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Intersection Cohomology of Rank One Local Systems for Arrangement Schubert Varieties

Shuo Lin
University of Massachusetts Amherst

INTERSECTION COHOMOLOGY OF RANK ONE LOCAL SYSTEMS FOR ARRANGEMENT SCHUBERT VARIETIES

A Dissertation Presented

by

SHUO LIN

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Approved as to style and content by:

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Tom Braden, Chair

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Paul Gunnells, Member

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Eric Sommers, Member

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Tigran Sedrakyan, Member

____
Robin Young, Department Head
Mathematics and Statistics

To my teachers

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ABSTRACT

INTERSECTION COHOMOLOGY OF RANK ONE LOCAL SYSTEMS FOR ARRANGEMENT SCHUBERT VARIETIES

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SHUO LIN, B.S., HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

M.S., NATIONAL CENTRAL UNIVERSITY

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Tom Braden

In this thesis we study the intersection cohomology of arrangement Schubert varieties with coefficients in a rank one local system on a hyperplane arrangement complement. We prove that the intersection cohomology can be computed recursively in terms of certain polynomials, if a local system has only ± 1 monodromies. In the case where the hyperplane arrangement is generic central or equivalently the associated matroid is uniform and the local system has only ± 1 monodromies, we prove that the intersection cohomology is a combinatorial invariant. In particular when the hyperplane arrangement is associated to the uniform matroid of rank $n-1$ over n elements, and the local system has ± 1 monodromies, we can give a closed formula for the intersection cohomology.

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CHAPTER 1

INTRODUCTION

This thesis investigates the intersection cohomology of an "arrangement Schubert variety" with coefficients in a rank one local system on a hyperplane arrangement complement.

Let V be a vector space over complex numbers $\mathbb{C}$, with a central hyperplane arrangement $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ such that $H_1 \cap H_2 \cap \cdots \cap H_n = \{0\}$. The arrangement Schubert variety $Y(V)$ associated to V is defined to be the closure of V via the embedding

$$V \hookrightarrow V/H_1 \times V/H_2 \times \cdots \times V/H_n \cong \mathbb{C}^n \subset (\mathbb{CP}^1)^n.$$

It is a singular space in general, so that intersection cohomology is a suitable topological tool in this context.

For the above vector space V, we also define an associated matroid $M(V)$ on the ground set $[n] = \{1, 2, ..., n\}$, which is characterized by the condition that $F \subset [n]$ is a flat of $M(V)$ if and only if there exists a vector $(x_1, x_2, ..., x_n)$ in the image of $V \hookrightarrow V/H_1 \times V/H_2 \times \cdots \times V/H_n$ such that $F = \{i \mid x_i = 0\}$. We will see that the combinatorics of $M(V)$ encodes a lot of geometry of $Y(V)$ in Chapter 4.[1]

[1]Conversely, assume that M is realizable over $\mathbb{C}$, realized by a spanning set of vectors $\{w_1, w_2, ..., w_n\}$ in a vector space $W \cong \mathbb{C}^{\mathrm{rk}(M)}$. This induces a surjective map $\mathbb{C}^n \to W$, whose

The arrangement Schubert variety, an analogue of the Schubert varieties in the flag variety of a semisimple algebraic group, has been studied by many people. It was first studied by Ardila and Boocher, who showed that the arrangement Schubert variety has an algebraic cell decomposition [AB16, Theorem 1.3] (which is the coarse stratification we describe later in Chapter 4). Huh and Wang [HW17] used the intersection cohomology of the arrangement Schubert variety to prove the Dowling-Wilson top-heavy conjecture and Rota's unimodal conjecture for matroids in the realizable case.

In the paper [EPW16], Elias, Proudfoot, and Wakefield introduced the Kazhdan-Lusztig polynomial $P_M(t)$ of a matroid M, in analogy with Kazhdan-Lusztig polynomial in representation theory. In particular they showed that if a matroid M is realizable, then $P_M(t)$ is equal to the Poincaré polynomial in $t^{\frac{1}{2}}$ of the intersection cohomology of the reciprocal plane, which is the intersection of the arrangement Schubert variety and the affine chart $(\mathbb{C}^* \cup \{\infty\})^n \subset (\mathbb{CP})^n$ at $(\infty)^n$. Later Proudfoot, Xu and Young [PXY18] introduced the Z-polynomial $Z_M(t)$ as a weighted sum of the Kazhdan-Lusztig polynomials of all possible contractions of M. They showed that in the case where M is realizable, $Z_M(t)$ is equal to the Poincaré polynomial in $t^{\frac{1}{2}}$ of the intersection cohomology $IH^\bullet(Y)$ of an associated arrangement Schubert variety Y.

In fact $Z_M(t)$ and $P_M(t)$ can be computed by the following identity, which is the definition of $Z_M(t)$ [PXY18, Definition 2.1]:

$$Z_M(t) = \sum_{F \in \mathbb{L}(M)} t^{\operatorname{rk} F} P_{M_F}(t), \tag{1.1}$$

dual gives an injection $W^* \to \mathbb{C}^n$. The image of this dualizing map is a vector subspace of $\mathbb{C}^n$, isomorphic to $\mathbb{C}^{\operatorname{rk}(M)}$, whose closure inside $(\mathbb{CP}^1)^n$, the arrangement Schubert variety of the image is an associated arrangement Schubert variety to M (see [PXY18, Section 7] and [BV20, Section 1]).

where $\mathbb{L}(M)$ denotes the lattice of flats of M, $\operatorname{rk} F$ is the rank of the flat F, and M_F is the contraction matroid of M at F. For those who are not familiar with matroids, we discuss these terminologies in §2.4. For now, it is enough to know that the lattice of flats $\mathbb{L}(M_F)$ consists of the flats $\{G \setminus F \mid G \geq F\}$. There are two facts which are essential in computation for $Z_M(t)$ and $P_M(t)$. For any realizable matroid M,

(a) Since Y is compact, $IH^{\bullet}(Y; \mathbb{C})$ satisfies Poincaré duality, so $Z_M(t)$ is palindromic.

(b) If M is a matroid of rank 0, then $P_M(t) = 1$. If M has rank at least 1, then the degree of $P_M(t)$ is strictly less than $\frac{1}{2} \operatorname{rk} M$ because of the degree restriction of intersection cohomology.

Remark 1.1. *In fact, the constant term of $P_M(t)$ is equal to 1 for any matroid M (see [EPW16, Proposition 2.11]). Both $P_M(t)$ and $Z_M(t)$ vanish in odd degree if M is realizable (see [EPW16, Propsition 3.12] and [PXY18, Theorem 7.2]).*

With those two facts one can recursively compute $Z_M(t)$ and $P_M(t)$ for any realizable matroid M. For instance if $M = U_{n,n+k}$, the uniform matroid of rank n over $n + k$ elements, then the contraction M_F is a matroid of the same type, i.e. the uniform matroid $U_{\operatorname{crk} F, \operatorname{crk} F + k}$ (where crk means corank of flat) as long as F is not the ground set $[n] = \{1, 2, \ldots, n\}$. We will present some computation for the cases when $k = 1, 2$ in Section 7. For more examples, see the tables in Appendix.

Example 1. *We give an example here to illustrate how to obtain polynomials $P_M(t)$ and $Z_M(t)$ by an inductive argument.*

Suppose we want to find $P_M(t)$ and $Z_M(t)$ for $M = U_{3,4}$. Because of the degree restriction and the fact that the constant term of $P_M(t)$ is equal to 1, we have $P_{U_{1,2}}(t) = P_{U_{2,3}}(t) = 1$.

It is intuitive to represent a matroid by its lattice of flats. Through the whole article, we will omit the brackets for a flat so it does not look too bulky. The lattice of flats of $U_{3,4}$ *is shown in Figure 1 below.*

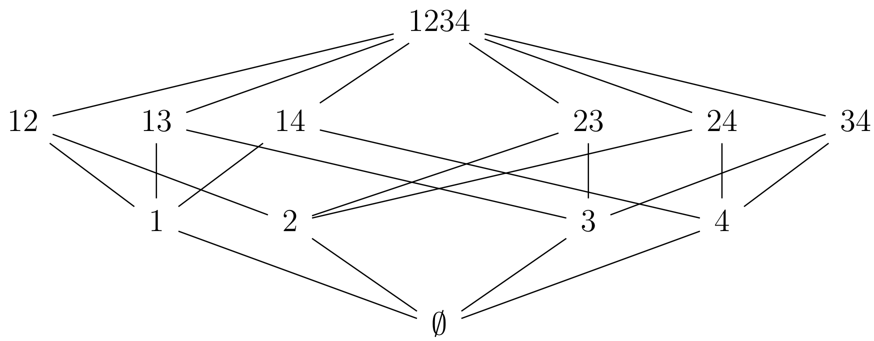
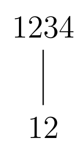
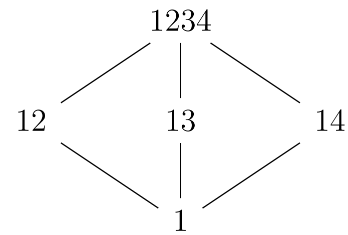


Figure 1: The lattice of flats of $U_{3,4}$

There are six rank 2 *flats in total, including* $12, 13, 14, 23, 24$ *and* 34*. At each rank* 2 *flat of* M*, the contraction matroid is isomorphic to* $U_{1,2}$*. For instance, the lattice of flats of* M_{12} *is*



There are four rank 1 flats in total, including $1, 2, 3$ and 4. At each rank 1 flat of M, the contraction matroid is isomorphic to $U_{2,3}$. For example, the lattice of flats of M_1 is



Note that $U_{1,2}$, $U_{2,3}$ and $U_{3,4}$ are all matroids of the type $U_{n,n+1}$.

From Equation (1.1), we know that

$$\begin{aligned} Z_M(t) &= t^3 + \binom{4}{2} t^2 \cdot P_{U_{1,2}} + \binom{4}{1} t \cdot P_{U_{2,3}} + P_M(t) \\ &= t^3 + 6t^2 + 4t + P_M(t) \end{aligned}$$

Because the degree of $P_M(t)$ is strictly less than $\frac{1}{2} \operatorname{rk} M = \frac{3}{2}$ and $Z_M(t)$ is palindromic, the only possibility is that $P_M(t) = 2t + 1$, and it follows that $Z_M(t) = t^3 + 6t^2 + 6t + 1$.

The geometry behind (1.1) will be explained here. The arrangement Schubert variety associated to V can be equipped with a coarse stratification $Y(V) = \coprod_{F \in \mathbb{L}(M(V))} S_F \cong \coprod_{F \in \mathbb{L}(M(V))} \mathbb{C}^{\operatorname{rk} F}$, due to Ardila and Boocher [AB16]. This cell decomposition implies a long exact sequence of compactly supported hypercohomology.

By the parity vanishing property [PXY18, Theorem 7.2], the long exact sequence splits. So the intersection cohomology of Y is isomorphic (although not canonically) to a direct sum

$$IH^\bullet(Y; \mathbb{C}) = \mathbb{H}_c^\bullet\big(Y; IC^\bullet(Y)\big) \cong \bigoplus_{S_F} \mathbb{H}_c^\bullet\big(S_F; IC^\bullet(Y)|_{S_F}\big). \tag{1.2}$$

We will show that in §4.4 (in a more general setting where the constant sheaf $\underline{\mathbb{C}}$ is replaced by a rank one local system on the hyperplane arrangement complement), at each point of S_F, there exists a neighborhood in Y which is isomorphic to a product of $\mathbb{C}^{\operatorname{rk} F}$ and a neighborhood of the most singular point of the arrangement Schubert variety Y_F, associated to the contraction matroid M_F. This product structure yields that

$$\mathbb{H}_c^\bullet\big(S_F; IC^\bullet(Y)|_{S_F}\big) \cong \mathbb{H}_c^\bullet\big(S_F; \mathbb{C}\big) \otimes H^\bullet\big(IC^\bullet(Y_F)_\infty\big). \tag{1.3}$$

In terms of the Poincaré polynomial, the algebraic isomorphism (1.3) together with Equation (1.2) implies Equation (1.1).

Remark 1.2. *We emphasize that when M is realizable the polynomials $P_M(t)$ and $Z_M(t)$ only depend on the matroid M, not on the choices of spanning vector spaces and arrangement Schubert varieties.*

1.1 Main results

Our goal in this article is to extend the results on computation for Kazhdan-Lusztig polynomials and Z-polynomials of realizable matroids in the papers [EPW16] and [PXY18] to the setting in which the constant sheaves on the hyperplane arrangement complements are replaced by rank one local systems.

For a rank one local system $\mathcal{L}$ on the hyperplane arrangement complement $U = V \setminus \bigcup_{1\leq i\leq n} H_i$, its monodromy around each hyperplane H_i is multiplication by a complex number a_i. Regarding the intersection cohomology of $Y(V)$, if we replace the constant sheaf $\underline{\mathbb{C}}_U$ on the arrangement complement by the local system $\mathcal{L}$, we add into new data of a set of complex numbers. So it is reasonable to expect that the intersection cohomology with coefficients in $\mathcal{L}$ will change in some certain pattern when one chooses different monodromy numbers around hyperplanes.

Our main theorem (Theorem 1.3) is a generalization of Equation (1.1).

Suppose that V is a vector subspace of $\mathbb{C}^n$ such that no coordinate hyperplane of $\mathbb{C}^n$ contains V. Under this assumption, the hyperplanes $H_i = \{x \in V \mid x_i = 0\}$ form a generic central arrangement of V. Now we explain new notations that we will use:

(i) Let $Z_{V,\mathcal{L}}(t)$ be shorthand for Poin $\left(IH^\bullet(Y(V);\mathcal{L}), t^{\frac{1}{2}}\right)$, the Poincaré polynomial of $IH^\bullet\big(Y(V);\mathcal{L}\big)$ in $t^{\frac{1}{2}}$. We call it the generalized Z-polynomial for V and $\mathcal{L}$.

(ii) The computation for the twisted intersection cohomology of a vector space stratified by generic central hyperplanes is a new problem for us, so it is worth giving new notation for that. We let $A_{V,\mathcal{L}}(t) = \text{Poin}\left(IH_c^\bullet(V;\mathcal{L}), t^{\frac{1}{2}}\right)$, the Poincaré polynomials of intersection cohomology with compact support $IH_c^\bullet(V;\mathcal{L})$ in $t^{\frac{1}{2}}$.

(iii) Lastly, we let $P_{V,\mathcal{L}} = \text{Poin}\left(H^\bullet\big(IC^\bullet(Y(V);\mathcal{L})_\infty\big), t^{\frac{1}{2}}\right)$ be the Poincaré polynomial of the stalk cohomology of $IC^\bullet\big(Y(V);\mathcal{L}\big)$ at the most singular point (∞^n). We call it the generalized Kazhdan-Lusztig polynomial for V and $\mathcal{L}$.

In the case of local system $\mathcal{L}$, we no longer have the parity vanishing property for intersection cohomology. Some examples that have nonvanishing intersection cohomology in odd degree can be found in the tables of the Appendix.

However, because of the additive group action of V on $Y = Y(V)$, we can imitate an argument by Kirwan [Kir88, Lemma 1.12, Lemma 2.8] to conclude that when $\mathcal{L}$ has ± 1 monodromies, i.e. all $a_i \in \{\pm 1\}$, the long exact sequence for intersection cohomology splits (see Theorem 5.1). So the intersection cohomology of Y is isomorphic (although not canonically) to a direct sum

$$IH^\bullet(Y;\mathcal{L}) \cong \mathbb{H}_c^\bullet\big(Y; IC^\bullet(Y;\mathcal{L})\big) \cong \bigoplus_{F\in\mathbb{L}(M(V))} \mathbb{H}_c^\bullet\big(S_F; IC^\bullet(Y;\mathcal{L})|_{S_F}\big). \tag{1.4}$$

It will be shown in §4.4 that there exists a local product structure of each stratum, which is a vector space V^F in tangent direction and a reciprocal plane of $Y(V_F)$ in normal direction. That implies a local product structure of the twisted intersection cohomology complex. We will identify its tangential factor $IC^\bullet(V^F;\mathcal{L}^F)$ and normal factor $IC^\bullet(V_F;\mathcal{L}_F)$ in §4.5. Analogous to Equation (1.3), there exists an isomorphism

$$\mathbb{H}_c^\bullet\big(S_F; IC^\bullet(Y;\mathcal{L})|_{S_F}\big) \cong IH_c^\bullet(V^F;\mathcal{L}^F) \otimes H^\bullet\big(IC^\bullet(V_F;\mathcal{L}_F)|_\infty\big). \tag{1.5}$$

In terms of Poincaré polynomials, using Equations (1.4) and (1.5) we obtain the following:

Theorem 1.3. *If $\mathcal{L}$ is a rank one local system with ± 1 monodromies around hyperplanes, then*

$$Z_{V,\mathcal{L}}(t) = \sum_{F\in\mathbb{L}(M(V))} A_{V^F,\mathcal{L}^F}(t)\cdot P_{V_F,\mathcal{L}_F}(t). \tag{1.6}$$

Remark 1.4. *In Equation (1.6), the polynomial $A_{V^F,\mathcal{L}^F}(t)$ replaces the term $t^{\operatorname{rk} F}$ in Equation (1.1).*

Remark 1.5. *The assumption on local system in Theorem 1.3 may be relaxed to the condition that the monodromies are given by the p-th roots of unity for some integer p.*

As before two facts give restrictions on the polynomial identity:

(A) Suppose $\mathcal{L}$ is a rank one local system with ± 1 monodromies around hyperplanes, then $\mathcal{L} \cong \mathcal{L}^{\vee}$. Since $Y(V)$ is compact, $IH^{\bullet}(Y(V);\mathcal{L})$ satisfies the Poincaré duality, so $Z_{V,\mathcal{L}}(t)$ is palindromic.

