On the Methodology of Valuation

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1. Introduction

The purpose of this paper is to consider the valuation methodology of uncertain stream of returns. Consider an economy in which a firm is owned by many shareholders through the stock market and lifespans of shareholders are different each other. Then, on what basis does a shareholder find a production plan optimal or not?

In a simple model of production without uncertainty, it is generally assumed that the objective of firm is the maximization of profit and shown that all shareholders unanimously agree to this criterion because the larger the profit is the larger the dividend per share is paid to them, the larger their incomes are, hence the higher utility they can enjoy provided that satiation is ruled out. Therefore, the profit maximization is also justified as firm's criterion of behavior from the viewpoint of consumers. If the model runs through time horizon, the term "profit" is replaced with the "discounted present value of profit stream" in the assumption above. In particular, a competitive equilibrium is known to be Pareto efficient in a wide variety of circumstances.

If the model specifies each good by its date and state of the world of delivery (contingency) in addition to its physical characteristics and provides a complete set of contingent markets, introducing time and uncertainty changes nothing essential as uncertainty in profit can be fully insured by certainty equivalents.[(1)] However, common observation suggests that markets are not complete in the real world. Being a production unit to yield outputs from inputs and fixed capital stock, the modern corporation is regarded as an economic institution to be owned by consumers through the stock market. A consumer is called a shareholder of firm j if he holds a positive number of shares of the firm. In such a stock market economy, stocks play two roles for consumers: (1) the ownership role, namely shareholding entiles a consumer participation in the decision making process of firm, and (2) the asset role, namely the claim to dividend through which consumers are able to reallocate risks and uncertainties on their wealth over time.

In an incomplete market, the maximization of profit or market value of firm does not make sense because profit or market value is no longer a nonstochastic number, rather a random variable depending on unknown states of the world in future dates as well as other factors common to the certainty case. Since there are many shareholders in each firm and each of them has different forecast about future environments (especially prices) and attitude toward risk, they generally disagree about the choice of production plan although they may be in agreement about the way in which the valuation of production plan depends on the selected production plan.

This paper answers the valuation problem by searching a rational methodology of valuation. The organization of the paper is as follows: after the model is formulated in the next section, the valuation methodology is discussed in section 3. Section 4 gives the proof.

2. A Model

The time horizon is divided into periods of equal duration, the

(1) For detailed discussion, see Arrow and Debreu (1954) and Debreu (1959).

length of which is assumed to be equal to the production period. The state of the world ω in future dates is unkown at present. Let Ω donote the set of states of the world. In each period, the economy is finite, namely there exist $(\ell-1)$ produced goods, a homogeneous labor servece (primary factor of production), I consumers and J firms, where ℓ, I and J are positive integers $(\ell\geq 2)$.(2) Despite that it makes the analysis harder [see Hart (1977, footnote 5)], more than two goods are assumed to exist because price uncertainty cannot be discussed in a one good model.

Take a typical period, say period t. Each firm, indexed $j=1,\ldots,J$, yields output $y^j(t)\in R^\ell$ which is the result of last period production. At the beginning of period t, firm j is endowed with fixed capital stock $K^j(t)\in R^\ell$ and outstanding number of shares, and has a borrowing $b^j(t-1)$ to be redeemed in this period. In a production model without fixed capital stock or with only circulating capital, all of the market value of firm, the value of shares in stock market and the value of production plan are indentical as such a model is essentially a two period model where all firms are liquidated at the end of the second period. On the other hand, the introduction of fixed capital not only makes these values diverse but also requires longer horizon beyond second period, hence the analysis becomes complicated.

The relationship between input and output is described by a production possibility set Y^j which depends on fixed capital K^j,(3)

(2. 1) $\quad (x^j(t-1),y^j(t))\in Y^j(K^j(t-1))$ for all t.

It is assumed that

(2) This does not intend to assume that all economic agents have the same length of planning horizon. Generally, some consumer lives longer than other and consumers as a whole have shorter horizon than firms as a whole.

