Available online at www.sciencedirect.com

ScienceDirect

Journal of Economic Theory 214 (2023) 105745

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

Coherence without rationality at the zero lower bound ☆

Guido Ascari [a,b,*], Sophocles Mavroeidis [c,d], Nigel McClung [e]

[a] University of Pavia, Italy
[b] De Nederlandsche Bank, Netherlands
[c] University of Oxford, United Kingdom
[d] INET Oxford, United Kingdom
[e] Bank of Finland, Finland

Received 29 July 2022; final version received 5 October 2023; accepted 8 October 2023
Available online 18 October 2023

Abstract

Standard rational expectations models with an occasionally binding zero lower bound constraint either admit no solutions (incoherence) or multiple solutions (incompleteness). This paper shows that deviations from full-information rational expectations mitigate concerns about incoherence and incompleteness. Models with no rational expectations equilibria admit self-confirming equilibria involving the use of simple mis-specified forecasting models. Completeness and coherence are restored if expectations are adaptive or if agents are less forward-looking due to some information or behavioral friction. In the case of incompleteness, the E-stability criterion selects an equilibrium.

JEL classification: C62; E4; E52

Keywords: Incompleteness; Incoherence; Expectations; Zero lower bound

☆ We thank Seppo Honkapohja, Martin Ellison, Klaus Adam, Alexandre Kohlhas, and James Moberly for insightful comments. We also thank seminar participants at the Bank of Finland and De Nederlandsche Bank and participants at the CEPR European Summer Symposium in International Macroeconomic (ESSIM) 2022 conference, the 2022 Barcelona Summer Forum Workshop on Expectations in Dynamic Macroeconomic Models, the Dynare 2022 Conference, 2022 Bank of Finland-CEPR conference, and ASSA 2023. The views expressed in this paper are those of the authors and not necessarily those of the Bank of Finland or De Nederlandsche Bank.

* Corresponding author at: Department of Economics, University of Pavia, Via San Felice 5, 27100 Italy.
E-mail addresses: guido.ascari@unipv.it (G. Ascari), sophocles.mavroeidis@economics.ox.ac.uk (S. Mavroeidis), nigel.mcclung@bof.fi (N. McClung).

https://doi.org/10.1016/j.jet.2023.105745
0022-0531/

The irrationality of a thing is no argument against its existence, rather a condition of it.
[Friedrich Nietzsche, "Human, All Too Human: A Book for Free Spirits", 1878.]

1. Introduction

In the last 15 years since the Great Financial Crisis, central banks in Western economies had to face the problem of a zero (or effective) lower bound (ZLB) on the nominal interest rate. This spurred a very large and important literature on the topic. At least from the seminal contribution by Benhabib et al. (2001), it is well-known that rational expectations (RE) models with a ZLB on the nominal interest rate generally admit multiple equilibria and also multiple steady states. However, the stochastic element in the ZLB literature is often very stylized with one single (often discount factor) shock that occurs only once and has either a stochastic or a known duration.

More recently, Ascari and Mavroeidis (2022, henceforth AM) highlight an even more serious concern regarding this type of models when stochastic shocks hit the economy, a standard assumption in macroeconomic models. They show that in models featuring a ZLB constraint, a stochastic environment and RE, equilibrium existence is not generic, i.e., the model is incoherent, and when these models do admit an equilibrium, they generally admit more equilibria than previously acknowledged, i.e., the model is incomplete.[1] Specifically, AM derive conditions for existence of a rational expectations equilibrium (REE), and for existence and uniqueness of a minimum state variable (MSV) equilibrium for dynamic forward-looking models with occasionally binding constraints. These conditions are difficult to interpret. Therefore, AM highlight a different and more fundamental problem in models with occasionally binding constraints and standard stochastic shocks than the ones already noted in the literature in this class of models, such as the indeterminacy of REE equilibria in linear models and/or multiplicity of steady states. Section 3 reviews the AM results in more detail.

Given that a model without an equilibrium cannot be of any use, this paper points to a possible route to tackle the incoherence problem: abandoning the full-information RE assumption. We show that the problem of incoherence and incompleteness hinges on the assumption that agents have RE. Non-existence of REE is by itself a compelling and novel reason to investigate the possibility of non-rational equilibria. Indeed, one of the main results from this paper is that a standard New Keynesian model with the ZLB constraint can fail to yield a REE and still admit other types of self-confirming equilibria. To illustrate this point, we consider two distinct equilibrium concepts which have been associated with different types of deviations from full-information RE.

First, we investigate one of the most studied deviations from RE, that is, adaptive learning as typified by Evans and Honkapohja (2001). Adaptive learning agents have imperfect knowledge about the economy's structure, but learn to forecast macroeconomic variables by recursively estimating the parameters of a subjective forecasting model using simple statistical tools like least squares. A classic question examined in adaptive learning applications is whether agents eventually learn to forecast rationally, and hence whether the learning economy converges to a REE. However, given that we are interested in cases where a REE does not exist, we assume

[1] Following AM we will use the terms incoherence and incompleteness to mean the non-existence of equilibria and the multiplicity of equilibria, respectively. Hence, a model is coherent if it admits at least one equilibrium, and complete if the equilibrium is unique.

that agents learn by recursively estimating forecasting models that are mis-specified and under-parameterized relative to the forecasting models that agents would have in a REE. Under this assumption, we derive analytically conditions for the economy to settle on a self-confirming equilibrium in which agents make optimal forecasts within their class of forecasting rule. This form of self-confirming equilibrium, which is distinct from REE, is often labeled *restricted perceptions equilibrium (RPE)* in the learning literature (e.g. see Evans and Honkapohja (2001) or Branch (2022)). Importantly, we prove that a RPE can exist when the RE model is incoherent and hence no REE exists.

Second, we consider bounded rationality as a possible deviation from RE. Boundedly rational agents are less forward-looking than rational agents, for instance because they are myopic à la Gabaix (2020), have imperfect common knowledge as in Angeletos and Lian (2018), or have finite planning horizons similar to Woodford and Xie (2022). In this setting, too, a unique *bounded rationality equilibrium (BRE)* may exist, even if a REE does not. Hence, both adaptive learning and bounded rationality might alleviate, under certain conditions, the coherence problem of the standard NK model with a ZLB constraint. Finally, we also investigate the implications of combining the two deviations from rationality.

The derivation of an adaptive learning RPE and BRE in an incoherent REE framework is the central contribution of the paper. In this respect, some remarks are noteworthy.

First, adaptive learning can ensure completeness and coherence all by itself. Specifically, we prove that a unique *temporary* equilibrium always exists in our model with a ZLB constraint and adaptive learning agents, provided that agents do *not* observe current endogenous variables before market clearing takes place—a very common assumption in the learning literature.

Second, a RPE emerges as a self-confirming equilibrium, even if the underlying model does not admit a REE. The learning literature has typically focused on the question of whether a REE can be learnable, because the underlying model admits a REE solution. Here, instead, we investigate whether adaptive learning can generate self-confirming equilibria even when a REE does not exist. When agents do *not* observe current endogenous variables, expectations are predetermined, and a temporary equilibrium always exists, but it is not necessarily self-confirming. To the best of our knowledge, our finding that self-confirming adaptive learning equilibria exist when there is no REE is a novel and intriguing addition to the literature.

Third, and related to the previous point, whenever the NK model does not admit a REE, it is impossible for agents to form self-confirming beliefs about the dynamics of inflation and output (i.e., as implied by a standard MSV in our simple model). The economy can easily diverge into a deflationary spiral if agents attempt to learn these dynamics using simple statistical techniques. Hence, while it is a curse to be smart, it is a blessing to be simple-minded, because the non-rationality of agents' beliefs can save the economy from spiralling out of control and lead it to a coherent and complete self-confirming RPE.

Fourth, the source of the problem of rational incoherence can be intuitively explained in terms of income and substitution effects, following Bilbiie (2022). A similar intuition is behind the so-called "forward guidance puzzle" and its proposed solutions that hinge on weakening agents' forward-lookingness (e.g., Del Negro et al., 2023; McKay et al., 2016b; Angeletos and Lian, 2018; Gabaix, 2020; Woodford and Xie, 2022; Eusepi et al., 2021). Hence, we show that weakening the 'rationality' of agents kills several birds with one stone, because it simultaneously solves different problems highlighted by the literature (forward-guidance puzzle, belief-driven liquidity traps, existence of an equilibrium) that share the same mechanism as a common source.

Fifth, a basic takeaway from the existence analysis is that the baseline NK model with RE is incoherent *when negative shocks are sufficiently large in magnitude or sufficiently persistent*, but can still admit RPE or BRE. A fundamentals-driven RE liquidity trap must, therefore, be relatively short-lived compared to the duration of actual liquidity trap events experienced by Japan, the Euro Area and the U.S., because persistent shocks would make the RE model incoherent. This is not true for the RPE or BRE, where a liquidity trap can be highly persistent. In this sense, one could argue that a RPE or a BRE could explain why the economy did not blow up after a large shock such as the Great Financial Crisis.

Finally, a second contribution of the paper concerns the stability properties of these equilibria under learning, that is, the issue of whether RPE and REE can emerge from a process of learning. Following the adaptive learning literature, we employ the expectational stability or "E-stability" criterion to select an equilibrium that may arise through an economy-wide adaptive learning process in which agents recursively update the parameters of their subjective forecasting models using simple statistical techniques such as least squares. We find there is a unique E-stable RPE when a RPE exists. Similarly, only one MSV REE can be E-stable.

After a brief literature review, the paper proceeds as follows. Section 2 introduces a simple model of the ZLB that nests our different assumptions about expectations formation as special cases. Section 3 illustrates the problem of rational incoherence and the possibility of irrational coherence. Section 4 shows how adaptive learning resolves incompleteness issues, and also discusses the plausibility of the RPE concept. Section 5 concludes. The proofs of all the Propositions can be found in the Appendix.

1.1. Literature review

This paper contributes to an already large literature about deviations from RE and the ZLB. Earlier work on adaptive learning at the ZLB studied monetary and fiscal policies that can prevent an economy with learning agents from getting stuck in a liquidity trap (Evans et al., 2008; Benhabib et al., 2014; Evans et al., 2022b),[2] unconventional policies such as forward guidance (Cole, 2021; Eusepi et al., 2021), "make-up" strategies such as price level targeting (Honkapohja and Mitra, 2020) or average inflation targeting (Honkapohja and McClung, 2021). Christiano et al. (2018) show that the E-stability criterion selects one of multiple equilibria of a model with a transitory demand shock that can drive the economy into a liquidity trap. This finding is closely related to our result about E-stability of REE in the case of incompleteness. However, their model assumes that the economy returns to a steady state after the shock dissipates, whereas our framework allows for multiple, recurring liquidity trap episodes, consistent with the recurrence of ZLB events in the U.S. and elsewhere. Thus, we extend insights from Christiano et al. (2018) to models with recurring demand shocks. More generally, the above mentioned papers do not consider existence and stability of equilibria of models with recurring, fundamentals-driven liquidity traps.

