THE BASIS NUMBER OF THE STRONG PRODUCT OF PATHS AND CYCLES WITH BIPARTITE GRAPHS

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Abstract. The basis number of a graph G is defined to be the least integer d such that there is a basis $\mathcal{B}$ of the cycle space of G such that each edge of G is contained in at most d members of $\mathcal{B}$. MacLane [13] proved that a graph G is planar if and only if the basis number of G is less than or equal to 2. Ali [3] proved that the basis number of the strong product of a path and a star is less than or equal to 4. In this work,

(1) We give an appropriate decomposition of trees.
(2) We give an upper bound of the basis number of a cycle and a bipartite graph.
(3) We give an upper bound of the basis number of a path and a bipartite graph.

This is a generalization of Ali's result [3].

1. Introduction. Unless otherwise specified, all graphs considered here are connected, finite, undirected, and simple. We start by introducing the definitions of the following basic product graphs. Let G and H be two graphs.

(1) The direct product $G^* = G \wedge H$ has vertex-set $V(G) \times V(H)$ and edge-set $E(G^*) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$.
(2) The cartesian product $G^* = G \times H$ has vertex-set $V(G^*) = V(G) \times V(H)$ and edge-set $E(G^*) = \{(u_1, u_2)(v_1, v_2) \mid u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)\}$.
(3) The strong product $G^* = G \otimes H$ has vertex-set $V(G^*) = V(G) \times V(H)$ and edge set $E(G^*) = \{(u_1, u_2)(v_1, v_2) \mid u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$ or $u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$.
(4) The semi-strong product $G^* = G \bullet H$ has vertex set $V(G^*) = V(G) \times V(H)$ and edge set $E(G^*) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)\}$.
(5) The lexicographic product $G^* = G[H]$ has vertex set $V(G^*) = V(G) \times V(H)$ and edge set $E(G) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)\}$.

It is clear to see that the cartesian, the direct, and the strong are commutative products and the lexicographic and the semi-strong are non-commutative products. Moreover, $G \times H \subset G \otimes H \subset G[H]$ and $G \wedge H \subset G \bullet H \subset G \otimes H$.

Given a graph G, let $e_1, e_2, \dots, e_{|E(G)|}$ be an ordering of its edges. Then a subset S of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \dots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integer numbers modulo 2. The vectors in $(Z_2)^{|E(G)|}$, which correspond to the cycles in G, generate a subspace called the cycle space of G and are denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r$, where r is the number of connected components.

Definition 1.1. A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a d-fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. The basis number, $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. The required basis of G is a basis $\mathcal{B}$ of $b(G)$-fold.

Let G and H be two graphs. Let $\varphi\colon G \to H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of G. Then $\{\varphi(c) \mid c \in \mathcal{B}\}$ is called the corresponding (required) basis of $\mathcal{B}$ in H.

Ali [3] investigated the basis number of the strong product of some special graphs. In fact, he proved that $b(C \otimes C^*)$ and $b(P \otimes S)$ are less than or equal to 4 for any two cycles C and C^*, and for any path P and star S.

In this work, we shall be primarily concerned with giving an upper bound of the basis number of the strong product of a cycle (path) with a bipartite graph, which were unavailable even in the simple setting.

From now on, H_a stands for the copy $a \times H$, and $\mathcal{B}_H$ stands for the required basis of H.

2. Known Upper Bounds.

In this section, we list the most important known upper bounds of the basis numbers of graphs. The first important result concerning the basis number of a graph was obtained in 1937 by MacLane [13], who proved the following result.

Theorem 2.1. (MacLane) The basis number of a graph is less than or equal to 2 if and only if G is planar.

The global upper bound of a graph G and the existence of large basis numbers were given in 1981 by Schmeichel [14], who gave the following results.

Theorem 2.2. (Schmeichel) Let G be a graph with genus $\gamma(G)$. Then $b(G) \leq 2\gamma(G) + 2$.

Theorem 2.3. (Schmeichel) For any integer r, there is a graph with basis number greater than or equal to r.

Schmeichel [14] and Ali [1] gave upper bounds of the basis number of complete graphs, complete bipartite graphs, and complete multipartite graphs as in the following two theorems.

Theorem 2.4. (Schmeichel) $b(K_n) \leq 3$ and $b(K_{m,n}) \leq 4$, where K_n is the complete graph of order n and $K_{m,n}$ is the complete bipartite graph of order $m + n$.

