A strongly polynomial-time algorithm for weighted general factors with three feasible degrees

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Abstract

General factors are a generalization of matchings. Given a graph G with a set $\pi(v)$ of feasible degrees, called a degree constraint, for each vertex v of G, the general factor problem is to find a (spanning) subgraph F of G such that $\deg_F(v) \in \pi(v)$ for every v of G. When all degree constraints are symmetric Δ-matroids, the problem is solvable in polynomial time. The weighted general factor problem is to find a general factor of the maximum total weight in an edge-weighted graph. Strongly polynomial-time algorithms are only known for weighted general factor problems that are reducible to the weighted matching problem by gadget constructions.

In this paper, we present the first strongly polynomial-time algorithm for a type of weighted general factor problems with real-valued edge weights that is provably not reducible to the weighted matching problem by gadget constructions.

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1 Introduction

A matching in an undirected graph is a subset of the edges that have no vertices in common, and it is perfect if its edges cover all vertices of the graph. Graph matching is one of the most studied problems both in graph theory and combinatorial optimization, with beautiful structural results and efficient algorithms described, e.g., in the monograph of Lovász and Plummer [34] and in relevant chapters of standard textbooks [39, 30]. In particular, the *weighted (perfect) matching problem* is to find a (perfect) matching of the maximum total weight for a given graph of which each edge is assigned a weight. This problem can be solved in polynomial time by the celebrated Edmonds' blossom algorithm [12, 13]. Since then, a number of more efficient algorithms have been developed [17, 31, 28, 6, 19, 24, 21, 20, 23, 26]. Table III of [8] gives a detailed review of these algorithms.

The *f-factor problem* is a generalization of the perfect matching problem in which one is given a non-negative integer $f(v)$ for each vertex $v \in V$ of $G = (V, E)$. The task is to find a (spanning) subgraph $F = (V_F, E_F)$ of G such that $\deg_F(v) = f(v)$ for every $v \in V$.[1] The case $f(v) = 1$ for every $v \in V$ is the perfect matching problem. This problem, as well as the weighted version, can be solved efficiently by a gadget reduction to the perfect matching problem [14]. In addition, Tutte gave a characterization of graphs having an f-factor [42], which generalizes his characterization theorem for perfect matchings [41]. Subsequently, the

[1] In graph theory, a graph factor is usually a spanning subgraph. Here, without causing ambiguity, we allow F to be an arbitrary subgraph including the empty graph and we adapt the convention that $\deg_F(v) = 0$ if $v \in V \setminus V_F$.

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study of graph factors has attracted much attention with many variants of graph factors, e.g., b-matchings, $[a, b]$-factors, (g, f)-factors, parity (g, f)-factors, and anti-factors introduced, and various types of characterization theorems proved for the existence of such factors. We refer the reader to the book [1] and the survey [36] for a comprehensive treatment of the developments on the topic of graph factors.

In the early 1970s, Lovász introduced a generalization of the above factor problems [32, 33], for which we will need a few definitions. For any nonnegative integer n, let $[n]$ denote $\{0, 1, \ldots, n\}$. A degree constraint D of arity n is a subset of $[n]$.[2] We say that a degree constraint D has a gap of length k if there exists $p \in D$ such that $p+1, \ldots, p+k \notin D$ and $p+k+1 \in D$. An instance of the *general factor problem* (GFP) [32, 33] is given by a graph $G = (V, E)$ and a mapping π that maps every vertex $v \in V$ to a degree constraint $\pi(v) \subseteq [\deg_G(v)]$ of arity $\deg_G(v)$. The task is to find a subgraph, if one exists, F of G such that $\deg_F(v) \in \pi(v)$ for every $v \in V$. The case $\pi(v) = \{0, 1\}$ for every $v \in V$ is the matching problem, and the case $\pi(v) = \{1\}$ for every $v \in V$ is the perfect matching problem. Lovász showed that the GFP is NP-complete when the degree constraint $\{0, 3\}$ of arity 3 (and gap 2) occurs [33]. Later, answering a question of Lovász, Cornuéjols showed that the GFP is solvable in polynomial time if each degree constraint has gaps of length at most 1 [5].

In this paper, we consider the *weighted general factor problem* (WGFP) where each edge is assigned a real-valued weight and the task is to find a general factor of the maximum total weight. Since the unweighted version is already hard when a degree constraint with a gap of length more than 1 occurs [33], we only need to consider the WGFP where each degree constraint has gaps of length at most 1. Some cases of the WGFP are reducible to the weighted matching or perfect matching problem by gadget constructions, and hence are polynomial-time solvable. In these cases, the degree constraints are called *matching-realizable* (see Definition 18). For instance, the degree constraint $D = [b]$ where $b > 0$, for b-matchings is matching realizable [43]. The weighted b-matching problem is interesting in its own right in combinatorial optimization and has been well studied with many elaborate algorithms developed [37, 35, 18, 2, 22]. Besides b-matchings, Cornuéjols showed that the *parity interval* constraint $D = \{g, g+2, \ldots, f\}$ where $f \geq g \geq 0$ and $f \equiv g \mod 2$, is matching realizable [5], and Szabó showed that the *interval* constraint $D = \{g, g+1, \ldots, f\}$ where $f \geq g \geq 0$, for (g, f)-factors is matching realizable [40]. Thus, the WGFP where each degree constraint is an interval or a parity interval is reducible to the weighted matching problem (with some vertices required to have degree exactly 1) and hence solvable in polynomial-time by Edmonds' algorithm, although Szabó gave a different algorithm for this problem [40]. By reducing the WGFP with interval and parity interval constraints to the weighted (g, f)-factor problem, a faster algorithm was obtained in [9] based on Gabow's algorithm [18].

In [40], Szabó further conjectured that the WGFP is solvable in polynomial time without requiring each degree constraint being an interval or a parity interval, as long as each degree constraint has gaps of length at most 1. To prove the conjecture, a natural question is then the following: *Are there other WGFPs that are polynomial-time solvable by a gadget reduction to weighted matchings?* In other words, *are there other degree constraints that are matching realizable?* In this paper, we show that the answer is *no.*

▶ **Theorem 1.** *A degree constraint with gaps of length at most* 1 *is matching realizable if and only if it is an interval or a parity interval.*

[2] We always associate a degree constraint with an arity. Two degree constraints are different if they have different arities although they may be the same set of integers.

Previous results beyond matchings realizable degree constraints With the answer for the above question being negative, new algorithms need to be devised for the WGFP with degree constraints that are not intervals or parity intervals. Unlike the weighted matching problem and the weighted b-matching problem for which various types of algorithms have been developed, only a few algorithms have been presented for the more general and challenging WGFP: For the cardinality version of WGFP, i.e., the WGFP where each edge is assigned weight 1, Dudycz and Paluch introduced a polynomial-time algorithm for this problem with degree constrains having gaps of length at most 1, which leads to a pseudo-polynomial-time algorithm for the WGFP with non-negative integral edge weights [9].

The algorithm in [9] is based on a structural result showing that if a factor is not optimal, then a factor of larger weight can be found by a local search, which can be done in polynomial time. However, it is not clear how much larger the weight of the new factor is. In order to get an optimal factor, the algorithm needs to repeat local searches iteratively until no better factors can be found, and the number of local searches is bounded by the total edge weight, which makes the algorithm pseudo-polynomial-time. Later, in an updated version [10], by carefully assigning edge weights, the algorithm was improved to be weakly polynomial-time with a running time $O(\log Wmn^6)$, where W is the largest edge weight, m is the number of edges and n is the number of vertices.

Our main contribution Independently of [10], in this paper, we make a step towards a *strongly* polynomial-time algorithm for the WGPF. Let $p \geq 0$ be an arbitrary integer. Consider the following two types of degree constraints $\{p, p+1, p+3\}$ and $\{p, p+2, p+3\}$ (of arbitrary arity). We will call them *type-1* and *type-2* respectively. These are the "smallest" degree constraints that are not matching realizable.

▶ **Theorem 2.** *There is a strongly polynomial-time algorithm for the* WGFP *with real-valued edge weights where each degree constraint is an interval, a parity interval, a type-1, or a type-2 (of arbitrary arities). The algorithm runs in time $O(n^6)$ for a graph with n vertices.*

In particular, this gives a tractability result for the WGFP with degree constraints that are provably not matching realizable, thus going beyond existing algorithms. The algorithm is a recursive algorithm by reducing the problem to a smaller sub-problem of itself by fixing the parity of degree constraints on vertices. Its correctness is based on a delicate structural result, which is stronger than that of [10].[3] Equipped with this result, our algorithm can directly find an *optimal* factor (not just a better one) of an instance of a larger size by performing only one local search from an optimal factor of a smaller instance. Here, what is important is not how to find a better one by a local search (the main result of [10]), it is how to ensure that the better one obtained by only one local search is actually optimal under certain assumptions. This is the key to making our algorithm strongly polynomial. In addition, as a by-product, we give a simple proof of the result of [10] for the special case of WGFP with interval, parity interval, type-1 and type-2 degree constraints by reducing the problem to WGFP on subcubic graphs and utilizing the equivalence between 2-vertex connectivity and 2-edge connectivity of subcubic graphs.

Let D be a degree constraint of arity at most 3. If $D \neq \{0, 3\}$ then D is an interval, a parity interval, a type-1, or a type-2. Combining with the above-mentioned NP-hardness of the decision case [33], we obtain a complexity dichotomy for the WGFP on subcubic graphs.

[3] The result in [10] holds for the more general WGFP with all degree constraints having gaps of length at most 1, while our result only works for the WGFP with interval, parity interval, type-1 and type-2 degree constraints.

▶ **Corollary 3.** *The* WGFP *on subcubic graphs is strongly polynomial-time solvable if the degree constraint* $\{0, 3\}$ *of arity three does not occur. Otherwise, it is NP-hard.*

Relation with edge constraint satisfaction problems The edge constraint satisfaction problem (CSP) is a type of CSPs in which every variable appears in exactly two constraints [27, 15]. For the edge-CSP on the Boolean domain, Feder showed that the problem is NP-complete if a constraint that is not a Δ-matroid occurs, except for those that are tractable by Schaefer's dichotomy theorem [38]. In a subsequent line of work [7, 25, 16, 11], tractability of the Boolean edge-CSP has been established for special classes of Δ-matroids, most recently for even Δ-matroids [29]. A complete complexity classification for the Boolean edge-CSP is still open with the conjecture that all Δ-matroids are tractable. The graph factor problem is a special case of the Boolean edge-CSP where every constraint is symmetric (i.e, the value of the constraint only depends on the Hamming weight of its input). For a degree constraint (or a symmetric constraint), it is a Δ-matroid if and only if it has gaps of length at most 1. Thus, the above conjecture holds for the symmetric Boolean edge-CSP by Cornuéjols' result on the general factor problem [5]. A complexity classification for the weighted Boolean edge-CSP is certainly a more challenging goal: The complexity of the weighted Boolean edge-CSP with even Δ-matroids as constraints is still open. Our result in Theorem 2 gives a tractability result for the weighted Boolean edge-CSP with certain symmetric Δ-matroids as constraints, and our result in Corollary 3 establishes a complexity dichotomy for the weighted Boolean edge-CSP with symmetric constraints of arity no more than 3.

Organization In Section 2, we present basic definitions and notation. In Section 3, we describe our algorithm and give a structural result for the WGFP which ensures the correctness and the strongly polynomial-time running time of our algorithm. In Section 4, we introduce basic augmenting subgraphs as an analogy of augmenting paths for weighed matchings and give a proof of the structural result. The proof is based on a result regarding the existence of certain basic factors for subcubic graphs, for which we give a proof sketch in Section 5. Finally, we discuss matching realizability and its relation with Δ-matroids in Appendix A. All omitted proofs can be found in the full paper, which is attached after the references of this extended abstract.

2 Preliminaries

Let $\mathcal{D}$ be a (possibly infinite) set of degree constraints.

▶ **Definition 4.** *The* weighted general factor problem *parameterized by* $\mathcal{D}$*, denoted by* WGFP($\mathcal{D}$)*, is the following computational problem. An instance is a triple* $\Omega = (G, \pi, \omega)$*, where* $G = (V, E)$ *is a graph,* $\pi : V \to \mathcal{D}$ *assigns to every* $v \in V$ *a degree constraint* $D_v \in \mathcal{D}$ *of arity* $\deg_G(V)$*, and* $\omega : E \to \mathbb{R}$ *assigns to every* $e \in E$ *a real-valued weight* $w(e) \in \mathbb{R}$*. The task is to find, if one exists, a general factor* F *of* G *such that the total weight of edges in* F *is maximized.*

The general factor problem GFP($\mathcal{D}$) *is the decision version of* WGFP($\mathcal{D}$)*; i.e., deciding whether a general factor exists or not.*

Suppose that $\Omega = (G, \pi, \omega)$ is a WGFP instance. If F is a general factor of G under π, then we say that F is a factor of Ω, denoted by $F \in \Omega$. In terms of this inclusion relation, Ω can be viewed as a set of subgraphs of G. We extend the edge weight function ω to subgraphs of G. For a subgraph H of G, its weight $\omega(H)$ is $\sum_{e \in E(H)} \omega(e)$ ($\omega(H) = 0$ if H is the empty graph). If H contains an isolated vertex v, then $\omega(H) = \omega(H')$, where H' is the graph obtained from H by removing v. Moreover, $H \in \Omega$ if and only if $H' \in \Omega$. In the following, without other specification, we always assume that a factor does not contain any isolated vertices. The optimal value of Ω, denoted by $\mathrm{Opt}(\Omega)$, is $\max_{F\in\Omega} \omega(F)$. We define $\mathrm{Opt}(\Omega) = -\infty$ if Ω has no factor. A factor F of Ω is *optimal* in Ω if $\omega(F) = \mathrm{Opt}(\Omega)$.

For a WGFP instance $\Omega' = (G', \pi', \omega')$, where $G' \subseteq G$[4] and ω' is the restriction of ω on the edges of G', we say Ω' is a *sub-instance* of Ω, denoted by $\Omega' \subseteq \Omega$, if $F \in \Omega$ for every $F \in \Omega'$. In particular, Ω' is a subset of Ω by viewing them as two sets of subgraphs of G. If $\Omega' \subseteq \Omega$, then $\mathrm{Opt}(\Omega') \le \mathrm{Opt}(\Omega)$. For two WGFP instances $\Omega_1 = (G, \pi_1, \omega)$ and $\Omega_2 = (G, \pi_2, \omega)$, we use $\Omega_1 \cup \Omega_2$ to denote the union of factors of these two instances, i.e., $\Omega_1 \cup \Omega_2 = \{F \subseteq G \mid F \in \Omega_1 \text{ or } F \in \Omega_2\}$, and $\Omega_1 \cap \Omega_2$ to denote the intersection, i.e., $\Omega_1 \cap \Omega_2 = \{F \subseteq G \mid F \in \Omega_1 \text{ and } F \in \Omega_2\}$. Note that $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2$ are sets of subgraphs of G and may not define WGFP instances on G.

We use $\mathcal{G}_1$ and $\mathcal{G}_2$ to denote the set of degree constraints that are intervals and parity intervals, respectively, and $\mathcal{T}_1$ and $\mathcal{T}_2$ to denote the set of degree constraints that are type-1 and type-2, respectively. Let $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. In this paper, we study the problem WGFP($\mathcal{G} \cup \mathcal{T}$).

Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two subgraphs of G. The symmetric difference graph $H_1 \Delta H_2$ is the induced subgraph of G induced by the edge set $E_1 \Delta E_2$. Note that there are no isolated vertices in a symmetric difference graph. When $E_1 \cap E_2 = \emptyset$, we may write $H_1 \Delta H_2$ as $H_1 \cup H_2$. When $E_2 \subseteq E_1$, we may write $H_1 \Delta H_2$ as $H_1 \backslash H_2$. A *subcubic* graph is defined to be a graph where every vertex has degree $1, 2$ or 3. Unless stated otherwise, we use V_G and E_G to denote the vertex set and the edge set of a graph G, respectively.

3 Algorithm

We give a recursive algorithm for the problem WGFP($\mathcal{G} \cup \mathcal{T}$), using the problems WGFP($\mathcal{G}$) and the decision problem GFP($\mathcal{G} \cup \mathcal{T}$) as oracles.

Given an instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathcal{G} \cup \mathcal{T}$), we define the following sub-instances of $\Omega = (G, \pi, \omega)$ that will be used in the recursion. Recall that V_G denotes the vertex set of the graph G. Let T_Ω denote the set $\{v \in V_G \mid \pi(v) \in \mathcal{T}\}$. (We may omit the subscript Ω of T_Ω when it is clear from the context.)

For every vertex $v \in T_\Omega$, we split the instance Ω in two by splitting the degree constraint $\pi(v)$ in two parity intervals. More precisely, we define

$$D^0_v = \{p_v + 1, p_v + 3\} \text{ and } D^1_v = \{p_v\} \qquad \text{if} \quad \pi(v) = \{p_v, p_v + 1, p_v + 3\} \in \mathcal{T}_1;$$
$$D^0_v = \{p_v, p_v + 2\} \text{ and } D^1_v = \{p_v + 3\} \qquad \text{if} \quad \pi(v) = \{p_v, p_v + 2, p_v + 3\} \in \mathcal{T}_2.$$

We have $D^0_v, D^1_v \in \mathcal{G}_2$. For $i \in \{0, 1\}$ and $v \in T_\Omega$, we define $\Omega^i_v = (G, \pi^i_v, \omega)$ to be the sub-instance of Ω where $\pi^i_v(x) = \pi(x)$ for every $x \in V_G \backslash \{v\}$ and $\pi^i_v(v) = D^i_v$. Then, for every $v \in T_\Omega$, we have $\Omega^0_v \cap \Omega^1_v = \emptyset$ and $\Omega^0_v \cup \Omega^1_v = \Omega$. Moreover, $T_{\Omega^0_v} = T_{\Omega^1_v} = T_\Omega \backslash \{v\}$.

Let F be a factor of Ω. Similarly to above, one can partition Ω into $2^{|T_\Omega|}$ many sub-instances according to F such that each one is an instance of WGFP($\mathcal{G}$) – for each $v \in T_\Omega$, we choose one of the two splits of $\pi(v)$ as above. (We note that the algorithm will not consider all exponentially many sub-instances.) In detail, for every vertex $v \in T_\Omega$, we define $D^F_v = D^i_v$

[4] We use the term "subgraph" and notation $G' \subseteq G$ throughout for the standard meaning of a "normal" subgraph i.e., if $G = (V', E')$ and $G = (V, E)$ then $G' \subseteq G$ means $V' \subseteq V$ and $E' \subseteq E$.

where $\deg_F(v) \in D_v^i$ as follows:

$$
\begin{aligned}
D_v^F &= \{p_v\} && \text{if} \quad \pi(v) = \{p_v, p_v+1, p_v+3\} \in \mathcal{T}_1 \ \text{ and } \ \deg_F(v) = p_v, \\
D_v^F &= \{p_v+1, p_v+3\} && \text{if} \quad \pi(v) = \{p_v, p_v+1, p_v+3\} \in \mathcal{T}_1 \ \text{ and } \ \deg_F(v) \neq p_v; \\
D_v^F &= \{p_v+3\} && \text{if} \quad \pi(v) = \{p_v, p_v+2, p_v+3\} \in \mathcal{T}_2 \ \text{ and } \ \deg_F(v) = p_v+3, \\
D_v^F &= \{p_v, p_v+2\} && \text{if} \quad \pi(v) = \{p_v, p_v+2, p_v+3\} \in \mathcal{T}_2 \ \text{ and } \ \deg_F(v) \neq p_v+3.
\end{aligned}
$$

By definition, $\deg_F(v) \in D_v^F \subseteq \pi(v)$ and $D_v^F \in \mathcal{G}_2$. In fact, D_v^F is the maximal set such that $\deg_F(v) \in D_v^F \subseteq \pi(v)$ and $D_v^F \in \mathcal{G}_2$. One can also check that for every $v \in T_\Omega$, $\pi(v) \backslash D_v^F \in \mathcal{G}_2$, and moreover for every $p \in D_v^F$ and $q \in \pi(v) \backslash D_v^F$, $p \not\equiv q \mod 2$.

For every $W \subseteq T_\Omega$, we define $\Omega_W^F = (G, \pi_W^F, \omega)$ to be the sub-instance of Ω where $\pi_W^F(v) = \pi(v) \backslash D_v^F$ for $v \in W$, $\pi_W^F(v) = D_v^F$ for $v \in T_\Omega \backslash W$, and $\pi_W^F(v) = \pi(v)$ for $v \in V \backslash T_\Omega$. Then for every W, Ω_W^F is an instance of WGFP($\mathcal{G}$). Moreover, we have $\cup_{W \subseteq T} \Omega_W^F = \Omega$ and $\Omega_{W_1}^F \cap \Omega_{W_2}^F = \emptyset$ for every $W_1 \neq W_2$. Thus, $\{\Omega_W^F\}_{W \subseteq T_\Omega}$ is a partition of Ω (viewed as a set of subgraphs of G). When $W = \emptyset$, we write Ω_W^F as Ω^F.

Our algorithm is given in Algorithm 1.

Algorithm 1 Finding an optimal factor for an instance of WGFP($\mathcal{G} \cup \mathcal{T}$)

Function `Decision`:
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathcal{G} \cup \mathcal{T}$).
  Output : A factor of Ω, or "No" if Ω has no factor.

Function `Optimization`:
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathcal{G}$).
  Output : An optimal factor of Ω, or "No" if Ω has no factor.

Function `Main`:
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathcal{G} \cup \mathcal{T}$).
  Output : An optimal factor $F \in \Omega$, or "No" if Ω has no factor.

  $T \leftarrow \{v \in V \mid \pi(v) \in \mathcal{T}\}$;
  if T is the empty set **then**
    return `Optimization` (Ω);
  else
    Arbitrarily pick $u \in T$;
    if `Decision` (Ω_u^0) returns "No" **then**
      return `Main` (Ω_u^1);
    else
      $F^{\text{opt}} \leftarrow$ `Main` (Ω_u^0);
      foreach $v \in T$ **do**
        `// Elements of` T `can be traversed in an arbitrary order.`
        $W \leftarrow \{u\} \cup \{v\}$;
        if `Optimization`($\Omega_W^{F^{\text{opt}}}$) $\neq$ "No" **then** $F' \leftarrow$ `Optimization`($\Omega_W^{F^{\text{opt}}}$);
        if $\omega(F') > \omega(F^{\text{opt}})$ **then** $F^{\text{opt}} \leftarrow F'$;
      end
      return F^{opt};
    end
  end

The key that makes our algorithm running in strongly polynomial time is the following structural result (Theorem 5) for the problem WGFP($\mathcal{G} \cup \mathcal{T}$). It says that given an optimal factor F of Ω_u^0 for some $u \in T_\Omega$, if F is not optimal in Ω, then we can directly find an optimal factor of Ω by searching at most n sub-instances of Ω which are in WGFP($\mathcal{G}$). Note that the number of searches is independent of the edge weights. Thus, the problem of finding an optimal factor in Ω can be reduced to finding an optimal factor in Ω_u^0, where there is one fewer vertex u with constraints in $\mathcal{T}$. By recursively reducing an instance to another with fewer vertices with constraints in $\mathcal{T}$, we eventually get an instance of WGFP($\mathcal{G}$) which can be solved in polynomial-time. This leads to a strongly polynomial time algorithm for finding an optimal factor.

▶ **Theorem 5.** *Suppose that $\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathcal{G} \cup \mathcal{T}$)*, F is a factor of Ω and F is optimal in Ω_u^0 for some $u \in T_\Omega$. Then a factor F' is optimal in Ω if and only if $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2.*

In other words, if F is not optimal in Ω, then there is an optimal factor of Ω which belongs to Ω_W^F for some W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or $|W| = 2$.

▶ Remark 6. This result is *stronger* than the main result (Theorem 2) of [10], and it is *not* simply implied by [10]. To clarify this, we give a simple proof outline of Theorem 5 here.

In order to prove Theorem 5, it suffices to prove the direction that if $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2, then F' is optimal in Ω. We prove this by contradiction. Suppose that F' is not optimal in Ω, and F^* is an optimal factor of Ω. Then, $\omega(F^*) > \omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$. Also, $\omega(F^*) \notin \Omega_u^0$ since $\omega(F^*) > \omega(F) = \mathrm{Opt}(\Omega_u^0)$. Thus, $\deg_{F^*}(u) \not\equiv \deg_F(u) \mod 2$.

By [10], a *canonical path* $M \subseteq F\Delta F^*$ with positive weight[5] can be found, and then $F\Delta M$ is a factor of Ω with larger weight than F and $F\Delta M \in \Omega_W^F$ for some $W \subseteq T_\Omega$ where $|W| \leq 2$. However, this does not lead to a contradiction. To get a contradiction, we need to show that the positive weighted canonical path M (a basic augmenting subgraph) further satisfies $\deg_M(u) \equiv 0 \mod 2$. Then, $\deg_{F\Delta M}(u) \equiv \deg_F(u) \mod 2$. Thus, $F\Delta M$ is a factor with larger weight than F and $F\Delta M \in \Omega_u^0$, which contradicts with F being optimal in Ω_u^0.

The existence of a basic augmenting subgraph M satisfying $\deg_M(u) \equiv 0 \mod 2$ is formally stated in the second property of Lemma 12. The main technical part of the paper (Section 5.2 of the full paper) is devoted to prove it. In Section 5 of this short version, we give an example to illustrate the proof ideas. The existence of such a basic augmenting subgraph is highly non-trivial. In fact, it does *not* hold anymore after a subtle change of the condition "F is optimal in $\Omega_u^{\mathbf{0}}$" to "F is optimal in $\Omega_u^{\mathbf{1}}$" for some $u \in T_\Omega$. We give the following example (see Figure 1) to show this.

