thus $x_2 = y_1$ is a good choice, since on the intersection of order 2 both of them are acting trivially. Thus the depth of C is 3 in G.

□

Remark 4.7. Note that elementary abelian subgroups of type $(2, 2, 2)$ have combinatorial depth 3 in $Sz(8)$, since these subgroups are centers of some Sylow 2-subgroups of $Sz(8)$. Calculations with GAP [8] show that elementary abelian subgroups of type $(2, 2, 2)$ have combinatorial depth 4 in $Sz(32)$ and in $Sz(128)$.

Proposition 4.8. *The combinatorial depth of the Frobenius groups $B_i = N_G(A_i)$ in G is 4, for $i = 1, 2$.*

Proof. We prove the statement for $i = 1$, the proof for $i = 2$ is similar. First we prove that the depth is at most 4. If $B_1 \cap B_1^{x_1} \cap B_1^{x_2} = 1$ then we want to prove that there exists an element $y_1 \in G$ such that $B_1 \cap B_1^{y_1} = 1$. Let us suppose that $B_1 \cap B_1^x \neq 1$. Since $N_G(A_1) = N_G(B_1) = B_1$, if $x \notin B_1$ then by Theorem 3.2 b) we have that $B_1 \cap B_1^x \cap A_1 = 1$. Thus $B_1 \cap B_1^x$ contains an involution e^x such that $e \in B_1$. Since the involutions in B_1 are conjugate inside B_1, there is an element $y \in B_1$ such that $e^x = e^y$. Then $xy^{-1} \in C_G(e)$ and $x \in C_G(e)B_1$. Thus the number of elements x such that $e \in B_1$ lies in the conjugate B_1^x is $|C_G(e)B_1| = |C_G(e)A_1| \le |F||A_1| = q^2|A_1|$. In order to get an upper bound on the number of elements $x \in G$ such that $B_1 \cap B_1^x$ is nontrivial, we have to multiply this by the number of involutions in B_1, which is $|A_1|$. Thus the upper bound is at most $q^2|A_1|^2$ and since $|A_1| < 2q$, we have that $q^2|A_1|^2 < q^2(q^2 + 1)(q - 1) = |G|$. Thus there is an element y_1 such that $B_1 \cap B_1^{y_1} = 1$.

If both of x_i are in B_1, then $y_1 = x_1$ is a good choice. If some of the x_i is not inside B_1, then the intersection cannot contain elements of A_1, since this is a TI-set. Then $B_{x_1,x_2} := B_1 \cap B_1^{x_1} \cap B_1^{x_2}$ is either of order 4 or of order 2. If it is of order 4 then either $B_1 \cap B_1^{x_1}$ or $B_1 \cap B_1^{x_2}$ is of order 4. We choose $y_1 = x_1$ or $y_1 = x_2$ in this case to get the order 4 intersection. If there is no intersection of order 4, then there must be an intersection of order 2. If the intersection B_{x_1,x_2} is of order 2, then if $B_1 \cap B_1^{x_1}$ is of order 2 then we choose $y_1 = x_1$. If it is of order 4, then the x_2 cannot be in B_1 and $B_1 \cap B_1^{x_2}$ is of order 2, Then choose $y_1 = x_2$. If $B_1 \cap B_1^{x_1}$ and $B_1 \cap B_1^{x_2}$ are both of order 4, then their intersection cannot be of order 2, since in a Frobenius group two complements have trivial intersection. Thus we have proved that the depth of B_1 is at most 4. Now we prove that the depth cannot be 3. Let $B_1 \cap B_1^{x_2} = C$ be of order 4. This happens e.g. if x_2 is inside the centre of the Sylow 2-subgroup of G containing C, but not inside C. Let $x_1 \in B_1$ such that $C^{x_1} \neq C$. We want to prove that there is no y_1 such that $B_1 \cap B_1^{y_1} = C$ and $c^{x_1} = c^{y_1}$ for every $c \in C$. Let us suppose by contradiction that such y_1 would exist. Then $C^{y_1} \neq C$. There exists a subgroup $C_0 \leq B_1$ such that $C_0^{y_1} = C$. Let c_0, c be involutions of C_0 and C respectively. Then $c_0^{y_1}, c^{y_1} \in B_1$, since $c^{y_1} = c^{x_1} \in B_1$. Since $\langle c_0 \rangle A_1$ is a Frobenius group of index 2 in B_1, $c \in \langle c_0 \rangle A_1$, and $cc_0 \in A_1$. Thus similarly $(cc_0)^{y_1} = c^{y_1} c_0^{y_1} \in A_1$. Since A_1 is a TI-set and $y_1 \notin B_1$, this is a contradiction. Hence the depth of B_1 is 4. □

Proposition 4.9. *The combinatorial depth of $B_0 = N_G(H)$ is 4 in G.*

Proof. We know from Theorem 3.1 that $|B_0| = 2(q-1)$. First we prove that the depth of B_0 is at most 4. By Theorem 3.2, H is also a TI-subgroup of G, so $B_0 \cap B_0^{x_1} \cap B_0^{x_2}$ is of order 1 or 2, if x_1, x_2 are not both in B_0. If it is of order 1 then we have to find an element y_1 such that $B_0 \cap B_0^{y_1} = 1$. We use similar calculations as in Proposition 4.8. If $x \notin B_0 = N_G(H)$ then $B_0 \cap B_0^x \cap H = 1$. Thus if $B_0 \cap B_0^x \neq 1$ then this intersection contains an involution e^x such that $e \in B_0$. Since the involutions of B_0 are conjugate in B_0, there exists an element $y \in B_0$ such that $e^x = e^y$. Then $xy^{-1} \in C_G(e)$ and $x \in C_G(e)B_0$. Since the number of involutions in B_0 is $|H| = q-1$ and $|C_G(e)B_0|(q-1) = q^2(q-1)^2 < |G|$, there exists an element $y_1 \in G$ such that $B_0 \cap B_0^{y_1} = 1$.

