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N-Fold Matrix Factorizations

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N-FOLD MATRIX FACTORIZATIONS

by

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The study of matrix factorizations began when they were introduced by Eisenbud; they have since been an important topic in commutative algebra. Results by Eisenbud, Buchweitz, and Yoshino relate matrix factorizations to maximal Cohen-Macaulay modules over hypersurface rings. There are many important properties of the category of matrix factorizations, as well as tensor product and hom constructions. More recently, Backelin, Herzog, Sanders, and Ulrich used a generalization of matrix factorizations – so called N-fold matrix factorizations – to construct Ulrich modules over arbitrary hypersurface rings. In this dissertation we build up the theory of N-fold matrix factorizations, proving analogues of many known properties of the classical setting. We also obtain tensor product and internal hom constructions using a special type of roots of unity and combinatorial results from Heller and Stephan. Finally, we prove generalizations of two of Eisenbud's landmark results for the classical setting in the context of 3-fold matrix factorizations.

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Chapter 1

Introduction

Commutative Algebra is the branch of algebra studying commutative rings and their modules. There are many special classes of rings whose associated categories of modules are of interest to describe, such as regular rings, hypersurface rings, or more generally complete intersections. Further, there are many special classes of modules whose properties are of particular importance.

One such class of modules are Maximal Cohen-Macaulay (MCM) modules. While these are defined using the abstract notions of depth and dimension, the following central result of Eisenbud (see [5], [4], [14]) relates the category of MCM modules over a hypersurface ring to the more concrete notion of matrix factorizations over a related regular ring – that is, a finitely generated projective $\mathbb{Z}/2\mathbb{Z}$-graded module over a regular ring Q equipped with a degree 1 endomorphism d satisfying $d^2 = f \cdot \mathrm{id}$ for f a non-zero-divisor. Matrix factorizations in fact form a category, $\mathsf{MF}(Q, f)$, with morphisms taken to be degree preserving Q-linear maps that commute with the differential.

Theorem (Eisenbud (1980)). *Let Q be a regular ring, f a non-zero divisor, and $R = Q/(f)$. Let $\mathsf{MCM}(R)$ denote the category of MCM modules over R.*

There exists an equivalence of categories between (a suitable quotient of) $\mathsf{MF}(Q, f)$ and $\mathsf{MCM}(R)$ given by $(d_1, d_2) \mapsto \operatorname{coker}(d_1)$.

Further, this induces an equivalence between the homotopy category of $\mathsf{MF}(Q, f)$ and the stable category of $\mathsf{MCM}(R)$.

This led to the study of matrix factorizations in their own right. The second part of the above theorem requires a notion of homotopy equivalence in $\mathsf{MF}(Q, f)$, which turns out to coincide with the definition one expects from viewing a matrix factorization as a "complex-like" object. Similarly, when Q is local there is an obvious notion of minimality, and just as for complexes, matrix factorizations can be decomposed into a direct sum of contractible and minimal parts.

This close connection between matrix factorizations and complexes leads to other familiar constructions involving matrix factorizations. Given matrix factorizations C, D of f and g, respectively, one can construct the objects $\operatorname{Hom}(C, D) \in \mathsf{MF}(Q, g-f)$ and $C \otimes D \in \mathsf{MF}(Q, f + g)$ (see, e.g., [15]). Further, these constructions preserve contractibility and form an adjoint pair.

Additionally, the category $\mathsf{MF}(Q, f)$ is known to be Frobenius ([11]) and the Krull-Schmidt theorem holds in $\mathsf{MF}(Q, f)$ ([15]).

Within the class of MCM modules there is a subclass called Ulrich modules – MCM modules with minimal number of generators equal to multiplicity (also called Maximally Generated or Linear MCM modules) – which are of particular interest to us. First introduced by Ulrich in [13], the existence of Ulrich modules over a ring has some unexpected applications, though the question of whether or not a ring has an Ulrich module has proven difficult. For example, the existence of Ulrich modules over homogeneous hypersurface rings implies that every homogeneous polynomial f in $k[X_1, ..., X_n]$ has a power f^m which is the determinant of a matrix with linear entries (see [3], [2]). Though in general existence of Ulrich modules over arbitrary rings is unknown, it is known through the series of papers ([2], [1], [3]) that all hypersurface rings have Ulrich modules. These proofs relied on a construction generalizing matrix factorizations – so called N-fold matrix factorizations – with special properties.

In chapter 2, we develop the theory of N-fold matrix factorizations. Foremost, we focus on generalizing known results for classical matrix factorizations. Central to many of the results is the appropriate generalization of homotopy equivalence and contractibility discussed in sections 2.2 and 2.6, especially the characterization of contractible objects below.

Theorem. *Let $(Q, \mathfrak{m}, k)$ be a regular local ring and $f \in \mathfrak{m}$ a non-zero-divisor. An object P in $\mathsf{N\text{-}MF}(Q, f)$ is contractible if and only if P is isomorphic to a direct sum of discs; that is, $P \cong \bigoplus_{n=0}^{N-1} D_n(f)^{m_n}$ for some nonnegative integers m_n.*

A notion of homotopy equivalence encourages us to consider the homotopy category of N-fold matrix factorizations. In particular, in section 2.4 we define an internal Hom construction whose homology relates to the homotopy category. In section 2.5, we discuss a tensor product construction which is adjoint to the internal Hom and preserves homotopy equivalence.

Focusing on the local case, we find multiple candidates for a notion of minimal N-fold matrix factorizations. We discuss the two extremes, which we call "weakly" and "strongly" minimal, in section 2.7. Weak minimality exhibits the expected decomposition property – that every object decompose into a direct sum of minimal and contractible objects. Strongly minimal objects are more difficult to find, although these are the N-fold matrix factorizations arising in [2], [1], and [3] when constructing Ulrich modules over hypersurface rings.

We conclude the chapter with two significant properties of the category $\mathsf{N}\text{-}\mathsf{MF}(Q, f)$ when Q is local. In section 2.8, we characterize the projective and injective objects, concluding that $\mathsf{N}\text{-}\mathsf{MF}(Q, f)$ is a Frobenius category and showing that the stable category coincides with the homotopy category. In section 2.9, we turn to unique decomposition, and find that under the additional assumption that Q be Henselian, the category $\mathsf{N}\text{-}\mathsf{MF}(Q, f)$ is a Krull-Schmidt category, and hence has unique direct sum decompositions into indecomposable objects.

Finally, in chapter 3 we restrict our attention to the case $N = 3$. Section 3.1 is devoted to the following analogue of Eisenbud's correspondence stated above.

Theorem. *Let Q be a regular ring (not necessarily local), $f \in Q$ a non-zero-divisor, and $R = Q/(f)$.*

There exists an equivalence of additive categories between (a suitable quotient of) $3\text{-}\mathsf{MF}(Q, f)$ and $\mathcal{E}(\mathsf{MCM}(R))$, the category of short exact sequences of MCM R-modules.

Moreover, this induces an equivalence between the homotopy category of $3\text{-}\mathsf{MF}(Q, f)$ and the stable category of $\mathcal{E}(\mathsf{MCM}(R))$.

In section 3.2 we focus on a related result of Eisenbud in [5], his theory of "higher homotopies." We generalize this to a method of constructing a 3-fold matrix factorization from a short exact sequence of arbitrary R-modules and a chosen free resolution.

Chapter 2

N-Fold Matrix Factorizations

In this chapter, we develop the category of N-fold matrix factorizations, and extend many well known properties of Eisenbud's classical matrix factorizations to this setting. Throughout, Q will be a regular ring, f a non-zero-divisor, and R the quotient ring $Q/(f)$.

We will be examining a class of $\mathbb{Z}/N\mathbb{Z}$-graded objects in this chapter. Recall that a module M over a graded ring S is graded if there is an abelian group direct sum decomposition $M = \bigoplus M_i$ satisfying $S_i M_j \subseteq M_{i+j}$, where the index set is any monoid.

The primary scenario in this paper will be when $S = Q$ is trivially graded (that is, all elements of Q are homogeneous of degree 0) and the graded index set is the group $\mathbb{Z}/N\mathbb{Z}$. In this case, M is a finite direct sum of abelian groups (in fact, of Q-modules). We will still index graded pieces in $\mathbb{Z}$ with the identification that $M_i = M_j$ when $i \equiv j \mod N$.

Degree i morphisms of $\mathbb{Z}/N\mathbb{Z}$-graded modules are then Q-linear maps sending homogeneous elements degree m to homogeneous elements of degree $m + i$. Here we allow i to be any integer, and use the identification of graded pieces modulo N as before.

2.1 The Category of N-fold Matrix Factorizations

Definition 2.1.1 ([3])**.** Let Q be a regular ring and $f \in Q$. An *N-fold matrix factorization of f* is a finitely generated projective $\mathbb{Z}/N\mathbb{Z}$-graded Q-module P equipped with a degree 1 endomorphism (the differential) d_P satisfying $d_P^N = f \cdot \mathrm{id}$.

Alternatively, one can view an N-fold matrix factorization as a collection of N finitely generated projective modules $P_1, ..., P_N$ and maps $d_i : P_i \to P_{i+1}$ and $d_N : P_N \to P_1$ satisfying

$$f \cdot \mathrm{id} = d_N \cdot \cdots \cdot d_1 = d_{N-1} \cdot \cdots \cdot d_1 \cdot d_N = \cdots = d_1 \cdot d_N \cdot \cdots \cdot d_2.$$

Definition 2.1.2. The *category of N-fold matrix factorizations* of f, denoted $\mathsf{N\text{-}MF}(Q, f)$, can be constructed with objects as defined above and morphisms α being degree 0 Q-linear maps satisfying $\alpha d = d\alpha$.

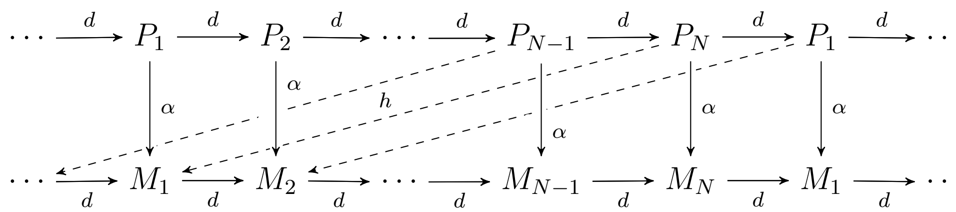
2.2 Homotopy

The appropriate notion of homotopy in $\mathsf{N\text{-}MF}(Q, f)$ is somewhat more complex than the classical case. We follow the similar situation found in [6].

Definition 2.2.1. A morphism $\alpha : P \to M$ in $\mathsf{N\text{-}MF}(Q, f)$ is *null homotopic* if there exists a degree 1 map $h : P \to M$ such that

$$\alpha = \sum_{i=0}^{N-1} d_M^{N-1-i} h d_P^i.$$

Pictorially, α is the sum of every possible route from P_i to M_i using the differentials d and homotopy map h in the diagram below.



We say two maps α, β are homotopic, denoted $\alpha \sim \beta$, if $\beta - \alpha$ is null homotopic.

The following two lemmas show that there is a well defined notion of a homotopy category.