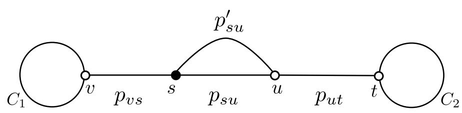

Figure 1 An example that violates Theorem 5 when F is optimal in Ω_u^1 instead of Ω_u^0

In this instance, $\pi(u) = \pi(v) = \pi(t) = \{0, 1, 3\}$ (denoted by hollow nodes) and $\pi(s) = \{0, 2, 3\}$ (denoted by the solid node), and $\omega(C_1) = \omega(p_{vs}) = \omega(p_{su}) = \omega(p'_{su}) = \omega(p_{ut}) = \omega(C_2) = 1$. Inside the cycles C_1 and C_2, and the paths p_{vs}, p_{su}, p_{ut}, and p'_{su}, there are other vertices of degree 2 with the degree constraint $\{0, 2\}$ so that the graph G is simple. We omit these vertices of degree 2 in Figure 1. In this case, $T_\Omega = \{u, v, s, t\}$. Consider the sub-instance $\Omega_u^1 = (G, \pi_u^1, \omega)$. We have $\pi_u^1(u) = D_u^1 = \{0\}$ since $\pi(u) = \{0, 1, 3\}$. Then, the

[5] See definition 3 of [10]. They are defined as basic augmenting subgraphs (Definition 11) in this paper.

only factor F of Ω_u^1 is the empty graph (assuming there are no isolated vertices in factors), and F is not optimal in Ω. Also, the only optimal factor of Ω is the graph G and $G \in \Omega_{T_\Omega}^F$ where $|T_\Omega| = 4$. Clearly, $\deg_G(u) \not\equiv \deg_F(u) \mod 2$. One can check that for any factor F' of Ω with larger weight than F, $\deg_{F'}(u) \not\equiv \deg_F(u) \mod 2$. In other words, there is no basic augmenting subgraph M such that $\deg_M(u) \equiv 0 \mod 2$. Moreover, one can check that in this case, Theorem 5 also does not hold. In other words, the existence of a basic augmenting subgraph satisfying $\deg_M(u) \equiv 0 \mod 2$ is crucial for the correctness of Theorem 5.

Using Theorem 5, we now prove that Algorithm 1 is correct.

▶ **Lemma 7.** *Given an instance $\Omega = (G, \pi, \omega)$ of* WGFP($\mathcal{G}, \mathcal{T}$)*, Algorithm 1 returns either an optimal factor of Ω, or "No" if Ω has no factor.*

Proof. Recall that for an instance $\Omega = (G, \pi, \omega)$, we define $T_\Omega = \{v \in V_G \mid \pi(v) \in \mathcal{T}\}$ where V_G is the vertex set of G . We prove the correctness by induction on the $|T_\Omega|$.

If $|T_\Omega| = 0$, Ω is an instance of WGFP($\mathcal{G}$). Algorithm 1 simply returns `Optimization` (Ω). By the definition of the function `Optimization`, the output is correct.

Suppose that Algorithm 1 returns correct results for all instances Ω' of WGFP($\mathcal{G}, \mathcal{T}$) where $|T_{\Omega'}| = k$. We consider an instance Ω of WGFP($\mathcal{G}, \mathcal{T}$) where $|T_\Omega| = k+1$. Algorithm 1 first calls the function `Decision` (Ω_u^0) for some arbitrary $u \in T$.

We first consider the case that `Decision` (Ω_u^0) returns "No". By the definition, Ω_u^0 has no factor. Moreover, since $\Omega = \Omega_u^0 \cup \Omega_u^1$, we have $F \in \Omega$ if and only if $F \in \Omega_u^1$. Then, a factor $F \in \Omega_u^1$ is optimal in Ω if and only if it is optimal in Ω_u^1. Note that Ω_u^1 is an instance of WGFP($\mathcal{G}, \mathcal{T}$) where $|T_{\Omega_u^1}| = k$. By the induction hypothesis, Algorithm 1 returns a correct result `Main` (Ω_u^1) for the instance Ω_u^1, which is also a correct result for the instance Ω.

Now, we consider the case that `Decision` (Ω_u^0) returns a factor of Ω_u^0. Then, `Main` (Ω_u^0) returns an optimal factor F of Ω_u^0. After the loop (lines 13 to 17) in Algorithm 1, we get a factor F^{opt} of Ω such that $\omega(F^{\text{opt}}) \geq \text{Opt}(\Omega_W^F)$ for every $u \in W \subseteq T_\Omega$ where $|W| = 1$ (when $u = v$) or $|W| = 2$ (when $u \neq v$) and $\omega(F^{\text{opt}}) \geq \omega(F)$. By Theorem 5, F^{opt} is an optimal factor of Ω. Thus, Algorithm 1 returns a correct result. ◀

Now, we consider the time complexity of Algorithm 1. The size of an instance is defined to be the number of vertices of the underlying graph of the instance.

▶ **Lemma 8.** *Run Algorithm 1 on an instance $\Omega = (G, \pi, \omega)$ of size n. Then,*

- *the algorithm will stop the recursion after at most n recursive steps;*
- *the algorithm will call* `Decision` *at most n many times, call* `Optimization` *at most $\frac{n(n+1)}{2} + 1$ many times, and perform at most $\frac{n(n+1)}{2}$ many comparisons;*
- *the algorithm runs in time $O(n^6)$.*

Proof. Let $\Omega^k = \{G, \pi^k, \omega\}$ be the instance after k many recursive steps. Here $\Omega^0 = \Omega$. Recall that $T_{\Omega^k} = \{v \in V \mid \pi^k(v) \in \mathcal{T}\}$. For an instance Ω^k with $|T_{\Omega^k}| > 0$, the recursive step will then go to the instance $(\Omega^k)_u^0$ or $(\Omega^k)_u^1$ for some $u \in T_{\Omega^k}$. Thus, $\Omega^{k+1} = (\Omega^k)_u^0$ or $(\Omega^k)_u^1$. In both cases, $T_{\Omega^{k+1}} = T_{\Omega^k} \backslash \{u\}$ and hence $|T_{\Omega^{k+1}}| = |T_{\Omega^k}| - 1$. By design, the algorithm will stop the recursion and return `Optimization` (Ω^m) when it reaches an instance Ω^m with $|T_{\Omega^m}| = 0$. Thus, #recursive steps $= m = |T_\Omega| - 0 \leq |V| = n$. To prove the second item, we consider the number of operations inside the recursive step for the instance $\Omega^k = \{G, \pi^k, \omega\}$. Note that $k \leq n$ and $|T_{\Omega^k}| = |T_\Omega| - k \leq n - k$. If $|T_{\Omega^k}| = 0$, then the algorithm will simply call `Optimization` once. If $|T_{\Omega^k}| > 0$, then inside the recursive step, the algorithm will call `Decision` once, and call `Optimization` once or $|T_{\Omega^k}|$ many times depending on the answer of `Decision`. Moreover, in the later case, the algorithm will also perform

$|T_{\Omega^k}|$ many comparisons. Thus, we have #calls of `Decision` $= \sum_{|T_{\Omega^k}|>0} 1 = \sum_{i=1}^{|T_\Omega|} 1 = |T_\Omega| \leq n$, #calls of `Optimization` $\leq 1 + \sum_{|T_{\Omega^k}|>0} |T_{\Omega^k}| = 1 + \sum_{i=1}^{|T_\Omega|} i \leq \frac{n(n+1)}{2} + 1$, and #comparisons $\leq \sum_{|T_{\Omega^k}|>0} |T_{\Omega^k}| \leq \frac{n(n+1)}{2}$. Let $t_{\texttt{Main}}(n)$ denote the running time of Algorithm 1 on an instance of size n, and $t_{\texttt{Dec}}(n)$ and $t_{\texttt{Opt}}(n)$ denote the running time of algorithms for functions `Decision` and `Optimization`, respectively. Then, $t_{\texttt{Dec}}(n) = O(n^4)$ by the algorithm in [5] and $t_{\texttt{Opt}}(n) = O(n^4)$ by the algorithm in [9]. Thus, $t_{\texttt{Main}}(n) \leq nt_{\texttt{Dec}}(n) + \frac{n(n+1)+2}{2} t_{\texttt{Opt}}(n) + \frac{n(n+1)}{2} = O(n^6)$. ◀

4 Proof of Theorem 5

In this section, we give a proof of Theorem 5. The general strategy is that starting with a non-optimal factor F of an instance $\Omega = (G, \omega, \pi)$, we want to find a subgraph H of G such that by taking the symmetric difference $F \Delta H$, we get another factor of Ω with larger weight. The existence of such subgraphs is trivial (Lemma 10). However, the challenge is how to find one efficiently. As an analogy of augmenting paths in the weighted matching problem, we introduce basic augmenting subgraphs (Definition 11) for the weighted graph factor problem, which can be found efficiently. We will show that given a non-optimal factor F, a basic augmenting subgraph always exists (Lemma 12, property 1). Then, we can efficiently improve the factor F to another factor with larger weight. As shown in [10], this already gave a weakly-polynomial time algorithm. However, the existence of basic augmenting subgraphs is not enough to get a strongly polynomial-time algorithm, which requires the number of improvement steps being independent of edge weights. Thus, in order to prove Theorem 5, which leads to a strongly polynomial-time algorithm, we further establish that there exists a basic augmenting subgraph that satisfies certain stronger properties under suitable assumptions (Lemma 12, property 2). This result will imply Theorem 5.

▶ **Definition 9** (F-augmenting subgraphs). *Suppose that F is a factor of an instance $\Omega = (G, \pi, \omega)$. A subgraph H of G is F-augmenting if $F \Delta H \in \Omega$ and $\omega(F \Delta H) - \omega(F) > 0$.*

▶ **Lemma 10.** *Suppose that F is a factor of an instance Ω. If F is not optimal in Ω, then there exists an F-augmenting subgraph.*

Proof. Since F is not optimal, there is some $F' \in \Omega$ such that $\omega(F') > \omega(F)$. Let $H = F \Delta F'$. We have $F \Delta H = F' \in \Omega$ and $\omega(H) = \omega(F') - \omega(F) > 0$. Thus, H is F-augmenting. ◀

Recall that for an instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathcal{G}, \mathcal{T}$), T_Ω is the set $\{x \in V_G \mid \pi(v) \in \mathcal{T}\}$. For two factors $F, F^* \in \Omega$, we define $T_\Omega^{F \Delta F^*} = \{v \in T_\Omega \mid \deg_{F \Delta F^*}(v) \equiv 1 \mod 2\}$.

▶ **Definition 11** (Basic augmenting subgraphs). *Suppose that F and F^* are factors of an instance $\Omega = (G, \pi, \omega)$ and $\omega(F) < \omega(F^*)$. An F-augmenting subgraph $H = (V_H, E_H)$ is* (F, F^*)-basic *if $H \subseteq F \Delta F^*$, $|V_H^{\text{odd}}| \leq 2$, and $V_H^{\text{odd}} \cap T_\Omega \subseteq T_\Omega^{F \Delta F^*}$ where $V_H^{\text{odd}} = \{v \in V_H \mid \deg_H(v) \equiv 1 \mod 2\}$.*

▶ **Lemma 12.** *Suppose that F and F^* are two factors of an instance $\Omega = (G, \pi, \omega)$.*

1. *If $\omega(F^*) > \omega(F)$, then there exists an (F, F^*)-basic subgraph.*
2. *If $\omega(F^*) > \text{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega^{F \Delta F^*}$ with $|W| \leq 2$, and $T_\Omega^{F \Delta F^*}$ contains a vertex u such that $F \in \Omega_u^0$ (i.e., $\deg_F(u) \in D_u^0$), then there exists an (F, F^*)-basic subgraph H where $\deg_H(u) \equiv 0 \mod 2$.*

▶ Remark 13. The first property of Lemma 12 implies the following: a factor $F \in \Omega$ is optimal if and only if $\omega(F) \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ with $|W| \leq 2$. This is a special case of the main result (Theorem 2) of [10] where the authors consider the WGFP for all constraints with gaps of length at most 1. The second property of Lemma 12 is more refined than the first property and it implies our main result (Theorem 5). In this paper, as a by-product of the proof of property 2, we give a simple proof of Theorem 2 of [10] for the special case WGFP($\mathcal{G} \cup \mathcal{T}$) based on certain properties of subcubic graphs.

Using the second property of Lemma 12, we can prove Theorem 5.

▶ **Theorem** (Theorem 5). *Suppose that F is a factor of an instance $\Omega = (G, \pi, \omega)$, and F is optimal in Ω_u^0 for some $u \in T_\Omega$. Then a factor F' is optimal in Ω if and only if $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2.*

Proof. If F' is optimal in Ω, then clearly $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2. Thus, to prove the theorem, it suffices to prove the other direction. Since $\omega(F') \geq \omega(F)$ and F is optimal in Ω_u^0, we have $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $u \notin W$ and $|W| \leq 2$. Also, since $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2, we have $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$. For a contradiction, suppose that F' is not optimal in Ω. Let F^* be an optimal factor of Ω. Then, $\omega(F^*) > \omega(F')$. Thus, $\omega(F^*) > \omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$. Also, $^* \notin \Omega_u^0$ since $\omega(F^*) > \omega(F)$ and F is optimal in Ω_u^0. Thus, $\deg_{F^*}(u) \not\equiv \deg_F(u) \mod 2$. Then, $T_\Omega^{F\Delta F^*}$ contains the vertex u such that $F \in \Omega_u^0$. By Lemma 12, there exists an (F, F^*)-basic subgraph H where $\deg_H(u) \equiv 0 \mod 2$. Let $F'' = F\Delta H$. Then $F'' \in \Omega$ and $\omega(F'') > \omega(F)$. Also, $F'' \in \Omega_u^0$ since $\deg_{F''}(u) \equiv \deg_F(u) \mod 2$. This is a contradiction with F being optimal in Ω_u^0. ◀

Now it suffices to prove Lemma 12. By a type of normalization maneuver, we can transfer any instance of WGFP($\mathcal{G}, \mathcal{T}$) to an instance of WGFP($\mathcal{G}, \mathcal{T}$) defined on subcubic graphs, called a key instance (Definition 14). Recall that a subcubic graph is a graph where every vertex has degree 1, 2 or 3. For key instances, there are five possible forms of basic augmenting subgraphs, called basic factors (Definition 15). Then, the crux of the proof of Lemma 12 is to establish the existence of certain basic factors of key instances (Theorem 16). For a proof of Lemma 12 using Theorem 16, please refer to the proof of Lemma 4.4 in the full paper.

▶ **Definition 14** (Key instance). *A* key instance *$\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathcal{G}, \mathcal{T}$) *where G is a subcubic graph, and for every $v \in V_G$, $\pi(v) = \{0, 1\}$ if $\deg_G(v) = 1$, $\pi(v) = \{0, 2\}$ if $\deg_G(v) = 2$, and $\pi(v) = \{0, 1, 3\}$ (i.e., type-1) or $\{0, 2, 3\}$ (i.e., type-2) if $\deg_G(v) = 3$. We say a vertex v of degree 3 is of type-1 or type-2 if $\pi(v)$ is type-1 or type-2 respectively. We say a vertex v of any degree is 1-feasible or 2-feasible if $1 \in \pi(v)$ or $2 \in \pi(v)$ respectively.*

▶ **Definition 15** (Basic factor). *Let Ω be a key instance. A factor of Ω is a* basic factor *if it is in one of the following five forms: a* path, *a* cycle, *a* tadpole graph *(i.e., a graph consisting of a cycle and a path such that they intersect at one endpoint of the path), a* dumbbell graph *(i.e., a graph consisting of two vertex disjoint cycles and a path such that the path intersects with each cycle at one of its endpoints), and a* theta graph *(i.e., a graph consisting of three vertex disjoint paths with the same two endpoints).*

▶ **Theorem 16.** *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance.*

1. *If $\omega(G) > 0$, then there is a basic factor F of Ω such that $\omega(F) > 0$.*
2. *If $\omega(G) > 0$, $\omega(G) > \omega(F)$ for every basic factor F of Ω, and G contains a vertex u with $\deg_G(u) = 1$ or $\deg_G(u) = 3$ and $\pi(u) = \{0, 2, 3\}$, then there is a basic factor F^* of Ω such that $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \mod 2$. (Recall that $\deg_{F^*}(u) = 0$ if $u \notin V_{F^*}$.)*

▶ Remark 17. For the second property of Theorem 16, the requirement of $\pi(u) = \{0, 2, 3\}$ when $\deg_G(u) = 3$ is crucial. Consider the instance $\Omega = (G, \pi, \omega)$ as shown in Figure 1. Note that Ω is a key instance. and $\pi(u) = \{0, 1, 3\}$. In this case where $\pi(u) = \{0, 1, 3\}$, it can be checked that the second property does *not* hold.

5 Proof Sketch of Theorem 16

In this section, we give a proof sketch of Theorem 16 and we focus on the proof of the second property using the first property. Omitted proofs can be found in Section 5 of the full paper.

Proof sketch. By property 1 of Theorem 16, there exists at least one basic factor of Ω such that its weight is positive. Among all such basic factors, we pick an F such that $\omega(F)$ is the largest. Consider the graph $G' = G\backslash F$, i.e., the subgraph of G induced by the edge set $E_G\backslash E_F$. We consider the instance $\Omega' = (G', \pi', \omega')$ where for every $x \in V_{G'}$, $\pi'(x) = \{0, 1\}$ if $\deg_{G'}(x) = 1$, $\pi'(x) = \{0, 2\}$ if $\deg_{G'}(x) = 2$ and $\pi'(x) = \pi(x)$ if $\deg_{G'}(x) = 3$, and ω' is the weight function ω restricted to G'. Note that Ω' is also a key instance, but it is not necessarily a sub-instance of Ω. Since $\omega(G) > \omega(F)$, we have $\omega'(G') = \omega(G') = \omega(G) - \omega(F) > 0$. Without causing ambiguity, we may simply write ω' as ω in the instance Ω'. By property 1 of Theorem 16, there exists a basic factor F' of Ω' such that $\omega(F') > 0$. Since $E_{F'} \subseteq E_G\backslash E_F$, F and F' are edge-disjoint. Let $H = F \cup F'$, which is the subgraph of G induced by the edge set $E_F \cup E_{F'}$. We will show that we can find a subgraph F^* of H such that F^* is the desired basic factor of Ω satisfying $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \mod 2$.

First, we show that H is a factor of Ω. Let $V_\cap = V_F \cap V_{F'}$. We show that for every $x \in V_H\backslash V_\cap$, $\deg_H(x) \in \pi(x)$. If $x \in V_F\backslash V_\cap$, then $\deg_H(x) = \deg_F(x)$. Since $F \in \Omega$, $\deg_F(x) \in \pi(x)$. Then, $\deg_H(x) \in \pi(x)$. If $x \in V_{F'}\backslash V_\cap$, then $\deg_H(x) = \deg_{F'}(x)$. Since $x \notin V_F$ and $G' = G\backslash F$, $\deg_{G'}(x) = \deg_G(x)$. Then, by the definition of Ω', we have $\pi'(x) = \pi(x)$. Since F' is a factor of Ω', $\deg_{F'}(x) \in \pi'(x)$. Thus, $\deg_H(x) \in \pi(x)$. Now, we consider vertices in $V_\cap$. Since F and F' are edge disjoint, for every $x \in V_\cap$ we have $\deg_H(x) = \deg_F(x) + \deg_{F'}(x) \le \deg_G(x) \le 3$. Also, $\deg_F(x), \deg_{F'}(x) \ge 1$ since F and F' are subcubic graphs which have no isolated vertices.

- If $\deg_F(x) = 1$, then $1 \in \pi(x)$. The vertex x is 1-feasible. Thus, $\deg_G(x) \ne 2$. Since $\deg_G(x) > \deg_F(x) = 1$, $\deg_G(x) = 3$. Then, $\deg_{G'}(x) = \deg_G(x) - \deg_F(x) = 2$, $\pi'(x) = \{0, 2\}$ and $\deg_{F'}(x) = 2$.
- If $\deg_F(x) = 2$, then $\deg_G(x) = 3$ since $\deg_G(x) > \deg_F(x)$. Then, $\deg_{G'}(x) = \deg_G(x) - \deg_F(x) = 1$, $\pi'(x) = \{0, 1\}$ and $\deg_{F'}(x) = 1$.

Thus, for every $x \in V_\cap$, $\deg_H(x) = \deg_F(x) + \deg_{F'}(x) = 3 \in \pi(x)$. Thus, H is a factor of Ω.

Then, we finish the proof by a careful analysis of possible forms of F and F', and possible intersection vertices in $V_\cap$. Here, we give an example where F is a tadpole graph and $\deg_F(u) = 3$ to illustrate this. Since $\deg_F(u) = 3$, by assumption, $\pi(u) = \{0, 2, 3\}$. Also, since $\deg_F(v) = 1 \in \pi(v)$, v is 1-feasible.

Consider possible vertices in $V_\cap$. Recall that for every $x \in V_\cap$, $\deg_F(x) = 1$ and $\deg_{F'}(x) = 2$, or $\deg_F(x) = 2$ and $\deg_{F'}(x) = 1$. Since $\deg_F(u) = 3 = \deg_G(u)$, we have $u \notin V_\cap$. Also, consider the possible forms of F'. We show that F' is not a cycle. For a contradiction, suppose that F' is a cycle. Then, all vertices of F' have degree 2. Thus, the only possible vertex in $V_\cap$ is v. If $V_\cap = \emptyset$, then for every $x \in V_{F'}$, $\deg_{F'}(x) = \deg_H(x) \in \pi(x)$. Thus, F' is a basic factor of Ω where $\omega(F') > 0$ and $\deg_{F'}(u) = 0$. We are done. Otherwise, $V_\cap = \{v\}$. Then, $\deg_F(v) = 1$ and $\deg_{F'}(v) = 2$. Since F is a tadpole graph, the graph H is a dumbbell graph where u and v are the two vertices of degree 3. Thus, H is a basic factor of Ω. Since $\omega(F') > 0$, we have $\omega(H) = \omega(F) + \omega(F') > \omega(F)$ which leads to a contraction with F being a basic factor with the largest weight. Thus, F' is a basic factor which is not a cycle. By Definition 15, F' contains exactly two vertices of odd degree, denoted by s and t. Then, we have $V_\cap \subseteq \{v, s, t\}$.

Recall that F is a tadpole graph consisting of a path and a cycle. We use C to denote the cycle part of F, and V_C denotes its vertex set. Consider $\{s,t\} \cap V_C$. Now, we handle possible subcases according to intersection vertices appearing in V_C. There are three subcases. Below, for two points x and y, we use p_{xy} or p'_{xy} to denote a path with endpoints x and y.

1. $\{s,t\} \subseteq V_C$. Then, $\deg_F(s) = \deg_F(t) = \deg_C(s) = \deg_C(t) = 2$. In this case, $\deg_H(u) = \deg_H(s) = \deg_H(t) = 3$ and $\pi(u) = \pi(s) = \pi(t) = \{0,2,3\}$. Also, $\deg_{F'}(s) = \deg_{F'}(t) = 1$. Thus, F' is a path with endpoints s and t. Note that in this case, it is possible that $v \in V_{F'}$. If $v \in V_{F'}$, then $\deg_H(v) = 3$ and $\pi(v) = \{0,1,3\}$; otherwise, $\deg_H(v) = 1$ and $\pi(v) = \{0,1\}$. The points u, s, and t split C into three paths, p_{us}, p_{st}, p_{tu}. Then, $C = p_{us} \cup p_{st} \cup p_{tu}$. (See Figure 2.) If $\omega(C) > 0$, then we are done since C is a basic factor of Ω and $\deg_C(u) = 2$. Thus, we may assume that $\omega(C) \leq 0$.

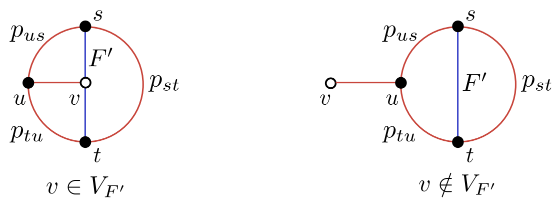

Figure 2 The two possible forms of graph H when $\{s,t\} \in V_C$. Hollow nodes denote 1-feasible vertices, and solid nodes denote 2-feasible vertices; red-colored lines denote paths in F, and blue-colored lines denote paths in F'.

Consider the graph $H_1 = H \backslash p_{st} = (F \backslash p_{st}) \cup F'$. Note that $V_{H_1} = (V_H \backslash V_{p_{st}}) \cup \{s,t\}$. For every $x \in V_{H_1} \backslash \{s,t\}$, we have $\deg_{H_1}(x) = \deg_H(x) \in \pi(x)$ since H is a factor of Ω. Also, $\deg_{H_1}(s) = 2 \in \pi(s)$ and $\deg_{H_1}(t) = 2 \in \pi(t)$. Thus, H_1 is a factor of Ω. Also, H_1 is a tadpole graph if $\deg_H(v) = 1$ or a theta graph if $\deg_H(v) = 3$. Thus, in both cases, H_1 is a *basic* factor of Ω. Since F is a basic factor of Ω with the largest weight, we have $\omega(F) \geq \omega(H_1) = \omega(F) - \omega(p_{st}) + \omega(F')$. Thus, $\omega(p_{st}) \geq \omega(F') > 0$. Since $\omega(C) = \omega(p_{st}) + \omega(p_{us}) + \omega(p_{tu}) \leq 0$, $\omega(p_{us}) + \omega(p_{tu}) < 0$. Without loss of generality, we may assume that $\omega(p_{us}) < 0$. Then, consider the graph $H_2 = H \backslash p_{us}$. Similarly, one can check that H_2 is a factor of Ω, and $\deg_{H_2}(u) = 2$. Also, H_2 is a tadpole graph if $\deg_H(v) = 1$, or a theta graph if $\deg_H(v) = 3$. Thus, H_2 is a basic factor of Ω. Moreover, $\omega(H_2) = \omega(H) - \omega(p_{us}) > 0$. We are done.