If $B_{0\{x_1,x_2\}}$ is of order 2 then either $B_0 \cap B_0^{x_1}$ or $B_0 \cap B_0^{x_2}$ is of order 2. In the first case again let $y_1 = x_1$. In the second case let $y_1 = x_2$. We prove that the depth cannot be 3. Let $C := B_0 \cap B_0^{x_2}$ be of order 2, e.g. let $x_2 \in Z(S) \backslash B_0$, where $S \in Syl_2(G)$ intersecting B_0. Let $x_1 \in B_0 \backslash C$. Assume by contradiction that there exists an element $y_1 \in G$ such that $C^{x_1} = C^{y_1}$ and $B_0 \cap B_0^{y_1} = C$. Since $C^{x_1} \neq C$ thus $C^{y_1} \neq C$. There exists a subgroup C_0 of order 2 in B_0 such that $C_0^{y_1} = C$ and $C^{y_1} = C^{x_1} \leq B_0$. Let c_0, c be the unique involutions in C and C_0, respectively. Then $c_0 c \in H$ and $(c_0 c)^{y_1} = {c_0}^{y_1} c^{y_1} \in H$. However, H is a TI-subgroup and $H \cap H^{{y_1}^{-1}}$ contains a nontrivial element, so $y_1 \in B_0$, which is a contradiction. Thus the depth of B_0 cannot be 3. □

Remark 4.10. For finding an element $y \in G$ such that $B_0 \cap B_0^y = 1$ we also could have used a similar argument to that of Lemma 2.1 in [4].

We now study the depth of a Suzuki subgroup G_1 of the Suzuki group G.

Remark 4.11. First we remark that $N_G(G_1) = G_1$, since G does not contain a subgroup, where G_1 is a normal subgroup, by Theorem 3.1. The intersections of the form $G_1 \cap G_1^u$ have the following properties:

1. If $G_1 \cap G_1^u$ contains a nontrivial cyclic subgroup C of odd order then it also contains a maximal cyclic subgroup L of G_1 such that $C \leq L$.
   To see this, by Theorem 3.2, C is contained in a maximal cyclic subgroup L of G_1 of odd order. It is also contained in a conjugate of L^u in G_1^u, say in L^{uy}. Since L is a characteristic subgroup of a TI-subgroup of G it is also TI-subgroup in G. But then $L = L^{uy}$, and so L is contained in the intersection.
2. The intersection $G_1 \cap G_1^u$ cannot be a Suzuki group $Sz(s_1)$ for $s_1 < s$.
   To see this, we assume by contradiction that the intersection would be $Sz(s_1)$, where $s_1 > 2$. Then there would be a maximal cyclic subgroup in it of order

$s_1 - 1$ and this would be contained in a maximal cyclic subgroup of G_1. Since $s_1|s$, this maximal cyclic subgroup is of order $s-1$ which cannot be contained in the smaller $Sz(s_1)$, since $s-1 \not| |Sz(s_1)|$, which contradicts the fact that by 1. a maximal cyclic subgroup of this order must be contained in the intersection. If $s_1 = 2$ then $|Sz(s_1)| = 20$ and its cyclic subgroup of order 5 is contained in a maximal cyclic group of odd order of G_1, which is contained in the intersection $G_1 \cap G_1^u$ by part 1. Since the intersection is $Sz(s_1)$ this maximal cyclic subgroup of G_1 should be of order 5, which cannot happen.

3. The intersection $G_1 \cap G_1^u$ cannot be a dihedral group.
   To see this, let us suppose by contradiction that $G_1 \cap G_1^u$ is a dihedral group. Then it cannot be a 2-group of order at least 8, because a Suzuki 2-group does not contain such a subgroup. Let $e \in G_1 \cap G_1^u$ be an involution. Let $e \in S_1 \in Syl_2(G_1)$ and $e \in S_2 \in Syl_2(G_1^u)$ thus $S_1 \neq S_2$. Then S_1, S_2 are conjugate in G. Since the Sylow 2-subgroups of G are TI sets and S_1 intersects S_2 they must be inside the same Sylow 2-subgroup S of G. If $S_1^k = S_2$ then $k \in N_G(S)$. Moreover k cannot be in $N_G(Z(S_1))$ since $Z(S_1)$ is an elementary abelian subgroup of order at least 8, which cannot lie in a dihedral group. Also k cannot be a 2-element, since then it would centralize all the involutions in S, and then $Z(S_1)$ would be in the intersection. If k is a $2'$-element, then it belongs to some complement H in the Frobenius group $N_G(S) = HS$. This group contains $N_G(Z(S_1))$. Thus $N_G(Z(S_1)) = (SH) \cap N_G(Z(S_1)) = S(H \cap N_G(Z(S_1))) = SH_1$, by the Dedekind identity (modularity). So the complement H_1 can be chosen as a subgroup of H. Let us take an involution $e_1 \in S_1$ such that $e_1^k = e$. Since $N_{G_1}(S_1) \leq N_G(Z(S_1))$, by the Dedekind identity $N_{G_1}(S_1) = H_2 S_1$ and $H_2 \leq H_1^s \leq H^s$ for some $s \in S$. Since H_2 is sharply 1-transitive on the involutions of S_1, there is an element $l \in H_2$ such that $e_1^l = e$. However, since $e_1^k = e$ thus $e_1^{k^s} = e$ and H^s is sharply 1-transitive on the involutions of $Z(S)$, we have that $l = k^s$. This is a contradiction, since k^s does not normalize $Z(S_1)$, however $l \in N_G(Z(S_1))$.

