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COUNTDOWN GAMES, AND SIMULATION ON (SUCCINCT) ONE-COUNTER NETS

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Abstract. We answer an open complexity question by Hofman, Lasota, Mayr, Totzke (LMCS 2016) for simulation preorder on the class of succinct one-counter nets (*i.e.*, one-counter automata with no zero tests where counter increments and decrements are integers written in binary); the problem was known to be PSPACE-hard and in EXPSPACE. We show that all relations between bisimulation equivalence and simulation preorder are EXPSPACE-hard for these nets; simulation preorder is thus EXPSPACE-complete. The result is proven by a reduction from reachability games whose EXPSPACE-completeness in the case of succinct one-counter nets was shown by Hunter (RP 2015), by using other results. We also provide a direct self-contained EXPSPACE-completeness proof for a special case of such reachability games, namely for a modification of countdown games that were shown EXPTIME-complete by Jurdzinski, Sproston, Laroussinie (LMCS 2008); in our modification the initial counter value is not given but is freely chosen by the first player.

We also present an alternative proof for the upper bound by Hofman et al. In particular, we give a new simplified proof of the belt theorem that yields a simple graphic presentation of simulation preorder on (non-succinct) one-counter nets and leads to a polynomial-space algorithm (which is trivially extended to an exponential-space algorithm for succinct one-counter nets).

1. Introduction

One-counter automata (OCA), *i.e.*, finite automata equipped with a nonnegative counter, are studied as one of the simplest models of infinite-state systems. They can be viewed as a special case of Minsky counter machines, or as a special case of pushdown automata. In general, OCA can test the value of the counter for zero, *i.e.*, some transitions could be enabled only if the value of the counter is zero. One-counter nets (OCN) are a "monotonic"

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subclass of OCA where every transition enabled for zero is also enabled for nonzero values. As usual, we can consider deterministic, nondeterministic, and/or alternating versions of OCA and/or OCN. The basic versions are *unary*, where the counter can be incremented and decremented by one in one step, while in the *succinct* versions the possible changes can be arbitrary integers (but fixed for a given transition); as usual, the changes are assumed to be written in binary in a description of a given automaton. (Remark. In some papers the term "unary" is used differently, for machines with a single-letter input alphabet.)

Problems that have been studied on OCA and OCN include reachability, equivalence, model checking, and also different kinds of games played on these automata. One of the earliest results showed decidability of (language) equivalence for deterministic OCA [VP75]. The open polynomiality question in [VP75] was positively answered in [BGJ13], by showing that shortest distinguishing words of non-equivalent deterministic OCA have polynomial lengths. (We can refer, e.g., to [CCH+19] for precise bounds in some related cases.)

Later other behavioural equivalences (besides language equivalence) were studied. Most relevant for us is the research started by Abdulla and Čerāns who showed in [Av98] that simulation preorder on one-counter nets is decidable. An alternative proof of this fact was given in [JMS99]; it was also noted that simulation equivalence is undecidable for OCA. A relation to bisimulation problems was shown in [JKM00]. Kučera showed some lower bounds in [Kuč00]; Mayr [May03] showed the undecidability of weak bisimulation equivalence on OCN.

Simulation preorder on one-counter nets turned out PSPACE-complete: the lower bound was shown by Srba [Srb09], and the upper bound by Hofman, Lasota, Mayr, and Totzke [HLMT16]. It was also shown in [HLMT16] that deciding weak simulation on OCN can be reduced to deciding strong simulation on OCN, and thus also solved in polynomial space. (Strong) bisimulation equivalence on OCA is also known to be PSPACE-complete [BGJ14] (which also holds for a probabilistic version of OCA [FJKW18]). We note that PSPACE-membership of problems for the unary case easily yields EXPSPACE-membership for the succinct (binary) case.

Succinct (and parametric) OCA were considered, e.g., in [HKOW09], where reachability on succinct OCA was shown to be NP-complete. Games studied on OCA include, e.g., parity games on one-counter processes (with test for zero) [Ser06], and are closely related to counter reachability games (e.g. [Rei16]). Model checking problems on OCA were studied for many types of logics, *e.g.*, LTL [DLS08], branching time logics [GL10], or first-order logics [GMT09]. DP-lower bounds for some model-checking (and also equivalence checking) problems were shown in [JKMS04]. A recent study [ABHT20] deals with parameterized universality problems for one-counter nets.

An involved result by Göller, Haase, Ouaknine, Worrell [GHOW10] shows that model checking a fixed CTL formula on succinct one-counter automata is EXPSPACE-hard. The proof is nontrivial, using two involved results from complexity theory. The technique of this proof was referred to by Hunter [Hun15], to derive EXPSPACE-hardness of reachability games on succinct OCN.

Our contribution. In this paper we close a complexity gap for the simulation problem on succinct OCN that was mentioned in [HLMT16], noting that there was a PSPACE lower bound and an EXPSPACE upper bound for the problem. We show EXPSPACE-hardness (and thus EXPSPACE-completeness) of the problem, using a defender-choice technique (cf., e.g., [JS08]) to reduce reachability games to any relation between simulation preorder and bisimulation equivalence. Further contributions are explained below in more detail.

As already mentioned, the EXPSPACE-hardness of reachability games on succinct OCNs was shown in [Hun15] by using [GHOW10]. Here we present a direct proof of EXPSPACE-hardness (and completeness) even for a special case of reachability games, which we call the "existential countdown games". It is a mild relaxation of the countdown games from [JSL08] (or their variant from [Kie13]), which is an interesting EXPTIME-complete problem. We thus provide a simple EXPSPACE-hardness proof (in fact, by a master reduction via a natural intermediate problem dealing with ultimately periodic words) that is independent of [Hun15] (and of the involved technique from [GHOW10] used by [Hun15]).

We now give an informal sketch of the results for countdown games. The left-hand part

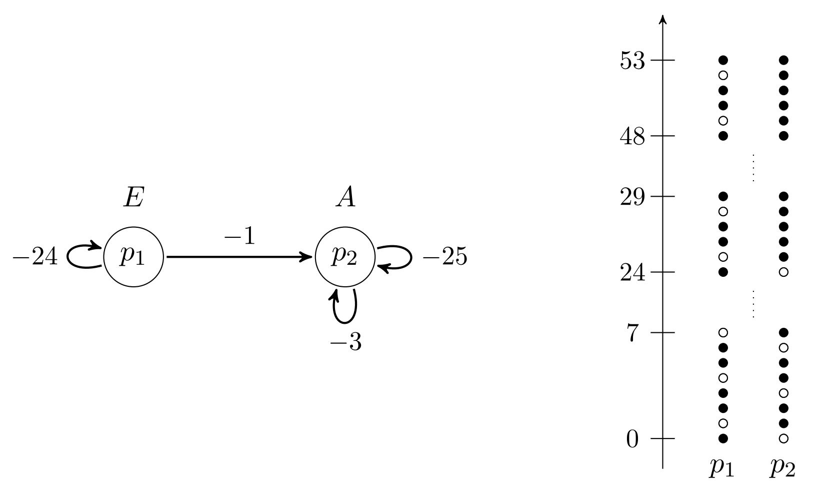


FIGURE 1. A simple countdown game (with $p_{win} = p_2$), and its solution.

of Figure 1 shows an example of a very simple countdown game. It is, in fact, a special finite automaton, with Eve's states, in our case just p_1, and Adam's states, in our case just p_2. The game assumes a nonnegative counter whose value is modified by transitions; in the *countdown* games the counter is only *decreased* by transitions. E.g., in the configuration $p_1(185)$, i.e. in the situation where the current state is p_1 and the current counter value is 185, Eve can choose the transition $p_1 \xrightarrow{-24} p_1$, which changes the current configuration to $p_1(185 - 24)$, i.e. to $p_1(161)$, or the transition $p_1 \xrightarrow{-1} p_2$, which changes the current configuration to $p_2(184)$. In $p_2(184)$ Adam has two choices, either going to $p_2(159)$ or to $p_2(181)$. Since the counter cannot become negative, in $p_2(17)$ Adam has one choice only, necessarily reaching $p_2(2)$ by several steps, where the respective play finishes. Eve's goal is that the play finishes in $p_{win}(0)$ for a distinguished state p_{win}; in our example we put $p_{win} = p_2$.

It is obvious that an initial configuration $p_2(n)$ is winning for Eve iff $n \in \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$ as partly depicted in the right-hand part of Figure 1 by the white points. (The black points thus correspond to the configurations where Adam has a winning strategy; e.g., $p_2(7)$ is winning for Adam.) An initial configurations $p_1(n)$ is winning for Eve (i.e., Eve has a strategy guaranteeing reaching $p_2(0)$), iff $n \bmod 3 = 1$.

Our concrete example is simple, but also in the general case, with states $p_1, p_2, \ldots, p_k$ (distributed between Eve and Adam), it is straightforward to stepwise fill the respective "black-white-points table" in the bottom-up fashion that we now describe. The "points" $p_1(0), p_2(0), \ldots, p_k(0)$ are black except of $p_i(0)$ where $p_i = p_{win}$ since $p_{win}(0)$ is white. If we have filled the rows $0, 1, \ldots, \ell$ (hence each $p_i(j)$ where $i \in \{1, 2, \ldots, k\}$ and $j \in \{0, 1, \ldots, \ell\}$ has been determined to be black or white), it is trivial to fill the row $\ell{+}1$: $p_i(\ell{+}1)$ becomes white iff p_i belongs to Eve and there is a move from $p_i(\ell{+}1)$ to an already established white point (in the rows $0, 1, \ldots, \ell$), or p_i belongs to Adam, there is a move from $p_i(\ell{+}1)$, and each such move leads to an already established white point.

It is thus clear that the question if $p(n)$ is Eve's win is in EXPTIME: we fill the "rows" for $0, 1, 2, \ldots, n$, using exponential time and space in the input size, since n and the countdown values are given in binary or in decimal notation. We also note that if m is the maximum countdown value (which is 25 in Figure 1), the row $\ell \geq m$ is determined by the rows $\ell{-}1, \ell{-}2, \ldots, \ell{-}m$. It is thus clear that deciding the "existential version" of the countdown games, i.e. the question, given a state p, if there is n such that $p(n)$ is winning for Eve, can be decided in exponential space. (When filling the table in the bottom-up fashion, we can always keep just last m rows in memory.) We can also observe that the black-white table is thus (ultimately) periodic, with an at most double-exponential period; the period is indeed double-exponential in concrete cases, as we also show in this paper.

The lower bounds might look more surprising: deciding if $p(n)$ is Eve's win is EXPTIME-complete [JSL08], while deciding if there is n such that $p(n)$ is Eve's win is EXPSPACE-complete, as we show in this paper. In fact, these lower bounds (EXPTIME-hardness and EXPSPACE-hardness) can be also relatively easily established, when looking at the "bottom-up" computation-table of an exponential-space Turing machine in a convenient way, as is depicted in and discussed around Figure 5.

Regarding the simulation problem, we give an example of a succinct one-counter net in Figure 2; it is also a finite automaton equipped with a nonnegative counter. Now there is no Eve or Adam, and the counter changes associated with transitions can be also nonnegative; moreover, the transitions have action-labels (a, b in our example). On the set

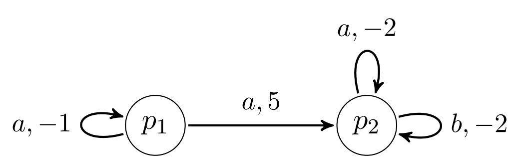


Figure 2. An example of a succinct one-counter net.

of all configurations $p(n)$, the *simulation preorder* $\preceq$ is the maximal relation such that for each pair $p(m) \preceq q(n)$ and each move $p(m) \xrightarrow{a} p'(m')$ there is a move $q(n) \xrightarrow{a} q'(n')$ (with the same label a) such that $p'(m') \preceq q'(n')$. In our example we can note that $p_2(3) \not\preceq p_1(58)$, since the move $p_2(3) \xrightarrow{b} p_2(1)$ cannot be answered by any b-transition from $p_1(58)$; on the other hand, we have $p_2(1) \preceq p_1(58)$, since no transition is enabled in $p_2(1)$. Slightly more subtle is to note that $p_1(0) \preceq p_2(6)$: the move $p_1(0) \xrightarrow{a} p_2(5)$ is answered by $p_2(6) \xrightarrow{a} p_2(4)$, and we check that $p_2(5) \preceq p_2(4)$.

We can also think in terms of games here. In the simulation game a position is not just one configuration, but a pair $(p(m), q(n))$ of configurations. The first player, called Attacker (or Spoiler), chooses a move $p(m) \xrightarrow{a} p'(m')$ (Attacker loses if there is no such move), and the other player, called Defender (or Duplicator), answers by some $q(n) \xrightarrow{a} q'(n')$ (with the same label a); Defender loses if there is no such answer. The play then continues with the next round, with the current pair $(p'(m'), q'(n'))$. An infinite play is deemed to be winning for Defender. It is standard to observe that $p(m) \not\preceq q(n)$ iff Attacker has a winning strategy from $(p(m), q(n))$ (and $p(m) \preceq q(n)$ iff Defender has a winning strategy from $(p(m), q(n))$).

Given a succinct one-counter net, we can represent the relation $\preceq$ by the respective "black-white colourings" $C_{\langle p,q\rangle}$ of the integer points in the first quadrant of the plane, for each ordered pair of states $\langle p, q\rangle$; white points correspond to Attacker's wins, black points correspond to Defender's wins. In our example we have four ordered pairs of states $((p_1, p_1),(p_1, p_2),(p_2, p_1),(p_2, p_2))$, and the respective four colourings are depicted in Figure 3. E.g., $p_2(1) \preceq p_1(n)$ and $p_2(2) \not\preceq p_1(n)$ for all n, $p_1(0) \not\preceq p_2(5)$ and $p_1(0) \preceq p_2(6)$, etc.

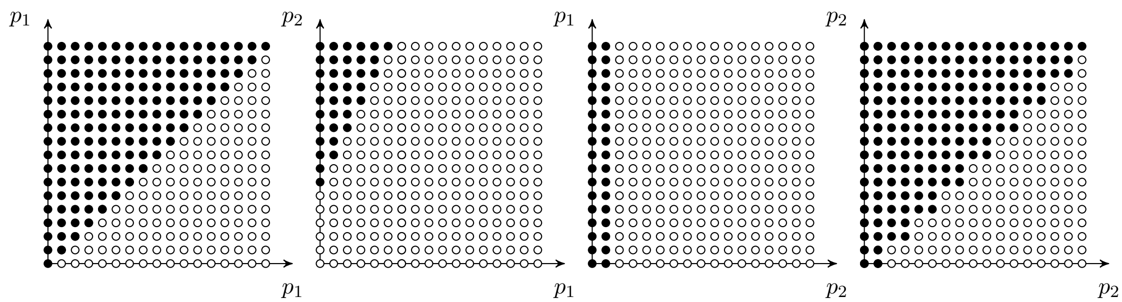


FIGURE 3. A depiction of the simulation preorder $\preceq$ related to the net from Figure 2.

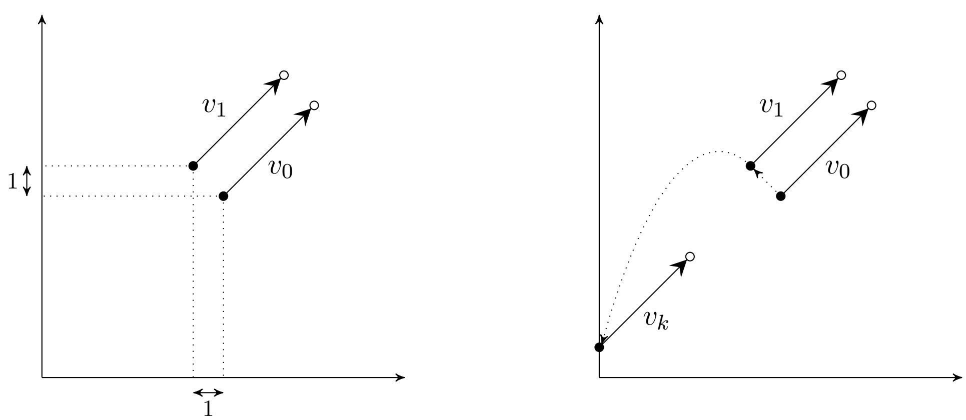
We can easily observe the general monotonicity: if $p(m) \preceq q(n)$ then $p(m') \preceq q(n')$ for all $m' \leq m$ and $n' \geq n$. Our example also suggests that the black-white frontier in each colouring is contained in a "linear belt", with a rational (or infinite) slope and a certain width. Such a belt (two parallel lines) in $C_{\langle p_1,p_2\rangle}$ in Figure 3 can be described as follows: the frontier points, which we can define as the rightmost black points in each row, have coordinates $(0, 6), (0, 7), (1, 8), (1, 9), (2, 10), (2, 11), \cdots$ and are contained in a belt with the slope $\frac{2}{1}$ and the vertical width 1. (A more general form is depicted in Figure 10.)

Contemplating a bit, it is not difficult to get an intuition captured by a "belt theorem", claiming that also in any general case of (succinct) one-counter nets the frontier in each plane is contained in a linear belt; moreover, it is intuitively obvious that each frontier is periodic, from some row onwards. (In this paper, we also show that the periods can be, and are at most, double-exponential.) By the previous figures and discussions, one can also easily get an intuition that deciding the countdown games, even in their (EXPSPACE-complete) existential form, could be reduced to deciding the simulation preorder on (succinct) one-counter nets; finding a solution like the one depicted in Figure 1 seems intuitively simpler than finding a solution like the one depicted in Figure 3. This is confirmed in this paper, by a reduction based on a "defender-choice technique" that in particular enables to mimic Adam's choices in a countdown game by Defender's choices in the corresponding simulation game.

The above mentioned belt theorem is important for the upper bounds, namely for showing that deciding simulation preorder is in PSPACE for one-counter nets where the counter changes are presented in unary, and in EXPSPACE for succinct one-counter nets.

Proving the belt theorem had turned out surprisingly difficult; we also contribute to this topic here, as we explain below. The (non)succinctness of the one-counter nets plays no important role in this discussion, so we restrict ourselves to the cases in which the counter can change by at most 1 in one move. The *qualitative form* of the belt theorem (claiming just the existence of belts, not caring about their slopes, widths, and positions) followed from [Av98] where an involved mechanism of two-player games was used; this qualitative result was shown in [JMS99] by another technique, based rather on "geometric" ideas. The *quantitative form*, stating that the linear belts, assumed to start in the origin $(0, 0)$, have the slopes and widths that can be presented as (fractions of) numbers with polynomially bounded values, was shown in [HLMT16], by enhancing the technique of games from [Av98]; this is the crux of the PSPACE-membership of simulation preorder for OCN [HLMT16] (which also yields the EXPSPACE-membership for succinct OCN).

In this paper we give a new self-contained proof for both the qualitative and quantitative versions of the belt theorem. One important new ingredient is a simple observation that we call a *black-white vector travel*. We describe it here, using Figure 4, but the description can be safely skipped if the reader finds it too technical; it is later captured by Proposition 5.6 and Corollary 5.16. For any vector v_0 with a positive slope and with a black start and



(A) A neighbour smaller-rank black-white vector. (B) A vector travel.

FIGURE 4. Black-white vector travel.

a white end in some colouring $C_{\langle p_0,q_0\rangle}$, like the vector v_0 in Figure 4, there are vectors $v_1, v_2, \dots, v_k$ of the same size and slope as v_0, each v_i being a black-white vector in some colouring $C_{\langle p_i,q_i\rangle}$, such that

- for all $i \in \{1, 2, \dots, k\}$, v_i is a neighbour of v_{i-1}, i.e., the x-coordinates of the starts of v_i and v_{i-1} differ by at most 1, and the same constraint holds for their y-coordinates; and
- the start of v_k is on the vertical axis.

This can be easily verified: in the pair of configurations corresponding to the white end of v_0 Attacker has an optimal transition $p_0 \xrightarrow{a,z} p_1$, for which each Defender's response $q_0 \xrightarrow{a,z'} q$ establishes a "white" pair with a smaller rank, i.e., closer to Attacker's final win. Attacker can perform the transition $p_0 \xrightarrow{a,z} p_1$ also in the pair of configurations corresponding to the black start of v_0, if this start is not on the vertical axis (which would entail that Attacker's counter is zero); there is at least one Defender's response $q_0 \xrightarrow{a,z_1} q_1$ establishing another "black" pair. Hence by changing the x-coordinate of v_0 by z and its y-coordinate by z_1 we get the vector v_1 that is black-white in the colouring $C_{\langle p_1, q_1\rangle}$. Etc.

This black-white vector travel, together with other enhancements, allowed us to substantially simplify the proof of the qualitative belt theorem from [JMS99]. Moreover, the quantitative version can be now derived from the qualitative version by a few simple observations.

Organization of the paper. Section 2 gives the basic definitions. In Section 3 we show that the "existential" countdown games are EXPSPACE-complete (which also yields an alternative proof for the known EXPTIME-completeness of countdown games). Section 4 describes the reductions from reachability games to (bi)simulation relations, in a general framework and then in the framework of succinct OCN. Section 5 contains new proofs for both the qualitative and quantitative versions of the belt theorem (leading to the PSPACE membership for OCN). Section 6 shows that the period of the belts in the succinct case can be double-exponential. We finish with some additional remarks in Section 7.

2. Basic Definitions

By $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_{>0}$ we denote the sets of integers, of nonnegative integers, and of positive integers, respectively. We use $[i, j]$, where $i, j \in \mathbb{Z}$, for denoting the set $\{i, i{+}1, \ldots, j\}$.

Labelled transition systems and (bi)simulations. A *labelled transition system*, an *LTS* for short, is a tuple

$$\mathcal{L} = (S, Act, (\xrightarrow{a})_{a\in Act})$$

where S is the set of *states*, Act is the set of *actions*, and $\xrightarrow{a} \subseteq S \times S$ is the set of *a-transitions* (transitions labelled with a), for each $a \in Act$. We write $s \xrightarrow{a} t$ instead of $(s, t) \in \xrightarrow{a}$. By $s \xrightarrow{a}$ we denote that a is *enabled in* s, *i.e.*, $s \xrightarrow{a} t$ for some t.

Given $\mathcal{L} = (S, Act, (\xrightarrow{a})_{a\in Act})$, a relation $R \subseteq S \times S$ is a *simulation* if for every $(s, s') \in R$ and every $s \xrightarrow{a} t$ there is $s' \xrightarrow{a} t'$ such that $(t, t') \in R$; if, moreover, for every $(s, s') \in R$ and every $s' \xrightarrow{a} t'$ there is $s \xrightarrow{a} t$ such that $(t, t') \in R$, then R is a *bisimulation*.

The union of all simulations (on S) is the maximal simulation, denoted $\preceq$; it is a preorder, called *simulation preorder*. The union of all bisimulations is the maximal bisimulation, denoted $\sim$; it is an equivalence, called *bisimulation equivalence* (or *bisimilarity*). We obviously have $\sim \subseteq \preceq$.

We can write $s_1 \preceq s_2$ or $s_1 \sim s_2$ also for states s_1, s_2 from different LTSs $\mathcal{L}_1$, $\mathcal{L}_2$, in which case the LTS arising by the disjoint union of $\mathcal{L}_1$ and $\mathcal{L}_2$ is (implicitly) referred to.

It is useful to think in terms of two-player turn-based games, played by Attacker and Defender (or Spoiler and Duplicator). A round of the *simulation game* from a (current) pair (s, s') proceeds as follows: Attacker chooses a transition $s \xrightarrow{a} t$, and Defender responds with some $s' \xrightarrow{a} t'$ (for the action a chosen by Attacker); the play then continues with another round, now from the current pair (t, t'). If a player has no legal move in a round, then the other player wins; infinite plays are deemed to be Defender's wins. It is standard that $s \not\preceq s'$ iff Attacker has a winning strategy from (s, s').

The *bisimulation game* is analogous, but in any round starting from (s, s') Attacker can choose to play $s \xrightarrow{a} t$ or $s' \xrightarrow{a} t'$, and Defender has to respond with some $s' \xrightarrow{a} t'$ or $s \xrightarrow{a} t$, respectively. Here we have $s \not\sim s'$ iff Attacker has a winning strategy from (s, s').

Stratified simulation, and ranks of pairs of states. Given $\mathcal{L} = (S, Act, (\xrightarrow{a})_{a\in Act})$, we use (transfinite) induction to define the relations $\preceq_\lambda$ where λ ranges over the class *Ord* of ordinals. We put $\preceq_0 = S \times S$. For $\lambda > 0$ we have $s \preceq_\lambda s'$ if for each transition $s \xrightarrow{a} t$ and each $\lambda' < \lambda$ there is a transition $s' \xrightarrow{a} t'$ where $t \preceq_{\lambda'} t'$.