2. $\{s,t\} \cap V_C = \{s\}$ or $\{t\}$. Without loss of generality, we may assume that $s \in V_C$. Then, $\deg_H(u) = \deg_H(s) = 3$ and $\pi(u) = \pi(s) = \{0,2,3\}$. If $\omega(C) > 0$, then we are done since C is a basic factor of Ω and $\deg_C(u) = 2$. Thus, we may assume that $\omega(C) \leq 0$. Vertices s and u split C into two paths p_{us} and p'_{us}. Since $\omega(C) = \omega(p_{us}) + \omega(p'_{us}) \leq 0$, among them at least one is non-positive. Without loss of generality, we assume that $\omega(p_{us}) \leq 0$. Consider the graph $H' = H \backslash p_{us}$. We have $\omega(H') = \omega(H) - \omega(p_{us}) > 0$, and $\deg_{H'}(u) = 2$. Similar to the above case, one can check that H' is a factor of Ω. However, it is not clear whether H' is a *basic* factor of Ω. Consider the sub-instance $\Omega'_H = (H', \pi_{H'}, \omega_{H'})$ of Ω defined on the subgraph H' of G where $\pi_{H'}(x) = \pi(x) \cap [\deg_{H'}(x)] \subseteq \pi(x)$ for every $x \in V_{H'}$ and $\omega_{H'}$ is the restriction of ω on $E_{H'}$ (we may write $\omega_{H'}$ as ω for simplicity). Since $\omega(H') > 0$, by property 1 of Theorem 16, there is a basic factor $F^* \in \Omega_{H'}$ such that $\omega(F^*) > 0$. Then, $\deg_{F^*}(u) \in \pi_{H'}(u) = \{0,2\}$. Now, F^* is a basic factor of Ω.

3. $\{s,t\} \cap V_C = \emptyset$. In this case, the cycle C does not intersect with F'. Then, by viewing the cycle C as an enlargement of the vertex u, this case is similar to the case that F is a path with endpoints u and v, which is proved separately. Please refer to the full paper for its proof. ◀

A Δ-Matroids and Matching Realizability

A Δ-matroid is a family of sets obeying an axiom generalizing the matroid exchange axiom. Formally, a pair $M = (U, \mathcal{F})$ is a Δ-matroid if U is a finite set and $\mathcal{F}$ is a collection of subsets of U satisfying the following: for any $X, Y \in \mathcal{F}$ and any $u \in X \Delta Y$ in the symmetric difference of X and Y, there exits a $v \in X \Delta Y$ such that $X \Delta \{u, v\}$ belongs to $\mathcal{F}$ [3]. A Δ-matroid is *symmetric* if, for every pair of $X, Y \subseteq U$ with $|X| = |Y|$, we have $X \in \mathcal{F}$ if and only if $Y \in \mathcal{F}$. A Δ-matroid is *even* if for every pair of $X, Y \subseteq U$, $|X| \equiv |Y| \mod 2$.

Suppose that $U = \{u_1, u_2, \ldots, u_n\}$. A subset $V \subseteq U$ can be encoded by a binary string α_V of n-bits where the i-th bit of α_V is 1 if $u_i \in V$ and 0 if $u_i \notin V$. Then, a Δ-matroid $M = (U, \mathcal{F})$ can be represented by a relation R_M of arity $|U|$ which consists of binary strings that encode all subsets in $\mathcal{F}$. Such a representation is unique up to a permutation of variables of the relation. A degree constraint D of arity n can be viewed as an n-ary symmetric relation which consists of binary strings with the Hamming weight d for every $d \in D$. By the definition of Δ-matroids, it is easy to check that a degree constraint D (as a symmetric relation) represents a Δ-matroid if and only if D has all gaps of length at most 1.

▶ **Definition 18** (Matching Gadget). *A gadget using a set $\mathcal{D}$ of degree constraints consists of a graph $G = (U \cup V, E)$ where $\deg_G(u) = 1$ for every $u \in U$ and there are no edges between vertices in U, and a mapping $\pi : V \to \mathcal{D}$. A* matching gadget *is a gadget where $\mathcal{D} = \{\{0, 1\}, \{1\}\}$. A degree constraint D of arity n is* matching realizable *if there exists a matching gadget $(G = (U \cup V, E), \pi : V \to \{\{0, 1\}, \{1\}\})$ such that $|U| = n$ and for every $k \in [n]$, $k \in D$ if and only if for every $W \subseteq U$ with $|W| = k$, there exists a matching $F = (V_F, E_F)$ of G such that $V_F \cap U = W$ and for every $v \in V$ where $\pi(v) = \{1\}$, $v \in V_F$.*

The definition of matching realizability can be extended to a relation R of arity n by requiring the set U of n vertices in a matching gadget to represent the n variables of R. If R is realizable by a matching gadget $G = (U \cup V, E)$, then for every $\alpha \in \{0, 1\}^n$, $\alpha \in R$ if and only if there is a matching $F = (V_F, E_F)$ of G such that $V_F \cap U$ is exactly the subset of U encoded by α (i.e., for every $u_i \in U$, $u_i \in V_F$ if and only if $\alpha_i = 1$), and for every $v \in V$ where $\pi(v) = \{1\}$, $v \in V_F$. Note that the matching realizability of a relation is invariant under a permutation of its variables. We say that a Δ-matroid is matching realizable if the relation representing it is matching realizable.[6]

The following result generalizes Lemma A.1 of [29].

▶ **Lemma 19.** *Suppose that $M = (U, \mathcal{F})$ is a matching realizable Δ-matroid, and $V_1, V_2 \in \mathcal{F}$. Then, $V_1 \Delta V_2$ can be partitioned into single variables $S_1, \ldots, S_k$ and pairs of variables $P_1, \ldots, P_\ell$ such that for every $P = S_{i_1} \cup \cdots \cup S_{i_r} \cup P_{j_1} \cup \cdots \cup P_{j_t}$ ($\{i_1, \ldots, i_r\} \subseteq [k], \{j_1, \ldots, j_t\} \subseteq [\ell]$), $V_1 \Delta P \in \mathcal{F}$ and $V_2 \Delta P \in \mathcal{F}$.*

▶ **Theorem 20.** *A degree constraint D of gaps of length at most 1 is matching realizable if and only if all its gaps are of the same length 0 or 1.*

Proof. By the gadget constructed in the proof of [5, Theorem 2], if a degree constraint has all gaps of length 1 then it is matching realizable.[7] We give the following gadget (Figure 3)

[6] This definition of matching realizability for Δ-matroids is different with the one that is usually used for even Δ-matroids [4, 11, 29], in which the gadget is only allowed to use the constraint $\{1\}$ for perfect matchings, and hence the resulting Δ-matroid must be even.

[7] We remark that [5] includes gadgets for other types of degree constraints, including type-1 and type-2, but only under a more general notion of gadget constructions that involve edges and triangles. The gadget that only involves edges is a matching gadget defined in this paper.

to realize a degree constraint D with all gaps of length 0, which generalizes the gadget in [43]. Suppose that $D = \{p, p+1, \ldots, p+r\}$ of arity n where $n \geq p+r \geq p \geq 0$. Consider the following graph $G = (U \cup V, E)$: U consists of n vertices of degree 1, and V consists of two parts V_1 with $|V_1| = n$ and V_2 with $|V_2| = n-p$; the induced subgraph $G(V)$ of G induced by V is a complete bipartite graph between V_1 and V_2, and the induced subgraph $G(U \cup V_1)$ of G induced by $U \cup V_1$ is a bipartite perfect matching between U and V_1. Every vertex in V_1 is labeled by the constraint $\{1\}$. There are r vertices in V_2 labeled by $\{0, 1\}$ and the other $n-p-r$ vertices in V_2 labeled by $\{1\}$. One can check that this gadget realizes D.

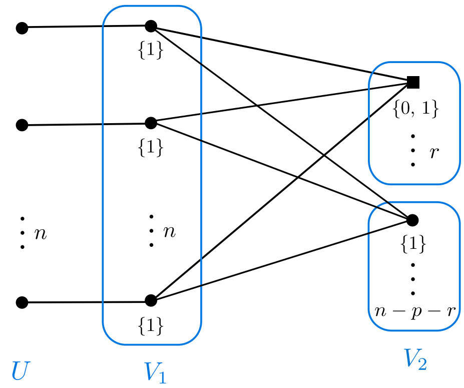

Figure 3 A matching gadget realizing $D = \{p, p+1, \ldots, p+r\}$ of arity n

For the other direction, without loss of generality, we may assume that $\{p, p+1, p+3\} \subseteq D$ and $p+2 \notin D$. Since D has gaps of length at most 1, it can be associated with a symmetric Δ-matroid $M = (U, \mathcal{F})$. Then, there is $V_1 \in \mathcal{F}$ with $|V_1| = p$ and $V_2 \in \mathcal{F}$ with $|V_2| = p+3$. Since M is symmetric, we may pick $V_2 = V_1 \cup \{v_1, v_2, v_3\}$ for some $\{v_1, v_2, v_3\} \cap V_1 = \emptyset$. Let $S = V_1 \Delta V_2 = \{v_1, v_2, v_3\}$. By Lemma 19, S can be partitioned into single variables and/or pairs of variables such that for any union P of them, $V_2 \backslash P \in \mathcal{F}$. Since $|S| = 3$, there exists at least a single variable x_i in the partition of S such that $V_2 \backslash \{v_i\} \in \mathcal{F}$. Note that $|V_2 \backslash \{v_i\}| = p+2$. Thus, $p+2 \in D$. A contradiction. ◀

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The full paper is attached.

A Strongly Polynomial-Time Algorithm for Weighted General Factors with Three Feasible Degrees*

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Abstract

General factors are a generalization of matchings. Given a graph G with a set $\pi(v)$ of feasible degrees, called a degree constraint, for each vertex v of G, the general factor problem is to find a (spanning) subgraph F of G such that $\deg_F(v) \in \pi(v)$ for every v of G. When all degree constraints are symmetric Δ-matroids, the problem is solvable in polynomial time. The weighted general factor problem is to find a general factor of the maximum total weight in an edge-weighted graph. Strongly polynomial-time algorithms are only known for weighted general factor problems that are reducible to the weighted matching problem by gadget constructions.

In this paper, we present the first strongly polynomial-time algorithm for a type of weighted general factor problems with real-valued edge weights that is provably not reducible to the weighted matching problem by gadget constructions.

1 Introduction

A matching in an undirected graph is a subset of the edges that have no vertices in common, and it is perfect if its edges cover all vertices of the graph. Graph matching is one of the most studied problems both in graph theory and combinatorial optimization, with beautiful structural results and efficient algorithms described, e.g., in the monograph of Lovász and Plummer [LP09] and in relevant chapters of standard textbooks [Sch03, KV18]. In particular, the *weighted (perfect) matching problem* is to find a (perfect) matching of the maximum total weight for a given graph of which each edge is assigned a weight. This problem can be solved in polynomial time by the celebrated Edmonds' blossom algorithm [Edm65a, Edm65b]. Since then, a number of more efficient algorithms have been developed [Gab74, Law76, Kar76, CM78, Gab85, GMG86, GGS89, Gab90, GT91, HK12]. Table III of [DP14] gives a detailed review of these algorithms.

The *f-factor problem* is a generalization of the perfect matching problem in which one is given a non-negative integer $f(v)$ for each vertex $v \in V$ of $G = (V, E)$. The task is to find a (spanning) subgraph $F = (V_F, E_F)$ of G such that $\deg_F(v) = f(v)$ for every $v \in V$.[1] The

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[1]In graph theory, a graph factor is usually a spanning subgraph. Here, without causing ambiguity, we allow F to be an arbitrary subgraph including the empty graph and we adapt the convention that $\deg_F(v) = 0$ if $v \in V \setminus V_F$.

case $f(v) = 1$ for every $v \in V$ is the perfect matching problem. This problem, as well as the weighted version, can be solved efficiently by a gadget reduction to the perfect matching problem [EJ70]. In addition, Tutte gave a characterization of graphs having an f-factor [Tut52], which generalizes his characterization theorem for perfect matchings [Tut50]. Subsequently, the study of graph factors has attracted much attention with many variants of graph factors, e.g., b-matchings, $[a, b]$-factors, (g, f)-factors, parity (g, f)-factors, and anti-factors introduced, and various types of characterization theorems proved for the existence of such factors. We refer the reader to the book [AK11] and the survey [Plu07] for a comprehensive treatment of the developments on the topic of graph factors.

In the early 1970s, Lovász introduced a generalization of the above factor problems [Lov70, Lov72], for which we will need a few definitions. For any nonnegative integer n, let $[n]$ denote $\{0, 1, \ldots, n\}$. A degree constraint D of arity n is a subset of $[n]$.[2] We say that a degree constraint D has a gap of length k if there exists $p \in D$ such that $p+1, \ldots, p+k \notin D$ and $p+k+1 \in D$. An instance of the *general factor problem* (GFP) [Lov70, Lov72] is given by a graph $G = (V, E)$ and a mapping π that maps every vertex $v \in V$ to a degree constraint $\pi(v) \subseteq [\deg_G(v)]$ of arity $\deg_G(v)$. The task is to find a subgraph, if one exists, F of G such that $\deg_F(v) \in \pi(v)$ for every $v \in V$. The case $\pi(v) = \{0, 1\}$ for every $v \in V$ is the matching problem, and the case $\pi(v) = \{1\}$ for every $v \in V$ is the perfect matching problem. Lovász showed that the GFP is NP-complete when the degree constraint $\{0, 3\}$ of arity 3 (and gap 2) occurs [Lov72]. Later, answering a question of Lovász, Cornuéjols showed that the GFP is solvable in polynomial time if each degree constraint has gaps of length at most 1 [Cor88].

In this paper, we consider the *weighted general factor problem* (WGFP) where each edge is assigned a real-valued weight and the task is to find a general factor of the maximum total weight. Since the unweighted version is already hard when a degree constraint with a gap of length more than 1 occurs [Lov72], we only need to consider the WGFP where each degree constraint has gaps of length at most 1. Some cases of the WGFP are reducible to the weighted matching or perfect matching problem by gadget constructions, and hence are polynomial-time solvable. In these cases, the degree constraints are called *matching-realizable* (see Definition A.1). For instance, the degree constraint $D = [b]$ where $b > 0$, for b-matchings is matching realizable [Tut54]. The weighted b-matching problem is interesting in its own right in combinatorial optimization and has been well studied with many elaborate algorithms developed [Pul73, Mar79, Gab83, Ans87, GS13]. Besides b-matchings, Cornuéjols showed that the *parity interval* constraint $D = \{g, g+2, \ldots, f\}$ where $f \geq g \geq 0$ and $f \equiv g \mod 2$, is matching realizable [Cor88], and Szabó showed that the *interval* constraint $D = \{g, g+1, \ldots, f\}$ where $f \geq g \geq 0$, for (g, f)-factors is matching realizable [Sza09]. Thus, the WGFP where each degree constraint is an interval or a parity interval is reducible to the weighted matching problem (with some vertices required to have degree exactly 1) and hence solvable in polynomial-time by Edmonds' algorithm, although Szabó gave a different algorithm for this problem [Sza09]. By reducing the WGFP with interval and parity interval constraints to the weighted (g, f)-factor problem, a faster algorithm was obtained in [DP18] based on Gabow's algorithm [Gab83].

In [Sza09], Szabó further conjectured that the WGFP is solvable in polynomial time without requiring each degree constraint being an interval or a parity interval, as long as each degree constraint has gaps of length at most 1. To prove the conjecture, a natural question is then the following: *Are there other WGFPs that are polynomial-time solvable by a gadget reduction to weighted matchings?* In other words, *are there other degree constraints that are matching realizable?* In this paper, we show that the answer is *no*.

[2]We always associate a degree constraint with an arity. Two degree constraints are different if they have different arities although they may be the same set of integers.

Theorem 1.1. *A degree constraint with gaps of length at most* 1 *is matching realizable if and only if it is an interval or a parity interval.*

Previous results beyond matchings realizable degree constraints With the answer for the above question being negative, new algorithms need to be devised for the WGFP with degree constraints that are not intervals or parity intervals. Unlike the weighted matching problem and the weighted b-matching problem for which various types of algorithms have been developed, only a few algorithms have been presented for the more general and challenging WGFP: For the cardinality version of WGFP, i.e., the WGFP where each edge is assigned weight 1, Dudycz and Paluch introduced a polynomial-time algorithm for this problem with degree constrains having gaps of length at most 1, which leads to a pseudo-polynomial-time algorithm for the WGFP with non-negative integral edge weights [DP18].

The algorithm in [DP18] is based on a structural result showing that if a factor is not optimal, then a factor of larger weight can be found by a local search, which can be done in polynomial time. However, it is not clear how much larger the weight of the new factor is. In order to get an optimal factor, the algorithm needs to repeat local searches iteratively until no better factors can be found, and the number of local searches is bounded by the total edge weight, which makes the algorithm pseudo-polynomial-time. Later, in an updated version [DP21], by carefully assigning edge weights, the algorithm was improved to be weakly polynomial-time with a running time $O(\log Wmn^6)$, where W is the largest edge weight, m is the number of edges and n is the number of vertices.

Our main contribution Independently of [DP21], in this paper, we make a step towards a *strongly* polynomial-time algorithm for the WGPF. Let $p \geq 0$ be an arbitrary integer. Consider the following two types of degree constraints $\{p, p+1, p+3\}$ and $\{p, p+2, p+3\}$ (of arbitrary arity). We will call them *type-1* and *type-2* respectively. These are the "smallest" degree constraints that are not matching realizable.

Theorem 1.2. *There is a strongly polynomial-time algorithm for the* WGFP *with real-valued edge weights where each degree constraint is an interval, a parity interval, a type-1, or a type-2 (of arbitrary arities). The algorithm runs in time $O(n^6)$ for a graph with n vertices.*

In particular, this gives a tractability result for the WGFP with degree constraints that are provably not matching realizable, thus going beyond existing algorithms. The algorithm is a recursive algorithm by reducing the problem to a smaller sub-problem of itself by fixing the parity of degree constraints on vertices. Its correctness is based on a delicate structural result, which is stronger than that of [DP21].[3] Equipped with this result, our algorithm can directly find an *optimal* factor (not just a better one) of an instance of a larger size by performing only one local search from an optimal factor of a smaller instance. Here, what is important is not how to find a better one by a local search (the main result of [DP21]), it is how to ensure that the better one obtained by only one local search is actually optimal under certain assumptions. This is the key to making our algorithm strongly polynomial. In addition, as a by-product, we give a simple proof of the result of [DP21] for the special case of WGFP with interval, parity interval, type-1 and type-2 degree constraints by reducing the problem to WGFP on subcubic graphs and utilizing the equivalence between 2-vertex connectivity and 2-edge connectivity of subcubic graphs.

Let D be a degree constraint of arity at most 3. If $D \neq \{0, 3\}$ then D is an interval, a parity interval, a type-1, or a type-2. Combining with the above-mentioned NP-hardness of the decision case [Lov72], we obtain a complexity dichotomy for the WGFP on subcubic graphs.

[3] The result in [DP21] holds for the more general WGFP with all degree constraints having gaps of length at most 1, while our result only works for the WGFP with interval, parity interval, type-1 and type-2 degree constraints.

Corollary 1.3. *The* WGFP *on subcubic graphs is strongly polynomial-time solvable if the degree constraint* $\{0, 3\}$ *of arity three does not occur. Otherwise, it is NP-hard.*

Relation with edge constraint satisfaction problems The edge constraint satisfaction problem (CSP) is a type of CSPs in which every variable appears in exactly two constraints [Ist97, Fed01]. For the edge-CSP on the Boolean domain, Feder showed that the problem is NP-complete if a constraint that is not a Δ-matroid occurs, except for those that are tractable by Schaefer's dichotomy theorem [Sch78]. In a subsequent line of work [DF03, GIM03, FF06, DK15], tractability of the Boolean edge-CSP has been established for special classes of Δ-matroids, most recently for even Δ-matroids [KKR18]. A complete complexity classification for the Boolean edge-CSP is still open with the conjecture that all Δ-matroids are tractable. The graph factor problem is a special case of the Boolean edge-CSP where every constraint is symmetric (i.e, the value of the constraint only depends on the Hamming weight of its input). For a degree constraint (or a symmetric constraint), it is a Δ-matroid if and only if it has gaps of length at most 1. Thus, the above conjecture holds for the symmetric Boolean edge-CSP by Cornuéjols' result on the general factor problem [Cor88]. A complexity classification for the weighted Boolean edge-CSP is certainly a more challenging goal: The complexity of the weighted Boolean edge-CSP with even Δ-matroids as constraints is still open. Our result in Theorem 1.2 gives a tractability result for the weighted Boolean edge-CSP with certain symmetric Δ-matroids as constraints, and our result in Corollary 1.3 establishes a complexity dichotomy for the weighted Boolean edge-CSP with symmetric constraints of arity no more than 3.

Organization In Section 2, we present basic definitions and notation. In Section 3, we describe our algorithm and give a structural result for the WGFP which ensures the correctness and the strongly polynomial-time running time of our algorithm. In Section 4, we introduce basic augmenting subgraphs as an analogy of augmenting paths for weighed matchings and give a proof of the structural result. The proof is based on a result regarding the existence of certain basic factors for subcubic graphs, for which we give a proof sketch in Section 5. Finally, we discuss matching realizability and its relation with Δ-matroids in Appendix A.

2 Preliminaries

Let $\mathscr{D}$ be a (possibly infinite) set of degree constraints.

Definition 2.1. *The* weighted general factor problem *parameterized by* $\mathscr{D}$*, denoted by* WGFP($\mathscr{D}$)*, is the following computational problem. An instance is a triple* $\Omega = (G, \pi, \omega)$*, where* $G = (V, E)$ *is a graph,* $\pi : V \to \mathscr{D}$ *assigns to every* $v \in V$ *a degree constraint* $D_v \in \mathscr{D}$ *of arity* $\deg_G(V)$*, and* $\omega : E \to \mathbb{R}$ *assigns to every* $e \in E$ *a real-valued weight* $w(e) \in \mathbb{R}$*. The task is to find, if one exists, a general factor* F *of* G *such that the total weight of edges in* F *is maximized.*

The general factor problem GFP($\mathscr{D}$) *is the decision version of* WGFP($\mathscr{D}$)*; i.e., deciding whether a general factor exists or not.*

Suppose that $\Omega = (G, \pi, \omega)$ is a WGFP instance. If F is a general factor of G under π, then we say that F is a factor of Ω, denoted by $F \in \Omega$. In terms of this inclusion relation, Ω can be viewed as a set of subgraphs of G. We extend the edge weight function ω to subgraphs of G. For a subgraph H of G, its weight $\omega(H)$ is $\sum_{e \in E(H)} \omega(e)$ ($\omega(H) = 0$ if H is the empty graph). If H contains an isolated vertex v, then $\omega(H) = \omega(H')$, where H' is the graph obtained from H by removing v. Moreover, $H \in \Omega$ if and only if $H' \in \Omega$. In the following, without other specification, we always assume that a factor does not contain any isolated vertices. The optimal value of Ω, denoted by $\mathrm{Opt}(\Omega)$, is $\max_{F \in \Omega} \omega(F)$. We define $\mathrm{Opt}(\Omega) = -\infty$ if Ω has no factor. A factor F of Ω is *optimal* in Ω if $\omega(F) = \mathrm{Opt}(\Omega)$. For a WGFP instance $\Omega' = (G', \pi', \omega')$, where $G' \subseteq G$[4] and ω' is the restriction of ω on the edges of G', we say Ω' is a *sub-instance* of Ω, denoted by $\Omega' \subseteq \Omega$, if $F \in \Omega$ for every $F \in \Omega'$. In particular, Ω' is a subset of Ω by viewing them as two sets of subgraphs of G. If $\Omega' \subseteq \Omega$, then $\text{Opt}(\Omega') \leq \text{Opt}(\Omega)$.

For two WGFP instances $\Omega_1 = (G, \pi_1, \omega)$ and $\Omega_2 = (G, \pi_2, \omega)$, we use $\Omega_1 \cup \Omega_2$ to denote the union of factors of these two instances, i.e., $\Omega_1 \cup \Omega_2 = \{F \subseteq G \mid F \in \Omega_1 \text{ or } F \in \Omega_2\}$, and $\Omega_1 \cap \Omega_2$ to denote the intersection, i.e., $\Omega_1 \cap \Omega_2 = \{F \subseteq G \mid F \in \Omega_1 \text{ and } F \in \Omega_2\}$. Note that $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2$ are just sets of subgraphs of G and may not define WGFP instances on G.

We use $\mathscr{G}_1$ and $\mathscr{G}_2$ to denote the set of degree constraints that are intervals and parity intervals, respectively, and $\mathscr{T}_1$ and $\mathscr{T}_2$ to denote the set of degree constraints that are type-1 and type-2, respectively. Let $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2$ and $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$. In this paper, we study the problem WGFP($\mathscr{G} \cup \mathscr{T}$).

Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two subgraphs of G. The symmetric difference graph $H_1 \Delta H_2$ is the induced subgraph of G induced by the edge set $E_1 \Delta E_2$. Note that there are no isolated vertices in a symmetric difference graph. When $E_1 \cap E_2 = \emptyset$, we may write $H_1 \Delta H_2$ as $H_1 \cup H_2$. When $E_2 \subseteq E_1$, we may write $H_1 \Delta H_2$ as $H_1 \backslash H_2$.

A *subcubic* graph is defined to be a graph where every vertex has degree $1, 2$ or 3. Unless stated otherwise, we use V_G and E_G to denote the vertex set and the edge set of a graph G, respectively.

Definition 2.2 (2-vertex-connectivity). *A connected graph G is 2-vertex-connected (or 2-connected) if it has more than 2 vertices and remains connected by removing any vertex.*

Menger's Theorem gives an equivalent definition of 2-connectivity, cf. [Die10] for a proof.