Theorem 2.5. (Ali) $b(K_{n(m)}) \leq 9$, where $K_{n(m)}$ is the complete multipartite graph of order mn.

In 1982, Banks and Schmeichel [6] proved the following result.

Theorem 2.6. (Banks and Schmeichel) $b(Q_n) \leq 4$, where Q_n is the n-cube.

Many authors have studied the basis number of graph products. The cartesian product of any two graphs was studied by Ali and Marougi [4] and Alsardary and Wojciechowski [5].

Theorem 2.7. (Ali and Marougi) If G and H are two connected disjoint graphs, then $b(G \times H) \leq \max\{b(G) + \triangle(T_H), b(H) + \triangle(T_G)\}$, where T_H and T_G are spanning trees of H and G, respectively, such that the maximum degrees $\triangle(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of H and G.

Theorem 2.8. (Alsardary and Wojciechowski) For every $d \geq 1$ and $n \geq 2$, we have $b(K_n^d) \leq 9$, where K_n^d is a d times cartesian product of the complete graph K_n.

A tree T consisting of n equal order paths $\{P^{(1)}, P^{(2)}, \dots, P^{(n)}\}$ is called an n-special star if there is a vertex, say v, such that v is an end vertex for each path in $\{P^{(1)}, P^{(2)}, \dots, P^{(n)}\}$ and $V(P^{(i)}) \cap V(P^{(j)}) = \{v\}$ for each $i \neq j$ (see [9]). Ali [2] and Jaradat [11] gave an upper bound for the basis number of the semi-strong and the direct products of some special graphs. They proved the following results.

Theorem 2.9. (Jaradat) Let G be a bipartite graph and C_n be a cycle. Then $b(G \bullet C_n) \leq 4 + b(G)$. Moreover, $b(G \bullet C_n) \leq 3 + b(G)$ if G has a spanning tree containing no subgraph isomorphic to a 3-special star of order 7.

Theorem 2.10. (Ali) For any two cycles C_n and C_m, $b(C_n \wedge C_m) \leq 3$.

Theorem 2.11. (Jaradat) For each bipartite graph G and H, $b(G \wedge H) \leq 5 + b(G) + b(H)$.

Theorem 2.12. (Jaradat) Let P_m and θ_n be a path and a theta graph of order m and n, respectively. Then $b(P_m \wedge \theta_n) \leq 3$.

Moreover, Jaradat [9] classified trees with respect to the basis number of their direct product with paths of order greater than or equal to 5 and provided necessary and sufficient conditions for the direct product of two trees, $T_1 \wedge T_2$, to be non-planar as in the following results.

Theorem 2.13. (Jaradat) Let T be a tree and P be a path of order ≥ 5. Then $b(T \wedge P) \leq 3$. Moreover,

(1) $b(T \wedge P) = 0$ if and only if $|V(T)| \leq 2$.
(2) $b(T \wedge P) = 1$ if and only if $|V(T)| = 3$.
(3) $b(T \wedge P) = 2$ if and only if $|V(T)| \geq 4$ and T has no subgraph isomorphic to a 3-special star of order 7.
(4) $b(T \wedge P) = 3$ if and only if T has a subgraph isomorphic to a 3-special star of order 7.

Theorem 2.14. (Jaradat) For any two trees T_1 and T_2 such that $|V(T_1)|$, $|V(T_2)| \geq 5$, we have that $T_1 \wedge T_2$ is non-planar ($b(T_1 \wedge T_2) > 2$) if and only if one of the following holds:

(i) $\Delta(T_1) \geq 3$ and $\Delta(T_2) \geq 3$.
(ii) One of them is a path and the other contains a subgraph isomorphic to a 3-special star of order 7.

The lexicographic product was studied by Jaradat [12]. He obtained the following result.

Theorem 2.15. (Jaradat) For each two connected graphs G and H, $b(G[H]) \leq \max\{4, 2\Delta(G) + b(H), 2 + b(G)\}$.

3. A Certain Decomposition of a Tree. It should be mentioned that establishing an upper bound of the basis number of the strong product of a cycle (path) and a bipartite graph cannot be done using existing methods, because bipartite graphs have no uniform forms. Therefore, we shall give a certain decomposition which decomposes a tree (which has no uniform form) into stars and paths of order 2. Let T be a tree of order ≥ 3 and let $EV = \{v \in V(T) \mid v$ is an end vertex of T and either of the following (i) or (ii) holds$\}$.