A significant strand of the adaptive learning literature focuses on self-confirming "misspecification equilibria" that can emerge if agents recursively learn to forecast using a misspecified forecasting rule. In a misspecification equilibrium, agents do not understand the true equilibrium law of motion for economic variables, but observable macroeconomic outcomes nonetheless confirm their subjective beliefs about specific statistical properties of the economy. RPE is a special case of misspecification equilibrium involving a "simple" under-parameterized forecasting model that omits some variables which affect the macroeconomic dynamics. In a RPE, agents

[2] See also Evans and McGough (2018) for a related discussion on interest rate pegs and adaptive learning.

forecast optimally within their class of forecasting rules in the sense that forecast errors are orthogonal to their forecasting model. The properties of RPE and misspecification equilibria, as well as their emergence through adaptive learning, have been explored in Branch (2006), Branch (2022), Evans and Honkapohja (2001), Marcet and Sargent (1989), Evans et al. (1993), Branch and Evans (2006a), Branch and Evans (2006b), Bullard et al. (2008), Evans and McGough (2020) and Evans et al. (2022a), Hommes and Sorger (1997), Hommes and Zhu (2014), Branch and Gasteiger (2018), among many others. Empirical support for RPE and related misspecification equilibria comes from experiments involving monetary sticky price economies (Adam, 2007) and analysis of survey and macroeconomic data involving estimation of New Keynesian frameworks (Hommes et al., forth.).[3]

A number of earlier works, including Angeletos and Lian (2018), Gabaix (2020) and Woodford and Xie (2022), study BRE and issues related to the ZLB. Among other things, these papers show that deviations from RE that make agents less forward-looking than rational agents can resolve the so-called NK paradoxes of the ZLB, such as the prediction that forward guidance announcements can have arbitrarily large effects on the economy ("forward guidance puzzle"). Importantly, contributions to this literature typically treat the ZLB regime as arising from a transitory shock, usually with a known duration, after which time the economy returns to steady state forever. Models employing shocks with known duration are not susceptible to the issues of equilibrium existence and multiplicity that we study here. Our contribution, therefore, is to embed bounded rationality into models with recurring stochastic shocks, and to show that these deviations from RE resolve the problem of incoherence and incompleteness identified by AM.

Finally, Mertens and Ravn (2014), Nakata and Schmidt (2019, 2022), and Bilbiie (2022), among others, study conditions for the existence of both fundamentals-driven and confidence-driven liquidity trap equilibria, which are caused by fundamental shocks to the economy and non-fundamental (sunspot) shocks, respectively.[4] One takeaway from these papers is that the fundamentals-driven liquidity trap equilibrium is unlikely to exist if shocks are too persistent, but sunspot equilibria can feature very persistent liquidity traps. However, to our knowledge, confidence-driven liquidity trap equilibria have only been derived in coherent models (i.e. models that admit at least one MSV solution). An incoherent model fails to admit confidence-driven liquidity trap equilibria, and tight restrictions on the support of *fundamental* shocks are necessary for existence of both MSV and confidence-driven liquidity trap equilibria.

2. Model and expectations formation mechanisms

We employ a model that nests the simple New Keynesian model as well as reflects the reduced-form of the alternative bounded rationality models explored by Gabaix (2020), Angeletos and Lian (2018), Woodford and Xie (2022):

$$x_t = M\hat{E}_t x_{t+1} - \sigma(i_t - N\hat{E}_t \pi_{t+1}) + \epsilon_t, \tag{1}$$

$$\pi_t = \lambda x_t + M_f \beta \hat{E}_t \pi_{t+1}, \tag{2}$$

$$i_t = \max\{\psi \pi_t, -\mu\}, \tag{3}$$

[3] See also Slobodyan and Wouters (2012), Ormeno and Molnár (2015), Beshears et al. (2013), Assenza et al. (2021), and Branch and Gasteiger (2018) for additional empirical support for small misspecified forecasting rules.

[4] Additionally, Bianchi et al. (2021) study implications of fundamentals-driven liquidity traps in a nonlinear New Keynesian model.

where x_t is the output gap, i_t the nominal interest rate and π_t is the inflation rate. If $M = N = M_f = 1$, the model nests the simple three-equation New Keynesian model of Woodford (2003) where (1) is the Euler equation, (2) is the NK Phillips Curve and (3) the monetary policy rule, described by the simplest Taylor rule but with a ZLB constraint. The model is log-linearized around the zero inflation steady state and $0 < \beta < 1$, $0 < \sigma, \lambda, \mu$, and $\psi > 1$ (i.e. the "Taylor principle" holds). Bounded rationality implies, instead, $0 < M, N, M_f \leq 1$. Note that $\hat{E}$ denotes (possibly non-rational) expectations and $\hat{E} = E$ denotes model-consistent (rational) expectations.

We follow earlier work, including Eggertsson and Woodford (2003), Nakata and Schmidt (2019), Christiano et al. (2018), and AM, and assume that the demand shock, ϵ_t, follows a two-state Markov process with transition matrix:

$$K := \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix},$$

with $0 < p = Pr(\epsilon_t = \epsilon_1 | \epsilon_{t-1} = \epsilon_1) \leq 1$, $0 < q = Pr(\epsilon_t = \epsilon_2 | \epsilon_{t-1} = \epsilon_2) \leq 1$. If we assume $q = 1$ and $\epsilon_2 = 0$, similar to Eggertsson and Woodford (2003) or Christiano et al. (2018), then we have a model in which a transitory shock, $\epsilon_t = \epsilon_1 \neq 0$, displaces the economy from steady state, but the economy eventually returns to the absorbing steady state of the model when $\epsilon_t = \epsilon_2 = 0$. In the standard RE version of the model there are two non-stochastic steady states: one with zero inflation, and one with zero nominal interest rates. However, equilibrium inflation and output in the temporary state ($\epsilon_t = \epsilon_1$) depend on whether agents have full-information RE or whether they are boundedly rational in some way.

We consider three models of expectations formation. First, agents have full-information RE in the special case of the model with no discounting in the Euler equation and Phillips curve (1)-(3) and model-consistent expectations.

Definition 1. Agents have **full-information rational expectations (RE)** if and only if $\hat{E} = E$ and $M = M_f = N = 1$ in the NK model given by Equations (1)-(3).

A REE, defined in Section 3, is a solution of the model (1)-(3) obtained under these assumptions. In keeping with the literature, we treat full-information RE as the benchmark model of expectations formation, against which we compare ZLB dynamics under alternative expectations formation mechanisms. Particular attention is paid to the possibility that agents do not have full knowledge about the structure of the economy, and consequently expectations can be model-inconsistent (i.e., $\hat{E} \neq E$). The adaptive learning literature in particular studies agents with imperfect knowledge who learn to forecast the law of motion for aggregate variables using standard statistical tools like least squares. In this setting, imperfect knowledge can imply model-inconsistent expectations, but the focus of a large swath of this literature is whether agents can form self-confirming beliefs, either by learning a REE, or some non-rational, self-confirming equilibrium if their subjective forecasting models are mis-specified with respect to the rational forecasting models.

Definition 2. Agents have **imperfect knowledge** if and only if $\hat{E} \neq E$; $M = M_f = N = 1$ in the NK model given by Equations (1)-(3).

Definition 2 follows the "Euler equation approach" to imperfect knowledge, which treats the Euler equation form of the first-order conditions of agents' optimization problem under RE, (1)-(2), as agents' subjective decision rules under imperfect knowledge. The alternative is the so-called "infinite horizon approach" of Preston (2005) according to which optimizing learning agents with imperfect knowledge learn to forecast the path of interest rates, output and inflation.[5] Therefore, our definition of imperfect knowledge involves both non-rational beliefs and sub-optimal decision-making, in keeping with a large literature on imperfect knowledge and learning. Our main conclusion that imperfect knowledge can lead to coherence when the model is rationally incoherent continues to hold under infinite-horizon learning.[6]

We can deviate from RE without relaxing the assumption that agents have full knowledge about the structure of their economic environment. For instance, Gabaix (2020) derives a model in which households and firms are relatively myopic due to cognitive limitations. In this setting, myopia implies a change in the model structure in the form of discounting in the aggregate demand curve (1) (i.e., $M < 1$) and additional discounting in the Phillips curve (2) (i.e. $M_f < 1$). However, nothing in Gabaix's (2020) model prevents agents from having full knowledge about the world they inhabit, and therefore nothing prevents these boundedly rational agents from having model-consistent expectations. Hence, Gabaix's (2020) behavioral model shows how we can deviate from full-information RE without sacrificing the assumption that agents have perfect knowledge. Bounded rationality models by Angeletos and Lian (2018) and Woodford and Xie (2022) may also lead to reduced-form structural models with additional discounting in the structural equations. If M, M_f or N is less than one, we say that agents are boundedly rational.

Definition 3. Agents are said to be **boundedly rational** if and only if $\hat{E} = E$ and $min\{M, M_f, N\} < 1$.

3. Coherence: existence of an equilibrium

To put the whole paper into context, it is worth clarifying the main contributions of AM. While the stochastic element in the literature on the ZLB is often very stylized, featuring one single (often discount factor) shock that occurs only once and has either a stochastic or a known duration, AM consider the general problem of the conditions for existence and uniqueness of equilibria in dynamic forward-looking models with RE when some variables are subject to occasionally binding constraints, like in the ZLB case, and when recurrent stochastic shocks hit the economy, a standard assumption in macroeconomic models. AM propose to use a method based on Gourieroux et al. (1980) that studied this problem in the context of simultaneous equations models with endogenous regime switching, and derived conditions for existence and uniqueness of solutions, which Gourieroux et al. (1980) label as coherency conditions. The problem of existence of equilibria, i.e., coherence, in more standard stochastic environments commonly used in macroeconomic models is obviously fundamental and a first-order concern for this literature.[7]

5 See Bullard and Eusepi (2014) for comparison of Euler equation learning and infinite horizon learning.

6 For brevity, we give those results in Online Appendix B.1.

7 Even though there is a large and expanding literature on solution algorithms for such models, (see e.g., Fernández-Villaverde et al., 2015; Guerrieri and Iacoviello, 2015; Gust et al., 2017; Aruoba et al., 2018, 2021; Eggertsson et al., 2021), there are no general conditions for existence of equilibria for this class of models, as say, the Blanchard-Kahn conditions for standard linear dynamic RE models. Moreover, NK models with a ZLB are often presented as (log)linear approximations around an equilibrium of some originally nonlinear model, whose existence needs to be checked as an obvious precondition of the analysis. A number of theoretical papers provide sufficient conditions for existence of MSV equilibria in NK models (see Eggertsson, 2011; Boneva et al., 2016; Armenter, 2018; Christiano et al., 2018; Nakata, 2018; Nakata and Schmidt, 2019), while AM provide both necessary and sufficient conditions that can be applied more generally.