Theorem 2.3 (Menger's Theorem). *A connected graph G is 2-connected if and only if for any two vertices of G, there exists two vertex disjoint paths connecting them (i.e., there is a cycle containing these two vertices).*

Definition 2.4 (Bridge and 2-edge-connectivity). *A bridge of a connected graph is an edge whose deletion makes the graph disconnected. A connected graph is 2-edge-connected if it has no bridge.*

The following theorem is the edge version of Menger's Theorem.

Theorem 2.5. *A connected graph G is 2-edge-connected if and only if for any two vertices of G, there exists two edge disjoint paths connecting them.*

If two paths connecting a pair of vertices are vertex-disjoint, then they are also edge-disjoint. Thus, 2-vertex-connectivity implies 2-edge-connectivity. For subcubic graphs, one can check that two edge-disjoint paths are also vertex-disjoint. Thus, for subcubic graphs, 2-vertex-connectivity is equivalent to 2-edge-connectivity. In particular, we have the following result.

Lemma 2.6. *If a connected subcubic graph is not 2-connected, then it contains a bridge.*

The following fact regarding 2-connected graphs will also be used.

Lemma 2.7. *Let $G = (V_G, E_G)$ be a 2-connected graph, $H = (V_H, E_H) \subseteq G$, and $u \in V_H$. If $\deg_H(u) = 2 < \deg_G(u) = 3$, then there exists a path $p_{uw} = (V_{p_{uw}}, E_{p_{uw}}) \subseteq G$ with endpoints u and w for some $w \in V_H$ such that $E_{p_{uw}} \cap E_H = \emptyset$.*

[4]We use the term "subgraph" and notation $G' \subseteq G$ throughout for the standard meaning of a "normal" subgraph i.e., if $G = (V', E')$ and $G = (V, E)$ then $G' \subseteq G$ means $V' \subseteq V$ and $E' \subseteq E$.

Proof. Since $\deg_H(u) = 2 < \deg_G(u) = 3$, there is an edge $e_{vu} = (v, u) \in E_G$ incident to u such that $e_{vu} \notin E_H$. If $v \in V_H$, then the edge e_{vu} is the desired path. Thus, we may assume that $v \notin V_H$. Since G is 2-connected, there is a path p_{vu} with endpoints v and u such that $e_{vu} \notin E_{p_{vu}}$. Since $u \in V_H$, $V_{p_{vu}} \cap V_H \neq \emptyset$. Let w be the first vertex in the path p_{vu} (within the order of traversing the path from v to u) belonging to V_H. Then, $w \neq u$ since $e_{vu} \notin E_{p_{vu}}$ and $\deg_G(u) = 3$. Also, $w \neq v$ since $v \notin V_H$. Let $p_{vw} \subsetneq p_{vu}$ be the segment with endpoints v and w. Then, $E_{p_{vw}} \cap E_H = \emptyset$. Let p_{uw} be the path consisting of e_{vu} and p_{vw}. It has endpoints $u, w \in V_H$, and $E_{p_{uw}} \cap E_H = \emptyset$. $\square$

3 Algorithm

We give a recursive algorithm for the problem WGFP($\mathscr{G} \cup \mathscr{T}$), using the problems WGFP($\mathscr{G}$) and the decision problem GFP($\mathscr{G} \cup \mathscr{T}$) as oracles. Given an instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G} \cup \mathscr{T}$), we define the following sub-instances of $\Omega = (G, \pi, \omega)$ that will be used in the recursion. Recall that V_G denotes the vertex set of the underlying graph G. Let T_Ω denote the set $\{v \in V_G \mid \pi(v) \in \mathscr{T}\}$. (We may omit the subscript Ω of T_Ω when it is clear from the context.)

For every vertex $v \in T_\Omega$, we split the instance Ω in two by splitting the degree constraint $\pi(v)$ in two parity intervals. More precisely, we define

$$D_v^0 = \{p_v + 1, p_v + 3\} \text{ and } D_v^1 = \{p_v\} \quad \text{if} \quad \pi(v) = \{p_v, p_v + 1, p_v + 3\} \in \mathscr{T}_1;$$
$$D_v^0 = \{p_v, p_v + 2\} \text{ and } D_v^1 = \{p_v + 3\} \quad \text{if} \quad \pi(v) = \{p_v, p_v + 2, p_v + 3\} \in \mathscr{T}_2.$$

We have $D_v^0, D_v^1 \in \mathscr{G}_2$. For $i \in \{0, 1\}$ and $v \in T_\Omega$, we define $\Omega_v^i = (G, \pi_v^i, \omega)$ to be the sub-instance of Ω where $\pi_v^i(x) = \pi(x)$ for every $x \in V_G \backslash \{v\}$ and $\pi_v^i(v) = D_v^i$. Then, for every $v \in T_\Omega$, we have $\Omega_v^0 \cap \Omega_v^1 = \emptyset$ and $\Omega_v^0 \cup \Omega_v^1 = \Omega$. Moreover, $T_{\Omega_v^0} = T_{\Omega_v^1} = T_\Omega \backslash \{v\}$.

Let F be a factor of Ω. Similarly to above, one can partition Ω into $2^{|T_\Omega|}$ many sub-instances according to F such that each one is an instance of WGFP($\mathscr{G}$) – for each $v \in T_\Omega$, we choose one of the two splits of $\pi(v)$ as above. (We note that the algorithm will not consider all exponentially many sub-instances.) In detail, for every vertex $v \in T_\Omega$, we define $D_v^F = D_v^i$ where $\deg_F(v) \in D_v^i$ as follows:

$$
\begin{aligned}
D_v^F &= \{p_v\} && \text{if} \quad \pi(v) = \{p_v, p_v + 1, p_v + 3\} \in \mathscr{T}_1 \text{ and } \deg_F(v) = p_v,\\
D_v^F &= \{p_v + 1, p_v + 3\} && \text{if} \quad \pi(v) = \{p_v, p_v + 1, p_v + 3\} \in \mathscr{T}_1 \text{ and } \deg_F(v) \neq p_v;\\
D_v^F &= \{p_v + 3\} && \text{if} \quad \pi(v) = \{p_v, p_v + 2, p_v + 3\} \in \mathscr{T}_2 \text{ and } \deg_F(v) = p_v + 3,\\
D_v^F &= \{p_v, p_v + 2\} && \text{if} \quad \pi(v) = \{p_v, p_v + 2, p_v + 3\} \in \mathscr{T}_2 \text{ and } \deg_F(v) \neq p_v + 3.
\end{aligned}
$$

By definition, $\deg_F(v) \in D_v^F \subseteq \pi(v)$ and $D_v^F \in \mathscr{G}_2$. In fact, D_v^F is the maximal set such that $\deg_F(v) \in D_v^F \subseteq \pi(v)$ and $D_v^F \in \mathscr{G}_2$. One can also check that for every $v \in T$, $\pi(v) \backslash D_v^F \in \mathscr{G}_2$, and moreover for every $p \in D_v^F$ and $q \in \pi(v) \backslash D_v^F$, $p \not\equiv q \mod 2$.

For every $W \subseteq T_\Omega$, we define $\Omega_W^F = (G, \pi_W^F, \omega)$ to be the sub-instance of Ω where

$$
\begin{aligned}
\pi_W^F(v) &= \pi(v) \backslash D_v^F && \text{for} \quad v \in W,\\
\pi_W^F(v) &= D_v^F && \text{for} \quad v \in T_\Omega \backslash W,\\
\pi_W^F(v) &= \pi(v) && \text{for} \quad v \in V \backslash T_\Omega.
\end{aligned}
\tag{1}
$$

By definition, for every W, Ω_W^F is an instance of WGFP($\mathscr{G}$). Moreover, we have $\cup_{W \subseteq T} \Omega_W^F = \Omega$ and $\Omega_{W_1}^F \cap \Omega_{W_2}^F = \emptyset$ for every $W_1 \neq W_2$. Thus, $\{\Omega_W^F\}_{W \subseteq T_\Omega}$ is a partition of Ω (viewed as a set of subgraphs of G). When $W = \emptyset$, we write Ω_W^F as Ω^F, and when $W = \{s\}$ or $W = \{s, t\}$, we write Ω_W^F as Ω_s^F or $\Omega_{s,t}^F$ respectively for simplicity.

Algorithm 1: Finding an optimal factor for an instance of WGFP($\mathscr{G} \cup \mathscr{T}$)

Function `Decision`**:**
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G} \cup \mathscr{T}$).
  Output: A factor of Ω, or "No" if Ω has no factor.

Function `Optimization`**:**
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G}$).
  Output: An optimal factor of Ω, or "No" if Ω has no factor.

Function `Main`**:**
  Input : An instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G} \cup \mathscr{T}$).
  Output: An optimal factor $F \in \Omega$, or "No" if Ω has no factor.

  $T \leftarrow \{v \in V \mid \pi(v) \in \mathscr{T}\}$;
  if T is the empty set **then**
    return `Optimization` (Ω);
  else
    Arbitrarily pick $u \in T$;
    if `Decision` (Ω_u^0) returns "No" **then**
      return `Main` (Ω_u^1);
    else
      $F^{\text{opt}} \leftarrow$ `Main` (Ω_u^0);
      foreach $v \in T$ **do**
        `// Elements of T can be traversed in an arbitrary order.`
        $W \leftarrow \{u\} \cup \{v\}$;
        if `Optimization`$(\Omega_W^{F^{\text{opt}}}) \neq$ "No" **then** $F' \leftarrow$ `Optimization`$(\Omega_W^{F^{\text{opt}}})$;
        if $\omega(F') > \omega(F^{\text{opt}})$ **then** $F^{\text{opt}} \leftarrow F'$;
      end
      return F^{opt};
    end
  end

Our algorithm is given in Algorithm 1.

The key that makes our algorithm running in strongly polynomial time is the following structural result (Theorem 3.1) for the problem WGFP($\mathscr{G} \cup \mathscr{T}$). It says that given an optimal factor F of Ω_u^0 for some $u \in T_\Omega$, if F is not optimal in Ω, then we can directly find an optimal factor of Ω by searching at most n sub-instances of Ω which are in WGFP($\mathscr{G}$). Note that the number of searches is independent of the edge weights. Thus, the problem of finding an optimal factor in Ω can be reduced to finding an optimal factor in Ω_u^0, where there is one fewer vertex u with constraints in $\mathscr{T}$. By recursively reducing an instance to another with fewer vertices with constraints in $\mathscr{T}$, we eventually get an instance of WGFP($\mathscr{G}$) which can be solved in polynomial-time. This leads to a strongly polynomial time algorithm for finding an optimal factor.

Theorem 3.1. *Suppose that $\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathscr{G} \cup \mathscr{T}$)*, F is a factor of Ω and F is optimal in Ω_u^0 for some $u \in T_\Omega$. Then a factor F' is optimal in Ω if and only if $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or $|W| = 2$.*

In other words, if F is not optimal in Ω, then there is an optimal factor of Ω which belongs to Ω_W^F for some W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or $|W| = 2$.

Remark 3.2. *This result is* stronger *than the main result (Theorem 2) of [DP21], and it is* not *simply implied by [DP21]. To clarify this, we give a simple proof outline of Theorem 3.1 here.*

In order to prove Theorem 3.1, it suffices to prove the direction that if $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2, then F' is optimal in Ω. We prove this by contradiction. Suppose that F' is not optimal in Ω, and F^* is an optimal factor of Ω. Then, $\omega(F^*) > \omega(F') \geq \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$. Also, $\omega(F^*) \notin \Omega_u^0$ since $\omega(F^*) > \omega(F) = \mathrm{Opt}(\Omega_u^0)$. Thus, $\deg_{F^*}(u) \not\equiv \deg_F(u) \mod 2$.

By [DP21], a canonical path *$M \subseteq F \Delta F^*$ with positive weight[5] can be found, and then $F \Delta M$ is a factor of Ω with larger weight than F and $F \Delta M \in \Omega_W^F$ for some $W \subseteq T_\Omega$ where $|W| \leq 2$. However, this does not lead to a contradiction. To get a contradiction, we need to show that the positive weighted canonical path M (a basic augmenting subgraph) further satisfies $\deg_M(u) \equiv 0 \mod 2$. Then, $\deg_{F\Delta M}(u) \equiv \deg_F(u) \mod 2$. Thus, $F \Delta M$ is a factor with larger weight than F and $F \Delta M \in \Omega_u^0$, which contradicts with F being optimal in Ω_u^0.*

The existence of a basic augmenting subgraph M satisfying $\deg_M(u) \equiv 0 \mod 2$ is formally stated in the second property of Lemma 4.4. The main technical part of the paper (Section 5.2 of the full paper) is devoted to prove it. In Section 5 of this short version, we give an example to illustrate the proof ideas. The existence of such a basic augmenting subgraph is highly non-trivial. In fact, it does not *hold anymore after a subtle change of the condition "F is optimal in $\Omega_u^{\mathbf{0}}$" to "F is optimal in $\Omega_u^{\mathbf{1}}$" for some $u \in T_\Omega$. We give the following example (see Figure 1) to show this.*

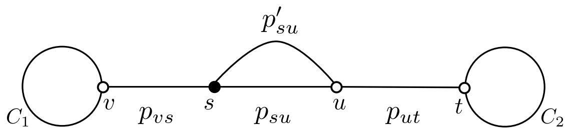

Figure 1: An example that violates Theorem 3.1 when F is optimal in Ω_u^1 instead of Ω_u^0

In this instance, $\pi(u) = \pi(v) = \pi(t) = \{0, 1, 3\}$ (denoted by hollow nodes) and $\pi(s) = \{0, 2, 3\}$ (denoted by the solid node), and $\omega(C_1) = \omega(p_{vs}) = \omega(p_{su}) = \omega(p'_{su}) = \omega(p_{ut}) = \omega(C_2) = 1$. Inside the cycles C_1 and C_2, and the paths p_{vs}, p_{su}, p_{ut}, and p'_{su}, there are other vertices of degree 2 with the degree constraint $\{0, 2\}$ so that the graph G is simple. We omit these vertices of degree 2 in Figure 1. In this case, $T_\Omega = \{u, v, s, t\}$. Consider the sub-instance $\Omega_u^1 = (G, \pi_u^1, \omega)$. We have $\pi_u^1(u) = D_u^1 = \{0\}$ since $\pi(u) = \{0, 1, 3\}$. Then, the only factor F of Ω_u^1 is the empty graph (assuming there are no isolated vertices in factors), and F is not optimal in Ω. Also, the only optimal factor of Ω is the graph G and $G \in \Omega_{T_\Omega}^F$ where $|T_\Omega| = 4$. Clearly, $\deg_G(u) \not\equiv \deg_F(u) \mod 2$. One can check that for any factor F' of Ω with larger weight than F, $\deg_{F'}(u) \not\equiv \deg_F(u) \mod 2$. In other words, there is no basic augmenting subgraph M such that $\deg_M(u) \equiv 0 \mod 2$. Moreover, one can check that in this case, Theorem 3.1 also does not hold. In other words, the existence of a basic augmenting subgraph satisfying $\deg_M(u) \equiv 0 \mod 2$ is crucial for the correctness of Theorem 3.1.

Using Theorem 3.1, we now prove that Algorithm 1 is correct.

Lemma 3.3. *Given an instance $\Omega = (G, \pi, \omega)$ of* $\mathrm{WGFP}(\mathscr{G}, \mathscr{T})$*, Algorithm 1 returns either an optimal factor of Ω, or "No" if Ω has no factor.*

Proof. Recall that for an instance $\Omega = (G, \pi, \omega)$, we define $T_\Omega = \{v \in V_G \mid \pi(v) \in \mathscr{T}\}$ where V_G is the vertex set of G . We prove the correctness by induction on the $|T_\Omega|$.

If $|T_\Omega| = 0$, Ω is an instance of $\mathrm{WGFP}(\mathscr{G})$. Algorithm 1 simply returns `Optimization` (Ω). By the definition of the function `Optimization`, the output is correct.

[5] See definition 3 of [DP21]. They are defined as basic augmenting subgraphs (Definition 4.3) in this paper.

Suppose that Algorithm 1 returns correct results for all instances Ω' of WGFP$(\mathscr{G}, \mathscr{T})$ where $|T_{\Omega'}| = k$. We consider an instance Ω of WGFP$(\mathscr{G}, \mathscr{T})$ where $|T_\Omega| = k+1$. Algorithm 1 first calls the function `Decision` (Ω_u^0) for some arbitrary $u \in T$.

We first consider the case that `Decision` (Ω_u^0) returns "No". By the definition, Ω_u^0 has no factor. Moreover, since $\Omega = \Omega_u^0 \cup \Omega_u^1$, we have $F \in \Omega$ if and only if $F \in \Omega_u^1$. Then, a factor $F \in \Omega_u^1$ is optimal in Ω if and only if it is optimal in Ω_u^1. Note that Ω_u^1 is an instance of WGFP$(\mathscr{G}, \mathscr{T})$ where $|T_{\Omega_u^1}| = k$. By the induction hypothesis, Algorithm 1 returns a correct result `Main` (Ω_u^1) for the instance Ω_u^1, which is also a correct result for the instance Ω.

Now, we consider the case that `Decision` (Ω_u^0) returns a factor of Ω_u^0. Then, `Main` (Ω_u^0) returns an optimal factor F of Ω_u^0. After the loop (lines 13 to 17) in Algorithm 1, we get a factor F^{opt} of Ω such that $\omega(F^{\text{opt}}) \geq \text{Opt}(\Omega_W^F)$ for every $u \in W \subseteq T_\Omega$ where $|W| = 1$ (when $u = v$) or $|W| = 2$ (when $u \neq v$) and $\omega(F^{\text{opt}}) \geq \omega(F)$. By Theorem 3.1, F^{opt} is an optimal factor of Ω. Thus, Algorithm 1 returns a correct result. □

Now, we consider the time complexity of Algorithm 1. The size of an instance is defined to be the number of vertices of the underlying graph of the instance.

Lemma 3.4. *Run Algorithm 1 on an instance $\Omega = (G, \pi, \omega)$ of size n. Then,*

- *the algorithm will stop the recursion after at most n recursive steps;*
- *the algorithm will call* `Decision` *at most n many times, call* `Optimization` *at most $\frac{n(n+1)}{2} + 1$ many times, and perform at most $\frac{n(n+1)}{2}$ many comparisons;*
- *the algorithm runs in time $O(n^6)$.*

Proof. Let $\Omega^k = \{G, \pi^k, \omega\}$ be the instance after k many recursive steps. Here $\Omega^0 = \Omega$. Recall that $T_{\Omega^k} = \{v \in V \mid \pi^k(v) \in \mathscr{T}\}$. For an instance Ω^k with $|T_{\Omega^k}| > 0$, the recursive step will then go to the instance $(\Omega^k)_u^0$ or $(\Omega^k)_u^1$ for some $u \in T_{\Omega^k}$. Thus, $\Omega^{k+1} = (\Omega^k)_u^0$ or $(\Omega^k)_u^1$. In both cases, $T_{\Omega^{k+1}} = T_{\Omega^k} \backslash \{u\}$ and hence $|T_{\Omega^{k+1}}| = |T_{\Omega^k}| - 1$. By design, the algorithm will stop the recursion and return `Optimization` (Ω^m) when it reaches an instance Ω^m with $|T_{\Omega^m}| = 0$. Thus, #recursive steps $= m = |T_\Omega| - 0 \leq |V| = n$.

To prove the second item, we consider the number of operations inside the recursive step for the instance $\Omega^k = \{G, \pi^k, \omega\}$. Note that $k \leq n$ and $|T_{\Omega^k}| = |T_\Omega| - k \leq n - k$. If $|T_{\Omega^k}| = 0$, then the algorithm will simply call `Optimization` once. If $|T_{\Omega^k}| > 0$, then inside the recursive step, the algorithm will call `Decision` once, and call `Optimization` once or $|T_{\Omega^k}|$ many times depending on the answer of `Decision`. Moreover, in the later case, the algorithm will also perform $|T_{\Omega^k}|$ many comparisons. Thus,

$$\#\text{calls of } \texttt{Decision} = \sum_{|T_{\Omega^k}|>0} 1 = \sum_{i=1}^{|T_\Omega|} 1 = |T_\Omega| \leq n.$$

$$\#\text{calls of } \texttt{Optimization} \leq 1 + \sum_{|T_{\Omega^k}|>0} |T_{\Omega^k}| = 1 + \sum_{i=1}^{|T_\Omega|} i \leq \frac{n(n+1)}{2} + 1.$$

$$\#\text{comparisons} \leq \sum_{|T_{\Omega^k}|>0} |T_{\Omega^k}| \leq \frac{n(n+1)}{2}$$

Let $t_{\texttt{Main}}(n)$ denote the running time of Algorithm 1 on an instance of size n, and $t_{\texttt{Dec}}(n)$ and $t_{\texttt{Opt}}(n)$ denote the running time of algorithms for the functions `Decision` and `Optimization`, respectively. Then, $t_{\texttt{Dec}}(n) = O(n^4)$ by the algorithm in [Cor88] and $t_{\texttt{Opt}}(n) = O(n^4)$ by the algorithm in [DP18]. Thus, $t_{\texttt{Main}}(n) \leq n t_{\texttt{Dec}}(n) + \frac{n(n+1)+2}{2} t_{\texttt{Opt}}(n) + \frac{n(n+1)}{2} = O(n^6)$. □

4 Proof of Theorem 3.1

In this section, we give a proof of Theorem 3.1. The general strategy is that starting with a non-optimal factor F of an instance $\Omega = (G, \omega, \pi)$, we want to find a subgraph H of G such that by taking the symmetric difference $F\Delta H$, we get another factor of Ω with larger weight. The existence of such subgraphs is trivial (Lemma 4.2). However, the challenge is how to find one efficiently. As an analogy of augmenting paths in the weighted matching problem, we introduce basic augmenting subgraphs (Definition 4.3) for the weighted graph factor problem, which can be found efficiently. We will show that given a non-optimal factor F, a basic augmenting subgraph always exists (Lemma 4.4, property 1). Then, we can efficiently improve the factor F to another factor with larger weight. As shown in [DP18], this already gave a weakly-polynomial time algorithm. However, the existence of basic augmenting subgraphs is not enough to get a strongly polynomial-time algorithm, which requires the number of improvement steps being independent of edge weights. Thus, in order to prove Theorem 3.1, which leads to a strongly polynomial-time algorithm, we further establish that there exists a basic augmenting subgraph that satisfies certain stronger properties under suitable assumptions (Lemma 4.4, property 2). This result will imply Theorem 3.1.

Definition 4.1 (F-augmenting subgraphs)**.** *Suppose that F is a factor of an instance $\Omega = (G, \pi, \omega)$. A subgraph H of G is F-augmenting if $F\Delta H \in \Omega$ and $\omega(F\Delta H) - \omega(F) > 0$.*

Lemma 4.2. *Suppose that F is a factor of an instance Ω. If F is not optimal in Ω, then there exists an F-augmenting subgraph.*

Proof. Since F is not optimal, there is some $F' \in \Omega$ such that $\omega(F') > \omega(F)$. Let $H = F\Delta F'$. We have $F\Delta H = F' \in \Omega$ and $\omega(H) = \omega(F') - \omega(F) > 0$. Thus, H is F-augmenting. □

Recall that for an instance $\Omega = (G, \pi, \omega)$ of WGFP($\mathscr{G} \cup \mathscr{T}$), T_Ω is the set $\{v \in V_G \mid \pi(v) \in \mathscr{T}\}$. For two factors $F, F^* \in \Omega$, we define $T_\Omega^{F\Delta F^*} = \{v \in T_\Omega \mid \deg_{F\Delta F^*}(v) \equiv 1 \mod 2\} = \{v \in T_\Omega \mid \deg_F(v) \not\equiv \deg_{F^*}(v) \mod 2\}$.

Definition 4.3 (Basic augmenting subgraphs)**.** *Suppose that $\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathscr{G}, \mathscr{T}$)*, and F and F^* are factors of Ω with $\omega(F) < \omega(F^*)$. An F-augmenting subgraph $H = (V_H, E_H)$ is (F, F^*)*-basic *if $H \subseteq F\Delta F^*$, $|V_H^{\text{odd}}| \le 2$, and $V_H^{\text{odd}} \cap T_\Omega \subseteq T_\Omega^{F\Delta F^*}$ where $V_H^{\text{odd}} = \{v \in V_H \mid \deg_H(v) \equiv 1 \mod 2\}$.*

Lemma 4.4. *Suppose that $\Omega = (G, \pi, \omega)$ is an instance of* WGFP($\mathscr{G} \cup \mathscr{T}$)*, and F and F^* are two factors of Ω.*

1. *If $\omega(F^*) > \omega(F)$, then there exists an (F, F^*)-basic subgraph.*
2. *If $\omega(F^*) > \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega^{F\Delta F^*}$ with $|W| \le 2$, and $T_\Omega^{F\Delta F^*}$ contains a vertex u such that $F \in \Omega_u^0$ (i.e., $\deg_F(u) \in D_u^0$), then there exists an (F, F^*)-basic subgraph H where $\deg_H(u) \equiv 0 \mod 2$.*

Remark 4.5. *The first property of Lemma 4.4 implies the following: a factor $F \in \Omega$ is optimal if and only if $\omega(F) \ge \mathrm{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ with $|W| \le 2$. This is a special case of the main result (Theorem 2) of [DP18] where the authors consider the* WGFP *for all constraints with gaps of length at most 1. The second property of Lemma 4.4 is more refined than the first property and it implies our main result (Theorem 3.1). In this paper, as a by-product of the proof of property 2, we give a simple proof of Theorem 2 of [DP18] for the special case* WGFP($\mathscr{G} \cup \mathscr{T}$) *based on certain properties of cubic graphs.*

Using the second property of Lemma 4.4, we can prove Theorem 3.1.