Lemma 2.2.2. *Homotopy is an equivalence relation.*

Proof. It is clear that $\alpha \sim \alpha$ by $h = 0$. If $\alpha \sim \beta$ with homotopy h, then $\beta \sim \alpha$ with homotopy $-h$. Finally, if $\alpha \sim \beta$ with homotopy h_1 and $\beta \sim \gamma$ with homotopy h_2, then $\alpha \sim \beta$ with homotopy $h_1 + h_2$, noticing that $\alpha - \gamma = (\alpha - \beta) + (\beta - \gamma)$. □

Lemma 2.2.3. *Composition respects homotopy – precisely, if $\alpha_1 \sim \beta_1$ and $\alpha_2 \sim \beta_2$ then $\alpha_1\alpha_2 \sim \beta_1\beta_2$.*

Proof. We first notice that if $\alpha \sim \beta$ with homotopy h, then $\alpha\gamma \sim \beta\gamma$ with homotopy $h\gamma$ (because γ commutes with d) and $\gamma\alpha \sim \gamma\beta$ with homotopy γh.

Then $\alpha_1\alpha_2 \sim \alpha_1\beta_2 \sim \beta_1\beta_2$. □

Definition 2.2.4. The *homotopy category of N-fold matrix factorizations of f*, denoted by $\underline{\mathsf{N\text{-}MF}}(Q, f)$ is formed from $\mathsf{N\text{-}MF}(Q, f)$ as follows: the objects are left unchanged, while the set of morphisms is defined to be the quotient of the set of morphisms in $\mathsf{N\text{-}MF}(Q, f)$ by the homotopy equivalence relation.

That is, for objects $P, M \in \mathsf{N\text{-}MF}(Q,f)$,

$$\underline{\mathrm{Hom}}_{\mathsf{N\text{-}MF}(Q,f)}(P,M) := \frac{\mathrm{Hom}_{\mathsf{N\text{-}MF}(Q,f)}(P,M)}{\sim}.$$

Along with this definition of homotopy comes the natural notion of contractibility.

Definition 2.2.5. An N-fold matrix factorization P is contractible if $\mathrm{id}_P \sim 0$.

2.3 Distinguished Primitive Roots of Unity

When defining the tensor product (and internal Hom) of complexes and classical matrix factorizations, the definitions use a sign (see [15]). In this section, we generalize this to an appropriate root of unity and outline many of the needed combinatorial results. For completeness, we record the necessary definitions and lemmas from Heller and Stephan ([8]), including some proofs omitted in their paper.

Definition 2.3.1 ([8], 3.1)**.** Let S be any commutative unital ring, and fix $\zeta \in S$. Let $[\cdot]_\zeta : \mathbb{N} \to S$ denote the function $[n]_\zeta = \sum_{k=0}^{n-1} \zeta^k$ with the convention that $[0]_\zeta = 0$.

The ζ-factorial $[n]_\zeta!$ of n is defined by

$$[n]_\zeta! = \prod_{k=1}^{n} [k]_\zeta.$$

For integers $n \geq m \geq 0$, the ζ-binomial coefficient $\binom{n}{m}_\zeta \in S$ is defined inductively by

$$\begin{cases} \binom{n}{0}_\zeta = \binom{n}{n}_\zeta = 1 \\ \binom{n+1}{m+1}_\zeta = \binom{n}{m}_\zeta + \zeta^{m+1}\binom{n}{m+1}_\zeta \text{ for } n-1 \geq m \geq 0 \end{cases}.$$

Note 2.3.2. *In the case $S = \mathbb{Z}$, $\zeta = 1$, the map $[\cdot]_\zeta : \mathbb{N} \to \mathbb{Z}$ is the inclusion, and the ζ-factorial and ζ-binomial coefficients coincide with ordinary factorial and binomial coefficients, respectively.*

Definition 2.3.3 ([8], 3.3)**.** A *distinguished primitive Nth root of unity of S* is an element $\zeta \in S$ such that $[N]_\zeta = 0$ and $[n]_\zeta$ is invertible for all $1 \leq n \leq N-1$.

Note that if $\zeta \neq 1$, such an element is indeed a primitive Nth root of unity, as

$$\zeta^n - 1 = (\zeta - 1)(\zeta^{n-1} + \zeta^{n-2} + \cdots + \zeta^0) = (\zeta - 1)[n]_\zeta$$

which is 0 for $n = N$ and nonzero for $1 \leq n \leq N-1$.

Example 2.3.4. If S contains an algebraically closed field whose characteristic does not divide N, then the distinguished primitive Nth roots of unity in S coincide with the ordinary primitive Nth roots of unity – those elements s for which $i = N$ is the smallest power of s for which $s^i = 1$.

If $N = p$ a prime, and S has characteristic p, then $\zeta = 1$ is a distinguished primitive Nth root of unity.

If $N = pm$ for some prime p and integer $m > 1$, and S has characteristic p, then S has no distinguished primitive Nth roots of unity, because $[p]_\zeta = 0$ for all $\zeta \in S$.

Lemma 2.3.5 ([8], 3.5)**.** *Let S be a commutative unital ring and let ζ be a distinguished primitive Nth root of unity of S. Then*

$$\binom{n}{m}_\zeta = \frac{[n]_\zeta!}{[n-m]_\zeta![m]_\zeta!}$$

for all $N \geq n \geq m \geq 0$ with the convention that $\frac{[N]_\zeta!}{[N]_\zeta!} = 1$.

Proof. For $m = n$ and $m = 0$, we certainly have $\binom{n}{m}_\zeta = 1$. Then by induction,

$$\begin{aligned}
\binom{n+1}{m+1}_\zeta &= \binom{n}{m}_\zeta + \zeta^{m+1}\binom{n}{m+1}_\zeta \\
&= \frac{[n]_\zeta!}{[n-m]_\zeta![m]_\zeta!} + \zeta^{m+1}\frac{[n]_\zeta!}{[n-m-1]_\zeta![m+1]_\zeta!} \\
&= \frac{[n]_\zeta!}{[n-m-1]_\zeta![m]_\zeta!}\left(\frac{1}{[n-m]_\zeta} + \zeta^{m+1}\frac{1}{[m+1]_\zeta}\right) \\
&= \frac{[n]_\zeta!}{[n-m-1]_\zeta![m]_\zeta!}\left(\frac{[m+1]_\zeta + \zeta^{m+1}[n-m]_\zeta}{[n-m]_\zeta[m+1]_\zeta}\right) \\
&= \frac{[n]_\zeta!}{[n-m]_\zeta![m+1]_\zeta!}\left(\sum_{k=0}^{m}\zeta^k + \sum_{k=m+1}^{n}\zeta^k\right) \\
&= \frac{[n+1]_\zeta!}{[n-m]_\zeta![m+1]_\zeta!}.
\end{aligned}$$

□

Lemma 2.3.6. *Let S be a commutative unital ring and let ζ be a distinguished primitive Nth root of unity of S. Then*

$$\binom{N-1}{m}_\zeta = (-1)^m\zeta^{-\binom{m+1}{2}}$$

for all $0 \leq m \leq N-1$

Proof. We proceed by induction on m. The case $m = 0$ trivially holds.

For $m > 0$, by definition,

$$\binom{N-1}{m}_\zeta = \zeta^{-m}\left(\binom{N}{m}_\zeta - \binom{N-1}{m-1}_\zeta\right).$$

By Lemma 2.3.5, $\binom{N}{m}_\zeta = 0$. Applying our inductive hypothesis then gives us

$$\binom{N-1}{m}_\zeta = -\zeta^{-m}\left((-1)^{m-1}\zeta^{-\binom{m}{2}}\right) = (-1)^m\zeta^{-\binom{m+1}{2}}.$$

□

The case $\zeta = 1$, $S = \mathbb{Z}$ of the following identity is a form of the Chu-Vandermonde identity.

Lemma 2.3.7 ([8], 3.6)**.** *Let S be a commutative unital ring and let ζ be a distinguished primitive Nth root of unity of S. For all $0 \le s \le t < u$ the identity*

$$\zeta^{(s+1)t} \sum_{i=s}^{t} \binom{u-1-i}{u-1-t}_{\zeta} \binom{i}{s}_{\zeta} \zeta^{-i(s+1)} = \binom{u}{u+s-t}_{\zeta}$$

holds.

Proof. Fix an s.

If $t = s$, then the identity holds trivially for any u.

We prove the case $u = t+1$ by induction on t. The base case $t = s$ is already satisfied. We seek to prove

$$\sum_{i=s}^{t} \binom{i}{s}_{\zeta} \zeta^{(t-i)(s+1)} = \binom{t+1}{s+1}_{\zeta}.$$

Indeed,

$$\begin{aligned}\sum_{i=s}^{t} \binom{i}{s}_{\zeta} \zeta^{(t-i)(s+1)} &= \binom{t}{s}_{\zeta} + \sum_{i=s}^{t-1} \binom{i}{s}_{\zeta} \zeta^{(t-i)(s+1)} \\ &= \binom{t}{s}_{\zeta} + \zeta^{s+1} \sum_{i=s}^{t-1} \binom{i}{s}_{\zeta} \zeta^{(t-1-i)(s+1)} \\ &= \binom{t}{s}_{\zeta} + \zeta^{s+1} \binom{t}{s+1}_{\zeta} \\ &= \binom{t+1}{s+1}_{\zeta}.\end{aligned}$$

Now we prove the identity for any $u > t+1$ and $t > s$, assuming it holds for both $(u-1, t)$ and $(u-1, t-1)$.

$$
\begin{aligned}
\zeta^{(s+1)t} & \sum_{i=s}^{t} \binom{u-1-i}{u-1-t}_{\zeta} \binom{i}{s}_{\zeta} \zeta^{-i(s+1)} \\
& = \zeta^{(s+1)t} \left(\binom{t}{s}_{\zeta} \zeta^{-t(s+1)} \right. \\
& \qquad \left. + \sum_{i=s}^{t-1} \left(\left(\binom{u-2-i}{u-2-t}_{\zeta} + \zeta^{u-1-t} \binom{u-2-i}{u-1-t}_{\zeta} \right) \binom{i}{s}_{\zeta} \zeta^{-i(s+1)} \right) \right) \\
& = \zeta^{(s+1)t} \left(\sum_{i=s}^{t} \binom{u-2-i}{u-2-t}_{\zeta} \binom{i}{s}_{\zeta} \zeta^{-i(s+1)} \right. \\
& \qquad \left. + \sum_{i=s}^{t-1} \zeta^{u-1-t} \binom{u-2-i}{u-1-t}_{\zeta} \binom{i}{s}_{\zeta} \zeta^{-i(s+1)} \right) \\
& = \binom{u-1}{u-1+s-t}_{\zeta} + \zeta^{u-t+s} \binom{u-1}{u+s-t}_{\zeta} \\
& = \binom{u}{u+s-t}_{\zeta}. \qquad \square
\end{aligned}
$$

2.4 An Internal Hom

Our first use of the distinguished primitive roots of unity comes in defining an internal Hom for the category $\mathsf{N}\text{-}\mathsf{MF}(Q, f)$.