(B) The degree of $P_{V,\mathcal{L}}(t)$ is strictly less than $\frac{1}{2}\dim V$ because of the degree restriction of intersection cohomology.

With these facts and Equation (1.6), assuming that the polynomial $A_{V^F,\mathcal{L}^F}(t)$ are known we can recursively compute $Z_{V,\mathcal{L}}(t)$ and $P_{V,\mathcal{L}}(t)$.

When the monodromies of a local system $\mathcal{L}$ are multiplication by ± 1 and the hyperplane arrangement in V is generic central, we are able to compute $A_{V,\mathcal{L}}(t)$ and conclude that $A_{V,\mathcal{L}}(t)$ is a combinatorial invariant in §3.4. As a consequence, $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ are also combinatorial invariants.

In particular when V is of dimension $n-1$ in $\mathbb{C}^n$, or $M(V)$ is the uniform matroid $U_{n-1,n}$, not only can we recursively compute $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$, but we can also give closed formulas for them in Theorem 6.4. In that case, both $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ only have integer degree terms, in other words the corresponding intersection cohomology vanishes in odd degree. But this is not true for the uniform matroid $U_{n-2,n}$ (see the tables in the Appendix).

Naturally we are inclined to ask if the three polynomials $A_{V,\mathcal{L}}(t)$, $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ are still combinatorial invariants if we remove the restriction that $\mathcal{L}$ only has ± 1 monodromies and the hyperplane arrangement in V is generic central.

Conjecture 1.6. *The polynomials $A_{V,\mathcal{L}}(t)$, $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ are combinatorial invariants.*

1.2 Structure of the paper

Chapter 1 is the overview of the thesis.

In Chapter 2, we collect the basic tools that we use through the article. Section 2.1 is a quick review of intersection cohomology theory. Section 2.2 is about the localization of constructible sheaf complex with respect to multiplicative group action. In Section 2.3, we recall a fundamental result on the topology of hyperplane arrangement complements. We also discuss the terminologies that we need from matroid theory in Section 2.4.

Chapter 3 discusses for the computaion of $A_{V,\mathcal{L}}(t)$. We study the intersection cohomology of a vector space with coefficients in a rank one local system.

We discuss the geometry of arrangement Schubert varieties in Chapter 4. The main results are the local product structure of a stratum (see Proposition 4.4) and the product structure of the restricted intersection cohomology complex on a stratum (see Theorem 4.6). We will show how to determine the tangential data V^F (resp. the normal data V_F) with associated $\mathcal{L}^F$ (resp. $\mathcal{L}_F$).

In Chapter 5 we prove that the long exact sequence for intersection cohomology of an arrangement Schubert variety breaks down to short exact sequences so that $Z_{V,\mathcal{L}}(t)$ can be computed by a summation of products of $A_{V^F,\mathcal{L}^F}(t)$ and $P_{V_F,\mathcal{L}_F}(t)$. We conclude that if the hyperplane arrangement in V is generic central and $\mathcal{L}$ has only ± 1 monodries around hyperplanes, then $A_{V,\mathcal{L}}(t)$, $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ are all combinatorially invariant (see Theorem 5.3 and Theorem 5.5). We conjecture this statement is still true if the assumption on $\mathcal{L}$ and hyperplane arrangement of V is removed.

Chapter 6 includes one of our main results, the closed formulas for polynomials $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ in the uniform matroid $U_{n-1,n}$ case.

We give some concrete examples in Chapter 7. Appendix includes polynomial tables for $U_{n-1,n}$ and $U_{n-2,n}$.

CHAPTER 2

PRELIMINARIES

2.1 A few things about intersection cohomology

2.1.1 Introduction

In this section, we will collect some basic results from the theory of intersection cohomology that are needed for the purposes of this paper. We are not going to write down the formal definitions of concepts, but refer interested readers to the standard references [GM83], [Bea08, V] and [CGJ92].

One of the motivations in the development of intersection cohomology theory of Goresky and MacPherson was to obtain a generalized version of Poincaré duality for singular spaces, where the ordinary Poincaré duality fails. (See the theorem on the duality of intersection chains $IC^{\bullet}$ and its corollaries below.)

For an n-dimensional piecewise linear (PL) topological stratified pseudomanifold X equipped with a local system $\mathcal{L}$ on the open stratum, one defines $IH_i^{\overline{p}}(X;\mathcal{L})$, called **perversity $\overline{p}$ intersection homology** with coefficients in $\mathcal{L}$, as the i-th homology of the intersection chain complex $IC_{\bullet}^{\overline{p}}(X;\mathcal{L})$; this is a subcomplex of the PL locally finite chains of X with coefficients in $\mathcal{L}$. The intersection chains are those chains ξ characterized by the condition that ξ and its boundary $\partial\xi$ meet strata of stratification in sets of suitably small dimensions controlled by the perversity function $\overline{p}$, which is a parameter measuring the deviation of chain from transversality. Since this condition is local, there exists a cochain complex of fine sheaves $IC^{\bullet}_{\overline{p}}(X;\mathcal{L})$ such that $\Gamma\big(U, IC^{i}_{\overline{p}}(X;\mathcal{L})\big) = IC^{\overline{p}}_{n-i}(U;\mathbb{Q})$ for open subsets U of X.[1] The hypercohomology of $IC^{\bullet}_{\overline{p}}(X;\mathcal{L})$ is called **perversity $\overline{p}$ intersection cohomology** with coefficients in $\mathcal{L}$, and we have $\mathbb{H}^{i}\big(X; IC^{\bullet}_{\overline{p}}(X;\mathcal{L})\big) \cong IH^{p}_{n-i}(X;\mathcal{L})$. Its compactly supported hypercohomology is the intersection homology for corresponding finite chains, and we write $\mathbb{H}^{i}_{c}\big(X; IC^{\bullet}_{\overline{p}}(X;\mathcal{L})\big) \cong IH^{\overline{p}}_{c,n-i}(X;\mathcal{L})$. Following the Borel indexing schemes, we use the notations

$$IH^{\bullet}_{\overline{p}}(X;\mathcal{L}) \coloneqq \mathbb{H}^{\bullet}\big(X; IC^{\bullet}_{\overline{p}}(X;\mathcal{L})\big),$$
$$IH^{\bullet}_{c,\overline{p}}(X;\mathcal{L}) \coloneqq \mathbb{H}^{\bullet}_{c}\big(X; IC^{\bullet}_{\overline{p}}(X;\mathcal{L})\big).$$

A more profound definition of intersectiom cohomology, which is independent of the PL structure and choice of stratification, involves giving criteria that uniquely characterize the sheaf complex $IC^{\bullet}_{p}(X;\mathcal{L})$ up to quasi-isomorphism, hence as an object in the constructible bounded derived category $D^{b}_{c}(X)$. The sheaf theoretic point of view allow us to use the functorial apparatus. (See section §2.2.)

2.1.2 Elements that we need

We will list some standard results that we need from the intersection cohomology theory, without giving proofs.

The following is a stratification free characterization of $IC^{\bullet}_{\overline{p}}(X;\mathcal{L})$, which implies that intersection (co)homology is a topological invariant.

Theorem 2.1 (Uniqueness Theorem, [GM83, §4.1]). *For any topological pseudomanifold X of dimension n, there exists a constructible bounded sheaf complex*

[1] For the sheaf complex we follow the Borel indexing scheme, which is different from the one used by Goresky and Macpherson by a shift of degree n.

$IC^{\bullet}_{\overline{p}}(X;\mathcal{L})$ in $D^b_c(X)$ which is uniquely characterized up to canonical isomorphism in $D^b_c(X)$ by the following axioms:

(a) Normalization: There exist a closed subset $\Sigma \subset X$ and a local system $\mathcal{L}$ on $X \setminus \Sigma$ such that $IC^{\bullet}_{\overline{p}}(X;\mathcal{L})|_{X\setminus\Sigma} = \mathcal{L}$.

(b) Lower bound: $\mathcal{H}^j\big(IC^{\bullet}_{\overline{p}}(X;\mathcal{L})\big) = 0$ for all $j < 0$.

(c) Support condition: For all $j > 0$, $\dim\big\{x \in X \mid \mathcal{H}^j\big(i_x^*\big(IC^{\bullet}_{\overline{p}}(X;\mathcal{L})\big)\big) \neq 0\big\} \leq n - \overline{p}^{-1}(j)$, with $i_x : \{x\} \hookrightarrow X$ denoting the point inclusion.

(d) Cosupport condition: For all $j < n$, $\dim\big\{x \in X \mid \mathcal{H}^j\big(i_x^!\big(IC^{\bullet}_{\overline{p}}(X;\mathcal{L})\big)\big) \neq 0\big\} \leq n - \overline{q}^{-1}(n-j)$, where $\overline{q}$ is the complementary perversity to $\overline{p}$.

Because $IC^{\bullet}_{\overline{p}}(X;\mathcal{L})$ is constructible, its stalk cohomology at a point can be computed as the hypercohomology of a small distinguished open neighborhood around the point. For a stratified topological pseudomanifold, the calculation can be reduced further to the link of the point as follows.

Let X be an n-dimensional stratified topological pseudomanifold and $\mathcal{L}$ be a local system on the top stratum U. Let $x \in S_{n-k}$ a point in the stratum of codimension k, and $U \cong \mathbb{R}^{n-k} \times \overset{o}{c}(L)$ be a distinguished neighborhood of x with link L (a stratified space of dimension $k-1$ satisfying certain conditions compatible with the stratification of X). Then we have

Proposition 2.2 (Calculation of the Local Intersection Cohomology)**.** *The stalk cohomology at $x \in S_{n-k}$ of $IC^{\bullet}_{\overline{p}}(X;\mathcal{L})$ is*

$$\begin{aligned} H^j\big(IC^{\bullet}_{\overline{p}}(X;\mathcal{L})_x\big) &\cong \mathbb{H}^j\big(U; IC^{\bullet}_{\overline{p}}(X;\mathcal{L})\big) \\ &\cong \begin{cases} \mathbb{H}^j\big(L; IC^{\bullet}_{\overline{p}}(L;\mathcal{L}|_{L\cap U})\big) & \text{if } j \leq \overline{p}(k), \\ 0 & \text{if } j > \overline{p}(k). \end{cases} \end{aligned}$$

For a proof for the above result, one may see [Bea08, Lemma V.3.15]. In [GM83, §2.4] Goresky and MacPherson give a more geometric approach to the calculation.

One of the most important properties of intersection chain complex is the duality between $IC^{\bullet}_{\overline{p}}$ and $IC^{\bullet}_{\overline{q}}$, when $\overline{p}$ and $\overline{q}$ are complementary perversities. (See [GM83, §5.3] and [Bea08, V. §9.B].)

Theorem 2.3 (Duality on $IC^{\bullet}$). *Suppose k is a field. Let X be k-orientable stratified topological pseudomanifold of dimension n and $\mathcal{L}$ be a local system of finite dimensional k-vector space, on the nonsingular open set of stratification. The dual local system $\mathcal{L}^{\vee}$ has stalks $Hom(\mathcal{L}_x, k)$, and the Verdier dual of $\mathcal{L}$ denoted by $\mathbb{D}\mathcal{L}$ has an isomorphism $\mathbb{D}\mathcal{L} \cong \mathcal{L}^{\vee}[n]$ in $D^b_c(X)$.*

If $\overline{p}$ and $\overline{q}$ are complementary perversities, then there exists an isomorphism in $D^b_c(X)$:

$$IC^{\bullet}_{\overline{p}}(X; \mathcal{L}^{\vee})[n] \cong \mathbb{D}IC^{\bullet}_{\overline{q}}(X; \mathcal{L}).$$

In this article we are mostly interested in the cases that $k = \mathbb{C}$.

Upon applying hypercohomology, one has the following

Corollary 2.4 (Generalized Poincaré Duality). *Under the same asssumption of the above theorem*

$$IH^{n-j}_{\overline{p}}(X; \mathcal{L}^{\vee}) \cong IH^{j}_{c,\overline{q}}(X; \mathcal{L})^{\vee}.$$

Corollary 2.5 (Duality with Middle Perversity). *If it is assumed further that X has only even codimensional strata, then there exists a unique self-complementary perversity $\overline{m}$, called the* ***middle perversity****. If moreover X is compact (e.g. a complex projective variety) with $\overline{p}$ and $\overline{q}$ being both the middle perversity $\overline{m}$, then there exist an isomorphism*

$$IH^{n-j}_{\overline{m}}(X; \mathcal{L}^{\vee}) \cong IH^{j}_{\overline{m}}(X; \mathcal{L})^{\vee}.$$

CONVENTION. Since this point, what we will use are all with the middle perversity, so we just simply use notations with scripts $\overline{m}$ omitted.

The Künneth formula for intersection cohomology is another tool we need. In this article, we restrict ourselves to the case that the middle perversity is taken, and sheaves have field coefficients. Suppose X and Y are topological pseudomanifolds, but are not necessarily compact. Let $\mathcal{L}_X$ and $\mathcal{L}_Y$ be local systems of vector spaces on the nonsingular parts of X and Y. Let π_1 and π_2 be the projections of $X \times Y$ to X and Y respectively. For the proof of the following theorem, see [CGJ92, Proposition 2, Remark 5].

Theorem 2.6 (Künneth Formula)**.** *Under the assumptions above, there exists an quasi-isomorphism,*

$$\pi_1^*\big(IC^\bullet(X;\mathcal{L}_X)\big) \otimes \pi_2^*\big(IC^\bullet(Y;\mathcal{L}_Y)\big) \cong IC^\bullet(X \times Y;\mathcal{L}_X \boxtimes \mathcal{L}_Y),$$

which implies that

$$IH_c^\bullet(X;\mathcal{L}_X) \otimes IH_c^\bullet(Y;\mathcal{L}_Y) \cong IH_c^\bullet(X \times Y;\mathcal{L}_X \boxtimes \mathcal{L}_Y).$$

We state the decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber (see [BBD82]).

Theorem 2.7 (BBDG Decomposition Theorem)**.** *Suppose $f : X \to Z$ is a proper map between complex algebraic varieties. Then there exist closed subvarieties Z_α of Z, local systems $\mathcal{L}_\alpha$ on the nonsingular part of Z_α and a quasi-isomorphism*

$$f_*IC^\bullet(X;\mathbb{C}) \cong \bigoplus_{\alpha}^{\textit{finite}} (i_\alpha)_*IC^\bullet(Z_\alpha;\mathcal{L}_\alpha)[-l_\alpha]$$

where $i_\alpha : Z_\alpha \to Z$ is the inclusion of the closed subvariety Z_α, and l_α is a suitable integer. Taking hypercohomology, it follows that

$$IH^i(X;\mathbb{C}) \cong \bigoplus_{\alpha}^{\textit{finite}} IH^{i-l_\alpha}(Z_\alpha;\mathcal{L}_\alpha).$$

An important application is that when f is a resolution of singularities of Z, a unique one of the direct summands will be $IC^\bullet(Z;\mathbb{C})$. It follows that $IH^\bullet(Z;\mathbb{C})$ is one of the direct summands of $IH^\bullet(X;\mathbb{C}) \cong H^\bullet(X;\mathbb{C})$, where the intersection cohomology and the cohomology coincide as X is smooth.

2.2 Localization with respect to a multiplicative group action

The following technical result will be needed in §4.5, where we prove Proposition 4.6, the product decomposition of the restricted intersection cohomology of each stratum of an arrangement Schubert variety. For a proof, see [Bra03, Lemma 6, Lemma 7].

Lemma 2.8 (Attracting Lemma)**.** *Let X be a variety equipped with an attracting $\mathbb{G}_m$-action. Let $Z = X^{\mathbb{G}_m}$ be the set of fixed points. Let $i : Z \hookrightarrow X$ be the inclusion map, and $p : X \to Z$ be the attracting map given by $p(x) = \lim_{t\to 0} t \cdot x$. For any $\mathcal{F}^\bullet \in D^b_c(X)$, there are natural maps*

$$p_*\mathcal{F}^\bullet \to i^*\mathcal{F}^\bullet \quad \textit{and} \quad i^!\mathcal{F}^\bullet \to p_!\mathcal{F}^\bullet.$$

If $\mathcal{F}^\bullet$ is $\mathbb{G}_m$-constructible, these maps are isomorphisms.