We note that $\preceq_{\lambda'} \supseteq \preceq_\lambda$ when $\lambda' \leq \lambda$, and that $\preceq = \bigcap_{\lambda\in Ord} \preceq_\lambda$. For each pair $(s, s') \notin \preceq$, we define its *rank* $\text{RANK}(s, s')$ as the least ordinal λ such that $s \not\preceq_\lambda s'$. We note in particular that $\text{RANK}(s, s') = 1$ iff s enables an action a (i.e., $s \xrightarrow{a}$) that is not enabled in s' (i.e., $s' \not\xrightarrow{a}$).

Remark 2.1. We use such a general definition for the purpose of the general reduction presented in Section 4. Otherwise we consider just (special cases of) LTSs that are *image-finite* (i.e., in which the sets $\{t \mid s \xrightarrow{a} t\}$ are finite for all $s \in S$, $a \in Act$); in such systems we have $\preceq = \bigcap_{i\in\mathbb{N}} \preceq_i$ and $\text{RANK}(s, s') \in \mathbb{N}$ for each $(s, s') \notin \preceq$.
We could define the analogous concepts for bisimulation equivalence as well.

One-counter nets (OCNs and SOCNs), and their associated LTSs. A *labelled one-counter net*, or just a *one-counter net* or even just an *OCN* for short, is a triple

$$\mathcal{N} = (Q, Act, \delta),$$

where Q is the finite set of *control states*, *Act* the finite set of *actions*, and $\delta \subseteq Q \times Act \times \{-1, 0, +1\} \times Q$ is the set of (*labelled transition*) *rules*. By allowing δ to be a finite subset of $Q \times Act \times \mathbb{Z} \times Q$, and presenting $z \in \mathbb{Z}$ in the rules (q, a, z, q') in binary, we get a *succinct one-counter net*, or a *SOCN* for short. A rule (q, a, z, q') is usually presented as $q \xrightarrow{a,z} q'$.

Each OCN or SOCN $\mathcal{N} = (Q, Act, \delta)$ has the *associated LTS*

$$\mathcal{L}_{\mathcal{N}} = (Q \times \mathbb{N}, Act, (\xrightarrow{a})_{a\in Act}) \tag{2.1}$$

where $(q, m) \xrightarrow{a} (q', n)$ iff $q \xrightarrow{a,n-m} q'$ is a rule in δ. We often write a state (q, m), which is also called a *configuration*, in the form $q(m)$, and we view m as a value of a nonnegative counter. A rule $q \xrightarrow{a,z} q'$ thus induces transitions $q(m) \xrightarrow{a} q'(m+z)$ for all $m \geq \max\{0, -z\}$.

We remark that *one-counter automata* extend one-counter nets by the ability to test zero, i.e., by transitions that are enabled only if the counter value is zero.

Reachability games (r-games), winning areas, ranks of states. We are interested in reachability games played in LTSs associated with (succinct) one-counter nets, but we first define the respective notions generally.

By a *reachability game*, or an *r-game* for short, we mean a tuple

$$\mathcal{G} = (V, V_\exists, \rightarrow, \mathcal{T}),$$

where V is the set of *states* (or *vertices*), $V_\exists \subseteq V$ is the set of *Eve's states*, $\rightarrow \subseteq V \times V$ is the *transition relation* (or the set of *transitions*), and $\mathcal{T} \subseteq V$ is the set of *target states*. By *Adam's states* we mean the elements of $V_\forall = V \smallsetminus V_\exists$.

Eve's winning area is $Win_\exists = \bigcup_{\lambda\in Ord} W_\lambda$, for *Ord* being the class of ordinals, where the sets $W_\lambda \subseteq V$ are defined inductively as follows.

We put $W_0 = \mathcal{T}$; for $\lambda > 0$ we put $W_{<\lambda} = \bigcup_{\lambda'<\lambda} W_{\lambda'}$, and we stipulate:

a) if $s \notin W_{<\lambda}$, $s \in V_\exists$, and $s \rightarrow \bar{s}$ for some $\bar{s} \in W_{<\lambda}$, then $s \in W_\lambda$;
b) if $s \notin W_{<\lambda}$, $s \in V_\forall$, and we have $\emptyset \neq \{\bar{s} \mid s \rightarrow \bar{s}\} \subseteq W_{<\lambda}$, then $s \in W_\lambda$.

(If (a) applies, then λ is surely a successor ordinal.)

For each $s \in Win_\exists$, by $\textsc{rank}(s)$ we denote (the unique) λ such that $s \in W_\lambda$. A transition $s \to \bar{s}$ is *rank-reducing* if $\textsc{rank}(s) > \textsc{rank}(\bar{s})$. We note that for any $s \in Win_\exists$ with $\textsc{rank}(s) > 0$ we have: if $s \in V_\exists$, then there is at least one rank-reducing transition $s \to \bar{s}$ (in fact, $\textsc{rank}(s) = \textsc{rank}(\bar{s})+1$ in this case); if $s \in V_\forall$, then there is at least one transition $s \to \bar{s}$ and all such transitions are rank-reducing. This entails that $Win_\exists$ is the set of states from which Eve has a strategy that guarantees reaching (some state in) $\mathcal{T}$ when Eve is choosing a next transition in Eve's states and Adam is choosing a next transition in Adam's states.

Remark 2.2. We are primarily interested in the games that have (at most) countably many states and are finitely branching (the sets $\{\bar{s} \mid s \to \bar{s}\}$ are finite for all s). In such cases we have $\textsc{rank}(s) \in \mathbb{N}$ for each $s \in Win_\exists$. We have again introduced the general definition for the purpose of the reduction in Section 4.

Reachability games on succinct one-counter nets. We now define specific r-games, presented by SOCNs with partitioned control-state sets; these succinct one-counter nets are *unlabelled*, which means that the set of actions can be always deemed to be a singleton.

By a *succinct one-counter net reachability game*, a *socn-r-game* for short, we mean a tuple

$$\mathcal{N} = (Q, Q_\exists, \delta, p_{win})$$

where Q is the finite set of *(control) states*, $Q_\exists \subseteq Q$ is the set of *Eve's (control) states*, $p_{win} \in Q$ is the *target (control) state*, and $\delta \subseteq Q \times \mathbb{Z} \times Q$ is the finite set of *(transition) rules*. We often present a rule $(q, z, q') \in \delta$ as $q \xrightarrow{z} q'$. By *Adam's (control) states* we mean the elements of $Q_\forall = Q \smallsetminus Q_\exists$. A socn-r-game $\mathcal{N} = (Q, Q_\exists, \delta, p_{win})$ has the *associated r-game*

$$\mathcal{G}_\mathcal{N} = (Q \times \mathbb{N}, Q_\exists \times \mathbb{N}, \to, \{(p_{win}, 0)\}) \tag{2.2}$$

where $(q, m) \to (q', n)$ iff $q \xrightarrow{n-m} q'$ is a rule (in δ). We often write $q(m)$ instead of (q, m) for states of $\mathcal{G}_\mathcal{N}$.

We define the problem Socn-Rg (to decide succinct one-counter net r-games) as follows:

Name: Socn-Rg
Instance: a socn-r-game $\mathcal{N}$ (with integers z in rules $q \xrightarrow{z} q'$ written in binary), and a control state p_0.
Question: is $p_0(0) \in Win_\exists$ in the game $\mathcal{G}_\mathcal{N}$?

Remark 2.3. We have defined the target states (in $\mathcal{G}_\mathcal{N}$) to be the singleton set $\{p_{win}(0)\}$. There are other natural variants (e.g., one in [Hun15] defines the target set $\{p(0) \mid p \neq p_0\}$) that can be easily shown to be essentially equivalent.

The EXPSPACE-hardness of Socn-Rg was announced in [Hun15], where an idea of a proof is sketched, also using a reference to an involved result [GHOW10] (which is further discussed in Section 7). In Section 3 we give a direct self-contained proof that does not rely on [Hun15] or involved techniques from [GHOW10], and that even shows that Socn-Rg is EXPSPACE-hard already in the special case that slightly generalizes the countdown games from [JSL08]. (The EXPSPACE-membership follows from [Hun15], but we add a short proof to be self-contained.)

Countdown games. We define a *countdown game* as a socn-r-game $\mathcal{N} = (Q, Q_\exists, \delta, p_{win})$, where in every rule $q \xrightarrow{z} q'$ in δ we have $z < 0$. The problem Cg is defined as follows:

Name:	Cg (countdown games)
Instance:	a countdown game $\mathcal{N}$ (with integers in rules written in binary), and an initial configuration $p_0(n_0)$ where $n_0 \in \mathbb{N}$ (n_0 in binary).
Question:	is $p_0(n_0) \in \mathit{Win}_\exists$?

The problem Cg (in an equivalent form) was shown EXPTIME-complete in [JSL08]. Here we define an existential version, *i.e.*the problem Ecg:

Name:	Ecg (existential countdown games)
Instance:	a countdown game $\mathcal{N}$ and a control state p_0.
Question:	is there some $n \in \mathbb{N}$ such that $p_0(n) \in \mathit{Win}_\exists$?

We note that Ecg can be viewed as a subproblem of Socn-Rg: given an instance of Ecg, it suffices to add a fresh Eve's state p_0' and rules $p_0' \xrightarrow{1} p_0'$, $p_0' \xrightarrow{0} p_0$; the question then is if $p_0'(0) \in \mathit{Win}_\exists$.

3. EXPSPACE-Completeness of Existential Countdown Games

In this section we prove the following new theorem.

Theorem 3.1. Ecg *(existential countdown game) is EXPSPACE-complete.*

Our EXPSPACE-hardness proof of ECG is, in fact, a particular instance of a simple general method. A slight modification also yields a proof of EXPTIME-hardness of countdown games that is an alternative to the proof in [JSL08] and in [Kie13].

Theorem 3.2. [JSL08] Cg *is EXPTIME-complete.*

In the rest of this section we present the proofs of Theorems 3.1 and 3.2 together, since they only differ by a small detail. We start with the upper bounds since these are obvious.

3.1. ECG is in EXPSPACE (and CG in EXPTIME).

Let us consider an Ecg-instance $\mathcal{N} = (Q, Q_\exists, \delta, p_{win})$, p_0. By $\textsc{m}$ we denote the maximum value by which the counter can be decremented in one step (i.e., $\textsc{m} = \max\{\,|z|\,;\text{ there is some } q \xrightarrow{z} q' \text{ in } \delta\,\}$); the value $\textsc{m}$ is at most exponential in the size of the instance $\mathcal{N}, p_0$.

We can stepwise construct $W(0), W(1), W(2), \ldots$ where $W(j) = \{q \in Q \mid q(j) \in \mathit{Win}_\exists\}$; the Ecg-instance $\mathcal{N}, p_0$ is positive iff $p_0 \in W(j)$ for some j. We have $W(0) = \{p_{win}\}$, and for determining $W(n)$ ($n \geq 1$) it suffices to know the segment $W(n{-}\textsc{m}')$, $W(n{-}\textsc{m}'{+}1)$, $\ldots, W(n{-}1)$ where $\textsc{m}' = \min\{\textsc{m}, n\}$. Hence, during the construction of $W(j)$, $j = 0, 1, 2, \ldots$, it suffices to remember just the segment $W(j{-}\textsc{m}')$, $W(j{-}\textsc{m}'{+}1)$, $\ldots, W(j{-}1)$ (where $\textsc{m}' \leq \textsc{m}$). Obviously, if for some j, j', where $\textsc{m} \leq j < j'$, the segments $W(j{-}\textsc{m}), \ldots, W(j{-}1)$ and $W(j'{-}\textsc{m}), \ldots, W(j'{-}1)$ are the same (*i.e.*, if $W(j{-}k) = W(j'{-}k)$ for each $k = 1, 2, \ldots, \textsc{m}$), then also $W(j{+}k) = W(j'{+}k)$ for each $k \in \mathbb{N}$, so the sequence repeats itself with a period $j' - j$. By the pigeonhole principle, this surely happens for some j, j' such that $j < j' \leq \textsc{m} + 2^{|Q| \cdot \textsc{m}}$ because there are at most $2^{|Q| \cdot \textsc{m}}$ segments of length $\textsc{m}$. This means that in the algorithm that checks whether $p_0 \in W(j)$ for some $j \in \mathbb{N}$, the computation can be stopped after constructing $W(\textsc{m} + 2^{|Q| \cdot \textsc{m}})$. The size of a binary counter serving to count till the (at most double-exponential) value $\mathrm{M} + 2^{|Q|\cdot \mathrm{M}}$ is at most exponential. Therefore ECG belongs to EXPSPACE.

It is also clear that CG is in EXPTIME, since for the instance $\mathcal{N}, p_0(n_0)$ we can simply construct $W(0), W(1), W(2), \ldots, W(n_0)$.

3.2. **ECG is EXPSPACE-hard (and CG EXPTIME-hard).** In principle, we use a "master" reduction. We fix an arbitrary language L in EXPSPACE, in an alphabet Σ (hence $L \subseteq \Sigma^*$), decided by a (deterministic) Turing machine $\mathcal{M}$ in space $2^{p(n)}$ for a fixed polynomial p. For any word $w \in \Sigma^*$, $|w| = n$, there is the respective computation of $\mathcal{M}$ using at most $m = 2^{p(n)}$ tape cells, which is accepting iff $w \in L$. Our aim is to show a construction of a countdown game $\mathcal{N}^{\mathcal{M}}_{w,m}$, with a specified control state p_0, such that there is $k \in \mathbb{N}$ for which $p_0(k) \in \mathit{Win}_\exists$ if, and only if, $\mathcal{M}$ accepts w. The construction of $\mathcal{N}^{\mathcal{M}}_{w,m}$ will be polynomial, in the size $n = |w|$; this will establish that ECG is EXPSPACE-hard. (In fact, this polynomial construction easily yields a logspace-reduction, but this detail is unimportant at our level of discussion.)

In fact, the same construction of $\mathcal{N}^{\mathcal{M}}_{w,m}$ will also show EXPTIME-hardness of CG. In this case we assume that L is decided by a Turing machine $\mathcal{M}$ in time (and thus also space) $2^{p(n)}$, and we construct a concrete exponential value n_0 guaranteeing that $p_0(n_0) \in \mathit{Win}_\exists$ iff $\mathcal{M}$ accepts w.

Construction informally. The construction of the countdown game $\mathcal{N}^{\mathcal{M}}_{w,m}$ elaborates an idea that is already present in [CKS81] (in Theorem 3.4) and that was also used, e.g., in [JS07]. We first present the construction informally.

Figure 5 presents an accepting computation of $\mathcal{M}$, on a word $w = a_1 a_2 \ldots a_n$; it starts in the initial control state q_0 with the head scanning a_1. The computation is a sequence of configurations $C^w_0, C^w_1, \ldots, C^w_t$, where C^w_t is accepting (since the control state is q_{acc}). We assume that $\mathcal{M}$ never leaves cells $0 \ldots m{-}1$ of its tape during the computation. Hence each C^w_i can be presented as a word of length m over the alphabet $\Delta = (Q \times \Gamma) \cup \Gamma$ where Q and Γ are the set of control states and the tape alphabet of $\mathcal{M}$, respectively; by $\Box \in \Gamma$ we denote the special blank tape symbol. We refer to the (bottom-up) presentation of $C^w_0, C^w_1, \ldots, C^w_t$ depicted in Figure 5 as to a *computation table*.

Given m, each number $k \in \mathbb{N}$ determines the cell j in the "row" i (i.e., in the potential C^w_i) where $i = k \div m$ ($\div$ being integer division) and $j = k \bmod m$; we refer by $\textsc{cell}(k)$ to this cell j in the row i.

For $k > m$, if $\textsc{cell}(k)$ is in the computation table ($k \leq t \cdot m$ in Figure 5), then the symbol β in $\textsc{cell}(k)$ in the table is surely determined by the symbols $\beta_1, \beta_2, \beta_3$ in the cells $\textsc{cell}(k{-}m{-}1)$, $\textsc{cell}(k{-}m)$, $\textsc{cell}(k{-}m{+}1)$ (and by the transition function of the respective Turing machine $\mathcal{M}$); see Figure 5 for illustration (where also the cases β' and β'' on the "border" are depicted). The transition function of $\mathcal{M}$ allows us to define which *triples* $(\beta_1, \beta_2, \beta_3)$ are *eligible for* β, i.e., those that can be in the cells $\textsc{cell}(k{-}m{-}1)$, $\textsc{cell}(k{-}m)$, $\textsc{cell}(k{-}m{+}1)$ when β is in $\textsc{cell}(k)$ (these triples are independent of k, assuming $k > m$).

Let us now imagine a game between Eve and Adam where Eve, given w, claims that w is accepted by $\mathcal{M}$, in space m (in our case $m = 2^{p(|w|)}$ for a respective polynomial p). Eve does not present a respective accepting computation table but she starts a play by producing a tape-symbol x and a number $k_0 \in \mathbb{N}$, i.e. sets a counter to k_0, claiming that $\textsc{cell}(k_0)$ in the computation table contains (q_{acc}, x) (in Figure 5 the correct values are $k_0 = t \cdot m$

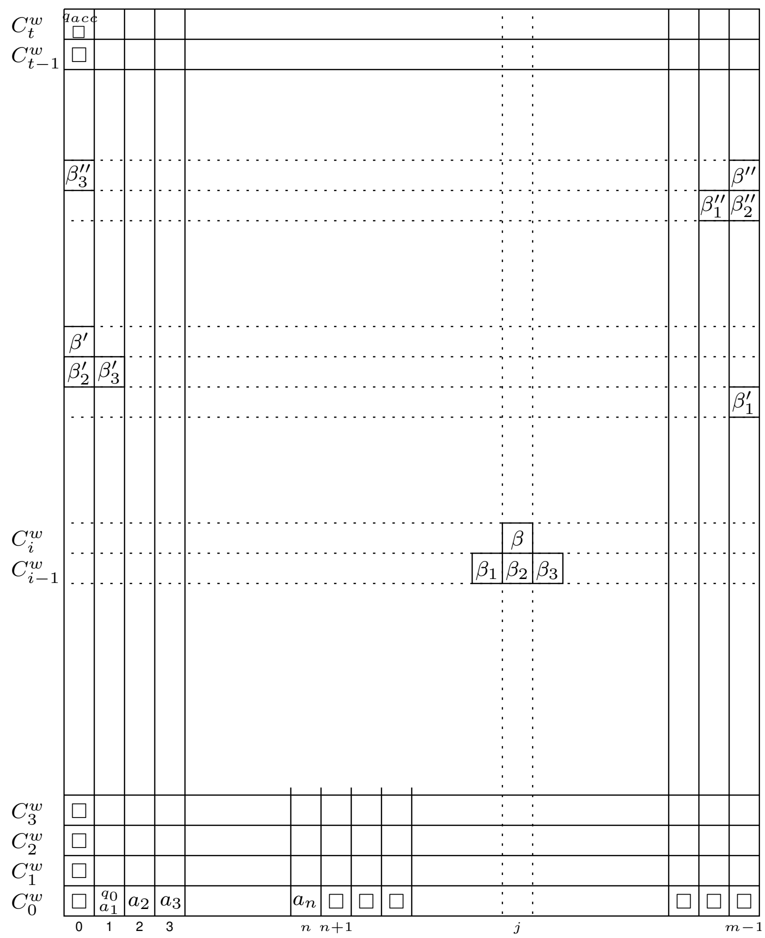


FIGURE 5. A computation table of $\mathcal{M}$ on a word $w = a_1 a_2 \dots a_n$ (in the bottom-up fashion).

and $x = \square$). Then the play proceeds as follows. If Eve claims that β is the symbol in $\textsc{cell}(k)$ for the current counter value k, while also claiming that $k > m$, then she decreases the counter by $m{-}2$ and produces a triple $(\beta_1, \beta_2, \beta_3)$ that is eligible for β, thus claiming that $\textsc{cell}(k{-}m{-}1), \textsc{cell}(k{-}m), \textsc{cell}(k{-}m{+}1)$ contain $\beta_1, \beta_2, \beta_3$, respectively. Adam then decreases the counter either by 3, asking to verify the claim that $\textsc{cell}(k{-}m{-}1)$ contains β_1, or by 2, asking to verify that $\textsc{cell}(k{-}m)$ contains β_2, or by 1, asking to verify that $\textsc{cell}(k{-}m{+}1)$ contains β_3. Eve then again produces an eligible triple for the current symbol, etc., until claiming that the counter value is $k \leq m$ (which can be contradicted by Adam when he is able to decrease the counter by $m{+}1$). The last phase just implements checking if the symbol claimed for $\textsc{cell}(k)$ corresponds to the initial configuration.

It is clear that Eve has a winning strategy in this game if w is accepted by $\mathcal{M}$ (in space m). If w is not accepted, then Eve's first claim does not correspond to the real computation table. Moreover, if a claim by Eve is incorrect, then at least one claim for any respective eligible triple is also incorrect (as can be easily checked), hence Adam can be always asking to verify incorrect claims, which is revealed when the consistency with the initial configuration is verified in the end. Hence Eve has no winning strategy if w is not accepted by $\mathcal{M}$.

In the formal construction presented below we proceed by introducing an intermediate auxiliary problem (in two versions) that allows us to avoid some technicalities in the construction of countdown games $\mathcal{N}^{\mathcal{M}}_{w,m}$. Roughly speaking, instead of the computation of $\mathcal{M}$ on w we consider the computation of $\mathcal{M}_w$ on the empty word, which first writes w on the tape and then invokes $\mathcal{M}$; checking the consistency with the (empty) initial configuration is then technically easier to handle in the constructed countdown game.

Construction formally. Now we formalize the above idea, using the announced intermediate problem.

By a *sequence description* we mean a tuple $\mathcal{D} = (\Delta, D, m)$, where Δ is its finite alphabet, always containing two special symbols # and $\Box$ (and other symbols), $D : \Delta^3 \to \Delta$ is its *description function*, and $m \geq 3$ is its *initial length*. The sequence description $\mathcal{D} = (\Delta, D, m)$ defines the infinite sequence $\mathcal{S}_\mathcal{D}$ in Δ^ω, i.e. the function $\mathcal{S}_\mathcal{D} : \mathbb{N} \to \Delta$, that is defined inductively as follows:

- $\mathcal{S}_\mathcal{D}(0) = \#$,
- $\mathcal{S}_\mathcal{D}(1) = \mathcal{S}_\mathcal{D}(2) = \cdots = \mathcal{S}_\mathcal{D}(m) = \Box$,
- for $i > m$ we have $\mathcal{S}_\mathcal{D}(i) = D(\beta_1, \beta_2, \beta_3)$ where $\beta_1 = \mathcal{S}_\mathcal{D}(i{-}m{-}1)$, $\beta_2 = \mathcal{S}_\mathcal{D}(i{-}m)$, and $\beta_3 = \mathcal{S}_\mathcal{D}(i{-}m{+}1)$.

The two versions of the announced intermediate problem are defined as follows:

NAME: SEQ (sequence problem)
INSTANCE: A sequence description $\mathcal{D} = (\Delta, D, m)$, $n_0 \in \mathbb{N}$, and $\beta_0 \in \Delta$ (with m and n_0 written in binary).
QUESTION: Is $\mathcal{S}_\mathcal{D}(n_0) = \beta_0$?

NAME: ESEQ (existential sequence problem)
INSTANCE: A sequence description $\mathcal{D} = (\Delta, D, m)$, and $\beta_0 \in \Delta$ (with m written in binary).
QUESTION: Is there $i \in \mathbb{N}$ such that $\mathcal{S}_\mathcal{D}(i) = \beta_0$?

Our informal discussion around Figure 5 (including the remark on $\mathcal{M}_w$ starting with the empty tape) makes (almost) clear that

- SEQ is EXPSPACE-complete, and
- ESEQ is EXPTIME-complete.

Remark 3.3. In fact, we would get the same complexity results even if we restricted D in $\mathcal{D} = (\Delta, D, m)$ to $D : \Delta^2 \to \Delta$ and defined $\mathcal{S}_\mathcal{D}(i) = D(\mathcal{S}_\mathcal{D}(i{-}m{-}1, \mathcal{S}_\mathcal{D}(i{-}m))$ for $i > m$. (We would simulate $\mathcal{M}$ by a Turing machine $\mathcal{M}'$ that only moves to the right in each step, while working on a circular tape of length m.) But this technical enhancement would not simplify our construction of the countdown games $\mathcal{N}^{\mathcal{M}}_{w,m}$, in fact.