Proposition 4.12. *Let $q = 2^{2m+1}$, let $s = 2^{2p+1}$ with $s^t = q$, let $G_1 = Sz(s) < G = Sz(q)$. . Let $\pi \in Aut(GF(q))$ such that $\pi : x \mapsto x^{2^{m+1}}$. Then the restriction of π to $GF(s)$ is the automorphism of $GF(s)$ mapping x to $x^{2^{p+1}}$. Let*

$$u(a,b) := \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a\pi & 1 \end{pmatrix}$$

Let us suppose that $S := \{u(a,b)\ a,b \in GF(q)\}$ and $S_1 = \{u(a,b)|a,b \in GF(s)\}$. (So $S_1 \in Syl_2(G_1)$, $S \in Syl_2(G)$ and $S_1 < S$.)

Then $N_S(Z(S_1)u(1,0)) = \{u(c,d)|c,d \in GF(q), c + c\pi \in GF(s)\} = \{u(c,d)|c \in GF(s), d \in GF(q)\}$ and $N_S(S_1) = \{u(c,d)|c,d \in GF(q), a(c\pi) + c(a\pi) \in GF(s)$ for every *$a \in GF(s)\}$. Furthermore, $N_S(Z(S_1)u(1,0)) = N_S(S_1)$ and $S > N_S(S_1)$. Moreover, $N_S(S_1) = N_S(Z(S_1)x)$ also holds for every nontrivial coset $Z(S_1)x$ in S_1.*

Proof. The restriction of π to $GF(s)$ is an automorphism of it having the property that the square of it raises every element to the second power. Since such an automorphism is unique it has to act the way as it is written in the Proposition.

Since multiplication in S is defined by $u(a,b)u(c,d) = u(a+c, c(a\pi) + b + d)$, thus $u(a,b)^{-1} = u(a, b + a(a\pi))$ and $Z(S_1) = \{u(0,b)|b \in GF(s)\}$. Furthermore

$u(0,b)u(1,0) = u(1,b)$. Similarly $u(c, d + c(c\pi))u(1,b)u(c,d) = u(1, b + c + (c\pi))$, hence $N_S(Z(S_1)u(1,0)) = \{u(c,d) | c,d \in GF(q), c + (c\pi) \in GF(s)\}$. To calculate the normalizer of S_1 we note that $u(a,b) \in S_1$ iff $a,b \in GF(s)$. We have to calculate when $u(c, d+c(c\pi))u(a,b)u(c,d) = u(c+a, d+c(c\pi)+b+c(a\pi))u(c,d) = u(a, d + b + c(c\pi) + c(a\pi) + d + (c+a)(c\pi)) = u(a, b + c(a\pi) + a(c\pi))$ belongs to S_1. This happens if and only if $c(a\pi) + a(c\pi) \in GF(s)$ for every $a \in GF(s)$, thus $N_S(S_1) = \{u(c,d) | c,d \in GF(q), c(a\pi) + a(c\pi) \in GF(s)$ for every $a \in GF(s)\}$.

Obviously $N_S(S_1) \leq N_S(Z(S_1)u(1,0))$. We want to prove that if $c \in GF(q)$ and $c + (c\pi) \in GF(s)$ then $c \in GF(s)$. This would imply that the inequality cannot be strict. The map $id + \pi$ is $GF(2)$-linear from $GF(q)$ into itself. Since π has no nontrivial fixed points, the kernel of $id+\pi$ is $GF(2)$. This map is not surjective, the identity of $GF(q)$ is not in the image of $id+\pi$. For if $x\pi + x = 1$ then $x\pi = x+1$ and $x^2 = x\pi^2 = x+2 = x$ and thus $x \in \{0,1\}$. Since the image of 0 and 1 under $id+\pi$ is zero, 1 is not in the image of $id+\pi$. Thus $GF(q) = Im(id+\pi) \oplus GF(2)$ and similarly $GF(s) = Im(id+\pi)_{GF(s)} \oplus GF(2)$. However, $Im(id+\pi)_{GF(s)} = Im(id+\pi) \cap GF(s)$ by the Dedekind identity. Let $c \in GF(q) \backslash GF(s)$. Then if $c(id + \pi)$ would be in $GF(s)$ then there would be an element $d \in GF(s)$ such that $d(id + \pi) = c(id + \pi)$. But then $d - c \in GF(2)$, so $d \in GF(s)$, which is a contradiction.