2.3 Topology of hyperplane arrangement complement

We collect a standard result on the topology of hyperplane complements in this section. Good references include [Dim17] and [C$^+$09].

Proposition 2.9. *Suppose $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ is a hyperplane arrangement in $\mathbb{C}^d$, with the complement $M(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{1\le i\le n} H_i$. Then*

$$H_1(M(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}\{\sigma_1, \sigma_2, \ldots, \sigma_n\},$$

where σ_i is a cycle represented by an elementary loop γ_i around the hyperplane H_i, for $1 \le i \le n$.

Let $\mathcal{L}$ be a rank one local system on $M(\mathcal{A})$ with a monodromy representation $\rho : \pi_1(M(\mathcal{A})) \to \mathbb{C}^*$. As $\mathbb{C}^*$ is abelian, ρ factors through $H_1(M(\mathcal{A}), \mathbb{Z})$. Hence $\mathcal{L}$ is determined a n-tuple $(a_1, a_2, \ldots, a_n) \in (\mathbb{C}^*)^n$, where $a_i = \rho(\gamma_i)$ is the local monodromy around the hyperplane H_i.

2.4 Review of matroids

We collect the basic terminologies from matroid theory here. For further details, readers may refer to the book [Oxl11].

Matroids are combinatorial objects that generalize the notion of linear independence for vectors. There can be axiomatized in a number of equivalent ways. The rank function and flats are the two axiomatizations we consider in this article. We first explain these two notions for a finite set of vectors in a vector space.

Example 2. *Let E be a finite set of vectors in a vector space. The* ***rank function*** *of E is the integer-valued function on the power set of E that assigns to each subset $S \subset E$ the dimension of the span of the vectors in S. A subset $F \subset E$ is a* ***flat*** *if every vector in E that lies in the span of the vectors in F is already contained in F. It turns out that these two pieces of data are equivalent.*

Let E be a finite set. A **matroid** M on E is defined by one of the following two axiomatizations:

1. A function $\mathrm{rk} : \mathcal{P}(E) \to \mathbb{Z}$ named rank for which
   - If S is a subset of E, then $0 \leq \mathrm{rk}(S) \leq |S|$.
   - If S, T are subsets of E for which $S \subset T$, then $\mathrm{rk}(S) \leq \mathrm{rk}(T)$.
   - If S, T are subsets of E, then $\mathrm{rk}(S \cap T) \leq \mathrm{rk}(S) + \mathrm{rk}(T) - \mathrm{rk}(S \cup T)$.
2. A collection of subsets of E are called flats for which
   - The set E is flat.
   - If F_1 and F_2 are flats, then $F_1 \cap F_2$ is a flat.
   - If F is a flat, then any element of $E \setminus F$ is contained in exactly one flat that is minimal among flats properly containing F.

The set E is called the **ground set** of M. The **rank** of M denoted by $\mathrm{rk}(M)$, is defined to be $\mathrm{rk}(E)$. The **corank** of a subset $S \subset E$ denoted by $\mathrm{crk}(S)$, is defined to be $\mathrm{rk}(M) - \mathrm{rk}(S)$.

The flats of M under the order of inclusion, form a lattice, denoted by $\mathbb{L}(M)$.

Let F be a flat of M. The **localization** of M at F denoted by M^F, is the matroid on F whose flats are the flats of M contained in F. The **contraction** of M at F denoted by M_F, is the matroid on $E \setminus F$ whose flats are $\{G \setminus F \mid G \geq F\}$.

If the ground set E of a matroid M is a finite set of vectors in a vector space, then it is not hard to verify that both the rank function and the flats of E satisfy these axioms. We call M a **linear matroid** on E. Two matroids are isomorphic if there exists a one-to-one correspondence between their ground sets which preserves the additional structure in the obvious way. We say a matroid is **realizable** if there exists a field k for which it is isomorphic to the linear matroid on a set of vectors in a vector space over k. In this article, we are mainly interested in the case that $k = \mathbb{C}$.

Example 3. *Uniform matroids form the main example we consider through this article. The uniform matroid* $U_{d,d+m}$ *is the matroid on the ground set* $E = [d+m] = \{1, 2, \ldots, d+m-1, d+m\}$ *such that* E *and the subsets of* E *with* d *or fewer elements are flats. It is realizable over* $\mathbb{C}$ *(or any infinite field).*

CHAPTER 3

INTERSECTION COHOMOLOGY OF A VECTOR SPACE WITH COEFFICIENTS IN A LOCAL SYSTEM ON AN ARRANGEMENT COMPLEMENT

One ingredient in our computation for the intersection of an arrangement Schubert variety is the compactly supported intersection cohomology of the vector space part, or the associated polynomial $A_{V,\mathcal{L}}(t)$. In this section, we will present that how to solve the general problem under the following assumptions. Suppose V is a vector space of complex dimension d, and

1. there is a **generic central hyperplane arrangement** $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ of V, in the sense that all the hyperplanes H_i pass through the origin point, and
$$\dim H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_j} = \begin{cases} d-j & \text{if } j \leq d, \\ 0 & \text{if } j > d. \end{cases}$$
Note that this assumption can be characterized by matroid that $M(V)$ is the uniform matroid of rank d over the ground set $\{1, 2, \ldots, n\}$;

2. A rank one local system $\mathcal{L}$ on the arrangement complement $V \setminus \bigcup_{1\leq i\leq n} H_i$ is given by numbers ± 1 for each hyperplane, so $\mathcal{L} \cong \mathcal{L}^\vee$.

We will start with some computations for the complex line with the one point stratification. First, one can ignore the point in stratification that have trivial monodromies and delete the points from X that have -1 monodromies, without affecting the intersection cohomology. This trick will be discussed in §3.1.

Example 4. *Let $X = (\mathbb{C}, \{0\})$, the complex line with the one-point stratification $\mathbb{C} \supset \{0\}$, where 0 denotes the origin point. Let $\mathcal{L}_{+1}$ be the rank one local system on the punctured disk $\mathbb{C}^*$, with the monodromy determined by $+1$ around the origin. In that case we can ignore the stratification (see §3.1 below). In fact, as the monodromy around the origin point is trivial, the intersection cohomology complex is quasi-isomorphic to the constant sheaf $\underline{\mathbb{C}}_X$. Thus the intersection cohomology is the same as the cohomology:*

$$IH^i(X; \mathcal{L}_{+1}) = \begin{cases} \mathbb{C}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

By the generalized Poincaré duality (see Corollary 2.4), the intersection cohomology with compact support is

$$IH^i_c(X; \mathcal{L}_{+1}) = \begin{cases} \mathbb{C}, & \text{if } i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Note that in both cases the intersection cohomology vanishes in odd degrees.

Example 5. *For the same space $X = (\mathbb{C}, \{0\})$ as in Example 4, now we consider instead the rank one local system $\mathcal{L}_{-1}$ with the monodromy determined by -1 around the origin. In that case, $IC^\bullet(X; \mathcal{L}_{-1})$ is the pushforward of $\mathcal{L}_{-1}$ from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$ from the Deligne's complex construction (see [GM83] or [Bea08]). The twisted intersection cohomology of X is the same as the twisted cohomology of $\mathbb{C} \setminus \{0\}$. Since $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to a circle, we only need to compute the twisted* *cohomology $H^\bullet(S^1;\mathcal{L}_{-1})$, which can be done using the cochain complex from the CW decomposition of a circle, with one single 0-cell in degree zero, and one single 1-cell in degree one (for computation details, a good reference is [C+09, Chapter 8]):*

$$\cdots \to \mathbb{C} \xrightarrow{\Gamma-\mathrm{id}} \mathbb{C} \to \cdots,$$

in the boundary map Γ is the monodromy -1. Thus

$$H^i(S^1;\mathcal{L}_{-1}) = \begin{cases} \ker(\Gamma-\mathrm{id}) = 0, & \text{if } i=0, \\ \operatorname{coker}(\Gamma-\mathrm{id}) = 0, & \text{if } i=1, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $IH^i(X;\mathcal{L}_{-1})$ vanishes for all i, and so does the intersection cohomology with compact support.

More generally, we have the following proposition:

Proposition 3.1. *Let X be the complex plane $\mathbb{C}$ with some distinguished points as its singularity, and $\mathcal{L}$ a rank one local system on X with monodromies of either $+1$ or -1 around those points. Suppose that $n > 1$ and there are n points around which the monodromies are -1, then the intersection cohomology is*

$$IH^i(X;\mathcal{L}) = \begin{cases} \mathbb{C}^{n-1}, & \text{if } i=1, \\ 0, & \text{otherwise.} \end{cases}$$

If all the monodromies are multiplication by -1, then the intersection cohomology is the same as cohomology of the complement.

Proof. One can think of the space X as $\mathbb{C}$ with n points deleted which is homotopy equivalent to a wedge product of n circles, and the twisted intersection cohomology is the same as the twisted cohomology. Consider the cochain complex, consisting of one 0-cell in degree zero, n 1-cells in degree one, and zero in other degrees with a boundary map $\Gamma - \mathrm{id}$ associated to the monodromy data $\Gamma = (-1, -1, \ldots, -1)$ in that case:

$$\cdots \to \mathbb{C} \xrightarrow{\Gamma - \mathrm{id}} \mathbb{C}^n \to \cdots$$

The twisted intersection cohomology is the same as the twisted cohomology:

$$H^i\Big(\bigvee_n S^1; \mathcal{L}_{-1}\Big) = \begin{cases} \ker(\Gamma - \mathrm{id}) = 0, & \text{if } i = 0, \\ \operatorname{coker}(\Gamma - \mathrm{id}) = \mathbb{C}^{n-1}, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

◇

3.1 Hyperplanes with trivial monodromy can be ignored

A very useful trick is that, if any hyperplane has $+1$ monodromy, then we can remove it from the arrangement without changing the intersection cohomology of the vector space. This can be proved by using the following cohomology sheaf stalk calculation: Let X be a pseudomanifold and $x \in X$ with U_x a distinguished open neighborhood of x. If $\mathcal{F}^\bullet$ is a cohomologically locally constant complex of sheaves with finitely generated stalks, i.e. $\mathcal{F}^\bullet$ is a constructible sheaf complex (for example intersection cohomology complex), then

$$\mathcal{H}^i(\mathcal{F}^\bullet)_x \cong \mathbb{H}^i(U_x; \mathcal{F}^\bullet).$$

Let us go back the case of vector space V, stratified by the intersections of the hyperplanes from the arrangement $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ (these intersections are also called geometric flats). Let $IC^\bullet(V; \mathcal{L})$ be the intersection cohomology complex on V with coefficients in $\mathcal{L}$. Now we assume x is a point of some hyperplane H_k with $+1$ monodromy, but not lying on other hyperplanes. So its distinguished open neighborhood U_x is isomorphic to $\mathbb{R}^{2(d-1)} \times \overset{o}{c}S^1$ where S^1 is the link at x. By the above cohomology sheaf stalk calculation, and Proposition 2.2 we have

$$\begin{aligned}
\mathcal{H}^i\big(IC^\bullet(V;\mathcal{L})\big)_x &\cong \mathbb{H}^i\big(U_x; IC^\bullet(V;\mathcal{L})\big) \\
&= IH^i(U_x;\mathcal{L}) \\
&\cong \begin{cases} H^i(S^1;\mathcal{L}_{+1}), & \text{if } i \leq 0, \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \mathbb{C}, & \text{if } i = 0, \\ 0, & \text{otherwise .} \end{cases}
\end{aligned}$$

This tells us that cohomology stalk of $IC^\bullet(V;\mathcal{L})$ is the constant $\mathbb{C}$ in degree 0 at each generic point of H_n. Let $\mathcal{L}'$ be the rank one local system on $V \setminus \bigcup_{i\neq k} H_i$ with the same monodromies as those of $\mathcal{L}$ around the hyperplanes except H_k. Note that $IC^\bullet(V;\mathcal{L})$ and $IC^\bullet(V;\mathcal{L}')$ both satisfy the characterization axioms of Theorem 2.1 with respect to the stratification induced by $\mathcal{A}' = \mathcal{A} \setminus \{H_k\}$. Therefore $IC^\bullet(V;\mathcal{L})$ and $IC^\bullet(V;\mathcal{L}')$ are canonically isomorphic in the derived category $D^b_c(V)$ and we can take off the hyperplane H_k of trivial monodromy from the arrangement without changing the intersection cohomology complex on V. With this observation we obtain the following proposition immediately:

Proposition 3.2. *Let V be a vector space of complex dimension d, stratified by the hyperplanes of generic central arrangement. Let $IC^\bullet(V;\mathcal{L})$ be the intersection cohomology complex with coefficients in a local system $\mathcal{L}$. If all the hyperplanes* *have* $+1$ *monodromy, then*

$$IH^i_c(V;\mathcal{L}) = \begin{cases} \mathbb{C}, & \text{if } i = 2(d-1), \\ 0, & \text{otherwise.} \end{cases}$$

By Poincaré duality,

$$IH^i(V;\mathcal{L}) = \begin{cases} \mathbb{C}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

If there exists at least one hyperplane of -1 *monodromy and the number of hyperplanes of* -1 *monodromy is not greater than* d*, then* $IH^i_c(V;\mathcal{L})$ *and* $IH^i(V;\mathcal{L})$ *vanishes in all degrees* i*.*

Proof. If all the monodromies are given by $+1$, we apply the above "ignorance trick" inductively to conclude that $IC^\bullet(V;\mathcal{L})$ is isomorphic to the constant sheaf $\underline{\mathbb{C}}_V$ on V in $D^b_c(V)$. In fact all the stalks of $IC^\bullet(V;\mathcal{L})$ at each hyperplane (including the origin point) is $\mathbb{C}$. It follows that the (compactly supported) intersection cohomology and the usual (compactly supported) cohomology coincide.

Next assume there exist at least one monodromy of -1, and the number of hyperplanes of -1 monodromy is not greater than the dimension of V. We take off or add into hyperplanes of $+1$ if necessary so that there are d hyperplanes in the end, but $IC^\bullet(V,\mathcal{L})$ remains the same in the derived category. Note that now V is isomorphic to a product of d copies of $\mathbb{C}$, with each as a stratified space carrying a rank one local system on $\mathbb{C}^*$ with monodromy of either $+1$ or -1 around the origin point.

With the above observation, using Theorem 2.6 we obtain the following Künneth formula on the intersection cohomology complexes

$$\begin{aligned} IC^\bullet(V;\mathcal{L}) &\cong IC^\bullet(\mathbb{C}\times\cdots\times\mathbb{C};\mathcal{L}) \\ &\cong IC^\bullet(\mathbb{C};\mathcal{L}_{+1})\boxtimes IC^\bullet(\mathbb{C};\mathcal{L}_{+1})\cdots\boxtimes IC^\bullet(\mathbb{C};\mathcal{L}_{-1}), \end{aligned} \tag{3.1}$$

where $\mathcal{L}_{\pm 1}$ denote the rank one local system on $\mathbb{C}^*$ with monodromy of ± 1. The last component of the exterior tensor product comes from a monodromy -1 around some hyperplane. It follows that

$$IH_c^\bullet(V;\mathcal{L}) \cong IH_c^\bullet(\mathbb{C};\mathcal{L}_{+1})\otimes IH_c^\bullet(\mathbb{C};\mathcal{L}_{+1})\cdots\otimes IH_c^\bullet(\mathbb{C};\mathcal{L}_{-1}). \tag{3.2}$$

The last component of the right hand side is zero, hence $IH_c^\bullet(V;\mathcal{L})$ is zero, so is $IH^\bullet(V;\mathcal{L})$. $\diamond$

3.2 Arinkin-Varchenko argument

Proposition 3.2 partially solves our problem, if there are not too many -1 monodromy hyperplanes. When the number of -1 monodromy exceeds the dimension of the ambient vector space, we cannot apply the Künneth formula. The work of Arinkin and Varchenko [AV12] helps us resolve the difficulty. Roughly speaking, with their results one can blow-up at the origin, then reduce the intersection cohomology of a vector space with a hyperplane arrangement to the cohomology of a induced projective hyperplanes arrangement complement of the exceptional divisor, which can be thought as a vector space minus a finite number of hyperplanes in general position.

In the next section, we will see that a formula of Hattori (see Equation (3.3)) can be applied to compute the cohomology of a vector space minus a finite number of hyperplanes in general position. So we are able to answer the question in the beginning of the section: how to find the rank one local system intersection cohomology of a vector space stratified by central generic hyperplanes.

Now we start with notation. Let V be the vector space of dimension d with a generic central hyperplane arrangement $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$. Denote by D the union of all hyperplanes in V. Let $\mathcal{L}$ be a rank one local system on the hyperplane arrangement complement $U = V \setminus D$ with all monodromies given by -1, and let $j : U \to V$ be the open inclusion.