Theorem (Theorem 3.1)**.** *Suppose that* $\Omega = (G, \pi, \omega)$ *is an instance of* WGFP$(\mathscr{G} \cup \mathscr{T})$*,* F *is a factor of* Ω *and* F *is optimal in* Ω_u^0 *for some* $u \in T_\Omega$*. Then a factor* F' *is optimal in* Ω *if and only if* $\omega(F') \geq \omega(F)$ *and* $\omega(F') \geq \text{Opt}(\Omega_W^F)$ *for every* W *where* $u \in W \subseteq T_\Omega$ *and* $|W| = 1$ *or* 2*.*

Proof. If F' is optimal in Ω, then clearly $\omega(F') \geq \omega(F)$ and $\omega(F') \geq \text{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2. Thus, to prove the theorem, it suffices to prove the other direction. Since $\omega(F') \geq \omega(F)$ and F is optimal in Ω_u^0, we have $\omega(F') \geq \text{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $u \notin W$ and $|W| \leq 2$. Also, since $\omega(F') \geq \text{Opt}(\Omega_W^F)$ for every W where $u \in W \subseteq T_\Omega$ and $|W| = 1$ or 2, we have $\omega(F') \geq \text{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$.

For a contradiction, suppose that F' is not optimal in Ω. Let F^* be an optimal factor of Ω. Then, $\omega(F^*) > \omega(F')$. Thus, $\omega(F^*) > \omega(F') \geq \text{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega$ where $|W| \leq 2$. Also, $\omega(F^*) \notin \Omega_u^0$ since $\omega(F^*) > \omega(F)$ and F is optimal in Ω_u^0. Thus, $\deg_{F^*}(u) \not\equiv \deg_F(u) \mod 2$. Then, $T_\Omega^{F\Delta F^*}$ contains the vertex u such that $F \in \Omega_u^0$. By Lemma 4.4, there exists an (F, F^*)-basic subgraph H where $\deg_H(u) \equiv 0 \mod 2$. Let $F'' = F \Delta H$. Then $F'' \in \Omega$ and $\omega(F'') > \omega(F)$. Also, $F'' \in \Omega_u^0$ since $\deg_{F''}(u) \equiv \deg_F(u) \mod 2$. This is a contradiction with F being optimal in Ω_u^0. □

Now it suffices to prove Lemma 4.4. By a type of normalization maneuver, we can transfer any instance of WGFP$(\mathscr{G}, \mathscr{T})$ to an instance of WGFP$(\mathscr{G}, \mathscr{T})$ defined on subcubic graphs, called a key instance (Definition 4.6). Recall that a subcubic graph is a graph where every vertex has degree 1, 2 or 3. For key instances, there are five possible forms of basic augmenting subgraphs, called basic factors (Definition 4.7). Then, the crux of the proof of Lemma 4.4 is to establish the existence of certain basic factors of key instances (Theorem 4.8).

Definition 4.6 (Key instance)**.** *A* key instance $\Omega = (G, \pi, \omega)$ *is an instance of* WGFP$(\mathscr{G}, \mathscr{T})$ *where* G *is a subcubic graph, and for every* $v \in V_G$*,* $\pi(v) = \{0, 1\}$ *if* $\deg_G(v) = 1$*,* $\pi(v) = \{0, 2\}$ *if* $\deg_G(v) = 2$*, and* $\pi(v) = \{0, 1, 3\}$ *(i.e., type-1) or* $\{0, 2, 3\}$ *(i.e., type-2) if* $\deg_G(v) = 3$*. We say a vertex* $v \in V_G$ *of degree* 3 *is of type-1 or type-2 if* $\pi(v)$ *is type-1 or type-2 respectively. We say a vertex* $v \in V_G$ *of any degree is 1-feasible or 2-feasible if* $1 \in \pi(v)$ *or* $2 \in \pi(v)$ *respectively.*

Definition 4.7 (Basic factor)**.** *Let* Ω *be a key instance. A factor of* Ω *is a* basic factor *if it is in one of the following five forms.*

1. *A path, i.e., a tree with two vertices of degree 1 (called endpoints) and all other vertices, if there exists any, of degree 2.*
2. *A cycle, i.e., a graph consisting of two vertex disjoint paths with the same two endpoints.*
3. *A tadpole graph, i.e., a graph consisting of a cycle and a path such that they intersect at one endpoint of the path.*
4. *A dumbbell graph, i.e., a graph consisting of two vertex disjoint cycles and a path such that the path intersects with each cycle at one of its endpoints.*
5. *A theta graph (i.e., a graph consisting of three vertex disjoint paths with the same two endpoints) where one vertex of degree 3 is of type-1, and the other vertex of degree 3 is of type-2.*

Theorem 4.8. *Suppose that* $\Omega = (G, \pi, \omega)$ *is a key instance.*

1. *If* $\omega(G) > 0$*, then there is a basic factor* F *of* Ω *such that* $\omega(F) > 0$*.*

2. *If $\omega(G) > 0$, $\omega(G) > \omega(F)$ for every basic factor F of Ω, and G contains a vertex u with $\deg_G(u) = 1$ or $\deg_G(u) = 3$ and $\pi(u) = \{0,2,3\}$, then there is a basic factor F^* of Ω such that $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \mod 2$. (Recall that $\deg_{F^*}(u) = 0$ if $u \notin V_{F^*}$).*

Remark 4.9. *For the second property of Theorem 4.8, the requirement of $\pi(u) = \{0,2,3\}$ when $\deg_G(u) = 3$ is crucial. Consider the instance $\Omega = (G,\pi,\omega)$ as shown in Figure 1. It is easy to that Ω is a key instance. In this case, it can be checked that $\omega(G) = 6 > 0$ and $\omega(G) > \omega(F)$ for every basic factor F of Ω. However, there is no basic factor F^* of Ω such that $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \bmod 2$. Thus, the second property does* not *hold for a vertex u where $\deg_G(u) = 3$ and $\pi(u) = \{0,1,3\}$.*

We will now describe the normalization maneuver, and use Theorem 4.8 to prove Lemma 4.4.

Proof of Lemma 4.4. Recall that F and F^* are two factors of the instance $\Omega = (G,\pi,\omega)$ (not necessarily a key instance). Consider the subgraph $G_\Delta = F\Delta F^*$ of G. $G_\Delta = (V_{G_\Delta}, E_{G_\Delta})$ is not necessarily a subcubic graph. In order to invoke Theorem 4.8, we modify G_Δ to a subcubic graph G^s, and construct a key instance $\Omega^s = (G^s, \pi^s, \omega^s)$ on it.

For every $v \in V_{G_\Delta}$, we consider the set of edges incident to v in G_Δ, denoted by E_v. Since $G_\Delta = F\Delta F^*$, we have $E_v \subseteq E_{G_\Delta} = E_F \Delta E_{F^*}$, where E_F and E_{F^*} are the edge sets of the factors F and F^* respectively. If there is a pair of edges $e, e^* \in E_v$ such that $e \in E_F$ and $e^* \in E_{F^*}$, then we perform the following *separation* operation for this pair of edges. Suppose that $e = (v,u)$ and $e^* = (v,u^*)$; we add a new vertex v^1 to the graph, and replace the edges e and e^* by (v^1,u) and (v^1,u^*) respectively. We label the vertex v^1 (of degree 2) by $\pi^s(v^1) = \{0,2\}$. With a slight abuse of notation, we may still use e and e^* to denote these two new edges, and also use E_{G_Δ} to denote the set of all edges of the new graph.

For each E_v, keep doing the separation operations for pairs of edges of which one is in E_F and the other is in E_{F^*} until all the remaining edges in E_v are in E_F or in E_{F^*} We use $E_v^{\mathfrak{r}}$ to denote the set of remaining edges. It is possible that $E_v^{\mathfrak{r}}$ is empty. Let $P_v^1, \dots, P_v^k$ be the pairs of edges that have been separated, and $v^1, \dots, v^k$ be the added vertices (k can be zero). Note that all these new vertices are of degree 2, and are labeled by $\{0,2\}$. Now, we have the partition $E_v = P_v^1 \cup \cdots \cup P_v^k \cup E_v^{\mathfrak{r}}$. Let $r = |E_v^{\mathfrak{r}}|$. Then $r = |\deg_F(v) - \deg_{F^*}(v)|$. Note that r is even if $\pi(v) \in \mathscr{G}_2$, and $r \le 3$ if $\pi(v) \in \mathscr{T}$. We deal with edges in $E_v^{\mathfrak{r}}$ according to r and $\pi(v)$.

- If $r = 0$, then v is an isolated vertex in the current graph, and we simply remove it. Consider an arbitrary subgraph H of the original G_Δ induced by a union of some pairs of edges in $P_v^1, \dots, P_v^k$. Then, for the subgraph $F\Delta H$ of G_Δ, we have
$$\deg_{F\Delta H}(v) = \deg_F(v) \in \pi(v).$$
- If $r \neq 0$ and $\pi(v) \in \mathscr{G}_1$, then we replace the vertex v with r many new vertices, and replace the r many edges incident to v by r many edges incident to these new vertices such that each vertex has degree 1. We label every new vertex by $\{0,1\}$. Suppose that $L = \min\{\deg_F(v), \deg_{F^*}(v)\}$ and $U = \max\{\deg_F(v), \deg_{F^*}(v)\}$. Since $\pi(v) \in \mathscr{G}_1$, $\{L, L+1, \dots, U\} \subseteq \pi(v)$. Consider an arbitrary subgraph $H \subseteq G_\Delta$ induced by a union of some pairs of edges in $P_v^1, \dots, P_v^k$ and a subset of $E_v^{\mathfrak{r}}$. Then, for the subgraph $F\Delta H$ of G_Δ, we have
$$\deg_{F\Delta H}(v) \in \{L, L+1, \dots, U\} \in \pi(v).$$
- If $r \neq 0$ and $\pi(v) \in \mathscr{G}_2 \backslash \mathscr{G}_1$, then we replace the vertex v with $r/2$ many vertices, and replace the r many edges incident to v by r many edges incident to these new vertices such that each vertex has degree 2. (We can partition these r many edges into arbitrary

pairs.) We label every new vertex by $\{0, 2\}$. Suppose that $L = \min\{\deg_F(v), \deg_{F^*}(v)\}$ and $U = \max\{\deg_F(v), \deg_{F^*}(v)\}$. Since $\pi(v) \in \mathscr{G}_2$, $\{L, L+2, \ldots, U\} \subseteq \pi(v)$. Consider an arbitrary subgraph $H \subseteq G_\Delta$ induced by a union of some pairs of edges in $P_v^1, \ldots, P_v^k$ and an even-size subset of E_v^{r}. Then, for the subgraph $F\Delta H$ of G_Δ, we have

$$\deg_{F\Delta H}(v) \in \{L, L+2, \ldots, U\} \in \pi(v).$$

- If $r \neq 0$ and $\pi(v) \in \mathscr{T}$, then there are three subcases. If $r = 1$, then v has degree 1 in the current graph. We label it by $\pi^s(v) = \{0, 1\}$. If $r = 2$, then v has degree 2 in the current graph. We label it by $\pi^s(v) = \{0, 2\}$. If $r = 3$, then v has degree 3 in the current graph. We label it by $\pi^s(v) = \{0, 1, 3\}$ if $\deg_F(v) \in D_v^1$, and $\pi^s(v) = \{0, 2, 3\}$ if $\deg_F(v) \in D_v^0$. Consider an arbitrary subgraph $H \subseteq G_\Delta$ induced by a union of some pairs of edges in $P_v^1, \ldots, P_v^k$ and a subset I of E_v^{r} where $|I| \subseteq \pi^s(v)$. Then, for the subgraph $F\Delta H$ of G_Δ, we have
$$\deg_{F\Delta H}(v) \in \pi(v).$$

Now, we get a subcubic graph $G^s = (V_{G^s}, E_{G^s})$ from G_Δ. Each vertex v in G_Δ is replaced by a set of new vertices in G^s, denoted by $S(v)$.

- If $\pi(v) \in \mathscr{G}_1$, then $S(v)$ consists of vertices of degree 2 or 1.
- If $\pi(v) \in \mathscr{G}_2$, then $S(v)$ consists of vertices of degree 2.
- If $\pi(v) \in \mathscr{T}$, then $S(v)$ consists of vertices of degree 2 and possibly a vertex of degree r where $r = |\deg_F(v) - \deg_{F^*}(v)| \leq 3$ (there is no such a vertex if $r = 0$). In particular, if $\deg_F(v) - \deg_{F^*}(v) \equiv 0 \mod 2$, then $S(v)$ consists of vertices of degree 2.

In all cases, we have $\deg_{G_\Delta}(v) = \sum_{x \in S(v)} \deg_{G^s}(x)$. Each edge (u, v) in G_Δ is replaced by an edge $(u^s, v^s) \in G_\Delta$ where $u^s \in S(u)$ and $v^s \in S(v)$. Once we get G^s from G_Δ, it is clear that there is a natural one-to-one correspondence between edges in G^s and edges in G_Δ. Without causing ambiguity, when we say an edge or an edge set in G^s, we may also refer it to the corresponding edge or edge set in G_Δ.

As we constructed G^s, we have already defined the mapping π^s which labels each vertex in G^s with a degree constraint. For $x \in V_{G^s}$, we have $\pi^s(x) = \{0, 1\}$ if $\deg_{G^s}(x) = 1$, $\pi^s(x) = \{0, 2\}$ if $\deg_{G^s}(x) = 2$, and $\pi^s(x) = \{0, 1, 3\}$ or $\{0, 2, 3\}$ if $\deg_{G^s}(x) = 3$. Moreover, as we have discussed above, for a vertex $v \in V_{G_\Delta}$ and a subgraph $H \subseteq G_\Delta$ induced be a set E of edges incident to v in G_Δ, we have $\deg_{F\Delta H}(v) \in \pi(v)$ if $\deg_{H^s}(x) \in \pi^s(x)$ for every $x \in S(v)$ where H^s is the subgraph of G^s induced by the edge set E (viewed as edges in G^s).

Now, we define the function ω^s for edges in G^s as follow. Recall that for every edge in G^s, its corresponding edge in G_Δ is either in the factor F or the factor F^* but not in both since $G_\Delta = F\Delta F^*$. For $e \in E_{G^s}$, we define $\omega^s(e) = \omega(e)$ if $e \in E_{F^*}$ and $\omega^s(e) = -\omega(e)$ if $e \in E_F$. We can extend ω^s to any subgraph of G^s by defining its weight to be the total weight of all its edges. Then, for any subgraph $H^s \subseteq G^s$, $\omega^s(H^s) = \omega(F\Delta H) - \omega(F)$ where H is the subgraph of G_Δ corresponding to H^s. In particular, $\omega^s(G^s) = \omega(F^*) - \omega(F) > 0$. Thus, we get a key instance $\Omega^s = (G^s, \pi^s, \omega^s)$ where $\omega^s(G^s) > 0$.

Suppose that F^s is a factor of G^s with $\omega^s(F^s) > 0$. We consider the subgraph H of G_Δ induced by the edge set E_{F^s} (viewed as edges in G_Δ). We show that H is an (F, F^*)-basic subgraph of G. We have $H \subseteq G_\Delta = F\Delta F^*$. As we have discussed above, for every vertex $v \in V_{F\Delta H}$, $\deg_{F\Delta H}(v) \in \pi(v)$. Thus, $F\Delta H \in \Omega$. Also, $\omega^s(F^s) = \omega(F\Delta H) - \omega(F) > 0$. Then, H is an F-augmenting subgraph. For every $v \in V_H$, $\deg_H(v) = \sum_{x \in S(v)} \deg_{F^s}(x)$. Then, $\deg_H(v)$ is odd only if there is a vertex $x \in S(v)$ such that $\deg_{F^s}(x)$ is odd. Thus, the number of odd vertices in H is no more than the number of odd vertices in F^s. Since F^s is a basic factor, it has at most 2 vertices of odd degree. Thus, H has at most 2 vertices of odd degree. Moreover, for a vertex $v \in V_H \cap T_\Omega$, if $\deg_F(v) \equiv \deg_{F^*}(v) \mod 2$, then $S(v)$ consists of vertices of degree 2. Thus, $\deg_{F^s}(x) \in \{0,2\}$ for every $x \in S(v)$. Then, $\deg_H(v) = \sum_{x\in S(v)} \deg_{F^s}(x)$ is even. Thus, for a vertex $v \in V_H \cap T_\Omega$, $\deg_H(v)$ is odd only if $\deg_F(v) \not\equiv \deg_{F^*}(v) \mod 2$. Then, $V_H^{\text{odd}} \cap T_\Omega \subseteq T_\Omega^{F\Delta F^*}$ where $V_H^{\text{odd}} = \{v \in V_H \mid \deg_H(v) \equiv 1 \mod 2\}$. Thus, H is an (F, F^*)-basic subgraph of G.

By the first part of Theorem 4.8, there exists a basic factor $F^s \in \Omega^s$ with $\omega^s(F^s) > 0$. Thus, there exists an (F, F^*)-basic subgraph $H \subseteq G$ induced by the edge set E_{F^s}. The first part is done.

Now, we prove the second part. Suppose that $\omega(F^*) > \text{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega^{F\Delta F^*}$ where $|W| \le 2$, and $T_\Omega^{F\Delta F^*}$ contains a vertex u where $\deg_F(u) \in D_u^0$. Consider the instance Ω^s. First, we prove that $\omega^s(G^s) > \omega(F^s)$ for every basic factor F^s of Ω^s. For a contradiction, suppose that there is some $F^s \in \Omega^s$ such that $\omega^s(G^s) \le \omega(F^s)$. Still consider the subgraph H of G_Δ inducted by E_{F^s}. We know that H is an (F, F^*)-basic subgraph of G and $\omega^s(F^s) = \omega(F\Delta H) - \omega(F)$. Let $W = V_H^{\text{odd}} \cap T_\Omega$. Then, $W \subseteq T_\Omega^{F\Delta F^*}$ and $|W| \le 2$. For every $x \in W$, since $\deg_H(x)$ is odd, we have $\deg_{F\Delta H}(x) \not\equiv \deg_F(x) \mod 2$, and then $\deg_{F\Delta H}(x) \in \pi(x)\backslash D_v^F$. For every $x \in T_\Omega\backslash W$, since $\deg_H(x)$ is even, we have $\deg_{F\Delta H}(x) \equiv \deg_F(x) \mod 2$ and then $\deg_{F\Delta H}(x) \in D_v^F$. Consider the sub-instance $\Omega_W^F = (G, \pi_W^F, \omega)$ of Ω (see Equation (1) for the definition of Ω_W^F). Then, $F\Delta H \in \Omega_W^F$. Thus, $\omega(F\Delta H) \le \text{Opt}(\Omega_W^F)$. Since

$$\omega^s(G^s) = \omega(F^*) - \omega(F) \le \omega^s(F^s) = \omega(F\Delta H) - \omega(F),$$

we have $\omega(F^*) \le \omega(F\Delta H)$. Then, $\omega(F^*) \le \text{Opt}(\Omega_W^F)$. A contradiction with the assumption that $\omega(F^*) > \text{Opt}(\Omega_W^F)$ for every $W \subseteq T_\Omega^{F\Delta F^*}$ where $|W| \le 2$. Thus, $\omega^s(G^s) > \omega^s(F^s)$ for every basic factor F^s of Ω^s.

Since $T_\Omega^{F\Delta F^*}$ contains a vertex u where $\deg_F(u) \in D_u^0$. Consider the vertex set $S(u)$ in G^s that corresponds to u. Since $u \in T_\Omega^{F\Delta F^*}$, $\deg_F(u) \not\equiv \deg_{F^*}(u) \mod 2$. Thus, $S(u)$ consists of vertices of degree 2 and a vertex u^s of degree $\deg_{G^s}(u^s) = |\deg_F(u) - \deg_{F^*}(u)|$ which is 1 or 3. If $|\deg_F(u) - \deg_{F^*}(u)| = 3$, then $\pi^s(u^s) = \{0,2,3\}$ since $\deg_F(u) \in D_u^0$. Thus, G^s contains a vertex u^s where $\deg_{G^s}(u^s) = 1$ or $\deg_{G^s}(u^s) = 3$ and $\pi^s(u^s) = \{0,2,3\}$. Then, by the second part of Theorem 4.8, there is a basic factor $F^s \in \Omega^s$ such that $\omega^s(F^s) > 0$ and $\deg_{F^s}(u_s) \equiv 0 \mod 2$. Again, consider the subgraph H of G_Δ inducted by E_{F^s}. We have proved that H is an (F, F^*)-basic subgraph of G. Also,

$$\deg_H(u) = \sum_{x\in S(u)\backslash\{u^s\}} \deg_{F^s}(x) + \deg_{F^s}(u^s) \equiv 0 \mod 2$$

since $\deg_{F^s}(x) \in \pi^s(x) = \{0,2\}$ for every $x \in S(u)\backslash\{u^s\}$, and $\deg_{F^s}(u_s) \equiv 0 \mod 2$. Thus, there is an (F, F^*)-basic subgraph H of G such that $\deg_H(u) \equiv 0 \mod 2$. □

5 Proof of Theorem 4.8

We first prove the first property (restated in Lemma 5.6), and then prove the second property (restated in Lemma 5.7) using the first property. In this section, for two points x and y, we use p_{xy}, p'_{xy} or p''_{xy} to denote a path with endpoints x and y. Recall that $V_{p_{xy}}$ and $E_{p_{xy}}$ denotes the vertex set and the edge set of p_{xy} respectively.

5.1 Proof of the first property

Lemma 5.1. *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance with $\omega(G) > 0$. If G is not connected, then there is a factor $F \in \Omega$ such that $\omega(F) > 0$ and $|E_F| < |E_G|$.*

Proof. Suppose that G_1 is a connected component of G, and $G_2 = G\Delta G_1$ is the rest of the graph. Note that G_1 and G_2 are both factors of G. By the definition of subcubic graphs, there are no isolated vertices in G. Thus, neither G_1 nor G_2 is a single vertex. Then, $|E_{G_1}|, |E_{G_2}| \geq 1$. Since E_G is the disjoint union of E_{G_1} and E_{G_2}, $|E_{G_1}|, |E_{G_2}| < |E_G|$, and $\omega(G) = \omega(G_1)+\omega(G_2)$. Since $\omega(G) > 0$, among $\omega(G_1)$ and $\omega(G_2)$, one is positive. Thus, we are done. □

Lemma 5.2. *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance with $\omega(G) > 0$. Then, there is a factor $F \in \Omega$ such that $\omega(F) > 0$ and $|E_F| < |E_G|$ if one of the following conditions holds:*

1. *There is a path $p_{uv} \subseteq G$ with endpoints u and v where u and v are the only two vertices in p_{uv} of type-2 (i.e., $\deg_G(u) = \deg_G(v) = 3$ and $\pi(u) = \pi(v) = \{0, 2, 3\}$) and $\omega(p_{uv}) \leq 0$.*
2. *There is a cycle $C \subseteq G$ where no vertex is of type-2 and $\omega(C) \leq 0$.*

Proof. Suppose that the first condition holds. Consider the subgraph $F = G\backslash p_{uv}$ of F. Then, $|E_F| = |E_G| - |E_{p_{uv}}| < |E_G|$, and $\omega(F) = \omega(G) - \omega(p_{uv}) \geq \omega(G) > 0$. Now we only need to show that F is a factor of Ω. The vertex set V_F consists of three parts:

$$V_1 = V_G\backslash V_{p_{uv}}, \quad V_2 = \{x \in V_{p_{uv}}\backslash\{u, v\} \mid \deg_G(x) = 3\}, \quad \text{and} \quad V_3 = \{u, v\}.$$

Since u and v are the only two vertices of type-2 in p_{uv}, for every $x \in V_2$, x is of type-1 (i.e., $\pi(x) = \{0, 1, 3\}$). Then, for every $x \in V_F$, we have $\deg_F(x) = \deg_G(x) \in \pi(x)$ if $x \in V_1$, $\deg_F(x) = 1 \in \pi(x)$ if $x \in V_2$, and $\deg_F(x) = 2 \in \pi(x)$ if $x \in V_3$. Thus, F is a factor of G. We are done.

Suppose that the second condition holds. Consider the subgraph $F = G\backslash C$. Then $|E_F| < |E_G|$ and $\omega(F) > 0$. Similar to the above proof, one can check that F is a factor Ω. We are done. □

Lemma 5.3. *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance with $\omega(G) > 0$, G is not a basic factor of Ω and $C \subseteq G$ is a cycle. Let k be the number of type-1 vertices and ℓ be the number of type-2 vertices in C. If $k \neq 1$ and $\ell \neq 1$, then there is a factor $F \in \Omega$ such that $\omega(F) > 0$ and $|E_F| < |E_G|$.*

Proof. We prove this lemma in two cases depending on whether $\omega(C) > 0$ or $\omega(C) \leq 0$.

We first consider the case that $\omega(C) > 0$. If $k = 0$, then all vertices in C are 2-feasible (see Definition 4.6). Thus, C is a factor of Ω. Since G is not a basic factor of Ω, we have $G \neq C$. Also, since G has no isolated vertices, $C \subsetneq G$ implies that $|E_C| < |E_G|$. We are done. Thus, we may assume that $k \geq 2$. Suppose that $\{u_1, u_2, \ldots, u_k\}$ are the type-1 vertices in C. We list them in the order of traversing the cycle starting from u_1 in an arbitrary direction. Then, these k many vertices split the cycle into k many paths $p_{u_1u_2}, \ldots, p_{u_ku_{k+1}}$ ($u_{k+1} = u_1$). For each path, all its vertices are 2-feasible except for its two endpoints which are 1-feasible. Thus, each path is a basic factor of G. We have $|E_{p_{u_iu_{i+1}}}| < |E_G|$ for every $i \in [k]$. Since

$$\omega(C) = \sum_{i=1}^{k} \omega(p_{u_iu_{i+1}}) > 0,$$

there is a path $p_{u_iu_{i+1}}$ such that $\omega(p_{u_iu_{i+1}}) > 0$. Thus, we are done.