Formally it suffices for us to claim just the lower bounds:

Proposition 3.4. ESEQ *is EXPSPACE-hard and* SEQ *is EXPTIME-hard.*

Proof. To show EXPSPACE-hardness of ESEQ, we assume an arbitrary fixed language $L \subseteq \Sigma^*$ in EXPSPACE, decided by a Turing machine $\mathcal{M}$ in space $2^{p(n)}$ for a polynomial p.

Using a standard notation, let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{acc}, q_{rej}\})$ where $\Sigma \subseteq \Gamma$, $\square \in \Gamma \setminus \Sigma$, $\# \notin \Gamma$, and $\delta : Q \times \Gamma \to Q \times \Gamma \times \{-1, 0, +1\}$ is the transition function of $\mathcal{M}$, satisfying $\delta(q_{acc}, x) = (q_{acc}, x, 0)$ and $\delta(q_{rej}, x) = (q_{rej}, x, 0)$ (hence an accepting or rejecting configuration is formally viewed as repeated forever). Moreover, w.l.o.g. we assume that the computation of $\mathcal{M}$ on $w = a_1 a_2 \cdots a_n \in \Sigma^*$ starts with w written in tape cells $1, 2, \ldots, n$ with the head scanning the cell 1 (the control state being q_0), the computation never leaves the cells $0, 1, \ldots, m{-}1$ for $m = 2^{p(n)}$, never rewrites $\square$ in the cell 0, and the state q_{acc}, q_{rej} can only be entered when the head is scanning the cell 0 (as is also depicted in Figure 5); we also assume that p is such that $m = 2^{p(n)}$ satisfies $m > n$ and $m \geq 3$.

Given $w \in \Sigma^*$, we now aim to show a polynomial construction of $\mathcal{D}_w = (\Delta_w, D_w, m)$, where $m = 2^{p(|w|)}$, such that $w \in L$ iff there is $i \in \mathbb{N}$ such that $\mathcal{S}_{\mathcal{D}_w}(i) = (q_{acc}, \square)$. (Hence w is reduced to the Eseq-instance $\mathcal{D}_w$, β_0 where $\beta_0 = (q_{acc}, \square)$.)

As already suggested in the previous discussion, for $w = a_1 a_2 \cdots a_n$ we first construct a Turing machine M_w that starts with the empty tape while scanning the cell 0 in its initial state q'_0, then by moving to the right it writes $w = a_1 a_2 \cdots a_n$ in the cells $1, 2, \ldots, n$, after which it moves the head to the cell 1 and enters q_0, thus invoking the computation of $\mathcal{M}$ on w.

When we consider the computation of M_w on the empty word as the sequence $S = C_0 C_1 C_2 \ldots$ of configurations of length m, where $m = 2^{p(|w|)}$, and we view the symbol $(q'_0, \square)$ as #, then we observe that $S(0) = \#$, $S(1) = S(2) = \cdots = S(m) = \square$, and for $i > m$ the symbol $S(i)$ is determined by $S(i{-}m{-}1)$, $S(i{-}m)$, $S(i{-}m{+}1)$ and the transition function of $\mathcal{M}_w$, independently of the actual value of i. The symbols in Δ_w and the function D_w, guaranteeing $\mathcal{S}_{\mathcal{D}_w} = S$, are thus obvious.

A polynomial reduction from L to Eseq is therefore clear, yielding EXPSPACE-hardness of Eseq.

In the case of Seq, we assume that $\mathcal{M}$ deciding L works in time (and thus also space) $2^{p(n)}$; to the Eseq-instance $\mathcal{S}_{\mathcal{D}_w}$, $\beta_0 = (q_{acc}, \square)$ constructed to w as above we simply add $n_0 = m^2$ (for $m = 2^{p(|w|)}$), to get a Seq-instance. Here it is clear that $w \in L$ iff $\mathcal{S}_{\mathcal{D}_w}(n_0) = \beta_0$. This yields EXPTIME-hardness of Seq. $\square$

We now show polynomial (in fact, logspace) reductions from Eseq to Ecg and from Seq to Cg. Again, we present both reductions together.

Given a sequence description $\mathcal{D} = (\Delta, D, m)$ (where $\#, \square \in \Delta$), we construct the countdown game

$$\mathcal{N}_{\mathcal{D}} = (\overline{Q}, \overline{Q}_{\exists}, \delta, p_{win})$$

where

- $\overline{Q}_{\exists} = \{p_{win}, p_{bad}, p_1\} \cup \{s_\beta \mid \beta \in \Delta\}$,
- $\overline{Q}_{\forall} = \{p_2\} \cup \{\, t_{(\beta_1, \beta_2, \beta_3)} \mid \beta_1, \beta_2, \beta_3 \in \Delta\}$

(recall that $\overline{Q} = \overline{Q}_{\exists} \cup \overline{Q}_{\forall}$), and the set δ consists of the rules in Figure 6 (for all $\beta, \beta_1, \beta_2, \beta_3 \in \Delta$). (Note that the rules for states $s_\square$ and $s_\#$ also include the rules of the form (5).)

The idea is that the configuration $s_\beta(k)$ of $\mathcal{N}_{\mathcal{D}}$ should "claim" that $\mathcal{S}_{\mathcal{D}}(k) = \beta$ (and Eve should have a winning strategy from $s_\beta(k)$ iff this claim is correct). For technical reasons we add 2 to the counter, hence it is $s_\beta(k{+}2)$ that "claims" $\mathcal{S}_{\mathcal{D}}(k) = \beta$.

Lemma 3.5. *For each $\beta \in \Delta$ we have $s_\beta(0) \notin Win_\exists$, $s_\beta(1) \notin Win_\exists$, and for each $k \in \mathbb{N}$ we have $s_\beta(k{+}2) \in Win_\exists$ iff $\mathcal{S}_{\mathcal{D}}(k) = \beta$.*

States	Rules	
p_{win} (∃)	—	
p_{bad} (∃)	—	
p_1 (∃)	$p_1 \xrightarrow{-1} p_1 \qquad p_1 \xrightarrow{-1} p_{win}$	(1)
p_2 (∀)	$p_2 \xrightarrow{-1} p_1 \qquad p_2 \xrightarrow{-(m+2)} p_{bad}$	(2)
$s_\square$ (∃)	$s_\square \xrightarrow{-1} p_2$	(3)
$s_\#$ (∃)	$s_\# \xrightarrow{-2} p_{win}$	(4)
s_β (∃)	$s_\beta \xrightarrow{-(m-2)} t_{(\beta_1,\beta_2,\beta_3)}$ when $D(\beta_1,\beta_2,\beta_3)=\beta$	(5)
$t_{(\beta_1,\beta_2,\beta_3)}$ (∀)	$t_{(\beta_1,\beta_2,\beta_3)} \xrightarrow{-3} s_{\beta_1}$ $t_{(\beta_1,\beta_2,\beta_3)} \xrightarrow{-2} s_{\beta_2}$ $t_{(\beta_1,\beta_2,\beta_3)} \xrightarrow{-1} s_{\beta_3}$	(6)

FIGURE 6. Rules of $\mathcal{N}_\mathcal{D}$.

Proof. We start by noting the following facts that are easy to check (recall that Eve wins iff the configuration $p_{win}(0)$ is reached):

a) $p_{win}(k) \in Win_\exists$ iff $k=0$; $p_{bad}(k) \notin Win_\exists$ for all $k \in \mathbb{N}$;
b) $p_1(k) \in Win_\exists$ iff $k \geq 1$; $p_2(k) \in Win_\exists$ iff $k \in [2, m+1]$;
c) for each $\beta \in \Delta$, $s_\beta(0) \notin Win_\exists$ and $s_\beta(1) \notin Win_\exists$;
d) $s_\#(2) \in Win_\exists$ (recall that $\mathcal{S}_\mathcal{D}(0) = \#$);
e) $s_\square(k+2) \in Win_\exists$ for each $k \in [1, m]$
(recall that $\mathcal{S}_\mathcal{D}(1) = \mathcal{S}_\mathcal{D}(2) = \cdots = \mathcal{S}_\mathcal{D}(m) = \square$).

The statement of the lemma for $s_\beta(0)$ and $s_\beta(1)$ follows from the fact (c). To prove that $s_\beta(k+2) \in Win_\exists$ iff $\mathcal{S}_\mathcal{D}(k) = \beta$, we proceed by induction on k:

- *Base case* $k \in [0, m]$: First we note that Eve cannot win in $s_\beta(k+2)$ by playing any rule of the form (5) because either this cannot be played at all, or

$$s_\beta(k+2) \xrightarrow{-(m-2)} t_{(\beta_1,\beta_2,\beta_3)}(k')$$

where $k' \leq 4$, and Adam either cannot continue in $t_{(\beta_1,\beta_2,\beta_3)}(k')$, or he can play to $s_{\beta_\ell}(k'')$ for some $\ell \in \{1,2,3\}$ with $k'' < 2$, which is losing for Eve (fact (c)). Now it easily follows from the facts (d) and (e) that Eve wins in $s_\beta(k+2)$ exactly in those cases where either $\beta = \#$ and $k = 0$ (fact (d)), or $\beta = \square$ and $k \in [1, m]$ (fact (e)).
- *Induction step* for $k > m$: Eve cannot win in $s_\beta(k+2)$ by playing a rule of the form (3) or (4), so she is forced to use a rule from (5), *i.e.*, to play a transition of the form

$$s_\beta(k+2) \xrightarrow{-(m-2)} t_{(\beta_1,\beta_2,\beta_3)}(k')$$

where $D(\beta_1,\beta_2,\beta_3) = \beta$ and $k' = k-m+4 \geq 5$. By this, she chooses a triple $(\beta_1,\beta_2,\beta_3)$ where β_1, β_2, β_3 are symbols supposedly occurring on positions $k-m-1$, $k-m$, and $k-m+1$ in $\mathcal{S}_\mathcal{D}$. Now Adam can challenge some of the symbols β_1, β_2, β_3 by choosing

$\ell \in \{1,2,3\}$ and playing

$$t_{(\beta_1,\beta_2,\beta_3)}(k') \xrightarrow{-(4-\ell)} s_{\beta_\ell}(k'')$$

where $k'' = k - m + \ell \geq 2$. Either β is correct (*i.e.*, $\mathcal{S}_\mathcal{D}(k) = \beta$), and then Eve can choose correct β_1, β_2, β_3, or β is incorrect (*i.e.*, $\mathcal{S}_\mathcal{D}(k) \neq \beta$), and then at least one of β_1, β_2, β_3 must be also incorrect. Since, by the induction hypothesis, Adam can win in $t_{(\beta_1,\beta_2,\beta_3)}(k')$ iff one of β_1, β_2, β_3 is incorrect (by choosing the corresponding move for this incorrect β_ℓ), Eve can win in $s_\beta(k{+}2)$ iff β is correct. From this we derive that $s_\beta(k{+}2) \in \mathit{Win}_\exists$ iff $\mathcal{S}_\mathcal{D}(k) = \beta$. □

Hence, for a given Eseq-instance $\mathcal{D}$, β_0 (where $\mathcal{D} = (\Delta, D, m)$) there is $i \in \mathbb{N}$ such that $\mathcal{S}_\mathcal{D}(i) = \beta_0$ iff there is $k \in \mathbb{N}$ such that $s_{\beta_0}(k) \in \mathit{Win}_\exists$ for $\mathcal{N}_\mathcal{D}$. Moreover, $\mathcal{S}_\mathcal{D}(n_0) = \beta_0$ iff $s_{\beta_0}(n_0{+}2) \in \mathit{Win}_\exists$. Recalling Proposition 3.4, we have thus established the lower bounds in Theorems 3.1 and 3.2.

4. Reachability Game Reduces to (Bi)simulation Game

We show a reduction for general r-games, and then apply it to the case of socn-r-games. This yields a logspace reduction of Socn-Rg to behavioural relations between bisimulation equivalence and simulation preorder.

Recalling the EXPSPACE-hardness of Socn-Rg (from [Hun15], or from the stronger statement of Theorem 3.1), the respective lemmas (Lemma 4.2 and 4.3) will yield the following theorem (which also answers the respective open question from [HLMT16]):

Theorem 4.1. *For succinct labelled one-counter net (SOCNs), deciding membership problem in any relation containing bisimulation equivalence and contained in simulation preorder (of the associated LTSs) is EXPSPACE-hard.*

4.1. Reduction in a General Framework.

We start with an informal introduction to the reduction, which is an application of the technique called "Defender's forcing" in [JS08].

Any r-game $\mathcal{G} = (V, V_\exists, \to, \mathcal{T})$ gives rise to the LTS $\mathcal{L} = (V, \mathit{Act}, (\xrightarrow{a})_{a \in \mathit{Act}})$ where $\mathit{Act} = \{a_{\langle s,\bar{s}\rangle} \mid s \to \bar{s}\} \cup \{a_{win}\}$, $\xrightarrow{a_{\langle s,\bar{s}\rangle}} = \{(s,\bar{s})\}$, and $\xrightarrow{a_{win}} = \{(s,s) \mid s \in \mathcal{T}\}$; hence each transition gets its unique action-label, and each target state gets a loop labelled by Eve's winning action a_{win}. Let $\mathcal{L}'$ be a copy of $\mathcal{L}$ with the state set $V' = \{s' \mid s \in V\}$ but without the action a_{win}. We thus have $s \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}$ in $\mathcal{L}$ iff $s' \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}'$ in $\mathcal{L}'$, and for $s \in \mathcal{T}$ we have $s \not\precsim s'$ in the disjoint union of $\mathcal{L}$ and $\mathcal{L}'$ (since a_{win} is enabled in s but not in s'). If $V_\exists = V$ (in each state it is Eve who chooses the next transition), then we easily observe that

$$s \in \mathit{Win}_\exists \text{ entails } s \not\precsim s' \text{ and } s \notin \mathit{Win}_\exists \text{ entails } s \sim s'. \tag{4.1}$$

A technical problem is how to achieve (4.1) in the case $V_\exists \neq V$; in this case also Adam's choices in the r-game have to be faithfully mimicked in the (bi)simulation game, where it is now Defender who should force the outcome of the relevant game rounds. This is accomplished by adding "intermediate" states and transitions, and the "choice action" a_c, as depicted in Figure 8 (and discussed later in detail); Attacker must let Defender to really choose since otherwise Defender wins by reaching a pair (s,s) with the equal sides (where $s \sim s$).

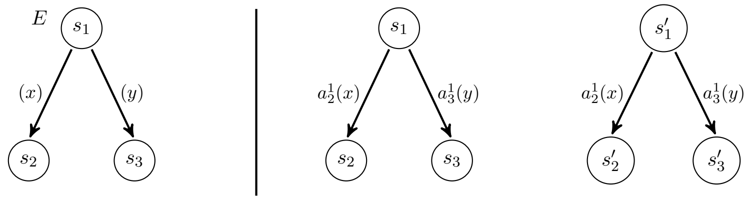


FIGURE 7. Eve's state s_1 in $\mathcal{G}$ (left) is mimicked by the pair (s_1, s_1') in $\mathcal{L}(\mathcal{G})$ (right); it is thus Attacker who chooses (s_2, s_2') or (s_3, s_3') as the next current pair.

Now we formalize the above sketch. We assume an r-game $\mathcal{G}$, and we define a "mimicking" LTS $\mathcal{L}(\mathcal{G})$ (the enhanced union of the above LTSs $\mathcal{L}$ and $\mathcal{L}'$). In illustrating Figures 7 and 8 we now ignore the bracketed parts of transition-labels; hence, e.g., in Figure 7 we can see the transition $s_1 \to s_2$ in $\mathcal{G}$ on the left and the (corresponding) transitions $s_1 \xrightarrow{a_2^1} s_2$ and $s_1' \xrightarrow{a_2^1} s_2'$ in $\mathcal{L}(\mathcal{G})$ on the right. Let $\mathcal{G} = (V, V_\exists, \to, \mathcal{T})$, where $V_\forall = V \smallsetminus V_\exists$; we define $\mathcal{L}(\mathcal{G}) = (S, Act, (\xrightarrow{a})_{a\in Act})$ as follows. We put

$$S = V \cup V' \cup \{\langle s, \bar{s}\rangle \mid s \in V_\forall, s \to \bar{s}\} \cup \{\langle s, X\rangle \mid s \in V_\forall, X = \{\bar{s} \mid s \to \bar{s}\} \neq \emptyset\}$$

where $V' = \{s' \mid s \in V\}$ is a "copy" of V. (In Figure 8 we write, e.g., s_3^1 instead of $\langle s_1, s_3\rangle$, and s_{23}^1 instead of $\langle s_1, \{s_2, s_3\}\rangle$.)

We put $Act = \{a_c, a_{win}\} \cup \{a_{\langle s,\bar{s}\rangle} \mid s \to \bar{s}\}$ and define $\xrightarrow{a}$ for $a \in Act$ as follows. If $s \in V_\exists$ and $s \to \bar{s}$, then $s \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}$ and $s' \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}'$ (in Figure 7 we write, e.g., a_3^1 instead of $a_{\langle s_1,s_3\rangle}$). If $s \in V_\forall$ and $X = \{\bar{s} \mid s \to \bar{s}\} \neq \emptyset$, then:

a) $s \xrightarrow{a_c} \langle s, X\rangle$, and $s \xrightarrow{a_c} \langle s, \bar{s}\rangle$, $s' \xrightarrow{a_c} \langle s, \bar{s}\rangle$ for all $\bar{s} \in X$ (cf. Figure 8 where $s = s_1$ and $X = \{s_2, s_3\}$; a_c is a "choice-action");
b) for each $\bar{s} \in X$ we have $\langle s, X\rangle \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}$ and $\langle s, \bar{s}\rangle \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}'$; moreover, for each $\bar{\bar{s}} \in X \smallsetminus \{\bar{s}\}$ we have $\langle s, \bar{s}\rangle \xrightarrow{a_{\langle s,\bar{\bar{s}}\rangle}} \bar{\bar{s}}$ (e.g., in Figure 8 we thus have $s_2^1 \xrightarrow{a_2^1} s_2'$ and $s_2^1 \xrightarrow{a_3^1} s_3$).

For each $s \in \mathcal{T}$ we have $s \xrightarrow{a_{win}} s$ (for special a_{win} that is not enabled in s').

Lemma 4.2. *For an r-game $\mathcal{G} = (V, V_\exists, \to, \mathcal{T})$, and the LTS $\mathcal{L}(\mathcal{G}) = (S, Act, (\xrightarrow{a})_{a\in Act})$, the following conditions hold for every $s \in V$ and every relation ρ satisfying $\sim \subseteq \rho \subseteq \preceq$:*

a) *if $s \in Win_\exists$ (in $\mathcal{G}$), then $s \not\preceq s'$ (in $\mathcal{L}(\mathcal{G})$) and thus $(s, s') \notin \rho$;*
b) *if $s \notin Win_\exists$, then $s \sim s'$ and thus $(s, s') \in \rho$.*

Proof. a) For the sake of contradiction suppose that there is $s \in Win_\exists$ such that $s \preceq s'$; we consider such $s \in Win_\exists$ with the least rank. We note that $\textsc{rank}(s) > 0$, since $s \in \mathcal{T}$ entails $s \not\preceq s'$ due to the transition $s \xrightarrow{a_{win}} s$. If $s \in V_\exists$, then let $s \to \bar{s}$ be a rank-reducing transition. Attacker's move $s \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}$, from the pair (s, s'), must be responded with $s' \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}'$; but we have $\bar{s} \not\preceq \bar{s}'$ by the "least-rank" assumption, which contradicts the assumption $s \preceq s'$. If $s \in V_\forall$, then $X = \{\bar{s} \mid s \to \bar{s}\}$ is nonempty (since $s \in Win_\exists$) and $\textsc{rank}(\bar{s}) < \textsc{rank}(s)$ for all $\bar{s} \in X$. For the pair (s, s') we now consider Attacker's move $s \xrightarrow{a_c} \langle s, X\rangle$. Defender can choose $s' \xrightarrow{a_c} \langle s, \bar{s}\rangle$ for any $\bar{s} \in X$ (recall that $\textsc{rank}(\bar{s}) < \textsc{rank}(s)$). In the current pair

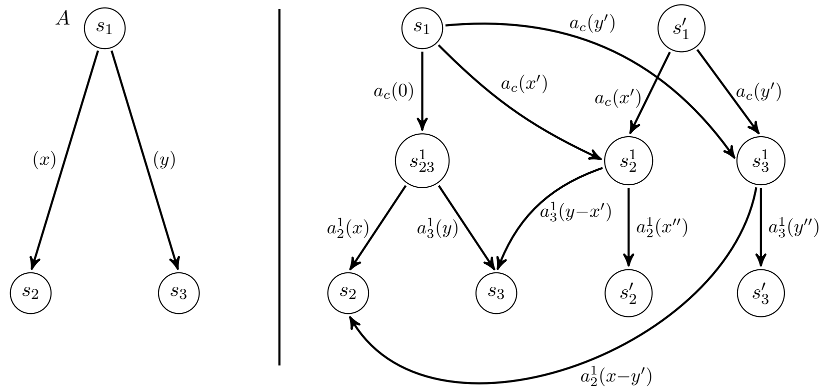


Figure 8. In (s_1, s'_1) it is, in fact, Defender who chooses (s_2, s'_2) or (s_3, s'_3) (when Attacker avoids pairs with equal states); to take the counter-changes into account correctly, we put $x' = \min\{x, 0\}$, $x'' = \max\{x, 0\}$, and $y' = \min\{y, 0\}$, $y'' = \max\{y, 0\}$ (hence $x = x'+x''$ and $y = y'+y''$).

$(\langle s, X\rangle, \langle s, \bar{s}\rangle)$ Attacker can play $\langle s, X\rangle \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}$, and this must be responded by $\langle s, \bar{s}\rangle \xrightarrow{a_{\langle s,\bar{s}\rangle}} \bar{s}'$. But we again have $\bar{s} \not\preceq \bar{s}'$ by the "least-rank" assumption, which contradicts $s \preceq s'$.

b) It is easy to verify that the following set is a bisimulation in $\mathcal{L}(\mathcal{G})$:

$$I \cup \{(s, s') \mid s \in V \smallsetminus \mathit{Win}_\exists\} \cup \{(\langle s, X\rangle, \langle s, \bar{s}\rangle) \mid s \in V_\forall \smallsetminus \mathit{Win}_\exists, \bar{s} \in V \smallsetminus \mathit{Win}_\exists\}$$

where $I = \{(s, s) \mid s \in S\}$. □

We note that the transitions $s_1 \xrightarrow{a_c} s^1_2$ and $s_1 \xrightarrow{a_c} s^1_3$ in Figure 8 could be omitted if we only wanted to show that $s \in \mathit{Win}_\exists$ iff $s \not\preceq s'$.

4.2. **Socn-Rg Reduces to Behavioural Relations on SOCNs.** We now note that the LTS $\mathcal{L}(\mathcal{G}_\mathcal{N})$ "mimicking" the r-game $\mathcal{G}_\mathcal{N}$ associated with a socn-r-game $\mathcal{N}$ (recall (2.2)) can be presented as $\mathcal{L}_{\mathcal{N}'}$ for a SOCN $\mathcal{N}'$ (recall (2.1)) that is efficiently constructible from $\mathcal{N}$:

Lemma 4.3. *There is a logspace algorithm that, given a socn-r-game $\mathcal{N}$, constructs a SOCN $\mathcal{N}'$ such that the LTSs $\mathcal{L}(\mathcal{G}_\mathcal{N})$ and $\mathcal{L}_{\mathcal{N}'}$ are isomorphic.*

Proof. We again use Figures 7 and 8 for illustration; now s_i are viewed as control states and the bracketed parts of edge-labels are counter-changes (in binary).

Given a socn-r-game $\mathcal{N} = (Q, Q_\exists, \delta, p_{win})$, we first consider the r-game

$$\mathcal{N}^{csg} = (Q, Q_\exists, \to, \{p_{win}\})$$

("the control-state game of $\mathcal{N}$") arising from $\mathcal{N}$ by *forgetting the counter-changes*; hence $q \to \bar{q}$ iff there is a rule $q \xrightarrow{z} \bar{q}$. In fact, we will assume that there is at most one rule $q \xrightarrow{z} \bar{q}$ in δ (of $\mathcal{N}$) for any pair $(q, \bar{q}) \in Q \times Q$; this can be achieved by harmless modifications.

We construct the (finite) LTS $\mathcal{L}(\mathcal{N}^{csg})$ ("mimicking" $\mathcal{N}$). Hence each $q \in Q$ has the copies q, q' in $\mathcal{L}(\mathcal{N}^{csg})$, and other states are added (as also depicted in Figure 8 where s_i

are now in the role of control states); there are also the respective labelled transitions in $\mathcal{L}(\mathcal{N}^{csg})$, with labels $a_{\langle q,\bar{q}\rangle}$, a_c, a_{win}.