(i) v is adjacent to a vertex of degree 2.
(ii) v is adjacent to a vertex of degree greater than or equal to 3, say v^*, and every adjacent vertex to v^*, with possibly one exception, is of degree 1. It was shown in [9] that $EV \neq \emptyset$.

Let $\{H_i\}_{i=1}^{m}$ be a set of graphs. Then the decomposition $\cup_{i=1}^{m} H_i$ is defined to be the graph induced by the union of vertices and edges of $H_1, H_2, \ldots, H_m$.

Proposition 3.1. Let T be a tree of order ≥ 2. Then T can be decomposed into $\cup_{i=1}^{r} S_i$ of subgraphs $S_1, S_2, \ldots, S_r$ for some integer r, such that the following holds:

(i) For each $i \geq 1$, S_i is either a star or a path of order 2 and S_1 is a path incedent with an end vertex.
(ii) $E(S_i) \cap E(S_j) = \phi$ for each $i \neq j$.
(iii) $\cup_{i=1}^{r} E(S_i) = E(T)$.
(iv) For each $v \in V(T)$, if $d_T(v) \geq 2$, then $|\{i : v \in V(S_i)\}| = 2$, and if $d_T(v) = 1$, then $|\{i : v \in V(S_i)\}| = 1$.
(v)

$$V(S_i) \cap (\cup_{j=1}^{i-1} V(S_j)) = v_1^{(i)},$$

where

$$d_{S_i}(v_1^{(i)}) = \max_{v \in V(S_i)} d_{S_i}(v), d_{\cup_{j=1}^{i-1} S_j}(v_1^{(i)}) = 1$$

for each $i = 2, 3, \ldots, r$, and $v_1^{(i)} \neq v_1^{(j)}$ for each $i \neq j$.

Proof. We shall proceed by induction on the order of T. If T is a tree of order 2, then T is a path of order 2 and so we take $S_1 = T$. Assume T is a tree of order $n+1$ and $v \in EV$. Set $T^{'} = T - v$. Then, by the inductive step, $T^{'} = \cup_{i=1}^{r} S_i^{'}$, which satisfies the required conditions. Let $vv^* \in E(T)$. We now consider two cases.

Case a. $d_T(v^*) = 2$. Then we decompose T into $\cup_{i=1}^{r+1} S_i$, where $S_i = S_i^{'}$ for $1 \leq i \leq r$ and $S_{r+1} = vv^*$ if v^* is not incident with $S_1^{'}$; otherwise, $S_1 = vv^*$ and $S_{i+1} = S_i^{'}$ for $1 \leq i \leq r$.

Case b. $d_T(v^*) \geq 3$. Then there are $1 \leq s, t \leq r$ such that v^* belongs only to $V(S_s^{'})$ and $V(S_t^{'})$. Since $d_T(v^*) \geq 3$ and $v \in EV$, there is an end vertex, say v_1^*, such that $v^* v_1^*$ is an edge that belongs to only one of $E(S_s^{'})$ and $E(S_t^{'})$, say $E(S_s^{'})$. We have two subcases to consider.

Subcase b1. $S_s^{'} \neq S_1^{'}$. Then we decompose T into $\cup_{i=1}^{r} S_i$, where $S_i = S_i^{'}$ for $i \neq s$ and $S_s = S_s^{'} \cup vv^*$. Note that if $S_s^{'}$ is an edge, then S_s is a star of order 3 and if $S_s^{'}$ is a star of order l, then S_s is a star of order $l+1$.

Subcase b2. $S_s^{'} = S_1^{'}$. Then $t > 1$ and so we decompose T into $\cup_{i=1}^{r} S_i$, where $S_i = S_i^{'}$ for $i \neq t$ and $S_t = S_t^{'} \cup vv^*$. Similarly, if $S_t^{'}$ is an edge, then S_t is a star of order 3 and if $S_t^{'}$ is a star of order l, then S_t is a star of order $l+1$.

It is clear that the above decomposition, whether in Case a or Case b, satisfies the required conditions of the proposition. The proof is complete.

4. Strong Product of Cycles and Bipartite Graphs. In this section, we give an upper bound of the basis number of a path (cycle) and a bipartite graph.

Lemma 4.1. For every tree T of order ≥ 2 and path P_2 of order equal to 2, $b(P_2 \otimes T) = 2$.