It is easy to see that $S > N_S(S_1)$, since $|S| = q^2$, however $|N_S(S_1)| = qs$. If $Z(S_1)x$ is another nontrivial coset of $Z(S_1)$ in S_1 then since by [10, Thm. 6.8 in Ch. VIII], $N_{G_1}(S_1)$ acts transitively on cosets of $S_1/Z(S_1)$, there is an element $h \in N_{G_1}(S_1)$ such that $(Z(S_1)u(1,0))^h = Z(S_1)x$. Then $N_S(S_1)^h = N_S(Z(S_1)u(1,0))^h = N_S(Z(S_1)x)$. However, $N_{G_1}(S_1) \leq N_G(S)$, so it normalizes $N_S(S_1)$. Thus $N_S(Z(S_1)x) = N_S(S_1)^h = N_S(S_1)$. So we are done. □

Proposition 4.13. *The intersection of two conjugates of $S_1 \in Syl_2(G_1)$ in G can be either 1 or S_1 or $Z(S_1)$, and all these intersections occur.*

Proof. Since $S_1 < S \in Syl_2(G)$ and S is TI, if we take u outside the normalizer of S, then $S_1 \cap S_1^u = 1$. Let $u \in N_G(S_1)$. Then $S_1 \cap S_1^u = S_1$.

Let us suppose that $S_1 \in Syl_2(G_1)$ and $S_1 < S \in Syl_2(G)$, $1 \neq S_1 \cap S_1^u = Z < Z(S_1)$. Then $u \in N_G(S)$. In fact u is of odd order, and it belongs to a complement K in the Frobenius group $N_G(S)$. By the Dedekind identity the complement K_1 of the subgroup $N_G(Z(S_1)) = K_1 S$ can be chosen to be a subgroup of K. Let $1 \neq x \in Z$. Then $o(x) = 2$. Thus there exists an element $y \in Z(S_1)$ such that $y^u = x$. Since the Frobenius complement K_2 of $N_{G_1}(S_1) = K_2 S$ acts sharply 1-transitively on the involutions of S_1, there exists an element $n \in K_2$ such that $y^n = x$. There is an element $s \in S$ such that $n_1 = n^s \in K_1 \leq K$. Then $y^{n_1} = x$ and $y^u = x$. Thus $u{n_1}^{-1} \in C_G(y) \leq S$. On the other hand it belongs to K. Hence $u = n_1$. However, $u = n_1 \in N_G(Z(S_1))$ and hence $S_1 \cap S_1^u \geq Z(S_1)$, which contradicts to our assumption.

Thus the intersection cannot be a nontrivial proper subgroup of $Z(S_1)$.

By the previous argument we also have that if $S_1 \cap S_1^u \neq 1$ then $S_1 \cap S_1^u \geq Z(S_1)$. Thus $u \in N_G(S)$. Now we want to show that if $S_1 \cap S_1^u > Z(S_1)$ then $S_1 \cap S_1^u = S_1$.

Let us suppose now that the $Z(S_1) < S_1 \cap S_1^u = L < S_1$. First we note that $Z(S_1) \leq Z(S_1)^u = Z(S_1^u) < S_1^u$, but since their order is the same $Z(S_1) = Z(S_1)^u$. Let $Z(S_1)x \in S_1 \cap S_1^u$, where $o(x) = 4$. Then since u^{-1} fixes $Z(S_1)$ it takes the coset of x to another coset in S_1, say $Z(S_1)x^{u^{-1}} = Z(S_1)y$, where $y \in S_1$ has order 4. By [10, Thm. 6.8 in Ch. VIII.] the complement of $N_G(S)$ acts sharply 1-transitively on

$S/Z(S)$ and the complement of $N_{G_1}(S_1)$ acts sharply 1-transitively on $S_1/Z(S_1)$. Thus there is an element $h \in N_{G_1}(S_1) \leq N_G(S)$ such that $(Z(S_1)x)^h = Z(S_1)y$. But then $hu \in N_G(Z(S_1)x)$ and hence $hu \in N_G(Z(S)x)$. However, since $hu \in N_G(S)$ and the elements of the complement act sharply 1-transitively on $S/Z(S)$, $hu \in S$. Thus $hu \in N_S(Z(S_1)x) = N_S(S_1)$ by Proposition 4.12. Thus u normalizes S_1, which is a contradiction.