There are two main takeaways from AM. First, the question of coherence is a nontrivial problem in models with a ZLB constraint and AM were only able to provide some general results for a limited class of models. A typical New Keynesian (NK) model with a ZLB constraint is not generically coherent both when the Taylor rule is active and when monetary policy is optimal under discretion. The restrictions on the support of the shocks that are needed to restore an equilibrium are difficult to interpret because they are asymmetric and because they depend both on the structural parameters and on the past values of the state variables. AM show that the assumption of orthogonality of structural shocks is incompatible with coherence, because if a model admits multiple shocks, their support restrictions cannot be independent from each other. Second, imposing the (somewhat awkward) support restrictions needed to guarantee existence of a solution causes another serious problem: multiplicity of MSV solutions, i.e., incompleteness.[8] AM show the existence of many MSV solutions, possibly up to 2^k MSV equilibria, where k is the number of (discrete) states that the exogenous variables can take, for example, using a k-state approximation of an AR(1) process. While the literature on the ZLB has recognized the possibility of multiple steady states and/or multiple equilibria, and of sunspots solutions due either to indeterminacy or to belief-driven fluctuations between the two steady states, this is a novel source of multiplicity, that concerns 'fundamental' solutions, i.e., MSV ones. This is particularly relevant because numerical solution algorithms usually search for a solution of this type. The multiplicity of MSV solutions arises from the interaction between RE and the non-linear nature of the problem, as we will show below. Our paper investigates whether relaxing the full-information RE assumption could alleviate the problems highlighted by AM by breaking this interaction.

3.1. Rationality without coherence

We start by assuming full-information RE to illustrate the problem of incoherence. For simplicity, we focus on MSV REE, but some of the insights from our paper can be extended to study non-fundamental "sunspot" equilibria which feature extraneous volatility. Since our model, (1)-(3), is a purely forward looking model with a two-state discrete-valued exogenous shock, the MSV REE law of motion for $Y_t = (x_t, \pi_t)'$ will assume the form $Y_t = \mathbf{Y}_j$ where $Y_t = \mathbf{Y}_1$ if $\epsilon_t = \epsilon_1$ and $Y_t = \mathbf{Y}_2$ otherwise.

Definition 4. Rational expectations equilibrium (REE). $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2')'$ is a rational expectations equilibrium if and only if $\mathbf{Y}_j$ solves (1)-(3) given $\hat{E}_t(Y_{t+1}|\epsilon_t = \epsilon_j) = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t = \epsilon_j)\mathbf{Y}_1 + Pr(\epsilon_{t+1} = \epsilon_2|\epsilon_t = \epsilon_j)\mathbf{Y}_2$, for $j = 1, 2$.

There are up to four MSV REE of (1)-(3). First, there is a possible solution in which interest rates are always positive ("PP" solution). Then, there is a potential solution with binding ZLB if and only if $\epsilon_t = \epsilon_1$, which we refer to as the "ZP" solution. Analogously, there could be a "PZ" solution with binding ZLB if and only if $\epsilon_t = \epsilon_2$. Finally, it is possible that the ZLB is always binding ("ZZ" solution). We add a superscript i to $\mathbf{Y}$ to distinguish between the REE (i.e. $\mathbf{Y}^i$ where $i = PP, ZP, PZ, ZZ$). Following AM, if at least one of the four possible REE exist then the model is coherent.

[8] In AM, an MSV equilibrium is defined as usually intended, that is, as a function of the state variables of the model. However, an incoherent model could in principle admit other types of equilibria, but, to the best of our knowledge, no work in the literature, including AM, has found them. We use the terminology MSV and REE interchangeably in the case of incoherence.

Proposition 1. *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. A rational expectations equilibrium (REE) exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{REE}$, where $\bar{\epsilon}_{REE}$ is a constant that depends on the model's parameters, defined in Equation* (A.3) *in Appendix A.1.*

Proposition 1 generalizes Proposition 5 of AM to the case with $q < 1$. It establishes that under the conventional assumption that the Taylor rule (3) satisfies the Taylor Principle and recurrent demand shocks, we need to restrict the magnitude of the shocks, ϵ_t, to get a REE. For a solution to exist, ϵ_1 cannot be too negative (i.e. the shock cannot be too "big", in absolute value). The lower bound on ϵ_1, denoted as $\bar{\epsilon}_{REE}$, is increasing in p for standard parameters, which means that a model with more persistent shocks requires tighter restrictions on the magnitude of the shocks for an equilibrium to exist. This explains why fundamentals-driven liquidity trap cannot be persistent in a REE. A "big" shock is needed to take the economy into a liquidity trap, but then, for a REE to exist, it cannot be persistent. Thus, the model is not generically coherent; solutions only exist for special calibrations of the shock process and solutions do not exist if the shocks are too persistent (i.e. p is very high) or if the shock is big (ϵ_1 is very low).

Intuition from a special case While Proposition 1 deals with the case with $q < 1$, the assumption that the high demand state is absorbing ($q = 1$) and equal to zero ($\epsilon_2 = 0$) is helpful for intuition.[9] Under this assumption, the economy under full-information RE either returns to the steady state with zero inflation (i.e. $\pi_t = x_t = i_t = 0$) or the steady state with zero interest rates (i.e. $i_t = -\mu$, $\pi_t = -\mu < 0$ $x_t = -\mu(1-\beta)/\lambda < 0$). The "temporary state" value of output when $\epsilon_t = \epsilon_1 < 0$ (assuming for brevity that we go back to the zero-inflation steady state) is given by:

$$x_t = \nu(p) E_t x_{t+1} - \sigma \max\{\frac{\psi\lambda}{1-\beta p} x_t, -\mu\} + \epsilon_1, \tag{4}$$

$$\nu(p) := \left(1 + \frac{\lambda\sigma}{1-\beta p}\right) > 1, \tag{5}$$

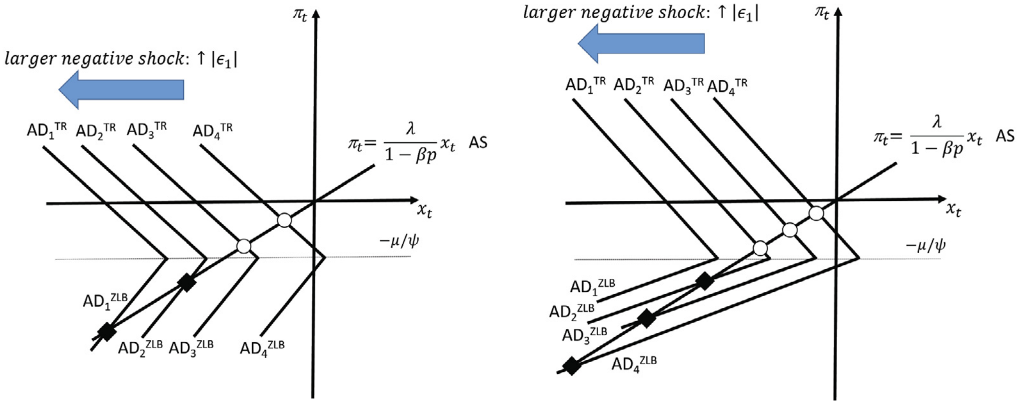
which we obtain by substituting the Phillips curve and Taylor rule into (1). From (4), it is apparent that for any p, sufficiently low values of ϵ_1 preclude unconstrained interest rates. Thus, for a sufficiently negative demand shock, output will be given by:

$$x_t = \frac{1}{1 - p\nu(p)} (\sigma\mu + \epsilon_1), \tag{6}$$

if a solution of the model exists at all. However, if the negative demand shock is sufficiently persistent, so that $p\nu(p) > 1$, then x_t and therefore temporary inflation, $\pi_t = \frac{\lambda}{1-\beta p} x_t$ are decreasing in ϵ_1. This implies that sufficiently large ϵ_1 will increase x_t and π_t, precluding existence of a solution in which the ZLB binds. Therefore, for a solution to exist we need to either restrict p to be small enough to ensure $p\nu(p) < 1$, which in turn implies a solution for any ϵ_1, or, alternatively, we need to restrict ϵ_1 to be close to zero.

Fig. 1a illustrates the determination of demand for the case $p\nu(p) < 1$. It can be seen that a solution exists for any ϵ_1. Fig. 1b illustrates equilibrium determination when $p\nu(p) > 1$. It is apparent that two solutions exist if ϵ_1 is small, but no solution if ϵ_1 is large in magnitude. In this

[9] The assumption $q = 1$ is standard in the literature (e.g., Eggertsson and Woodford, 2003; Christiano et al., 2018; Bilbiie, 2022). To explain the intuition, we borrow heavily from AM and Bilbiie (2022).



(a) $p\nu(p) < 1$

(b) $p\nu(p) > 1$

Note: "AS" ("AD") stands for aggregate supply (demand) curve; "ZLB" stands for zero-lower-bound regime; "TR" stands for Taylor rule. The "AD" is piecewise linear depending on whether the ZLB is binding (AD^{ZLB}) or slack (AD^{TR}). White dots (black squares) indicate equilibria with a positive (zero) interest rate.

Fig. 1. Incoherence and Income vs. Substitution.

case, the model is generally incoherent, while, if we impose support restrictions, i.e., $\epsilon_1 > \bar{\epsilon}_{REE}$, the model is incomplete. The issue of incompleteness will be tackled in Section 4.[10]

How should we interpret this restriction on p and ϵ_1? Following Bilbiie (2022), there are two effects of the demand shock, ϵ_1, when interest rates are pegged at zero. First, a larger demand shock (i.e., a more negative value of ϵ_1) raises real interest rates given a fixed nominal rate, inducing households to save more. This intertemporal substitution effect should put downward pressure on inflation and output. At the same time, $\nu(p) > 1$ implies strong income effects at the ZLB; current income, x_t, responds by *more* than proportionally to an increase in expected future output, $E_t x_{t+1}$. For high values of p, an exogenous increase in real interest rates (via lower ϵ_1) raises demand and inflation through this income effect. In the case where $p\nu(p) > 1$, the income effect dominates the substitution effect, and the negative demand shock has the counter-intuitive effect of raising inflation at the ZLB, while lowering inflation away from the ZLB (see the black squares and white dots respectively in Fig. 1b). In this scenario, we need to make sure that ϵ_1 is not *too* negative. On the other hand, if $p\nu(p) < 1$ then intertemporal substitution effects dominate and a larger negative shock (more negative ϵ_1) pushes down inflation and output, which in turn ensures that a solution with a binding ZLB always exists.

In sum, we can discuss the problem of incoherence in our model in terms of income and substitution effects. RE implies that agents are entirely forward-looking, which in turn allows for a scenario where income effects dominate substitution effects. Tight restrictions on the persistence parameter, p, are necessary to avoid such cases, while restrictions on ϵ_1 are essential to ensure equilibrium existence when income effects are strong. Much of the rest of this paper investigates whether deviations from RE can ensure that these substitution effects dominate income effects

[10] In fact two or four solutions exist in the two cases, respectively, depending on whether one assumes the economy returns to the zero-inflation steady state—as in Figs. 1a and 1b—or one assumes the economy goes to the permanent liquidity trap steady state—not depicted in Figs. 1a and 1b. Moreover, the figures express visually the way the condition $p\nu(p) \lesseqqgtr 1$ relates to the relative slope of the AS and the AD curve under the ZLB. See AM.

when $p\nu(p) > 1$, thus opening up the possibility that non-rational solutions exist even when rational solutions do not.

3.2. Coherence without rationality

We now turn to the question of what happens if no REE exists. Specifically, we investigate the possible existence of non-rational equilibria. First, we look at the case of imperfect knowledge as in Definition 2. Agents with imperfect knowledge are assumed to recursively estimate simple subjective forecasting models in the spirit of the adaptive learning literature. We assess existence of *temporary equilibria* when agents are learning. Then, we ask if there exists an adaptive learning process that could generate an equilibrium where agents expectations are confirmed. We show that a self-confirming RPE may emerge as the outcome of an adaptive learning process where agents use an under-parameterized forecasting rule and attempt to forecast period-ahead inflation and output using their estimates of the long-run average of both variables. Second, bounded rationality does not need to imply imperfect knowledge, and so it is important to consider what happens when agents are boundedly rational as in Definition 3. It turns out that bounded rationality in the form of discounting $(M, M_f, N < 1)$ can imply an even more complete resolution of the problem of incoherence than RPE, provided that the discount factors are exogenously given and do not depend on the magnitude of the shock.