Proof. To prove that $b(P_2 \otimes T) \leq 2$, by MacLane's Theorem, it suffices to show that $P_2 \otimes T$ is planar. Let $T = \bigcup_{j=1}^{r} S_j$ as in Proposition 3.1, and let $V(S_j) = \{v_1^{(j)}, v_2^{(j)}, \ldots, v_{n_j}^{(j)}\}$ such that $d_{S_j}(v_1^{(j)}) = n_j - 1$. Let $V(P_2) = \{x, y\}$. For each $j = 1, 2, \ldots, r$, define the following set of cycles.

$$\begin{aligned}
\mathcal{A}_j = \{a_j^{(1_1)} &= (x, v_1^{(j)})(y, v_1^{(j)})(x, v_2^{(j)})(x, v_1^{(j)}),\\
a_j^{(2_1)} &= (y, v_1^{(j)})(y, v_2^{(j)})(x, v_2^{(j)})(y, v_1^{(j)}),\\
a_j^{(3_1)} &= (x, v_1^{(j)})(y, v_2^{(j)})(x, v_2^{(j)})(x, v_1^{(j)})\},\\
\mathcal{B}_j^{(1)} = \{a_j^{(1_i)} &= (y, v_1^{(j)})(y, v_{i+1}^{(j)})(x, v_{i+1}^{(j)})(y, v_1^{(j)}) \mid i = 2, 3, \ldots, n_j - 1\},\\
\mathcal{B}_j^{(2)} = \{a_j^{(2_i)} &= (x, v_1^{(j)})(x, v_{i+1}^{(j)})(y, v_1^{(j)})(y, v_i^{(j)})(x, v_1^{(j)})\\
&\qquad \mid i = 2, 3, \ldots, n_j - 1\},
\end{aligned}$$

and

$$\mathcal{B}_j^{(3)} = \{a_j^{(3_i)} = (x, v_1^{(j)}), (x, v_{i+1}^{(j)})(y, v_{i+1}^{(j)})(x, v_1^{(j)}) \mid i = 2, 3, \ldots, n_j - 1\}.$$

Set $\mathcal{B}_j^{'} = A_j \cup \mathcal{B}_j^{(1)} \cup \mathcal{B}_j^{(2)} \cup \mathcal{B}_j^{(3)}$. It is straightforward to see that $\mathcal{B}_j^{'}$ is a 2-fold basis of $\mathcal{C}(P_2 \otimes S_j)$. And so, $P_2 \otimes S_j$ is a planar subgraph for each $j = 1, 2, \ldots, r$. In fact, the cycles of $\mathcal{B}_j^{'}$ are the finite faces of $P_2 \otimes S_j$ for each $j = 1, 2, \ldots, r$. Moreover, we can draw $P_2 \otimes S_j$ in such a way that $(x, v_1^{(j)})(y, v_1^{(j)})$ is a common edge of the infinite face of $P_2 \otimes S_j$ and the finite face $a_j^{(1_1)}$ for each $j = 1, 2, \ldots, r$. By (v) of Proposition 2.1, we have that $V(S_1) \cap V(S_2) = v_1^{(2)}$. Hence,

$$E(P_2 \otimes S_1) \cap E(P_2 \otimes S_2) = \{(x, v_1^{(2)})(y, v_1^{(2)})\} = E(a_1^{(3_1)}) \cap E(a_2^{(1_1)}).$$

Thus, the subgraph $P_2 \otimes S_2$ can be drawn without crossing in the face $a_1^{(3_1)}$. Therefore, the subgraph $P_2 \otimes (S_1 \cup S_2)$ is planar. Similarly, by (v) and (vi) of Proposition 2.1, $V(S_1 \cup S_2) \cap V(S_3) = v_1^{(3)}$ and $v_1^{(3)} \neq v_1^{(2)}$. Moreover, there is $1 \leq i_2 \leq n_2$ such that $v_1^{(3)} = v_{i_2}^{(2)}$. Hence,

$$E(P_2 \otimes (S_1 \cup S_2)) \cap E(P_2 \otimes S_3) = \{(x, v_1^{(3)})(y, v_1^{(3)})\} = E(a_3^{(1_1)}) \cap E(a_2^{(3_{i_2})}).$$