It remains to add the counter changes (integer increments and decrements in binary), to create the required SOCN $\mathcal{N}'$. For $q \in Q_\exists$ this adding is simple, as depicted in Figure 7: if $q \xrightarrow{z} \bar{q}$ (in $\mathcal{N}$), then we simply extend the label $a_{\langle q,\bar{q}\rangle}$ in $\mathcal{L}(\mathcal{N}^{csg})$ with z; for $q \xrightarrow{a_{\langle q,\bar{q}\rangle}} \bar{q}$ and $q' \xrightarrow{a_{\langle q,\bar{q}\rangle}} \bar{q}'$ in $\mathcal{L}(\mathcal{N}^{csg})$ we get $q \xrightarrow{a_{\langle q,\bar{q}\rangle},z} \bar{q}$ and $q' \xrightarrow{a_{\langle q,\bar{q}\rangle},z} \bar{q}'$ in $\mathcal{N}'$.

For $q \in Q_\forall$ (where $Q_\forall = Q \smallsetminus Q_\exists$) it is tempting to the same, i.e. to extend the label $a_{\langle q,\bar{q}\rangle}$ with z when $q \xrightarrow{z} \bar{q}$, and extend a_c with 0. But this might allow cheating for Defender: she could thus mimic choosing a transition $q(k) \xrightarrow{x} \bar{q}(k{+}x)$ even if $k{+}x < 0$. This is avoided by the modification that is demonstrated in Figure 8 (by $x = x'{+}x''$, etc.); put simply: Defender must immediately prove that the transition she is choosing to mimic is indeed performable. Formally, if $X = \{\bar{q} \mid q \to \bar{q}\} \neq \emptyset$ (in $\mathcal{L}(\mathcal{N}^{csg})$), then in $\mathcal{N}'$ we put $q \xrightarrow{a_c,0} \langle q, X\rangle$ and $\langle q, X\rangle \xrightarrow{a_{\langle q,\bar{q}\rangle},z} \bar{q}$ for each $q \xrightarrow{z} \bar{q}$ (in $\mathcal{N}$); for each $q \xrightarrow{z} \bar{q}$ we also define $z' = \min\{z, 0\}$, $z'' = \max\{z, 0\}$ and put $q' \xrightarrow{a_c,z'} \langle q, \bar{q}\rangle$, $\langle q, \bar{q}\rangle \xrightarrow{a_{\langle q,\bar{q}\rangle},z''} \bar{q}'$. Then for any pair $q \xrightarrow{\bar{z}} \bar{q}$, $q \xrightarrow{\bar{\bar{z}}} \bar{\bar{q}}$ where $\bar{q} \neq \bar{\bar{q}}$ we put $\langle q, \bar{q}\rangle \xrightarrow{a_{\langle q,\bar{\bar{q}}\rangle},\bar{\bar{z}}-\bar{z}'} \bar{\bar{q}}$.

Finally, $p_{win} \xrightarrow{a_{win}} p_{win}$ in $\mathcal{L}(\mathcal{N}^{csg})$ is extended to $p_{win} \xrightarrow{a_{win},0} p_{win}$ in $\mathcal{N}'$. □

We have thus finished the proof of Theorem 4.1.

5. Structure of simulation preorder on one-counter nets

In this section we give a new self-contained proof clarifying the structure of simulation preorder $\preceq$ on the LTS $\mathcal{L}_\mathcal{N}$ associated with a given one-counter net $\mathcal{N}$; this will also yield a polynomial-space algorithm generating a description of $\preceq$ on $\mathcal{L}_\mathcal{N}$ that can be used to decide if $p(m) \preceq q(n)$.

We first show a natural graphic presentation of the relation $\preceq$, and in Section 5.1 we show its linear-belt form; the result is captured by the *belt theorem*. The proof is inspired by [JMS99] but is substantially different. A main new ingredient is the notion of so called down-black and up-white lines, and their limits in which we do not a priori exclude lines with irrational slopes; this allows us to avoid many technicalities used in previous proofs, while the presented "geometric" ideas should be straightforward, and transparent due to the respective figures.

In Section 5.2 we prove that the slopes and the widths of belts are presented/bounded by small integers. Though this *quantitative belt theorem* is in principle equivalent to the respective theorem proved in [HLMT16], our proof is again conceptually different; using the novel notions, the quantitative characteristics are derived from the (qualitative) belt theorem easily.

For completeness, in Section 5.3 we briefly recall the idea from the previous papers that shows how the achieved structural results yield a polynomial-space algorithm generating a description of $\preceq$ for a given one-counter net.

In the sequel we assume a fixed OCN $\mathcal{N} = (Q, Act, \delta)$ if not said otherwise. By $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$ we denote the sets of reals, of nonnegative reals, and of positive reals, respectively. We also use ∞ for an infinite amount (in particular for the slope of a vertical line); hence $\alpha < \infty$ for all $\alpha \in \mathbb{R}$. By $\langle \gamma, \gamma'\rangle$, where $\gamma \in \mathbb{R}_{\geq 0}$ and $\gamma' \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, we denote the set $\{\alpha \in \mathbb{R}_{\geq 0} \cup \{\infty\} \mid \gamma \leq \alpha \leq \gamma'\}$; in particular, $\langle 0, \infty\rangle = \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Monotonic black-white presentation of the simulation preorder $\preceq$. For each pair $(p,q) \in Q \times Q$ we define the (black-white) *colouring* $C_{\langle p,q \rangle}$ of the integer points in the first quadrant of the plane $\mathbb{R} \times \mathbb{R}$; we put $C_{\langle p,q \rangle} : \mathbb{N} \times \mathbb{N} \to \{\text{black}, \text{white}\}$ where

$$C_{\langle p,q \rangle}(m,n) = \begin{cases} \text{black} & \text{if } p(m) \preceq q(n), \\ \text{white} & \text{if } p(m) \not\preceq q(n). \end{cases}$$

We recall the definition of $\textsc{rank}(s,s')$ in the paragraph "Stratified simulation, and ranks of pairs of states" in Section 2; this yields the definition of $\textsc{rank}(p(m), q(n))$ for our fixed OCN $\mathcal{N} = (Q, Act, \delta)$. Hence for each $(p,q) \in Q \times Q$ we have that each white point (m,n) in $C_{\langle p,q \rangle}$ has an associated finite rank, namely $\textsc{rank}(p(m), q(n)) \in \mathbb{N}$.

The next proposition captures the trivial fact that "*black is upwards- and leftwards-closed*" and "*white is downwards- and rightwards-closed*" (as is also depicted in Figure 3 in Introduction, which also shows the "linear-belt form" of the colourings).

Proposition 5.1 (Black and white monotonicity)**.**
If $C_{\langle p,q \rangle}(m,n) = \text{black}$*, then* $C_{\langle p,q \rangle}(m',n') = \text{black}$ *for all* $m' \leq m$ *and* $n' \geq n$*. Hence if* $C_{\langle p,q \rangle}(m,n) = \text{white}$*, then* $C_{\langle p,q \rangle}(m',n') = \text{white}$ *for all* $m' \geq m$ *and* $n' \leq n$*.*

Proof. Since OCNs are monotonic in the sense that $p(m) \xrightarrow{a} q(n)$ implies $p(m{+}i) \xrightarrow{a} q(n{+}i)$ for all $i \in \mathbb{N}$, we will easily verify that the relation

$$R = \big\{ \big(p(m'), q(n')\big) \mid p(m) \preceq q(n) \text{ for some } m \geq m' \text{ and } n \leq n' \big\}$$

is a simulation relation. Since $\preceq \;\subseteq R$, we will thus get $R = \;\preceq$, and the proof will be finished. To verify that R is indeed a simulation, we consider $\big(p(m'), q(n')\big) \in R$ and fix $m \geq m'$ and $n \leq n'$ so that $p(m) \preceq q(n)$. If $p(m') \xrightarrow{a} p'(m'{+}i)$ (for $i \in \{-1, 0, +1\}$), then $p(m) \xrightarrow{a} p'(m{+}i)$. Since $p(m) \preceq q(n)$, we have $q(n) \xrightarrow{a} q'(n{+}j)$ where $j \in \{-1, 0, +1\}$ and $p'(m{+}i) \preceq q'(n{+}j)$. Hence $q(n') \xrightarrow{a} q'(n'{+}j)$, and $\big(p'(m'{+}i), q'(n'{+}j)\big) \in R$. $\square$

5.1. **Belt theorem.** Below we state the (qualitative) belt theorem, illustrated in Figure 10, after introducing the needed notions. We can remark that the validity of the belt theorem can be easily intuitively anticipated (by a quick thought about the problem) but it has turned out surprisingly hard to be rigorously proven. (It can be also intuitively anticipated that the colouring inside the belt determined by the lines ℓ_L, ℓ_R in Figure 10 is ultimately periodic, but this is only used in Section 5.3.)

Lines, slopes, axes, points, points above or below lines, points on the left of or on the right of vertical lines. (See Figure 9.) A *vertical line* ℓ (in the plane $\mathbb{R} \times \mathbb{R}$) is the set $\ell = \{(x,y) \mid y \in \mathbb{R}\}$ for a fixed $x \in \mathbb{R}$; we put $\textsc{slope}(\ell) = \infty$. A *non-vertical line* ℓ is the set $\ell = \{(x,y) \mid y = \gamma \cdot x + c\}$ for some fixed $\gamma \in \mathbb{R}$ and $c \in \mathbb{R}$; here $\textsc{slope}(\ell) = \gamma$. A *line* is either a vertical line or a non-vertical line. By a *γ-line*, $\gamma \in \mathbb{R} \cup \{\infty\}$, we mean a line whose slope is γ. As expected, by the *horizontal axis*, $\textsc{h-axis}$, we mean the 0-line that goes through the *origin* $(0,0)$; the *vertical axis*, $\textsc{v-axis}$, is the ∞-line that goes through $(0,0)$.

By *points* we mean the pairs $(m,n) \in \mathbb{N} \times \mathbb{N}$ (hence only the integer points in the first quadrant of the plane $\mathbb{R} \times \mathbb{R}$), unless explicitly stated that we consider all integer points (elements of $\mathbb{Z} \times \mathbb{Z}$). Given a non-vertical line $\ell = \{(x,y) \mid y = \gamma \cdot x + c\}$, a *point* (m,n) is *above* ℓ if $n \geq \gamma \cdot m + c$, and it is *below* ℓ if $n \leq \gamma \cdot m + c$. Given a vertical line $\ell = \{(x,y) \mid y \in \mathbb{R}\}$, a *point* (m,n) is *above* ℓ, or also *on the left of* ℓ, if $m \leq x$, and it is

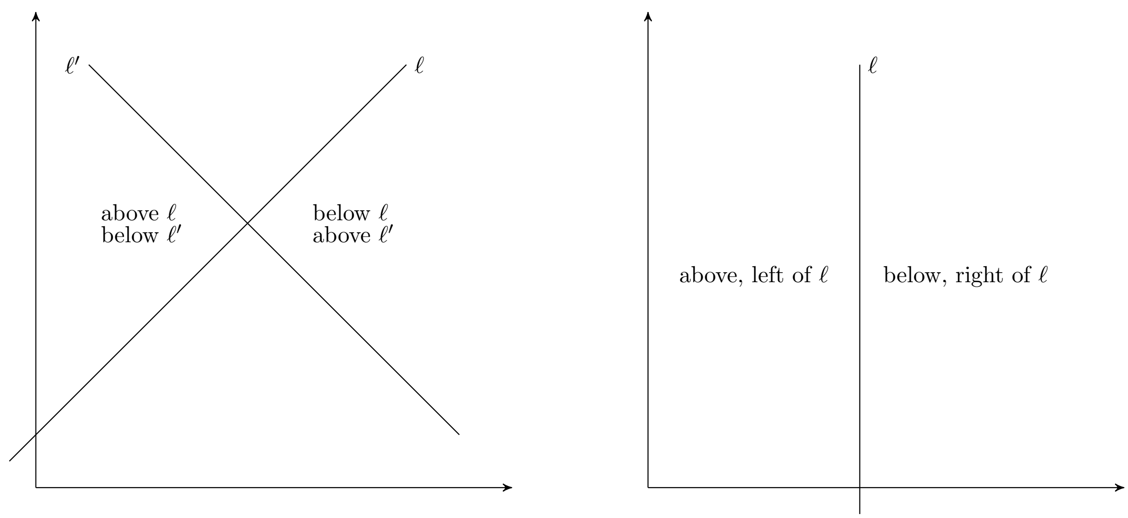


FIGURE 9. Areas “above ℓ”, “below ℓ”, “left of ℓ”, and “right of ℓ”.

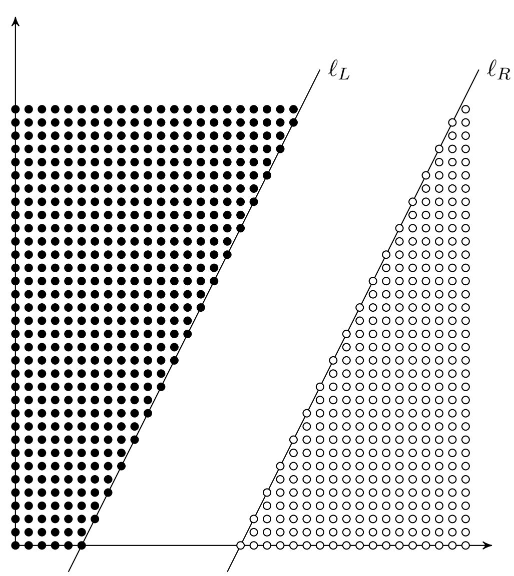


FIGURE 10. Illustration of the belt theorem, for one colouring $C = C_{\langle p,q\rangle}$.

below ℓ, or also *on the right of* ℓ, if $m \geq x$. A point P is *strictly above* ℓ if P is above ℓ but not on ℓ; similarly, P is *strictly below* ℓ if P is below ℓ but not on ℓ.

Theorem 5.2 (Belt theorem)**.** *For each $(p, q) \in Q \times Q$, and its respective colouring $C = C_{\langle p,q\rangle}$, there are (a slope) $\gamma \in \langle 0, \infty\rangle$ that is rational or ∞, and two (parallel) γ-lines ℓ_L and ℓ_R such that all points above ℓ_L are black in C and all points below ℓ_R are white in C. (See Figure 10.)*

We recall that we do not discuss how the mapping $C_{\langle p,q\rangle}$ looks inside the belt depicted in Figure 10; only in Section 5.3 we refer to the fact that it is ultimately periodic there.

Now we introduce further notions useful in the proof of the belt theorem.

Down-black and up-white lines and slopes, down-limits α_C and up-limits β_C. A *line* ℓ, with $\text{SLOPE}(\ell) \in \langle 0, \infty \rangle$, is *down-black in* a colouring $C = C_{\langle p,q \rangle}$ if there are infinitely many points below ℓ that are black in C. A (nonnegative) *slope* $\gamma \in \langle 0, \infty \rangle$ is *down-black in* C if there is a γ-line ℓ that is down-black in C.

By the *down-limit of* C we mean the value

$$\alpha_C = \inf \{ \gamma \mid \gamma \text{ is down-black in } C \}$$

(where $\inf \emptyset = \infty$). We say that α is a *down-limit* if $\alpha = \alpha_C$ for some $C = C_{\langle p,q \rangle}$. For a down-limit α, by α*-colourings* we mean the colourings C such that $\alpha_C = \alpha$.

A *line* ℓ is *up-white in* a colouring $C = C_{\langle p,q \rangle}$ if there are infinitely many points above ℓ that are white in C. A (nonnegative) *slope* $\gamma \in \langle 0, \infty \rangle$ is *up-white in* C if there is a γ-line ℓ that is up-white in C. By the *up-limit of* C we mean the value

$$\beta_C = \sup \{ \gamma \mid \gamma \text{ is up-white in } C \}$$

(where $\sup \emptyset = 0$).

We highlight the following trivial fact.

Proposition 5.3.

- *If γ is down-black in C, then each $\gamma' \in \langle \gamma, \infty \rangle$ is down-black in C; moreover, if $\gamma' > \gamma$, then each γ'-line is down-black in C.*
- *If γ is up-white in C, then each $\gamma' \in \langle 0, \gamma \rangle$ is up-white in C; moreover, if $\gamma' < \gamma$, then each γ'-line is up-white in C.*

We note that we cannot a priori exclude that some α_C is irrational and/or not down-black; similarly we cannot exclude that β_C differs from α_C, and that it is irrational and/or not up-white. But we immediately note a simple fact:

Proposition 5.4. *For each $(p, q) \in Q \times Q$, $\beta_{C_{\langle p,q \rangle}} \geq \alpha_{C_{\langle p,q \rangle}}$.*

Proof. Let $\alpha_C > \beta_C$ for $C = C_{\langle p,q \rangle}$, and let $\alpha_C > \gamma > \beta_C$; hence γ is not down-black and not up-white. Thus for each γ-line ℓ we have that almost all points (i.e., all but finitely many) below ℓ are white and almost all points above ℓ are black; by an obvious shift of ℓ we deduce that γ is up-white and/or down-black after all, which is a contradiction. □

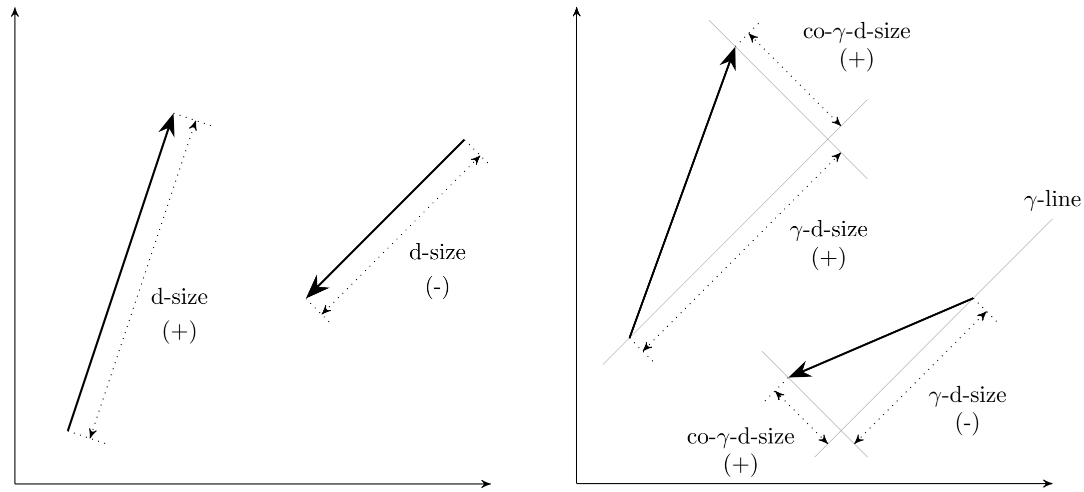
We now state a lemma that trivially entails Theorem 5.2: the slope γ claimed for C in Theorem 5.2 is equal to α_C (which is claimed to be equal to β_C by the lemma).

Lemma 5.5. *For each $(p, q) \in Q \times Q$ and the respective colouring $C = C_{\langle p,q \rangle}$ the following conditions hold:*

(1) *α_C is rational or ∞,*
(2) *$\alpha_C = \beta_C$, and*
(3) *there are α_C-lines ℓ_L and ℓ_R such that all points above ℓ_L are black in C, and all points below ℓ_R are white in C.*

Before proving the lemma we introduce a main technical ingredient of the proof, namely Proposition 5.7 and Corollary 5.9, preceded by the needed notions and by a simple, yet very useful, observation (Proposition 5.6).

We can remark that these technical ingredients can hardly be "intuitively anticipated"; they have been "distilled" from the overall nontrivial proof, as useful technical claims. Some intuition about a possible proof strategy might be perhaps got by looking at the simulation games described in the papers [Av98] and [HLMT16], but our proof is, in fact, not so tightly



(A) The d-size of a vector. (B) The d-sizes related to $\gamma \in \langle 0, \infty\rangle$.

Figure 11

related to the simulation problem and could be given in a more general framework of specific tiling problems (as we also recall in Section 7).

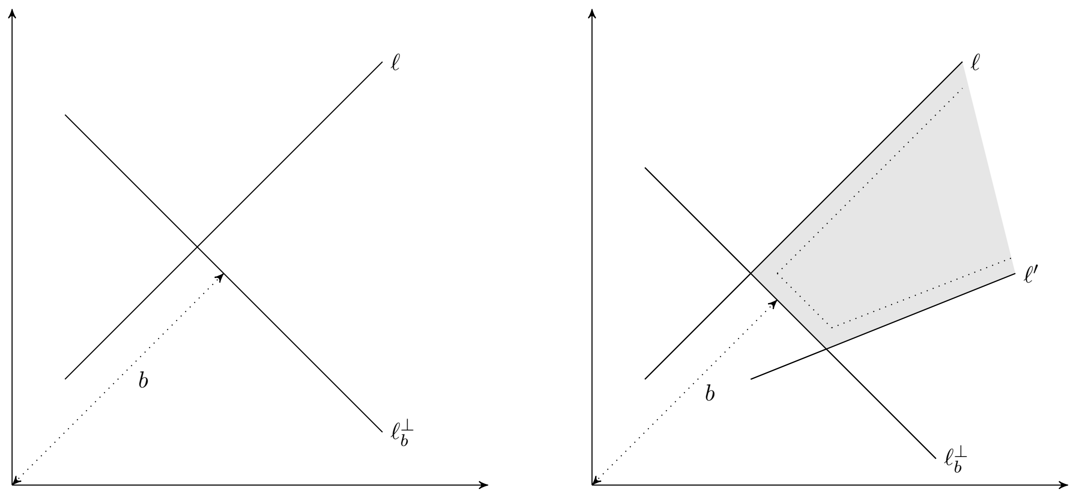
Vectors, their slopes and d-sizes. (See Figure 11a.) We view a *vector* v as a pair $(\textsc{start}(v), \textsc{end}(v))$ where $\textsc{start}(v) \in \mathbb{N} \times \mathbb{N}$ is the *start-point* of v and $\textsc{end}(v) \in \mathbb{N} \times \mathbb{N}$ is the *end-point* of v. The *slope of a vector* v, $\textsc{slope}(v)$, is defined when $\textsc{start}(v) \neq \textsc{end}(v)$, in which case it is the slope of the line going through both $\textsc{start}(v)$ and $\textsc{end}(v)$. The *d-size of* v ("*d*" stands for *direction*) is the Euclidean distance of the points $\textsc{start}(v)$ and $\textsc{end}(v)$ with the positive sign $(+)$ or the negative sign $(-)$: if $\textsc{start}(v) \neq \textsc{end}(v)$, then we consider the line ℓ that is perpendicular to v and goes through $\textsc{start}(v)$; if $\textsc{slope}(\ell) \neq \infty$ and $\textsc{end}(v)$ is above ℓ, or $\textsc{slope}(\ell) = \infty$ and $\textsc{end}(v)$ is to the right of ℓ, then the d-size of v is positive, and otherwise the d-size of v is negative.

Black-white vectors. A *vector* v is *black-white in a colouring* C if $\textsc{start}(v)$ is black and $\textsc{end}(v)$ is white in C.

Neighbour points and vectors. A *point* (m', n') is a *neighbour of a point* (m, n) if $|m' - m| \leq 1$ and $|n' - n| \leq 1$ (which includes the case $m' = m$, $n' = n$). A vector v' is a *neighbour vector of* a vector v if $\textsc{start}(v')$ is a neighbour of $\textsc{start}(v)$ and v, v' have the same slopes and sizes (hence v' is a "small shift" of v).

The next proposition is the announced simple observation. It states that for any vector v where $\textsc{start}(v)$ is not on the vertical axis, $\textsc{end}(v)$ is not on the horizontal axis, and v is black-white in some $C_{\langle p,q\rangle}$ there is its neighbour vector v' that is black-white in some $C_{\langle p',q'\rangle}$ and the rank of the white end-point of v' in $C_{\langle p',q'\rangle}$ is smaller than the rank of the white end-point of v in $C_{\langle p,q\rangle}$. (Recall the vectors v_0, v_1 in Figure 4a in Introduction, and the discussion of ranks $\textsc{rank}(p(m), q(n))$ between the definition of $C_{\langle p,q\rangle}(m, n)$ and Proposition 5.1.)

Proposition 5.6 (Neighbour black-white vector with smaller rank)**.** *Let*



(A) The perpendicular line $\ell_b^\perp$.