3.2.1. Restricted perceptions

The model (1)-(3) has a single state variable, ϵ_t, which follows a regime-switching process. Consequently, the REE law of motion for output and inflation is a regime-switching intercept—see Definition 4. Rational agents are assumed to know the functional form of the REE solution. However, agents without RE could fail to grasp the structure of the REE, particularly so in the case of incoherence when no such equilibrium exists. Consequently, they might try to forecast inflation and output using an under-parameterized forecasting model which omits the state variable, ϵ_t. Agents with these restricted perceptions instead try to forecast the *unconditional* mean of output and inflation:

$$\hat{E}_t Y_{t+j} = Y_t^e = Y_{t-1}^e + t^{-1}\left(Y_{t-k} - Y_{t-1}^e\right), \tag{7}$$

where Y_t^e is the agents' most recent least squares estimate of the unconditional mean of $Y = (x, \pi)'$ using all data available from $t = 0, \ldots, t-k$ where $k = 0$ if agents have current information and $k = 1$ if agents have lagged information and only observe endogenous variables after markets clear. We assume a decreasing gain parameter equal to t^{-1}, but more generally the gain parameter could be a small constant, $g_y \in (0, 1]$ for $y = x, \pi$ ("constant-gain learning"), or a mix of constant-gain and decreasing-gain learning as in Marcet and Nicolini (2003).

If we substitute (7) into the model (1)-(3) with $M = M_f = N = 1$ then we have the following result.

Proposition 2. *The model (1)-(3) with $M = M_f = N = 1$ and expectations formed according to (7) with $k = 1$ is coherent and complete for all $\sigma, \lambda, \psi > 0$.*

Coherence and completeness means in this context that the model admits a "temporary equilibrium", that is, it has a unique solution for the endogenous variables Y_t for any given $p, q, \epsilon_1, \epsilon_2$, provided that Y_t is not observed contemporaneously (i.e. $k = 1$). We consider this to be an inherently significant finding. From a theoretical perspective, it shows that relying on the lagged information assumption, commonly employed in the adaptive learning literature, suffices to solve the coherence problem in a NK model with a ZLB constraint.[11] Intuitively, learning implies that expectations are predetermined, and this simplifies the task of computing the market clearing equilibrium allocation relative to the nontrivial fixed point problem needed to solve for the REE. From an empirical perspective, inflation has been mostly low but stable during and after the Great Recession, contrary to the prediction of deflationary spirals in an RE model. This proposition could provide a possible account of this period, so that inflation is actually determined by a temporary equilibrium, where agents update their beliefs based on an under-parameterized forecast rule as data becomes available with a lag.

Though a temporary equilibrium for the economy always exists, learning agents do not have expectations that are necessarily consistent with the data they observe. An equilibrium, instead, is a *self-confirming equilibrium* if the learning agents' subjective inflation and output forecasts coincide with the true unconditional means of inflation and output, that is if:

$$\hat{E}_t Y_{t+j} = E(Y) = \bar{q}\hat{\mathbf{Y}}_2 + (1-\bar{q})\hat{\mathbf{Y}}_1,$$

where $Y = (x, y)'$, $\hat{\mathbf{Y}}_j$ is Y_t when $\epsilon_t = \epsilon_j$ and $\bar{q} = Pr(\epsilon_t = \epsilon_2) = (1-p)/(2-p-q)$. If the agents form conditional forecasts using the unconditional mean of inflation and output (i.e. if $\hat{E}_t Y_{t+j} = E(Y)$) then agents' beliefs about the long-run averages of inflation and output are true and self-confirming only if $\hat{\mathbf{Y}}_j$ solves (1)-(3) given $\hat{E}_t Y_{t+j} = E(Y) = \bar{q}\hat{\mathbf{Y}}_2 + (1-\bar{q})\hat{\mathbf{Y}}_1$ and $\epsilon_t = \epsilon_j$ for $j = 1, 2$.

Definition 5. ***Restricted perceptions equilibrium (RPE).*** *$\hat{\mathbf{Y}} = (\hat{\mathbf{Y}}_1', \hat{\mathbf{Y}}_2')'$ is a restricted perceptions equilibrium if and only if (i) $\hat{\mathbf{Y}}_j$ solves (1)-(3) given $E_t Y_{t+1} = \bar{\mathbf{Y}} := \bar{q}\hat{\mathbf{Y}}_2 + (1-\bar{q})\hat{\mathbf{Y}}_1$ and $\epsilon_t = \epsilon_j$ for $j = 1, 2$; and (ii) $E(Y_t) = \bar{\mathbf{Y}}$.*[12]

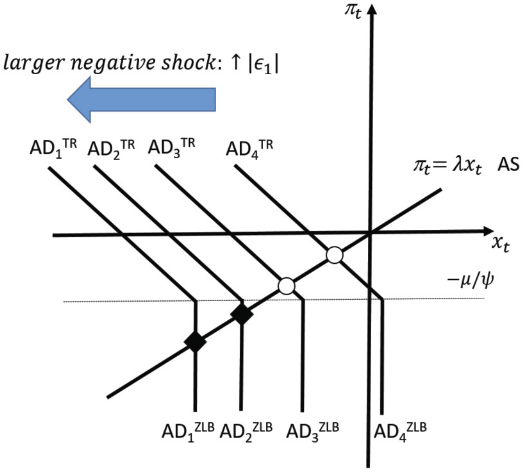
There are four possible RPE of (1)-(3) indexed by $i = PP, ZP, PZ, ZZ$, which are analogous to the REE discussed earlier. In a RPE, agents have "restricted perceptions" in the sense that they omit key fundamental state variables from their forecasting models, that is, they use an under-parameterized forecast rule. In our simple model, ϵ_t is the only state variable. Consequently, the natural under-parameterized forecast rule for this model omits ϵ_t as (7) does. This RPE concept also makes the analysis tractable, leading to the following useful result.

Proposition 3. *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. Then:*

i. A restricted perceptions equilibrium (RPE) exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{RPE}$, where $\bar{\epsilon}_{RPE}$ depends on the model's parameters, see Equation (A.5) in Appendix A.3, and satisfies $\bar{\epsilon}_{RPE} = -\infty$ if $q = 1$.
ii. $\bar{\epsilon}_{REE} \geq \bar{\epsilon}_{RPE}$ if and only if $p + q \geq 1$.

[11] If $k = 0$ then a temporary equilibrium can fail to exist for small values of t with decreasing-gain, or sufficiently large constant gain parameters. Therefore, under contemporaneous information we need to restrict the magnitude of the gain parameter to get a solution. Evans and McGough (2018) document that constant-gain learning models with contemporaneous information can lead to unreasonable predictions when interest rates are pegged. Proposition 2 is a complementary result that favors the lagged information assumption.

[12] See Evans and Honkapohja (2001, sec. 3.6 and 13.1), Branch (2006) and Branch (2022) for a thorough discussion of the RPE concept.



Note: "AS" ("AD") stands for aggregate supply (demand) curve; "ZLB" stands for zero-lower-bound regime; "TR" stands for Taylor rule. The "AD" is piecewise linear depending on whether the ZLB is binding (AD^{ZLB}) or slack (AD^{TR}). White dots (black squares) indicate equilibria with a positive (zero) interest rate.

Fig. 2. Restricted Perceptions Equilibrium.

Proposition 3 is one of the main results of this paper. It tells us that models with persistent shocks (i.e. $p + q > 1$) admit non-rational equilibria but *not* rational equilibria if $\epsilon_1 \in [\bar{\epsilon}_{RPE}, \bar{\epsilon}_{REE})$.[13] Thus we can gain traction in an otherwise incoherent model of the ZLB by assuming restricted perceptions.

As in the case of REE, it is useful to study RPE when $q = 1$ and $\epsilon_2 = 0$ to develop intuition, see Fig. 2. In this case, we have $\bar{q} = 1$ and so the RPE forecast is simply equal to one of the two non-stochastic steady states of the model. Substituting the forecast consistent with the economy reverting to the zero inflation steady state into the model—so $\hat{E}_t x_{t+1} = \hat{E}_t \pi_{t+1} = 0$ in (1)-(3)—and solving for equilibrium output in the temporary state with $\epsilon_t = \epsilon_1$ gives: $x_t = \sigma\mu + \epsilon_1$, assuming the ZLB binds. Thus, effectively the perceived p is equal to zero and the slope of the aggregate demand curve becomes vertical in the temporary state under a ZLB. It follows that a RPE exists for *any* p and ϵ_1. No support restrictions for the shock distribution are needed. Restricted perceptions ensure that the income effects of raising real rates do not dominate the substitution effects, and thus equilibrium is ensured for any values of p and ϵ_1, in accordance with Proposition 3.

3.2.2. Bounded rationality

Assuming bounded rationality in the form of discounting ($M, M_f, N < 1$) yields the following proposition that illustrates how deviations from RE ameliorate incoherence concerns, as in Proposition 3.

Proposition 4. *Consider (1)-(3) and suppose* $min\{M, M_f, N\} < 1$ *and* $\epsilon_2 \geq 0$. *Then:*

[13] We note that $Corr(\epsilon_t \epsilon_{t-1}) = \left(E(\epsilon_t \epsilon_{t-1}) - [E(\epsilon_t)]^2\right) / (E(\epsilon_t^2) - [E(\epsilon_t)]^2) = p + q - 1$. If $p + q = 1$, then there is no distinction between the REE and RPE because ϵ_t is i.i.d.

i. *A bounded-rationality equilibrium (BRE) exists if and only if* $\epsilon_1 \geq \bar{\epsilon}_{BR}$, *for some constant* $\bar{\epsilon}_{BR}$ *that depends on the model's parameters (see Equation* (A.8) *in Appendix A.4).*
ii. *If* $(M-1)(1-M_f\beta)+\lambda\sigma N<0$ *then* $\bar{\epsilon}_{BR}=-\infty$.

Again, we can understand the coherence result in terms of the income and substitution effects of shocks that raises real interest rates at the ZLB. Assume $q=1$ and $\epsilon_2=0$. The BRE value of output in the temporary state binding ZLB is given by:

$$x_t = \nu^{BR}(p)E_t x_{t+1} - \sigma \max\{\frac{\psi\lambda}{1-M_f\beta p}x_t, -\mu\} + \epsilon_1, \tag{8}$$

$$\nu^{BR}(p) := \left(M + N\frac{\lambda\sigma}{1-\beta M_f p}\right).$$

In this bounded rationality model, output at the ZLB is, therefore, given by

$$x_t = \frac{1}{1-p\nu^{BR}(p)}(\sigma\mu+\epsilon_1). \tag{9}$$

Clearly, substitution effects dominate income effects if and only if $p\nu^{BR}(p)<1$, similar to the RE case. However, unlike the RE case, we have $\nu^{BR}(p)<1$ for any p if and only if

$$(M-1)(1-M_f\beta)+\lambda\sigma N<0,$$

which is the condition in Proposition 4. Therefore, myopia can ensure that substitution effects dominate income effects for any p (i.e., implying existence of a MSV solution for any p and ϵ_1).