Thus, the subgraph $P_2 \otimes S_3$ can be drawn without crossing in the face $a_2^{(3_{i_2})}$. Therefore, the subgraph $P_2 \otimes (S_1 \cup S_2 \cup S_3)$ is planar. Continuing in this procedure, we have that $P_2 \otimes (\cup_{j=1}^{r-1} S_j)$ is planar and there are $2 \leq s \leq r-1$ and $1 \leq i_s \leq n_s$ such that $P_2 \otimes S_r$ can be drawn without crossing in the face $a_s^{(3i_s)}$. Therefore, $P_2 \otimes T = P_2 \otimes (\cup_{j=1}^{r} S_j)$ is a planar graph. Now, the inequality $b(P_2 \otimes T) \geq 2$ holds because any spanning set for $\mathcal{C}(P_2 \otimes T)$ is not an edge-pairwise disjoint of cycles (see [9]). The proof is complete.

Let G be a planar graph, then by Euler's formula

$$|E(G)| - |V(G)| + 2 = f,$$

where f is the number of faces. Hence,

$$\dim \mathcal{C}(G) = f - 1 = \text{ the number of finite faces.}$$

Therefore, it is easy to see that the set of all finite faces forms a 2-fold basis. Throughout this work, we consider $\mathcal{B}_{P_2}$ to be the basis which consists of the finite faces of the planar graph $P_2 \otimes T$.

Theorem 4.1. For each path P of order ≥ 3 and tree T of order ≥ 4, $3 \leq b(P \otimes T) \leq 4$.

Proof. Let $P = a_1 a_2 \ldots a_{|V(P)|}$ and $P_2^{(k)} = a_k a_{k+1}$. Set $\mathcal{B}(P \otimes T) = \bigcup_{k=1}^{|V(P)|-1} \mathcal{B}_{P_2^{(k)}}$. We shall proceed by mathematical induction on the order of P to show that $\mathcal{B}$ is linearly independent. If P is a path of order 2, then $\mathcal{B} = \mathcal{B}_{P_2^{(1)}}$ and so, by Lemma 3.1, $\mathcal{B}(P \otimes T)$ is linearly independent. Assume that P is a path of order $h+1$ (i.e. $|V(P)| = h+1$). Let $P' = a_1 a_2 \ldots a_{|V(P)|-1}$. Then P' is a path of order $h = |V(P)| - 1$. Note that

$$\mathcal{B}(P \otimes T) = \left(\bigcup_{k=1}^{|V(P)|-2} \mathcal{B}_{P_2^{(k)}} \right) \cup \mathcal{B}_{P_2^{(|V(P)|-1)}} = \mathcal{B}(P' \otimes T) \cup \mathcal{B}_{P_2^{(|V(P)|-1)}}.$$

By the inductive step and Lemma 3.1, each of

$$\mathcal{B}(P' \otimes T) = \bigcup_{k=1}^{|V(P)|-2} \mathcal{B}_{P_2^{(k)}} \text{ and } \mathcal{B}_{P_2^{(|V(P)|-1)}}$$

is linearly independent. Now we show that the cycles of $\mathcal{B}(P' \otimes T)$ are linearly independent of cycles of $\mathcal{B}_{P_2^{(|V(P)|-1)}}$. Note that,

$$E(\mathcal{B}_{P_2^{(m)}}) \cap E(\mathcal{B}_{P_2^{(n)}}) = \begin{cases} E(T_{a_n}), & \text{if } m = n-1, \\ E(T_{a_m}), & \text{if } m = n+1, \\ \emptyset, & \text{if } |n-m| > 1. \end{cases} \qquad (1)$$

Thus, if

$$\sum_{i=1}^{t} c_i + \sum_{i=1}^{f} d_i = 0 \pmod 2,$$

where $\{d_i\}_{i=1}^{t} \subseteq \cup_{k=1}^{|V(P)|-2} \mathcal{B}_{P_2^{(k)}}$ and $\{c_i\}_{i=1}^{f} \in \mathcal{B}_{P_2^{(|V(P)|-1)}}$, then

$$E(c_1 \oplus c_2 \oplus \cdots \oplus c_t) = E(d_1 \oplus d_2 \oplus \cdots \oplus d_f).$$