(3) This production possibility set differs from that of Arrow-Debreu-McKenzie [see references in footnote 1 as well as McKenzie (1959)] in two aspects: (i) fixed capital stock and (ii) time structure which involves uncertainty.

(a. 1) The production possibility set $Y^j(.)$ defined by (2. 1) is continuous with respect to fixed capital stock. For given level of fixed capital, the set $Y^j(K^j(t))$ is convex and closed.

(a. 2) The labor input is necessary for all production processes, namely, $x^j_\ell(t)>0$ for all j and t.

(a. 3) The fixed capital stock does not depreciate. No consolidation or split is made on the outstanding number of shares.

The continuity of (a. 1) is needed in *Theorem* below. The latter half of (a. 1) is usual assumption on technology. (a. 2) guarantees positive value of consumer's endowment and implies

$$Y^j(t) \cap \{\tilde{y} \in R^{2\ell} \mid \tilde{y} \leq 0\} = \{0\}.$$

Assumption (a. 3) is made for simplicity, which means

(2. 2) $K^j(t) = K^j(t-1) + \Delta K^j(t-1) = \sum_{\tau=0}^{t-1} K^j(\tau)$

where $\Delta K^j(\tau)$ is the investment on fixed capital made in period τ and $\Delta K^j(0)$ is given.

(2. 3) the outstanding number of shares $= \sum_{\tau=0}^{t-1} n^j(\tau)$

where $n^j(\tau)$ is the number of new shares issued in period τ and $n^j(0)$ is given.

The firm makes decisions on input purchase $x^j(t)$, investment $\Delta K^j(t)$ and its financing. Three methods are assumed to be available for financing: they are through

(i) retained profit given by

$$p(t)y^j(t) - b^j(t-1) - d^j(t)\sum_{\tau=0}^{t-1} n^j(\tau)$$

where $d^j(t)$ is the dividend per share paid in period t,

(ii) public offering of new shares which amounts to

$$q^j(t)\, n^j(t)$$

where $q^j(t)$ is the share price and $n^j(t)$ the number of newshares issued in period t,

(iii) borrowing in the form of one period loan from consumers $r(t)b^j(t)$
where $r(t)$ is the borrowing price such that $[1-r(t)]/r(t)$ is the rate of interest on the loan.

The assumption on the borrowing is made for simplicity, that is, one dollar is paid back in next period for each $r(t)$ dollar borrowed today. The financial plan is expressed by a vector $(d^j(t),\ n^j(t),\ b^j(t)) \in R^3_+$ of dividend, flotation and borrowing. Let

$$\rho^j(t) \equiv (x^j(t),\ \Delta K^j(t),\ d^j(t), n^j(t),\ b^j(t)) \in R^{2\ell+3}_+$$

and call this package of production, investment and finance plans a policy of firm j in period t. A feasible policy must satisfy the budget constraint,

$$(2.\ 4) \qquad -p(t)\ \{y^j(t)-x^j(t)-\Delta K^j(t)\}\ +b^j(t-1)+d^j(t)\sum_{\tau=0}^{t-1} n^j(\tau)$$
$$-q^j(t)\ n^j(t)-r(t)\ b^j(t) \leq 0.$$

Define a set of feasible policies

$$(2.\ 5) \qquad B^j(t) \equiv \{\rho^j(t) \in R^{2\ell+3} \mid (2.\ 4) \text{ is satisfied for given } (p(t),\ q(t),\ r(t)) \text{ and } d^j(t)\}.$$