Not only does $(M-1)(1-M_f\beta)+\lambda\sigma N<0$ ensure coherence in the case of bounded rationality, it also ensures existence of a unique BRE ("completeness"), as formalized in the following proposition.

Proposition 5. *Consider the model given by (1)-(3) and assume* $\psi>1$. *A unique bounded rationality equilibrium (BRE) exists for any* p, q, ϵ_1 *and* $\epsilon_2 \geq 0$ *if and only if* $(M-1)(1-M_f\beta)+\lambda\sigma N<0$. *Further, there exist* $\epsilon^{PP,BR}$ *and* $\epsilon^{ZP,BR}$ *such that* $\epsilon^{PP,BR}>\epsilon^{ZP,BR}$ *and*

i. *The PP solution is the unique BRE if and only if* $\epsilon_1>\epsilon^{PP,BR}$.
ii. *The ZP solution is the unique BRE if and only if* $\epsilon^{PP,BR}\geq\epsilon_1>\epsilon^{ZP,BR}$.
iii. *The ZZ solution is the unique BRE if and only if* $\epsilon_1\leq\epsilon^{ZP,BR}$.

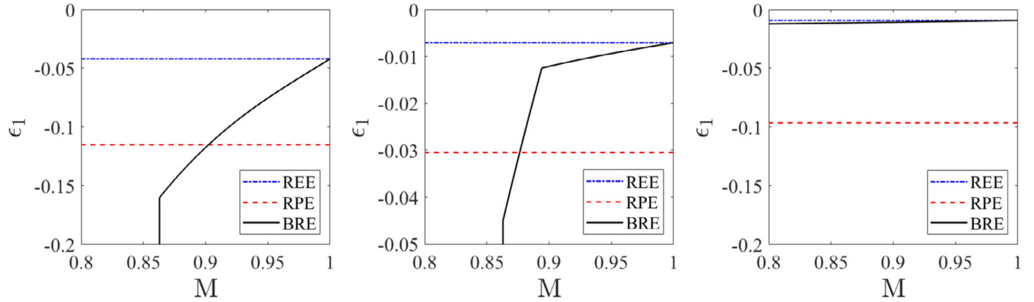
Although the condition $(M-1)(1-M_f\beta)+\lambda\sigma N<0$ completely mitigates concerns about incoherence and incompleteness, it requires a rather high degree of discounting in the Euler and Phillips curve equations. As it turns out, the condition is satisfied by Gabaix's preferred calibration: $M=0.85$, $M_f=0.8$, $N=1$, $\beta=0.99$, $\lambda=0.11$, $\sigma=0.2$. For that calibration, we have:

$$(M-1)(1-M_f\beta)+\lambda\sigma N=-0.0092<0.$$

On the other hand, it is not satisfied for the calibration in McKay et al. (2016a): $M=0.97$, $M_f=N=1$, $\beta=0.99$, $\lambda=0.02$, $\sigma=0.375$. That calibration yields:

$$(M-1)(1-M_f\beta)+\lambda\sigma N=0.0072>0.$$

Thus bounded rationality offers a full solution of the problems of incoherence and incompleteness for some, but not all, calibrations featured in the literature.



(a) (b) $p = 0.965$ (c) $\lambda = 0.2$

Note: The area above the dash-dotted-blue (dashed-red) curve depicts values of ϵ_1 for which at least one REE (RPE) exists. The area above the solid-black curve depicts values of ϵ_1 and $M = M_f$ for which at least one BRE exists. Other parameter values: $\beta = 0.99$, $\sigma = 1$, $\lambda = 0.02$, $q = 0.98$, $p = 0.85$, $N = 1$, $\epsilon_2 = 0.01$.

Fig. 3. Region of Coherence of the REE, RPE, and of the BRE. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

3.2.3. BRE, RPE and coherence

Bounded rationality and imperfect knowledge constitute two distinct departures from RE that are widely discussed in the literature, and they both mitigate concerns about coherence. In this regard, several points are worth considering.

First, bounded rationality might seem to provide a more robust resolution to the problem relative to imperfect knowledge, as coherence can be ensured for any assumption about p, q and ϵ_t if M, M_f, N are sufficiently small. However, this need not be the case if prices are relatively flexible or if agents choose their discount factors optimally as in Moberly (2022).

To illustrate the importance of price rigidity, Fig. 3 depicts different combinations of values for the negative shock, ϵ_1, and for the bounded rationality discount factor, M, that yield coherence in the REE, RPE and BRE cases. The dash-dotted-blue and dashed-red lines depict $\bar{\epsilon}_{REE}$ and $\bar{\epsilon}_{RPE}$, respectively, and the solid-black line depicts $\bar{\epsilon}_{BRE}$ for different values of ϵ_1 and $M = M_f$. Panels (a), (b) and (c) show that the difference between $\bar{\epsilon}_{REE}$, $\bar{\epsilon}_{RPE}$, and $\bar{\epsilon}_{BR}$ can be substantial. Panel (a) shows that larger values of M can rule out existence of BRE in cases where a RPE exists. Panel (b) shows that the same result holds even if the expected duration of the low-demand state is calibrated to match the duration of the 2008-2015 U.S. ZLB episode (i.e. $p = 0.965$ implies an expected duration of 28 quarters). However, if $M < 0.86$ in the calibrated model then $(M-1)(1-M_f\beta) + \lambda\sigma N < 0$ and $\bar{\epsilon}_{BRE} = -\infty$. Panel (c) reveals that in addition to small M, a high degree of price stickiness (small λ) is necessary for the BRE approach to provide a more complete solution to the incoherence problem than the RPE concept. For high values of λ even heavy cognitive discounting in the Euler equation and Phillips curve will not resolve the problem of incoherence.[14] The so-called "curse of flexibility" is therefore a much more pronounced problem for both REE and BRE than for RPE. When considered alongside the theoretical literature on state-dependent models, and the empirical evidence on the time-variation of the frequency of price-setting, both of which indicate that the flexibility of prices might vary with economic conditions, one might expect that in deep recessions where the ZLB

[14] For any M, M_f, N, there is always a large enough value of the product $\lambda\sigma$ to ensure that $(M-1)(1-M_f\beta)+\lambda\sigma N > 0$. Thus, price rigidity and the intertemporal elasticity of substitution play a key role in the existence of BRE.

is binding persistently, prices should be more flexible and thus λ should be high, making the solution provided by BRE less robust.

BRE also may not exist if agents are assumed to choose their discount factors optimally. Thus far, in keeping with most of the literature on the bounded rationality approach by Gabaix (2020), we have kept fixed the cognitive parameters M, M_f, N. However, the degree of attention of agents should be endogenous, and agents might pay more attention when the economy is subject to large shocks, as in deep recessions where the ZLB is binding persistently. Online Appendix B.2 employs the approach developed by Moberly (2022) to endogenize the degree of attention in the Gabaix (2020) model. In Moberly (2022), firms and households face a cost of paying attention, as in Gabaix (2020), and they choose discount factors, $M_{f,\epsilon_t}, M_{\epsilon_t}$ in order to balance the loss of not paying attention with the cost of paying attention. Online Appendix B.2 shows that in this case the shock must be bounded for a solution to exist. Intuitively, it is optimal to pay full attention ($M_{f,\epsilon_t} = M_{\epsilon_t} = 1$) when the shock ϵ_1 is sufficiently large in magnitude. However, a solution does not exist when the shock is large and discount factors are high (see Proposition 4). Online Appendix B.2 details this important caveat, showing that whether bounded rationality solves the problem of incoherence hinges on whether discount factors are predetermined or fixed.

Second, the results above cast doubt on whether the BRE concept can provide a robust solution to the coherence problem, motivating the consideration of alternative departures from RE, that is, imperfect knowledge/adaptive learning. However, it is important to note that the two deviations are not mutually exclusive, and some recent papers have combined imperfect knowledge with myopia or versions of bounded rationality. For example, Hajdini (2022) studies the expectations of myopic agents who have misspecified forecasting models; Meggiorini and Milani (2021) estimates a model that combines adaptive learning and myopia; and Audzei and Slobodyan (2022) derives restricted perceptions equilibrium in an environment that combines adaptive learning and Gabaix's sparse rationality. Similarly, it is possible to combine the two deviations from RE in our model.

Definition 6. Agents have **bounded rationality and imperfect knowledge** if $\hat{E} \neq E$; $\max\{M, M_f, N\} < 1$ in the NK model given by Equations (1)-(3).

The analysis in Appendix A.6 shows that an environment with boundedly rational agents who have imperfect knowledge could admit a bounded rationality RPE.

Definition 7. Bounded rationality restricted perceptions equilibrium (BR-RPE). $\hat{\mathbf{Y}} = (\hat{\mathbf{Y}}_1', \hat{\mathbf{Y}}_2')'$ is a restricted perceptions equilibrium if and only if (i) $\hat{\mathbf{Y}}_j$ solves (1)-(3) given M, M_f, N, $E_t Y_{t+1} = \bar{\mathbf{Y}} := \bar{q}\hat{\mathbf{Y}}_2 + (1-\bar{q})\hat{\mathbf{Y}}_1$ and $\epsilon_t = \epsilon_j$ for $j = 1, 2$; and (ii) $E(Y_t) = \bar{\mathbf{Y}}$.

There are four possible BR-RPE of (1)-(3) indexed by $i = PP, ZP, PZ, ZZ$, which are analogous to the BRE and RPE discussed earlier. Suitable restrictions on the model ensure existence of BR-RPE.

Proposition 6. *Consider (1)-(3) and suppose* $min\{M, M_f, N\} < 1$ *and* $\epsilon_2 \geq 0$. *Then:*

i. A bounded-rationality restricted-perceptions equilibrium (BR-RPE) exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{BR,RPE}$, *for some constant* $\bar{\epsilon}_{BR,RPE}$ *that depends on the model's parameters, see Equation (A.10) in Appendix A.6.*

ii. If $(M-1)(1-M_f\beta) + \lambda\sigma N < 0$, *then* $\bar{\epsilon}_{BR,RPE} = -\infty$.

iii. *If* $(M-1)(1-M_f\beta)+\lambda\sigma N\geq 0$ *and* $p+q\geq 1$ *or if* $(M-1)(1-M_f\beta)+\lambda\sigma N<0$*, then* $\bar{\epsilon}_{BR}\geq\bar{\epsilon}_{BR,RPE}$.

The condition for BR-RPE existence in Proposition 6 is weaker than the condition for BRE existence when the shocks are persistent ($p+q>1$). Thus, the two deviations from RE are not redundant, and combining them leads to a less restricted resolution to the incoherence problem than either assumption alone given that standard calibrations in the literature assume persistent shocks.