Therefore by (1), the ring sums $c_1 \oplus c_2 \oplus \cdots \oplus c_t$ and $d_1 \oplus d_2 \oplus \cdots \oplus d_f$ are subgraphs of $T_{a_{(|V(P)|-1)}}$, which contradicts the fact that any linear combination of cycles of a linearly independent set is a cycle or an edge disjoint union of cycles. Thus, $\mathcal{B}(P \otimes T)$ is linearly independent. Since

$$|\mathcal{B}_{P_2^{(k)}}| = 3|E(T)|,$$

we have

$$|\mathcal{B}(P \otimes T)| = \sum_{k=1}^{|E(P)|} \mathcal{B}_{P_2^{(k)}} = 3|E(T)||E(P)| = \dim \mathcal{C}(P \otimes T).$$

Therefore, $\mathcal{B}(P \otimes T)$ is a basis for $\mathcal{C}(P \otimes T)$. Now let $e \in E(P \otimes T)$. If $e \in E(P \wedge T)$, then $f_{\mathcal{B}(P \otimes T)}(e) \leq 2$ and if $e \in E(P \times T)$, then $f_{\mathcal{B}(P \otimes T)}(e) \leq 4$. Thus, $\mathcal{B}(P \otimes T)$ is a 4-fold basis. So, $b(P \otimes T) \leq 4$. Since $|V(T)| \geq 3$, T contains either a path of order 4, P_4, or a star of order 4, S_4. Thus, at least one of $P \otimes P_4$ and $P \otimes S_4$ is a subgraph of $P \otimes T$. It is easy to see that $P_4 \otimes P_3$ and $S_4 \otimes P_3$ contain subgraphs homeomorphic to K_5 and $K_{3,3}$, respectively. Thus, $P \otimes T$ is non planar. And so, by MacLane's Theorem, $b(P \otimes T) \geq 3$. The proof is complete.

By specializing the tree in Theorem 4.1 into a star, we have the following result [3].

<u>Corollary 4.1</u>. For any path P of order ≥ 3 and a star S of order ≥ 4 we have $3 \leq b(P \otimes S) \leq 4$.

<u>Lemma 4.2</u>. For each tree T and cycle C we have that $3 \leq b(C \otimes T) \leq 4$.

<u>Proof</u>. Let $C = a_1 a_2 \ldots a_{|V(C)|} a_1$ and $\mathcal{B}^{**} = \cup_{k=1}^{|V(C)|} \mathcal{B}_{P_2^{(k)}}$, where $P_2^{(k)} = a_k a_{k+1}$ for $k = 1, 2, \ldots, |V(C)| - 1$ and $P_2^{(|V(C)|)} = a_{|V(C)|} a_1$. By Theorem 4.1, $\mathcal{B}^* = \cup_{k=1}^{|V(C)|-1} \mathcal{B}_{P_2^{(k)}}$ is linearly independent. Since each cycle of $\mathcal{B}_{P_2^{(|V(C)|)}}$ must contain at least one edge of $E(P_2^{(|V(C)|)} \wedge T) \cup E(P_2^{(|V(C)|)} \times N)$ which is not in any cycle of $E(\cup_{k=1}^{|V(C)|-1} \mathcal{B}_{P_2^{(k)}})$, where N

is the null graph with $V(N) = V(T)$, it follows that $\mathcal{B}^{**} = \cup_{k=1}^{|V(C)|} \mathcal{B}_{P_2^{(k)}}$ is linearly independent. Let uv be any edge of T such that u is an end vertex. Let Q be a $|V(C)|$-cycle defined as follows: if $|V(C)|$ is odd, then we take

$$Q = \{(a_1, u)(a_2, v)(a_3, u)(a_4, v) \dots (a_{|V(C)|-1}, v)(a_{|V(C)|}, u)(a_1, u)\},$$

and if $|V(C)|$ is even, then we take

$$Q = \{(a_1, u)(a_2, v)(a_3, u)(a_4, v) \dots (a_{|V(C)|-1}, u)(a_{|V(C)|}, v)(a_1, u)\}.$$

We now show that Q is linearly independent from the cycles of $\mathcal{B}^*$. Assume

$$Q = \sum_{i=1}^{\alpha_1} c_{1_i} + \dots + \sum_{i=1}^{\alpha_{|V(C)|}} c_{|V(C)|_i} \pmod 2,$$

where $c_{k_i} \in \mathcal{B}_{P_2^{(k)}}$. Thus,

$$\sum_{i=1}^{\alpha_1} c_{1_i} = Q + \sum_{i=1}^{\alpha_2} c_{2_i} + \dots + \sum_{i=1}^{\alpha_{|V(C)|}} c_{|V(C)|_i} \pmod 2.$$