Definition 2.4.1. Let Q be a regular ring (not necessarily local), $f, g \in Q$ non-zero-divisors, and $\zeta \in Q$ a distinguished primitive Nth root of unity. Let C and D be objects in $\mathsf{N}\text{-}\mathsf{MF}(Q, f)$ and $\mathsf{N}\text{-}\mathsf{MF}(Q, g)$, respectively. We define an internal Hom as the $\mathbb{Z}/N\mathbb{Z}$-graded projective module

$$\mathcal{H}om^{\zeta}(C, D) := \prod_{i=0}^{N-1} \operatorname{Hom}_Q(C, D)_i$$

where $\operatorname{Hom}_Q(C, D)_i$ denotes the graded module homomorphisms of degree i, with differential ∂ given on homogeneous α by

$$\partial(\alpha) = d_D \alpha - \zeta^{|\alpha|} \alpha d_C.$$

Lemma 2.4.2. *As defined above, $\mathcal{H}om^{\zeta}(C,D)$ is in* $\mathsf{N}\text{-}\mathsf{MF}(Q, g-f)$.

Proof. We claim that

$$\partial^k(\alpha) = \sum_{m=0}^{k} (-1)^{k-m} \zeta^{\binom{k-m}{2}+(k-m)|\alpha|} \binom{k}{m}_{\zeta} d_D^m \alpha d_C^{k-m}$$

for all $k \geq 0$.

We proceed by induction on k. When $k=0$, we have $\partial^0(\alpha) = \alpha$ as required. Now suppose the formula holds for $k-1$. Then

$$\begin{aligned}
\partial^k(\alpha) &= \partial \circ \partial^{k-1}(\alpha) \\
&= \partial \left(\sum_{m=0}^{k-1} (-1)^{k-1-m} \zeta^{\binom{k-1-m}{2}+(k-1-m)|\alpha} \binom{k-1}{m}_{\zeta} d_D^m \alpha d_C^{k-1-m} \right) \\
&= \sum_{m=0}^{k-1} (-1)^{k-1-m} \zeta^{\binom{k-1-m}{2}+(k-1-m)|\alpha|} \binom{k-1}{m}_{\zeta} d_D^{m+1} \alpha d_C^{k-1-m} \\
&\qquad - \zeta^{|\alpha|+k-1} \sum_{m=0}^{k-1} (-1)^{k-1-m} \zeta^{\binom{k-1-m}{2}+(k-1-m)|\alpha|} \binom{k-1}{m}_{\zeta} d_D^m \alpha d_C^{k-m} \\
&= \sum_{m=1}^{k} (-1)^{k-m} \zeta^{\binom{k-m}{2}+(k-m)|\alpha|} \binom{k-1}{m-1}_{\zeta} d_D^m \alpha d_C^{k-m} \\
&\qquad + \sum_{m=0}^{k-1} (-1)^{k-m} \zeta^{\binom{k-1-m}{2}+k-1-m+(k-m)|\alpha|} \zeta^m \binom{k-1}{m}_{\zeta} d_D^m \alpha d_C^{k-m} \\
&= d_D^k \alpha + (-1)^k \zeta^{\binom{k}{2}+k|\alpha|} \alpha d_C^k \\
&\qquad + \sum_{m=1}^{k-1} (-1)^{k-m} \zeta^{\binom{k-m}{2}+(k-m)|\alpha|} \binom{k}{m}_{\zeta} d_D^m \alpha d_C^{k-m} \\
&= \sum_{m=0}^{k} (-1)^{k-m} \zeta^{\binom{k-m}{2}+(k-m)|\alpha|} \binom{k}{m}_{\zeta} d_D^m \alpha d_C^{k-m}
\end{aligned}$$

which proves the claim.

Now setting $k = N$, and using Lemma 2.3.5, we find that

$$\partial^N(\alpha) = g\alpha + 0 + (-1)^N \zeta^{\binom{N}{2} + N|\alpha|}\alpha f.$$

By Lemma 2.3.6, $\zeta^{\binom{N}{2}} = (-1)^{N-1} \cdot \binom{N-1}{N-1}_\zeta^{-1} = (-1)^{N-1}$.

Therefore, $(-1)^N \zeta^{\binom{N}{2}} = -1$ and hence $\partial^N(\alpha) = (g - f)\alpha$. $\square$

In fact, $\mathcal{H}om^\zeta(C, -)$ and $\mathcal{H}om^\zeta(-, D)$ are additive functors between the appropriate categories of N-fold matrix factorizations, which follows from the same property of the Hom functor of Q-modules.

Discussion of the internal Hom naturally leads to the scenario in the following lemma – objects in $\mathsf{N\text{-}MF}(Q, 0)$. N-fold matrix factorizations of 0 are a special collection of N-complexes (see, e.g. [6]) with projective modules and a $\mathbb{Z}/N\mathbb{Z}$-grading. For N-complexes – sequences of Q-modules and Q-linear maps d satifying $d^N = 0$ – there are $N - 1$ possible notions of boundaries, cycles, and homology.

Definition 2.4.3 ([6])**.** Let P be an N-complex with differential d. For $t = 1, 2, ..., N-1$ we define the following collections of modules:

The *amplitude t cycles in degree n*, denoted ${}_tZ_n(P)$, is the submodule $\ker(d^t)$ of P_n.

The *amplitude t boundaries in degree n*, denoted ${}_tB_n(P)$, is the submodule $\mathrm{im}(d^t)$ of P_n.

The *amplitude t homology in degree n*, denoted ${}_tH_n(P)$, is the quotient ${}_tZ_n(P)/{}_{N-t}B_n(P)$.

While there are relationships between different amplitude homology modules, especially on their vanishing, discussed in [6], we will focus our attention only on the case $t = 1$. In this case, we will simplify notation, using $Z_n(P)$, $B_n(P)$, and $H_n(P)$ to denote the cycles, boundaries, and homology, respectively, in degree n.

Using the preceding notation, the following lemma shows one way in which the behavior of the internal Hom mirrors the case of the internal Hom of chain complexes.

Lemma 2.4.4. *Let C, D be objects in $\mathsf{N\text{-}MF}(Q, f)$, so that $\mathcal{H}om^{\zeta}(C, D) \in \mathsf{N\text{-}MF}(Q, 0)$. Then $Z_0\left(\mathcal{H}om^{\zeta}(C, D)\right)$ is $\operatorname{Hom}_{\mathsf{N\text{-}MF}(Q,f)}(C, D)$ and $B_0\left(\mathcal{H}om^{\zeta}(C, D)\right)$ is the set of all null homotopic maps from C to D. Therefore, $H_0\left(\mathcal{H}om^{\zeta}(C, D)\right)$ is $\underline{\operatorname{Hom}}_{\mathsf{N\text{-}MF}(Q,f)}(C, D)$, the morphisms in the homotopy category.*

Proof. The first claim is clear, as for $\alpha \in \ker(\partial)$ of degree 0, we have $d_D\alpha - \alpha d_C = 0$, which exactly gives the definition of morphisms in $\mathsf{N\text{-}MF}(Q, f)$.

Now suppose α is a boundary of degree 0. That is, there is some β of degree $-N + 1 \equiv 1 \mod N$ such that $\alpha = \partial^{N-1}(\beta)$. Using the formula for ∂^k in Lemma 2.4.2, we have

$$\alpha = \sum_{m=0}^{N-1} (-1)^{N-1-m} \zeta^{\binom{N-1-m}{2} + (N-1-m)(-N+1)} \binom{N-1}{m}_{\zeta} d_D^m \beta d_C^{N-1-m}.$$

Applying Lemma 2.3.6,

$$\alpha = \sum_{m=0}^{N-1} (-1)^{N-1} \zeta^{\binom{N-1-m}{2} + (N-1-m)(-N+1) - \binom{m+1}{2}} d_D^m \beta d_C^{N-1-m}.$$

Straightforward simplification of the exponent of ζ yields

$$\alpha = \sum_{m=0}^{N-1} (-1)^{N-1} \zeta^{-\binom{N}{2}} d_D^m \beta d_C^{N-1-m}.$$

Again applying Lemma 2.3.6, and noticing that $\binom{N-1}{N-1}_{\zeta} = 1$ yields

$$\alpha = \sum_{m=0}^{N-1} d_D^m \beta d_C^{N-1-m}$$

so α is null homotopic with homotopy given by β. ∎

Finally, we will need to know that the internal Hom preserves contractibility.

Lemma 2.4.5. *If $X \in \mathsf{N\text{-}MF}(Q, f)$ is contractible, then $\mathcal{H}om^\zeta(X, P)$ and $\mathcal{H}om^\zeta(P, X)$ are contractible for any $P \in \mathsf{N\text{-}MF}(Q, g)$.*

Proof. Suppose $\mathrm{id}_X = \sum_{i=0}^{N-1} d^{N-1-i} h d^i$ for some h. Then since $\mathcal{H}om^\zeta(P, -)$ is additive, the induced map $\mathcal{H}om^\zeta(P, h) : \mathcal{H}om^\zeta(P, X) \to \mathcal{H}om^\zeta(P, X)$ shows $\mathrm{id}_{\mathcal{H}om^\zeta(P,X)} \sim 0$.

Similarly, the induced map $\mathcal{H}om^\zeta(h, P)$ shows $\mathrm{id}_{\mathcal{H}om^\zeta(X,P)} \sim 0$. □

2.5 Tensor Product

Similarly, we can define a tensor product of N-fold matrix factorizations.

Definition 2.5.1. Let Q be a regular ring (not necessarily local), $f, g \in Q$ non-zero-divisors, and $\zeta \in Q$ a distinguished primitive Nth root of unity. Let C and D be objects in $\mathsf{N\text{-}MF}(Q, f)$, and $\mathsf{N\text{-}MF}(Q, g)$, respectively. Let ζ be a distinguished primitive Nth root of unity. We define a tensor product as the $\mathbb{Z}/N\mathbb{Z}$-graded projective module given in degree i by

$$(C \otimes^\zeta_{\mathsf{N\text{-}MF}(Q)} D)_i := \bigoplus_{a+b \equiv i \mod N} C_a \otimes_Q D_b$$

with differential ∂ given on homogeneous simple tensors $(x \otimes y)$ by

$$\partial(x \otimes y) = d_C(x) \otimes y + \zeta^{|x|} x \otimes d_D(y).$$

When the context is clear, we will use $\otimes^\zeta$ to denote this tensor product, for simplicity.

Lemma 2.5.2. *As defined above, $C \otimes^{\zeta} D$ is in* $\mathsf{N\text{-}MF}(Q, f+g)$.

Proof. We claim that

$$\partial^k(x \otimes y) = \sum_{m=0}^{k} \zeta^{|x|(k-m)} \binom{k}{m}_q d_C^m(x) \otimes d_D^{k-m}(y)$$

for all $k \geq 0$ and simple tensors of homogeneous elements $x \otimes y$.