Then we consider the case that $\omega(C) \leq 0$. If $\ell = 0$, then $C \subseteq G$ is cycle with no type-2 vertices. By Lemma 5.2, we are done. Thus, we may assume that $\ell \geq 2$. Suppose that $\{v_1, v_2, \ldots, v_\ell\}$ are the type-2 vertices in C. We list them in the order of traversing the cycle starting from v_1 in an arbitrary direction. Then, these ℓ many vertices split the cycle into ℓ

many paths $p_{v_1v_2}, \ldots, \ldots, p_{v_\ell v_{\ell+1}}$ $(v_{\ell+1} = v_1)$. For each path, it has no vertex of type-2 except for its two endpoints which are of type-2. Since

$$\omega(C) = \sum_{i=1}^{k} \omega(p_{v_i v_{i+1}}) \leq 0,$$

there is a path $p_{v_iv_{i+1}}$ such that $\omega(p_{v_iv_{i+1}}) \leq 0$. Thus, there is a path $p_{v_iv_{i+1}} \subseteq G$ where v_i and v_{i+1} are the only two vertices of type-2 in $p_{v_iv_{i+1}}$ and $\omega(p_{v_iv_{i+1}}) \leq 0$. Then, by Lemma 5.2, we are done. □

Lemma 5.4. *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance with $\omega(G) > 0$, and G is not a basic factor of Ω. If G is 2-connected, then there is a factor $F \in \Omega$ such that $\Omega(F) > 0$ and $|E_F| < |E_G|$.*

Proof. Since G is 2-connected, it contains at least three vertices and it contains no vertex of degree 1. Consider the number of type-1 vertices in G. There are three cases.

- G has no type-1 vertex.

  Since G is 2-connected, there is a cycle $C \subseteq G$. Clearly, C has no type-1 vertex. If C has exactly one type-2 vertex, denoted by v, then v is the only vertex in C such that $\deg_G(v) = 3$. Then, there is an edge $e \in E_G$ incident to v such that $e \notin E_C$. It is easy to see that e is a bridge of G, a contradiction with G being 2-connected. Thus, C has no type-2 vertex, or it has at least two type-2 vertices. Then, by Lemma 5.3, we are done.

- G has exactly one type-1 vertex.

  Let u be the type-1 vertex of G. Since G is 2-connected, there is a cycle $C \subseteq G$ containing the vertex u. Since $\deg_C(u) = 2 < \deg_G(u) = 3$, by Lemma 2.7, there is a path $p_{uw} \subseteq G$ with endpoints $u, w \in V_C$ such that $E_{p_{uw}} \cap E_C = \emptyset$.

  Consider the subgraph $H = p_{uw} \cup C$ of G. H is a theta graph where $\deg_H(u) = \deg_H(w) = 3$. All vertices of H are 2-feasible except for u which is 1-feasible. Note that H is a basic factor of Ω. Since G is not a basic factor of Ω, $H \neq G$. Also since G is connected, there exists an edge $e_{ts} = (t, s)$ incident to a vertex $s \in V_H$ such that $e_{ts} \notin E_H$. Clearly, s is a vertex of type-2, $\deg_G(s) = 3$ and $\deg_H(s) = 2$. Then, by Lemma 2.7, there is a path p_{sr} with endpoints $s, r \in V_H$ such that $E_{p_{sr}} \cap E_H = \emptyset$. Clearly, $\deg_G(r) = 3$ and r is a vertex of type-2. Since $s, r \in V_H$ and H is a theta graph which is 2-connected, we can find a path $p'_{sr} \subseteq H$ with endpoints s and r such that the only type-1 vertex u in H is not in p'_{sr}. Consider the cycle $C' = p_{sr} \cup p'_{sr}$. It has no vertex of type-1, and it has at least two vertices s and r of type-2. By Lemma 5.3, we are done.

- G has at least two type-1 vertices.

  Since G is 2-connected and it contains at least two type-1 vertices, we can find a cycle $C \subseteq G$ that contains at least two type-1 vertices. Consider the number of type-2 vertices in C. If the number is not 1, then by Lemma 5.3, we are done. Thus, we may assume that C contains exactly one vertex of type-2, denoted by v. Since G is 2-connected and $\deg_G(v) = 3 > \deg_C(v) = 2$, we can find a path p_{vu} for some $u \in V_C$ such that $E_{p_{vu}} \cap E_C = \emptyset$. We have $\deg_G(u) = 3$. Since v is the only vertex of type-2 in C, u is a vertex of type-1. Vertices v and u split C into two paths p'_{vu} and p''_{vu}. Since C contains at least two type-1 vertices, there exists some $w \in V_C$ where $w \neq u$ such that w is of type-1. Also, $w \neq v$ since v is of type-2. Since $w \in V_C = V_{p'_{vu}} \cup V_{p''_{vu}}$ and $V_{p'_{vu}} \cap V_{p''_{vu}} = \{u, v\}$, without loss of generality, we may assume that $w \in V_{p'_{vu}}$.

Consider the path p_{vu}. If p_{vu} contains at least two vertices of type-2, then the cycle $C' = p_{vu} \cup p'_{vu}$ contains at least two vertices of type-2 and at least two vertices u and w of type-1. Then, by Lemma 5.3, we are done. Thus, we may assume that v is the only vertex of type-2 in p_{vu}. Consider the theta graph $H = p_{vu} \cup C$. Then v is the only vertex of type-2 in H. Note that $w \in V_H$, $\deg_H(w) = 2 < \deg_G(w) = 3$. Since G is 2-connected, by Lemma 2.7, we can find a path p_{ws} for some $s \in V_H$ such that $E_{p_{ws}} \cap E_H = \emptyset$. Clearly $s \neq v$. Then, s is of type-1 since v is the only vertex of type-2 in H.

Consider the number of type-2 vertices in p_{ws}. Suppose that there is no vertex of type-2 in p_{ws}. Since H is 2-connected and H contains only one vertex v of type 2, we can find a path $p'_{ws} \subseteq H$ such that p'_{ws} does not contain the vertex v of type-2. Then, the cycle $p_{ws} \cup p'_{ws}$ has no vertex of type-2 and at least two vertices w and s of type-1. By Lemma 5.3, we are done. Otherwise, there is at least one vertex of type-2 in p_{ws}. Since H is 2-connected, we can find a path $p''_{ws} \subseteq H$ such that p''_{ws} contains the vertex v of type-2. Then, the cycle $p_{ws} \cup p''_{ws}$ has at least two vertices of type-2 and at least two vertices w and s of type-1. By Lemma 5.3, we are done. □

Definition 5.5 (Induced sub-instance)**.** *For a key instance $\Omega = (G, \pi, \omega)$, and a factor $F \in \Omega$, the sub-instance of Ω induced by F, denoted by Ω_F, is a key instance (F, π_F, ω_F) defined on the subgraph F of G where $\pi_F(x) = \pi(x) \cap [\deg_F(x)] \subseteq \pi(x)$ for every $x \in V_F$ and ω_F is the restriction of ω on E_F (we may write ω_F as ω for simplicity).*

We are now ready to prove the first property of Theorem 4.8 as restated in the next lemma.

Lemma 5.6. *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance. If $\omega(G) > 0$, then there is a basic factor F of Ω such that $\omega(F) > 0$.*

Proof. We prove this lemma by induction on the number of edges in G.

If $|E_G| = 1$, then G is a single edge. Thus, G is a basic factor of Ω, and $\omega(G) > 0$. We are done.

We assume that the lemma holds for all key instances where the underlying graph has no more than n many edges. We consider a key instance $\Omega = (G, \pi, \omega)$ where $|E_G| = n + 1$.

If G is a basic factor of Ω, then clearly we are done. Thus, we may assume that G is not a basic factor of Ω. Suppose that we can find a factor $F \in \Omega$ such that $\omega(F) > 0$ and $|E_F| < |E_G| = n+1$. Then, consider the sub-instance Ω_F of Ω induced by F. Since $|E_F| < n+1$ and $\omega(F) > 0$, by the induction hypothesis, there is basic factor $F' \in \Omega_F$ such that $\omega(F') > 0$. Since $\Omega_F \subseteq \Omega$, $F' \in \Omega$. Then, we are done. Thus, in order to establish the inductive step, it suffices to prove that there is a factor $F \in \Omega$ such that $|E_F| < |E_G|$ and $\omega(F) > 0$.

By Lemmas 5.1 and 5.4, if G is not connected or G is 2-connected, then we are done. Thus, we may assume that G is a connected graph but not 2-connected. By Lemma 2.6, G contains at least a bridge. Fix such a bridge of G. Let p_{uv} be the path containing the bridge such that for every vertex $x \in V_{p_{uv}} \backslash \{u, v\}$, $\deg_G(x) = 2$ and $\deg_G(u), \deg_G(v) \neq 2$; observe that such a path exists and it is unique. In fact, the whole path can be viewed as a "long bridge" of the graph G. Then, $G \backslash p_{uv}$ is not connected and it has two connected components. Let $G_u \subseteq G \backslash p_{uv}$ be the part that contains u and $G_v \subseteq G \backslash p_{uv}$ be the part that contains v.

If both G_u and G_v are single vertices, then the graph G is a path. If both G_u and G_v are cycles, then G is a dumbbell graph. If one of G_u and G_v is a single vertex and the other one is a cycle, then G is a tadpole graph. In all these cases, G is a basic factor of Ω. A contradiction with our assumption. Thus, among G_u and G_v, at least one is neither a cycle nor a single vertex. Without loss of generality, we may assume that G_u is neither a cycle nor a single vertex.

Since G_u is not a single vertex, $\deg_G(u) \neq 1$. By assumption, $\deg_G(u) \neq 2$. Then $\deg_G(u) = 3$, and hence $\deg_{G_u}(u) = 2$. Let $e_1 = (u, w_1)$ and $e_2 = (u, w_2)$ be the two edges incident to u in

G_u. We slightly modify G_u to get a new graph. We replace the vertex u in G_u by two vertices u_1 and u_2, and replace the edges (u, w_1) and (u, w_1) in G_u by two new edges (u_1, w_1) and (u_2, w_2) respectively. We denote the new graph by G'. With a slight abuse of notations, we still use e_1 and e_2 to denote the edges (u_1, w_1) and (u_2, w_2) in G' respectively, and we say $E_{G_u} = E_{G'}$. Then, the edge weight function ω can be adapted to $E_{G'}$. We define the following instance $\Omega' = (G', \pi', \omega')$ where $\pi'(u_1) = \pi'(u_2) = \{0, 1\}$ and $\pi'(x) = \pi(x)$ for every $x \in V_{G'} \backslash \{u_1, u_2\}$, and $\omega'(e_1) = \omega(e_1) + \omega(G \backslash G_u)$, and $\omega'(e) = \omega(e)$ for every $e \in E_{G'} \backslash \{e_1\}$. In other words, we add the total weight of the subgraph $G \backslash G_u$ to the edge e_1. Then, $\omega'(G') = \omega(G) > 0$ and $|E_{G'}| = |E_{G_u}| < |E_G|$. By the induction hypothesis, there is a basic factor $F \in \Omega'$ such that $\omega'(F) > 0$. We will recover a factor of Ω from F such that it has positive weight and fewer edges than G. This will finish the proof of the inductive step.

There are four cases depending on the presence of e_1 and e_2 in F.

- $e_1, e_2 \notin E_F$. Then, $u_1, u_2 \notin V_F$. For every $x \in V_F$, $\deg_F(x) \in \pi'(x) = \pi(x)$. Thus, F is a basic factor of Ω. Clearly, $\omega(F) = \omega'(F) > 0$ and $|E_F| = |E_{F'}| < |E_G|$. We are done.

- $e_1 \in E_F$ and $e_2 \notin E_F$. We can view F as a subgraph of G_u by changing the edge (u_1, w_1) in G' back to the edge (u, w_1) in G_u. Then, the edge $(u, w_2) \notin E_F$. Consider the subgraph $H = F \cup (G \backslash G_u)$ of G. Since $(u, w_2) \notin E_F$, we have $(u, w_2) \notin E_H$. Then, $|E_H| < |E_G|$. Also, we have
$$\omega(H) = \omega(F) + \omega(G \backslash G_u) = \omega'(F) > 0.$$
The vertex set V_H consists of three parts $V_1 = V_F \backslash \{u\}$, $V_2 = \{u\}$, and $V_3 = V_{G \backslash G_u} \backslash \{u\}$. For every $x \in V_1$, $\deg_H(x) = \deg_F(x) \in \pi(x)$. For every $x \in V_3$, $\deg_H(x) = \deg_{G \backslash G_u}(x) = \deg_G(x) \in \pi(x)$. Now, we consider the vertex u.

  - If u is 2-feasible, then $\deg_H(u) = 2 \in \pi(x)$. Thus, H is a factor of Ω where $\omega(H) > 0$ and $|E_H| < |E_G|$.
  - If u is 1-feasible, then F and $G \backslash G_u$ both are factors of Ω since $\deg_F(u) = \deg_{G \backslash G_u}(u) = 1 \in \pi(u)$. Since $\omega(H) = \omega(F) + \omega(G \backslash G_u) > 0$, among $\omega(F)$ and $\omega(G \backslash G_u)$, at least one is positive. Also, $|E_F|, |E_{G \backslash G_u}| < |E_H| < |E_G|$. We are done.

- $e_2 \in E_F$ and $e_1 \notin E_F$. Again, we can view F as a subgraph of G_u where $(u, w_2) \in E_F$ and $(u, w_1) \notin E_F$. Then, we have $|E_F| < |E_{G_u}| < |E_G|$, and $\omega(F) = \omega'(F) > 0$.

  - If u is 1-feasible, then F is a factor of G where $|E_F| < |E_G|$ and $\omega(F) > 0$. We are done.
  - If u is 2-feasible, then G_u is a factor of Ω since $\deg_{G_u}(u) = 2$. If $\omega(G_u) > 0$, then we are done. Thus, we may assume that $\omega(G_u) \leq 0$. Then, $\omega(G \backslash G_u) = \omega(G) - \omega(G \backslash G_u) \geq \omega(G) > 0$. Still consider the subgraph $H = F \cup (G \backslash G_u)$. Then, H is a factor of Ω since $\deg_H(u) = 2 \in \pi(u)$. Also, $\omega(H) = \omega(F) + \omega(G \backslash G_u) > 0$ and $|E_H| < |E_G|$. We are done.

- $e_1, e_2 \in E_F$. Then, F (as a subgraph of G') contains two vertices u_1 and u_2 of degree 1. Since F is a basic factor, it is a path. Still we can view F as a subgraph of G_u by changing edges (u_1, w_1) and (u_2, w_2) in G' to edges (u, w_1) and (u, w_2) in G. Then, F is a cycle in G_u. Since G_u is not a cycle and it has no isolated vertices, $|E_F| < |E_{G_u}|$. Consider the subgraph $H = F \cup (G \backslash G_u)$ of G. We have $|E_H| < |E_G|$ and $\omega(H) = \omega(F) + \omega(G \backslash G_u) = \omega'(F) > 0$. Also, one can check that H is a factor of G no matter whether u is 1-feasible or 2-feasible since $\deg_H(u) = 3 \in \pi(u)$. We are done. □

5.2 Proof of the second property

Now we prove the second property of Theorem 4.8 using the first property (Lemma 5.6).

Lemma 5.7. *Suppose that $\Omega = (G, \pi, \omega)$ is a key instance, and u is a vertex of G where $\deg_G(u) = 1$ or $\deg_G(u) = 3$ and $\pi(u) = \{0, 2, 3\}$. If $\omega(G) > 0$ and $\omega(G) > \omega(F)$ for every basic factor F of Ω, then there is a basic factor F^* of Ω such that $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \mod 2$. (Recall that we agree $\deg_{F^*}(u) = 0$ if $u \notin V_{F^*}$.)*

Proof. By Lemma 5.6, there exists at least one basic factor of Ω such that its weight is positive. Among all such basic factors, we pick an F such that $\omega(F)$ is the largest. We have $0 < \omega(F) < \omega(G)$. If $\deg_F(u)$ is even, then we are done. Thus, we may assume that $\deg_F(u)$ is odd. Since F is a basic factor and it contains a vertex u of odd degree, F is not a cycle. By the definition of basic factors, F contains exactly one more vertex v of odd degree. Since F is a factor of Ω, $\deg_F(u) \subseteq \pi(u)$. Recall that $\deg_G(u) = 1$ or 3. If $\deg_G(u) = 1$, then $\pi(u) = \{0, 1\}$, and hence $\deg_F(u) = 1$. If $\deg_G(u) = 3$, then $\pi(u) = \{0, 2, 3\}$, and hence $\deg_F(u) = 3$. Thus, $\deg_F(u)$ always equals $\deg_G(u)$.

Consider the graph $G' = G \backslash F$, i.e., the subgraph of G induced by the edge set $E_G \backslash E_F$. Consider the instance $\Omega' = (G', \pi', \omega')$ where for every $x \in V_{G'}$, $\pi'(x) = \{0, 1\}$ if $\deg_{G'}(x) = 1$, $\pi'(x) = \{0, 2\}$ if $\deg_{G'}(x) = 2$ and $\pi'(x) = \pi(x)$ if $\deg_{G'}(x) = 3$, and ω' is the weight function ω restricted to G'. Note that Ω' is also a key instance, but it is not necessarily a sub-instance of Ω. Since $\omega(G) > \omega(F)$, we have $\omega'(G') = \omega(G') = \omega(G) - \omega(F) > 0$. Without causing ambiguity, we may simply write ω' as ω in the instance Ω'. By Lemma 5.6, there exists a basic factor F' of Ω' such that $\omega(F') > 0$. Since $E_{F'} \subseteq E_G \backslash E_F$, F and F' are edge-disjoint. Let $H = F \cup F'$, which is the subgraph of G induced by the edge set $E_F \cup E_{F'}$. We show that H is a factor of Ω.

Let $V_\cap = V_F \cap V_{F'}$. First we show that for every $x \in V_H \backslash V_\cap$, $\deg_H(x) \in \pi(x)$. If $x \in V_F \backslash V_\cap$, then $\deg_H(x) = \deg_F(x)$. Since $F \in \Omega$, $\deg_F(x) \in \pi(x)$. Then, $\deg_H(x) \in \pi(x)$. If $x \in V_{F'} \backslash V_\cap$, then $\deg_H(x) = \deg_{F'}(x)$. Since $x \notin V_F$ and $G' = G \backslash F$, $\deg_{G'}(x) = \deg_G(x)$. Then, by the definition of Ω', we have $\pi'(x) = \pi(x)$. Since F' is a factor of Ω', $\deg_{F'}(x) \in \pi'(x)$. Thus, $\deg_H(x) \in \pi(x)$. Now, we consider vertices in $V_\cap$. Since F and F' are edge disjoint, for every $x \in V_\cap$ we have $\deg_H(x) = \deg_F(x) + \deg_{F'}(x) \leq \deg_G(x) \leq 3$. Also, $\deg_F(x), \deg_{F'}(x) \geq 1$ since F and F' are subcubic graphs which have no isolated vertices.

- If $\deg_F(x) = 1$, then $1 \in \pi(x)$. The vertex x is 1-feasible. Thus, $\deg_G(x) \neq 2$. Since $\deg_G(x) > \deg_F(x) = 1$, $\deg_G(x) = 3$. Then, $\deg_{G'}(x) = \deg_G(x) - \deg_F(x) = 2$, $\pi'(x) = \{0, 2\}$ and $\deg_{F'}(x) = 2$.
- If $\deg_F(x) = 2$, then $\deg_G(x) = 3$ since $\deg_G(x) > \deg_F(x)$. Then, $\deg_{G'}(x) = \deg_G(x) - \deg_F(x) = 1$, $\pi'(x) = \{0, 1\}$ and $\deg_{F'}(x) = 1$.

Thus, for every $x \in V_\cap$, $\deg_H(x) = \deg_F(x) + \deg_{F'}(x) = 3 \in \pi(x)$. Thus, H is a factor of Ω.

Consider the sub-instance $\Omega_H = (H, \pi_H, \omega_H)$ of Ω induced by H (we will write ω_H as ω for simplicity). We will show that we can find a a basic factor F^* of Ω_H such that $\omega(F^*) > 0$ and $\deg_{F^*}(u) \equiv 0 \mod 2$. Clearly, F^* is also a factor of Ω.

Consider the set $V_\cap$ of intersection points. If $V_\cap = \emptyset$, then for every $x \in V_{F'}$, $\deg_{F'}(x) = \deg_H(x) \in \pi(x)$. Thus, F' is a basic factor of Ω where $\omega(F') > 0$ and $\deg_{F'}(u) = 0$. That is, F' is the desired F^*. We are done. Thus, we may assume that $V_\cap$ is non-empty. For every $x \in V_\cap$, $\deg_F(x) = 1$ and $\deg_{F'}(x) = 2$, or $\deg_F(x) = 2$ and $\deg_{F'}(x) = 1$. Recall that F is a basic factor containing two vertices u, v of odd degree, and $\deg_F(u) = \deg_G(u)$. Clearly, $u \notin V_\cap$.

We consider the possible forms of F and F'. Recall that F is not a cycle. We show that F' is also not a cycle. For a contradiction, suppose that F' is a cycle. Then, all vertices of F' have degree 2. Thus, the only possible vertex in $V_\cap$ is v. Since $V_\cap$ is non-empty, $V_\cap = \{v\}$. Then,

$\deg_F(v) = 1$ and $\deg_{F'}(v) = 2$. If $\deg_F(u) = 1$, then F is a path. The graph H is a tadpole graph where v is the only vertex of degree 3. If $\deg_F(u) = 3$, then F is a tadpole graph. The graph H is a dumbbell graph where v and u are the two vertices of degree 3. In both cases, H is a basic factor of Ω. Since $\omega(F') > 0$, we have $\omega(H) = \omega(F) + \omega(F') > \omega(F)$ which leads to a contraction with F being a basic factor with the largest weight. Thus, F' is a basic factor which is not a cycle. Then, it contains exactly two vertices s, t of odd degree. Then, $V_\cap \subseteq \{v, s, t\}$.

We consider the graph H depending on the forms of F and F', and the vertices in $V_\cap$. There are 5 main cases.

I. F is a path.

II. F is a tadpole graph and $\deg_F(u) = 3$.

III. F is a tadpole graph and $\deg_F(u) = 1$.

IV. F is a dumbbell graph.

V. F is a theta graph.

Recall that for two points x and y, we use p_{xy}, p'_{xy} or p''_{xy} to denote a path with endpoints x and y. We also use q_{xy^3} or q'_{xy^3} to denote a tadpole graph where x is the vertex of degree 1 and y is the vertex of degree 3, and θ_{xy} to denote a theta graph where x and y are the two points of degree 3. In the following Figures 2 to 12, we use hollow nodes to denote 1-feasible vertices, solid nodes to denote 2-feasible vertices, semisolid nodes to denote vertices that are possibly 1-feasible or 2-feasible, red-colored lines to denote paths in F, and blue-colored lines to denote paths in F'.

Case I: F is a path. There are 4 subcases depending on the form of F'.

I.1 F and F' are both paths. Then, $V_\cap \subseteq \{v, s, t\}$. There are 5 subcases: $V_\cap = \{v\}$, $V_\cap = \{s\}$ or $\{t\}$, $V_\cap = \{v, s\}$ or $\{v, s\}$, $V_\cap = \{s, t\}$, and $V_\cap = \{v, s, t\}$.

(a) $V_\cap = \{v\}$.

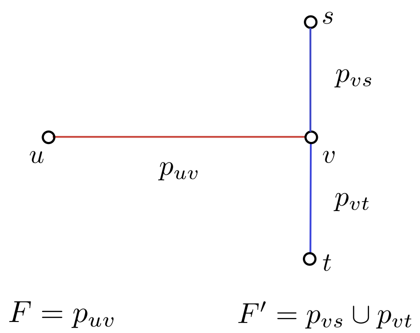

Figure 2: The graph H in Case I.1.(a)

In this case, $\deg_H(u) = \deg_H(s) = \deg_H(t) = 1$, $\deg_H(v) = 3$, and $\pi(v) = \{0, 1, 3\}$ since $\deg_F(v) = 1 \in \pi(v)$. The graph H consists of three edge-disjoint paths p_{uv}, p_{vs} and p_{vt}. Then, $F = p_{uv}$ and $F' = p_{vs} \cup p_{vt}$. (See Figure 2.)
Since $\omega(F') = \omega(p_{vs}) + \omega(p_{vt}) > 0$, among $\omega(p_{vs})$ and $\omega(p_{vt})$, at least one is positive. Without loss of generality, we may assume that $\omega(p_{vs}) > 0$. Since u does not appear in p_{vs}, we have $\deg_{p_{vs}}(u) = 0$. For every vertex x in p_{vs} where $x \neq v$, $\deg_{p_{vs}}(x) = \deg_H(x) \in \pi(x)$. Also, $\deg_{p_{vs}}(v) = 1 \in \pi(v)$. Thus, the path p_{vs} is a basic factor of Ω where $\omega(p_{vs}) > 0$ and $\deg_{p_{vs}}(u) = 0$.

(b) $V_\cap = \{s\}$ or $\{t\}$.
These two cases are symmetric. We only consider the case that $V_\cap = \{s\}$.