Now turn to consumer side of the model. Each consumer, indexed $i=1, \ldots, I$, lives for finite number of periods. Let T_i denote the final period of his life. At the beginning of period t, consumer i has initial assets $(\sum_{\tau=0}^{t-1} s^i(\tau),\ b^i(t-1)) \in R^{J+1}$ of various stocks and lending, where

$$s^i(\tau) \equiv (s^i_1(\tau),\ \ldots,\ s^i_J(\tau)) \in R^J$$

is a vector of shares he bought in period τ. In addition to recieve dividends on shareholding and redemption of lending, he supplies $|x^i_\ell(t)|$ units of labor and demands consumption goods $(x^i_1(t), \ldots, x^i_{\ell-1}(t))$ and assets $(s^i(t),\ b^i(t))$ in period t. Let

$x^i\ (t) \equiv (x_1^i(t), \ldots, x_{\ell-1}^i(t),\ x_\ell^i(t)) \in R_+^{\ell-1} \times R_-.$

Then, his budget constraint is given by

(2. 6) $p(t)\ x^i(t) + q(t)\ s^i(t) + r(t)\ b^i(t)\ - d(t) \sum_{\tau=0}^{t-1} s^i\ (\tau)$

$- b^i(t-1) \leq 0$

where $q(t) \equiv (q^1(t), \ldots, q^J(t))$ is the vector of share prices and $d(t) \equiv (d^1(t), \ldots, d^J(t))$ is the vector of dividends.

Note that the dividend of period t is paid to the initial shareholders, that is consumers who hold shares at the end of period (t−1). Consumer i has his subjective expectation in addition to preference relation $\geq_i$ over concumption stream, which is defined as

(2. 7) $\Psi^i: p \times \Omega \rightarrow \mathcal{M}(p \times \ldots \times p, \mathcal{B}(p \times \ldots \times p))$

where $\mathcal{M}(p \times \ldots \times p, \mathcal{B}(p \times \ldots \times p))$ is the set of all probability measures on $\overbrace{p \times \ldots \times p}^{T_i - t - 1}$ and its Borel σ field.

It is assumed that

(b. 1) The expectation mapping Ψ^i defined by (2.7) is continuous in the weak topology.

(b. 2) The preference relation $\geq_i$ defined over consumption streams $(R^\ell \times \ldots \times R^\ell)$ is continuous, convex and strictly monotone.

(b. 3) No consumer can take a short position in the stock market and borrow.

It is known that assumptions (b. 1) and (b. 2) imply the existence of expected utility function qiven by

(2. 8) $v^i\ (x^i(t),\ p(t)) = \int u^i\ (x^i(1),\ \ldots,\ x^i(T_i))\ d\Psi^i(p(t))$

which is continuous, concave, strictly monotone and bounded.(4)

Assumption (b. 3) implies

(4) Since it is not the purpose here, the derivation of expected utility function is not given. See Grandmont (1972, 1677).

(2.9) $\sum_{\tau=0}^{t-1} s^i(\tau) \geq 0$ and $b^i(t) \geq 0$ for all t.

Being made for simplicity, (b. 3) can be relaxed if the possibility of bankruptcy is ruled out.

The behavior of consumer i in period t can be expressed by a vector $(x^i(t), s^i(t), b^i(t)) \in R_+^{\ell+1} \times R_-$ of consumption demands, labor supply and portfolio choices (shares and lending). His behavior is said to be feasible if the vector satisfies the budget constraint (2.6). Define the set of feasible behaviors of consumer i,

(2.10) $$B^i(t) \equiv \{(x^i(t), s^i(t), b^i(t)) \mid (2.6) \text{ holds for given } \sum_{\tau=0}^{t-1} s^i(\tau), b^j(t-1), (p(t), q(t), r(t)), d(t)\}$$

Consumers are assumed to behave so as to maximize own utility of consumption over life span subject to the feasibility of behavior (2.10). Since consumption stream $\{x^i(1), \ldots, x^i(t-1)\}$ upto period (t−1) is histrical data and cannot be changed, each consumer manipulates future stream $\{x^i(t), \ldots, x^i(T_i)\}$ and actually carries out only current consumption $x^i(t)$. In this sense, the equilibrium here is a temporary equilibrium.