(B) $\textsc{area}((\ell, b), \ell')$.

Figure 12

$$C_{\langle p,q\rangle}(m,n) = \text{black},\ C_{\langle p,q\rangle}(m',n') = \text{white}$$

where $m > 0$ *and* $n' > 0$. *Then there are* $p', q' \in Q$ *and* $i, j \in \{-1, 0, 1\}$ *such that*

$$C_{\langle p',q'\rangle}(m{+}i, n{+}j) = \text{black},\ C_{\langle p',q'\rangle}(m'{+}i, n'{+}j) = \text{white},$$

and $\textsc{rank}(p'(m'{+}i), q'(n'{+}j)) < \textsc{rank}(p(m'), q(n'))$.

Proof. Let the assumptions hold; we put $\textsc{rank}(p(m'), q(n')) = r \in \mathbb{N}$. We can thus fix a transition $p(m') \xrightarrow{a} p'(m'{+}i)$ (related to a rule $p \xrightarrow{a,i} p'$) such that for each $q(n') \xrightarrow{a} q'(n'{+}j)$ we have $\textsc{rank}(p'(m'{+}i), q'(n'{+}j)) < r$. Since $m > 0$, we have $p(m) \xrightarrow{a} p'(m{+}i)$, and since $p(m) \preceq q(n)$, there is a transition $q(n) \xrightarrow{a} q'(n{+}j)$ such that $p'(m{+}i) \preceq q'(n{+}j)$; since $n' > 0$, we also have $q(n') \xrightarrow{a} q'(n'{+}j)$. Hence $C_{\langle p',q'\rangle}(m{+}i, n{+}j) = \text{black}$, and $C_{\langle p',q'\rangle}(m'{+}i, n'{+}j) = \text{white}$, and $\textsc{rank}(p'(m'{+}i), q'(n'{+}j)) < r$. □

Now we aim to formulate a crucial ingredient of the proof of Lemma 5.5, namely Proposition 5.7 and its corollary, for which we need further notions.

Point-to-line distance, perpendicular lines $\ell_b^\perp$. For an integer point $\textsc{p}$ (in $\mathbb{Z} \times \mathbb{Z}$) and a line ℓ, by $\textsc{dist}(\textsc{p}, \ell)$ we mean the standard (Euclidean) distance of $\textsc{p}$ to ℓ. Given a line ℓ where $\textsc{slope}(\ell) \in \langle 0, \infty\rangle$ and a *level* (which might be also called a *bottom-level*) $b \in \mathbb{R}_{\geq 0}$, by $\ell_b^\perp$ we denote the line that is perpendicular to ℓ, intersects the horizontal and/or vertical axis in the first quadrant, and $\textsc{dist}((0,0), \ell_b^\perp) = b$. (Hence $(\ell')_b^\perp = \ell_b^\perp$ for any ℓ' that is parallel with ℓ. See Figure 12a.)

Areas, border and interior points of areas. By an *area* we mean just a set $A \subseteq \mathbb{N} \times \mathbb{N}$ of points. A point $\textsc{p} \in A$ is a *border point of* A if it has a neighbour point outside A (hence in $(\mathbb{N} \times \mathbb{N}) \smallsetminus A$); we put $\textsc{border}(A) = \{\textsc{p} \mid \textsc{p} \text{ is a border point of } A\}$. Each point $\textsc{p} \in A \smallsetminus \textsc{border}(A)$ is an *interior point of* A. (E.g., $(0,0)$ is an interior point of $A = \{(0,0), (0,1), (1,0), (1,1)\}$.)

Line-level-line areas $\text{AREA}((\ell,b),\ell')$ **and** $\text{AREA}(\ell,(b,\ell'))$**.** (See Figure 12b.) Given two lines ℓ,ℓ' where $\infty \geq \text{SLOPE}(\ell) \geq \text{SLOPE}(\ell') \geq 0$, and a bottom-level $b \in \mathbb{R}_{\geq 0}$, we define $\text{AREA}((\ell,b),\ell')$ as the set of points (in $\mathbb{N}\times\mathbb{N}$) that are below ℓ, above ℓ', and also above the line $\ell^{\perp}_{b}$; an exception is the case with $\text{SLOPE}(\ell)=0$ (hence $\text{SLOPE}(\ell')=0$ as well) where the points in $\text{AREA}((\ell,b),\ell')$ are to the right of (the vertical line) $\ell^{\perp}_{b}$.

By $\text{AREA}(\ell,(b,\ell'))$ we mean the set of points that are below ℓ, above ℓ', and also above the line $(\ell')^{\perp}_{b}$; again, an exception is the case with $\text{SLOPE}(\ell')=0$, where the points in $\text{AREA}(\ell,(b,\ell'))$ are to the right of (the vertical line) $(\ell')^{\perp}_{b}$.

In fact, we will not encounter the "pathological" case where $\text{SLOPE}(\ell)=\text{SLOPE}(\ell')$ and ℓ' is strictly above (to the left of) ℓ, in which case the sets $\text{AREA}((\ell,b),\ell')$ and $\text{AREA}(\ell,(b,\ell'))$ are empty. In the special case where $\text{SLOPE}(\ell)=\infty$ and $\text{SLOPE}(\ell')=0$, we only consider the sets $\text{AREA}((\ell,b),\ell')$ where $(\ell)^{\perp}_{b}$ coincides with ℓ', and the sets $\text{AREA}(\ell,(b,\ell'))$ where $(\ell')^{\perp}_{b}$ coincides with ℓ.

Special cases of line-level-line areas: $\text{AREA}(\leftarrow\ell)$ **and** $\text{AREA}(\ell\rightarrow)$**.** Given a line ℓ, with $\text{SLOPE}(\ell) \in \langle 0,\infty\rangle$, we put $\text{AREA}(\leftarrow\ell)=\{\,\text{P}\in\mathbb{N}\times\mathbb{N} \mid \text{P is above } \ell\,\}$, and $\text{AREA}(\ell\rightarrow)=\{\,\text{P}\in\mathbb{N}\times\mathbb{N} \mid \text{P is below } \ell\,\}$. (Hence $\text{AREA}(\leftarrow\ell)=\text{AREA}((\text{V-AXIS},0),\ell)$ and $\text{AREA}(\ell\rightarrow)=\text{AREA}(\ell,(0,\text{H-AXIS}))$.)

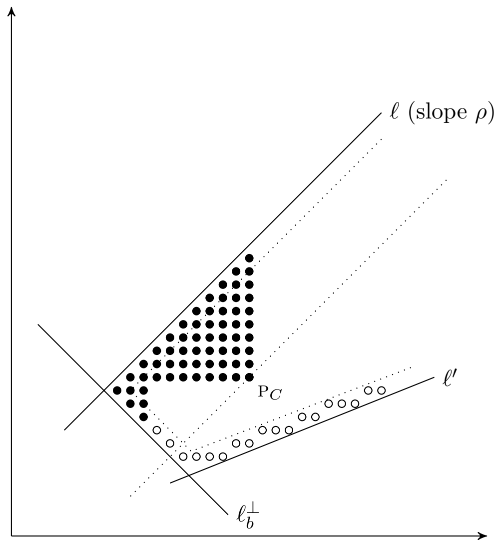
Border points of $\text{AREA}((\ell,b),\ell')$ **along** ℓ**, along** ℓ'**, and along** $\ell^{\perp}_{b}$**.** (See the dotted boundaries in Figure 12b.) Given a point $\text{P}\in\text{BORDER}(\text{AREA}((\ell,b),\ell'))$, we say that P *lies along* ℓ if $\text{P}\in\text{BORDER}(\text{AREA}(\ell\rightarrow))$, hence if P has a neighbour point (integer point in the first quadrant) strictly above ℓ; P *lies along* ℓ' if $\text{P}\in\text{BORDER}(\text{AREA}(\leftarrow\ell'))$. If $\text{P}\in\text{BORDER}(\text{AREA}((\ell,b),\ell'))$ does not lie along ℓ nor along ℓ', then we say that P *lies along* $\ell^{\perp}_{b}$.

D-sizes of vectors related to $\gamma\in\langle 0,\infty\rangle$**,** γ**-d-size and co-**γ**-d-size.** (See Figure 11b.) For $\gamma\in\langle 0,\infty\rangle$, the *$\gamma$-d-size of* v and the *co-γ-d-size of* v are the following real numbers. If $\text{START}(v)=\text{END}(v)$, then both γ-d-size and co-γ-d-size of v are 0. If $\text{START}(v)\neq\text{END}(v)$, then we consider the γ-line ℓ going through $\text{START}(v)$, and the line ℓ' that is perpendicular to ℓ and goes through $\text{END}(v)$. The γ-d-size of v is $\text{DIST}(\text{START}(v),\ell')$ (hence nonnegative) if $\text{SLOPE}(\ell')\neq\infty$ (hence $\gamma=\text{SLOPE}(\ell)>0$) and $\text{START}(v)$ is below ℓ', or if $\text{SLOPE}(\ell')=\infty$ and $\text{START}(v)$ is to the left of ℓ'; the γ-d-size of v is $-\text{DIST}(\text{START}(v),\ell')$ (hence nonpositive) otherwise. The co-γ-d-size of v is $\text{DIST}(\text{END}(v),\ell)$ (hence nonnegative) if $\text{SLOPE}(\ell)\neq\infty$ (hence $0\leq\gamma=\text{SLOPE}(\ell)<\infty$) and $\text{END}(v)$ is above ℓ, or if $\text{SLOPE}(\ell)=\infty$ and $\text{END}(v)$ is to the left of ℓ; the co-γ-d-size of v is $-\text{DIST}(\text{END}(v),\ell)$ (hence nonpositive) otherwise. (If $\gamma=\text{SLOPE}(v)$, then the γ-d-size of v coincides with the previously defined d-size of v.)

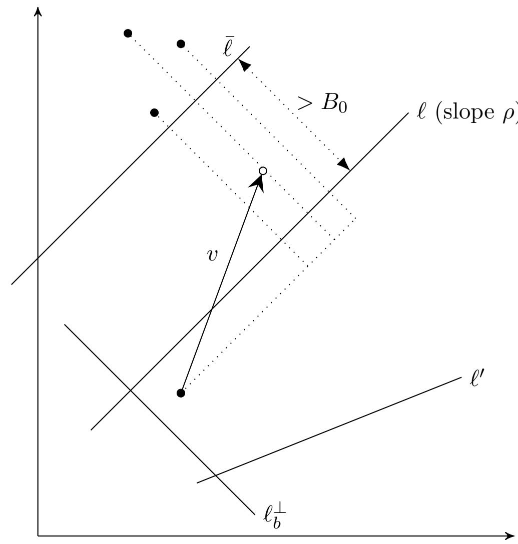
The following crucial technical proposition is illustrated in Figure 13. It will be used several times later to bound the up-limits under the described circumstances.

Proposition 5.7 (Bounding the up-limits β_C for a class $\mathcal{C}$ of colourings)**.**
We assume a nonempty set $\mathcal{C}$ of colourings (from the set $\{C_{\langle p,q\rangle} \mid p,q\in Q\}$), two lines ℓ,ℓ' and a level $b\in\mathbb{R}_{\geq 0}$ such that the following conditions are satisfied (cf. Figure 13a):

(1) $\infty\geq\text{SLOPE}(\ell)\geq\text{SLOPE}(\ell')\geq 0$*, and* $\text{AREA}((\ell,b),\ell')$ *has no point on the vertical axis;*
(2) *each colouring $C\notin\mathcal{C}$ is monochromatic (all-black or all-white) inside* $\text{AREA}((\ell,b),\ell')$*;*
(3) *for each $C\in\mathcal{C}$:*
    (a) *the points in* $\text{BORDER}(\text{AREA}((\ell,b),\ell'))$ *that lie along ℓ' are white in C,*



(A) Assumptions of Proposition 5.7.



(B) B_0 in the proof yields $\bar{\ell}$.

FIGURE 13

(b) *we can fix a point P_C such that $C(P_C)$ = black, P_C is an interior point of* $\text{AREA}((\ell,b),\ell')$*, and* $\text{DIST}(\mathrm{P}_C,\ell) \geq \text{BD}_C$ *where we put*

$$\text{BD}_C = \sup\{\,\text{DIST}(\mathrm{P},\ell) \mid \mathrm{P} \in \text{BORDER}(\text{AREA}((\ell,b),\ell')), C(\mathrm{P}) = \text{black}\}.$$

(Hence there is no point in $\text{BORDER}(\text{AREA}((\ell,b),\ell'))$ *that is black in C and has a larger distance from ℓ than P_C.)*

Then there is a line $\bar{\ell}$ with $\text{SLOPE}(\bar{\ell}) = \text{SLOPE}(\ell)$ *such that all points above $\bar{\ell}$ are black in each $C \in \mathcal{C}$.*

Before proving the proposition, we note its trivial corollary:

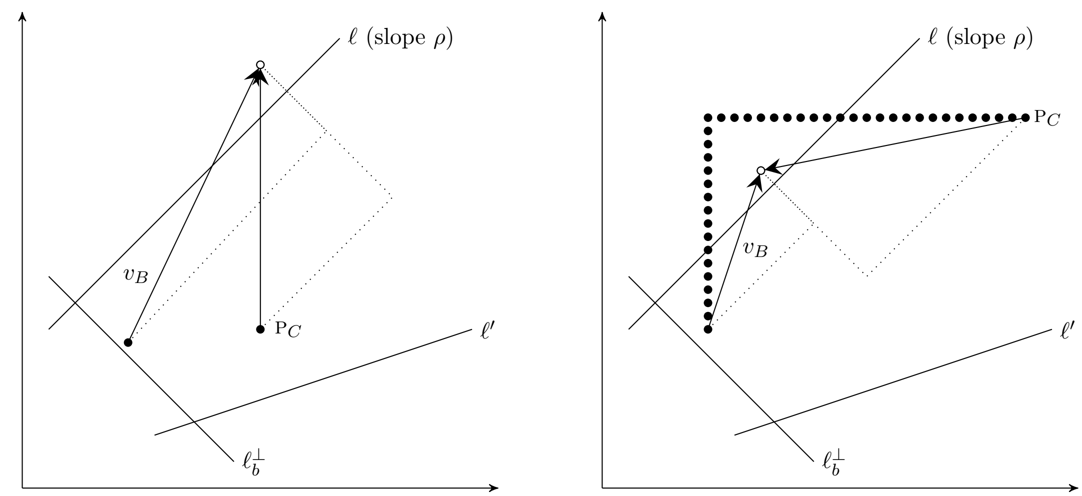
Corollary 5.8. *Under the conditions in Proposition 5.7,* $\text{SLOPE}(\ell) \geq \beta_C$ *for each $C \in \mathcal{C}$.*

Proof. To prove Proposition 5.7, we let its assumptions hold, and we put $\rho = \text{SLOPE}(\ell)$. If $\rho = \infty$, then the claim is trivial: there are no points above, i.e. left of, any vertical line $\bar{\ell}$ that does not intersect the first quadrant; such $\bar{\ell}$ trivially satisfies the claim. Hence we further assume

$$\infty > \rho = \text{SLOPE}(\ell) \geq \text{SLOPE}(\ell') \geq 0.$$

We also note that for almost all $m \in \mathbb{N}$ there is n such that $(m,n) \in \text{AREA}((\ell,b),\ell')$; this is obvious if $\text{SLOPE}(\ell) > \text{SLOPE}(\ell')$, and in the case $\rho = \text{SLOPE}(\ell) = \text{SLOPE}(\ell')$ it is guaranteed by the existence of interior points of $\text{AREA}((\ell,b),\ell')$ (they exist due to the points P_C): if $\mathrm{P} = (m,n)$ is interior in $\text{AREA}((\ell,b),\ell')$, then we have $(m,n{+}1) \in \text{AREA}((\ell,b),\ell')$ and for any $m' \geq m$ there is n' such that (m',n') lies below ℓ and above the parallel ρ-line going through P.

We call a *vector v eligible* if $\text{START}(v) \in \text{AREA}((\ell,b),\ell')$, v is black-white in some colouring, and both the ρ-d-size of v and the co-ρ-d-size of v are nonnegative. (We note that $\text{END}(v)$ can be outside $\text{AREA}((\ell,b),\ell')$, but then it necessarily lies strictly above ℓ.) E.g., v in Figure 13b is eligible.



(A) ρ-d-size of $(\mathrm{P}_C, \textsc{end}(v_B))$ is nonnegative. (B) ρ-d-size of $(\mathrm{P}_C, \textsc{end}(v_B))$ is negative.

Figure 14

To finish the proof, it suffices to show that the co-ρ-d-sizes of eligible vectors are bounded by some $B_0 \in \mathbb{R}_{\geq 0}$; the claimed line $\bar{\ell}$ then surely exists (as is illustrated in Figure 13b): we can take $\bar{\ell}$ above ℓ in the distance (to ℓ) greater than B_0 so that, moreover, $\bar{\ell}$ intersects V-AXIS above the vertical coordinates of all P_C, $C \in \mathcal{C}$.

For the sake of contradiction, we assume that the co-ρ-d-sizes of eligible vectors are not bounded; hence for every $B \in \mathbb{R}_{\geq 0}$ the set

$$E_B = \{\, v \mid v \text{ is eligible and the co-}\rho\text{-d-size of } v \text{ is greater than } B\,\}$$

is nonempty. We first note that there is some $B' \in \mathbb{R}_{\geq 0}$ such that for each $B \geq B'$ and each $v \in E_B$ we have that v is not black-white in any $C \notin \mathcal{C}$ (and thus must be black-white in some $C \in \mathcal{C}$).

We verify this claim by considering a fixed colouring $C \notin \mathcal{C}$, which is monochromatic in $\textsc{area}((\ell, b), \ell')$, and by noting that for each $B \in \mathbb{R}_{\geq 0}$ and each $v \in E_B$ we have:

- If C is all-white in $\textsc{area}((\ell, b), \ell')$, then $C(\textsc{start}(v)) = \text{white}$ (and thus v is not black-white in C).
- If C is all-black in $\textsc{area}((\ell, b), \ell')$ (and thus $C(\textsc{start}(v)) = \text{black}$), and $C(\textsc{end}(v)) = \text{white}$, then $\textsc{end}(v)$ is outside $\textsc{area}((\ell, b), \ell')$, hence strictly above ℓ; let us consider this case, where $\textsc{end}(v) = (m, n)$. Then $C(m, n') = \text{white}$ for all $n' \leq n$ (by Proposition 5.1). This entails that for this m there is no n' such that $(m, n') \in \textsc{area}((\ell, b), \ell')$. As discussed above, this can be the case only for finitely many m, which entails that there are only finitely many eligible vectors that are black-white in some $C \notin \mathcal{C}$. Hence the existence of B' is clear.

Now for each $B \geq B'$ we fix a vector $v_B \in E_B$ with the least possible rank of its white end; thus v_B is black-white in some $C_{\langle p,q \rangle} \in \mathcal{C}$, $\textsc{end}(v_B) = (m, n)$, and $\textsc{rank}(p(m), q(n))$ is the least possible, when considering all $v \in E_B$ and all $C \in \mathcal{C}$. Since $\textsc{start}(v_B)$ is in $\textsc{area}((\ell, b), \ell')$, it is not on the vertical axis, and thus Proposition 5.6 entails that $\textsc{start}(v_B) \in \textsc{border}(\textsc{area}((\ell, b), \ell'))$ (otherwise v_B has a neighbour vector that is also in E_B and its white end has a lesser rank).

For any fixed $B \geq B'$, by 3(b) we thus have $\text{DIST}(\text{START}(v_B), \ell) \leq \text{DIST}(\text{P}_C, \ell)$ for the respective colouring $C = C_{\langle p,q \rangle} \in \mathcal{C}$ (in which $\text{END}(v_B)$ has the least possible rank); the vector $(\text{P}_C, \text{END}(v_B))$, which is also black-white in C, thus has the co-ρ-d-size also greater than B (since it is not smaller than the co-ρ-d-size of v_B). If the vector $(\text{P}_C, \text{END}(v_B))$ were eligible (as is depicted in Figure 14a), thus belonging to E_B, Proposition 5.6 would yield a contradiction (since P_C is an interior point of $\text{AREA}((\ell, b), \ell')$, the vector $(\text{P}_C, \text{END}(v_B))$ would have a neighbour vector that is also in E_B and its white end has a lesser rank). Hence the ρ-d-size of $(\text{P}_C, \text{END}(v_B))$ is negative (as depicted in Figure 14b); this entails that $\rho = \text{SLOPE}(\ell) > 0$ (since in the case $\rho = 0$ the fact $C(\text{P}_C) = \text{black}$ would entail $C(\text{END}(v_B)) = \text{black}$, by Proposition 5.1, which contradicts the assumption $C(\text{END}(v_B)) = \text{white}$).

The fact that $\rho = \text{SLOPE}(\ell) > 0$ and the ρ-d-size of $(\text{P}_C, \text{END}(v_B))$ is negative entails that $\text{END}(v_B)$ is strictly below the line that is perpendicular to ℓ and goes through P_C; moreover, $\text{END}(v_B)$ is below the horizontal line going through P_C since $C(\text{P}_C) = \text{black}$ and $C(\text{END}(v_B)) = \text{white}$ (see Figure 14b). There are thus only finitely many points that can be $\text{END}(v_B)$, independently of the chosen $B \geq B'$. In other words, the set $\{\text{END}(v_B) \mid B \geq B'\}$ is finite, which also entails that the set $\{v_B \mid B \geq B'\}$ is finite (recall that v_B is eligible, and thus the ρ-d-size of v_B is nonnegative). This contradicts our choice of v_B that entails that the co-ρ-d-size of v_B is greater than B, for each $B \in \mathbb{R}_{\geq 0}$.

The co-ρ-d-sizes of eligible vectors are thus indeed bounded by some $B_0 \in \mathbb{R}_{\geq 0}$. □

Besides Corollary 5.8, which is trivial, we also derive the next corollary, showing that if the assumptions of Proposition 5.7 are slightly strengthened, then we get a strict upper bound on the up-limits β_C.

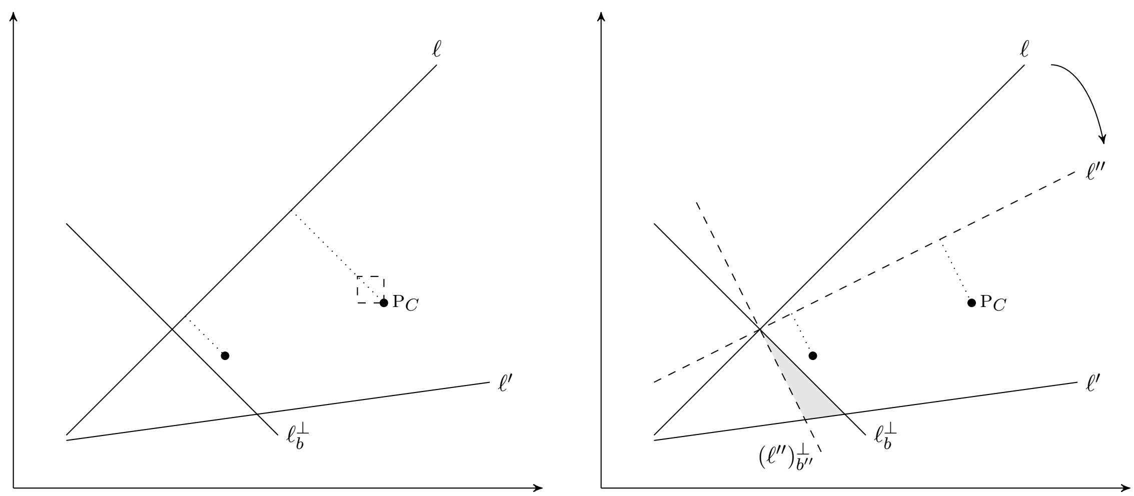
Corollary 5.9 (Bounding the up-limits β_C strictly)**.**
If $\mathcal{C}, \ell, \ell', b$ satisfy the conditions of Proposition 5.7 and, moreover, $\text{SLOPE}(\ell) > \text{SLOPE}(\ell')$ and for each $C \in \mathcal{C}$ we can choose P_C (which is an interior point of $\text{AREA}((\ell, b), \ell')$) so that it has no neighbour point on ℓ and the inequality in 3(b) is strict ($\text{DIST}(\text{P}_C, \ell) > \text{BD}_C$), then $\text{SLOPE}(\ell) > \beta_C$ for each $C \in \mathcal{C}$.

Proof. Let the described conditions be satisfied (in fact, such a situation is depicted already in Figure 13a; Figure 15a makes clear that all neighbour points of P_C are strictly below ℓ).