The intersection can also be $Z(S_1)$, e.g. let $u \in S \backslash N_S(S_1)$, then $S_1 > S_1 \cap S_1^u \geq Z(S_1)$, so the intersection is $Z(S_1)$. □

Corollary 4.14. *Let $S_1 \in Syl_2(G_1)$, where $G_1 = Sz(s)$ is a Suzuki subgroup of $G = Sz(q)$, where $s^t = q$ and $q = 2^{2m+1}$. Then the combinatorial depth $d_c(S_1, G) = 3$.*

Proof. Since S_1 is not normal in G, $d_c(S_1, G) > 2$. If $S_1 \cap S_1^{x_1} \cap S_1^{x_2} = 1$ then y_1 can be chosen outside $N_G(S)$, for $S_1 < S \in Syl_2(G)$. If $S_1 \cap S_1^{x_1} \cap S_1^{x_2} = S_1$ then $y_1 = x_1$ is a good choice. If $S_1 \cap S_1^{x_1} \cap S_1^{x_2} = Z(S_1)$ then if $S_1 \cap S_1^{x_1} = Z(S_1)$ then we choose $y_1 = x_1$ if $S_1 \cap S_1^{x_1} = S_1$ and x_1 is a 2-element in $N_G(S_1)$, then it centralizes $Z(S_1)$, so we choose $y_1 \in S \backslash N_S(S_1)$. This is possible since $S > N_S(S_1)$, by Proposition 4.12. On the other hand, if x_1 is an element of odd order in $N_G(S_1) = K_1 S_2 \leq N_G(S) = KS$, then we observe that $N_G(Z(S_1)) = \tilde{K}_1 S$, where $|K_1| = |\tilde{K}_1| = |Z(S_1)| - 1$, by the sharply 1-transitivity of the action on the involutions by K_1 and $\tilde{K}_1$. We may suppose that $x_1 \in K_1$. Let us choose y_1 in a complement $\tilde{K}_1$ of $N_G(Z(S_1))$ such that $\tilde{K}_1 \not\leq N_G(S_1)$, and y_1 is conjugate to x_1 in $N_G(Z(S_1))$. Such a complement $\tilde{K}_1$ exists, since $N_G(Z(S_1))$ has more complements than $N_G(S_1)$. Then $y_1 = {x_1}^s$, where $s \in S$, thus the action of y_1 and x_1 on $Z(S_1)$ is the same. However $S_1 \cap {S_1}^{y_1} = Z(S_1)$, since this intersection contains $Z(S_1)$ and does not contain S_1. Thus by Proposition 4.13 gives us that the intersection is $Z(S_1)$.

Now we show that $1 < S_{1 x_1, x_2} = Z < Z(S_1)$ cannot hold. Let us suppose now that $1 \neq S_1 \cap S_1^{x_1} \cap S_1^{x_2} = Z < Z(S_1)$. Then $S_1 \cap S_1^{x_1} = Z(S_1)$ must hold and $Z(S_1) \cap S_1^{x_2} = Z$. Then $1 < Z(S_1)^{x_2^{-1}} \cap S_1 \leq S_1^{x_2^{-1}} \cap S_1$ and since it is a proper subgroup of S_1, by Proposition 4.13, $S_1^{x_2^{-1}} \cap S_1 = Z(S_1)$. Thus $S_1 \cap S_1^{x_2} = Z(S_1)^{x_2} = Z(S_1)$ which cannot be the case, since $S_{1\{x_1,x_2\}}$ is a proper subgroup of $Z(S_1)$. Thus the above intersection cannot be $1 \neq Z < Z(S_1)$. It is easy to see that the chosen elements y_1 in each case act the same way on $S_{1\{x_1,x_2\}}$ as x_1. Thus $d_c(S_1, G) = 3$. □

Corollary 4.15. *Let the notations be as in Corollary 4.14. If the intersection ${G_1} \cap {G_1}^x$ contains S_1 then this intersection cannot contain an element of odd order, except for the case when this intersection is G_1.*

Proof. Let $G_1 \cap G_1^x \geq S_1$. Then $S_1, S_1^{x^{-1}} \leq G_1$. Thus there exists an element $g_1 \in G_1$ such that $S_1^{g_1} = S_1^{x^{-1}}$ and so $S_1^{g_1 x} = S_1$ and $n = g_1 x \in N_G(S_1)$. Thus ${G_1}^x = {G_1}^n$. So we may assume that already at the beginning $x \in N_G(S_1)$. If the intersection $G_1 \cap G_1^x$ would contain an element of odd order, then by Remark 4.11/1, it would also contain a maximal cyclic subgroup of odd order in G_1. Let us denote such a maximal cyclic subgroup by K_1. If $K_1 \not\leq N_{G_1}(S_1)$ then $\langle K_1, S_1 \rangle = G_1$, since no proper subgroup of G_1 contains both K_1 and S_1. Thus if the intersection is a proper subgroup of G_1 then $K_1 \leq N_{G_1}(S_1)$ and $N_{G_1}(S_1) = K_1 S_1$. So we may assume that $G_1 \cap G_1^x = K_1 S_1$. We know that $x \in N_G(S_1)$. If x is a 2-element, then since $K_1^{x^{-1}} \leq N_G(S_1)$ and $K_1^{x^{-1}} \leq G_1$, thus $K_1^{x^{-1}} \leq N_{G_1}(S_1) = K_1S_1$. Hence we have that ${K_1}^{x^{-1}} = {K_1}^{s_1}$ for some $s_1 \in S_1$. Thus $K_1 = K_1^{s_1x}$. Since $x \in N_{G_1}(S_1) \leq N_G(S)$, both s_1 and x belong to S, so their product is an element of S normalizing K_1, which is impossible, since $N_G(S)$ is a Frobenius group. Let now x be of odd order in $N_G(S_1)$. We have that $K_1^x \leq N_G(S_1)$ and $K_1^x \leq G_1^x$. Since $K_1, K_1^x \leq N_{G_1^x}(S_1)$, they are conjugate, so there is an element $y \in N_{G_1^x}(S_1)$ such that $K_1 = K_1^{xy}$. Thus $xy \in N_G(K_1)$. However $xy \in N_G(S_1) \leq N_G(S)$, so it belongs to a complement K_2 of $N_G(S_1)$ containing K_1. Since K_1 acts sharply 1-transitively on the involutions of S_1, K_2 cannot be bigger than K_1, otherwise some involution would be centralized by a $2'$-element, which is not possible in a Suzuki group. Hence $xy \in K_1$. But $x \notin G_1^x, y \in G_1^x$ thus $xy \notin G_1^x$, contradicting the fact that $K_1 \leq G_1^x$. Thus $G_1 \cap G_1^x$ is a 2-group. □