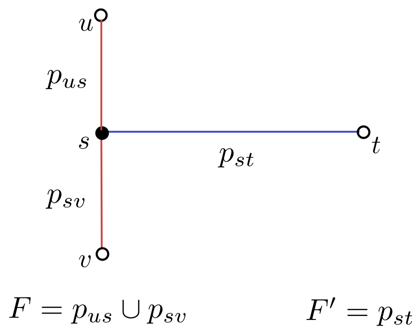

Figure 3: The graph H in Case I.1.(b)

In this case, $\deg_H(s) = 3$, $\deg_H(u) = \deg_H(v) = \deg_H(t) = 1$, and $\pi(s) = \{0, 2, 3\}$ since $\deg_{F'}(s) = 1$ and $\deg_F(s) = 2 \in \pi(s)$. The graph H consists of three edge-disjoint paths p_{us}, p_{sv} and p_{st}. Then, $F' = p_{st}$ and $F = p_{us} \cup p_{sv}$. (See Figure 3.)
Note that $\omega(p_{st}) = \omega(F') > 0$. Let $p_{ut} = p_{us} \cup p_{st}$ be the path with endpoints u and t. For every vertex x in p_{ut} where $x \neq s$, we have $\deg_{p_{ut}}(x) = \deg_H(x) \in \pi(x)$. Also, $\deg_{p_{ut}}(s) = 2 \in \pi(s)$. Thus, p_{ut} is a basic factor of Ω. Then, $\omega(F) \geq \omega(p_{ut})$ since F is a basic factor of Ω with the largest weight $\omega(F)$. Then,

$$\omega(F) = \omega(p_{us}) + \omega(p_{sv}) \geq \omega(p_{us}) + \omega(p_{st}) = \omega(p_{ut}).$$

Thus, $\omega(p_{sv}) \geq \omega(p_{st}) > 0$. Let $p_{vt} = p_{sv} \cup p_{st}$ be the path with endpoints v and t. Then,

$$\omega(p_{vt}) = \omega(p_{sv}) + \omega(p_{st}) > 0.$$

Since u is not in p_{vt}, $\deg_{p_{vt}}(u) = 0$. Similar to the proof of $p_{ut} \in \Omega$, we have $p_{vt} \in \Omega$. Thus, the path p_{vt} is a basic factor of Ω where $\omega(p_{vt}) > 0$ and $\deg_{p_{vt}}(u) = 0$.

(c) $V_\cap = \{v, s\}$ or $\{v, t\}$.
These two cases are symmetric. We only consider the case that $V_\cap = \{v, s\}$.

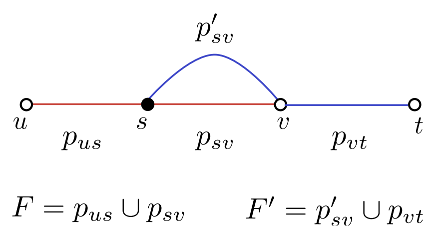

Figure 4: The graph H in Case I.1.(c)

In this case, $\deg_H(v) = \deg_H(s) = 3$, $\deg_H(u) = \deg_H(t) = 1$, $\pi(v) = \{0, 1, 3\}$ since $\deg_F(v) = 1 \in \pi(v)$, and $\pi(s) = \{0, 2, 3\}$ since $\deg_F(s) = 2 \in \pi(s)$. The point s splits F into two paths p_{us} and p_{sv}. Then, $F = p_{us} \cup p_{sv}$. The point v splits F' into two paths p'_{sv} and p_{vt}. Then, $F' = p'_{sv} \cup p_{vt}$. (See Figure 4.)
Consider the path $p'_{uv} = p_{us} \cup p'_{sv}$. Note that $\deg_{p'_{uv}}(s) = 2 \in \pi(s)$. Then, p'_{uv} is a basic factor of Ω. Since, F is a basic factor of Ω with the largest weight, we have

$$\omega(F) = \omega(p_{us}) + \omega(p_{sv}) \geq \omega(p_{us}) + \omega(p'_{sv}) = \omega(p'_{uv}).$$

Thus, $\omega(p_{sv}) \geq \omega(p'_{sv})$. Let F^* be the tadpole graph $q_{tv^3} = p_{sv} \cup p'_{sv} \cup p_{vt}$. Note that $\deg_{F^*}(s) = 2 \in \pi(s)$ and $\deg_{F^*}(v) = 3 \in \pi(v)$. Then, F^* is a basic factor of Ω and $\deg_{F^*}(u) = 0$. Also, the path p_{vt} is a basic factor of Ω since $\deg_{p_{vt}}(v) = 1 \in \pi(v)$, and $\deg_{p_{vt}}(u) = 0$. Then,

$$\omega(F^*)+\omega(p_{vt}) = \omega(p_{sv})+\omega(p'_{sv})+\omega(p_{vt})+\omega(p_{vt}) \geq 2(\omega(p'_{sv})+\omega(p_{vt})) = 2\omega(F') > 0.$$

Thus, among $\omega(F^*)$ and $\omega(p_{vt})$, at least one is positive. Thus, F^* or p_{vt} is a desired basic factor of Ω that satisfies the requirements.

(d) $V_\cap = \{s, t\}$.

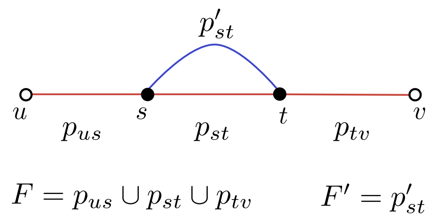

Figure 5: The graph H in Case I.1.(d)

In this case, $\deg_H(u) = \deg_H(v) = 1$, $\deg_H(s) = \deg_H(t) = 3$, and $\pi(s) = \pi(t) = \{0, 2, 3\}$. The points s and t split F into three paths. Without loss of generality, we may assume that s is closer to u and t is closer to v. Then, the three paths are p_{us}, p_{st}, and p_{tv}, and $F = p_{us} \cup p_{st} \cup p_{ts}$. Also, F' is a path with endpoints s and t, which is disjoint with p_{st}. (See Figure 5.)

Consider the path $p'_{uv} = p_{us} \cup F' \cup p_{tv}$. One can check that p'_{uv} is a basic factor of Ω. Then, $\omega(F) \geq \omega(p'_{uv})$. Thus, $\omega(p_{st}) \geq \omega(F') > 0$. Consider the cycle $F^* = F' \cup p_{st}$. Also, one can check that F^* is a basic factor of Ω. Moreover, $\omega(F^*) = \omega(F') + \omega(p_{st}) > 0$ and $\deg_{F^*}(u) = 0$. We are done.

(e) $V_\cap = \{v, s, t\}$.

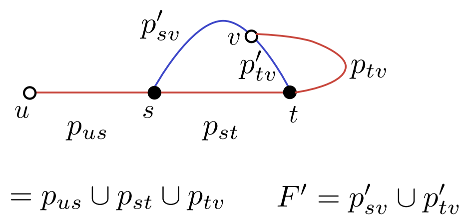

Figure 6: The graph H in Case I.1.(e)

In this case, $\deg_H(u) = 1$, $\deg_H(v) = \deg_H(s) = \deg_H(t) = 3$, $\pi(v) = \{0, 1, 3\}$, and $\pi(s) = \pi(t) = \{0, 2, 3\}$. The points s and t split F into three paths. Without loss of generality, we assume that they are p_{us}, p_{st}, and p_{tv}. Then, $F = p_{us} \cup p_{st} \cup p_{tv}$. The point v splits F' into two paths, p'_{sv} and p'_{tv}. Then, $F' = p'_{sv} \cup p'_{tv}$. (See Figure 6.)

Consider the path $p'_{uv} = p_{us} \cup p'_{sv}$. One can check that it is a basic factor of Ω. Since $\omega(F) \geq \omega(p'_{uv})$, we have

$$\omega(p_{st}) + \omega(p_{tv}) \geq \omega(p'_{sv}).$$

Consider the path $p''_{uv} = p_{us} \cup p_{st} \cup p'_{tv}$. One can check that it is also a basic factor of Ω. Since $\omega(F) \geq \omega(p''_{uv})$, we have

$$\omega(p_{tv}) \geq \omega(p'_{tv}).$$

Consider the tadpole graph $q_{uv^3} = p_{us} \cup p'_{sv} \cup p'_{tv} \cup p_{tv}$. One can check that it is also a basic factor of Ω. Since $\omega(F) \geq \omega(q_{uv^3})$, we have

$$\omega(p_{st}) \geq \omega(p'_{sv}) + \omega(p'_{tv}).$$

Sum up the above three inequalities, and we have

$$2(\omega(p_{st}) + \omega(p_{tv})) \geq 2(\omega(p'_{sv}) + \omega(p'_{tv})) = 2\omega(F') > 0.$$

Consider the theta graph $F^* = p_{st} \cup p_{tv} \cup p'_{sv} \cup p'_{tv}$. Still one can check that it is a basic factor of Ω. Moreover,

$$\omega(F^*) = \omega(p_{st}) + \omega(p_{tv}) + \omega(p'_{sv}) + \omega(p'_{tv}) > 0$$

and $\deg_{F^*}(u) = 0$. We are done.

We are done with Case I.1 where F and F' are both paths.

I.2. F is a path and F' is a tadpole graph. Without loss of generality, we may assume that $\deg_{F'}(s) = 1$ and $\deg_{F'}(t) = 3$. In other words, F' consists of a path with endpoints s and t, and a cycle C containing the vertex t. Then, $V_\cap \subseteq \{v, s\}$. There are three subcases: $V_\cap = \{v\}$, $V_\cap = \{s\}$, and $V_\cap = \{v, s\}$.

(a) $V_\cap = \{v\}$.

In this case, $\deg_H(u) = \deg_H(s) = 1$, $\deg_H(v) = \deg_H(t) = 3$, $\pi(v) = \{0, 1, 3\}$, and $\pi(t) = \{0, 1, 3\}$ or $\{0, 2, 3\}$. There are two subcases depending on whether the intersection point v appears in the path part or the cycle part of F'.

i. v appears in the path part.

Note that for every $x \in V_C \backslash \{t\}$, $\deg_H(x) = 2$, and $\deg_H(t) = 3$. We say such a cycle with exactly one vertex of degree 3 in H is a *dangling* cycle in H. Let e_t be the edge incident to t where $e_t \notin E_C$. We call the vertex t the *connecting point* of C, and the edge e_t the *connecting bridge* of C.

Consider the graph $H' = H \backslash C$. Notice that $\deg_{H'}(x) = \deg_H(x)$ for every $x \in V_{H'} \backslash \{t\}$ and $\deg_{H'}(t) = 1$. Consider the instance $\Omega_{H'} = (H', \pi_{H'}, \omega_{H'})$ where $\pi_{H'}(x) = \pi_H(x)$ for every $x \in V_{H'} \backslash \{t\}$ and $\pi_{H'}(t) = \{0, 1\}$, and $\omega_{H'}(e) = \omega(e)$ for every $e \in E_{H'} \backslash \{e_t\}$ and $\omega_{H'}(e_t) = \omega(e_t) + \omega(C)$. In other words, the instance $\Omega_{H'}$ is obtained from Ω_H by contracting the dangling cycle C to its connecting point t and adding the total weight of C to its connecting bridge e_t. Clearly, $\Omega_{H'}$ is a key instance and $\omega_{H'}(H') = \omega(H') + \omega(C) = \omega(H) > 0$.

For every factor $K' \in \Omega_{H'}$, we can recover a factor $K \in \Omega_H$ from K' as follows: $K = K'$ if $e_t \notin E_{K'}$ and $K = K' \cup C$ if $e_t \in E_K$. One can check that K is a factor of Ω_H, and $\omega(K) = \omega_{H'}(K')$. If $e_t \notin K$, then $K' = K$. Clearly, K' is a basic factor of $\Omega_{H'}$ if and only if K is a basic factor of Ω. Now, suppose that $e_t \in K$. Remember that $\deg_{H'}(t) = 1$. Then, K' is a path with t as an endpoint if and only if $K = K' \cup C$ is a tadpole graph with t as the vertex of degree 3, and K' is a tadpole with t as the vertex of degree 1 if and only if $K = K' \cup C$ is a dumbbell graph. Thus, K' is a basic factor of $\Omega_{H'}$ if and only if K is a basic factor of Ω_H.

Notice that the instance Ω'_H has a similar structure to the instance Ω_H in Case I.1.(a). By replacing the vertex v in Case I.1.(a) by the cycle C (and re-arranging the weights between the cycle C and its connecting bridge), one can check that the proof of Case I.1.(a) works here. Note that after this replacement, the path p_{vt} in Case I.1.(a) becomes a tadpole graph q_{vt^3} which is still a basic factor.

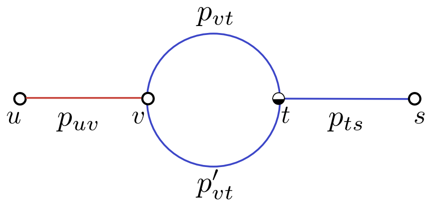

Figure 7: The graph H in Case I.2.(a)

ii. v appears in the cycle part.
Together with the point t, the point v splits the cycle in F' into two paths p_{vt} and p'_{vt}. Let p_{ts} denote the path in F' with endpoints t and s. Then, $F' = p_{vt} \cup p'_{vt} \cup p_{ts}$. (See Figures 7.)

- If $\pi(t) = \{0, 1, 3\}$, then the paths p_{vt}, p'_{vt} and p_{ts} are all basic factors of Ω. Moreover, the vertex u does not appear in any of these paths. Also, since $\omega(F') = \omega(p_{vt}) + \omega(p'_{vt}) + \omega(p_{ts}) > 0$, there is at least one path with positive weight. We are done.
- If $\pi(t) = \{0, 2, 3\}$, then the tadpole graph $q_{uv^3} = F \cup p_{vt} \cup p'_{vt}$ is a basic factor or Ω. Since $\omega(F) \geq \omega(q_{uv^3})$, we have $\omega(p_{vt}) + \omega(p'_{vt}) \leq 0$. Without loss of generality, we assume that $\omega(p'_{vt}) \leq 0$. Consider the path $F^* = p_{vt} \cup p_{ts}$. We have $\deg_{F^*}(u) = 0$, and F^* is a basic factor of Ω. Since

$$\omega(F') = \omega(p_{vt}) + \omega(p'_{vt}) + \omega(p_{ts}) = \omega(F^*) + \omega(p'_{vt}) > 0$$

and $\omega(p'_{vt}) \leq 0$, we have $\omega(F^*) > 0$. We are done.

(b) $V_\cap = \{s\}$.
Still, the cycle C in F' is a dangling cycle with the connecting point t. We can contract C to t and add the weight $\omega(C)$ to its connecting bridge. Then, this case is similar to Case I.1.(b). By replacing the vertex t in Case I.1.(b) by the C, one can check that the proof of Case I.1.(b) works here. Note that the path p_{st} in Case I.1.(b) is replaced by a tadpole graph q_{st^3} which is still a basic factor.

(c) $V_\cap = \{v, s\}$.
In this case, $\deg_H(u) = 1$, $\deg_H(v) = \deg_H(s) = \deg_H(t) = 3$, $\pi(v) = \{0, 1, 3\}$, $\pi(s) = \{0, 2, 3\}$, and $\pi(t) = \{0, 1, 3\}$ or $\{0, 2, 3\}$. There are two subcases depending on whether the intersection point v appears in the path part or the cycle part of the tadpole graph F'. Note that there is only one way for the intersection point s to appear in the path F, and s always splits F into two paths p_{us} and p_{st}.

i. v appears in the path part.
Still, the cycle C is a dangling cycle with the connecting point t. This case is similar to Case I.1.(c). By replacing the vertex t in Case I.1.(c) by the cycle C, one can check that the proof of Case I.1.(c) works here. Note that the path p_{vt} and the tadpole graph $F^* = q_{tv^3}$ in Case I.1.(c) are replaced by a tadpole graph and a dumbbell graph respectively. Both are still basic factors.

ii. v appears in the cycle part.
Together with the point t, the point v splits the cycle in F' into two parts p_{vt} and p'_{vt}. Let p_{st} denote the path in F' with endpoints s and t. Then, $F' = p_{st} \cup p_{vt} \cup p'_{vt}$. (See Figure 8.)

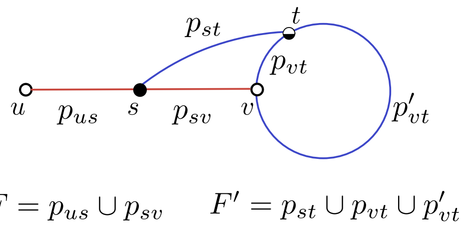

Figure 8: The graph H in Case I.2.(c)

- If $\pi(t) = \{0, 1, 3\}$, then the paths p_{vt} and p'_{vt} are both basic factors of Ω. If $\omega(p_{vt}) > 0$ or $\omega(p'_{vt}) > 0$, then we have a basic factor satisfying the requirements. Thus, we may assume $\omega(p_{vt}), \omega(p'_{vt}) \le 0$. Since $\omega(F') = \omega(p_{st}) + \omega(p_{vt}) + \omega(p'_{vt}) > 0$, we have $\omega(p_{st}) > 0$. Consider the path $p_{ut} = p_{us} \cup p_{st}$. It is a basic factor of Ω. Since F is a basic factor of Ω with the largest weight,

$$\omega(F) = \omega(p_{us}) + \omega(p_{sv}) \ge \omega(p_{us}) + \omega(p_{st}) = \omega(p_{ut}).$$

  Then, $\omega(p_{sv}) \ge \omega(p_{st}) > 0$.
  Consider the path $p''_{vt} = p_{sv} \cup p_{st}$. It is a basic factor of Ω and $\deg_{p''_{vt}}(u) = 0$. Also, $\omega(p''_{vt}) = \omega(p_{sv}) + \omega(p_{st}) > 0$. We are done.
- If $\pi(t) = \{0, 2, 3\}$, then the tadpole graph $q_{uv^3} = F \cup p_{vt} \cup p'_{vt}$ is a basic factor of Ω. Since $\omega(F) \ge \omega(q_{uv^3})$, we have $\omega(p_{vt}) + \omega(p'_{vt}) \le 0$. Since $\omega(F') > 0$, we have $\omega(p_{st}) > 0$. Consider the path $p'_{uv} = p_{us} \cup p_{st} \cup p_{vt}$. It is a basic factor of Ω. Since $\omega(F) \ge \omega(p'_{uv})$, we have

$$\omega(p_{sv}) \ge \omega(p_{st}) + \omega(p_{vt}).$$

  Similarly, consider the path $p''_{uv} = p_{us} \cup p_{st} \cup p'_{vt}$. We have

$$\omega(p_{sv}) \ge \omega(p_{st}) + \omega(p'_{vt}).$$

  Sum up the above two inequalities, we have

$$2\omega(p_{sv}) \ge 2\omega(p_{st}) + \omega(p_{vt}) + \omega(p'_{vt}) \ge 2\omega(p_{st}) > 0.$$

  Consider the theta graph $F^* = p_{sv} \cup F'$. Note that F^* is a basic factor of Ω and $\deg_{F^*}(u) = 0$. Also, $\omega(F^*) = \omega(p_{sv}) + \omega(F') > 0$. We are done.

We are done with Case I.2 where F is a path and F' is a tadpole graph.

I.3. F is a path and F' is a dumbbell graph. Then, $V_\cap = \{v\}$.

Let C_s and C_t be the two cycles in F' that contain vertices s and t respectively. Clearly, among V_{C_s} and V_{C_t}, there exists at least one such that it does not contain the intersection point v. Notice that vertices s and t are symmetric in this case. Without loss of generality, we may assume that $v \notin V_{C_s}$. Then, $V_{C_s} \cap V_F = \emptyset$. Thus, C_s is a dangling cycle with the connecting point s. Then, this case is similar to Case I.2.(a). By replacing the vertex s in Case I.2.(a) by the cycle C_s, one can check that the proof of Case I.2.(a) works here.

I.4. F is a path and F' is a theta graph. Then, $V_\cap = \{v\}$.

In this case, $\deg_H(u) = 1$, $\deg_H(v) = \deg_H(s) = \deg_H(t) = 3$, and $\pi(v) = \{0, 1, 3\}$. Since F' is a theta graph and $\deg_{F'}(s) = \deg_{F'}(t) = 3$, without loss of generality, we may

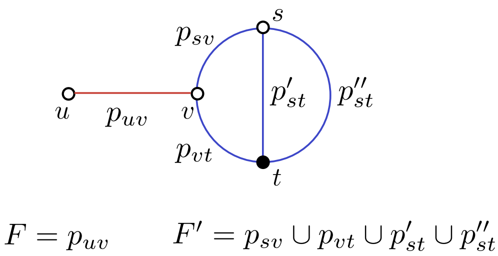

Figure 9: The graph H in Case I.4

assume that $\pi(s) = \{0, 1, 3\}$ and $\pi(t) = \{0, 2, 3\}$. F' consists of three paths p_{st}, p'_{st} and p''_{st}. Without loss of generality, we may assume that v appears in the path p_{st} and it splits the path into two paths p_{sv} and p_{vt}. (see Figure 9.)

Consider the paths $p'_{sv} = p'_{st} \cup p_{vt}$ and $p''_{sv} = p''_{st} \cup p_{vt}$, and the tadpole graph $q_{vs^3} = p_{sv} \cup p'_{st} \cup p''_{st}$. It can be checked that p'_{sv}, p''_{sv} and q_{vs^3} are all basic factors of H. The vertex u does not appear in any of them. Also,

$$\omega(p_{sv}) + \omega(p'_{sv}) + \omega(p''_{sv}) + \omega(q_{vs^3}) = 2\omega(F') > 0.$$

Then, among them at least one is positive. Thus, we can find a basic factor of Ω satisfying the requirements.

We are done with Case I where F is a path.

Case II: F is a tadpole graph and $\deg_F(u) = 3$. By assumption, $\pi(u) = \{0, 2, 3\}$. Also, since $\deg_F(v) = 1 \in \pi(v)$, v is 1-feasible. Let C be the cycle part of F. Consider $\{s, t\} \cap V_C$. Here, we discuss possible cases depending on intersection vertices belonging to V_C instead of the entire set $V_\cap$ of vertices points as in Case I. There are three subcases.

II.1 $\{s, t\} \cap V_C = \emptyset$.

In this case, $\deg_H(x) = 2$ for every $x \in V_C \backslash \{u\}$. Thus, C is a dangling cycle with in connecting point u in H. Then, the case is similar to Case I. By replacing the vertex u in Case I by the cycle C, one can check that the proof of Case I works here. Note that after the above replacement, a path containing u as an endpoint in Case I becomes a tadpole graph containing the cycle C, and a tadpole graph containing u as the vertex of degree 1 in Case I becomes a dumbbell graph.

II.2 $\{s, t\} \cap V_C = \{s\}$ or $\{t\}$.

Without loss of generality, we may assume that $s \in V_C$. Then, $\deg_H(u) = \deg_H(s) = 3$ and $\pi(u) = \pi(s) = \{0, 2, 3\}$. If $\omega(C) > 0$, then we are done since C is a basic factor of Ω and $\deg_C(u) = 2$. Thus, we may assume that $\omega(C) \le 0$. Vertices s and u split C into two paths p_{us} and p'_{us}. Since $\omega(C) = \omega(p_{us}) + \omega(p'_{us}) \le 0$, among them at least one is non-positive. Without loss of generality, we assume that $\omega(p_{us}) \le 0$.

Consider the graph $H' = H \backslash p_{us}$. Note that $V_{H'} = (V_H \backslash V_{p_{us}}) \cup \{u, s\}$. For every $x \in V_{H'} \backslash \{u, s\}$, we have $\deg_{H'}(x) = \deg_H(x) \in \pi(x)$ since H is a factor of Ω. Also, $\deg_{H'}(u) = 2 \in \pi(u)$ and $\deg_{H'}(s) = 2 \in \pi(s)$. Thus, H' is a factor of Ω. Also, $\omega(H') = \omega(H) - \omega(p_{us}) > 0$. However, it is not clear whether H' is a *basic* factor of Ω. Consider the sub-instance $\Omega'_H = (H', \pi_{H'}, \omega)$ of Ω induced by the factor H'. Since $\omega(H') > 0$, by Lemma 5.6, there is a basic factor $F^* \in \Omega_{H'}$ such that $\omega(F^*) > 0$. Then, $\deg_{F^*}(u) \in \pi_{H'}(u) = \{0, 2\}$. Clearly, F^* is also a basic factor of Ω. We are done.

Note that this proof works no matter whether F' is a path or a tadpole graph, and whether $v \in V_\cap$ or $t \in V_\cap$. In fact, this proof also works when F is a dumbbell graph as long as s (or symmetrically t) is the only vertex in $V_{F'}$ appearing in the cycle C of F that contains the vertex u.

II.3 $\{s,t\} \subseteq V_C$.

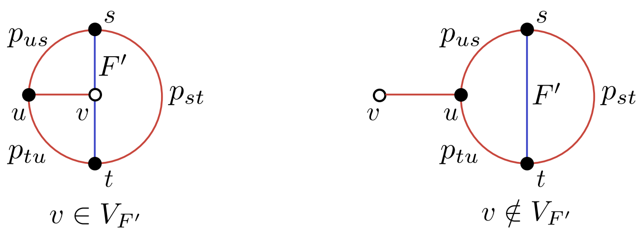

Figure 10: The two possible forms of graph H in Case II.3.

In this case, $\deg_H(u) = \deg_H(s) = \deg_H(t) = 3$ and $\pi(u) = \pi(s) = \pi(t) = \{0,2,3\}$. Also, $\deg_{F'}(s) = \deg_{F'}(t) = 1$. Thus, F' is a path with endpoints s and t. Note that in this case, it is possible that $v \in V_{F'}$. If $v \in V_{F'}$, then $\deg_H(v) = 3$ and $\pi(v) = \{0,1,3\}$; otherwise, $\deg_H(v) = 1$ and $\pi(v) = \{0,1\}$. The points u, s, and t split C into three paths, p_{us}, p_{st}, p_{tu}. Then, $C = p_{us} \cup p_{st} \cup p_{tu}$. (See Figure 10.) If $\omega(C) > 0$, then we are done. Thus, we may assume that $\omega(C) \leq 0$.