Since $\rho = \text{SLOPE}(\ell) > \text{SLOPE}(\ell') \geq 0$, we can very slightly rotate ℓ to the right around the intersection-point of ℓ and $\ell_b^{\perp}$ (see Figure 15b), by which we get ℓ'' with a slope ρ'', $\rho > \rho'' \geq \text{SLOPE}(\ell')$, so that the resulting $\text{AREA}((\ell'', b''), \ell')$ ($\ell_b^{\perp}$ has rotated to $(\ell'')_{b''}^{\perp}$) is a subset of $\text{AREA}((\ell, b), \ell')$ (in other words, the rotation of $\ell_b^{\perp}$ to $(\ell'')_{b''}^{\perp}$ is so small that there is no [integer] point in $\text{AREA}((\ell'', b''), \ell') \smallsetminus \text{AREA}((\ell, b), \ell')$, i.e. in the grey triangle in Figure 15b) and so that the original points P_C are interior points also in $\text{AREA}((\ell'', b''), \ell')$ while the conditions of Proposition 5.7 (in particular $\text{DIST}(\text{P}_C, \ell) \geq \text{BD}_C$) are kept. (To be very precise, in the case where $\text{SLOPE}(\ell) = \infty$ and $\text{SLOPE}(\ell') = 0$ we recall our stipulation that $\ell_b^{\perp}$ coincides with ℓ'; the discussed slight rotation then trivially has the property that there is no point in $\text{AREA}((\ell'', b''), \ell') \smallsetminus \text{AREA}((\ell, b), \ell')$.) Hence Proposition 5.7 entails that $\rho'' \geq \beta_C$, and thus $\rho > \beta_C$ (since $\rho > \rho''$). □

Proof of Lemma 5.5. We denote the down-limits, i.e. the elements of the set $\{\alpha_C \mid C = C_{\langle p,q \rangle}, (p,q) \in Q \times Q\}$) by $\alpha_1, \alpha_2, \cdots, \alpha_k$ ($k \leq |Q|^2$) so that

$$0 \leq \alpha_1 < \alpha_2 \cdots < \alpha_k \leq \infty. \tag{5.1}$$



(A) Assumptions of Corollary 5.9 (the unit square shows that P_C has no neighbour on ℓ).

(B) Rotation is so slight that the grey area contains no integer points, except possibly on $\ell_b^\perp$.

FIGURE 15

We now assume that the statement of Lemma 5.5 holds for the colourings related to $\alpha_1, \alpha_2, \dots, \alpha_{i-1}$ ($i \in \{1, 2, \dots, k\}$), and prove the statement for the α_i-colourings. The proof is given by Claims 5.10 – 5.15.

Claim 5.10. For each α_i-colouring C we have $\beta_C = \alpha_i$ (hence the down-limit $\alpha_C = \alpha_i$ and the up-limit β_C coincide).

Proof. If $\alpha_i = \infty$, then the claim is trivial (since $\infty \geq \beta_C \geq \alpha_C$, by Proposition 5.4); hence we assume $\infty > \alpha_i$. We fix some slope ρ where $\infty > \rho > \alpha_i$ and $\alpha_{i+1} > \rho$ if $i < k$. If $\alpha_i = 0$, then we put $\rho' = 0$, and if $\alpha_i > 0$, then we fix ρ' so that $\alpha_i > \rho' > 0$ and $\rho' > \alpha_{i-1}$ if $i > 1$. We fix a ρ-line ℓ and a ρ'-line ℓ', both going through the origin $(0, 0)$. (We can recall ℓ and ℓ' from Figure 13a, and imagine that they go through $(0, 0)$.) By $\mathcal{C}$ we denote the set of all α_i-colourings, and note:

- for each $C \in \mathcal{C}$ (where $\alpha_C = \alpha_i$), there are infinitely many black points in $\textsc{area}((\ell, 0), \ell')$, since the ρ-line ℓ must be down-black in C due to the fact $\rho > \alpha_i = \alpha_C$ (recall Proposition 5.3), but almost all border points of $\textsc{area}((\ell, 0), \ell')$ along ℓ' (i.e. those having neighbours strictly below ℓ') are white: if $\alpha_i = \alpha_C > \rho'$, then no ρ'-line can be down-black (in particular, no shift of ℓ' to the left can be down-black), and if $\alpha_i = \rho' = 0$, then ℓ' is the horizontal axis and almost all points on ℓ' are interior, having no neighbours (i.e. neighbour integer points in the first quadrant) outside $\textsc{area}((\ell, 0), \ell')$, since $\rho = \textsc{slope}(\ell) > 0$;
- for each $C \notin \mathcal{C}$ (where $\alpha_C \neq \alpha_i$) we have:
  - if $\alpha_C > \alpha_i$, hence $\alpha_C = \alpha_j$ for some $j > i$, then only finitely many points of $\textsc{area}((\ell, 0), \ell')$ are black in C (since $\alpha_C > \rho = \textsc{slope}(\ell)$, and thus ℓ cannot be down-black in C);
  - if $\alpha_C < \alpha_i$, hence $\alpha_C = \alpha_j$ for some $j < i$, then only finitely many points of $\textsc{area}((\ell, 0), \ell')$ are white in C (here $\alpha_i > \rho' = \textsc{slope}(\ell') > \alpha_C = \alpha_j$, and we use that $\beta_C = \alpha_C = \alpha_j$ by the induction hypothesis of our proof of Lemma 5.5, which entails that ℓ' is not up-white in C).

Hence we can choose a sufficiently large level $b \in \mathbb{R}_{\geq 0}$ so that the set $\mathcal{C}$ (of α_i-colourings) and ℓ, ℓ', b satisfy the conditions 1, 2, and 3(a) of Proposition 5.7; in particular, the value

$$\text{BD}_C = \sup \{ \text{DIST}(\text{P}, \ell) \mid \text{P} \in \text{BORDER}(\text{AREA}((\ell, b), \ell')), C(\text{P}) = \text{black} \}$$

is finite for each $C \in \mathcal{C}$.

Since $\text{SLOPE}(\ell) = \rho > \alpha_i$, we can fix some ρ'' satisfying $\rho > \rho'' > \alpha_i$ and also fix the ρ''-line ℓ'' going through $(0,0)$. The line ℓ'' is down-black in all $C \in \mathcal{C}$ (by Proposition 5.3) and the fact $\text{SLOPE}(\ell) > \text{SLOPE}(\ell'') > \text{SLOPE}(\ell')$ entails that for each $C \in \mathcal{C}$ there are infinitely many points in $\text{AREA}((\ell, b), \ell')$ that are below ℓ'' and black in C. Hence for each $C \in \mathcal{C}$ the value $\sup \{ \text{DIST}(\text{P}, \ell) \mid \text{P} \in \text{AREA}((\ell, b), \ell'), C(\text{P}) = \text{black}\}$ is infinite (in each $C \in \mathcal{C}$, the distance of the black points in $\text{AREA}((\ell, b), \ell')$ to ℓ increases above any bound when the level increases).

Thus the condition 3(b) of Proposition 5.7 is also satisfied (for each $C \in \mathcal{C}$, the point P_C can be chosen in a bigger distance from ℓ than the finite value BD_C, as is also depicted in Figure 13a, so even the assumptions of Corollary 5.9 are satisfied); Proposition 5.7 thus entails that $\rho \geq \beta_C$ for each $C \in \mathcal{C}$. In fact, $\rho > \alpha_i$ could be chosen arbitrarily close to α_i; we have thus shown $\rho \geq \beta_C$ for each $\rho > \alpha_i$. By recalling that $\beta_C \geq \alpha_C$ (Proposition 5.4), we derive that $\beta_C = \alpha_C = \alpha_i$ for each α_i-colouring C. □

Claim 5.11. There is an α_i-line ℓ_R such that all points below ℓ_R are white in all α_i-colourings.

Proof. If $\alpha_i = 0$ (which can happen for $i = 1$), then the claim is trivial (we can consider a horizontal line ℓ_R strictly below H-AXIS, below which there are no points [i.e. integer points in the first quadrant]).

So we assume $\alpha_i > 0$ and we fix an α_i-line ℓ such that all points below ℓ are white in the maximum number of α_i-colourings. Let $\mathcal{C}$ be the set of remaining α_i-colourings; hence each α_i-line is down-black in each $C \in \mathcal{C}$ (which has not been so far excluded though $\alpha_C = \alpha_i$ is the down-limit of C). We assume $\mathcal{C} \neq \emptyset$ (otherwise we are done by putting $\ell_R = \ell$), and show a contradiction.

Each $C \in \mathcal{C}$ has black points below ℓ in unbounded distances from ℓ (since each α_i-line is down-black in each $C \in \mathcal{C}$). We fix some ℓ' such that $\alpha_i = \text{SLOPE}(\ell) > \text{SLOPE}(\ell') \geq 0$, and $\text{SLOPE}(\ell') > \alpha_{i-1}$ if $i > 1$. (We can again look at Figure 13a.) It is now a routine to verify that there is some $b \in \mathbb{R}_{\geq 0}$ such that $\mathcal{C}$ and ℓ, ℓ', b satisfy the assumptions of Proposition 5.7, and even the assumptions of Corollary 5.9. This entails that $\text{SLOPE}(\ell) = \alpha_i > \beta_C$, for each $C \in \mathcal{C}$. But since $\mathcal{C}$ is a set of α_i-colourings (hence $\alpha_C = \alpha_i$ for each $C \in \mathcal{C}$), we must have $\beta_C \geq \alpha_i$, for each $C \in \mathcal{C}$ (by Proposition 5.4). Hence $\mathcal{C} = \emptyset$ after all. □

We note that ℓ_R from Claim 5.11 can be shifted to the right, which entails a useful corollary (for which we recall that $\text{AREA}(\leftarrow\ell) = \text{AREA}((\text{V-AXIS}, 0), \ell)$ and that all border points of $\text{AREA}(\leftarrow\ell)$ lie along ℓ):

Corollary 5.12. *There is an α_i-line ℓ_R such that all border points of* $\text{AREA}(\leftarrow\ell_R)$ *are white in all α_i-colourings.*

We note that the claim also comprises the case $\alpha_i = 0$ with ℓ_R being below H-AXIS, in which case $\text{AREA}(\leftarrow\ell_R)$ has no border points at all.

Now we aim to show that there is an α_i-line ℓ_L such that all points above ℓ_L are black in all α_i-colourings, and that α_i is rational or ∞. (By this the proof of Lemma 5.5 will be finished.) Since the case $\alpha_i = \infty$ is trivial, we further assume that $\infty > \alpha_i$.

We will use the notion of the rightmost down-black α_i-lines ℓ_C for α_i-colourings C, after which we again apply Proposition 5.7. But, in fact, we can not a priori assume that the rightmost down-black α_i-lines exist and that α_i is rational, so we partition and analyse the class $\mathcal{C}$ of α_i-colourings without any such assumptions. We define the line ℓ_C as the "right-hand limit" for the respective down-black lines, if some α_i-lines that are down-black in C exist, which does not exclude the case that ℓ_C itself is not down-black.

Partition of α_i-colourings by rightmost down-black lines ℓ_C. Let $\mathcal{C}$ be the set of all α_i-colourings. For technical convenience we use a fixed α_i-line ℓ_R guaranteed by Claim 5.11 (or Corollary 5.12), to define the partition of $\mathcal{C}$ into the following classes:

- $\mathcal{C}_1$ contains each $C \in \mathcal{C}$ for which there is an α_i-line that is down-black in C; for each $C \in \mathcal{C}_1$ we define the *rightmost down-black line* ℓ_C as the α_i-line whose distance from ℓ_R is the infimum of distances of the α_i-lines that are down-black in C. We further partition $\mathcal{C}_1$ as follows:
  - $\mathcal{C}_{11}$ contains all $C \in \mathcal{C}_1$ for which ℓ_C is down-black;
  - $\mathcal{C}_{12}$ contains all $C \in \mathcal{C}_1$ for which ℓ_C is not down-black.
- $\mathcal{C}_2 = \mathcal{C} \smallsetminus \mathcal{C}_1$; hence for each $C \in \mathcal{C}_2$ and each α_i-line ℓ only finitely many points below ℓ are black in C.

We now show that there is a desired "left-bound-line" for all $C \in \mathcal{C}_1$, by Claim 5.14 that is preceded by a simple auxiliary fact in Claim 5.13; it turns out that $\mathcal{C}_{12}$ is empty, while we still do not exclude that $\mathcal{C}_{11}$ is empty as well. Then we show that $\mathcal{C}_2$ is empty (by Claim 5.15); hence the (nonempty) set $\mathcal{C}$ of α_i-colourings is equal to $\mathcal{C}_{11}$, and Claim 5.14 thus finishes the proof of Lemma 5.5 (including the fact that α_i is rational or ∞).

Claim 5.13 (auxiliary)**.**

(1) If $\gamma \in \mathbb{R}_{\geq 0}$ is rational, then for any γ-line ℓ we have:
  (a) ℓ either contains infinitely many integer points (in $\mathbb{Z} \times \mathbb{Z}$), or no integer point;
  (b) there exists the least value $\mu \in \mathbb{R}_{>0}$ such that $\mu = \textsc{dist}(\textsc{p}, \ell)$ for some integer point $\textsc{p}$ that is not on ℓ.

(2) If $\gamma \in \mathbb{R}_{>0}$ is irrational, then for each $B \in \mathbb{R}_{>0}$ there is a (tiny) real number $\mu > 0$ such that for any two γ-lines ℓ, ℓ' whose (Euclidean) distance is μ we have that the distance of any two different integer points lying between the lines ℓ and ℓ' is larger than B.

Proof. 1. For $\gamma = 0$ both parts of the claim are obvious; so we assume that $\gamma = \frac{\Delta_y}{\Delta_x}$ for $\Delta_x, \Delta_y \in \mathbb{N}_{>0}$, and consider a γ-line ℓ. We note that if an integer point $(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}$ is on ℓ, then the integer points $(z_1 + i \cdot \Delta_x, z_2 + i \cdot \Delta_y)$, for all $i \in \mathbb{Z}$, are on ℓ as well. Hence 1(a) is clear. Now we note that for any integer point $\textsc{p} = (z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}$ outside ℓ we have that $\textsc{dist}(\textsc{p}, \ell) = \textsc{dist}(\textsc{p}', \ell)$ for all (integer) points $\textsc{p}' = (z_1 + i \cdot \Delta_x, z_2 + i \cdot \Delta_y)$ where $i \in \mathbb{Z}$. Hence μ claimed in 1(b) is the least value in the set

$$\{ \textsc{dist}(\textsc{p}, \ell) \mid \textsc{p} = (z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}, \textsc{p} \notin \ell, 0 \leq z_2 \leq \Delta_y \},$$

which obviously exists (since it exists for each fixed $z_2 \in \{0, 1, \dots, \Delta_y\}$).

2. We fix an irrational $\gamma \in \mathbb{R}_{>0}$ and some $B \in \mathbb{R}_{>0}$. We note that for each vector $v = (\textsc{p}, \textsc{p}')$ where $\textsc{p}, \textsc{p}'$ are two different integer points and $\textsc{dist}(\textsc{p}, \textsc{p}') \leq B$ (by $\textsc{dist}(\textsc{p}, \textsc{p}')$ we refer to the Euclidean distance of points) we have that the absolute value of the co-γ-d-size of v is greater than zero (since $\textsc{slope}(v)$ is rational and thus differs from γ). Moreover, the

set

$$M_{\gamma,B} = \left\{ \mu' \;\middle|\; \begin{array}{l} \mu' \text{ is the absolute value of the co-}\gamma\text{-d-size of some } (\mathrm{P}, \mathrm{P}') \\ \text{where } \mathrm{P}, \mathrm{P}' \text{ are two different integer points and } \mathrm{DIST}(\mathrm{P}, \mathrm{P}') \leq B \end{array} \right\}$$

is obviously finite. We can thus choose μ satisfying $0 < \mu < \min M_{\gamma,B}$; such μ satisfies the claim. Indeed: if there were two γ-lines ℓ, ℓ' whose (Euclidean) distance is μ and two different integer points P, P' lying between ℓ and ℓ' and satisfying $\mathrm{DIST}(\mathrm{P}, \mathrm{P}') \leq B$, then the absolute value of the co-γ-d-size of the vector $(\mathrm{P}, \mathrm{P}')$ would belong to $M_{\gamma,B}$ and would be not bigger than μ — a contradiction. □

Claim 5.14. If $\mathcal{C}_1 \neq \emptyset$, then there is an α_i-line ℓ'_L such that all points above ℓ'_L are black in all colourings from $\mathcal{C}_1$; moreover, α_i is rational and $\mathcal{C}_{12}$ is empty. (We recall that $\infty > \alpha_i$, as stipulated after Corollary 5.12.)

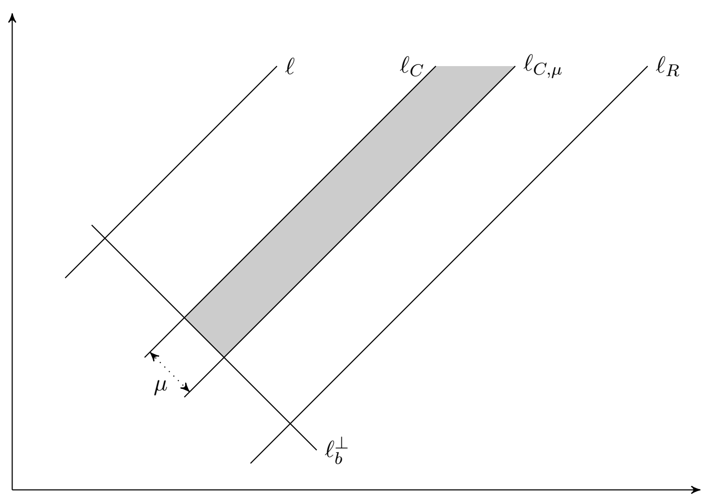
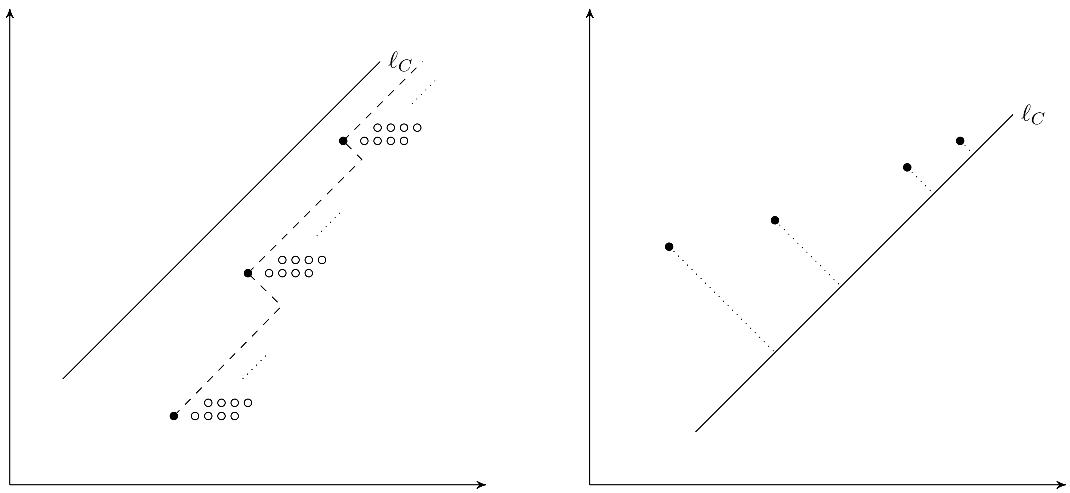


FIGURE 16. The situation considered in the proof of Claim 5.14.

Proof. We fix an α_i-line ℓ_R so that all points in BORDER(AREA($\leftarrow\ell_R$)) are white in each $C \in \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ (such ℓ_R exists by Corollary 5.12). We assume that $\mathcal{C}_1 \neq \emptyset$, and we fix an α_i-line ℓ sufficiently above all rightmost down-black lines ℓ_C, $C \in \mathcal{C}_1$ (defined after Corollary 5.12), so that AREA($(\ell, 0), \ell_C$) has infinitely many interior points for each $C \in \mathcal{C}_1$. (See Figure 16.) Finally we choose (a sufficiently large) $b \in \mathbb{R}_{\geq 0}$ so that AREA($(\ell, b), \ell_R$) does not intersect with the vertical axis and is monochromatic in each colouring $C \notin \mathcal{C}_1$; we can easily verify that such b exists since we have:

- for each (α_i-colouring) $C \in \mathcal{C}_2$ and for each colouring C for which $\alpha_C > \alpha_i$ there are only finitely many points below the α_i-line ℓ that are black in C (and we choose b so that these colourings are all-white in AREA($(\ell, b), \ell_R$));
- for each colouring C for which $\alpha_C < \alpha_i$, hence $\alpha_C = \alpha_j$ for some $j \in \{0, 1, \ldots, i-1\}$, we have $\alpha_C = \beta_C$ by the induction hypothesis of our proof of Lemma 5.5, and there are thus only finitely many points above the α_i-line ℓ_R that are white in C (and we choose b so that these colourings are all-black in AREA($(\ell, b), \ell_R$)).

Figure 16 depicts AREA($(\ell, b), \ell_R$), and ℓ_C just for one $C \in \mathcal{C}_1$; let us ignore the additional dark-grey belt and the line $\ell_{C,\mu}$ for this moment.



(A) $C \in \mathcal{C}_{11}$. (B) $C \in \mathcal{C}_{12}$.

FIGURE 17. A closer look at ℓ_C when α_C is irrational.

Let us now *assume that* α_i (the slope of ℓ, ℓ_C, ℓ_R) *is rational.* By using Claim 5.13(1) we easily deduce that for each $C \in \mathcal{C}_1$ the rightmost down-black line ℓ_C must be down-black; moreover, it contains infinitely many points that are black in C, while only finitely many points strictly below ℓ_C can be black in C. (If ℓ_C was not down-black then also a line arising by a tiny shift of ℓ_C to the left, by less than μ in Claim 5.13(1b), would be not down-black; this would contradict the definition of ℓ_C. On the other hand, if there were infinitely many points strictly below ℓ_C that are black in C, then a tiny shift of ℓ_C to the right would give us a down-black line, which also contradicts the definition of ℓ_C.)

Hence if α_i is rational, then $\mathcal{C}_{12}$ is empty, and thus $\mathcal{C}_1 = \mathcal{C}_{11}$. Moreover, we can simply increase b so that for each $C \in \mathcal{C}_1$ all the infinitely many points in AREA$((\ell_C, b), \ell_R)$ that are black in C are on the line ℓ_C. We can thus apply Proposition 5.7 (depicted in Figure 13) to $\mathcal{C}_1, \ell, \ell_R, b$; for each $C \in \mathcal{C}_1$, the point P$_C$ can be chosen as one of the infinitely many points on ℓ_C that are black in C and interior in AREA$((\ell, b), \ell_R)$ (hence not close to $\ell_b^{\perp}$). Proposition 5.7 gives us the claimed ℓ'_L.

To finish the proof, we now *assume that* α_i *is irrational* (which entails that any α_i-line contains at most one integer point), and we will lead this assumption to a contradiction with the assumption $\mathcal{C}_1 \neq \emptyset$. Recalling Figure 16, we aim to increase b so that not only Proposition 5.7, but even Corollary 5.9, could be applied; this will entail that $\alpha_C > \beta_C$ for some $C \in \mathcal{C}_1$, which is the desired contradiction.

Now we cannot assume that $\mathcal{C}_{12} = \emptyset$, but the colourings $C \in \mathcal{C}_{12}$ (where ℓ_C are not down-black) create no problem for our aim: We can increase b so that for each $C \in \mathcal{C}_{12}$ there are no points in AREA$((\ell_C, b), \ell_R)$ that are black in C. By definition of ℓ_C, now for each point P in AREA$((\ell, b), \ell_R)$ that is black in some $C \in \mathcal{C}_{12}$ (hence P is strictly above ℓ_C) there are infinitely many interior points P$'$ in AREA$((\ell, b), \ell_R)$ that are black in C and for which DIST$(\mathrm{P}', \ell_C) <$ DIST(P, ℓ_C), hence DIST$(\mathrm{P}, \ell) <$ DIST(P', ℓ). (Figure 17b aims to illustrate this, by depicting black points strictly above ℓ_C that "converge" to ℓ_C.) Hence for each $C \in \mathcal{C}_{12}$ and each sufficiently large b there surely is some P$_C$, $C($P$_C) =$ black, such that DIST$(\mathrm{P}, \ell) <$ DIST(P_C, ℓ) for all points P $\in$ BORDER(AREA$((\ell, b), \ell_R)$) that are black in C.