Proposition 4.16. *We use the notations of Corollary 4.14. If $G_1 \cap G_1^x$ contains an involution i, then this intersection is either $Z(S_1)$ or S_1 or G_1, where $i \in S_1 \in Syl_2(G_1)$.*

Proof. Let us suppose that the intersection $G_1 \cap G_1^x$ contains the involution i and this intersection is a proper subgroup of G_1. Let $S_2 \in Syl_2(G_1^x)$ containing i. Then S_1, S_2 are contained in the same Sylow 2-subgroup S of G and they are conjugate by an element $k \in N_G(S)$. If $k \in S$, then k centralizes $Z(S_1)$, and hence $Z(S_1) = Z(S_1)^k = Z(S_2)$ and thus $Z(S_1) \leq G_1 \cap G_1^x$. If k is of odd order, then let us take an involution $j \in S_1$ such that $j^k = i$. The subgroup $N_G(Z(S_1))$ is contained in $N_G(S) = SK$. We may suppose that k belongs to the complement K. By the Dedekind identity $N_G(Z(S_1)) = S(K \cap N_G(Z(S_1)) := SK_2$. However, $N_{G_1}(S_1) = S_1K_1 \leq N_G(S_1) \leq N_G(Z(S_1))$, thus there exists an element $s \in S$ such that ${K_1}^s \leq K_2$. We note that K_1 acts sharply 1-transitively on the involutions of $Z(S_1)$, and K_1^s is acting the same way, since elements of S centralize $Z(S_1)$. We will show that $K_2 = K_1^s$. An element in $K_2 \backslash K_1^s$ would take an involution to the same place as an element in K_1^s and their quotient would centralize that involution, which cannot happen in a Frobenius group. Thus $N_G(Z(S_1)) = SK_1^s$ and there is an element $h \in K_1^s$ such that $j^h = i$. But then hk^{-1} is in K and centralizes i, which can only happen if $h = k$, and hence $k \in N_G(Z(S_1))$. Thus, $Z(S_1) = Z(S_1)^k \leq S_1^k = S_2$ and hence $G_1 \cap {G_1}^x \geq Z(S_1)$.

We know that $Z(S_1), Z(S_1)^{x^{-1}} \leq G_1$. So there is an element $g_1 \in G_1$ such that $Z(S_1)^{g_1} = Z(S_1)^{x^{-1}}$. Thus, $Z(S_1)^{g_1x} = Z(S_1)$ and $g_1x \in N_{G_1}(Z(S_1))$. So we may assume that already originally x belongs to $N_G(Z(S_1))$. If $G_1 \cap G_1^x$ would contain an element of odd order then by Remark 4.11/1, it would also contain a maximal cyclic subgroup $H_1 \leq G_1$ of odd order. Because of the subgroup structure of G_1, if x does not normalize G_1 then H_1 normalizes $Z(S_1)$. Similarly, $H_1, H_1^x \leq N_{G_1^x}(Z(S_1)) \leq N_G(S)$. Hence H_1, H_1^x are conjugate in the Frobenius group $N_{G_1^x}(Z(S_1))$, i.e. there is an element $y \in N_{G_1^x}(Z(S_1))$ such that $H_1^{xy} = H_1$. Thus $xy \in N_G(H_1)$. However, $xy \in N_G(Z(S_1))$. If xy is an involution then it belongs to S, and then it cannot normalize H_1. If xy is of odd order then it belongs to H_1. However, $x \notin G_1^x$ and $y \in G_1^x$, thus $xy \notin G_1^x$. This contradicts the fact that $H_1 \leq G_1^x$.

Thus if $G_1 \cap G_1^x$ contains an involution then it is either $Z(S_1)$ or S_1, since it cannot contain an element of odd order. The intersection can be $Z(S_1)$ e.g. when $S_1 \cap S_1^x = Z(S_1)$. It happens e.g. when $x \in S \in Syl_2(G)$ containing S_1, but not normalizing S_1. Such an element exists, since $S > N_S(S_1)$ holds by Proposition

4.12. If for some $S_2 \in Syl_2(G_1)$ the equality $S_2^x = S_1$ would hold, then $S_2 \leq S$ and since $G_1 \cap S = S_1$, $S_1 = S_2$, which is not the case.