We proceed by induction on k. When $k = 0$, we have $\partial^0(x \otimes y) = x \otimes y$ as required. Now suppose the formula holds for $k-1$. Then

$$\begin{aligned}
\partial^k(x \otimes y) &= \partial \circ \partial^{k-1}(x \otimes y) \\
&= \partial \left(\sum_{m=0}^{k-1} \zeta^{|x|(k-1-m)} \binom{k-1}{m}_\zeta d_C^m(x) \otimes d_D^{k-1-m}(y) \right) \\
&= \sum_{m=0}^{k-1} \zeta^{|x|(k-1-m)} \binom{k-1}{m}_\zeta d_C^{m+1}(x) \otimes d_D^{k-1-m}(y) \\
&\qquad + \sum_{m=0}^{k-1} \zeta^{|x|+m} \zeta^{|x|(k-1-m)} \binom{k-1}{m}_\zeta d_C^m(x) \otimes d_D^{k-m}(y) \\
&= \sum_{m=1}^{k} \zeta^{|x|(k-m)} \binom{k-1}{m-1}_\zeta d_C^m(x) \otimes d_D^{k-m}(y) \\
&\qquad + \sum_{m=0}^{k-1} \zeta^{m} \zeta^{|x|(k-m)} \binom{k-1}{m}_\zeta d_C^m(x) \otimes d_D^{k-m}(y) \\
&= d_C^k(x) + \zeta^{|x|k} d_D^k(y) \\
&\qquad + \sum_{m=1}^{k-1} \zeta^{|x|(k-m)} \left(\binom{k-1}{m-1}_\zeta + \zeta^m \binom{k-1}{m}_\zeta \right) d_C^m(x) \otimes d_D^{k-m}(y) \\
&= \sum_{m=0}^{k} \zeta^{|x|(k-m)} \binom{k}{m}_\zeta d_C^m(x) \otimes d_D^{k-m}(y)
\end{aligned}$$

verifying the claim.

Now setting $k = N$ and noticing that $\binom{N}{m}_\zeta = 0$ for $1 \leq m \leq N-1$ by Lemma 2.3.5, we have

$$\partial^N(x \otimes y) = d^N(x) \otimes y + \zeta^{N|x|} x \otimes d^N(y) = f(x \otimes y) + g(x \otimes y) = (f+g)(x \otimes y). \qquad \square$$

Having verified that the tensor product is well defined, we first verify that the tensor product preserves homotopy equivalence.

Proposition 2.5.3. *Let C, C' be objects in $\mathsf{N\text{-}MF}(Q,f)$ and D in $\mathsf{N\text{-}MF}(Q,g)$. Suppose two maps $\alpha, \beta : C \to C'$ are homotopy equivalent with homotopy h. Then $\alpha \otimes^\zeta D, \beta \otimes^\zeta D : C \otimes^\zeta D \to C' \otimes^\zeta D$ are homotopy equivalent with homotopy $h \otimes^\zeta D$.*

Proof. We need to show for homogeneous x, y

$$\sum_{i=0}^{N-1} d^{N-1-i}_{C' \otimes^\zeta D} (h \otimes^\zeta D)\, d^i_{C \otimes^\zeta D}(x \otimes y) = \alpha(x) \otimes y - \beta(x) \otimes y.$$

Working from the left hand sum and applying the formula in Lemma 2.5.2 yields

$$\sum_{i=0}^{N-1} \sum_{m=0}^{N-1-i} \sum_{n=0}^{i} \zeta^{e(x,i,m,n)} \binom{i}{n}_\zeta \binom{N-1-i}{m}_\zeta d^m_{C'}\, h\, d^n_C(x) \otimes d_D^{N-1-m-n}(y)$$

where $e(x,i,m,n) = |x|(i-n) + (|x|+n+1)(-1-i-m)$. Then interchanging the order of summation, we have

$$\sum_{n=0}^{N-1} \sum_{m=0}^{N-1-n} \sum_{i=n}^{N-1-m} \zeta^{e(x,i,m,n)} \binom{i}{n}_\zeta \binom{N-1-i}{m}_\zeta d^m_{C'}\, h\, d^n_C(x) \otimes d_D^{N-1-m-n}(y).$$

Applying Lemma 2.3.7 with $s = n$, $t = N-1-m$, $u = N-1$ to the inner sum yields

$$\sum_{n=0}^{N-1} \sum_{m=0}^{N-1-n} \zeta^{|x|(N-1-m-n)} \binom{N}{n+m+1}_\zeta d^m_{C'}\, h\, d^n_C(x) \otimes d_D^{N-1-m-n}(y).$$

When $m < N-1-n$ then $n+m+1 < N$. Applying Lemma 2.3.5 in this case along with the facts that $[N]_\zeta = 0$ and $[\cdot]_\zeta$ is invertible for smaller inputs yields $\binom{N}{n+m+1}_\zeta = 0$. Therefore, the inner sum has only one non-zero term, leaving us with

$$\sum_{n=0}^{N-1} \zeta^0 \binom{N}{N}_\zeta d_{C'}^{N-1-n}\, h\, d_C^n(x) \otimes d_D^0(y) = \sum_{n=0}^{N-1} d_{C'}^{N-1-n}\, h\, d_C^n(x) \otimes y.$$

Finally, we apply the homotopy equivalence of α and β via h to obtain the desired result. □

A similar result holds for $D \otimes^\zeta \alpha, D \otimes^\zeta \beta$ with the homotopy given by the formula $x \otimes y \mapsto \zeta^{|x|} x \otimes h(y)$.

Finally, we note the expected relationship between $\mathcal{H}om^\zeta$ and $\otimes^\zeta$.

Proposition 2.5.4. *The functors* $-\otimes^\zeta C$ *and* $\mathcal{H}om^\zeta(C, -)$ *form an adjoint pair. That is, for* $B \in \mathsf{N\text{-}MF}(Q, f)$, $C \in \mathsf{N\text{-}MF}(Q, g)$, *and* $D \in \mathsf{N\text{-}MF}(Q, h)$, *there is an isomorphism*

$$\mathcal{H}om^\zeta(B \otimes^\zeta C, D) \cong \mathcal{H}om^\zeta(B, \mathcal{H}om^\zeta(C, D))$$

as objects in $\mathsf{N\text{-}MF}(Q, h-g-f)$, *and this isomorphism is natural in all three arguments.*

Proof. We first notice that the underlying module structure for $\otimes^\zeta$ and $\mathcal{H}om^\zeta$ are given by the module tensor and Hom functors. Thus, classical Hom-tensor adjunction holds.

Recall in the classical case the isomorphism φ is given by taking a morphism α in $\mathcal{H}om^\zeta(B \otimes^\zeta C, D)$ and mapping to the morphism β in $\mathcal{H}om^\zeta(B, \mathcal{H}om^\zeta(C, D))$ given by $\beta(b)(c) = \alpha(b \otimes c)$.

Notice that this isomorphism in fact preserves the $\mathbb{Z}/N\mathbb{Z}$-grading, yielding adjointness of the corresponding $\mathbb{Z}/N\mathbb{Z}$-graded module functors.

Finally, by unpacking the definition of the differentials of both objects, one can see that the isomorphism φ commutes with the differentials, yielding an isomorphism in the category $\mathsf{N\text{-}MF}(Q, h-g-f)$ as required.

Naturality similarly follows from the module setting. □

2.6 Contractibility

We now return our attentions to contractible matrix factorizations using the tools developed in sections 2.4 and 2.5. For the remainder of this chapter, we will require $(Q, \mathfrak{m}, k)$ to be a regular local ring with maximal ideal $\mathfrak{m}$ and residue field $Q/\mathfrak{m} = k$, $f \in \mathfrak{m}$ a non-zero-divisor, and $\zeta \in Q$ a distinguished primitive Nth root of unity.

We first cite the following technical result of Tribone on the structure of $\mathsf{N\text{-}MF}(Q, f)$.

Proposition 2.6.1 ([12], §2)**.** *The category $\mathsf{N\text{-}MF}(Q, f)$ is an exact category, with exact structure defined as follows: a short exact sequence in $\mathsf{N\text{-}MF}(Q, f)$ is a sequence*

$$P' \xrightarrow{\alpha} P \xrightarrow{\beta} P''$$

of matrix factorizations and morphisms so that the induced sequences

$$0 \to P'_i \xrightarrow{\alpha_i} P_i \xrightarrow{\beta_i} P'' \to 0$$

are short exact sequences of projective Q-modules for each i. The maps α and β form the kernel-cokernel pairs of the exact structure.

With this exact category structure, we can examine projective and injective objects.

Lemma 2.6.2. *If P in $\mathsf{N\text{-}MF}(Q,f)$ is contractible, then P is projective in $\mathsf{N\text{-}MF}(Q,f)$.*

Proof. We need to show $\operatorname{Hom}_{\mathsf{N\text{-}MF}(Q,f)}(P,-)$ is exact. In particular, given a short exact sequence

$$C' \to C \to C''$$

in $\mathsf{N\text{-}MF}(Q,f)$, we need to show the induced map

$$\operatorname{Hom}_{\mathsf{N\text{-}MF}(Q,f)}(P,C) \to \operatorname{Hom}_{\mathsf{N\text{-}MF}(Q,f)}(P,C'')$$

is surjective. Using Lemma 2.4.4, we realize this as showing the map

$$Z_0\left(\mathcal{H}om^{\zeta}(P,C)\right) \to Z_0\left(\mathcal{H}om^{\zeta}(P,C'')\right)$$

is surjective, where as before $Z_0 = {}_1Z_0$ denotes the amplitude 1 cycles in degree 0.

By Lemma 2.4.5, $\mathcal{H}om^{\zeta}(P,C'')$ is contractible in $\mathsf{N\text{-}MF}(Q,0)$.

We claim $\mathcal{H}om^{\zeta}(P,C'')$ is exact, in the sense that $B_i\left(\mathcal{H}om^{\zeta}(P,C'')\right) = Z_i\left(\mathcal{H}om^{\zeta}(P,C'')\right)$ for all i (in fact, this holds for each amplitude t cycles and boundaries). The containment $\operatorname{im}\partial^{N-1} \subseteq \ker\partial$ is clear because $\partial^N = 0$. And if $x \in \ker\partial$, then by the definition of contractibility,

$$x = \sum_{i=0}^{N-1} \partial^{N-1-i} h \partial^i(x) = \partial^{N-1}h(x)$$

so indeed, $x \in \operatorname{im}\partial^{N-1}$.

Finally, the underlying short exact sequence of modules

$$0 \to C' \to C \to C'' \to 0$$

is split because C'' is a projective module. Further, $\operatorname{Hom}(P,-)$ is additive, so the map $\operatorname{Hom}(P,C) \to \operatorname{Hom}(P,C'')$ is surjective.