Let $\pi : \widetilde{V} \to V$ be the blow-up at the origin of V. The preimage $\pi^{-1}(D)$ is a divisor in $\widetilde{V}$, which is locally a union of hyperplanes. The pull-back $\pi^*\mathcal{L}$ is a local system on the complement $\widetilde{V} \setminus \pi^{-1}(D)$, whose monodromy around each component of $\pi^{-1}(D)$ is either $+1$ or -1. Let $\widetilde{D}$ be the maximal divisor in $\widetilde{V}$, where $\pi^*\mathcal{L}$ has -1 monodromies around all of the connected components. Let $\widetilde{U} = \widetilde{V} \setminus \widetilde{D}$, and $\widetilde{U}_0 = \pi^{-1}(0) \cap \widetilde{U}$. The local system $\pi^*\mathcal{L}$ extends to a local system on $\widetilde{U}$ denoted by $\widetilde{\mathcal{L}}$.

When $H^i(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$ vanishes for degree $i > n-2$, we say the local system $\mathcal{L}$ on U satisfies **condition A** with respect to the resolution $\pi : \widetilde{V} \to V$ (see [AV12, Definition 1]). We aim to find out the intersection cohomology of V. We can achieve that by computing $H^i(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$ and $H^i(\widetilde{U}; \widetilde{\mathcal{L}})$ by using the following lemma.

Lemma 3.3. *If the local system $\mathcal{L}$ and its dual $\mathcal{L}^\vee$ with inverse monodromies on hyperplane arrangement complement U satisfy condition A with respect to a resolution $\pi : \widetilde{V} \to V$, then the intersection cohomology $IH^i(V; \mathcal{L}) \cong H^i(V; j_{!*}\mathcal{L})$ is isomorphic to $H^i(\widetilde{U}; \widetilde{\mathcal{L}})$.*

Proof. See [AV12, Theorem 1]. ◇

Since we only consider rank one local system with ± 1 monodromy, we have $\mathcal{L} \cong \mathcal{L}^\vee$. Both $\mathcal{L}$ and $\mathcal{L}^\vee$ satisfy condition A with respect to the blow-up resolution by the following argument. As $\widetilde{U}_0$ is an $n-2$ dimensional affine complex algebraic variety if it is nonempty and both $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}^\vee}$ are constructible sheaves, by Artin's vanishing theorem (see [Ach21, §2.6]), $H^i(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$ and $H^i(\widetilde{U}_0; \widetilde{\mathcal{L}^\vee}|_{\widetilde{U}_0})$ both vanish in degree $i > \dim \widetilde{U}_0 = n-2$. Using the above lemma, we conclude that $IH^i(V; \mathcal{L}) \cong H^i(\widetilde{U}; \widetilde{\mathcal{L}})$.

In fact we can further reduce the computation of the intersection cohomology of V to that of $\widetilde{U}_0$.

Lemma 3.4. *If $\mathcal{L}$ only has ± 1 monodromies, then $H^i(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$ and $H^i(\widetilde{U}; \widetilde{\mathcal{L}})$ are isomorphic. As a consequence, we have $IH^i(V; \mathcal{L}) \cong H^i(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$.*

Proof. If the number n of hyperplanes in V is odd, then $\widetilde{D}$ contains the exceptional divisor $\pi^{-1}(0)$, as its monodromy is given by a n-fold product of -1s, which is -1. (Since $\widetilde{U}_0$ is simply the empty set, $H^i(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$ is zero, $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ satisfy condition A trivially.) In this case, $\widetilde{U}$ is a $\mathbb{C}^*$-bundle over $B = \mathbb{CP}^{n-2} \setminus \{n \text{ generic hyperplanes}\}$ and the monodromy around a fiber is -1. Let $f : \widetilde{U} \to B$ be the projection. We will show that $f_* \widetilde{\mathcal{L}} \cong 0$, so $H^i(\widetilde{U}; \widetilde{\mathcal{L}}) = 0$. For any point $p \in B$, consider the fiber F_p at p. Let $i_p : p \to B$ and $i_{F_p} : F_p \to \widetilde{U}$ be the inclusion maps, and let $f_p : F_p \to p$ be the projection of the fiber onto p. Using base change, we have that $i_p^! f_* \widetilde{\mathcal{L}} \cong f_{p*} i_{F_p}^! \widetilde{\mathcal{L}}$. Note that $i_{F_p}^! \widetilde{\mathcal{L}} \cong i_{F_p}^* \widetilde{\mathcal{L}}[-2]$. Because $i_{F_p}^* \widetilde{\mathcal{L}} \cong 0$, we can conclude that $i_p^! f_* \widetilde{\mathcal{L}} \cong 0$. But for the stalk at p, $i_p^* f_* \widetilde{\mathcal{L}} \cong i_p^! f_* \widetilde{\mathcal{L}}[n-2] \cong 0$. It follows that $f_* \widetilde{\mathcal{L}} \cong 0$, and consequently $IH^i(V; \mathcal{L}) \cong H^i(\widetilde{U}; \widetilde{\mathcal{L}}) \cong 0$.

When the number n of hyperplanes in V is even, the exceptional divisor $\pi^{-1}(0)$ does not belong to $\widetilde{D}$, as its monodromy is given by an even product of -1s, which is $+1$. In fact $\widetilde{U}$ is a line bundle over B with $\widetilde{U}_0$ being its zero section, equipped with a $\mathbb{C}^*$-action which attracts the whole line bundle onto $\widetilde{U}_0$. Let $i : \widetilde{U}_0 \to \widetilde{U}$ be the inclusion map and let $p : \widetilde{U} \to \widetilde{U}_0$ be the projection map. Using the homotopy for constructible sheaves (see [Spr84, Proposition 1] and [Ach21, §2.10]), we have that $i^*\widetilde{\mathcal{L}} \cong p_*\widetilde{\mathcal{L}}$. It follows that $H^i(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0}) = H^i(\widetilde{U}_0; i^*\widetilde{\mathcal{L}}) \cong H^i(\widetilde{U}_0; p_*\widetilde{\mathcal{L}}) \cong H^i(\widetilde{U}; \widetilde{\mathcal{L}})$. $\diamond$

3.3 A formula of Hattori

Assume that there are n generic hyperplanes in a complex vector space V of dimension d, where $n > d$ (so the arrangement is noncentral). Let $\mathcal{L}$ be a nontrivial rank one local system on the arrangement complement U, with stalk $\mathbb{C}$, not necessarily with ± 1 monodromy around hyperplanes. In the paper [Hat75, Theorem 4], Hattori proved the following theorem:

Theorem 3.5. *The homology group $H_i(U; \mathcal{L})$ vanishes for $i \neq d$. The d-th homology group $H_d(U; \mathcal{L})$ is a $\mathbb{C}$-vector space of dimension*

$$\sum_{i=1}^{n-d} (-1)^{i+1} \binom{n}{d+i}. \tag{3.3}$$

3.4 Formula for polynomial $A_{V,\mathcal{L}}(t)$

With the preparation in Sections 4.1-4.3, we are ready to compute the polynomial $A_{V,\mathcal{L}}(t)$. Recall our notation that V is a vector space of complex dimension d, and $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ is a generic central hyperplane arrangement of V. In addition, we let $\mathcal{L}_{l,n-l}$ denote the rank one system with -1 monodromies around the hyperplanes $\{H_1, H_2, \ldots, H_l\}$ and $+1$ monodromies around the remaining $n-l$ hyperplanes. Notice that $(\mathcal{L}_{l,n-l})^\vee \cong \mathcal{L}_{l,n-l}$.

Proposition 3.6. *Let $d = \dim V$. If $1 \le l \le d$ or l is odd, then $A_{V,\mathcal{L}_{l,n-l}}(t) = 0$. If $l = 0$, then $A_{V,\mathcal{L}_{l,n-l}}(t) = t^d$. If $l > d$ and l is even, then we have*

$$A_{V,\mathcal{L}_{l,n-l}}(t) = \sum_{i=1}^{l-d} (-1)^{i+1} \binom{l-1}{d-1+i} t^{\frac{d+1}{2}}. \tag{3.4}$$

Proof. If $l \le d$, then by Proposition 3.2, we know that $A_{V,\mathcal{L}_{l,n-l}}(t) = 0$.

Also using Proposition 3.2, if $l = 0$, then one can remove all the hyperplanes with $+1$ monodromy without changing intersection cohomology, so $A_{V,\mathcal{L}_{l,n-l}}(t) = t^d$ as it is simply the Poincaré polynomial of compactly supported cohomology.

Assume that l is odd. Recall our notation in §3.2 that $\widetilde{U}_0$ the exceptional divisor at the origin with the nontrivial monodromy transforms of divisors removed, is the empty set in that case. By Lemma 3.4, the twisted intersection cohomology of V is zero, so $A_{V,\mathcal{L}_{l,n-l}}(t) = 0$.

Finally if $l > d$ and l is even, then $\widetilde{U}_0$ is a vector space of complex dimension $d - 1$, with generic $l - 1$ hyperplane removed, and around each of them the monodromy is -1. Let $\widetilde{\mathcal{L}}|_{\widetilde{U}_0}$ denote the pullback local system by blow-up restricted to $\widetilde{U}_0$. From Theorem 3.5, we have $H_\bullet(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$ is concentrated in degree $d - 1$, and $H_{d-1}(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$ is of dimension $\sum_{i=1}^{l-d} (-1)^{i+1} \binom{l-1}{d-1+i}$. So the cohomology $H^{d-1}(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$ as the dual is of the same dimension. From Lemma 3.4, letting $\mathcal{L} = \mathcal{L}^d_{l,n-l}$, it follows that $IH^\bullet(V; \mathcal{L})$ is concentrated in degree $d - 1$ and $IH^{d-1}(V; \mathcal{L}) \cong H^{d-1}(\widetilde{U}_0; \widetilde{\mathcal{L}}|_{\widetilde{U}_0})$. By the Poincaré duality, $IH^\bullet_c(V; \mathcal{L})$ is concentrated in degree $d + 1$ with $IH^{d+1}_c(V; \mathcal{L})$ being of dimension $\sum_{i=1}^{l-d} (-1)^{i+1} \binom{l-1}{d-1+i}$. Therefore the polynomial $A_{V,\mathcal{L}_{l,n-l}}(t)$ has the desired formula as Equation (3.4). ⋄

We see that as long as the hyperplane arrangement in V is generic central, $A_{V,\mathcal{L}_{l,n-l}}$ only depends on the dimension of V, the number of hyperplanes and the number of -1 monodromies. In other words, we have

Corollary 3.7. *$A_{V,\mathcal{L}_{l,n-l}}(t)$ is a combinatorial invariant.*

Example 6. *We present here the special case that V is a vector space of dimension $n-1$ and equipped with a generic central arrangement of n hyperplanes ($n \geq 3$), or equivalently $M(V)$ is the uniform matroid of rank $n-1$ over the ground set $[n] = \{1, 2, \ldots, n\}$. We will see later that $Z_{V,\mathcal{L}_{l,n-l}}(t)$ and $P_{V,\mathcal{L}_{l,n-l}}(t)$ can be written in some closed forms in this case (proved in Chapter 6).*

For ease of notation, in this case we simply write $A_{l,n-l}(t)$ for $A_{V,\mathcal{L}_{l,n-l}}(t)$. Similarly $Z_{l,n-l}(t)$ and $P_{l,n-l}(t)$ have the obvious meanings.

Using Proposition 3.6, one can verify that the polynomial is nonzero only if $l = 0$ or $l = n$. In fact,

$$A_{0,n}(t) = t^{n-1}$$

for any $n \geq 3$, and

$$A_{n,0}(t) = t^{\frac{n}{2}}$$

for any even number $n \geq 4$.

We also take $A_{0,1}(t) = A_{0,2}(t) = t$ by convention.

CHAPTER 4

GEOMETRY OF ARRANGEMENT SCHUBERT VARIETY

4.1 Arrangement Schubert variety and its associated matroid

Let M be a realizable matroid on the ground set $E = [n] = \{1, 2, \ldots, n\}$ and choose a realization $W = \mathrm{Span}\{w_1, w_2, \ldots, w_n\}$. We may assume M is a simple matroid, so every element in the spanning set is nonzero, and any two of them are linearly independent. Denote $L(M)$ the lattice of flats of M. With the chosen realization, there exists a surjective linear map f from $\mathbb{C}^n$ to W and its dual f^* is a linear injective map from W^* to $\mathbb{C}^n$. We denote the image of this dual map by V, a vector subspace of $\mathbb{C}^n$. Under this setting, a subset $F \subset E$ is a flat in $L(M)$ if and only $\{w_i\}_{i\in F}$ span a subspace in W such that adding any other w_j with $j \notin F$ to it results in a larger subspace.

Besides the above characterization of flats, we have one more which is useful in practice.

Proposition 4.1. *F is a flat if and only if there exists a vector $x = (x_1, x_2, \ldots, x_n) \in V$ such that $x_i = 0$ for $i \in F$, and $x_i \neq 0$ for $i \notin F$.*

Proof. In the above notations, we consider the matrix M_f given by the dual linear map f^* with respect to standard basis of $\mathbb{C}^n$. Reduce M_f to a row echelon form. As the matroid M is simple, each column of M_f is nonzero and every two columns of M_f are linearly independent, hence each column of the row echelon form is also nonzero, and every two columns of the row echelon are linearly independent. From this, using the first characterization and simple linear algebra, one can see that the second characterization holds. $\diamond$

Or one can start with a vector subspace $V \subset \mathbb{C}^n$ then obtain an associated matroid through geometric flats. Consider the coordinate hyperplanes $H_i = \{x \in V \mid x_i = 0\}$, where $1 \leq i \leq n$. The intersections of these hyperplanes form a poset, in which every element is called a geometric flat. Actually a matroid $M(V)$ on the ground set $[n]$ shows up here, whose flats are characterized by the condition that F is flat if and only if there exists a vector $x = (x_1, x_2, \ldots, x_n) \in V$ such that $x_i = 0$ for $i \in F$, and $x_i \neq 0$ for $i \notin F$.

The associated "**arrangement Schubert variety**" of a vector space $V \subset \mathbb{C}^n$, denoted by $Y(V)$, is defined to be the closure of V under the inclusion $V \subset \mathbb{C}^n \hookrightarrow (\mathbb{CP}^1)^n$. When the the vector space V and its associated matroid $M(V)$ are clear from the context, we simply use the notations Y and M to denote the arrangement Schubert variety and the matroid. The "**arrangement**" here refers to the hyperplane arrangement in V consisting of the coordinate hyperplanes $H_i = \{x \in V \mid x_i = 0\}$.

4.2 Coarse stratification by additive group action orbits indexed by flats

In fact an arrangement Schubert variety is very "linear". There exists a stratification for Y, given by the cells

$$S_F = \{x \in Y \mid x_i = \infty \text{ if and only if } i \notin F\},$$

where $F \subset E = [n]$ are flats in $L(M)$. In particular, the whole ground set E, the top flat gives the top cell S_E, which is the vector space V. For the other extreme, the bottom flat is the empty set $\emptyset$, and correspondingly $S_\emptyset$ is the most singular point $(\infty, \infty, \ldots, \infty)$ of an arrangement Schubert variety. We can see that $\overline{S_F} = \bigcup_{G \leq F} S_G$.

The group action $(V, +)$ on V given by vector addition naturally extends to the whole variety Y. Moreover, using the above characterization of a stratum S_F we see that the group action $(V, +)$ preserves the stratification, i.e. $V \cdot S_F = S_F$.

4.3 Fine stratification indexed by comparable flat pairs

Since we are going to use local systems on arrangement complements and investigate twisted intersection complexes, a fine stratification for the variety will be required.

As follows we take a fine stratification for an arrangement Schubert variety, given by the locally closed subsets

$$S_F^G = \{x \in Y \mid x_i = \infty \text{ if and only if } i \notin F, x_j = 0 \text{ if and only if } j \in G\},$$

where G is any flat contained in F, or $G \leq F$ with respect to the partial order given by inclusion. Note that $S_F^G \subset \overline{S_{F'}^{G'}}$ if and only if $G' \leq G \leq F \leq F'$.

In particular, if F is a nonempty flat, then for rank-one flats $\{i\} \subset F$, the locally closed subsets $S_F^{\{i\}}$ are the same as the intersections $\overline{H_i} \cap S_F$. Note that S_F is naturally a vector space of dimension $\operatorname{rk} F$, and these $S_F^{\{i\}}$ give a hyperplane arrangement of S_F. So the original hyperplane arrangement in V induces hyperplane arrangements in each S_F. Moreover, we will see that S_F is identified naturally with some quotient of V, so S_F can act additively as a subgroup of V, under a choice of embedding $S_F \hookrightarrow V$ (see the next subsection).

4.4 Local product structure along strata

We will show that each stratum is "very equisingular" in the sense that there exists a local product structure along it. Then for an intersection complex along a stratum with coefficients in a local system, it has a tensor product decomposition induced by the topology product structure of stratum, with one intersection complex in the tangential direction, and the other one in the normal direction.

We will start with a big Zariski open neighborhood U_F of S_F. It turns out that U_F has a nice product structure.