Therefore,

$$E(c_{1_1} \oplus c_{1_2} \oplus \dots \oplus c_{1_{\alpha_1}}) = E(Q \oplus c_{2_1} \oplus c_{2_2} \oplus \dots \oplus c_{|V(C)|_{\alpha_{|V(C)|}}}),$$

which is the edge set of a cycle or an edge disjoint union of cycles because $c_{1_1} \oplus c_{1_2} \oplus \dots \oplus c_{1_{\alpha_1}}$ is the ring sum of cycles of a linearly independent set. Since each $\mathcal{B}_{P_2^{(k)}}$ contains only one edge of Q which is either $(a_i, u)(a_{i+1}, v)$ or $(a_i, v)(a_{i+1}, u)$ or $(a_{|V(C)|}, v)(a_1, u)$ or $(a_{|V(C)|}, u)(a_1, u)$ and

$$E(\mathcal{B}_{P_2^{(1)}}) \cap E(\cup_{k=2}^{|V(C)|} \mathcal{B}_{P_2^{(k)}}) = E(T_{a_1}) \cup E(T_{a_2}),$$

it implies that the ring sum $Q \oplus c_{2_1} \oplus c_{2_2} \oplus \dots \oplus c_{2_{\alpha_2}} \oplus \dots \oplus c_{|V(C)|_{\alpha_{|V(C)|}}}$ must be a subgraph of the forest $T_{a_1} \cup T_{a_2} \cup (a_1, v)(a_2, u)$. This is a contradiction. Thus, $\mathcal{B}(C \otimes T) = \mathcal{B}^{**} \cup \{Q\}$ is linearly independent. Since

$$|\mathcal{B}(C \otimes T)| = \sum_{k=1}^{|V(C)|} |\mathcal{B}_{P_2^{(k)}}| + 1 = 3|E(T)||V(C)| + 1 = \dim \mathcal{C}(C \otimes T),$$

we conclude that $\mathcal{B}(C \otimes T)$ is a basis for $\mathcal{C}(C \otimes T)$. To complete the proof of the theorem we show that $f_{\mathcal{B}(C \otimes T)}(e) \leq 4$ for each $e \in E(C \otimes T)$.

(1) If $e \in T_{a_i}$, where i is neither 1 nor $|V(C)|$, then $f_{\mathcal{B}^*}(e) \leq 4$, $f_{\mathcal{B}_{P_2^{(|V(C)|)}}}(e) = 0$, and $f_{\{Q\}}(e) = 0$ and so $f_{\mathcal{B}(C\otimes T)}(e) \leq 4$.

(2) If $e \in T_{a_i}$, where $i = 1$ or $|V(C)|$, then $f_{\mathcal{B}^*}(e) \leq 2$, $f_{\mathcal{B}_{P_2^{(|V(C)|)}}}(e) \leq 2$, and $f_{\{Q\}}(e) = 0$ and so $f_{\mathcal{B}(C\otimes T)}(e) \leq 4$.

(3) If $e \in E(C \wedge T) \cup E(C \times N)$, then $f_{\mathcal{B}(C\otimes T)-\{Q\}}(e) \leq 2$ and $f_{\{Q\}}(e) \leq 1$ and so $f_{\mathcal{B}(C\otimes T)}(e) \leq 3$. Thus, $\mathcal{B}(C \otimes T)$ is of 4-fold. Let $e \in E(T)$. Then it is not difficult to see that $C \otimes e$ contains a subgraph homeomorphic to K_5. Hence, $C \otimes T$ is non planar. Thus, $b(C \otimes T) \geq 3$. The proof is complete.

The following proposition of Jaradat [12] will be needed in proving the following result.

Proposition 4.1. (Jaradat) Let G be a bipartite graph and P_2 be a path of order 2. Then $G \wedge P_2$ consists of two components, G_1 and G_2, each of which is isomorphic to G.

Theorem 4.2. For each bipartite graph H and cycle C, we have $3 \leq b(C \otimes H) \leq b(H) + 4$.