The intersection is S_1 if x normalizes S_1 but it does not normalize G_1. We can get such an element from $Z(S)$ since it cannot normalize G_1. To see, this, let us use [11, 3.12 Remarks c) in Ch. XI], telling that the outer automorphism group of G_1 is cyclic of odd order. We have that no element of $Z(S)$ centralizes G_1, since the centralizers of 2-elements are 2-groups. If $Z(S) \leq N_G(G_1)$ then then $Z(S)/Z(S_1) \leq Out(G_1)$, which is a contradiction. □

Proposition 4.17. *We use notations of Cor. 4.14. The intersection $G_1 \cap G_1^x$ can be up to conjugacy either 1, $Z(S_1)$, S_1, K_1, A_{1_1}, A_{2_1} or G_1, where K_1 is maximal cyclic of order $s-1$, A_{1_1}, A_{2_1} are maximal cyclic of orders $s+2r_1+1$ and $s-2r_1+1$, respectively, where $s = 2^{2p+1}$, $r_1 = 2^p$ and $q = s^t$. Moreover, these subgroups occur as intersections.*

Proof. If $G_1 \cap G_1^x \neq G_1$ contains a maximal cyclic subgroup of G_1 of odd order then it cannot contain other maximal cyclic subgroups, because of the subgroup structure. In fact one can get such an intersection if we take a cyclic subgroup $K_1 \leq G_1$ of order $s-1$. Then it is contained in a maximal cyclic subgroup K of order $q-1$ in G. Then $K \not\leq N_G(G_1)$, since otherwise G_1 would contain all the involutions of $S \in Syl_2(G)$ normalized by K, which is not the case. Let $x \in K \backslash N_G(G_1)$. Then $G_1 \cap G_1^x = K_1$. The subgroup A_{1_1} (and similarly A_{2_1}) is contained in a maximal cyclic subgroup of G of odd order. It is definitely bigger than $|A_{1_1}|$. Let us take an element x of such a maximal cyclic subgroup of odd order in G outside G_1, then $G_1 \cap G_1^x$ contains A_{1_1}. It cannot normalize G_1, since then together with G_1 it would generate a group, which cannot be a subgroup of G by Theorem 3.1. Thus the intersection is equal to A_{1_1}. Similarly one can contstruct the intersection to be A_{2_1}. The fact that $G_1 \cap {G_1}^x = 1$ occurs will be shown later in Proposition 4.21. The rest follows from Proposition 4.16. □

Corollary 4.18. *With the notations of Proposition 4.17, the intersection $G_1 \cap G_1^{x_1} \cap G_1^{x_2}$ can be 1, $Z(S_1)$, S_1, K_1, A_{1_1}, A_{2_1}, G_1 or some conjugate subgroups of G_1. The combinatorial depth $d_c(G_1, G)$ is 4.*

Proof. These subgroups occur as ${G_1}_{\{x_1,x_2\}}$, since by Proposition 4.17 they occur as $G_1 \cap {G_1}^{x_1}$. We have to show that there are no more possible triple intersections. However, these would be some subgroups of groups of the above types. On the other hand, $G_1 \cap G_1^{x_1}$ is one of the above types, hence $G_1 \cap {G_1}^{x_2}$ is either disjoint to it, or also one of these, hence their intersection is either 1 or one from the above list.

The combinatorial depth of G_1 cannot be 2 since it is not normal in G. We want to show that its combinatorial depth is at most 4. To see this, we observe that each subgroup which can be ${G_1}_{\{x_1,x_2\}}$ is already $G_1 \cap {G_1}^y$ for some suitable $y \in G$. This comes from Proposition 4.17 and the first part of this Corollary.

We now show that $d_c(G_1, G) \geq 4$. Let $G_1 \cap G_1^{x_1} \cap G_1^{x_2} = A_{1_1}$. Let us suppose that $A_{1_1} \neq A_{1_1}^{x_1} \leq G_1$. This happens e.g. when $x_1 \in G_1 \backslash N_{G_1}(A_{1_1})$ and $G_1 \cap {G_1}^{x_2} = A_{1_1}$. If the combinatorial depth of G_1 in G would be 3, then there would exist an element $y_1 \in G$ such that $G_1 \cap G_1^{y_1} = A_{1_1}$ and $A_{1_1}^{x_1} = A_{1_1}^{y_1}$. However, $A_{1_1}^{y_1} \leq G_1$, so $A_{1_1}^{y_1} \leq G_1 \cap G_1^{y_1} = A_{1_1}$, which is a contradiction. Thus $d_c(G_1, G)$ is at least 4. □

Remark 4.19. We observe that $N_G(G_1) = G_1$, since in a Suzuki group there is no larger subgroup where a smaller Suzuki group would be a normal subgroup.

Proposition 4.20. *Let G_1, S_1 be as in Cor. 4.14.*

a) *The set $\{x \in G | G_1 > G_1 \cap G_1^x \geq Z(S_1)^u,\ u \in G\}$ has size at most $(s-1)(q^2 - s^2)(s^2+1)^2$*

b) *With the notations of Prop. 4.17, the size of the set $\{x \in G | G_1 \cap G_1^x = K_1^u,\ u \in G\}$ is at most $\frac{(q-s)s^4(s^2+1)^2}{2}$.*

c) *With the notations of Prop. 4.17, the sets $\{x \in G | G_1 \cap G_1^x = A_{1_1}^u,\ u \in G\}$ and $\{x \in G | G_1 \cap G_1^x = A_{2_1}^u,\ u \in G\}$ have sizes $\frac{s^4(s-1)^2}{4}(s-2r_1+1)^2(q-s+2(r-r_1))$ and $\frac{s^4(s-1)^2)}{4}(s+2r_1+1)^2(q-s-2(r-r_1))$, respectively.*