We combine these facts in a simple diagram chase using the below commutative diagram to construct an element in $Z_0\left(\mathcal{H}om^{\zeta}(P,C)\right)$ which maps to any arbitrarily chosen element of $Z_0\left(\mathcal{H}om^{\zeta}(P,C'')\right)$ as required. Given $\alpha \in Z_0\left(\mathcal{H}om^{\zeta}(P,C'')\right)$, this lifts (along two maps) to an element of $\mathcal{H}om^{\zeta}(P,C)_{-N+1}$. Applying ∂^{N-1} constructs the desired element of $Z_0\left(\mathcal{H}om^{\zeta}(P,C)\right)$.

$$\begin{array}{ccccc}
\mathcal{H}om^{\zeta}(P,C)_{-N+1} & \xrightarrow{\partial^{N-1}} & Z_0\left(\mathcal{H}om^{\zeta}(P,C)\right) & & \\
\downarrow & & \downarrow & & \\
\mathcal{H}om^{\zeta}(P,C'')_{-N+1} & \xrightarrow{\partial^{N-1}} & Z_0\left(\mathcal{H}om^{\zeta}(P,C'')\right) & \longrightarrow & 0 \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

□

Lemma 2.6.3. *If P in $\mathsf{N\text{-}MF}(Q,f)$ is contractible, then P is injective in $\mathsf{N\text{-}MF}(Q,f)$.*

Proof. We need to show $\operatorname{Hom}_{\mathsf{N\text{-}MF}(Q,f)}(-,P)$ is exact. We proceed as in Lemma 2.6.2, showing for any short exact sequence

$$C' \to C \to C''$$

in $\mathsf{N\text{-}MF}(Q,f)$ the induced map

$$\operatorname{Hom}_{\mathsf{N\text{-}MF}(Q,f)}(C,P) \to \operatorname{Hom}_{\mathsf{N\text{-}MF}(Q,f)}(C',P)$$

is surjective. Following the arguments of Lemma 2.6.2, this leads us to the following commutative diagram and an equivalent diagram chase.

$$\begin{array}{ccccc}
\mathcal{H}om^{\zeta}(C,P)_{-N+1} & \xrightarrow{\partial^{N-1}} & Z_0\left(\mathcal{H}om^{\zeta}(C,P)\right) & & \\
\downarrow & & \downarrow & & \\
\mathcal{H}om^{\zeta}(C',P)_{-N+1} & \xrightarrow{\partial^{N-1}} & Z_0\left(\mathcal{H}om^{\zeta}(C',P)\right) & \longrightarrow & 0 \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

□

We use this result and the following definition to characterize the contractible objects in $\mathsf{N\text{-}MF}(Q, f)$.

Definition 2.6.4. The *disc* $D_n(f)$, for $1 \le n \le N$, is the object in $\mathsf{N\text{-}MF}(Q, f)$ defined as a $\mathbb{Z}/N\mathbb{Z}$-graded Q-module P with graded pieces $P_i = Q$ and differential d_i defined by

$$d_i : P_i \to P_{i+1} = \begin{cases} \text{id}, & i \neq n \\ f \cdot \text{id}, & i = n \end{cases} .$$

For example, $D_N(f)$ is pictured below.

$$\begin{array}{ccccccccccccc} \cdots & \xrightarrow{f} & P_1 & \xrightarrow{\text{id}} & P_2 & \xrightarrow{\text{id}} & \cdots & \xrightarrow{\text{id}} & P_{N-1} & \xrightarrow{\text{id}} & P_N & \xrightarrow{f} & P_1 & \xrightarrow{\text{id}} & \cdots \\ & & \| & & \| & & & & \| & & \| & & \| & & \\ & & Q & & Q & & & & Q & & Q & & Q & & \end{array}$$

Note 2.6.5. *Each disc $D_n(f)$ is contractible, with contracting homomorphism given on graded pieces by*

$$h_i : P_i \to P_{i+1} = \begin{cases} 0, & i \neq n \\ \text{id}, & i = n \end{cases} .$$

We also record the following results of Gillespie from [6], where S is any commutative unital ring.

Lemma 2.6.6 ([6], 3.2)**.** *Suppose we have a map $g : X \to Y$ of S-modules having a "splitting" $s : Y \to X$ satisfying $gsg = g$. Then $X = \ker g \oplus \operatorname{im} sg$. Moreover, the pair of maps (g, s) restrict to an isomorphism pair $g : \operatorname{im} sg \to \operatorname{im} g$ and $s : \operatorname{im} g \to \operatorname{im} sg$.*

Theorem 2.6.7 ([6], 3.3)**.** *A contractible N-complex (that is, a contractible object in $\mathsf{N\text{-}MF}(S,0)$) with differential d and contracting homotopy s has the form*

$$\ker(d) \oplus \bigoplus_{i=0}^{N-2} d^i s(\ker(d)).$$

Moreover, the differential d restricts to isomorphisms of the direct summands as follows:

$$d : d^i s(\ker(d)) \xrightarrow{\cong} d^{i+1} s(\ker(d)) \qquad 0 \le i \le N-3$$
$$d : d^{N-2} s(\ker(d)) \xrightarrow{\cong} \ker(d).$$

Using these results, we show the following.

Theorem 2.6.8. *Let $(Q, \mathfrak{m}, k)$ be a regular local ring and $f \in \mathfrak{m}$ a non-zero-divisor. An object P in $\mathsf{N\text{-}MF}(Q,f)$ is contractible if and only if P is isomorphic to a direct sum of discs; that is, $P \cong \bigoplus_{n=0}^{N-1} D_n(f)^{m_n}$ for some nonnegative integers m_n.*

Proof. The $\Leftarrow$ direction is clear by Note 2.6.5.

Now suppose $P \neq 0$ is contractible with contracting homotopy h. Reducing modulo $\mathfrak{m}$ gives us $\overline{P}$ a contractible object in $\mathsf{N\text{-}MF}(k,0)$. Then Theorem 2.6.7 gives us

$$\overline{P} \cong \ker(\overline{d}) \oplus \bigoplus_{i=0}^{N-2} \overline{d}^i \overline{h}(\ker(\overline{d})).$$

Since $\overline{P}$ is nonzero by Nakayama's Lemma, we may choose some $\overline{x_0} \in \overline{h}\ker(\overline{d})$ of degree ℓ for some ℓ. Then the isomorphisms given in Theorem 2.6.7 give us that $\overline{x_j} := \overline{d}^j(\overline{x_0})$ are nonzero for $0 \le j \le N-2$. Therefore, Nakayama's Lemma allows us to lift $\overline{x_0} \in \overline{P_i}$ to $x_0 \in P_i$, and each $\overline{x_j} \in \overline{P_{i+j}}$ to an element $x_j = d^j(x_0) \in P_{i+j}$ which is part of a basis for the free Q-module P_{i+j}.

We define a map $P \to D_i(f)$ by $x_j \mapsto 1$ and all other basis elements in each degree mapping to 0. This is clearly a surjective morphism in $\mathsf{N\text{-}MF}(Q, f)$. By Lemma 2.6.2, $D_i(f)$ is projective, so this map splits $P \cong D_i(f) \oplus P'$. Iterating this process yields a decomposition of P into a direct sum of discs, which is contractible by Note 2.6.5. □

2.7 Minimality

In the case $N = 2$ for matrix factorizations, or even more classically for chain complexes, there is a notion of minimality when Q is local – that the differential be contained in the maximal ideal. It can be shown that any 2-fold matrix factorization or chain complex decomposes as a direct sum of minimal and contractible matrices.

Further, the constructions of N-fold matrix factorizations for the purpose of constructing Ulrich modules in [2], [1], [3] involve objects of this type. In fact, they construct matrix factorizations P whose differential generates $\mathfrak{m}P$, a condition even stronger than simply minimality.

In the case of N-fold matrix factorizations, there are unfortunately multiple notions of minimality, and the stronger notion seems to impose a strictness making it difficult to produce strongly minimal matrix factorizations. In this section, we examine the two extreme notions of minimality and their properties.

Definition 2.7.1. An object P in $\mathsf{N\text{-}MF}(Q, f)$ is said to be *strongly minimal* if $\operatorname{im} d \subseteq \mathfrak{m}P$.

Similarly, P is said to be *weakly minimal* if $\operatorname{im} d^{N-1} \subseteq \mathfrak{m}P$.

Proposition 2.7.2. *Let $(Q, \mathfrak{m}, k)$ be a regular local ring and $f \in \mathfrak{m}$ a non-zero-divisor. An object P in $\mathsf{N\text{-}MF}(Q, f)$ is weakly minimal if and only if it has no non-zero contractible direct summands.*

Proof. If P has a non-zero contractible summand, then by Theorem 2.6.8 it in fact has a summand of the form $D_n(f)$ for some n. Since $D_n(f)$ is clearly not weakly minimal, so also P is not weakly minimal.

Conversely, suppose P is not weakly minimal. Then we have, for some n, $d^{N-1}(P_n) \not\subseteq \mathfrak{m}P_{n+N-1}$. By Nakayama's Lemma, there is a basis of P_n with an element e such that $d^i_P(e)$ is part of a basis for P_{n+i} for $0 \leq i \leq N-1$. We define a map $g : P \to D_n(f)$ by mapping $d^i_P(e)$ to 1 and all other basis elements in each degree to 0.

This is a surjective homomorphism of graded modules which clearly satisfies the commutativity condition of morphisms of matrix factorizations except for the basis element $d_P^{N-1}(e) \in P_{n+N-1}$. The necessary condition holds in this case as well:

$$gd_P(d_P^{N-1}(e)) = gd_P^N(e) = g(fe) = fg(e) = f = d_{D_i(f)}(1) = d_{D_i(f)}g(d_P^{N-1}(e)).$$

By Note 2.6.5 and Lemma 2.6.2, $D_n(f)$ is projective, so this surjection g splits, exhibiting a contractible summand of P. □

Corollary 2.7.3. *Any object P in $\mathsf{N\text{-}MF}(Q, f)$ can be written as $M \oplus C$ with M weakly minimal and C contractible.*

Proof. Factor out of P all contractible summands by the process in the proof of 2.7.2. The direct sum of these summands is C, and the complementary summand M has no contractible summands, hence is weakly minimal. □

Corollary 2.7.4. *The only weakly minimal and contractible object in $\mathsf{N\text{-}MF}(Q, f)$ is the zero object.* □

One might hope that weakly minimal matrix factorizations could be further decomposed into a strongly minimal summand and a not-strongly minimal summand in some meaningful way. The following example presents an obstruction to this goal.

Example 2.7.5. Let $Q = k[\![x]\!]$ with k a field, and $f = x^3$. The following is a 3-fold matrix factorization of f which is weakly but not strongly minimal.

$$Q^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x^2 & 0 \\ x & x \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x & 0 \\ -x & x \end{bmatrix}} Q^2$$

One might naturally hope for this to decompose as the direct sum of 3-fold matrix factorizations with a strongly minimal summand as shown below.

$$\begin{array}{c} Q \xrightarrow{1} Q \xrightarrow{x^2} Q \xrightarrow{x} Q \\ \oplus \\ Q \xrightarrow{x} Q \xrightarrow{x} Q \xrightarrow{x} Q \end{array}$$

However, straightforward (albeit tedious) computations show that no invertible degree 0 morphism of the underlying graded module can commute with the differentials.

2.8 Frobenius Category

Proposition 2.6.1 states that $\mathsf{N\text{-}MF}(Q, f)$ is an exact category. Further, Lemmas 2.6.2 and 2.6.3 exhibits a class of matrix factorizations which are both projective and injective. This leads to a natural question: do the classes of projectives and injectives coincide in $\mathsf{N\text{-}MF}(Q, f)$? This fact, along with $\mathsf{N\text{-}MF}(Q, f)$ having enough projectives and injectives would yield that $\mathsf{N\text{-}MF}(Q, f)$ is a Frobenius category.