Proof. Let $\mathcal{B}'$ be the basis of $C \otimes T$ as in Lemma 4.2, where T is a spanning tree of H. By Proposition 4.1, for each $e \in E(C)$, $e \wedge H$ consists of two components, each of which is isomorphic to H. Hence, we set $\mathcal{B}_e = \mathcal{B}_e^{(1)} \cup \mathcal{B}_e^{(2)}$, where $\mathcal{B}_e^{(1)}$ and $\mathcal{B}_e^{(2)}$ are the two corresponding required basis of $\mathcal{B}_H$ in the two copies of H in $e \wedge H$. Also, for each $i = 1, 2, \ldots, |V(C)|$, let $\mathcal{B}_{a_i}$ be the corresponding required basis of $\mathcal{B}_H$ in H_{a_i}. Since $E(\mathcal{B}_e^{(1)}) \cap E(\mathcal{B}_e^{(2)}) = \emptyset$, we have that $\mathcal{B}_e$ is linearly independent for each $e \in E(C)$. By the definition of the direct product, we have that $E(e \wedge H) \cap E(e' \wedge H) = \emptyset$, whenever $e \neq e'$. Thus, $E(\mathcal{B}_e) \cap E(\mathcal{B}_{e'}) = \emptyset$ for each $e, e' \in E(C)$ and $e \neq e'$. Therefore, $\cup_{e\in E(C)}\mathcal{B}_e$ is linearly independent. Since $e \wedge T$ is a forest (by Proposition 4.1) and since any linear combination of cycles of a linearly independent set is a cycle or an edge disjoint union of cycles, it follows that any linear combination of cycles of $\mathcal{B}_e$ must contain at least one edge of $e \wedge (H - T)$ and so any linear combination of cycles of $\cup_{e\in E(C)}\mathcal{B}_e$ must contain at least one edge of $E(C \wedge (H - T))$. Since $E(C \otimes T) \cap E(C \wedge (H - T)) = \emptyset$, we have $\mathcal{B}' \cup (\cup_{e\in E(C)}\mathcal{B}_e)$ is a linearly independent set. Clearly, $E(\mathcal{B}_{a_i}) \cap E(\mathcal{B}_{a_j}) = \emptyset$ for each $i \neq j$. Thus, $\cup_{i=1}^{|V(C)|}\mathcal{B}_{a_i}$ is linearly independent. Note that any linear combination of cycles of $\cup_{i=1}^{|V(C)|}\mathcal{B}_{a_i}$ contains at least one edge of $E(a_i \times (H - T))$ for some

i which is not in any cycle of $\mathcal{B}' \cup (\cup_{e\in E(C)}\mathcal{B}_e)$. Therefore, $\mathcal{B}(C \otimes H) = \mathcal{B}' \cup (\cup_{e\in E(C)}\mathcal{B}_e) \cup (\cup_{i=1}^{|V(C)|}\mathcal{B}_{a_i})$ is linearly independent. Now,

$$\begin{aligned}
|\mathcal{B}(C \otimes H)| &= |\mathcal{B}'| + \sum_{e\in E(C)} |\mathcal{B}_e| + \sum_{i=1}^{|V(C)|} |\mathcal{B}_{a_i}| \\
&= 3|E(T)||V(C)| + 1 + |V(C)| \dim \mathcal{C}(H) + 2|V(C)| \dim \mathcal{C}(H) \\
&= 3|V(C)|(|E(T)| + \dim \mathcal{C}(H)) + 1 \\
&= 3|V(C)||E(H)| + 1 \\
&= \dim \mathcal{C}(C \otimes H).
\end{aligned}$$

Thus, $\mathcal{B}(C \otimes H)$ is a basis for $\mathcal{C}(C \otimes H)$. It is easy to check out that $\mathcal{B}(C \otimes H)$ is a $(4 + b(H))$-fold basis. The proof is complete.

By adopting the same arguments as in Lemma 4.2 and Theorem 4.2 and applying them on $\mathcal{B}' \cup (\cup_{e\in E(P)}\mathcal{B}_e) \cup (\cup_{i=1}^{|V(P)|}\mathcal{B}_{a_i})$ we obtain the following result, where $\mathcal{B}'$ is the basis of $\mathcal{C}(P \otimes T)$ as in Theorem 4.1.

Corollary 4.2. If H is a bipartite graph of order ≥ 4 and P is a path of order ≥ 3, then $3 \leq b(P \otimes H) \leq b(H) + 4$.

Acknowledgment. The author would like to thank the referee for his very valuable comments.

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Mathematics Subject Classification (2000): 05C38, 05C75

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