Finally, it is well known that bounded rationality can attenuate the so-called "forward guidance puzzle" which is the counter-intuitive prediction that the macroeconomic effects of a promise to cut the interest rate in some future period, T, are strictly increasing in T. Theorem 1 in Online Appendix B.3 proves that the condition in Proposition 4.ii that ensures coherence/completeness in the occasionally-binding constraint framework, also rules out the forward guidance puzzle. Moreover, Propositions 10 and 11 in Online Appendix B.3 show that the forward guidance puzzle is also absent under imperfect knowledge with adaptive learning. Note that the forward guidance problem is a very different problem from the coherence problem highlighted in this section. First, forward-guidance is generated by a peg of the interest rate, while a peg would not be an issue for coherence, i.e., for the existence of an equilibrium. Second, forward guidance is often modeled as a fixed interest rate for a known duration (and a known duration of the negative deflationary shock) and then the policy would revert to a standard Taylor rule. Again, if the duration of the shock and of the peg is known, there is no issue of incoherence. Indeed, the model of forward guidance used in Gabaix (2020) and in Online Appendix B.3 is not susceptible to the problem of incoherence.[15] Thus, both deviations from RE help resolve various puzzles and paradoxes of the New Keynesian ZLB, in addition to resolving the problem of incoherence.

4. Learning to solve incompleteness: multiplicity of (MSV) solutions

We just saw that a BRE can ensure coherence and completeness with sufficient discounting, without any restrictions on the support of the shock. What about completeness in the REE and RPE cases? The coherence condition guarantees existence, but this generally implies a multiplicity of admissible MSV solutions in the case of RE (e.g., Ascari and Mavroeidis, 2022). Incompleteness is a problem that can only be solved using some criterion for selecting an equilibrium. Here we investigate whether learning can provide any guidance, that is, whether the "E-stability" criterion can select an equilibrium of the model as the outcome of an adaptive learning process.

4.1. Learning the REE

In order to derive the conditions under which a REE is E-stable, we first need to be precise about what it means for agents to be learning a REE. As in Section 2, adaptive learning agents have imperfect knowledge and cannot compute an equilibrium analytically. However, these agents make use of a subjective forecasting model or "perceived law of motion" (PLM) when making consumption, labor, savings and pricing decisions consistent with (1)-(2). If the

[15] See also Eusepi et al. (2021), Cole (2021), and Gibbs and McClung (2023) for more on forward guidance and adaptive learning considerations.

learning agents choose a PLM that is also consistent with how expectations are formed in a REE, then it is possible for learning agents to "learn" a REE if their beliefs about the PLM converge to RE, as beliefs are updated recursively using some statistical scheme for estimating the coefficients of the PLM and observable macro data.

Recall from Section 3.1 that our model admits four possible REE in which output and inflation follow a two-state process, which are indexed by superscript i to $\mathbf{Y}$, i.e. $\mathbf{Y}^i$ where $i = PP, ZP, PZ, ZZ$. Agents could conceivably learn one of these REE if their PLM for output and inflation is a two-state process which is estimated recursively using least squares. Consider the following model of learning, in which agents' PLM is a two-state process for inflation and output, like the REE, and beliefs about the state-contingent means are updated recursively using least squares:

$$Y^e_{j,t} = Y^e_{j,t-1} + t^{-1}\mathcal{I}_{j,t-1}\nu^{-1}_{j,t-1}\left(Y_{t-1} - Y^e_{j,t-1}\right), \tag{10}$$

$$\nu_{j,t} = \nu_{j,t-1} + t^{-1}\left(\mathcal{I}_{j,t-1} - \nu_{j,t-1}\right), \tag{11}$$

$$\hat{E}_t Y_{t+1} = Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y^e_{1,t} + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y^e_{2,t}, \tag{12}$$

where $j = 1, 2$, $k\nu_{j,k}$ is the number of periods for which $\epsilon_t = \epsilon_j$ up until time k, and $\mathcal{I}_{j,t} = 1$ if $\epsilon_t = \epsilon_j$ and $\mathcal{I}_{j,t} = 0$ otherwise (i.e. $\mathcal{I}_{j,t} = 1$ is the indicator function for state j). $Y^e_{j,t}$ is the agents' most recent estimate of the state-contingent average of Y_t when $\epsilon_t = \epsilon_j$. According to equation (10), agents revise their beliefs about the state-contingent average of Y in state j (i.e. $Y^e_{j,t}$) in the direction of their time-$t-1$ forecast error only if $\epsilon_{t-1} = \epsilon_j$ (otherwise, $Y^e_{j,t} = Y^e_{j,t-1}$). Equation (12) then gives agents' time-t forecast of period-ahead inflation and forecast. It is assumed that agents observe ϵ_t when forecasting at time-t and also that $Pr(\epsilon_{t+1}|\epsilon_t)$ coincides with the actual transition probabilities—e.g. agents know $Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t = \epsilon_1) = p$ and $Pr(\epsilon_{t+1} = \epsilon_2|\epsilon_t = \epsilon_2) = q$. After agents form time-t expectations, we obtain the time-t market-clearing equilibrium, Y_t, by substituting equation (12) into the model (1)-(3). The process repeats itself at time $t+1$ and so on.[16]

We are interested in knowing if $(Y^e_{1,t}, Y^e_{2,t}) \to (\mathbf{Y}^i_1, \mathbf{Y}^i_2)$ for some REE i as time goes on ($t \to \infty$) and agents' expectations evolve according to (10)-(12). We say that REE i is "stable under learning" if $(Y^e_{1,t}, Y^e_{2,t}) \to (\mathbf{Y}^i_1, \mathbf{Y}^i_2)$ almost surely. When might this convergence of subjective beliefs to RE occur? To make this question tractable, assume that $Y^e_t = (Y^{e'}_{1,t}, Y^{e'}_{2,t})'$ is sufficiently near REE i, such that the ZLB binds under adaptive learning if and only if the ZLB would bind in REE i. This implies the following actual law of motion for Y:

$$Y_t = A^i_t\left(Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t)Y^e_{1,t} + (1 - Pr(\epsilon_{t+1} = \epsilon_1|\epsilon_t))Y^e_{2,t}\right) + B^i_t, \tag{13}$$

for $i \in \{PP, PZ, ZP, ZZ\}$, where $A^{PP}_t = A_P$ and $B^{PP}_t = B_{P,t}$ for all t; $A^{ZZ}_t = A_Z$ and $B^{ZZ}_t = B_{Z,t}$ for all t; $A^{ZP}_t = A_P$ and $B^{ZP}_t = B_{P,t}$ if $\epsilon_t = \epsilon_2$ and $A^{ZP}_t = A_Z$ and $B^{ZP}_t = B_{Z,t}$ otherwise; $A^{PZ}_t = A_P$ and $B^{PZ}_t = B_{P,t}$ if $\epsilon_t = \epsilon_1$ and $A^{PZ}_t = A_Z$ and $B^{PZ}_t = B_{Z,t}$ otherwise, and

$$A_P := \begin{pmatrix} \frac{1}{\lambda\sigma\psi+1} & \frac{\sigma-\beta\sigma\psi}{\lambda\sigma\psi+1} \\ \frac{\lambda}{\lambda\sigma\psi+1} & \frac{\beta+\lambda\sigma}{\lambda\sigma\psi+1} \end{pmatrix} \qquad A_Z := \begin{pmatrix} 1 & \sigma \\ \lambda & \beta+\lambda\sigma \end{pmatrix}$$

[16] Closely related learning algorithms are used by Woodford (1990), Evans and Honkapohja (1994) and (Evans and Honkapohja, 2001, p.305-308) to study the E-stability of sunspot equilibria involving discrete-valued shocks, and by Evans and Honkapohja (1998) to study learnability of fundamental equilibria with exogenous shocks following a finite state Markov chain. We arrive at identical E-stability results if we alternatively assume least squares estimation of a PLM of the form: $Y^e_t = \hat{a} + \hat{b}\mathcal{I}_t$ where $\mathcal{I}_t = 1$ if $\epsilon_t = \epsilon_2$ and 0 otherwise.

$$B_{P,t} := \begin{pmatrix} \frac{\epsilon_t}{1+\lambda\psi\sigma} \\ \frac{\lambda\epsilon_t}{1+\lambda\psi\sigma} \end{pmatrix} \qquad\qquad B_{Z,t} := \begin{pmatrix} \epsilon_t + \sigma\mu \\ \lambda\epsilon_t + \lambda\sigma\mu \end{pmatrix}$$

Given beliefs that are local to RE beliefs, we assess the learnability of equilibrium using the E-stability principle. A REE i is said to be E-stable if it is a locally stable fixed point of the ordinary differential equation (ODE):

$$\frac{\partial \tilde{Y}^e}{\partial \tau} = H^i(\tilde{Y}^e), \qquad \text{where} \qquad H^i(\tilde{Y}^e) := \begin{pmatrix} Y_1^i(Y_1^e, Y_2^e) \\ Y_2^i(Y_1^e, Y_2^e) \end{pmatrix} - \begin{pmatrix} Y_1^e \\ Y_2^e \end{pmatrix}, \tag{14}$$

where τ is "notional" time, $Y_j^i(Y_1^e, Y_2^e)$ is the value of Y when $\epsilon_t = \epsilon_j$ as a function of expectations, $\tilde{Y}^e := (Y_1^{e'}, Y_2^{e'})'$. The relevant Jacobian for assessing the E-stability of REE i is: $DT_{Y^i} := \frac{\partial H^i(\tilde{Y}^e)}{\partial \tilde{Y}^e}|_{\tilde{Y}^e = \mathbf{Y}^i}$. A REE i is E-stable if the eigenvalues of DT_{Y^i} have negative real parts, see Evans and Honkapohja (2001).

There is an intuition for the link between the E-stability condition and stability of beliefs. The ODE (14) is an approximation of the dynamics of Y_t^e near the REE for large t, and it tells us that agents' expectations are revised in the direction of the forecast error, $\bar{Y}^i(Y^e) - Y^e$. If the roots of $DT_{\bar{Y}^i}$ have negative real parts, then agents' expectations about the unconditional means of inflation and output are also revised in the direction of their REE values.

We note the E-stability conditions applied to the REE of the occasionally binding constraint model are identical to the E-stability conditions applied to a model that features exogenous Markov-switching in the monetary policy stance driven entirely by ϵ_t (e.g., see Branch et al., 2013; McClung, 2020).[17] For example, the E-stability condition associated with the ZP equilibrium of (1)-(3) is the same condition associated with the MSV solution of a model that assumes $i_t = \psi\pi_t$ if $\epsilon_t = \epsilon_2$ and $i_t = -\mu$ if $\epsilon_t = \epsilon_1$ regardless of whether the ZLB binds.

Applying the E-stability conditions to the model at hand leads us to the conclusion that only one REE has the property of being E-stable (see Appendix A.7 for the proof).

Proposition 7. *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. Then:*

i. If $\epsilon_1 > \bar{\epsilon}_{REE}$, at most one E-stable rational expectations equilibrium (REE) exists.
ii. The E-stable REE is either the PP REE or the ZP REE.

Proposition 7 somewhat extends insights from Christiano et al. (2018) to models with recurring low demand states (i.e. $q < 1$). Thus Proposition 7 can be applied to study an economy such as the U.S. economy, which has visited the ZLB twice since 2007, following two distinct negative shocks to the economy. The result in Proposition 7 makes it clear that while multiple solutions exist, only one of them can be understood as the outcome of an adaptive learning process. Hence, incompleteness is resolved by E-stability.