Finally, merket clearing conditions are

(2.11) $$\sum_i x^j(t) - \sum_j \{ y^j(t) - x^j(t) - \Delta K^j(t)\} = 0,$$

(2.12) $$\sum_i \sum_{\tau=0}^{t} s^i(\tau) - \sum_{\tau=0}^{t} (n^1(\tau), \ldots, n^J(\tau)) = 0,$$

(2.13) $$\sum_j b^j(t) - \sum_i b^i(t) = 0.$$

The existence of a temporary equilibrium is assumed and not proven here since it is not the purpose of this paper. In fact, Kodaira (1977) showed the existence under assumptions (a. 1)−(a. 3) and (b. 1)−(b. 3) [see also Grandmont (1977) and Green (1973)].

3. Methodology of Valuation

This section discusses the methodology of valuation, an individual criterion of shareholding in an incomplete market. It is assumed that when the economy is large enough, each consumer demands and supplies such a small fraction of each firm's output and labor input that his utility is virtually unchanged by any change in policy of firm. On the other hand, each consumer may own a non negligible fraction of outstanding shares of a firm so the ownership effect caused by change in policy may not be small.

To define the judgement criterion of policy, a methodology should be first established to value each policy of production, investment and financing plans. If the return stream is certain, there is widespread agreement that the value of package of plans is determined as present discounted value of the stream. When the returns are uncertain, the substitution of expected returns or certainty equivalents is employed with discounting at some risk adjusted rate. But no explanation has been given why this is plausible. Furthermore, the matters are even worse if planning horizon varies in length from consumer to consumer. This section tries to give some evaluation method which is based on rational behavior of agent.

Facing a proposed policy $\rho^{j}(t)$ of firm j in period t, consumer i manipulates the dividend stream over his remaining life (till period T_i), which depends on the states of the world of future through his expectation Ψ^i. Let

$${}_i d^j(\tau,\ \omega) \quad \tau = t+1,\ \ldots,\ T_i \text{ and } \omega \in \Omega,$$

denote his expectation of dividend payment in period τ ($\tau > t$) if the state ω of the world is realized. Then his expected stream of dividends is written as

$$\{{}_i d^j(\tau,\ \omega)\}_{\tau=t+1,\ \cdots,\ T_i+1,\ \omega\in\Omega}$$

where ${}_i d^j\ (T_i+1,\ \omega)$ is the expected price of a share at the end of period T_i (scrap value).

The task here is to find some method to value the policy. Any method of evaluation should satisfy the following three conditions which Ross (1978) proposed.

(i) Linearity:

Suppose that the valuation of policy $\rho^1(t)$ of firm 1 is given by $\pi^i\ \rho^1(t)$ and $\rho^2(t)$ of firm 2 by $\pi^i\ \rho^2(t)$. Then, the portfolio $s^i(t)=(s_1^i(t),\ s_2^i(t))$ of consumer i is evaluated as

$$\pi^i(\rho^1(t)\,;s_1^i(t))+\pi^i(\rho^2(t);\ s_2^i(t))$$

$$=s_1^i(t)\ \ \pi^i\rho^1(t)+s_2^i(t)\ \ \pi^i\ \rho^2(t)$$

(ii) Positivity:

Let a policy $\rho^j(t)$ generate non negative stream $\{{}_id^j(\tau,\ \omega)\}$ of dividend payments. Then, the valuation is non negative ;

$\pi^i\ \rho^j(t)>0.$

(iii) Riskless valuation:

Imagine a policy $\rho^j(t)$ which yields certain one dollar of dividend at date $T(<T_i)$ and nothing at any other dates, namely

$$\{{}_id^j\ (\tau,\ \omega)\}_{\tau,\omega}=\{\overbrace{0,\ldots,0}^{T-t-1},\ 1,\overbrace{0,\ldots\ldots,0}^{T_i-T+1}\}$$

Then, the valuation is given as

$$\pi^i\ \rho^j(t)=r_i^{T+t}$$

where r_i is his subjective rate of discount (risk adjusted).