We denote by 0_F the "origin" of S_F, i.e. the point $(x_1, x_2, \ldots, x_n)$ with coordinates x_i being 0 if $i \in F$, and ∞ if $i \notin F$. In the fine stratification, $\{0_F\} = S_F^F$. Let U_F be the union of strata $\bigcup_{F \leq G} S_G$ and $\pi_F : U_F \to S_F$ be the projection given by

$$\pi_F(p) = q, \text{ where } q_i = \begin{cases} p_i, & \text{if } i \in F, \\ \infty, & \text{if } i \notin F. \end{cases}$$

We are going to show that U_F is isomorphic to the product of $\pi^{-1}(0_F)$ and S_F. But at first we have to check that π_F is well-defined, in other words the image of π_F lies in S_F. Moreover, it turns out that π_F is actually a surjective map.

Proposition 4.2. *The projection π_F is well-defined and surjective.*

Proof. First we show that the image is inside S_F. Suppose F is a flat. Then by Proposition 4.1 there exists a vector $y = (y_1, y_2, \ldots, y_n) \in V$ such that $y_i = 0$ for $i \in F$, and $y_i \neq 0$ for $i \notin F$. For an element $x = (x_1, x_2, \ldots, x_n) \in U_F$, we take the sum $x + ty$, where t varies in real numbers. As t tends to infinity, the coordinates of $x + ty$ not indexed by F tend to infinity, but the coordinates indexed by F are constant, hence $\pi_F(x) = \lim_{t\to\infty}(x + ty)$, an element of S_F.

Next we prove that the image of π_F is exactly S_F, so π_F is a surjective map. It will suffice to show that S_F is contained in the image of the restricted projection $\pi_F|_{S_E} = \pi_F|_V : V \to S_F$. Consider the inclusion $V \subset \mathbb{C}^n \hookrightarrow \mathbb{C}^F \times (\mathbb{CP}^1)^{E\setminus F} \hookrightarrow (\mathbb{CP}^1)^n$. Let $\overline{V}$ be the closure of V inside $\mathbb{C}^F \times (\mathbb{CP}^1)^{E\setminus F}$, which is exactly U_F. Note that $\pi_F(U_F) = \pi_F(\overline{V}) \subset \overline{\pi_F(V)} = \pi_F(V)$ as π_F is continuous and $\pi_F(V)$ is a closed set. Since $\pi_F(S_F) = S_F$, we have $S_F \subset \pi_F(U_F) \subset \pi_F(V)$. $\diamond$

The restriction $\pi_F|_V : V \to S_F$ is a surjective linear map, and its kernel is $V_F = \bigcap_{i\in F} H_i$. We can identify S_F with the quotient V/V_F, or view it as the image of V under the natural projection $\mathbb{C}^n \to \mathbb{C}^F$. It is worth keeping a notation V^F for S_F when we want to emphasize its vector space structure, rather than it being a stratum of the variety. Recall that $V^F = S_F$ as a vector space, has dimension $\operatorname{rk} F$, and is equipped with an hyperplane arrangement consisting of $\{S_F \cap \overline{H_i} \mid i \in F\}$.

Next we study the normal slices along each stratum S_F. Note that under the additive group action, two normal slices along a cell S_F are different only by a linear translation. The normal slice at 0_F the "origin" point of S_F, $\pi^{-1}(0_F)$ is the closure of V_F in $\{0\}^F \times (\mathbb{CP}^1)^{E\setminus F} \subset (\mathbb{CP}^1)^n$, which is isomorphic to the arrangement Schubert variety $Y(V_F)$ where V_F may be thought of as a vector subspace of $\mathbb{C}^{E\setminus F}$. Under the fine stratification of Y, $\pi^{-1}(0_F)$ is the closure of S_E^F in Y, which equals the union $\bigcup_{F\leq G_2\leq G_1} S_{G_1}^{G_2}$.

We want to show that there exists a product structure near each point of a stratum. We will take advantage of the additive group action $(V,+)$.

Proposition 4.3. *The union of strata* $U_F = \bigcup_{F\leq G} S_G$ *is isomorphic to the product* $\pi_F^{-1}(0_F)\times S_F$.

Proof. There exist splitting maps of the surjective map $\pi_F|_V : V \to S_F$, or embeddings of S_F into V. We choose one from these embeddings such that its image is a linear complement of $V_F = \bigcap_{i\in F} H_i$ in V, and denote it by $\sigma : S_F \to V$.

Using the additive group action, we define a map $\Theta_F : \pi^{-1}(0_F)\times S_F \to U_F$ given by

$$(a,b)\mapsto a+\sigma(b).$$

It is useful to note that $\pi_F(a+\sigma(b)) = \pi_F(a)+\pi_F(\sigma(b)) = b$. We define a map $\Phi_F : U_F \to \pi^{-1}(0_F)\times S_F$ given by

$$p\mapsto (p-\sigma(\pi_F(p)), \pi_F(p)).$$

It is not hard to see that Φ_F and Θ_F are the inverse of each other, so the isomorphism holds. $\diamond$

As we are interested in local systems and twisted intersection complexes on each factor of the product structure, we want not only a regular product but a stratification-preserving product for the fine stratification.

We denote M^F the **localization** matroid of M at the flat F and M_F the **contraction** matroid of M at the flat F. The lattice of flats $L(M^F)$ collects flats $G\leq F$, and the lattice of flats $L(M_F)$ collects flats $G\geq F$.

Proposition 4.4. *Locally near the point* 0_F, U_F *is a stratified product of* $\pi^{-1}(0_F)$ *and* S_F.

Proof. We have seen that $\pi^{-1}(0_F)$ is isomorphic to the arrangement Schubert variety $Y(V_F)$. As a vector space with a hyperplane arrangement $\{S_F \cap \overline{H_i},\}_{i\in F}$, S_F is stratified by the intersections of the hyperplanes.

The union $\bigcup_{F\leq G_2} S_{G_2}^F$ is an affine neighborhood of 0_F in $\pi^{-1}(0_F)$, in fact $\bigcup_{F\leq G_2} S_{G_2}^F = \pi^{-1}(0_F)$. Meanwhile the union $\bigcup_{G_1\leq F} S_F^{G_1}$ is S_F. For $G_1 \leq F \leq G_2$, however the product component $S_{G_2}^F \times S_F^{G_1}$ is not mapped into $S_{G_2}^{G_1}$, under the map Θ_F. The issue is that in $S_{G_2}^F$, the coordinates in $G_2 \setminus F$ are nonzero finite numbers, and in $\sigma(S_F^{G_1})$, the coordinates outside of F are also nonzero finite numbers. The points of the sum $S_{G_2}^F + \sigma(S_F^{G_1})$ possibly have zero coordinates in $G_2 \setminus F$, so then land outside of $S_{G_2}^{G_1}$. Here we recall from the proof of Proposition 4.3 that $\sigma : S_F \to V$ is a chosen splitting map of $\pi_F : V \to S_F$. Under the Zariski topology, Θ_F does not preserve stratification. We want to show that there exist topological neighborhoods of 0_F, $N \subset \pi^{-1}(0_F)$ and $N' \subset S_F$ so that $\Theta_F|_{N\times N'}$ respects strata.

We can choose a topological open neighborhood N of 0_F inside $\pi_F^{-1}(0_F)$, and a topological neighborhood $N'_\sigma \subset \sigma(S_F) \subset V$, and a number $\eta > 0$ such that for all $x \in N$, $2\eta < |x_i| < \infty$, and for all $y \in N'_\sigma$, $0 < |y_i| < \eta$, where $i \in G_2 \setminus F$. The condition that under the map Θ_F, the magnitudes $0 < \eta < |x_i + y_i| < \infty$ for all $i \in G_2 \setminus F$, fixes the above issue. Let $N' = \sigma^{-1}(N'_\sigma)$. The restricted $\Theta_F|_{N\times N'}$ respects the fine stratification. $\diamond$

4.5 Twisted intersection complex and product structure

Let $\mathcal{L}$ be a rank one local system on the biggest stratum of Y under the fine stratification, i.e. the hyperplane arrangement complement $S_E^{\emptyset}$. Suppose that the monodromy of $\mathcal{L}$ around the hyperplane H_i is given by a number a_i (see Proposition

2.9).

In this subsection we are going to show that at each point of the cell S_F, the intersection complex $IC^\bullet(Y;\mathcal{L})|_{S_F}$ has an exterior product structure, using the geometric product structure in Proposition 4.4.

We have seen that around the origin point 0_F of the stratum S_F, there exists a neighborhood U_F which is a product of the normal slice $\pi^{-1}(0_F)$ and S_F. To obtain an exterior product structure for the intersection complex $IC^\bullet(Y;\mathcal{L})|_{S_F}$, we need to figure out the local system $\mathcal{L}_F$ on $\pi^{-1}(0_F)$ in the normal direction and the local system $\mathcal{L}^F$ on S_F in the tangential direction. And then on the induced biggest stratum of $N_F \times N'_F$, $\mathcal{L} \cong \mathcal{L}_F \boxtimes \mathcal{L}^F$.

For our purpose, we modified the product structure bijection Θ in Proposition 4.3 to be a stratified isomorphism by restricting it to topological open sets as in Proposition 4.4. In particular, we can find topological open neighborhoods around 0_F, N_F inside S_E^F, and N'_F inside $S_F^\emptyset$, such that the image of the restriction $\Theta|_{N_F\times N'_F}$ lies entirely in $S_E^\emptyset$, hence does not hit any hyperplane H_i.

Note that N_F is homotopy equivalent to S_E^F via the deformation retraction, hence $H_1(N_F;\mathbb{Z}) \cong H_1(S_E^F;\mathbb{Z})$. Restricting the original hyperplane arrangement of V onto S_E^F, we see that each flat G such that $G > F$ and $\operatorname{rk} G = \operatorname{rk} F + 1$ gives a hyperplane $\bigcap_{i\in G} H_i$ on S_E^F. Hence $H_1(N_F;\mathbb{Z}) = \mathbb{Z}\{a_G \mid G > F, \operatorname{rk} G = \operatorname{rk} F + 1\}$ by Proposition 2.9. With similar arguments, we conclude that $H_1(N'_F;\mathbb{Z}) \cong H_1(S_F^\emptyset;\mathbb{Z}) = \mathbb{Z}\{b_i \mid i \in F\}$. We also recall that $H_1(S_E^\emptyset;\mathbb{Z}) = \mathbb{Z}\{\sigma_i \mid 1 \le i \le n\}$. The loop representatives for the generators in each homology group are clear from geometry.

Proposition 4.5. *Suppose that the monodromies of $\mathcal{L}$ are given by numbers a_i, for $1 \le i \le n$. The local system $\mathcal{L}_F$ on $N_F \subset S_E^F$ has monodromies $\prod_{i\in G\setminus F} a_i$,* *for each $G > F$ with $\operatorname{rk} G = \operatorname{rk} F + 1$, and the local system $\mathcal{L}^F$ on $N'_F \subset S^{\emptyset}_F$ has monodromies a_i for $i \in F$.*

Proof. We will figure out what the monodromies of $\mathcal{L}_F$ (respectively $\mathcal{L}^F$) are by embedding the elementary 1-cycles or loops around corresponding hyperplanes of N_F (respectively N'_F) into $S^{\emptyset}_E$.

Θ_F induces a map on homology groups,

$$(\Theta_F)_* : H_1(N_F; \mathbb{Z}) \times H_1(N'_F; \mathbb{Z}) \to H_1(S^{\emptyset}_E; \mathbb{Z}).$$

For each flat G with $G > F$ and $\operatorname{rk} G = \operatorname{rk} F + 1$, for instance we may choose a loop as a representative of the generator a_G in $H_1(N_F; \mathbb{Z})$, parameterized by θ in the form

$$\lambda_G = (\underbrace{0, \ldots, 0}_{F}, \underbrace{M_{k_1}e^{i\theta}, M_{k_2}e^{i\theta} \ldots, M_{k_p}e^{i\theta}}_{G \setminus F}, \underbrace{M_{l_1}, M_{l_2}, \ldots, M_{l_q}}_{E \setminus G})$$

where p and q are the numbers of $G \setminus F$ and $E \setminus G$ respectively, and M's are nonzero complex numbers with sufficiently large magnitudes (based on the choice of N_F), and satisfy some suitable linear relations as the loop lives in a vector subspace of $V \subset \mathbb{C}^n$, and G is a flat that covers the flat F.

On the other hand, we may choose a constant loop as a representative of the zero element in $H_1(N'_F; \mathbb{Z})$

$$\lambda'_F = (\underbrace{m, m, \ldots, m}_{F}, \underbrace{\infty, \infty, \ldots, \infty}_{E \setminus F})$$

where m is a complex number with a sufficiently small magnitude (based on the choice of N'_F). Recall from the construction in Proposition 4.4 that $\Theta_F(\lambda_G, \lambda'_F) = \lambda_G + \sigma(\lambda'_F)$ will live in $S^{\emptyset}_E$ as required with well chosen M's and m.

Let p_j be the projection from $(\mathbb{C}^*)^n$ to its j-th factor $\mathbb{C}^*$, for $1 \leq j \leq n$. Then

$$p_{j*}(\Theta_F)_*(a_G \times \{0\}) = \begin{cases} 1, & \text{if } j \in G \setminus F, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Geometrically, an elementary loop around the hyperplane $\cap_{i \in G} H_i$, say λ_G in S_E^F, after an additive translation by a small vector of V, is homologous to the composition of the elementary loops of σ_i around H_i, for $i \in G \setminus F$.

With an argument in the same flavor, we also conclude that

$$p_{j*}(\Theta_F)_*(\{0\} \times b_i) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Therefore $(\Theta_F)_*(a_G \times \{0\}) = \sum_{i \in G \setminus F} \sigma_i$ and $(\Theta_F)_*(\{0\} \times b_i) = \sigma_i$. Then the statement about the local system monodromies of $\mathcal{L}^F$ and $\mathcal{L}_F$ follows. $\diamond$

It is time to give the main result in this section:

Theorem 4.6. *Locally around 0_F, the restricted intersection cohomology complex $IC^\bullet(Y;\mathcal{L})|_{S_F}$ is isomorphic to the external product of $IC^\bullet\big(Y(V_F);\mathcal{L}_F\big)$ and $IC^\bullet(V^F;\mathcal{L}^F)$. As a consequence,*

$$\mathbb{H}_c^\bullet\big(S_F; IC^\bullet(Y;\mathcal{L})|_{S_F}\big) \cong H^\bullet\big(IC^\bullet(Y(V_F);\mathcal{L}_F)_\infty\big) \otimes IH_c^\bullet(V^F;\mathcal{L}^F). \tag{4.3}$$

Proof. With appropriately chosen neighborhoods as in Proposition 4.4, we see that $\Theta_F : \pi^{-1}(0) \times S_F \to U_F$ restricts to a stratified isomorphim from a product of neighborhoods $N \times N'$ to some open set $W = \Theta_F(N \times N') \subset U_F$. Recall that we use the notation V^F for S_F, when considering S_F as a vector space. Also V_F is the vector space $\bigcap_{i \in F} H_i$.

Let us consider the following commutative diagram

$$\begin{tikzcd}
& \{0_F\}\times N' \arrow[r, hook] \arrow[d, hook] & N\times N' \arrow[d, hook] & \\
\{0_F\}\times\{0\} \arrow[ur, hook, "j_0"] \arrow[r, hook, "j"] \arrow[d, "\theta_F"] & \{0_F\}\times V^F \arrow[r, hook] \arrow[d] & \pi^{-1}(0_F)\times V^F \arrow[d, "\Theta_F"] & \\
\{0_F\} \arrow[r, hook, "i"] & S_F \arrow[r, hook] & U_F \arrow[r, hook] & Y
\end{tikzcd}$$

where the lower three vertical arrows are isomorphims, in particular θ_F is the natural identification between two points induced by Θ_F, and the other arrows denote the obvious inclusions.

Regarding the intersection cohomology complexes, we have the isomorphisms by chasing the diagram

$$\begin{aligned}
\theta_F^* i^!\big(IC^\bullet(Y;\mathcal{L})|_{S_F}\big) &\cong j^!\big[\big(\Theta_F^* IC^\bullet(U_F;\mathcal{L})\big)|_{\{0_F\}\times V^F}\big] \\
&\cong j_0^!\big[\big(\Theta_F^* IC^\bullet(U_F;\mathcal{L})\big)|_{\{0_F\}\times N'}\big] \\
&\cong j_0^!\big[\big(IC^\bullet(N;\mathcal{L}_F)\boxtimes IC^\bullet(N';\mathcal{L}^F)\big)|_{\{0_F\}\times N'}\big] \\
&\cong j_0^!\big(IC^\bullet(N;\mathcal{L}_F)|_{0_F}\boxtimes IC^\bullet(N';\mathcal{L}^F)\big) \\
&\cong j^!\big(IC^\bullet(N;\mathcal{L}_F)|_{0_F}\boxtimes IC^\bullet(V^F;\mathcal{L}^F)\big)
\end{aligned} \tag{4.4}$$

where the third isomorphism comes from the Künneth formula for intersection cohomology complex (see Theorem 2.6), and the last isomorphism is induced from the upper left triangle in the commutative diagram.

Thinking about the multiplicative group action of $\mathbb{C}^*$ on Y, which is stratification preserving and induces the group action on S_F or V^F, then i and j are the inclusion of contracting points $\{0_F\}\times\{0\}$ and $\{0_F\}$ under corresponding $\mathbb{C}^*$-actions.