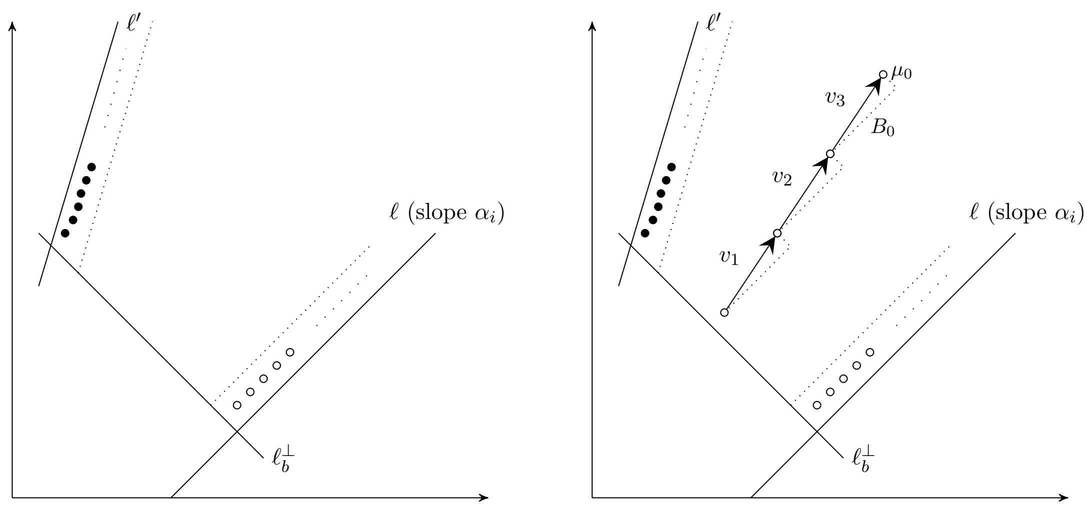
Now we look at the set $\mathcal{C}_{11}$, considering a fixed colouring $C \in \mathcal{C}_{11}$. By definition, the line ℓ_C (with the irrational slope α_i) is down-black in C; hence infinitely many points in $\textsc{area}((\ell_C, b), \ell_R)$ (recall Figure 16) are black in C, while there is at most one (integer) point on ℓ_C. By definition of ℓ_C, for any (however tiny) $\mu \in \mathbb{R}_{>0}$ there are only finitely many points in $\textsc{area}((\ell_{C,\mu}, b), \ell_R)$ that are black in C, where $\ell_{C,\mu}$ arises by shifting ℓ_C by μ to the right (see Figure 16); Figure 17a aims to illustrate this "converging" of black points to ℓ_C from below. Hence for any chosen $\mu \in \mathbb{R}_{>0}$ we can surely increase b so that all the infinitely many points in $\textsc{area}((\ell_C, b), \ell_R)$ that are black in C are in $\textsc{area}((\ell_C, b), \ell_{C,\mu})$ (i.e., in the grey belt between ℓ_C and $\ell_{C,\mu}$ in Figure 16).

If we manage to choose the above discussed μ and b so that for each $C \in \mathcal{C}_{11}$ there is no border point of $\textsc{area}((\ell, b), \ell_R)$ that is below ℓ_C and black in C (such a point could only be in the grey belt, i.e. in $\textsc{area}((\ell_C, b), \ell_{C,\mu})$, close to $\ell_b^\perp$), then we can surely choose the points $\textsc{p}_C$, required by Corollary 5.9 for $\mathcal{C}_1, \ell, \ell_R, b$, not only for all $C \in \mathcal{C}_{12}$ but also for all $C \in \mathcal{C}_{11}$ (for each $C \in \mathcal{C}_{11}$ we let $\textsc{p}_C$ be one of the infinitely points in the grey belt that are black in C). But this yields a contradiction since Corollary 5.9 in this case entails that $\textsc{slope}(\ell) = \alpha_i = \alpha_C > \beta_C$ for all $C \in \mathcal{C}_1$ (which contradicts $\beta_C \geq \alpha_C$ established by Proposition 5.4).

Now we show that we can indeed choose μ and b that yield the described contradiction. We can choose $\mu \in \mathbb{R}_{>0}$ sufficiently small so that any two different integer points in $\textsc{area}((\ell_C, b), \ell_{C,\mu})$ (i.e. in the grey belt) have very large distances, for any $C \in \mathcal{C}_{11}$. (This follows from Claim 5.13(2).) We can thus surely choose b so that there is no (integer) point that is a border point of $\textsc{area}((\ell, b), \ell_R)$ and lies in a grey belt (i.e. in $\textsc{area}((\ell_C, b), \ell_{C,\mu})$ for some $C \in \mathcal{C}_{11}$).

The assumptions that α_i is irrational and $\mathcal{C}_1 \neq \emptyset$ have thus yielded a contradiction. Hence the assumption $\mathcal{C}_1 \neq \emptyset$ entails that α_i is rational, $\mathcal{C}_{12} = \emptyset$ (hence $\mathcal{C}_1 = \mathcal{C}_{11}$), and there is the claimed line ℓ'_L. $\square$

Claim 5.15. $\mathcal{C}_2 = \emptyset$, and thus Claim 5.14 finishes the proof of Lemma 5.5.



(A) $\textsc{area}(\ell', (b, \ell))$ for proving Claim 5.15.

(B) An up-white slope larger than α_i.

Figure 18

Proof. For the sake of contradiction, we assume $\mathcal{C}_2 \neq \emptyset$, and recall that for each $C \in \mathcal{C}_2$ there is no α_i-line that is down-black in C (hence there are only finitely many points below any given α_i-line that are black in C). We also recall our assumption $\infty > \alpha_i$ (stipulated after Corollary 5.12).

Now we fix an α_i-line ℓ so that all points above ℓ are black in all $C \in \mathcal{C}_1$; the existence of such ℓ is trivial if $\mathcal{C}_1 = \emptyset$, and otherwise it is guaranteed by Claim 5.14. We also fix a line ℓ' where $\infty > \text{SLOPE}(\ell') > \text{SLOPE}(\ell) = \alpha_i$, and $\alpha_{i+1} > \text{SLOPE}(\ell')$ if $i < k$ (where k is the number of all down-limits, recall (5.1)); finally we fix some $b \in \mathbb{R}_{\geq 0}$ so that the following conditions are satisfied (see Figure 18a):

(1) $\text{AREA}(\ell', (b, \ell))$ is monochromatic for all $C \notin \mathcal{C}_2$;
(2) all border points of $\text{AREA}(\ell', (b, \ell))$ along ℓ' (having neighbours strictly above ℓ') are black in all $C \in \mathcal{C}_2$;
(3) all border points of $\text{AREA}(\ell', (b, \ell))$ along ℓ (having neighbours strictly below ℓ) are white in all $C \in \mathcal{C}_2$.

Any sufficiently large b will satisfy these conditions since we have:

- for each colouring C for which $\alpha_C > \alpha_i$, in which case $\alpha_C > \text{SLOPE}(\ell')$, there are only finitely many points below ℓ' that are black in C (and we choose b so that these colourings are all-white in $\text{AREA}(\ell', (b, \ell))$);
- for each (α_i-colouring) $C \in \mathcal{C}_1$ all points in $\text{AREA}(\ell', (b, \ell))$ are black in C by our choice of ℓ;
- for each colouring C for which $\alpha_C < \alpha_i$, in which case $\text{SLOPE}(\ell) > \alpha_C$ and $\alpha_C = \beta_C$ by the induction hypothesis of our proof of Lemma 5.5, there are only finitely many points above ℓ that are white in C (and we choose b so that these colourings are all-black in $\text{AREA}(\ell', (b, \ell))$);
- for each (α_i-colouring) $C \in \mathcal{C}_2$ we have $\alpha_i = \alpha_C = \beta_C$ (by Claim 5.10); hence above any line that is parallel with ℓ' (and whose slope is thus greater than β_C) there are only finitely many points that are white in C (and we can thus choose b so that the border points of $\text{AREA}(\ell', (b, \ell))$ along ℓ' are all black in all $C \in \mathcal{C}_2$);
- by definition of $\mathcal{C}_2$, for any $C \in \mathcal{C}_2$ and any α_i-line $\bar{\ell}$ (which is thus parallel with ℓ) there are only finitely many points below $\bar{\ell}$ that are black in C (and we can thus choose b so that the border points of $\text{AREA}(\ell', (b, \ell))$ along ℓ are all white in all $C \in \mathcal{C}_2$).

Let us now call a vector v to be a (B, μ)-*vector*, for $B \in \mathbb{R}_{\geq 0}$ and $\mu \in \mathbb{R}_{>0}$, if B is the α_i-d-size of v and μ is the co-α_i-d-size of v. We can see (looking at Figure 18 and recalling that each α_i-line is not down-black in any $C \in \mathcal{C}_2$) that we can fix some $\mu_0 > 0$ such that for any $B \in \mathbb{R}_{>0}$ there is $B_0 \geq B$ and a (B_0, μ_0)-vector v where $\text{START}(v) \in \text{BORDER}(\text{AREA}(\ell', (b, \ell)))$ and $C(\text{START}(v)) = C(\text{END}(v)) = \text{white}$ for some $C \in \mathcal{C}_2$ (which entails that $\text{END}(v)$ is in $\text{AREA}(\ell', (b, \ell))$). Moreover, we can even choose a sufficiently large B_0 so that

- there is a (B_0, μ_0)-vector v where $\text{START}(v) \in \text{BORDER}(\text{AREA}(\ell', (b, \ell)))$ and $C(\text{START}(v)) = C(\text{END}(v)) = \text{white}$ for some $C \in \mathcal{C}_2$;
- the slope of (B_0, μ_0)-vectors is strictly between $\text{SLOPE}(\ell')$ and $\text{SLOPE}(\ell)$;
- for each (B_0, μ_0)-vector v where $\text{START}(v) \in \text{BORDER}(\text{AREA}(\ell', (b, \ell)))$, and for each colouring C we have

$$\text{if } C(\text{START}(v)) = \text{white, then } C(\text{END}(v)) = \text{white.} \tag{5.2}$$

But then each (B_0, μ_0)-vector v with $\text{START}(v) \in \text{AREA}(\ell', (b, \ell))$ (not only with $\text{START}(v) \in \text{BORDER}(\text{AREA}(\ell', (b, \ell)))$) satisfies (5.2). Indeed, a violating (B_0, μ_0)-vector v with the least rank of its white start-point $\text{START}(v)$, where $\text{START}(v)$ is necessarily an interior point of $\text{AREA}(\ell', (b, \ell))$, leads to a contradiction by Proposition 5.6 applied to the opposite black-white vector (which would yield a violating (B_0, μ_0)-vector with a smaller rank of its start-point).

This entails that the slope of (B_0, μ_0)-vectors, which is larger than α_i, is up-white in each $C \in \mathcal{C}_2$ (consider the line containing white-white vectors $v_1, v_2, v_3, \ldots$ in Figure 18b); this is a contradiction since $\beta_C = \alpha_i$ for all $C \in \mathcal{C}_2$ (by Claim 5.10). Hence $\mathcal{C}_2 = \emptyset$ after all. □

5.2. **Quantitative belt theorem.** We have proven the belt theorem (Theorem 5.2), but we have not derived anything specific about the slopes, widths, and positions of the belts (one of which is depicted in Figure 10). Now we show that these parameters can be presented by polynomially bounded integers (in the size of the underlying one-counter net $\mathcal{N} = (Q, Act, \delta)$ that we have fixed).

We start with the *belt-slopes*, which is another name for the down-limits α_C (that coincide with the up-limits β_C by Lemma 5.5); we recall that we have ordered the set $\{\alpha_C \mid C = C_{\langle p,q\rangle}, (p, q) \in Q \times Q\}$, denoting its elements (in (5.1)) as

$$0 \leq \alpha_1 < \alpha_2 \cdots < \alpha_k \leq \infty. \tag{5.3}$$

Hence when we refer to a belt-slope α, we understand that $\alpha = \alpha_i$ for some $i \in [1, k]$. We can also recall that by an α-colouring we mean a colouring C ($C = C_{\langle p,q\rangle}$ for some $(p, q) \in Q \times Q$) for which $\alpha_C = \alpha$. Before clarifying that the belt-slopes are fractions of small integers (or ∞), which is shown by Proposition 5.17, we formulate a useful corollary of Proposition 5.6, which captures the black-white vector travel discussed around Figure 4 in Introduction. (The black-white vectors in Figure 4 are directed upwards but the corollary also handles the vectors directed downwards, like v_{i_1} in Figure 19.)

Corollary 5.16 (of Proposition 5.6)**.** *Any vector v_0 that is black-white in some C_0 gives rise to a sequence $v_0, v_1, \ldots, v_n$ where $\text{START}(v_n) \in \text{V-AXIS}$ or $\text{END}(v_n) \in \text{H-AXIS}$, and for $i = 0, 1, 2, \ldots, n-1$ we have that v_{i+1} is a neighbour vector of v_i that is black-white in some C_{i+1} and the rank of the white end of v_{i+1} in C_{i+1} is smaller than the rank of the white end of v_i in C_i.*

Proposition 5.17 (Belt slopes)**.** *If α, $\infty > \alpha > 0$, is a belt-slope, then $\alpha = \frac{\Delta_y}{\Delta_x}$ for some integers Δ_x, Δ_y from $\{1, 2, \cdots, |\mathcal{C}|\}$ where $\mathcal{C}$ is the set of α-colourings.*

Proof. Let $\mathcal{C}$ be the set of α-colourings, for some fixed belt-slope α where $\infty > \alpha > 0$, and let ℓ_L, ℓ_R be some α-lines such that all points above ℓ_L are black in all $C \in \mathcal{C}$ and all points below ℓ_R are white in all $C \in \mathcal{C}$; the belt theorem (Theorem 5.2) guarantees that such lines exist, and that α is rational.

By reasoning as in the proof of Claim 5.14 (when assuming that α_i is rational) we derive that for each $C \in \mathcal{C}$ there is the rightmost α-line ℓ_C that is down-black in C; we also recall that there are infinitely many points on ℓ_C that are black in C (but only finitely many points strictly below ℓ_C are black in C). Analogously we derive that there is the leftmost α-line ℓ'_C that is up-white in C; there are infinitely many points on ℓ'_C that are white in C. (The colourings depicted in Figure 3, related to the simple net in Figure 2, might suggest that ℓ'_C is below ℓ_C, but Figure 19 depicts a more typical situation.) By SPAN_C we denote the distance of ℓ_C and ℓ'_C, with the negative sign if ℓ'_C is below ℓ_C. We put

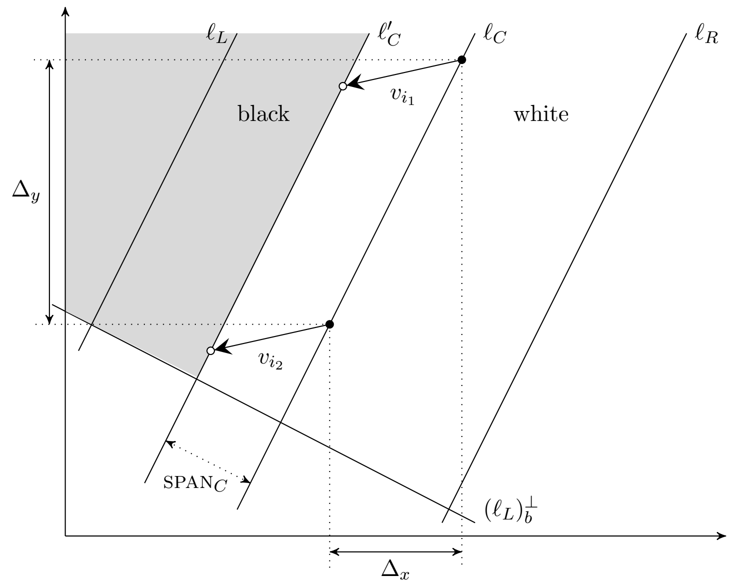


FIGURE 19. The vectors v_{i_1} and v_{i_2} imply $\alpha = \frac{\Delta_y}{\Delta_x}$ (where $\alpha = \text{SLOPE}(\ell_L)$).

$$s = \max \{ \text{SPAN}_C \mid C \in \mathcal{C} \}, \text{ and } \mathcal{C}_{\text{MS}} = \{ C \in \mathcal{C} \mid \text{SPAN}_C = s \}$$

(where MS refers to "Maximal Span").

Now we choose v_0 so that v_0 is black-white in some $C_0 \in \mathcal{C}_{\text{MS}}$, $\text{START}(v_0) \in \ell_{C_0}$, and $\text{END}(v_0) \in \ell'_{C_0}$; moreover, we choose v_0 sufficiently high in the respective α-belt so that each v_i in the prefix $v_0, v_1, v_2, \dots, v_{|\mathcal{C}_{\text{MS}}|}$ of the sequence guaranteed by Corollary 5.16 must necessarily satisfy that $\text{START}(v_i) \in \ell_{C_i}$ and $\text{END}(v_i) \in \ell'_{C_i}$ for some $C_i \in \mathcal{C}_{\text{MS}}$. Figure 19 depicts this "sufficiently high", where b is chosen so that

- for each $C \in \mathcal{C}$ the points strictly above ℓ'_C in $\text{AREA}(\text{V-AXIS}, (b, \ell'_C))$ are black in C and the points strictly below ℓ_C in $\text{AREA}((\ell_C, b), \text{H-AXIS})$ are white in C,
- and each colouring $C \notin \mathcal{C}$ (for which $\alpha_C \neq \alpha$) is monochromatic in $\text{AREA}((\ell_L, b), \ell_R)$.

By the pigeonhole principle, in the above discussed sequence $v_0, v_1, v_2, \dots, v_{|\mathcal{C}_{\text{MS}}|}$ we have some i_1, i_2 such that $0 \leq i_1 < i_2 \leq |\mathcal{C}_{\text{MS}}|$ and both $\text{START}(v_{i_1})$ and $\text{START}(v_{i_2})$ are on ℓ_C for the same $C \in \mathcal{C}_{\text{MS}}$ (and $\text{END}(v_{i_1})$ and $\text{END}(v_{i_2})$ are on ℓ'_C). Since $\text{SLOPE}(\ell_C) = \alpha$, and $\text{START}(v_{i_1})$, $\text{START}(v_{i_2})$ are two different points on ℓ_C, we deduce that α is the slope of the vector $(\text{START}(v_{i_1}), \text{START}(v_{i_2}))$. This entails that $\alpha = \frac{\Delta_y}{\Delta_x}$ where $\Delta_x, \Delta_y \in \{1, 2, \cdots, i_2 - i_1\}$; Δ_x is the absolute value of the 0-d-size [horizontal d-size] of the vector $(\text{START}(v_{i_1}), \text{START}(v_{i_2}))$, and Δ_y is the absolute value of its co-0-d-size [vertical d-size] (see Figure 19). □

To deal with the belt-widths and the belt positions, it is useful to define two quasi-orders on the set of colourings $\{C \mid C = C_{\langle p,q \rangle}, (p, q) \in Q \times Q\}$, based on the belt-slopes and the lines ℓ_L^C, ℓ_R^C defined below (which can differ from the lines ℓ_L, ℓ_R in Theorem 5.2 and ℓ'_C, ℓ_C used in the proof of Proposition 5.17). For technical reasons we also extend the notion of border points to *almost-border points* of particular areas.

Lines ℓ_L^C (leftmost with a white point) and ℓ_R^C (rightmost with a black point). For each colouring C we define ℓ_L^C as the leftmost α_C-line that contains a point that is white in C, and ℓ_R^C as the rightmost α_C-line that contains a point that is black in C; we put $\ell_L^C = \ell_R^C =$ H-AXIS if C is all-black (in which case $\alpha_C = 0$), and $\ell_L^C = \ell_R^C =$ V-AXIS if C is all-white (in which case $\alpha_C = \infty$). We note that Theorem 5.2 (depicted in Figure 10), which includes the fact that the belt-slopes α_C are rational or ∞, entails that the lines ℓ_L^C and ℓ_R^C are well-defined.

We again note that the simulation preorder depicted in Figure 3, related to our simple example in Figure 2, can give an impression that ℓ_L^C is typically below (to the right of) ℓ_R^C; in this case (whenever ℓ_L^C is below ℓ_R^C), the distance between ℓ_L^C and ℓ_R^C is clearly at most 1 (and the distance 1 is achieved in the colouring $C_{\langle p_2,p_1\rangle}$ in Figure 3). In more complicated examples, ℓ_L^C can be surely above (to the left of) ℓ_R^C; in Figure 19, ℓ_L^C would be between ℓ_L and ℓ'_C, and ℓ_R^C would be between ℓ_C and ℓ_R. In the case when ℓ_L^C is above ℓ_R^C the distance between ℓ_L^C and ℓ_R^C can be surely larger than 1; nevertheless it is polynomially bounded (in the size of Q of our fixed OCN $\mathcal{N} = (Q, Act, \delta)$), as we discuss below.

A counterclockwise quasi-order $\sqsubseteq_L$ and a clockwise quasi-order $\sqsubseteq_R$ on the set of colourings. The quasi-orders $\sqsubseteq_L$ and $\sqsubseteq_R$ on the set $\{ C \mid C = C_{\langle p,q\rangle}, (p,q) \in Q \times Q \}$ are defined as follows:

$$C \sqsubseteq_L C' \text{ if either } \alpha_C < \alpha_{C'}, \text{ or } \alpha_C = \alpha_{C'} \text{ and } \ell_L^C \text{ is below (to the right of) } \ell_L^{C'};$$

$$C \sqsubseteq_R C' \text{ if either } \alpha_C > \alpha_{C'}, \text{ or } \alpha_C = \alpha_{C'} \text{ and } \ell_R^C \text{ is above (to the left of) } \ell_R^{C'}.$$

By $C \sqsubset_L C'$ we denote that $C \sqsubseteq_L C'$ and $C' \not\sqsubseteq_L C$; the case $C \sqsubset_R C'$ is analogous.

We can recall the order $\alpha_1 < \alpha_2 \cdots < \alpha_k$ of belt-slopes (5.3). Hence $C \sqsubseteq_L C'$ iff $\alpha_C = \alpha_i$ and $\alpha_{C'} = \alpha_j$ where $i < j$, or $i = j$ and ℓ_L^C is below (to the right of) $\ell_L^{C'}$; analogously we can express $C \sqsubseteq_R C'$.

Almost-border points of AREA($\leftarrow\ell$) and AREA($\ell\rightarrow$). We say that P $\in$ AREA($\leftarrow\ell$) is an *almost-border point of* AREA($\leftarrow\ell$) if P has a neighbour point in AREA($\ell\rightarrow$) (hence P is a border point of AREA($\leftarrow\ell$) or has a neighbour point on ℓ). Similarly, P $\in$ AREA($\ell\rightarrow$) is an *almost-border point of* AREA($\ell\rightarrow$) if P has a neighbour point in AREA($\leftarrow\ell$).

The next proposition (Proposition 5.18) and its corollary (Corollary 5.19) show that the distances of lines ℓ_L^C and ℓ_R^C to the origin $(0,0)$ are "small", bounded by a polynomial in the number of colourings (hence in $|Q \times Q|$ for the underlying OCN $\mathcal{N} = (Q, Act, \delta)$); this also yields the announced polynomial bound on the distance between ℓ_L^C and ℓ_R^C (for any colouring $C = C_{\langle p,q\rangle}$).

Figure 20 (the notation on which will be explained below) aims to illustrate that ℓ_R^C cannot intersect H-AXIS in a large distance to the right from the origin $(0,0)$. This follows by Proposition 5.18(1) that entails that if the intersection of some ℓ_R^C with H-AXIS is to the right of $(0,0)$, then the distance of this intersection to $(0,0)$ is bounded by the product of a small number with the cardinality of the set $\{C' \mid C' \sqsubseteq_R C\}$ (as follows by a repeated use of Proposition 5.18(1)).

Proposition 5.18. *For each colouring C, and its respective lines ℓ_L^C and ℓ_R^C, we have:*

(1) *If $(0,0)$ is strictly above ℓ_R^C, then there is $C' \sqsubset_R C$ such that* AREA($\leftarrow\ell_R^{C'}$) *contains a point that is an almost-border point of* AREA($\leftarrow\ell_R^C$).