Proof. a) If the intersection $G_1 \cap {G_1}^x$ contains $Z(S_1)$ then there must be $S_2 \in Syl_2(G_1)$ such that $S_2^x = S_1$ and so $Z(S_2)^x = Z(S_1)$. By Sylow's theorem there exists an element $g_1 \in G_1$ such that $Z(S_1)^{g_1} = Z(S_2)$ and so $Z(S_1)^{g_1 x} = Z(S_1)$. Thus such an element x can be chosen from cosets of G_1 represented by some element of $N_G(Z(S_1))$. Two such cosets are different if the representing elements are in different cosets of $N_{G_1}(Z(S_1))$. Since $Z(S_1)^u$ must be a subgroup of G_1, the conjugating element can be chosen from G_1. Thus the set $\{x \in G | G_1 > G_1 \cap G_1 \geq Z(S_1)^u,\ u \in G\}$ has at most $(|N_G(Z(S_1)) : N_{G_1}(Z(S_1))| - 1)|G_1||G_1 : N_{G_1}(Z(S_1))|$ elements. We have seen in the proof of Prop. 4.16 that $|N_G(Z(S_1))| = (s-1)q^2, |N_{G_1}(Z(S_1))| = (s-1)s^2$, and the above product is $(\frac{(s-1)q^2}{(s-1)s^2} - 1)s^2(s^2+1)(s-1)(s^2+1)$. This is exactly what was stated.

b) By similar arguments as in part a), we have to calculate the number $|N_G(K_1) : N_{G_1}(K_1)| - 1)|G_1||G_1 : N_{G_1}(K_1)| = (\frac{2(q-1)}{2(s-1)} - 1)s^2(s^2+1)(s-1)s^2(s^2+1)/2$. This is exactly what was stated.

c) Similar to the proofs of parts a) and b).

□

Proposition 4.21. *We use notations of Prop 4.17. There exists an element $x \in G$ such that $G_1 \cap G_1^x = 1$.*

Proof. We have to show that the complement set of those elements $x \in G$ for which $G_1 \cap G_1^x \neq 1$ is nonempty. We use the results of the previous propositions. We note that $q = s^t$. We have to prove that $s^{2t}(s^t-1)(s^{2t}+1) > (s-1)(s^2+1)^2(s^{2t}-s^2) + s^4(s^2+1)^2(s^t-s)/2 + s^4(s-1)^2(s-2r_1+1)^2(s^t-s+2(r-r_1))/4 + s^4(s-1)^2(s+2r_1+1)^2(s^t-s-2(r-r_1))/4 + |G_1|$. Since $s \geq 8$, if $t = 3$ then the first part of the right hand side is bigger than than any other part on the right hand side. The left hand side is bigger than the first part of the right hand side multiplied by 5, so we are done. If $t \geq 5$ then the s^4 times of the first part of the right hand side is bigger than any other part on the right hand side, so it is enough to prove that $s^{2t}(s^t-1)(s^{2t}+1) > 5s^4(s-1)(s^2+1)^2(s^{2t}-s^2)$ which is obviously true. So we are done. □

5. The ordinary depth of subgroups of $Sz(q)$

As a consequece of the results of the previous section we have the following:

Corollary 5.1. *The ordinary depth of all subgroups of $G = Sz(q)$ mentioned in Theorem 4.1 is 3 except for $d(N_G(F), G)$ which is 5.*

Proof. Since G is simple, the depth of each nontrivial subgroup is at least 3. Since for subgroups B_0, B_1, B_2 and for proper Suzuki subgroups $Sz(s)$ there exist

conjugates, which intersect them trivially, these subgroups have ordinary depth 3 by Theorem 1.4. The subgroups in part b) of Theorem 4.1 have combinatorial depth 3, so they also have ordinary depth 3. Since F is TI, so by Theorem 1.4, the ordinary depth of each 2-subgroup (including those in part c)) is 3. The maximal subgroup $N_G(F)$ has ordinary depth at most 5 since $d(N_G(F), G) \leq d_c(N_G(F), G)$. If $d(N_G(F), G) \leq 4$ then by Theorem 1.3, $m(\chi) \leq 1$ for each $\chi \in \mathrm{Irr}(G)$. However $m(1_G) \geq 2$, since $1_{N_G(F)}$ is irreducible and if $\psi \in \mathrm{Irr}(N_G(F))$ containing F in its kernel, then $d(\psi, 1_{N_G(F)}) = 2$. Here we used [11, Thm 5.9, Thm 5.10 in Ch XI]. Thus $d(N_G(F), G) = 5$.
□

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Óbuda University, H-1034 Budapest, Bécsi út 96/B.
E-mail address: fobaba@t-online.hu

Department of Algebra, Budapest University of Technology and Economics, H-1521 Budapest, Műegyetem rkp. 3–9.
E-mail address: he@math.bme.hu

Department of Algebra, Budapest University of Technology and Economics, H-1521 Budapest, Műegyetem kp. 3–9.
E-mail address: petenyi.franciska@gmail.com