Linearity condition is a natural requirement of the stock market economy in which dividends are paid accordingly to the number of shares held. Positivity condition needs no explanation. Riskless valuation condition does not imply that a common discount rate applies for all individuals.

Theorem

If the stock market is in equilibrium, there exists a valuation method of policies in terms of observables (current prices). It satisfies all Ross's conditions on valuation. Futhermore, it is *continuous if assumptions (a. 1) and (b. 1) are satisfied.*

The proof is given in the next section. The rest of this section gives some notes.

(1) Since the valuation depends on subjective expectation of future, the valuation of policy generally differs among consumers. Hence, the valuation is not unique. The merits of the theorem are that there exists an agreement among shareholders of various life span and expectation, with respect to the methodology and that such a valuation satisfies linearity requirement as well as others.

Conversely, if all the valuations coincide among consumers, the stock market is complete.

(2) The evaluation is positive only for the marketed shares. In general, the valuation of non marketed shares can be negative.

(3) The simplest version of this method is given when all comsumers happen to look only one period ahead. The value of policy is, then, given by the (discounted value of) expected return, which is equal to the sum of expected dividend plus share price of next period [see Malinvaud (1972) and Kodaira (1977)].

4. The Proof

Let me start with some preparation of mathematics. A dividend payment depends on the state of the world of future date. Let $\mathcal{D}$ denote $\mathcal{L}_\infty$ space of $R_+ \times \Omega$, where the element of $\mathcal{D}$ stands for a stream $\{{}_i d^j(\tau, \omega)\}_{\tau=t+1,\cdots,T+1,\ \omega\in\Omega}$ of expected dividends per share upto period $T(>t)$ depending on the state of the world $\omega \in \Omega$ when a particular policy $\rho^j(t)$ is selected in period t, where $T(\leq T_i)$ is his planned horizon of shareholding. The functional space $\mathcal{D}$ is assumed to be endowed with a certain topology which is strong enough to ensure that the positive orthant $\mathcal{D}_+ \equiv \{d \in \mathcal{D} \mid d > 0\}$ is open [for example, the Mackey topology. See Bewley (1972)]. The functions ${}_i d^j$: $R_+ \times \Omega \rightarrow R$ are measurable, for each t, with respect to a given σ algebra.

Write the set of dividend functions of marketed stock as $\mathscr{D}_m$, a subset of $\mathscr{D}$. Let

$$f = f(\mathscr{D}_m)$$

denote the resulting linear space generated by $\mathscr{D}_m$. There exists a function π associated wiht $\mathscr{D}_m$ which assigns a scalor to each element of $\mathscr{D}_m$ (namely, dividend stream of policy $\rho^j(t)$). The extension theorem of Hahn and Banach tells that this function π has an extension to a correspondence on f [see, for example, Kolmogorov and Formin (1968, Theorem 3.2.4a)]. Provided that

(i) α is a bounded additive set valued function of $\mathscr{D}_m$,

(ii) $\pi(\alpha) \equiv \int_{\mathscr{D}_m} \pi(y)\, d\alpha$

(iii) $x(\alpha) \equiv \int_{\mathscr{D}_m} x\, d\alpha,\ x \in f,$

the extension of π on f is given by

$$\pi(x) = \{\pi(\alpha) \mid \text{for } \alpha \text{ such that } x = x(\alpha)\}.$$