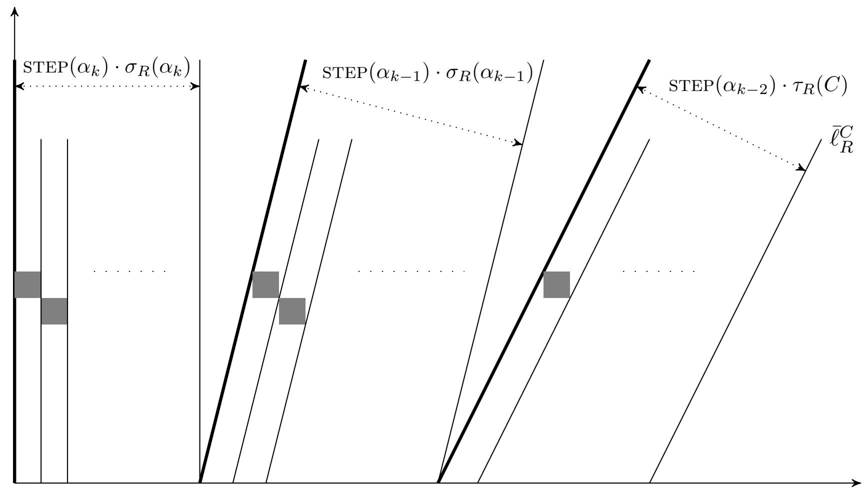


FIGURE 20. The actual ℓ_R^C is above (to the left of) the depicted $\bar{\ell}_R^C$; the grey squares have the size 1×1.

(2) *If* $(0,0)$ *is strictly below* ℓ_L^C, *then there is* $C' \sqsubset_L C$ *such that* $\text{AREA}(\ell_L^{C'}\!\rightarrow)$ *contains a point that is an almost-border point of* $\text{AREA}(\ell_L^C\!\rightarrow)$.

Proof. 1. Let us fix some C_0 such that $(0,0)$ is strictly above $\ell_R^{C_0}$ (see Figure 21); we will show that there is $C_0' \sqsubset_R C_0$ such that $\text{AREA}(\leftarrow\!\ell_R^{C_0'})$ contains a point that is an almost-border point of $\text{AREA}(\leftarrow\!\ell_R^{C_0})$.

We put $\alpha = \alpha_{C_0}$, and $\mathcal{C} = \{C \mid \alpha_C = \alpha \text{ and } C_0 \sqsubseteq_R C\}$. Let ℓ be the leftmost α-line that is up-white in some $C \in \mathcal{C}$; we fix such $\bar{C} \in \mathcal{C}$. Hence ℓ contains infinitely many points that are white in $\bar{C}$ but only finitely many points strictly above ℓ can be white in any colouring C such that $C_0 \sqsubseteq_R C$ (which includes all $C \in \mathcal{C}$). We can thus choose $b \in \mathbb{R}_{\geq 0}$ so that all points in $\text{AREA}(\text{V-AXIS}, (b, \ell))$ (i.e., in the grey area in Figure 21) that are strictly above ℓ are black in all C such that $C_0 \sqsubseteq_R C$ (hence those points can be white only in some $C \sqsubset_R C_0$).

In $\bar{C}$ we consider a black-white vector v_0 such that $\text{START}(v_0) \in \text{AREA}(\ell_R^{C_0}\!\rightarrow)$ (there must be a respective point that is black in $\bar{C}$ since $C_0 \sqsubseteq_R \bar{C}$) and $\text{END}(v_0) \in \ell$; we have infinitely many possibilities on ℓ for such $\text{END}(v_0)$ (see Figure 21).

We fix a sequence $v_0, v_1, \dots, v_n$ guaranteed by Corollary 5.16. There must be the least $i \in [1,n]$ such that $\text{START}(v_i)$ is strictly above $\ell_R^{C_0}$ (since $\text{START}(v_n)$ is on V-AXIS and thus strictly above $\ell_R^{C_0}$); hence $\text{START}(v_i)$ is an almost-border point of $\text{AREA}(\leftarrow\!\ell_R^{C_0})$ (see again Figure 21). We could choose v_0 with a sufficiently large α-d-size, so that $\text{END}(v_i)$ is necessarily in $\text{AREA}(\text{V-AXIS}, (b, \ell))$ (in the grey area in Figure 21); moreover, $\text{END}(v_i)$ is strictly above ℓ. Hence v_i is black-white in some $C_0' \sqsubset_R C_0$, which entails that $\text{START}(v_i)$ (which is an almost-border point of $\text{AREA}(\leftarrow\!\ell_R^{C_0})$) is in $\text{AREA}(\leftarrow\!\ell_R^{C_0'})$.

2. This condition is shown analogously as 1. Here we assume that $(0,0)$ is strictly below $\ell_L^{C_0}$, and put $\alpha = \alpha_{C_0}$ and $\mathcal{C} = \{C \mid \alpha_C = \alpha \text{ and } C_0 \sqsubseteq_L C\}$. Now ℓ is the rightmost α-line that is down-black in some $\bar{C} \in \mathcal{C}$. We choose v_0, with a sufficiently large absolute value of α-d-size, so that $\text{END}(v_0)$ is in $\text{AREA}(\leftarrow\!\ell_L^{C_0})$ and white in $\bar{C}$, and $\text{START}(v_0)$ is on ℓ and black in $\bar{C}$. $\square$

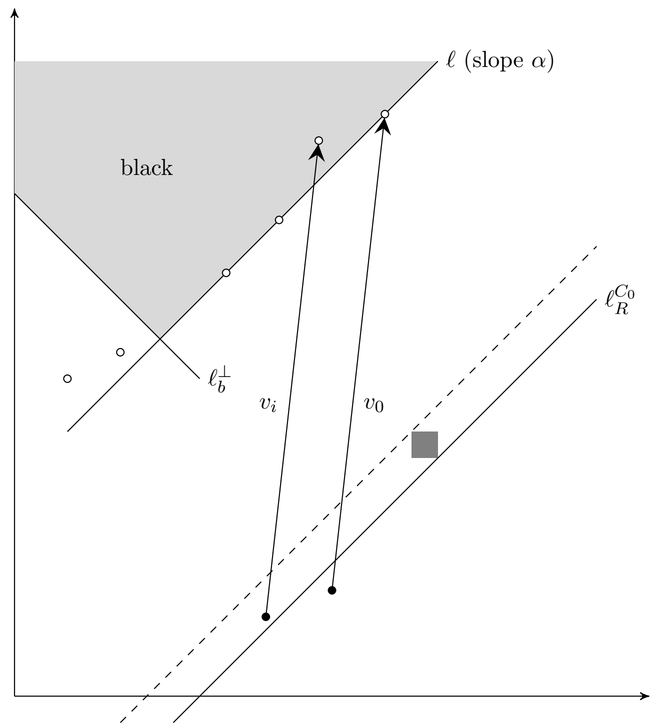


Figure 21. Vector v_0 that is black-white in $\bar{C}$ gives rise to v_i that is black-white in $C'_0 \sqsubset_R C_0$.

To formulate the announced corollary, we introduce further technical notions (that also appear in Figure 20).

Horizontal and vertical distances, values STEP(α), H-STEP(α), V-STEP(α), AND $\sigma_R(\alpha)$, $\sigma_L(\alpha)$, $\tau_R(C)$, $\tau_L(C)$.

- For a slope (a belt-slope in particular) α, $0 < \alpha < \infty$, let ℓ and ℓ' be two α-lines such that a unit square fits between them, with its top-left corner on ℓ and its bottom-right corner on ℓ' (see a grey square in Figure 20). By STEP(α) we mean the (Euclidean) distance of the lines ℓ and ℓ', by H-STEP(α) their horizontal distance (i.e., the distance of their intersections with H-AXIS), and by V-STEP(α) their vertical distance. We put STEP(∞) = H-STEP(∞) = 1, and STEP(0) = V-STEP(0) = 1.
- By H-DIST($(0,0),\ell$) (V-DIST($(0,0),\ell$)) we mean the distance of $(0,0)$ to the intersection of ℓ with H-AXIS (with V-AXIS).
- For a belt-slope α, by $\sigma_R(\alpha)$ we denote the number of equivalence classes of the equivalence $\equiv_R = \sqsubseteq_R \cap \sqsupseteq_R$ in the set of α-colourings. The value $\sigma_L(\alpha)$ is the number of classes of $\equiv_L = \sqsubseteq_L \cap \sqsupseteq_L$ in the set of α-colourings.
- For an α-colouring C, we define $\tau_R(C)$ to be the number of equivalence classes of $\equiv_R$ in the set $\{C' \mid C'$ is an α-colouring and $C' \sqsubseteq_R C\}$. The value $\tau_L(C)$ is the number of classes of $\equiv_L$ in the set $\{C' \mid C'$ is an α-colouring and $C' \sqsubseteq_L C\}$.

We note that $\text{STEP}(\alpha) \leq \sqrt{2}$ (with the equality for $\alpha = 1$). If α is a belt-slope and $\mathcal{C}$ is the set of α-colourings, then Proposition 5.17 entails that

$$\text{H-STEP}(\alpha) \leq 1 + |\mathcal{C}| \text{ if } \alpha > 0 \text{ and } \text{V-STEP}(\alpha) \leq 1 + |\mathcal{C}| \text{ if } \alpha < \infty. \tag{5.4}$$

Now we state the announced corollary of Proposition 5.18; we recall the order $\alpha_1 < \alpha_2 \cdots < \alpha_k$ of belt-slopes (5.3) and look at Figure 20.

Corollary 5.19. *For each α_i-colouring C (where $i \in [1, k]$) we have:*

(1) *if $(0,0)$ is above ℓ_R^C and $\alpha_i > 0$, then*

$$\text{H-DIST}((0,0), \ell_R^C) \ \leq \ \left(\sum_{j=i+1}^{k} \text{H-STEP}(\alpha_j) \cdot \sigma_R(\alpha_j) \right) + \text{H-STEP}(\alpha_i) \cdot \tau_R(C);$$

(2) *if $(0,0)$ is below ℓ_L^C and $\alpha_i < \infty$, then*

$$\text{V-DIST}((0,0), \ell_L^C) \ \leq \ \left(\sum_{j=1}^{i-1} \text{V-STEP}(\alpha_j) \cdot \sigma_L(\alpha_j) \right) + \text{V-STEP}(\alpha_i) \cdot \tau_L(C).$$

By Proposition 5.17, Corollary 5.19, and the fact (5.4) we get:

Theorem 5.20 (Quantitative belt theorem)**.** *The slopes, widths, and positions of belts from Theorem 5.2 can be presented by integers that are polynomial in $|Q|$, referring to the underlying OCN $\mathcal{N} = (Q, Act, \delta)$.*

We could describe the respective polynomials more specifically, but this is not necessary for deriving the PSPACE-membership for the simulation problem on one-counter nets, which is sketched in the next subsection.

5.3. **Polynomial-space algorithm.** Based on the basic belt theorem, a straightforward algorithm that decides whether $p(m) \preceq q(n)$ for a given OCN $\mathcal{N}$, or more generally constructs a description of $\preceq$ on the LTS $\mathcal{L}_{\mathcal{N}}$, was given in [JMS99, JKM00]; taking the "polynomial" slopes and widths of belts into account, we get polynomial-space algorithms [HLMT16]. In principle, we can use "brute force" and simply construct an initial rectangle of the planes $\mathbb{N} \times \mathbb{N}$, first assuming that all points are black in all colourings, and stepwise recolouring to white when finding points with ranks $1, 2, 3, \ldots$. It is easy to note that the colouring inside the belts is (ultimately) periodic, i.e., after a possible initial segment another segment repeats forever. The belt periods can be (and sometimes are) exponential, hence the above "brute-force" algorithm would also use exponential space to discover the repeated belt-segments and provide a complete description of the relation $\preceq$ (on the LTS $\mathcal{L}_{\mathcal{N}}$). Nevertheless, it is a technical routine to modify the algorithm so that it uses only a polynomial work-space.

6. Remark on Double-exponential Periods

We have mentioned the periodic colouring of the simulation-belts for OCNs, where the periods can be exponential. It is not so surprising then that these periods for SOCNs can be double-exponential, which we now discuss in more detail. In fact, by our constructions this is a corollary of the fact that the periods of sequences $\mathcal{S}_{\mathcal{D}}$ given by sequence descriptions $\mathcal{D} = (\Delta, D, m)$ can be double-exponential w.r.t. the size of $\mathcal{D}$ (where m is presented in binary); this also entails that the period in the sequence $W(0), W(1), W(2), \ldots$ for the countdown games (recall Section 3.1) can be double-exponential. Hence we concentrate just on the sequences $\mathcal{S}_\mathcal{D}$.

Given $\mathcal{D} = (\Delta, D, m)$ (where $D : \Delta^3 \to \Delta$), we recall that $\mathcal{S}_\mathcal{D} : \mathbb{N} \to \Delta$ satisfies $\mathcal{S}_\mathcal{D}(0) =$ #, $\mathcal{S}_\mathcal{D}(1) = \mathcal{S}_\mathcal{D}(2) = \cdots = \mathcal{S}_\mathcal{D}(m) = \Box$, and $\mathcal{S}_\mathcal{D}(i) = D(\mathcal{S}_\mathcal{D}(i{-}m{-}1), \mathcal{S}_\mathcal{D}(i{-}m), \mathcal{S}_\mathcal{D}(i{-}m{+}1))$ for $i > m$. It is thus obvious that there are $i_0, d \leq |\Delta|^{m+1}$, where $d \geq 1$, such that $\mathcal{S}_\mathcal{D}(i{+}d) = \mathcal{S}_\mathcal{D}(i)$ for each $i \geq i_0$ (in $\mathcal{S}_\mathcal{D}$, after an initial segment of length i_0, a segment of the period-length d is repeated forever). We now show that the least such d can be double-exponential.

Lemma 6.1. *For each $n \in \mathbb{N}$ there is a sequence description $\mathcal{D}$ of size polynomial in n such that for some $d \geq 2^{2^n}$ we have $\mathcal{S}_\mathcal{D}(i) =$ # iff i is a multiple of d; moreover, $\mathcal{S}_\mathcal{D}(i{+}d) = \mathcal{S}_\mathcal{D}(i)$ for each $i \geq 0$.*

Proof. We fix a Turing machine $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{acc}\})$, with $\Sigma = \{\mathtt{0}\}$ and $\Gamma \supseteq \{\mathtt{0}, \mathtt{1}, \Box, \$\}$ that behaves as follows. Starting from the configuration $q_0\mathtt{0}^n$, $\mathcal{M}$ writes the word $\$\mathtt{0}^m\$$ on the tape, where $m = 2^n$, while visiting only the cells $0, 1, \ldots, m{+}1$. Now, inside this space (the cells $0, 1, \ldots, m{+}1$), all $w \in \{\mathtt{0}, \mathtt{1}\}^m$ are stepwise generated; the tape content, always of length $m{+}2$, will stepwise become $\$\mathtt{0}\cdots\mathtt{000}\$$, $\$\mathtt{0}\cdots\mathtt{001}\$$, $\$\mathtt{0}\cdots\mathtt{010}\$$, $\$\mathtt{0}\cdots\mathtt{011}\$$, $\cdots\cdots$, $\$\mathtt{1}\cdots\mathtt{1}\$$. Finally $\mathcal{M}$ rewrites all cells with $\Box$, and halts at the cell 0 in q_{acc}. Hence $\mathcal{M}$ uses only cells $0, 1, \ldots, m{+}1$ and performs t steps where $t > 2^m = 2^{2^n}$.

Now for each $n \in \mathbb{N}$ we consider a Turing machine $\mathcal{M}_n$ that, when starting in q'_0 on the empty tape, first writes $\mathtt{0}^n$, then invokes the computation of $\mathcal{M}$ on $\mathtt{0}^n$, and when $\mathcal{M}$ halts (in q_{acc}), then $\mathcal{M}_n$ restores its initial configuration with the empty tape; the initial state q'_0 will be visited only at the start and at the end of the described computation.

Hence the computation $C_0, C_1, C_2, \ldots$ of $\mathcal{M}_n$ from q'_0 and the empty tape is infinite, visits only the cells $0, 1, \ldots, m{+}1$ for $m = 2^n$, and there is $d_0 \geq 2^{2^n}$ such that the control state q'_0 is visited exactly in the configurations C_i where i is a multiple of d_0. We also note that the size of $\mathcal{M}_n$ is $\mathcal{O}(n)$.

The infinite word $C_0C_1C_2\ldots$ (each C_i of length $m{+}2$) can be viewed as $\mathcal{S}_\mathcal{D}$ for a sequence description $\mathcal{D}$ of size polynomial with respect to n (recall the proof of Proposition 3.4). For $d = d_0 \cdot (m+2)$ we thus have $\mathcal{S}_\mathcal{D}(i) = \mathcal{S}_\mathcal{D}(i{+}d)$ for all $i \in \mathbb{N}$. Moreover, we identify # with the pair $(q'_0, \Box)$, which entails that $\mathcal{S}_\mathcal{D}(i) =$ # iff i is a multiple of d. ∎

7. Additional Remarks

One particular application of countdown games was shown by Kiefer [Kie13] who modified them to show EXPTIME-hardness of bisimilarity on BPA processes. Our EXPSPACE-complete modification does not seem easily implementable by BPA processes, hence the EXPTIME-hardness result in [Kie13] has not been improved here. (The known upper bound for bisimilarity on BPA is 2-EXPTIME.)

We also mention that the involved result in [GHOW10] shows that, given any fixed language L in EXPSPACE, for any word w (in the alphabet of L) we can construct a succinct one-counter automaton that performs a computation which is accepting iff $w \in L$. Such a computation needs to access concrete bits in the (reversed) binary presentation of the counter value. A straightforward direct access to such bits is destructive (the counter value is lost after the bit is read) but this can be avoided: instead of a "destructive reading" the computation just "guesses" the respective bits, and it is forced to guess correctly by a carefully constructed CTL formula that is required to be satisfied by the computation.

This result is surely deeper than the EXPSPACE-hardness of existential countdown games, though the former does not seem to entail the latter immediately.

Finally we note that the theorems and proofs in Section 5 could be formulated for specific tiling problems, without a direct reference to one-counter nets.

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References

[ABHT20] Shaull Almagor, Udi Boker, Piotr Hofman, and Patrick Totzke. Parametrized universality problems for one-counter nets. In *CONCUR 2020*, volume 171 of *LIPIcs*, pages 47:1–47:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.CONCUR.2020.47.

[Av98] Parosh Aziz Abdulla and Kārlis Čerāns. Simulation is decidable for one-counter nets (extended abstract). In *CONCUR '98*, volume 1466 of *LNCS*, pages 253–268. Springer, 1998. doi:10.1007/BFb0055627.

[BGJ13] Stanislav Böhm, Stefan Göller, and Petr Jančar. Equivalence of deterministic one-counter automata is NL-complete. In *STOC '13*, pages 131–140. ACM, 2013. doi:10.1145/2488608.2488626.

[BGJ14] Stanislav Böhm, Stefan Göller, and Petr Jančar. Bisimulation equivalence and regularity for real-time one-counter automata. *J. Comput. Syst. Sci.*, 80(4):720–743, 2014. Preliminary versions appeared at *CONCUR 2010* and *MFCS 2011*. doi:10.1016/j.jcss.2013.11.003.

[CCH+19] Dmitry Chistikov, Wojciech Czerwinski, Piotr Hofman, Michal Pilipczuk, and Michael Wehar. Shortest paths in one-counter systems. *Log. Methods Comput. Sci.*, 15(1), 2019. doi:10.23638/LMCS-15(1:19)2019.

[CKS81] Ashok K. Chandra, Dexter Kozen, and Larry J. Stockmeyer. Alternation. *J. ACM*, 28(1):114–133, 1981. doi:10.1145/322234.322243.

[DLS08] Stéphane Demri, Ranko Lazic, and Arnaud Sangnier. Model checking freeze LTL over one-counter automata. In *FOSSACS 2008*, volume 4962 of *LNCS*, pages 490–504. Springer, 2008. doi:10.1007/978-3-540-78499-9_34.

[FJKW18] Vojtech Forejt, Petr Jančar, Stefan Kiefer, and James Worrell. Game characterization of probabilistic bisimilarity, and applications to pushdown automata. *Log. Methods Comput. Sci.*, 14(4), 2018. doi:10.23638/LMCS-14(4:13)2018.

[GHOW10] Stefan Göller, Christoph Haase, Joël Ouaknine, and James Worrell. Model checking succinct and parametric one-counter automata. In *ICALP 2010*, volume 6199 of *LNCS*, pages 575–586. Springer, 2010. doi:10.1007/978-3-642-14162-1_48.

[GL10] Stefan Göller and Markus Lohrey. Branching-time model checking of one-counter processes. In *STACS 2010*, volume 5 of *LIPIcs*, pages 405–416. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2010. doi:10.4230/LIPIcs.STACS.2010.2472.

[GMT09] Stefan Göller, Richard Mayr, and Anthony Widjaja To. On the computational complexity of verifying one-counter processes. In *LICS 2009*, pages 235–244. IEEE Computer Society, 2009. doi:10.1109/LICS.2009.37.

[HKOW09] Christoph Haase, Stephan Kreutzer, Joël Ouaknine, and James Worrell. Reachability in succinct and parametric one-counter automata. In *CONCUR 2009*, volume 5710 of *LNCS*, pages 369–383. Springer, 2009. doi:10.1007/978-3-642-04081-8_25.

[HLMT16] Piotr Hofman, Slawomir Lasota, Richard Mayr, and Patrick Totzke. Simulation problems over one-counter nets. *Logical Methods in Computer Science*, 12(1), 2016. Preliminary versions appeared at *FSTTCS 2013*, *LICS 2013*, and in Totzke's Ph.D. thesis (2014). doi:10.2168/LMCS-12(1:6)2016.

[Hun15] Paul Hunter. Reachability in succinct one-counter games. In *RP 2015*, volume 9328 of *LNCS*, pages 37–49. Springer, 2015. doi:10.1007/978-3-319-24537-9_5.

[JKM00] Petr Jančar, Antonín Kučera, and Faron Moller. Simulation and bisimulation over one-counter processes. In *STACS 2000*, volume 1770 of *LNCS*, pages 334–345. Springer, 2000. doi:10.1007/3-540-46541-3_28.

[JKMS04] Petr Jančar, Antonín Kučera, Faron Moller, and Zdeněk Sawa. DP lower bounds for equivalence-checking and model-checking of one-counter automata. *Inf. Comput.*, 188(1):1–19, 2004. doi:10.1016/S0890-5401(03)00171-8.

[JMS99] Petr Jančar, Faron Moller, and Zdeněk Sawa. Simulation problems for one-counter machines. In *SOFSEM '99*, volume 1725 of *LNCS*, pages 404–413. Springer, 1999. doi:10.1007/3-540-47849-3_28.

[JS07] Petr Jančar and Zdeněk Sawa. A note on emptiness for alternating finite automata with a one-letter alphabet. *Inf. Process. Lett.*, 104(5):164–167, 2007. doi:10.1016/j.ipl.2007.06.006.

[JS08] Petr Jančar and Jiří Srba. Undecidability of bisimilarity by defender's forcing. *J. ACM*, 55(1):5:1–5:26, 2008. doi:10.1145/1326554.1326559.

[JSL08] Marcin Jurdzinski, Jeremy Sproston, and François Laroussinie. Model checking probabilistic timed automata with one or two clocks. *Logical Methods in Computer Science*, 4(3), 2008. doi:10.2168/LMCS-4(3:12)2008.

[Kie13] Stefan Kiefer. BPA bisimilarity is EXPTIME-hard. *Inf. Process. Lett.*, 113(4):101–106, 2013. doi:10.1016/j.ipl.2012.12.004.

[Kuč00] Antonín Kučera. Efficient verification algorithms for one-counter processes. In *ICALP 2000*, volume 1853 of *LNCS*, pages 317–328. Springer, 2000. doi:10.1007/3-540-45022-X_28.

[May03] Richard Mayr. Undecidability of weak bisimulation equivalence for 1-counter processes. In *ICALP 2003*, volume 2719 of *LNCS*, pages 570–583. Springer, 2003. doi:10.1007/3-540-45061-0_46.

[Rei16] Julien Reichert. On the complexity of counter reachability games. *Fundam. Inform.*, 143(3-4):415–436, 2016. doi:10.3233/FI-2016-1320.

[Ser06] Olivier Serre. Parity games played on transition graphs of one-counter processes. In *FOSSACS 2006*, volume 3921 of *LNCS*, pages 337–351. Springer, 2006. doi:10.1007/11690634_23.

[Srb09] Jiří Srba. Beyond language equivalence on visibly pushdown automata. *Logical Methods in Computer Science*, 5(1), 2009. doi:10.2168/LMCS-5(1:2)2009.

[VP75] Leslie G. Valiant and Mike Paterson. Deterministic one-counter automata. *J. Comput. Syst. Sci.*, 10(3):340–350, 1975. doi:10.1016/S0022-0000(75)80005-5.