4.2. Learning the RPE

We now turn to the question of learnability of RPE. Proposition 3 shows that a RPE can exist even if a REE does not. It turns out multiple RPE may exist when the restrictions in Proposition 3

[17] Mertens and Ravn (2014) also derive E-stability conditions for an equilibrium of a simple New Keynesian model with ZLB constraint, assuming a two-state discrete sunspot shock with an absorbing regime.

hold. Can one or more of these RPE emerge as the outcome an econometric learning process, similar to what we considered in the case of REE? The answer is yes. Here we show that the model may still admit one unique learnable, self-confirming RPE.

First, we must assume agents have a subjective PLM for output and inflation that is consistent with how expectations are formed in a RPE, which is given by equation (7). If we substitute (7) into the model and assume Y_t^e is sufficiently near RPE i then we have the following actual law of motion for Y:

$$Y_t = A_t^i Y_t^e + B_t^i, \tag{15}$$

where A_t^i and B_t^i are defined below equation (13).

We say that RPE i is stable under learning if $Y_t^e \to \bar{\mathbf{Y}}^i$ almost surely, where $\bar{\mathbf{Y}}^i$ denotes the unconditional mean of Y_t^i. Analogous to the discussion of E-stability of REE above, we say that RPE i is said to be E-stable if it is a locally stable fixed point of the ODE, $\partial Y^e/\partial\tau = h^i(Y^e)$, where $h^i(Y^e) = \bar{Y}^i(Y^e) - Y^e$ and $\bar{Y}^i(Y^e)$ is the unconditional mean of Y as a function of expectations, Y^e. Formally, E-stability obtains if the eigenvalues of the Jacobian, $DT_{\bar{Y}^i} := \frac{\partial h^i(Y_e)}{\partial Y^e}|_{Y^e=\bar{\mathbf{Y}}^i}$ have negative real parts. An E-stable RPE is stable under learning if agents estimate Y_t^e using least squares, as in (7), or related estimation routines.

Proposition 8. *Consider (1)-(3) and suppose $M = M_f = N = 1$, $\epsilon_2 \geq 0$. If $\epsilon_1 > \bar{\epsilon}_{RPE}$, then:*

i. There is a unique E-stable restricted perceptions equilibrium (RPE).
ii. The E-stable RPE is either the PP RPE or the ZP RPE.

Online Appendix B.5 shows that a unique E-stable BR-RPE exists in the case where agents both are boundedly rational and have imperfect knowledge and BR-RPE exist.

Proposition 8 indicates that agents can learn a unique RPE, but an attentive econometric agent might also detect that RPE beliefs are misspecified. Is the RPE therefore unreasonable? In the case of coherence we might doubt the plausibility of RPE on the basis that a learnable REE may exist (Proposition 7). However, incoherence precludes REE, and as shown in Online Appendix B.4, agents fail to form self-confirming expectations using a variety of different forecasting models that condition on the demand shock or lags of the endogenous variables in the case of incoherence. Further, the economy easily derails into a deflationary spiral when agents attempt to learn the RE-consistent dynamics of inflation and output when no REE exists, while RPE remain learnable (Proposition 8). Consequently, RPE provide coherent alternatives to REE in the case of rational incoherence by relaxing conditions for existence of a self-confirming equilibrium. In particular, learnable RPE exist when demand shocks are too persistent or large in magnitude, or prices are too flexible, to permit existence of REE. For standard model calibrations, this means that RPE can feature (recurring) ZLB episodes that are expected to last for over a decade, similar to the persistent ZLB events observed in Japan, or even Europe or the US. In contrast, RE ZLB events are implausibly short-lived and usually expected to last for less than 2 years under standard calibrations. Online Appendix B.6 provides the details of these results, alongside brief treatments of RPE in a model with continuous shocks, and an alternative equilibrium concept for incoherent models (Online Appendices B.7 and B.8, respectively). A complete treatment of alternative learnable non-rational equilibria is beyond the scope of this paper, but the existence of such equilibria is not relevant for our main result: rationally incoherent models can be non-rationally coherent.

5. Concluding remarks

Standard RE models with an occasionally binding zero lower bound (ZLB) constraint either admit no solutions (incoherence) or multiple solutions (incompleteness). This paper shows that the problem of incompleteness and incoherence hinges on the assumption of RE.

Models with no rational equilibria may admit self-confirming equilibria involving the use of simple mis-specified forecasting models. The main message of the paper from the existence analysis is that when negative shocks are sufficiently large in magnitude or sufficiently persistent, the baseline NK model is incoherent, but can admit RPE or BRE. Completeness and coherence can be restored if expectations are adaptive or if agents are less forward-looking due to some informational or behavioral friction.

In the case of multiple solutions, the E-stability criterion selects an equilibrium. A RPE can exist as a self-confirming equilibrium, even if the underlying model does not admit a REE. Thus, non-rationality of agents' beliefs can save the economy from blowing up into infinite deflationary spirals, while it yields persistent liquidity traps. These results highlight how deviations from RE help us understand persistent liquidity traps in theoretical models and interpret the recent episodes of liquidity traps in Japan, the Euro Area, and the U.S.

We leave room for future work. In particular, we used the RPE and BRE concepts to make our point simple and clear, and consequently we abstracted from other self-confirming equilibria that could emerge under adaptive learning, such as consistent expectations equilibrium or stochastic consistent expectations equilibrium. Similarly, we excluded other popular forms of non-rationality from our analysis, such as level-k reasoning, or social memory frictions as in Angeletos and Lian (2023).

Finally, we put a premium on analytical results and therefore we focus on a simple theoretical model. Future work could examine related issues in larger, empirically-relevant DSGE models. In that regard, the findings of this paper complement the conclusions of AM about the potential implications of incoherency for estimating models with occasionally binding constraints. In particular, AM discuss the potential identification and misspecification issues arising from using estimation methods that neglect incoherent or incomplete regions of the parameter space under RE. Convergence issues due to incoherence may lead researchers to impose overly restrictive prior distributions, further exacerbating these concerns. Estimating models under deviations from RE may alleviate incoherence and incompleteness issues, thus providing an argument for their use in applied work. It is, therefore, worth studying this issue further in empirical applications including the ZLB, such as Aruoba et al. (2018).

Declaration of competing interest

Declarations of interest: none.

Data availability

No data was used for the research described in the article.

Appendix A

We use the following definitions throughout the proofs: $a := \lambda\sigma$, $\hat{\pi}^i := (\pi_1^i, \pi_2^i)'$, $\rho := p + q - 1$, and e_j is the j-th column of the 2×2 identity matrix, I_2.

A.1. Proof of Proposition 1

Define $Q := I_2 - (1+\beta+\lambda\sigma)K + \beta K^2$.

Case $q<1$ Because $det(Q+\lambda\sigma\psi I_2) = a(\psi-1)(a(\psi-\rho)+(1-\rho)(1-\beta\rho)) > 0$, the PP solution is given by:

$$\hat{\pi}^{PP} = (Q+\lambda\sigma\psi I_2)^{-1}\begin{pmatrix}\lambda\epsilon_1\\ \lambda\epsilon_2\end{pmatrix}.$$

The PP solution exists if and only if $\psi\pi_j^{PP} > -\mu$ for $j=1,2$. We have:

$$\frac{\partial\pi_1^{PP}}{\partial\epsilon_1} = \frac{\lambda((1-q)(1+a-\rho\beta)+a(\psi-1))}{a(\psi-1)(a(\psi-\rho)+(1-\rho)(1-\beta\rho))} > 0,$$
$$\frac{\partial\pi_2^{PP}}{\partial\epsilon_1} = \frac{\lambda(1-q)(a-\beta\rho+1)}{a(\psi-1)(a(\psi-\rho)+(1-\rho)(1-\beta\rho))} > 0.$$

Thus, PP exists if and only if $\epsilon_1 > \epsilon^{PP} = \max\{\epsilon_1^{PP}, \epsilon_2^{PP}\}$, where ϵ_1^{PP} and ϵ_2^{PP} solve $\psi\pi_1^{PP} = -\mu$ and $\psi\pi_2^{PP} = -\mu$, respectively. We have

$$\epsilon_1^{PP} - \epsilon_2^{PP} = \frac{a(\psi-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)(a(\psi-\rho)+(1-\rho)(1-\beta\rho))}{\lambda(1-q)\psi(a-\beta\rho+1)(a(\psi-q)+(1-q)(1-\beta\rho))}$$

and hence $\epsilon_1^{PP} > \epsilon_2^{PP}$. Therefore, the PP solution exists if and only if $\epsilon_1 > \epsilon^{PP} = \epsilon_1^{PP}$, where

$$\epsilon^{PP} = \frac{a^2\mu(\psi-1)(\rho-\psi)}{\lambda\psi(1-(a+1)q+a\psi+\beta(q-1)\rho)} + \frac{a(\lambda\epsilon_2(p-1)\psi+\mu(\psi-1)(1-\rho)(\beta\rho-1)) - \lambda\epsilon_2(p-1)\psi(\beta\rho-1)}{\lambda\psi(1-(a+1)q+a\psi+\beta(q-1)\rho)}. \quad \text{(A.1)}$$

From above, $(Q+\lambda\sigma\psi I_2)^{-1}((\lambda\epsilon_1,\lambda\epsilon_2)')$ is a ZP solution if $\epsilon_1 = \epsilon^{PP}$. If $det(Q+\lambda\sigma\psi e_2 e_2') \neq 0$, then the ZP solution is given by

$$\hat{\pi}^{ZP} = \left(Q+\lambda\sigma\psi e_2 e_2'\right)^{-1}\begin{pmatrix}\lambda\epsilon_1+\lambda\sigma\mu\\ \lambda\epsilon_2\end{pmatrix}.$$

The ZP solution exists if and only if $\psi\pi_2^{ZP} > -\mu \geq \psi\pi_1^{ZP}$. From $\hat{\pi}^{ZP}$ we see that π_1^{ZP} and π_2^{ZP} are linear in ϵ_1 and

$$\frac{\partial\pi_1^{ZP}}{\partial\epsilon_1} = \frac{-\lambda((1-q)(a-\beta\rho+1)+a(\psi-1))}{a(a(p\psi-\rho)-(\beta\rho-1)((p-1)\psi+1-\rho)},$$
$$\frac{\partial\pi_2^{ZP}}{\partial\epsilon_1} = \frac{\lambda(q-1)(a-\beta\rho+1)}{a(a(p\psi-\rho)-(\beta\rho-1)((p-1)\psi+1-\rho)}.$$

Hence, $\frac{\partial\pi_1^{ZP}}{\partial\epsilon_1} > 0$ and $\frac{\partial\pi_2^{ZP}}{\partial\epsilon_1} > 0$ if and only if $den^{ZP} := -(a(p\psi-\rho)-(\beta\rho-1)((p-1)\psi+1-\rho)) = a^{-1}det(Q+\lambda\sigma\psi e_2 e_2') > 0$. Solving for ϵ_1^{ZP} and ϵ_2^{ZP} such that $\psi\pi_1^{ZP} = -\mu$ and $\psi\pi_2^{ZP} = -\mu$, respectively, we have

$$\epsilon_1^{ZP} - \epsilon_2^{ZP} = \frac{a(a\mu(\psi-1)+\lambda\epsilon_2\psi)den^{ZP}}{\epsilon_{\Delta ZP,den}},$$
$$\epsilon_{\Delta ZP,den} := (1-q)\lambda\psi(a-\beta\rho+1)((1-q)(a-\beta\rho+1)+a(\psi-1)) > 0.$$

Therefore, if $den^{ZP} > 0$ ($den^{ZP} < 0$) then $\epsilon_2^{ZP} < \epsilon_1 \leq \epsilon_1^{ZP}$ ($\epsilon_1^{ZP} \leq \epsilon_1 < \epsilon_2^{ZP}$) is necessary and sufficient for ZP existence. Further, $\epsilon_1^{ZP} = \epsilon^{PP}$ and

$$\epsilon_2^{ZP} = \frac{a^2\mu(\psi-1)\rho - \lambda\epsilon_2(p-1)\psi(\beta\rho-1) + a(\lambda\epsilon_2 p\psi + \mu(\psi-1)(1-\rho)(\beta\rho-1))}{\lambda(q-1)\psi(\beta\rho-a-1)}. \tag{A.2}$$

Finally, if $det(Q+\lambda\sigma\psi e_2e_2') = 0$ ($den^{ZP} = 0$) then $\epsilon^{PP} = \epsilon_2^{ZP}$, and a continuum of ZP solutions exists if $\epsilon_1 = \epsilon^{PP}$ and a ZP solution does not exist if $det(Q+\lambda\sigma\psi e_2e_2') = 0$ ($den^{ZP} = 0$) and $\epsilon_1 \neq \epsilon^{PP}$.