Tribone answers these questions in the affirmative in [12], Proposition 2.14 and Theorem 2.15. He further shows in Proposition 2.16 of [12] that the stable category of $\mathsf{N\text{-}MF}(Q, f)$ coincides with the homotopy category $\underline{\mathsf{N\text{-}MF}}(Q, f)$.

The results in this section were developed independently of Tribone, and the proofs presented are distinct from those in [12].

Lemma 2.8.1. *The category* N-MF(Q, f) *has enough projectives.*

Proof. Let M be any object in N-MF(Q, f) with differential d whose graded pieces are free modules of rank m. By Lemma 2.6.2 and Theorem 2.6.8, we know that direct sums of discs are projective. Let

$$P = \bigoplus_{n \in \mathbb{Z}/N\mathbb{Z}} D_n(f)^m.$$

We can then define a map $P \to M$ by first defining a map $D_n(f)^m \to M$ for each n. We define the map α_{n+1} in degree $n+1$ by sending a basis of $D_n(f)^m$ in degree $n+1$ to a basis of M in degree $n+1$. Then define $\alpha_{n+1+i} := d\alpha_{n+i}$ for $i = 1, ..., N-1$. This ensures the required commutativity to define a morphism of N-fold matrix factorizations.

Clearly, the constructed map $P \to M$ is surjective as required. □

Lemma 2.8.2. *The category* N-MF(Q, f) *has enough injectives.*

Proof. As in 2.8.1, let M be any object in N-MF(Q, f) with differential d whose graded pieces are free modules of rank m. By Lemma 2.6.3 and Theorem 2.6.8, we know that direct sums of discs are injective. Let

$$I = \bigoplus_{n \in \mathbb{Z}/N\mathbb{Z}} D_n(f)^m.$$

We define a map $M \to I$ by first defining maps $M \to D_n(f)^m$ for each n, then summing. We define this map α_n in degree n by sending a basis of M_n to the standard basis of Q^m. Then define $\alpha_{n-i} := \alpha_{n-1+i}d$ for $i = 1, ..., N-1$. This ensures the required commutativity to define a morphism of N-fold matrix factorizations.

The map $M \to I$ defined in this way is clearly injective as required. □

See also [12], 2.11 for alternate proofs of Lemmas 2.8.1 and 2.8.2.

Using these lemmas, the following theorem provides a characterization of the projective and injective objects of $\mathsf{N\text{-}MF}(Q, f)$.

Theorem 2.8.3. *Let Q be a regular local ring and $f \in Q$ a non-zero-divisor. For an object P in $\mathsf{N\text{-}MF}(Q, f)$, the following are equivalent:*

(i) P is contractible.

(ii) P is projective.

(iii) P is injective.

Proof. An alternate proof of the equivalence of (ii) and (iii) may be found in [12], 2.14.

$(i) \Rightarrow (ii)$ is Lemma 2.6.2.

$(ii) \Rightarrow (i)$: Suppose P is projective. Then by the proof of Lemma 2.8.1, there is a surjection $C \to P$ with C contractible. In particular, P is a direct summand of a contractible matrix factorization. By Corollary 2.7.3, C has no weakly minimal summands, so also P has no weakly minimal summands, so applying Corollary 2.7.3 again shows that P is contractible.

$(i) \Rightarrow (iii)$ is Lemma 2.6.3.

$(iii) \Rightarrow (i)$: Similar to the proof of $(ii) \Rightarrow (i)$, we can inject $P \to C$ with C contractible by Lemma 2.8.2, showing P is a direct summand of a contractible matrix factorization, and hence contractible by the same argument. □

Corollary 2.8.4. *Let Q be a regular local ring and $f \in Q$ a non-zero-divisor. The category* $\mathsf{N\text{-}MF}(Q, f)$ *is Frobenius, and the stable category of* $\mathsf{N\text{-}MF}(Q, f)$ *coincides with the homotopy category* $\underline{\mathsf{N\text{-}MF}}(Q, f)$*.*

Proof. By definition, a Frobenius category is an exact category with enough projectives (and injectives) in which the class of projective and injective objects coincides. Proposition 2.6.1, Lemma 2.8.2, and Theorem 2.8.3 provide these results for $\mathsf{N\text{-}MF}(Q, f)$. Finally, we observe further by Theorem 2.8.3 that the projective objects are the contractible objects, giving the stated equality of categories. □

The final result of this section is true more generally of the stable category of any Frobenius category, and applies both statements of the previous corollary.

Corollary 2.8.5 ([7], 2.6)**.** *The homotopy category* $\underline{\mathsf{N\text{-}MF}}(Q, f)$ *has a canonical triangulated structure.* □

2.9 Krull-Schmidt

Finally, we turn our attention to direct sum decompositions in $\mathsf{N\text{-}MF}(Q, f)$. Recall that a Krull-Schmidt category is a category in which every object decomposes into a finite direct sum of objects with (non-commutative) local endomorphism rings. In this section, we show that, under a mild additional assumption on the ring Q, the category $\mathsf{N\text{-}MF}(Q, f)$ is Krull-Schmidt, and deduce the analogue of the Krull-Remak-Schmidt Theorem.

Definition 2.9.1. An object $P \in \mathsf{N\text{-}MF}(Q, f)$ is *indecomposable* if, whenever $P \cong M \oplus N$ for objects $M, N \in \mathsf{N\text{-}MF}(Q, f)$, either M or N is the zero object.

Definition 2.9.2. A (not necessarily commutative) ring is *local* if the set of non-units forms a two-sided ideal.

The following result holds more generally for any additive category, see e.g. [10], 1.1. We provide the argument in this setting for completeness.

Lemma 2.9.3. *Let $P \in \mathsf{N\text{-}MF}(Q,f)$ be an object whose endomorphism ring* $\operatorname{End}_{\mathsf{N\text{-}MF}(Q,f)}(P)$ *is local. Then P is indecomposable.*

Proof. Suppose P has a nontrivial decomposition $P \cong M \oplus N$, and consider the endomorphisms $(\mathrm{id}_M, 0)$ and $(0, \mathrm{id}_N)$ of P. These are nontrivial idempotent elements, and hence non-units. However, their sum is id_P, so $\operatorname{End}_{\mathsf{N\text{-}MF}(Q,f)}(P)$ is not local. □

We adopt the following definition of Henselian from Leuschke and Wiegand and cite a helpful lemma, including the proof for completeness.

Definition 2.9.4 ([10], §1.2)**.** A local ring S is *Henselian* if, for every (not necessarily commutative) module finite S-algebra Λ, each idempotent of $\Lambda/\mathcal{J}(\Lambda)$ lifts to an idempotent of Λ (where $\mathcal{J}(-)$ denotes the Jacobson radical).

Note that every complete local ring is Henselian (Hensel's lemma, [10] 1.9).

Lemma 2.9.5 ([10], 1.7)**.** *Let $(S, \mathfrak{n})$ be a commutative ring and Λ a (not necessarily commutative) module-finite S-algebra. Then $\mathfrak{n}\Lambda \subseteq \mathcal{J}(\Lambda)$.*

Proof. Let $f \in \mathfrak{n}\Lambda$. Certainly, for any $\lambda \in \Lambda$, $(1-\lambda f)+(\lambda f) = 1$, so $(1-\lambda f)\Lambda + \mathfrak{n}\Lambda = \Lambda$. By Nakayama's lemma, $(1-\lambda f)\Lambda = \Lambda$ for all $\lambda \in \Lambda$, so the result holds. □

Adding the Henselian assumption, we now give the final steps to show that $\mathsf{N\text{-}MF}(Q,f)$ is a Krull-Schmidt category.

Proposition 2.9.6. *Let* $(Q, \mathfrak{m}, k)$ *be a Henselian regular local ring and* $f \in \mathfrak{m}$ *a non-zero-divisor, and* $P \in \mathsf{N\text{-}MF}(Q, f)$ *indecomposable. Then the endomorphism ring* $E := \mathrm{End}_{\mathsf{N\text{-}MF}(Q,f)}(P)$ *is local.*

Proof. Note that E is a subring of the endomorphism ring $\mathrm{End}_Q(P)$ of the underlying module. Since $\mathrm{End}_Q(P)$ is a module-finite Q-algebra, so also E is a module finite Q-algebra.

By Lemma 2.9.5, $E/\mathcal{J}(E)$ is a finite dimensional k-algebra. Therefore, $E/\mathcal{J}(E)$ is a semisimple Artinian ring, so by the Artin-Wedderburn structure theorem, $E/\mathcal{J}(E)$ is a product of division rings.

Now P indecomposable implies E has no nontrivial idempotents. Therefore, $E/\mathcal{J}(E)$ also has no nontrivial idempotents by the Henselian assumption. Thus we in fact have $E/\mathcal{J}(E)$ is a division ring, so E is local. □

Corollary 2.9.7. *When* Q *is a Henselian regular local ring, the category* $\mathsf{N\text{-}MF}(Q, f)$ *is a Krull-Schmidt category.*

Proof. The underlying module structure of every object in $\mathsf{N\text{-}MF}(Q, f)$ is a finitely generated free module. So the process of decomposing into proper direct summands must terminate after a finite number of steps. Therefore, every object decomposes into a finite direct sum of indecomposable objects, which by Proposition 2.9.6 have local endomorphism rings. □

The final corollary holds more generally for any Krull-Schmidt category by [9], 4.2 and following.

Corollary 2.9.8. *When* Q *is a Henselian regular local ring, every object in* $\mathsf{N\text{-}MF}(Q, f)$ *can be written uniquely (up to permutation and isomorphism) as a direct sum of indecomposable objects in* $\mathsf{N\text{-}MF}(Q, f)$. □

Chapter 3

3-Fold Matrix Factorizations

In this chapter we change our focus from the general case of N-fold matrix factorizations to the case $N = 3$. We return to the setting from the beginning of Chapter 2, with Q a regular (not necessarily local) ring, f a non-zero divisor, and R the quotient ring $Q/(f)$.

3.1 Generalizing Eisenbud's Correspondence

We seek to generalize Eisenbud's correspondence between (2-fold) matrix factorizations of f and MCM modules over R to the case of 3-fold matrix factorizations. We will first set up some notation.

Definition 3.1.1. Let $\mathcal{E}(\mathsf{MCM}(R))$ denote the category of short exact sequences of MCM R-modules, with morphisms defined to be triples of R-linear maps making the obvious diagrams commute.

Similarly, let $\mathcal{E}(\mathsf{PROJ}(R))$ denote the collection of short exact sequences of projective R-modules.

In this section, we will frequently refer to a 3-fold matrix factorization simply as a collection of maps. In particular, the matrix factorization with graded pieces P_i and differential $d_i : P_i \to P_{i+1}$ will be notated by the ordered triple (d_1, d_2, d_3).

The following definition generalizes the discs of Definition 2.6.4 to the non-local setting.

Definition 3.1.2. Let $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$ denote the collections of "trivial objects" in $3\text{-}\mathsf{MF}(Q, f)$ – objects isomorphic to $(f, \mathrm{id}, \mathrm{id})$, $(\mathrm{id}, f, \mathrm{id})$, $(\mathrm{id}, \mathrm{id}, f)$, respectively, with P_i projective.