Consider the inclusion $i : \{0_F\}\to S_F$. Denote by $p : S_F \to \{0_F\}$ the attracting map, with respect to the $\mathbb{C}^*$-action, which is a constant map in that case. Using the localization with respect to the $\mathbb{C}^*$-action, by Lemma 2.8, for any $\mathbb{C}^*$-constructible complex $\mathcal{F}^\bullet$ on S_F, $i^!\mathcal{F}^\bullet \cong p_!\mathcal{F}^\bullet$. Apply this identity to $IC^\bullet(Y;\mathcal{L})|_{S_F}$, then we have

$$i^!\big(IC^\bullet(Y;\mathcal{L})|_{S_F}\big) \cong p_!\big(IC^\bullet(Y;\mathcal{L})|_{S_F}\big).$$

Applying the cohomology functor to the both sides above, it follows that

$$H^\bullet\Big(i^!\big(IC^\bullet(Y;\mathcal{L})|_{S_F}\big)\Big) \cong \mathbb{H}_c^\bullet\big(S_F; IC^\bullet(Y;\mathcal{L})|_{S_F}\big). \tag{4.5}$$

Since θ_F is an identification between two points, the cohomology of left side of Equation (4.4) can be identified with $\mathbb{H}_c^\bullet\big(S_F; IC^\bullet(Y;\mathcal{L})|_{S_F}\big)$.

Similarly as what we did to the inclusion $i^!\big(IC^\bullet(Y;\mathcal{L})|_{S_F}\big)$, we can apply Lemma 2.8 to the second component of $j^!\big(IC^\bullet(N;\mathcal{L}_F)|_{0_F} \boxtimes IC^\bullet(V^F;\mathcal{L}^F)\big)$, then replace the last line of Equation (4.4) with

$$IC^\bullet\big(Y(V_F);\mathcal{L}_F\big)|_{0_F} \otimes p_! IC^\bullet(V^F;\mathcal{L}^F),$$

where we should notice that 0_F is the most singular point of the arrangement Schubert variety $Y(V_F)$.

Taking cohomology and using the Equation (4.4), we have

$$\mathbb{H}_c^\bullet\big(S_F; IC^\bullet(Y;\mathcal{L})|_{S_F}\big) \cong H^\bullet\big(IC^\bullet(Y(V_F);\mathcal{L}_F)|_\infty\big) \otimes IH_c^\bullet(V^F;\mathcal{L}^F). \tag{4.6}$$

$\diamond$

CHAPTER 5

LONG EXACT SEQUENCE FOR COARSE STRATIFICATION

Let V be a vector subspace in $\mathbb{C}^n$, with the associated matroid M and the associated arrangement Schubert variety Y. Let $\mathcal{L}$ be a rank one local system on the hyperplane arrangement complement $S_E^{\emptyset}$, with ± 1 monodromy around each hyperplane. For each $0 \leq k \leq \operatorname{rk} M$, $\coprod_{\operatorname{crk} F \leq k} S_F$ is an open subset of Y containing $\coprod_{\operatorname{crk} F = k} S_F$ as a closed subset with the open complement $\coprod_{\operatorname{crk} F < k} S_F$. To simplify notation, we define $Y(k) \coloneqq \coprod_{\operatorname{crk}(F) \leq k} S_F$, then $Y(k-1) = \coprod_{\operatorname{crk} F < k} S_F$. We let $Y(k, k-1) \coloneqq Y(k) \setminus Y(k-1) = \coprod_{\operatorname{crk} F = k} S_F$.

Let X be a general topological space with an open subset W. Let $j : W \to X$ be the open inclusion and let $i : X \setminus W \to X$ be the complementary closed inclusion. There is an attaching triangle in the derived category of sheaves of modules on X

$$j_! j^* \longrightarrow \mathrm{id} \longrightarrow i_* i^* \xrightarrow{[1]} .$$

Consider the triad, $Y(k)$ with the open subset $Y(k-1)$, and the closed complement $Y(k, k-1)$. Apply the above attaching triangle to $IC^\bullet(Y; \mathcal{L})|_{Y(k)}$, and then apply the compactly supported hypercohomology functor. We obtain the long exact sequence

$$\cdots \to \mathbb{H}_c^i(Y(k-1); IC^\bullet(Y;\mathcal{L})|_{Y(k-1)}) \to \mathbb{H}_c^i(Y(k); IC^\bullet(Y;\mathcal{L})|_{Y(k)}) \to$$
$$\mathbb{H}_c^i(Y(k,k-1); IC^\bullet(Y;\mathcal{L})|_{Y(k,k-1)}) \to \mathbb{H}_c^{i+1}(Y(k-1); IC^\bullet(Y;\mathcal{L})|_{Y(k-1)}) \to \cdots$$

Since $Y(k)$ is open, $\mathbb{H}_c^i\big(Y(k); IC^\bullet(Y;\mathcal{L})|_{Y(k)}\big) \cong IH_c^i\big(Y(k);\mathcal{L}\big)$. We can rewrite the above long exact sequence neatly as

$$\cdots \to IH_c^i\big(Y(k-1);\mathcal{L}\big) \to IH_c^i\big(Y(k);\mathcal{L}\big) \to$$
$$\mathbb{H}_c^i\big(Y(k,k-1); IC^\bullet(Y;\mathcal{L})|_{Y(k,k-1)}\big) \to IH_c^{i+1}\big(Y(k-1);\mathcal{L}\big) \to \cdots$$

Our computation of $IH^\bullet(Y;\mathcal{L}) = \mathbb{H}^\bullet\big(Y; IC^\bullet(Y;\mathcal{L})\big)$ will be given by an induction argument by adding one dimension at each time based on the following key theorem.

Theorem 5.1. *If Y is the arrangement Schubert variety of a vector space equipped with a rank one local system $\mathcal{L}$ on the hyperplane arrangement complement with monodromies of multiplication by ± 1, then the above long exact sequence breaks down into short exact sequences*

$$0 \to IH_c^i\big(Y(k-1);\mathcal{L}\big) \to IH_c^i\big(Y(k);\mathcal{L}\big) \to \mathbb{H}_c^i\big(Y(k,k-1); IC^\bullet(Y;\mathcal{L})|_{Y(k,k-1)}\big) \to 0.$$

Proof. To show the long exact sequence is breaking, it suffices to show that the maps

$$IH_c^i\big(Y(k-1);\mathcal{L}\big) \to IH_c^i\big(Y(k);\mathcal{L}\big)$$

are injective. Applying the Verdier duality, by Corollary 2.4 it will be enough to show that

$$IH^i\big(Y(k);\mathcal{L}^\vee\big) \to IH^i\big(Y(k-1);\mathcal{L}^\vee\big)$$

are all surjective. Note that $\mathcal{L}^\vee \cong \mathcal{L}$ under the monodromy assumption. (In fact we only need the condition that the class of local systems we consider is closed under duality. See Remark 5.2 after the proof.)

The natural $\mathbb{C}^*$-action on $\mathbb{C}$ given by multiplication extends to $\mathbb{CP}^1$, and then the extension induces a component-wise action on $(\mathbb{CP}^1)^n$. We also think of a double $\mathbb{C}^*$-action by squaring the natural $\mathbb{C}^*$ multiplication. Consider the map $\overline{\varphi} : (\mathbb{CP}^1)^n \to (\mathbb{CP}^1)^n$ given by

$$([x_i : y_i])_i \mapsto ([x_i^2 : y_i^2])_i \quad \text{for } i = 1, 2, \ldots, n.$$

We equip the left $(\mathbb{CP}^1)^n$ of the map $\overline{\varphi}$ with the natural $\mathbb{C}^*$-action, and the right one with the double $\mathbb{C}^*$-action, so that $\overline{\varphi}$ is an equivariant map. Since the arrangement Schubert variety Y is invariant under the group action, the restriction of the double $\mathbb{C}^*$-action on $(\mathbb{CP}^1)^n$ to Y is a $\mathbb{C}^*$-action on Y. Let Y' be the inverse image $\overline{\varphi}^{-1}(Y)$. The restriction of $\overline{\varphi}$ to Y', denoted by $\pi' = \overline{\varphi}|_{Y'} : Y' \to Y$, is a generic 2^n-to-1 cover map (except for the points with zero or infinity coordinates), which is also equivariant.

Let $\widetilde{\pi} : \widetilde{Y} \to Y'$ be an equivariant resolution of singularities of Y', with respect to the natural $\mathbb{C}^*$-action. Such a resolution exists by Hironaka's equivariant resolution of singularities theorem, announced by Hironaka in 1976 [Hir77, 9, Remark 8]. Denote by $\widetilde{S}_E^{\emptyset}$ the inverse image $\overline{\varphi}^{-1}(S_E^{\emptyset})$ and by π'' the restriction of $\overline{\varphi}$ to $\widetilde{S}_E^{\emptyset}$. Let $\varphi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ be the component-wise squaring $z_i \mapsto z_i^2$ for $1 \leq i \leq n$, which is also the restriction of $\overline{\varphi}$ to $(\mathbb{C}^*)^n$. We also write π for the composition of π' with $\tilde{\pi}$.

A commutative diagram is shown as follows

$$\begin{tikzcd}
 & & & \widetilde{Y} \arrow[d, "\widetilde{\pi}"] \arrow[dd, bend left=60, "\pi"] \\
(\mathbb{CP}^1)^n \arrow[d, "\overline{\varphi}"] & (\mathbb{C}^*)^n \arrow[l, hook'] \arrow[d, "\varphi"] & \widetilde{S}_E^{\emptyset} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\pi''"] & Y' \arrow[d, "\pi'"] \\
(\mathbb{CP}^1)^n & (\mathbb{C}^*)^n \arrow[l, hook'] & S_E^{\emptyset} \arrow[l, hook'] \arrow[r, hook] & Y
\end{tikzcd}$$

where the horizontal arrows are inclusions.

Now we are going to study the sheaf complex $\pi_* \underline{\mathbb{C}}_{\widetilde{Y}}$, in light of the BBDG decomposition theorem (see Theorem 2.7). Note that $\pi_* \underline{\mathbb{C}}_{\widetilde{Y}} = \pi'_* \widetilde{\pi}_* \underline{\mathbb{C}}_{\widetilde{Y}}$, since $\widetilde{\pi}$ is a resolution, by the decomposition theorem, one of the decomposition summands of $\widetilde{\pi}_* \underline{\mathbb{C}}_{\widetilde{Y}}$ is $IC^\bullet(Y'; \mathbb{C})$. On the nonsingular part $\widetilde{S}_E^\emptyset$ of Y', $IC^\bullet(Y'; \mathbb{C})$ is restricted to the constant local system $\underline{\mathbb{C}}_{\widetilde{S}_E^\emptyset}$.

Furthermore $\pi''_* \underline{\mathbb{C}}_{\widetilde{S}_E^\emptyset}$ is a semisimple local system of rank 2^n on $S_E^\emptyset$. This can be seen from the fact that the push-forward $\varphi_{i*} \underline{\mathbb{C}}_{\mathbb{C}^*}$ by $\varphi_i : z_i \mapsto z_i^2$ of the constant local system $\underline{\mathbb{C}}_{\mathbb{C}^*}$, is a rank two local system whose monodromy transformation is permuting the basis of a stalk. In other words, with some basis of a stalk $\mathbb{C}^2$, the monodromy transformation matrix can written as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with two eigenvalues $+1$ and -1. Hence $\varphi_{i*} \underline{\mathbb{C}}_{\mathbb{C}^*}$ can be decomposed as $\mathcal{L}_{+1} \oplus \mathcal{L}_{-1}$ on $\mathbb{C}^*$, the rank one local system with trivial monodromy and the rank one local system with -1 monodromy. With this fact in mind, we can find that

$$\pi''_* \underline{\mathbb{C}}_{\widetilde{S}_E^\emptyset} \cong \bigoplus_{\beta \in (\pm 1)^n} \mathcal{L}_\beta$$

where β ranges among the n-tuples whose coordinates are ± 1, and $\mathcal{L}_\beta$ is the rank one local system on $S_E^\emptyset$ with monodromy determined by β. Therefore $\mathcal{L}$ is one of these $\mathcal{L}_\beta$'s. By the BBDG decomposition theorem, $\pi'_* IC^\bullet(Y'; \mathbb{C})$ will have a direct summand $IC^\bullet(Y; \mathcal{L})$.

Summing up, $\pi_* \underline{\mathbb{C}}_{\widetilde{Y}} = \pi'_* \widetilde{\pi}_* \underline{\mathbb{C}}_{\widetilde{Y}}$ has one direct summand $IC^\bullet(Y; \mathcal{L})$ in its sheaf decomposition.

Restricting the resolution $\widetilde{\pi}$ to $\widetilde{Y}(k) = \widetilde{\pi}^{-1} \circ \overline{\varphi}^{-1}\big(Y(k)\big)$, we can find that $IC^\bullet\big(Y(k); \mathcal{L}\big)$ is a direct summand of $\pi_* \underline{\mathbb{C}}_{\widetilde{Y}(k)}$. Hence with a chosen decomposition we can find compatible projections from $\pi_* \underline{\mathbb{C}}_{\widetilde{Y}(k)}$ onto $IC^\bullet\big(Y(k); \mathcal{L}\big)$ for all k.

Let $j : Y(k-1) \to Y(k)$ denote the open inclusion. Applying the the adjunction morphism $\mathrm{id} \to j_* j^*$ to the projection $\pi_* \underline{\mathbb{C}}_{\widetilde{Y}(k)} \twoheadrightarrow IC^\bullet\big(Y(k); \mathcal{L}\big)$, and then applying the hypercohomology, we obtain the the commutative diagram

$$\begin{tikzcd} H^i\big(\widetilde{Y}(k);\mathbb{C}\big) \arrow[r] \arrow[d, two heads] & H^i\big(\widetilde{Y}(k-1);\mathbb{C}\big) \arrow[d, two heads] \\ IH^i\big(Y(k);\mathcal{L}\big) \arrow[r] & IH^i\big(Y(k-1);\mathcal{L}\big) \end{tikzcd}$$

where the vertical projections are surjective. In order to show that the bottom restriction $IH^i\big(Y(k);\mathcal{L}\big) \to IH^i\big(Y(k-1);\mathcal{L}\big)$ is surjective it suffices to show that the top arrow is surjecitve. It was proved in [Kir88, Lemma 1.12, Lemma 2.8] that the surjectiveness is guaranteed under an equivariant resolution. $\diamond$

Remark 5.2. *With a slightly modified proof, Theorem 5.1 can be easily generalized to a rank one local system with monodromies of multiplication by the p-th roots of unity, for some p. For example, the equivariant map on $(\mathbb{CP}^1)^n$ will be changed to be $z_i \mapsto z_i^p$, the generic cover map will be p^n-to-1, and the monodromy transformation matrix will be the matrix for the cyclic permutation $(123\cdots p)$.*

Note that $Y = Y(\operatorname{rk} M) = \coprod_{\operatorname{crk}(F)\leq \operatorname{rk} M} S_F = \coprod_k Y(k,k-1) = \coprod_{F\in L(M)} S_F$. Since Y is compact, $IH^i_c(Y;\mathcal{L}) = IH^i(Y;\mathcal{L})$. Since $S_\emptyset = (\infty,\infty,\dots,\infty)$ a proper subset in Y, one has that $\mathbb{H}^i_c\big(S_\emptyset; IC^\bullet(Y;\mathcal{L})|_{S_\emptyset}\big) \cong \mathbb{H}^i\big(IC^\bullet(Y;\mathcal{L})_\infty\big)$, where the right-hand side is the same as the cohomology of the stalk of intersection cohomology complex at the most singular point $(\infty,\infty,\dots,\infty)$.

An immediate consequence follows from Theorem 5.1 and Theorem 4.6:

Theorem 5.3. *If Y is the arrangement Schubert variety of a vector space V equipped with a rank one local system $\mathcal{L}$ on the hyperplane arrangement complement with monodromies $+1$ and -1, then there exists an isomorphism*

$$IH^\bullet(Y;\mathcal{L}) \cong \bigoplus_{F\in\mathbb{L}(M(V))} \mathbb{H}^\bullet_c\big(S_F; IC^\bullet(Y;\mathcal{L})|_{S_F}\big). \tag{5.1}$$

In terms of Poincaré polynomials, this isomorphism and Theorem 4.6 imply the identity

$$\begin{aligned} Z_{V,\mathcal{L}}(t) &= \sum_{F\in\mathbb{L}(M(V))} A_{V^F,\mathcal{L}^F}(t)\cdot P_{V_F,\mathcal{L}_F}(t) \\ &= A_{V,\mathcal{L}}(t) + \sum_{E>F>\emptyset} A_{V^F,\mathcal{L}^F}(t)\cdot P_{V_F,\mathcal{L}_F}(t) + P_{V,\mathcal{L}}(t). \end{aligned} \tag{5.2}$$

Remark 5.4. *Equation (5.2) still holds for local systems $\mathcal{L}$ with p-roots of unity for some integer p.*

Recall from §3.4 that if the hyperplane arrangement in V is generic central, then $A_{V,\mathcal{L}}(t)$ can be computed as a combinatorial invariant. We also have two facts:

(A) Since $\mathcal{L} \cong \mathcal{L}^\vee$ and $Y(V)$ is compact, $IH^\bullet(Y(V);\mathcal{L})$ satisfies the Poincaré duality, so $Z_{V,\mathcal{L}}(t)$ is palindromic.