Consider the graph $H_1 = H \backslash p_{st} = (F \backslash p_{st}) \cup F'$. Similar to the above Case II.2, one can check that H_1 is a factor of Ω. Also, H_1 is a tadpole graph if $\deg_H(v) = 1$ or a theta graph if $\deg_H(v) = 3$. Thus, in both cases, H_1 is a *basic* factor of Ω. Since F is a basic factor of Ω with the largest weight, we have

$$\omega(F) \geq \omega(H_1) = \omega(F) - \omega(p_{st}) + \omega(F').$$

Thus, $\omega(p_{st}) \geq \omega(F') > 0$. Since $\omega(C) = \omega(p_{st}) + \omega(p_{us}) + \omega(p_{tu}) \leq 0$, we have $\omega(p_{us}) + \omega(p_{tu}) < 0$. Without loss of generality, we may assume that $\omega(p_{us}) < 0$. Then, consider the graph $H_2 = H \backslash p_{us}$. Still, one can check that H_2 is a factor of Ω, and $\deg_{H_2}(u) = 2$. Also, H_2 is a tadpole graph if $\deg_H(v) = 1$, or a theta graph if $\deg_H(v) = 3$. Thus, H_2 is a basic factor of Ω. Moreover, $\omega(H_2) = \omega(H) - \omega(p_{us}) > 0$. We are done.

Case III: F is a tadpole graph and $\deg_F(v) = 3$. In this case, $\deg_F(u) = 1$, $\pi(u) = \{0,1\}$, $\deg_F(v) = 3$, and $\pi(v) = \{0,1,3\}$ or $\{0,2,3\}$. Recall that $\deg_H(u) = \deg_F(u) = 1$, and $u \notin V_\cap$. Let C be the cycle part of F. Still consider $\{s,t\} \cap V_C$. There are three subcases.

III.1 $\{s,t\} \cap V_C = \emptyset$.

In this case, $\deg_H(x) = 2$ for every $x \in V_C \backslash \{v\}$. Thus, in the graph H, the cycle C is a dangling cycle with the connecting point v. Then, the case is similar to Case I. For a graph H in Case I where $\deg_H(v) = 1$ (i.e., $v \notin V_\cap$), by replacing the vertex v by the cycle C, one can check that the proof of Case I works here.

III.2 $\{s,t\} \cap V_C = \{s\}$ or $\{t\}$.

Without loss of generality, we may assume that $s \in V_C$. Then $\deg_H(s) = 3$ and $\pi(s) = \{0,2,3\}$. Vertices s and v split C into two paths p_{vs} and p'_{vs}. Let p_{uv} be the path part in the tadpole graph F. There are two subcases depending on whether $t \in V_F$. Since $t \notin V_C$, $t \in V_F$ implies $t \in V_{p_{uv}}$.

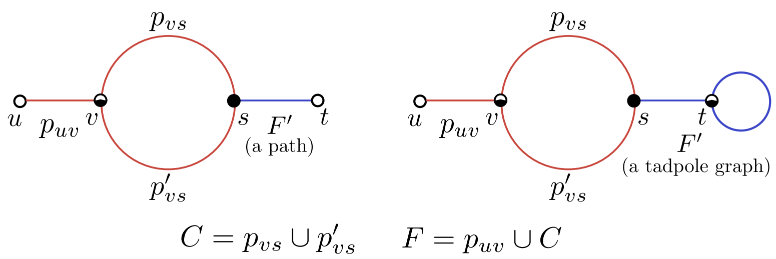

Figure 11: The two possible forms of graph H in Case III.2 where $t \notin V_{p_{uv}}$.

(a) $t \notin V_{p_{uv}}$.

In this case, $\deg_H(t) = 1$ or 3 depending on whether F' is a path or a tadpole graph respectively (See Figure 11). If F' is a tadpole graph, then the cycle in F' containing t is a dangling cycle in H with the connecting point t.

- If $\pi(v) = \{0, 1, 3\}$, then p_{uv} is a basic factor of Ω (this is true no matter whether $t \in V_{p_{uv}}$). Since F is a basic factor of Ω with the largest weight, $\omega(F) \geq \omega(p_{uv})$. Thus, $\omega(p_{vs}) + \omega(p'_{vs}) = \omega(C) = \omega(F) - \omega(p_{uv}) \geq 0$. Without loss of generality, we may assume that $\omega(p_{vs}) \geq 0$. Consider the graph $H' = F' \cup p_{vs}$. It is a path if F' is a path, or a tadpole graph of F' is a tadpole graph. Note that $\deg_{H'}(u) = 0 \in \pi(u)$, $\deg_{H'}(v) = 1 \in \pi(v)$, $\deg_{H'}(s) = 2 \in \pi(s)$, and $\deg_{H'}(t) = \deg_H(t) \in \pi(t)$. Also, for every $x \in V_{H'} \backslash \{u, v, s, t\}$, $\deg_{H'}(x) = \deg_H(x) \in \pi(x)$. Thus, H' is a basic factor of Ω. Also, $\omega(H') = \omega(F') + \omega(p_{vs}) > 0$. We are done.
- If $\pi(v) = \{0, 2, 3\}$, then the cycle C is a basic factor of Ω. Consider $H_1 = H \backslash p_{vs}$. Note that $\deg_{H_1}(u) = 1 \in \pi(u)$, $\deg_{H_1}(v) = 2 \in \pi(v)$, $\deg_{H_1}(s) = 2 \in \pi(s)$, and $\deg_{H_1}(t) = \deg_H(t) \in \pi(t)$. One can check that H_1 is a factor of Ω. Also, H_1 is either a path with endpoints u and s if F' is a path, or a tadpole graph with u being the vertex of degree 1 and t being the vertex of degree 3 if F' is a tadpole graph. Thus, H_1 is a basic factor of Ω. Since F is a basic factor with the largest weight,
$$\omega(F) \geq \omega(H_1) = \omega(F) - \omega(p_{vs}) + \omega(F').$$
Thus, $\omega(p_{vs}) \geq \omega(F') > 0$. Similarly, by considering $H_2 = H \backslash p'_{vs}$, we have $\omega(p'_{vs}) > 0$. Then, $\omega(C) = \omega(p_{vs}) + \omega(p'_{vs}) > 0$. Thus, C is a basic factor of Ω with positive weight and $\deg_C(u) = 0$.

(b) $t \in V_{p_{uv}}$.

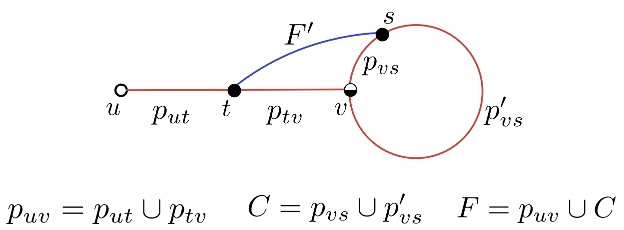

Figure 12: The graph H in Case III.2 where $t \in V_{p_{uv}}$.

In this case, $\deg_H(t) = 3$ and $\pi(t) = \{0, 2, 3\}$. F' is a path with endpoints s and t. The vertex t splits p_{uv} into two parts p_{ut} and p_{tv} (see Figure 12).

- If $\pi(v) = \{0, 1, 3\}$, then p_{uv} is a basic factor of Ω. Since $\omega(F) \geq \omega(p_{uv})$, we have $\omega(C) \geq 0$. Consider the path $p'_{uv} = p_{ut} \cup F' \cup p_{vs}$. It is also a basic factor of Ω.

Still, since $\omega(F) \geq \omega(p'_{uv})$, we have

$$\omega(p_{tv}) \geq \omega(F') + \omega(p_{vs}).$$

Similarly, by considering the path $p''_{uv} = p_{ut} \cup F' \cup p'_{vs}$, we have

$$\omega(p_{tv}) \geq \omega(F') + \omega(p'_{vs}).$$

Thus, $2\omega(p_{tv}) \geq 2\omega(F') + \omega(p_{vs}) + \omega(p'_{vs}) = 2\omega(F') + \omega(C) > 0$. Consider the theta graph $F^* = p_{tv} \cup C \cup F'$. Clearly, $\omega(F^*) > 0$. Then, F^* is a basic factor of Ω with $\deg_{F^*}(u) = 0$. We are done.

- If $\pi(v) = \{0, 2, 3\}$, then C is a basic factor of Ω. Consider $H_1 = H \backslash p_{vs}$. It is a tadpole graph with the vertex u of degree 1 and the vertex t of degree 3. Note that $\deg_{H_1}(u) = 1 \in \pi(u)$, $\deg_{H_1}(v) = 2 \in \pi(v)$, $\deg_{H_1}(s) = 2 \in \pi(s)$, and $\deg_{H_1}(t) = 3 \in \pi(t)$. One can check that H_1 is a basic factor of Ω. Thus, H_1 is a basic factor of Ω. Since F is a basic factor with the largest weight,

$$\omega(F) \geq \omega(H_1) = \omega(F) - \omega(p_{vs}) + \omega(F').$$

Thus, $\omega(p_{vs}) \geq \omega(F') > 0$. Similarly, by considering $H_2 = H \backslash p'_{vs}$, we have $\omega(p'_{vs}) > 0$. Then, $\omega(C) = \omega(p_{vs}) + \omega(p'_{vs}) > 0$. Thus, C is a basic factor of Ω with positive weight and $\deg_C(u) = 0$.

III.3 $\{s, t\} \in V_C$.

In this case, $\deg_H(u) = 1$, $\pi(u) = \{0, 1\}$, $\deg_H(v) = \deg_H(s) = \deg_H(t) = 3$, and $\pi(s) = \pi(t) = \{0, 2, 3\}$. Also, $\deg_{F'}(s) = \deg_{F'}(t) = 1$. Thus, F' is a path with endpoints s and t. Let $p_{st} \subseteq C$ be the path with endpoints t and s such that $v \notin V_{p_{st}}$.

Consider the tadpole graph $q_{uv^3} = (F \backslash p_{st}) \cup F'$. In other words, q_{uv^3} is the tadpole graph obtained from F by replacing the path p_{st} by F'. One can check that q_{uv^3} is also a basic factor of Ω. Since F is a basic factor of Ω with the largest weight,

$$\omega(F) \geq \omega(q_{uv^3}) = \omega(F) - \omega(p_{st}) + \omega(F').$$

Thus, $\omega(p_{st}) \geq \omega(F') > 0$. Consider the cycle $C' = p_{st} \cup F'$. Note that it is a basic factor of Ω. Also, $\deg_{C'}(u) = 0$ and $\omega(C') = \omega(p_{st}) + \omega(F') > 0$. We are done.

Case IV: F is a dumbbell graph. Let C_u and C_v be the two cycles of F containing vertices u and v respectively.

If $\{s, t\} \cap C_v = \emptyset$, then C_v is a dangling cycle in H with the connecting point v. This case is similar to Case II. For a graph H in Case II where $\deg_H(v) = 1$ (i.e., $v \notin V_\cap$), by replacing the vertex v by the cycle C_v, one can check that the proof of Case II works here.

If $\{s, t\} \cap C_u = \emptyset$, then C_u is a dangling cycle in H with the connecting point u. This case is similar to Case III. By replacing the vertex u in Case III by the cycle C_u, one can check that the proof of Case III works here.

If $\{s, t\} \cap C_u$ and $\{s, t\} \cap C_v$ are both non-empty, then without loss of generality, we may assume that $s \in C_u$ and $t \in C_v$. Thus, F' is a path with endpoints s and t. As we have mentioned in Case II.2, one can check that the proof of Case II.2 works here.

Case V: F is a theta graph. In this case, $\deg_H(u) = \deg_H(v) = 3$. By assumption, $\pi(u) = \{0, 2, 3\}$. Also, by the definition of theta graphs, $\pi(v) = \{0, 1, 3\}$. Then, $V_\cap \subseteq \{s, t\}$. There are two subcases.

V.1 $V_\cap = \{s\}$ or $\{t\}$.

Without loss of generality, we assume that $V_\cap = \{s\}$. Then, $\deg_H(s) = 3$ and $\pi(s) = \{0, 2, 3\}$. The theta graph F consists of three paths p_{uv}, p'_{uv} and p''_{uv}. Without loss of generality, we may assume that s appears in the path p_{uv} and it splits p_{uv} into two paths p_{us} and p_{sv}.

Consider the paths p_{sv}, $p'_{sv} = p'_{uv} \cup p_{su}$ and $p''_{sv} = p''_{uv} \cup p_{su}$, and the tadpole graph $q_{sv^3} = p_{sv} \cup p'_{uv} \cup p''_{uv}$. They are not factors of H since the degree of s is 1 in all these four graphs. However, by taking the union of F' with any one of them, we can get a basic factor of H and the degree of u in it is even. Since

$$\omega(p_{sv}) + \omega(p'_{sv}) + \omega(p''_{sv}) + \omega(q_{sv^3}) = 2\omega(F') > 0,$$

among them at least one is positive. Also, $\omega(F') > 0$. Then, by taking the union of it with F', we can find a basic factor of Ω satisfying the requirements.

V.2 $V_\cap = \{s, t\}$.

In this case, F' is a path with endpoints s and t. Since F is a theta graph which is 2-connected, we can find a path $p_{st} \subseteq F$ such that $v \notin V_{p_{st}}$. If $u \notin V_{p_{st}}$, then one can check that the theta graph $H' = (F \backslash p_{st}) \cup F'$ is also a basic factor of Ω. Since $\omega(F) \geq \omega(H')$, we have $\omega(p_{st}) \geq \omega(F') > 0$. Then, the cycle $C = p_{st} \cup F'$ is a basic factor of Ω where $\omega(C) = \omega(p_{st}) + \omega(F') > 0$ and $\deg_C(u) = 0$. We are done. Otherwise, $u \in V_{p_{st}}$. The vertex u splits p_{st} into two paths p_{us} and p_{ut}. Consider the theta graph $H' = (F \backslash p_{us}) \cup F'$, where $\deg_{H'}(v) = \deg_{H'}(t) = 3$, $\pi(v) = \{0, 1, 3\}$, and $\pi(t) = \{0, 2, 3\}$. One can check that H' is a basic factor of Ω. Since $\omega(F) \geq \omega(H') = \omega(F) - \omega(p_{us}) + \omega(F')$, we have $\omega(p_{us}) \geq \omega(F') > 0$. Similarly, by considering the theta graph $H'' = (F \backslash p_{ut}) \cup F'$, we have $\omega(p_{ut}) \geq \omega(F') > 0$. Then the cycle $C = p_{st} \cup F' = p_{us} \cup p_{ut} \cup F'$ is a basic factor of Ω where $\omega(C) = \omega(p_{us}) + \omega(p_{ut}) + \omega(F') > 0$ and $\deg_C(u) = 2$. We are done.

We have taken care of all possible cases and finished the proof. □

Combining Lemmas 5.6 and 5.7, we finished the proof of Theorem 4.8.

A Δ-Matroids and Matching Realizability

A Δ-matroid is a family of sets obeying an axiom generalizing the matroid exchange axiom. Formally, a pair $M = (U, \mathscr{F})$ is a Δ-matroid if U is a finite set and $\mathscr{F}$ is a collection of subsets of U satisfying the following: for any $X, Y \in \mathscr{F}$ and any $u \in X \Delta Y$ in the symmetric difference of X and Y, there exits a $v \in X \Delta Y$ such that $X \Delta \{u, v\}$ belongs to $\mathscr{F}$ [Bou87]. A Δ-matroid is *symmetric* if, for every pair of $X, Y \subseteq U$ with $|X| = |Y|$, we have $X \in \mathscr{F}$ if and only if $Y \in \mathscr{F}$. A Δ-matroid is *even* if for every pair of $X, Y \subseteq U$, $|X| \equiv |Y| \mod 2$.

Suppose that $U = \{u_1, u_2, \dots, u_n\}$. A subset $V \subseteq U$ can be encoded by a binary string α_V of n-bits where the i-th bit of α_V is 1 if $u_i \in V$ and 0 if $u_i \notin V$. Then, a Δ-matroid $M = (U, \mathscr{F})$ can be represented by a relation R_M of arity $|U|$ which consists of binary strings that encode all subsets in $\mathscr{F}$. Such a representation is unique up to a permutation of variables of the relation. A degree constraint D of arity n can be viewed as an n-ary symmetric relation which consists of binary strings with the Hamming weight d for every $d \in D$. By the definition of Δ-matroids, it is easy to check that a degree constraint D (as a symmetric relation) represents a Δ-matroid if and only if D has all gaps of length at most 1.

Definition A.1 (Matching Gadget)**.** *A* gadget *using a set $\mathscr{D}$ of degree constraints consists of a graph $G = (U \cup V, E)$ where $\deg_G(u) = 1$ for every $u \in U$ and there are no edges between vertices in U, and a mapping $\pi : V \to \mathscr{D}$. A* matching gadget *is a gadget where $\mathscr{D} = \{\{0,1\},\{1\}\}$. A degree constraint D of arity n is* matching realizable *if there exists a matching gadget $(G = (U \cup V, E), \pi : V \to \{\{0,1\},\{1\}\})$ such that $|U| = n$ and for every $k \in [n]$, $k \in D$ if and only if for every $W \subseteq U$ with $|W| = k$, there exists a matching $F = (V_F, E_F)$ of G such that $V_F \cap U = W$ and for every $v \in V$ where $\pi(v) = \{1\}$, $v \in V_F$.*

The definition of matching realizability can be extended to a relation R of arity n by requiring the set U of n vertices in a matching gadget to represent the n variables of R. If R is realizable by a matching gadget $G = (U \cup V, E)$, then for every $\alpha \in \{0,1\}^n$, $\alpha \in R$ if and only if there is a matching $F = (V_F, E_F)$ of G such that $V_F \cap U$ is exactly the subset of U encoded by α (i.e., for every $u_i \in U$, $u_i \in V_F$ if and only if $\alpha_i = 1$), and for every $v \in V$ where $\pi(v) = \{1\}$, $v \in V_F$. Note that the matching realizability of a relation is invariant under a permutation of its variables. We say that a Δ-matroid is matching realizable if the relation representing it is matching realizable.[6] The following is an equivalent definition for the matching realizability of Δ-matroids.

Lemma A.2. *If a Δ-matroid $M = (U, \mathscr{F})$ is matching realizable, then there is a graph $G = (U \cup W \cup X, E)$ where $\deg(v) = 1$ for every $v \in U \cup X$ and there are no edges between vertices in $U \cup X$, such that for every $V \subseteq U$, $V \in \mathscr{F}$ if and only if there exists $X_1 \subseteq X$ such that the induced subgraph of G induced by the vertex set $V \cup W \cup X_1$ (denoted by $G(V \cup W \cup X_1)$) has a* perfect *matching.*

With a slight abuse of notation, we also say the graph $G = (U \cup W \cup X, E)$ realizes M.

Proof. Let $G = (U \cup W, E)$ be the matching gadget realizing $M = (U, \mathscr{F})$. We construct the following graph G' from G. For every $x \in W$ with $\pi(x) = \{0,1\}$, we add a new edge incident to it. As the edge is added, a new vertex of degree of 1 is also added to the graph. We denote these new vertices by X and these new edges by E_X. Then, one can check that the graph $G' = (U \cup W \cup X, E \cup E_X)$ satisfies the requirements. □

The following result generalizes Lemma A.1 of [KKR18].

Lemma A.3. *Suppose that $M = (U, \mathscr{F})$ is a matching realizable Δ-matroid, and $V_1, V_2 \in \mathscr{F}$. Then, $V_1 \Delta V_2$ can be partitioned into single variables $S_1, \dots, S_k$ and pairs of variables $P_1, \dots, P_\ell$ such that for every $P = S_{i_1} \cup \cdots \cup S_{i_r} \cup P_{j_1} \cup \cdots \cup P_{j_t}$ ($\{i_1, \dots, i_r\} \subseteq [k], \{j_1, \dots, j_t\} \subseteq [\ell]$), $V_1 \Delta P \in \mathscr{F}$ and $V_2 \Delta P \in \mathscr{F}$.*

Proof. By Lemma A.2, there is a graph $G = (U \cup W \cup X, E)$ realizing M. Since $V_1, V_2 \in \mathscr{F}$, there exists $X_1 \subseteq X$ and $X_2 \subseteq X$ such that the induced subgraph $G(V_1 \cup W \cup X_1)$ has a perfect matching M_1, and $G(V_2 \cup W \cup X_2)$ has a perfect matching M_2. Let E_1 and E_2 be the edge sets of M_1 and M_2 respectively. Consider the graph $G' = (U \cup W \cup X, E_1 \Delta E_2)$. Since E_1 covers each vertex in $V_1 \cup W \cup X_1$ exactly once, and E_2 covers each vertex in $V_2 \cup W \cup X_2$ exactly once, for every $v \in (V_1 \cap V_2) \cup W \cup (X_1 \cap X_2)$ in G', $\deg(v) = 0$ or 2, and for every $v \in (V_1 \Delta V_2) \cup (X_1 \Delta X_2)$ in G', $\deg(v) = 1$. Thus, G' is a union of induced cycles and paths, where each path connects two vertices in $(V_1 \Delta V_2) \cup (X_1 \Delta X_2)$. For every vertex $u \in V_1 \Delta V_2$, if it is connected to another vertex $v \in V_1 \Delta V_2$ by a path in G', then we make $\{u, v\}$ a pair. Otherwise (i.e., u is connected to a vertex in $X_1 \Delta X_2$ by a path in G'), we make u a single variable. Then, $V_1 \Delta V_2$ can be partitioned into single variables $S_1, \dots, S_k$ and pairs $P_1, \dots, P_\ell$ according to the paths in G'.

[6] This definition of matching realizability for Δ-matroids is different with the one that is usually used for even Δ-matroids [Bou89, DK15, KKR18], in which the gadget is only allowed to use the constraint $\{1\}$ for perfect matchings, and hence the resulting Δ-matroid must be even.

Moreover, each path in G' is an alternating path with respect to both matchings M_1 and M_2. Pick a union of such paths (note that they are edge-disjoint). Suppose that there are r many paths that connect single variables in $S_{i_1}, \dots, S_{i_r}$ with variables in X, and t many paths that connect pairs $P_{j_1}, \dots, P_{j_t}$. Let $P = S_{i_1} \cup \cdots \cup S_{i_r} \cup P_{j_1} \cup \cdots \cup P_{j_t}$. After altering the matchings M_1 and M_2 according to these t many alternating paths, we obtain two new matchings that cover exactly $(V_1 \Delta P) \cup W \cup X_1'$ for some $X_1' \subseteq X$ and $(V_2 \Delta P) \cup W \cup X_2'$ for some $X_2' \subseteq X$ respectively. Thus, $V_1 \Delta P \in \mathscr{F}$ and $V_2 \Delta P \in \mathscr{F}$. □

Theorem A.4. *A degree constraint D of gaps of length at most 1 is matching realizable if and only if all its gaps are of the same length 0 or 1.*

Proof. By the gadget constructed in the proof of [Cor88, Theorem 2], if a degree constraint has all gaps of length 1 then it is matching realizable.[7] We give the following gadget (Figure 13) to realize a degree constraint D with all gaps of length 0, which generalizes the gadget in [Tut54]. Suppose that $D = \{p, p+1, \dots, p+r\}$ of arity n where $n \geq p + r \geq p \geq 0$. Consider the following graph $G = (U \cup V, E)$: U consists of n vertices of degree 1, and V consists of two parts V_1 with $|V_1| = n$ and V_2 with $|V_2| = n - p$; the induced subgraph $G(V)$ of G induced by V is a complete bipartite graph between V_1 and V_2, and the induced subgraph $G(U \cup V_1)$ of G induced by $U \cup V_1$ is a bipartite perfect matching between U and V_1. Every vertex in V_1 is labeled by the constraint $\{1\}$. There are r vertices in V_2 labeled by $\{0, 1\}$ and the other $n - p - r$ vertices in V_2 labeled by $\{1\}$. One can check that this gadget realizes D.

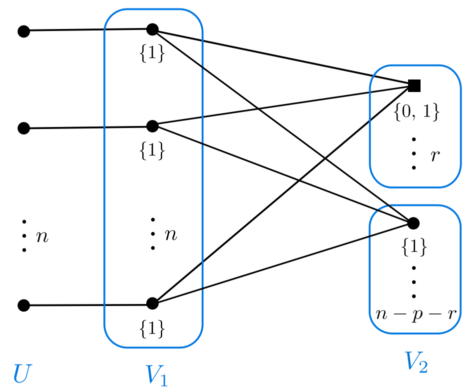

Figure 13: A matching gadget realizing $D = \{p, p+1, \dots, p+r\}$ of arity n

For the other direction, without loss of generality, we may assume that $\{p, p+1, p+3\} \subseteq D$ and $p + 2 \notin D$. Since D has gaps of length at most 1, it can be associated with a symmetric Δ-matroid $M = (U, \mathscr{F})$. Then, there is $V_1 \in \mathscr{F}$ with $|V_1| = p$ and $V_2 \in \mathscr{F}$ with $|V_2| = p + 3$. Since M is symmetric, we may pick $V_2 = V_1 \cup \{v_1, v_2, v_3\}$ for some $\{v_1, v_2, v_3\} \cap V_1 = \emptyset$. Let $S = V_1 \Delta V_2 = \{v_1, v_2, v_3\}$. By Lemma A.3, S can be partitioned into single variables and/or pairs of variables such that for any union P of them, $V_2 \backslash P \in \mathscr{F}$. Since $|S| = 3$, there exists at least a single variable x_i in the partition of S such that $V_2 \backslash \{v_i\} \in \mathscr{F}$. Note that $|V_2 \backslash \{v_i\}| = p+2$. Thus, $p + 2 \in D$. A contradiction. □

[7] We remark that [Cor88] includes gadgets for other types of degree constraints, including type-1 and type-2, but only under a more general notion of gadget constructions that involve edges and triangles. The gadget that only involves edges is a matching gadget defined in this paper.

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