For example, objects in $\mathscr{T}_1$ are isomorphic to an object of the form

$$\cdots \xrightarrow{\mathrm{id}} P_1 \xrightarrow{f} \underset{\substack{\| \\ P_1}}{P_2} \xrightarrow{\mathrm{id}} \underset{\substack{\| \\ P_1}}{P_3} \xrightarrow{\mathrm{id}} P_1 \xrightarrow{f} \cdots$$

with P_1 any projective Q-module.

We also want to briefly discuss quotient categories, using the definition of Yoshino in [14].

Definition 3.1.3. Let $\mathfrak{C}$ be an additive category, and let $\mathfrak{B}$ be a set of objects in $\mathfrak{C}$. We define the quotient category $\mathfrak{C}/\mathfrak{B}$ to be the category whose objects are the same as $\mathfrak{C}$, and morphisms from A to B in $\mathfrak{C}/\mathfrak{B}$ are elements of

$$\mathrm{Hom}_{\mathfrak{C}}(A, B)/\mathfrak{B}(A, B)$$

where $\mathfrak{B}(A, B)$ is the subgroup generated by all morphisms A to B which factor through direct sums of objects in $\mathfrak{B}$.

Notice that all objects in $\mathfrak{B}$ become isomorphic to the zero object in $\mathfrak{C}/\mathfrak{B}$.

With all of the necessary notation in place, we can now state the following generalization of Eisenbud's correspondence.

Theorem 3.1.4. *Let Q be a regular ring (not necessarily local), $f \in Q$ a non-zero-divisor, and $R = Q/(f)$. There exists an equivalence of additive categories*

$$\Psi : \mathsf{3\text{-}MF}(Q, f)/\{\mathscr{T}_3\} \to \mathcal{E}(\mathsf{MCM}(R))$$

given on objects by

$$\Psi(d_1, d_2, d_3) = \Big(0 \to \operatorname{coker} d_1 \to \operatorname{coker} d_2 d_1 \to \operatorname{coker} d_2 \to 0\Big)$$

and on morphisms by the obvious induced maps between respective cokernels.

Moreover, this induces an equivalence

$$\underline{\Psi} : \mathsf{3\text{-}MF}(Q, f)/\{\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3\} \to \mathcal{E}(\mathsf{MCM}(R))/\{\mathcal{E}(\mathsf{PROJ}(R))\}.$$

Before proving the theorem, we record some intermediate results necessary for verifying the functor Ψ is well-defined.

Lemma 3.1.5. *Given a 3-fold matrix factorization (d_1, d_2, d_3) of f, each of the cokernels $\operatorname{coker}(d_1), \operatorname{coker}(d_2), \operatorname{coker}(d_2 d_1)$ are MCM R-modules.*

Proof. Notice that $(d_1, d_3 d_2)$, $(d_2, d_1 d_3)$, and $(d_3, d_2 d_1)$ are each 2-fold matrix factorizations of f. Therefore, Eisenbud's correspondence gives that each cokernel is an MCM R-module. □

Lemma 3.1.6. *Given a 3-fold matrix factorization (d_1, d_2, d_3) of f, there is a short exact sequence*

$$0 \to \operatorname{coker}(d_1) \to \operatorname{coker}(d_2 d_1) \to \operatorname{coker}(d_2) \to 0.$$

Proof. Consider the commutative diagram with exact rows below.

$$\begin{array}{ccccccc}
 & & P_1 & \xrightarrow{d_1} & P_2 & \longrightarrow & \operatorname{coker}(d_1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_2 d_1} & & \downarrow{\scriptstyle d_2} & & \downarrow & & \\
0 & \longrightarrow & P_3 & \xrightarrow{\text{id}} & P_3 & \longrightarrow & 0 & &
\end{array}$$

The Snake Lemma gives an exact sequence

$$\ker(d_2 d_1) \to \ker(d_2) \to \operatorname{coker}(d_1) \to \operatorname{coker}(d_2 d_1) \to \operatorname{coker}(d_2) \to 0.$$

Since f is a non-zero divisor, $f = d_3 d_2 d_1$ is injective, so also d_2 and $d_2 d_1$ are injective, causing the above exact sequence to be the desired short exact sequence. □

Proof of Theorem 3.1.4. Lemmas 3.1.5 and 3.1.6 show that the functor Ψ is well-defined. We will show that Ψ is essentially surjective, full, and faithful.

Essentially Surjective: Given a short exact sequence of MCM R-modules

$$\mathcal{M}: \qquad 0 \to M' \to M \to M'' \to 0$$

we construct a 3-fold matrix factorization P so that $\Psi(P) = \mathcal{M}$.

Recall that $\text{pd}_Q(M) = 1$ for any MCM R-module M, as a consequence of the Auslander-Buchsbaum formula. So we can choose free resolutions $F'_\bullet$ of M', $F''_\bullet$ of M'' of length one, and construct a free resolution $F_\bullet$ of M using the horseshoe lemma to obtain the commutative diagram below with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1' & \xrightarrow{\alpha_1} & F_1 & \xrightarrow{\beta_1} & F_1'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota''} & & \\
0 & \longrightarrow & F_0' & \xrightarrow{\alpha_0} & F_0 & \xrightarrow{\beta_0} & F_0'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi''} & & \\
0 & \longrightarrow & M' & \xrightarrow[\alpha_{-1}]{} & M & \xrightarrow[\beta_{-1}]{} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

First, we claim the submodule $\alpha_0(F_0') + \iota(F_1)$ of F_0 is a projective Q-module. In fact, we claim it is the kernel of the surjective composition $\pi''\beta_0 : F_0 \to M''$. Assuming this holds, then there is a short exact sequence

$$(\dagger) \qquad 0 \to \alpha_0(F_0') + \iota(F_1) \to F_0 \to M'' \to 0$$

and $\alpha_0(F_0') + \iota(F_1)$ is projective because M'' has projective dimension 1 over Q.

The claim is verified by a diagram chase. Let $x \in F_0$ such that $\pi''\beta_0(x) = 0$. By the exactness of the right column, there exists a $y'' \in F_1''$ so that $\iota''(y'') = \beta_0(x)$. This lifts to an element $y \in F_1$ with $\beta_0\iota(y) = \beta_0(x)$. Then $x - \iota(y) \in \ker(\beta_0)$, so there is some $z \in F_0'$ with $\alpha_0(z) = x - \iota(y)$. Rearranging yields $x \in \alpha_0(F_0') + \iota(F_1)$, so $\ker(\pi''\beta_0) \subseteq \alpha_0(F_0') + \iota(F_1)$.

Conversely, if $x = \alpha_0(a) + \iota(b) \in \alpha_0(F_0') + \iota(F_1)$ then

$$\pi''\beta_0(x) = \pi''\beta_0(\alpha_0(a) + \iota(b)) = \pi''\beta_0\iota(b) = \beta_{-1}\pi\iota(b) = 0$$

so $\ker(\pi''\beta_0) = \alpha_0(F_0') + \iota(F_1)$ as claimed.

We can now define P to have graded pieces $P_1 = F_1$, $P_2 = \alpha_0(F_0') + \iota(F_1)$, $P_3 = F_0$. We define the maps d_1 to be ι composed with the inclusion and d_2 the inclusion into F_0. The map $d_3 : F_0 \to F_1$ is constructed as the unique lifting of the map of short exact sequences given by multiplication by f below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{\iota} & F_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow f & \swarrow d_3 & \downarrow f & & \downarrow f=0 & & \\ 0 & \longrightarrow & F_1 & \xrightarrow[\iota]{} & F_0 & \xrightarrow[\pi]{} & M & \longrightarrow & 0 \end{array}$$

We now examine $\Psi(P)$. We need to show $\Psi(P) \cong \mathcal{M}$ in $\mathcal{E}(\mathsf{MCM}(R))$. In particular, we need to show $\operatorname{coker}(d_1) \cong M'$, $\operatorname{coker}(d_2) \cong M''$, and $\operatorname{coker}(d_2 d_1) \cong M$. The second isomorphism was shown in (†), and the third is clear because $F_\bullet$ is a resolution of M. The first isomorphism follows from the sequence of isomorphisms below.

$$\frac{\alpha_0(F_0') + \iota(F_1)}{\iota(F_1)} \cong \frac{\alpha_0(F_0')}{\alpha_0(F_0') \cap \iota(F_1)} \cong \frac{F_0'}{F_1'} \cong M'$$

Finally, it is clear that these isomorphisms commute with the maps in the short exact sequences, forming the required isomorphism in $\mathcal{E}(\mathsf{MCM}(R))$.

Full: Suppose $\beta \in \operatorname{Hom}_{\mathcal{E}(\mathsf{MCM}(R))}(\Psi(P), \Psi(P'))$. That is, we have a commutative diagram of the form below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{coker}(d_1) & \longrightarrow & \operatorname{coker}(d_2 d_1) & \longrightarrow & \operatorname{coker}(d_2) & \longrightarrow & 0 \\ & & \downarrow \beta_1 & & \downarrow \beta_3 & & \downarrow \beta_2 & & \\ 0 & \longrightarrow & \operatorname{coker}(d_1') & \longrightarrow & \operatorname{coker}(d_2' d_1') & \longrightarrow & \operatorname{coker}(d_2') & \longrightarrow & 0 \end{array}$$

We need to construct a morphism $\alpha \in \mathrm{Hom}_{\text{3-MF}(Q,f)}(P, P')$ with $\Psi(\alpha) = \beta$. To this end, consider the commutative diagram of Q-modules with exact rows below. Define α_3 to be a lifting of β_3. This determines α_1 by restriction.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & P_3 & \longrightarrow & \operatorname{coker}(d_2 d_1) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha_1} & & \Big\downarrow{\scriptstyle \alpha_3} & & \Big\downarrow{\scriptstyle \beta_3} & & \\
0 & \longrightarrow & P_1' & \longrightarrow & P_3' & \longrightarrow & \operatorname{coker}(d_2' d_1') & \longrightarrow & 0
\end{array}$$

Since α_3 is also a lifting of β_2 as in the diagram below, we also can define α_2 by restriction.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & \operatorname{coker}(d_2) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha_2} & & \Big\downarrow{\scriptstyle \alpha_3} & & \Big\downarrow{\scriptstyle \beta_2} & & \\
0 & \longrightarrow & P_2' & \longrightarrow & P_3' & \longrightarrow & \operatorname{coker}(d_2') & \longrightarrow & 0
\end{array}$$

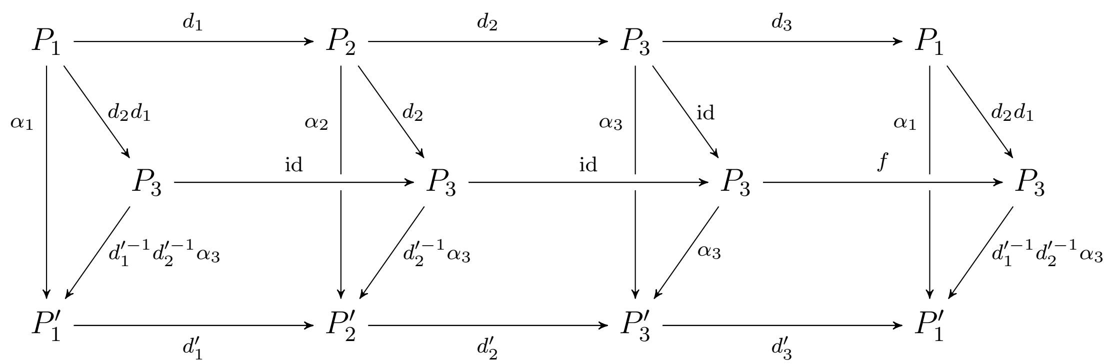
It remains to show that this choice of Q-module homomorphisms $\alpha_1, \alpha_2, \alpha_3$ forms a morphism of matrix factorizations. Precisely, we need to show $\alpha_{i+1} d_i = d_i' \alpha_i$ for $i = 1, 2, 3$.