(B) The degree of $P_{V,\mathcal{L}}(t)$ is strictly less than $\frac{1}{2}\dim V$ because of the degree restriction of intersection cohomology.

With these two facts and the formula for $A_{V,\mathcal{L}}(t)$ (see Proposition 3.6) we can recursively compute $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ by Equation (5.2). As a consequence we have the following

Theorem 5.5. *$P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ are combinatorial invariants when the hyperplane arrangement in V is generic central.*

Naturally we want to ask if the following conjecture is correct, without the assumption that the local system $\mathcal{L}$ only has ± 1 monodromies and the hyperplane arrangement in V is generic central.

Conjecture 5.6. *The polynomials $A_{V,\mathcal{L}}(t)$, $P_{V,\mathcal{L}}(t)$ and $Z_{V,\mathcal{L}}(t)$ are combinatorial invariants.*

CHAPTER 6

CLOSED FORMULAS FOR A UNIFORM CASE

Throughout this section we assume V is a vector space of dimension $n-1$ and equipped with a generic central arrangement of n hyperplanes ($n \geq 3$), or equivalently $M(V) = U_{n-1,n}$, the **uniform matroid** of rank $n-1$ over the ground set $E = [n] = \{1, 2, \ldots, n\}$. Recall from §3.4 that $\mathcal{L}_{l,n-l}$ denote the rank one system with -1 monodromies around the hyperplanes $\{H_1, H_2, \ldots, H_l\}$ and $+1$ monodromies around the remaining $n-l$ hyperplanes. Under this assumption, $Z_{V,\mathcal{L}_{l,n-l}}(t)$, $A_{V,\mathcal{L}_{l,n-l}}(t)$ and $P_{V,\mathcal{L}_{l,n-l}}(t)$ can be written down in closed form. We present these results in this section.

For ease of notation, we simply write $A_{l,k}(t)$ for $A_{V,\mathcal{L}_{l,k}}(t)$, or even $A_{l,k}$. Similarly we write $Z_{l,k}$ and $P_{l,k}$.

Remark 6.1. *The reason we require $n \geq 3$ is that if V is of dimension one, then there cannot exist a central hyperplane arrangement consisting of two hyperplanes, and V can be endowed with only one central arrangement, which is the origin. So the exceptions are $\mathbb{C}$ with ± 1 monodromy around the origin, which are Example 4 and Example 5.*

Lemma 6.2. *Assume $l+k \geq 3$. Then $P_{l+2,k}(t) = P_{l,k}(t) \cdot t$, and $Z_{l+2,k}(t) = Z_{l,k}(t) \cdot t$. If l is odd, then $P_{l,k}(t) = Z_{l,k}(t) = 0$.*

Proof. First we notice that in many cases the term $A_{V^F,\mathcal{L}^F}$ in Equation (5.2) is zero. In fact, if F is a middle flat, i.e. $E > F > \emptyset$, such that F contains a number $i \in [n]$ which is assigned with $a_i = -1$ monodromy, then by Proposition 4.5, on V^F the local system $\mathcal{L}^F$ has a -1 monodromy. Using a Künneth type argument as in Equation (3.1) and Equation (3.2), we know that the twisted intersection cohomology $IH_c^\bullet(V^F;\mathcal{L}^F) = 0$, consequently the polynomial $A_{V^F,\mathcal{L}^F}(t)$ is zero. This observation greatly helps us to reduce the amount of computation. Under the simplified notation, we have

$$Z_{l,k} = A_{l,k} + \sum_{j=0}^{k} \binom{k}{j} t^j \cdot P_{l,k-j}.$$

Here t^j appears as the A-polynomial for a vector space of dimension j, stratified by j hyperplanes, equipped with the local system with trivial monodromies.

We recall the results in Example 6. If $l \geq 3$ is odd and $k = 0$, then all the polynomials $A_{l,k}$ are zero. In this case, from Equation (5.2) of Corollary 5.3, as $Z_{l,0} = A_{l,0} + P_{l,0}$ is palindromic, $P_{l,0}$ is forced to be zero as well. Therefore the statement of lemma holds for any odd $l \geq 3$ and $k = 0$.

Next we assume $l \in 2\mathbb{Z}$ and $l \geq 2$. We are going to prove the statement by induction on k. First we show that the equations hold for $k = 0$. Using Equation (5.2) of Corollary 5.3 and Example 6 we have

$$Z_{l+2,0} = A_{l+2,0} + P_{l+2,0} = t^{\frac{l+2}{2}} + P_{l+2,0}$$

and

$$Z_{l,0} = A_{l,0} + P_{l,0} = t^{\frac{l}{2}} + P_{l,0}.$$

where many $A_{V^F,\mathcal{L}^F}$ terms in the middle are zero, except the first term and the last term, as we have analyzed at the beginning. Since $Z_{l,k}$ is a palindromic polynomial and the degree of $P_{l,k}$ is not greater than $\frac{l-2}{2}$, the only possibility is that $P_{l+2,0} = t^{\frac{l}{2}}$ and $P_{l,0} = t^{\frac{l-2}{2}}$. Therefore the statement holds when $l \in 2\mathbb{Z}$ and $k = 0$.

Now we prove the inductive step. Assume that the statement holds for integers from 0 to k.

If $l \geq 2$, then we have

$$Z_{l+2,k+1} = \sum_{0 \leq i \leq k} \binom{k+1}{k+1-i} t^{k+1-i} \cdot P_{l+2,i} + \binom{k+1}{0} t^0 \cdot P_{l+2,k+1}$$

and

$$Z_{l,k+1} = \sum_{0 \leq i \leq k} \binom{k+1}{k+1-i} t^{k+1-i} \cdot P_{l,i} + \binom{k+1}{0} t^0 \cdot P_{l,k+1}$$

where the binomial terms $\binom{k+1}{k+1-i} t^{k+1-i}$ are the A-polynomials of vector space of flats, by Equation (5.2) of Corollary 5.3.

When $l + k \leq 2$, $P_{l,k}$ is not defined. But we abuse notation slightly by setting $P_{0,1} = P_{0,2} = P_{2,0} = 1$, and $P_{1,0} = 0$ by convention. For the reason we do so, see Remark 6.1.

Matching up terms $P_{l+2,i}$ and $P_{l,i}$ from the above two summations, using the inductive assumption that $P_{l+2,i} = P_{l,i} \cdot t$ for $0 \leq i \leq k$ and the fact that $Z_{l,k}$ is palindromic, we see that $P_{l+2,k+1} = P_{l,k+1} \cdot t$, and consequently $Z_{l+2,k+1} = Z_{l,k+1} \cdot t$.

If $l = 0$, again using Equation (5.2) of Corollary 5.3, we have

$$Z_{2,k+1} = \binom{k+1}{k+1} t^{k+1} \cdot P_{2,0} + \binom{k+1}{k} t^k \cdot P_{2,1} + \sum_{i=2}^{k} \binom{k+1}{k+1-i} t^{k+1-i} \cdot P_{2,i} + \binom{k+1}{0} t^0 \cdot P_{2,k+1}$$

and

$$Z_{0,k+1} = t^k + \sum_{i=2}^{k} \binom{k+1}{k+1-i} t^{k+1-i} \cdot P_{0,i} + \binom{k+1}{0} t^0 \cdot P_{0,k+1}.$$

We claim that $P_{2,1} = 0$. One may find the concrete calculation in §7.3 later. We still can match up terms as follows: $\binom{k+1}{k+1} t^{k+1} \cdot P_{2,0}$ is equal t times the first term t^k in the second summation, and $P_{2,i} = P_{0,i} \cdot t$ for $2 \leq i \leq k$ by induction hypothesis.

Using the same reasoning as in the case of $l \geq 2$, we find that $P_{2,k+1}$ is forced to be $P_{0,k+1} \cdot t$, and then $Z_{2,k+1} = Z_{0,k} \cdot t$ follows. So the lemma is proved. $\diamond$

Remark 6.3. *The condition $l + k \geq 3$ is necessary in Lemma 6.2, as we have seen a counterexample that $P_{0,1} = 1$, but $P_{2,1} = 0 \neq t$. However, the condition is not quite strict since the identities of Lemma 6.2 hold for all other numbers l and k.*

With the help of Lemma 6.2, we can find all the polynomials $P_{l,k}$ and $Z_{l,k}$, once the extreme cases $P_{0,k}$ and $Z_{0,k}$ are known. From Equation (5.2), we find that the computation for $Z_{l,k}$ can be traced down to $P_{l,k}$ and A-polynomials. And we already know how to compute A-polynomials from Proposition 3.2 and Proposition 3.6.

Proudfoot, Wakefield and Young [PWY16, Theorem 1.2] show that the intersection cohomology of the **symmetric reciprocal plane** X, the closure of $\left\{x \in (\mathbb{C}^*)^n \mid \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 0\right\}$ in $\mathbb{C}^n$, vanishes in odd degree, and is nonzero only for $i < \frac{1}{2}(n-1)$. Its dimensions in even degrees are given by the formula

$$\dim IH^{2i}(X; \mathbb{C}) = \frac{1}{i+1}\binom{n-i-2}{i}\binom{n}{i}. \tag{6.1}$$

Suppose Y is the arrangement Schubert variety of the uniform matroid $U_{n-1,n}$. With the reciprocal of coordinates, $(0, 0, \ldots, 0)$ of X corresponds to the most singular point $(\infty, \infty, \ldots, \infty)$ of Y, so we note that $X = Y \setminus \bigcup_{1 \leq i \leq n} \overline{H_i}$ is an affine open neighborhood of the most singular point. Let $\mathcal{L} = \mathcal{L}_{0,n}$ be the constant sheaf $\underline{\mathbb{C}}_U$ on the hyperplane arrangement complement $U = V \setminus \bigcup_{1 \leq i \leq n} H_i$, i.e. every monodromy of $\mathcal{L}$ is given by $+1$. Note that the symmetric reciprocal plane X is a conic affine neighborhood of the attracting set $S_\emptyset = \{(\infty, \infty, \ldots, \infty)\}$ with respect to the natural $\mathbb{C}^*$-action. Using Lemma 2.8 we have

$$\begin{aligned} IH^\bullet(X; \mathbb{C}) &\cong \mathbb{H}^\bullet\big(IC^\bullet(X; \mathcal{L})_0\big) \\ &\cong \mathbb{H}^\bullet\big(IC^\bullet(Y; \mathcal{L})_\infty\big) \end{aligned}$$

where $IC^\bullet(X;\mathcal{L})_0$ is the stalk of $IC^\bullet(X;\mathcal{L})$ at the origin of X.

Recall that $P_{0,n}(t) = \mathrm{Poin}\big(H^\bullet\big(IC^\bullet(Y;\mathcal{L}_{0,n})_\infty\big), t^{\frac{1}{2}}\big)$. Applying Equation (6.1), we obtain

$$P_{0,n}(t) = \sum_{0\le i<\frac{1}{2}(n-1)} \frac{1}{i+1}\binom{n-i-2}{i}\binom{n}{i}t^i. \tag{6.2}$$

Finally we are ready to give closed formulas for $P_{l,k}(t)$ and $Z_{l,k}(t)$.

Theorem 6.4. *Closed formulas for $P_{l,k}(t)$ and $Z_{l,k}(t)$ are given as follows.*

1. Suppose $l+k\le 2$. Then $P_{0,1}(t) = P_{0,2}(t) = P_{2,0}(t) = 1$, $Z_{0,1}(t) = Z_{0,2}(t) = t+1$, and $P_{1,0}(t) = Z_{1,0}(t) = 0$.

2. Suppose $l+k\ge 3$. Except $P_{2,1}(t) = 0$ and $Z_{2,1} = t$, there are three cases:

(i) When l is odd, $P_{l,k}(t)$ and $Z_{l,k}(t)$ are both zero.

(ii) When l is even and $k\neq 0$, we have

$$\begin{aligned} P_{l,k}(t) &= P_{0,k}(t)\cdot t^{\frac{l}{2}} \\ &= \begin{cases} t^{\frac{l}{2}}, & \text{if } k=1,\\ \displaystyle\sum_{0\le i<\frac{1}{2}(k-1)} \frac{1}{i+1}\binom{k-i-2}{i}\binom{k}{i}t^{i+\frac{l}{2}}, & \text{if } k\ge 2. \end{cases} \end{aligned}$$

and

$$\begin{aligned} Z_{l,k}(t) &= Z_{0,k}(t)\cdot t^{\frac{l}{2}} \\ &= \begin{cases} (t+1)\cdot t^{\frac{l}{2}}, & \text{if } k=1,\\ \left(A_{0,k}(t) + \displaystyle\sum_{i=2}^{k}\binom{k}{k-i}t^{k-i}\cdot P_{0,i}(t)\right)\cdot t^{\frac{l}{2}}, & \text{if } k\ge 2, \end{cases} \\ &= \begin{cases} (t+1)\cdot t^{\frac{l}{2}}, & \text{if } k=1,\\ \left(t^{k-1} + \displaystyle\sum_{i=2}^{k}\binom{k}{k-i}t^{k-i}\cdot \sum_{0\le j<\frac{1}{2}(i-1)}\frac{1}{j+1}\binom{i-j-2}{j}\binom{i}{j}t^j\right)\cdot t^{\frac{l}{2}}, & \text{if } k\ge 2. \end{cases} \end{aligned}$$

(iii) When l is even, $l \geq 2$ and $k = 0$, we have

$$P_{l,0}(t) = t^{\frac{l}{2}-1},$$

and hence

$$Z_{l,0}(t) = t^{\frac{l}{2}} + t^{\frac{l}{2}-1}.$$

Proof. (1) These are the degenerate cases. See Examples 4 and 5.

(2) (i) and (ii) come from Lemma 6.2 and Equation (6.2).

(iii) is a consequence of Example 6 and the palindromy of the Z-polynomial. $\diamond$

Remark 6.5. *Proudfoot, Xu and Young [PXY18, Proposition 4.9] showed that the coefficient of t^i in $Z_{0,k}(t)$ is the Narayana number*

$$N(k, i+1) = \frac{1}{k}\binom{k}{i+1}\binom{k-1}{i+1}.$$

CHAPTER 7

CALCULATION EXAMPLES

7.1 Stratification and Lattice of flats

We present the coarse stratification and the corresponding lattice of flats of some arrangement Schubert varieties here. We let B_n denote the Boolean matroid of rank n, and $U_{n,n+k}$ denote the uniform matroid of rank n on $n+k$ elements.

Example 7. *Take $V = \{x_1 + x_2 + x_3 = 0\}$ in $\mathbb{C}^4$. In the natural stratification of $Y(V)$, $S_{1234} = V$ is the nonsingular part of $Y(V)$, the biggest stratum of dimension 3. There are four codimension 1 strata S_{123}, S_{14}, S_{24} and S_{34} (four flats of corank 1 respectively), and four codimension 2 strata S_1, S_2, S_3, S_4 (four flats of corank 2 accordingly). Finally $S_\emptyset$ corresponds to the most singular point $(\infty, \infty, \infty, \infty)$. The lattice of flats $L(M(V))$ is isomorphic to $L(U_{2,3}) \times L(B_1)$, presented in Figure 2. For the lattices of flats of restrictions, for instance $L(M(V)_1)$, $L(M(V)_2)$ and $L(M(V)_3)$ are all isomorphic to $L(U_{1,2}) \times L(B_1)$, and but $L(M(V)_4)$ is isomorphic to $L(U_{2,3}) \times L(B_0) \cong L(U_{2,3})$.*

Example 8. *Take $V = \{x_1 + x_2 + x_3 + x_4 = 0\}$ in $\mathbb{C}^4$. The associated matroid $M(V) = U_{3,4}$. The associated arrangement Schubert variety $Y(V)$ is the arrangement Schubert variety of $U_{3,4}$. In the natural stratification $S_{1234} = V$ is the nonsingular part of $Y(V)$, the biggest stratum of dimension 3. There are six codimension*

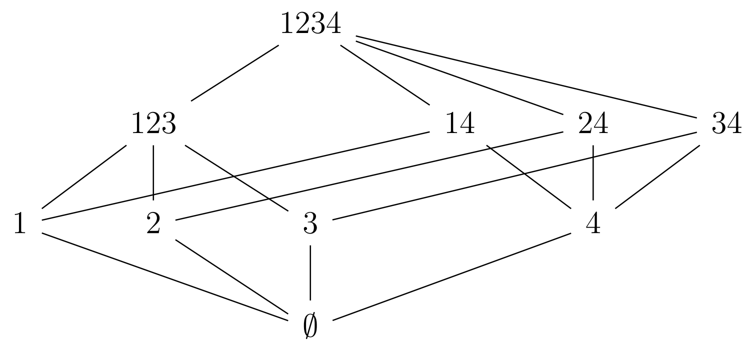


Figure 2: The lattice of flats of $M(V)$

1 *strata* $S_{12}, S_{13}, \ldots, S_{34}$ *(six flats of corank* 1 *accordingly), and four codimension* 2 *strata* S_1, S_2, S_3, S_4 *(four flats of corank* 2 *accordingly). Finally* $S_\emptyset$ *corresponds to the most singular point* $(\infty, \infty, \infty, \infty)$. *The lattice of flats of* $U_{3,4}$ *is shown in Figure 3.*

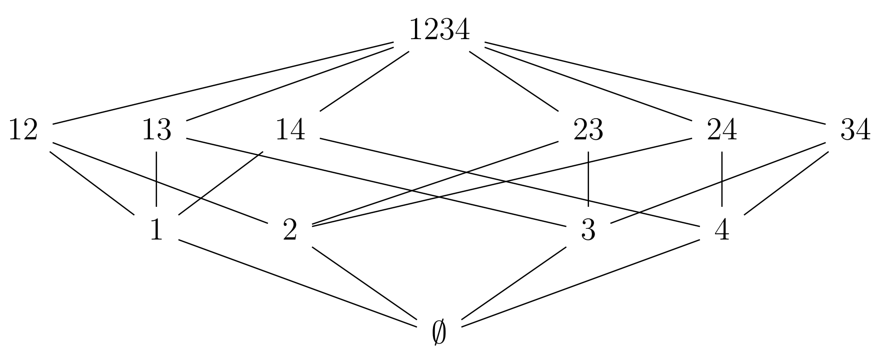


Figure 3: The lattice of flats of $U_{3,4}$

7.2 Computation for the rank one case

In this and next subsections, we present some concrete examples of low dimensions, the arrangement Schubert varieties of $U_{n-1,n}$ with $n = 2, 3$. These examples will illustrate how the recursive method works for general computations.