Now, Theorem of last section can be mathematically expressed as

Suppose that the valuation $\pi(\rho^j(t_j)) > 0$ for any (in $\mathscr{D}_m$) $\{{}_i d^j(\tau, \omega)\}_{\tau=t+1,\cdots,T+1,\ \omega\in\Omega}$ with ${}_i d^j(\tau,\omega) \geq 0$ and that there exists $\{{}_i d^{jo}(\tau, \omega)\}_{\tau=t+1,\cdots,T+1,\ \omega\in\Omega}$ with ${}_i d^{jo}(\tau, \omega) > 0$. Then, the valuation function π of policy $\rho^j(t)$ is linear in $\mathscr{D}_m$ and can be extended to $\pi^* \in \mathscr{D}^*$ for any strictly positive dividend stream $\{{}_i d^{jo}(\tau, \omega)\}$ such that $\pi^*(\rho^{jo}(t)) > 0$ for any $\{{}_i d^j(\tau, \omega)\} \in f$ with ${}_i d^j(\tau, \omega) \geq 0$, where $\mathscr{D}^*$ is a dual space of $\mathscr{D}$, namely be $(R_+ \times \Omega)$.

[Proof of Theorem] Let int $\mathscr{D}_+$ be an interior of positive orthant $\mathscr{D}_+$ and

$$\mathscr{D}_n \equiv \{{}_i d^j(\tau, \omega) \in \mathscr{D} \mid \text{for } \alpha \text{ such that } {}_i d^j(\tau, \omega) = {}_i d^j(\alpha) \text{ and that } \pi(\alpha) \leq 0\}.$$

Both int $\mathscr{D}_+$ and $\mathscr{D}_n$ are convex and non empty. Furthermore,

$$\text{int } \mathscr{D}_+ \cap \mathscr{D}_n = \emptyset.$$

Since int $\mathscr{D}_+$ is open, the separation theorem of Hahn and Banach [for example, see Kolmogorov and Formin (1968, Theorem 3.2.5)] can be applied to obtain $\pi^* \in \mathscr{D}^*$ such that

$$\pi^* a \leq b < \pi^* c$$

for all $a \in \mathscr{D}_n$, $b \in R$ and $c \in \text{int } \mathscr{D}_+$.

Since the origin belongs to $\mathscr{D}_n$ and to the closure of int $\mathscr{D}_+$, $b=0$. Hence, for all $\{_i d^j(\tau, \omega)\} \in f$ $\pi(\alpha) \leq 0$ implies $\pi^* \{_i d^j(\tau, \omega)\} \leq 0$. Normalize π^* such that

$$\pi^* \{_i d^j(\tau, \omega)\} = \pi(\alpha) < 0$$

for some dividend stream with $_i d^j(\tau, \omega) = {_i d^j}(\alpha)$, By definition of $\pi(.)$, for any g and h in R,

$$g\ \pi^*\ x + h\ \pi^*\ y \leq 0$$

if $g\pi(\alpha) + h\pi(\beta) \leq 0$ where $x = x(\alpha)$ and $y = y(\beta)$.

Therefore, for all $\{_i d^j(\tau, \omega)\} \in \mathscr{D}_m$,

$$\pi^* \{_i d^j(\tau, \omega)\} = \pi(\alpha)$$

where the element of divident stream $_i d^j(\tau, \omega) = {_i d^j}(\alpha)$. In particular, $\pi(\alpha)$ generates a function on $\mathscr{D}_m$ given with $\pi(\{_i d^j(\tau, \omega)\}) = \pi(\alpha)$ where the element $_i d^j(\tau, \omega) = {_i d^j}(\alpha)$. Then

$\pi^* \{_i d^j(\tau, \omega)\} > 0$ for all strictly positive stream.

Furthermore, non negativeness of the stream implies

$$\pi^* \{_i d^j(\tau, \omega)\} = \pi \{_i d^j(\tau, \omega)\} > 0$$

since π^* coincides with $\pi(.)$ on $\mathscr{D}_m$. This can be also written as $\pi(\rho^j(t))$ because the expected dividend stream $\{_i d^j(\tau, \omega)\}$ is originated by observing $\rho^j(t)$ and $(p(t), q(t), r(t))$.

Finally, the continuities of production possibility set (assumption (a. 1)) and expectation ((b. 1)) imply a continuous relation between a policy and dividend stream, hence the last part of the proposition follows. □

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