For $i = 1$, we first notice that $\alpha_2 d_1 = d_1' \alpha_1$ if and only if $d_2' \alpha_2 d_1 = d_2' d_1' \alpha_1$ because d_2' is injective. By construction, $d_2' \alpha_2 = \alpha_3 d_2$, so the required result holds if and only if $\alpha_3 d_2 d_1 = d_2' d_1' \alpha_1$. The final equality is true by construction.

This holds for $i = 2$ by construction.

For $i = 3$, we have $\alpha_1 d_3 = d_3' \alpha_3$ if and only if $d_2' d_1' \alpha_1 d_3 = d_2' d_1' d_3' \alpha_3$ because $d_2' d_1'$ is injective. Now $d_2' d_1' d_3' = f \cdot \mathrm{id}$ by definition, and multiplication by f commutes with α_3, so the result holds if and only if $d_2' d_1' \alpha_1 d_3 = \alpha_3 f$. By construction, $d_2' d_1' \alpha_1 = \alpha_3 d_2 d_1$, so the result holds if and only if $\alpha_3 d_2 d_1 d_3 = \alpha_3 f$, which holds by definition.

Faithful: Suppose $\alpha \in \mathrm{Hom}_{\mathsf{3\text{-}MF}(Q,f)}(P, P')$ and $\Psi(\alpha) = 0$. That is, $\mathrm{im}(\alpha_2) \subseteq \mathrm{im}(d_1')$ and $\mathrm{im}(\alpha_3) \subseteq \mathrm{im}(d_2'd_1') \subseteq \mathrm{im}(d_1')$. Then α factors through the trivial matrix factorization $(\mathrm{id}, \mathrm{id}, f)$ with graded pieces P_3' as shown below.



To see the induced equivalence, we notice the following:

$$\Psi(\mathrm{id}, f, \mathrm{id}) = 0 \to 0 \to P_3 \to P_3 \to 0$$

$$\Psi(f, \mathrm{id}, \mathrm{id}) = 0 \to P_2 \to P_2 \to 0 \to 0.$$

Further, any short exact sequence of projectives splits as

$$0 \to P' \to P \to P'' \to 0 = \begin{array}{c} 0 \to P' \to P' \to 0 \to 0 \\ \oplus \\ 0 \to 0 \to P'' \to P'' \to 0 \end{array}$$

so the images of $\mathscr{T}_1$ and $\mathscr{T}_2$ generate precisely $\mathcal{E}(\mathsf{PROJ}(R))$. □

3.2 Higher Homotopies

In [5], Eisenbud developed a theory of "higher homotopies", which may be used to construct a matrix factorization from an arbitrary R-module. In this section, we generalize this to a method of constructing a 3-fold matrix factorization from an arbitrary short exact sequence of R-modules.

Throughout, we will continue to have Q a regular ring, $f \in Q$ a non-zero-divisor, and $R = Q/(f)$. We begin with notation and definitions.

Definition 3.2.1. Let $\mathcal{E}(R - mod)$ denote the category of short exact sequences of R-modules, with morphisms defined to be triples of R-linear maps making the obvious diagrams commute.

Similarly, let $\mathcal{E}(\mathsf{Ch}(R))$ denote the category of short exact sequences of chain complexes of R-modules. A morphism of degree i in $\mathcal{E}(\mathsf{Ch}(R))$ is defined to be a triple of morphisms of chain complexes of degree i which is also degree-wise a morphism of short exact sequences of R-modules.

Theorem 3.2.2. *Let Q be a regular ring, $f \in Q$ a non-zero-divisor, $R = Q/(f)$, and let*

$$\mathcal{M}: \qquad 0 \to M' \xrightarrow{\iota_{-1}} M \xrightarrow{\pi_{-1}} M'' \to 0$$

be a short exact sequence of R-modules. If

$$\mathcal{F}_\bullet: \qquad 0 \to F'_\bullet \xrightarrow{\iota} F_\bullet \xrightarrow{\pi} F''_\bullet \to 0$$

is a Q-free resolution of $\mathcal{M}$ (by which we mean $F_\bullet$, $F'_\bullet$, and $F''_\bullet$ are Q-free resolutions of M, M', and M'', respectively, and form a short exact sequence of complexes), then there exist endomorphisms s_i of $\mathcal{F}_\bullet$ of degree $2i-1$ satisfying

(i) s_0 *is the differential of* $\mathcal{F}_\bullet$,

(ii) $s_1 s_0 + s_0 s_1 = f$, *and*

(iii) for all $i > 1$, $\sum_{j+k=i} s_j s_k = 0$.

Proof. We will notate by ι_i, π_i the maps $F'_i \to F_i$ and $F_i \to F''_i$, respectively. Further, let the triple $\underline{\delta_i} = (\delta'_i, \delta_i, \delta''_i)$ denote the differential of $\mathcal{F}_\bullet$ in degree i and $\underline{\varepsilon} = (\varepsilon', \varepsilon, \varepsilon'')$ denote the augmentation map $\mathcal{F}_\bullet \to \mathcal{M}$, as in the diagram below.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F_2' & \xrightarrow{\delta_2'} & F_1' & \xrightarrow{\delta_1'} & F_0' & \xrightarrow{\varepsilon'} & M' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle \iota_{-1}} & & \\
\cdots & \longrightarrow & F_2 & \xrightarrow{\delta_2} & F_1 & \xrightarrow{\delta_1} & F_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_{-1}} & & \\
\cdots & \longrightarrow & F_2'' & \xrightarrow{\delta_2''} & F_1'' & \xrightarrow{\delta_1''} & F_0'' & \xrightarrow{\varepsilon''} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

We choose $s_0 = \underline{\delta}$. Since M', M, M'' are annihilated by f, multiplication by f is null homotopic on each row of $\mathcal{F}_\bullet$. Choose s_1 to be a realization of this homotopy on $F_\bullet$, the resolution of M. This induces maps on the resolutions of M' and M'' using the degree-wise short exact sequences

$$0 \to F_i' \to F_i \to F_i'' \to 0,$$

so that we in fact have s_1 is a degree 1 endomorphism of $\mathcal{F}_\bullet$. By the definition of null homotopy, s_1 will satisfy condition (ii).

We construct s_i for $i > 1$ inductively. Set $e_i := \sum\limits_{\substack{j+k=i \\ j\neq i \\ k\neq i}} s_j s_k$. We claim $e_i s_0 = s_0 e_i$. Indeed, using our inductive hypothesis and condition (ii), as well as multiplication by f commuting with the Q-linear map s_{i-1}, we see

$$\begin{aligned}
e_i s_0 &= \sum_{\substack{j+k\neq i\\ j\neq i\\ k\neq i}} s_j s_k s_0 \\
&= \sum_{j=1}^{i-2}\sum_{k=i-j}^{i-1} s_j s_k s_0 + s_{i-1}s_1 s_0 \\
&= \sum_{j=1}^{i-1}\sum_{k=i-j}^{i=1} -s_j\left(s_0 s_k + \sum_{\substack{\ell+m=k\\ \ell\neq k\\ m\neq k}} s_\ell s_m\right) + s_{i-1}(f - s_0 s_1) \\
&= -(f - s_0 s_1)s_{i-1} + \sum_{j=2}^{i-2}\sum_{k=i-j}^{i-1}\sum_{\substack{\ell+m=j\\ \ell\neq j}} s_\ell s_m s_k - \sum_{j=1}^{i-2}\sum_{k=i-j}^{i-1}\sum_{\substack{\ell+m=k\\ \ell\neq k\\ m\neq k}} s_j s_\ell s_m \\
&\qquad + s_{i-1}f + \sum_{\substack{\ell+m=i-1\\ \ell\neq i-1}} s_\ell s_m s_1 \\
&= \sum_{\substack{j+k\neq i\\ j\neq i\\ k\neq i}} s_0 s_j s_k \\
&= s_0 e_i.
\end{aligned}$$

Now notice the induced map

$$M = \operatorname{coker}(\delta_1) \xrightarrow{-e_i} \operatorname{coker}(\delta_{2i}) \subseteq F_{2i-2}$$

is null homotopic, since M is annihilated by f. So there exists a map s_i of degree $2i-1$ on $F_\bullet$, which again induces maps on the resolutions of M' and M'', satisfying $-e_i = s_i s_0 + s_0 s_i$. Rearranging yields the desired result. □

In the setting of Theorem 3.2.2, we then can construct a 3-fold matrix factorization using the constructed maps s_i as outlined below.

Construction 3.2.3. We will first define a short exact sequence of (classical) matrix factorizations. Consider the following two short exact sequences of free modules

$$\mathcal{F}_{even}: \quad 0 \to F'_{even} \to F_{even} \to F''_{even} \to 0$$
$$\mathcal{F}_{odd}: \quad 0 \to F'_{odd} \to F_{odd} \to F''_{odd} \to 0$$

where F_{even}, F_{odd} denote the direct sums of the even degree and odd degree free modules in the resolution of M, respectively (and similarly for the "primed" versions). Note that these are short exact sequences of finitely generated free modules because Q is regular.

We define maps $A : \mathcal{F}_{even} \to \mathcal{F}_{odd}$ and $B : \mathcal{F}_{odd} \to \mathcal{F}_{even}$ by $A = \sum s_i = B$. By applying the properties of the family s_i in Theorem 3.2.2, we see $AB = f = BA$, so the pair of maps (A, B) is in fact a short exact sequence of 2-fold matrix factorizations.

The cokernel of $B : \mathcal{F}_{odd} \to \mathcal{F}_{even}$ is then, by Eisenbud's correspondence, a short exact sequence of MCM modules. Applying the equivalence of Theorem 3.1.4 to this short exact sequence yields a 3-fold matrix factorization.

Notice that, if $\mathcal{M}$ was a short exact sequence of MCM modules already, then s_1 and s_0 are the only non-zero maps constructed, $\mathcal{F}_{odd} = \mathcal{F}_1$, and $\mathcal{F}_{even} = \mathcal{F}_0$. In this case, the construction above gives the map $B = s_0$, so the 3-fold matrix factorization constructed is precisely one mapping to $\mathcal{M}$ under Theorem 3.1.4.

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