Consider the codimension 1 vector space $V = \{x_1 + x_2 = 0\}$ in $\mathbb{C}^2$ and the hyperplane arrangement $\mathcal{A} = \{H_i : x_i = 0,\ i = 1, 2\}$. Notice that $H_1 = H_2 = (0, 0)$. $Y = Y(V) \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is isomorphic to $\mathbb{CP}^1$. Let $\mathcal{L}_{+1}$ (respectively $\mathcal{L}_{-1}$) be the rank 1 local system on the hyperplane arrangement complement $U \cong \mathbb{C}^*$ with monodromy of $+1$ (respectively -1) around the origin. The lattice of flats of $U_{1,2}$ as in Figure 4 below, represents the stratification $Y = S_{12} \cup S_{\emptyset}$ where $S_{12} = V$ and $S_{\emptyset} = (\infty, \infty)$.

$$\begin{array}{c} 12 \\ | \\ \emptyset \end{array}$$

Figure 4: The lattice of flats of $U_{1,2}$

According to Example 4 and Example 5, $IH^i_c(V; \mathcal{L}_{+1}) = \mathbb{C}$, if $i = 2$, and vanishes otherwise; $IH^i_c(V; \mathcal{L}_{-1})$ is trivial for all i. Thus the stalk intersection cohomology at the infinity point is

$$H^i(IC^\bullet(Y; \mathcal{L}_{+1})_\infty) = \begin{cases} \mathbb{C}, & \text{if } i = 0, \\ 0, & \text{otherwise} . \end{cases}$$

Therefore $Z_{V,\mathcal{L}_{+1}}(t) = t+1$. Equivalently, the intersection cohomology $IH^\bullet(Y; \mathcal{L}_{+1})$ is as follows:

$$IH^i(Y; \mathcal{L}_{+1}) = \begin{cases} \mathbb{C}, & \text{if } i = 0 \text{ or } 2, \\ 0, & \text{otherwise.} \end{cases}$$

Using a similar argument, we conclude that $Z_{V,\mathcal{L}_{-1}}(t) = 0$, in other words $IH^\bullet(Y; \mathcal{L}_{-1})$ is zero. Notice that the above intersection cohomology groups all vanish in odd degree.

We collect the above data and make a table below.

Local systems	A-polynomials of S_{12}	P-polynomials of $S_\emptyset$	Z-polynomials of Y
$\mathcal{L}_{+1}$	t	1	$t+1$
$\mathcal{L}_{-1}$	0	0	0

Table 1: Polynomials for the rank one case

7.3 A rank 2 example

Since $n = 3$, things become more interesting. The arrangement Schubert variety of $U_{2,3}$ will be the first singular space we study.

Consider the codimension 1 space $V = \{x_1 + x_2 + x_3 = 0\}$ in $\mathbb{C}^3$ and the hyperplane arrangement $\mathcal{A} = \{H_i : x_i = 0,\ 1 \le i \le 3\}$. The lattice of flats of $U_{2,3}$ is showed in Figure 5 below. The associated arrangement Schubert variety $Y = Y(V)$ is stratified as $Y = \coprod_{F \in L(U_{2,3})} S_F$. Let $\mathcal{L}$ be a rank 1 local system on the

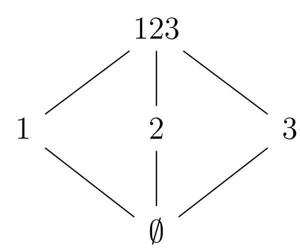


Figure 5: The lattice of flats of $U_{2,3}$

hyperplane arrangement complement with monodromies of either $+1$ or -1 around H_i's. There are essentially four cases to be considered in total: $\mathcal{L}_{0,3}, \mathcal{L}_{1,2}, \mathcal{L}_{2,1}$ and $\mathcal{L}_{3,0}$.

(a) $\mathcal{L} = \mathcal{L}_{0,3}$. From Example 6, we conclude that

$$IH_c^i(V; \mathcal{L}) = \begin{cases} \mathbb{C}, & \text{if } i = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the polynomial $V = S_{123}$ contributes is t^2. Using our polynomial notation in Chapter 6, that means $A_{0,3} = t^2$.

From the discussion in §4.4 and §4.5, we know that at each point of codimension one strata S_i, locally S_i is a product of the origin of $\mathbb{C}$ in tangent direction and the infinity point of the arrangement Schubert variety of $U_{1,2}$, and the local systems are trivial in both directions, so each S_i contributes $t \cdot 1$ from §7.2. Recalling the notation used in Chapter 6, we have

$$Z_{0,3} = t^2 + 3t \cdot 1 + P_{0,3}.$$

Because $Z_{0,3}$ is palindromic, $P_{0,3}$ has to be 1 under its degree restriction.

(b) $\mathcal{L} = \mathcal{L}_{1,2}$. Now the monodromy for H_1 is -1 but the monodromies around H_1 and H_2 are 1. By Proposition 3.6, we know that the $IH_c^\bullet(V;\mathcal{L})$ is trivial. In other words, $A_{1,2} = 0$.

The local topological product structure and the product structure of local systems on S_1, implies that S_1 contributes the polynomial 0, since the tangent data is space $\mathbb{C}$ plus local system $\mathcal{L}_{-1}$. For other two strata S_2 and S_3, they contribute the polynomial 0 too, because their normal data are both the infinity point of the arrangement Schubert variety of $U_{1,2}$, with local system $\mathcal{L}_{-1}$.

Then we conclude that $P_{1,2} = 0$, and hence $Z_{1,2} = 0$.

(c) $\mathcal{L} = \mathcal{L}_{2,1}$. Now the monodromies for H_1 and H_2 are both -1 but the monodromy around H_3 is 1. By Proposition 3.6, we know that the $IH_c^\bullet(V;\mathcal{L})$ is trivial. In other words, $A_{2,1} = 0$.

The strata S_2 and S_3 contribute the polynomial 0, as their tangent data are both space $\mathbb{C}$ plus local system $\mathcal{L}_{-1}$. But the stratum S_1 is locally a product of the origin of $\mathbb{C}$ in tangent direction and the infinity point of the arrangement Schubert variety of $U_{1,2}$, and the local systems are $\mathcal{L}_{+1}$ in both directions. Hence S_1 contributes the polynomial $t \cdot 1$.

Consequently $P_{2,1} = 0$, and then $Z_{2,1} = t$.

(d) $\mathcal{L} = \mathcal{L}_{3,0}$. By Proposition 3.6, we know that the $IH_c^\bullet(V;\mathcal{L})$ is trivial. In other words, $A_{3,0} = 0$.

Each stratum S_i contributes the polynomial 0 this time as their tangent data are all space $\mathbb{C}$ plus local system $\mathcal{L}_{-1}$.

Therefore $P_{3,0} = 0$, and $Z_{3,0} = 0$.

As a summary, we put the above data into Table 2 below for a future reference.

Local systems	A-polynomials of S_{123}	Poincaré polynomials of S_i	P-polynomials of $S_\emptyset$	Z-polynomials of Y
$\mathcal{L}_{0,3}$	t^2	$3 \cdot t$	1	$t^2 + 3t + 1$
$\mathcal{L}_{1,2}$	0	0	0	0
$\mathcal{L}_{2,1}$	0	t	0	t
$\mathcal{L}_{3,0}$	0	0	0	0

Table 2: Polynomials for $U_{2,3}$

7.4 Lines in a plane

As a comparison, we present an example of a family of line arrangements in $\mathbb{C}^2$ associated to the uniform matroid $U_{2,n}$, as $n \geq 3$ varies. This gives a simple example showing that for a hyperplane arrangement associated to a matroid other than $U_{d,d+1}$, the intersection cohomology may not vanish in odd degree.

Consider a central line arrangement in $\mathbb{C}^2$ consisting of n lines $L_1, L_2, \dots, L_n$. Suppose a rank one local system $\mathcal{L}$ is given by the data that around the first l lines, the monodromies are given by -1, and around the remaining $n - l$ lines, the monodromies are given by $+1$. Let Y be the closure of $\mathbb{C}^2 \hookrightarrow \prod_{1 \leq i \leq n} \mathbb{C}^2/L_i$ in $(\mathbb{CP}^1)^n$.

If l is odd, the intersection cohomology vanishes. If $l = 0$, the intersection cohomology coincides with the ordinary cohomology. If l is even and $l \geq 4$, then the intersection cohomology of the complex plane is given by

$$IH^i(Y;\mathcal{L}) = \begin{cases} 0, & \text{if } i = 4, \\ \mathbb{C}^\alpha, & \text{if } i = 3, \\ \mathbb{C}^{n-l}, & \text{if } i = 2, \\ \mathbb{C}^\alpha, & \text{if } i = 1, \\ 0, & \text{if } i = 0. \end{cases}$$

Here $\alpha = \sum_{j=1}^{l-2} (-1)^{j+1} \binom{l-1}{j+1}$, computed based on Proposition 3.6. So we see that the intersection cohomology does not vanish in degree 1 and degree 3.

When $n = 3$ and $l = 2$, the case is simply the arrangement associated to $U_{2,3}$, so there is nothing new but the example in §7.3.

APPENDIX

POLYNOMIALS FOR UNIFORM CASES

We include here the tables of P-polynomials and Z-polynomials for uniform matroids $U_{d,d+1}$ and $U_{d,d+2}$.

Table 3: P-polynomials for $U_{d,d+1}$ and $\mathcal{L}_{l,d+1-l}$

$(d,l) =$	$(1,0)$	$(2,0)$	$(2,2)$	$(3,0)$	$(3,2)$	$(3,4)$	$(4,0)$	$(4,2)$	$(4,4)$	$(5,0)$	$(5,2)$	$(5,4)$	$(5,6)$
1	1	1		1			1			1			
t				2	1	1	5	1		9	1		
t^2										5	2	1	1
t^3													

$(d,l) =$	$(6,0)$	$(6,2)$	$(6,4)$	$(6,6)$	$(7,0)$	$(7,2)$	$(7,4)$	$(7,6)$	$(7,8)$	$(8,0)$	$(8,2)$	$(8,4)$	$(8,6)$	$(8,8)$
1	1				1					1				
t	14	1			20	1				27	1			
t^2	21	5	1		56	9	1			120	14	1		
t^3					14	5	2	1	1	84	21	5	1	
t^4														

$(d,l) =$	$(9,0)$	$(9,2)$	$(9,4)$	$(9,6)$	$(9,8)$	$(9,10)$
1	1					
t	35	1				
t^2	225	20	1			
t^3	300	56	9	1		
t^4	42	14	5	2	1	1
t^5						

Table 4: Z-polynomials for $U_{d,d+1}$ and $\mathcal{L}_{l,d+1-l}$

$(d,l) =$	$(1,0)$	$(2,0)$	$(2,2)$	$(3,0)$	$(3,2)$	$(3,4)$	$(4,0)$	$(4,2)$	$(4,4)$	$(5,0)$	$(5,2)$	$(5,4)$	$(5,6)$
1	1	1		1			1			1			
t	1	3	1	6	1	1	10	1		15	1		
t^2		1		6	1	1	20	3	1	50	6	1	1
t^3				1			10	1		50	6	1	1
t^4							1			15	1		
t^5										1			

$(d,l) =$	$(6,0)$	$(6,2)$	$(6,4)$	$(6,6)$	$(7,0)$	$(7,2)$	$(7,4)$	$(7,6)$	$(7,8)$	$(8,0)$	$(8,2)$	$(8,4)$	$(8,6)$	$(8,8)$
1	1				1					1				
t	21	1			28	1				36	1			
t^2	105	10	1		196	15	1			336	21	1		
t^3	175	20	3	1	490	50	6	1	1	1176	105	10	1	
t^4	105	10	1		490	50	6	1	1	1764	175	20	3	1
t^5	21	1			196	15	1			1176	105	10	1	
t^6	1				28	1				336	21	1		
t^7					1					36	1			
t^8										1				

$(d,l) =$	$(9,0)$	$(9,2)$	$(9,4)$	$(9,6)$	$(9,8)$	$(9,10)$
1	1					
t	45	1				
t^2	540	28	1			
t^3	2520	196	15	1		
t^4	5292	490	50	6	1	1
t^5	5292	490	50	6	1	1
t^6	2520	196	15	1		
t^7	540	28	1			
t^8	45	1				
t^9	1					

Table 5: P-polynomials for $U_{d,d+2}$ and $\mathcal{L}_{l,d+2-l}$

$(d,l) =$	$(1,0)$	$(2,0)$	$(2,2)$	$(2,4)$	$(3,0)$	$(3,2)$	$(3,4)$	$(4,0)$	$(4,2)$	$(4,4)$	$(4,6)$
1	1	1			1			1			
$t^{\frac{1}{2}}$				2							
t					5	3	1	14	4		
$t^{\frac{3}{2}}$										2	4
t^2											

$(d,l)=$	$(5,0)$	$(5,2)$	$(5,4)$	$(5,6)$	$(6,0)$	$(6,2)$	$(6,4)$	$(6,6)$	$(6,8)$	$(7,0)$	$(7,2)$	$(7,4)$	$(7,6)$	$(7,8)$
1	1				1					1				
$t^{\frac{1}{2}}$														
t	28	5			48	6				75	7			
$t^{\frac{3}{2}}$			2				2					2		
t^2	21	10	3	1	98	30	4			288	63	5		
$t^{\frac{5}{2}}$							4	4	6			10	4	
t^3										84	35	10	3	1
$t^{\frac{7}{2}}$														

Table 6: Z-polynomials for $U_{d,d+2}$ and $\mathcal{L}_{l,d+2-l}$

$(d,l)=$	$(1,0)$	$(2,0)$	$(2,2)$	$(2,4)$	$(3,0)$	$(3,2)$	$(3,4)$	$(4,0)$	$(4,2)$	$(4,4)$	$(4,6)$
1	1	1			1			1			
$t^{\frac{1}{2}}$				2							
t	1	4	2		10	3	1	20	4		
$t^{\frac{3}{2}}$				2			2			2	4
t^2		1			10	3	1	45	12	2	
$t^{\frac{5}{2}}$										2	4
t^3					1			20	4		
$t^{\frac{7}{2}}$											
t^4								1			

$(d,l)=$	$(5,0)$	$(5,2)$	$(5,4)$	$(5,6)$	$(6,0)$	$(6,2)$	$(6,4)$	$(6,6)$	$(6,8)$	$(7,0)$	$(7,2)$	$(7,4)$	$(7,6)$	$(7,8)$
1	1				1					1				
$t^{\frac{1}{2}}$														
t	35	5			56	6				84	7			
$t^{\frac{3}{2}}$			2				2					2		
t^2	140	30	3	1	350	60	4			756	105	5		
$t^{\frac{5}{2}}$			6	4			12	4	6			20	4	
t^3	140	30	3	1	616	120	12	2		2058	350	30	3	1
$t^{\frac{7}{2}}$			2				12	4	6			40	12	6
t^4	35	5			350	60	4			2058	350	30	3	1
$t^{\frac{9}{2}}$							2					20	4	
t^5	1				56	6				756	105	5		
$t^{\frac{11}{2}}$												2		
t^6					1					84	7			
$t^{\frac{13}{2}}$														
t^7										1				

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