One can show that the PZ solution does not exist if $den^{PZ} := det(Q+\lambda\sigma\psi e_1e_1') = 0$. If $det(Q+\lambda\sigma\psi e_1e_1') \neq 0$, the PZ solution is given by

$$\hat{\pi}^{PZ} = \left(Q+\lambda\sigma\psi e_1e_1'\right)^{-1}\begin{pmatrix}\lambda\epsilon_1\\ \lambda\epsilon_2+\lambda\sigma\mu\end{pmatrix}.$$

The PZ solution exists if and only if $\psi\pi_1^{PZ} > -\mu \geq \psi\pi_2^{PZ}$. One can show

$$\frac{\partial\pi_1^{PZ}}{\partial\epsilon_1} = \frac{\lambda(1-(a+1)q+\beta(q-1)\rho)}{a(a(\rho-q\psi)-(\beta\rho-1)(\rho-1-q\psi+\psi))} = \frac{\lambda num_1^{PZ}}{den^{PZ}},$$
$$\frac{\partial\pi_2^{PZ}}{\partial\epsilon_1} = \frac{\lambda(1-q)(a-\beta\rho+1)}{a(a(\rho-q\psi)-(\beta\rho-1)(\rho-1-q\psi+\psi))} = \frac{\lambda num_2^{PZ}}{den^{PZ}}.$$

Clearly $num_2^{PZ} > 0$. Furthermore, if $num_1^{PZ} = 0$ then the PZ solution does not exist. Suppose $num_1^{PZ} \neq 0$ and $den^{PZ} \neq 0$. Solving for ϵ_1^{PZ} and ϵ_2^{PZ} such that $\psi\pi_1^{PZ} = -\mu$ and $\psi\pi_2^{PZ} = -\mu$, respectively, we have

$$\epsilon_1^{PZ} - \epsilon_2^{PZ} = \frac{(a\mu(\psi-1)+\lambda\epsilon_2\psi)den^{PZ}}{\lambda((1-q)(a-\beta\rho+1))\psi num_1^{PZ}}.$$

There are three cases to consider. First, if $den^{PZ} > 0$ (which implies $num_1^{PZ} > 0$ since $num_1^{PZ} = (a(\psi-1))^{-1}\left(den^{PZ} + a(1-p)(1-\beta\rho+a)\right) > 0$), then $\epsilon_1 > \epsilon_1^{PZ} > \epsilon_2^{PZ} \geq \epsilon_1$ is necessary for PZ existence, but not possible. Second, if $den^{PZ} < 0$ and $num_1^{PZ} > 0$, then $\epsilon_1 < \epsilon_1^{PZ} < \epsilon_2^{PZ} \leq \epsilon_1$ is necessary for PZ existence, but not possible. In the third case, $den^{PZ} < 0$ and $num_1^{PZ} < 0$, which implies $\epsilon_2^{PZ} < \epsilon_1^{PZ} < \epsilon_1$ is necessary and sufficient for PZ existence. One can show:

$$\epsilon_1^{PZ} - \epsilon^{PP} = \frac{a(p-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)(a-\beta\rho+1)}{(num_1^{PZ})\lambda((1-q)(1+a-\rho\beta)+a(\psi-1))} \geq 0,$$

if PZ exists (since this requires $num_1^{PZ} < 0$). Therefore, if PZ exists then $\epsilon_1 \geq \epsilon^{PP}$ and hence the PP or ZP solution also exists.

From above, $\left(Q+\lambda\sigma\psi e_2e_2'\right)^{-1}((\lambda\epsilon_1+\lambda\sigma\mu, \lambda\epsilon_2)')$ is a ZZ solution if $\epsilon_1 = \epsilon_2^{ZP}$ and $det(Q+\lambda\sigma\psi e_2e_2') \neq 0$. If $det(Q) \neq 0$, the ZZ solution is given by

$$\hat{\pi}^{ZZ} = (Q)^{-1}\begin{pmatrix}\lambda\epsilon_1+\lambda\sigma\mu\\ \lambda\epsilon_2+\lambda\sigma\mu\end{pmatrix}.$$

The ZZ solution exists if and only if $\psi\pi_j^{ZZ} \leq -\mu$ for $j = 1, 2$. One can show

$$\frac{\partial \pi_1^{ZZ}}{\partial \epsilon_1} = \frac{\lambda(1-(a+1)q+\beta(q-1)\rho)}{a(a\rho-(\rho-1)(\beta\rho-1))} = \frac{\lambda num_1^{ZZ}}{aden^{ZZ}},$$
$$\frac{\partial \pi_2^{ZZ}}{\partial \epsilon_1} = \frac{\lambda(1-q)(a-\beta\rho+1)}{a(a\rho-(\rho-1)(\beta\rho-1))} = \frac{\lambda num_2^{ZZ}}{aden^{ZZ}},$$

where $det(Q) = aden^{ZZ}$. Clearly, $num_2^{ZZ} > 0$. We can further show that $-num_1^{ZZ} = den^{ZZ} + (1-p)(1+a-\rho\beta) \geq den^{ZZ}$. Hence $den^{ZZ} > 0$ implies $num_1^{ZZ} < 0$. Solving for ϵ_1^{ZZ} and ϵ_2^{ZZ} such that $\psi\pi_1^{ZZ} = -\mu$ and $\psi\pi_2^{ZZ} = -\mu$, respectively, we have

$$\epsilon_1^{ZZ} - \epsilon_2^{ZZ} = \frac{aden^{ZZ}(a\mu(\psi-1)+\lambda\epsilon_2\psi)}{\lambda num_1^{ZZ}\psi(1-q)(a-\beta\rho+1)},$$

if $num_1^{ZZ} \neq 0$. There are the following cases to consider. First, if $den^{ZZ} > 0$ (which implies $num_1^{ZZ} < 0$) then ZZ existence requires $\epsilon_2^{ZZ} \geq \epsilon_1 \geq \epsilon_1^{ZZ}$. Second, if $den^{ZZ} < 0$ and $num_1^{ZZ} > 0$ then ZZ existence requires $\epsilon_1 \geq \epsilon_2^{ZZ} > \epsilon_1^{ZZ}$. In the third case, $den^{ZZ} < 0$ and $num_1^{ZZ} < 0$ then ZZ existence requires $\epsilon_1^{ZZ} \geq \epsilon_1 \geq \epsilon_2^{ZZ}$. If $num_1^{ZZ} = 0$ and $det(Q) \neq 0$ then a ZZ exists if and only if $\epsilon_1 \geq \epsilon_2^{ZZ}$. Finally, if $det(Q) = 0$ ($den^{ZZ} = 0$) and $\epsilon_1 = \epsilon_2^{ZP}$ then a continuum of ZZ solutions exists, and if $det(Q) = 0$ ($den^{ZZ} = 0$) and $\epsilon_1 \neq \epsilon_2^{ZP}$ then a ZZ solution does not exist. Now it can be shown that $\epsilon_2^{ZZ} = \epsilon_2^{ZP}$ and

$$\epsilon_1^{ZZ} - \epsilon^{PP} = \frac{a(p-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)(a-\beta\rho+1)}{\lambda num_1^{ZZ}((1-q)(1+a-\rho\beta)+a(\psi-1))} \geq 0,$$

if $num_1^{ZZ} < 0$. Hence ZZ existence and $\epsilon_1 > \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}$ implies ZP or PP existence.

From the analysis above, a REE exists only if $\epsilon_1 \geq \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}$. Further, if $\epsilon_1 \geq \epsilon^{PP}$ then a PP or ZP exists because $det(Q + \lambda\sigma\psi I_2) > 0$. If $\epsilon^{PP} > \epsilon_2^{ZP}$, then $det(Q + a\psi e_2 e_2') = aden^{ZP} \neq 0$ and therefore a PP, ZP or ZZ solution exists if, in addition, $\epsilon_1 \geq \epsilon_2^{ZP}$. We conclude that a REE exists if and only if

$$\epsilon_1 \geq \bar{\epsilon}_{REE} := \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}, \tag{A.3}$$

where ϵ^{PP} and ϵ_2^{ZP} are defined in (A.1) and (A.2), respectively.

Case $q = 1$ Here we show that Proposition 1 nests Proposition 5 of AM as a special case. Specifically, we compute the condition from $\lim_{q\to 1} \bar{\epsilon}_{REE}$ and show that this recovers the result in Proposition 5 of AM.[18] Define $\theta := \frac{(1-p)(1-p\beta)}{\lambda\sigma p} = \frac{(1-p)(1-p\beta)}{ap}$. From the preceding analysis, a REE exists if and only if $\epsilon_1 \geq \bar{\epsilon}_{REE} = \min\{\epsilon^{PP}, \epsilon_2^{ZP}\}$ where ϵ_2^{ZP} can be expressed as $\epsilon_2^{ZP} = \chi(1-q)^{-1}$. In the limit $q \to 1$ we have:

$$\epsilon^{PP} = \mu\left(\frac{a(p-\psi)}{\lambda\psi} - \frac{pa\theta}{\lambda\psi}\right) + \frac{\lambda\epsilon_2(p-1)(a-\beta p+1)}{a\lambda(\psi-1)},$$
$$\chi := \frac{(p(1+a+\beta)-p^2\beta-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)}{(1+a-p\beta)\psi\lambda}.$$

[18] Alternatively, we could repeat the preceding analysis in the model with $q = 1$, but this gives the same result. Mathematica routine available on request.

Now, $p(1+a+\beta)-1-p^2\beta<0$ if and only if $\theta>1$. Therefore, $\bar{\epsilon}_{REE}=\epsilon_2^{ZP}\to-\infty$ as $q\to1$ if $\theta>1$. We conclude that any value of ϵ_1 ensures existence of a solution when $\theta>1$ and $q=1$. If $\theta<1$, then $\chi\to+\infty$ and $\bar{\epsilon}_{REE}=\epsilon^{PP}$, and $\epsilon_2^{ZP}\geq\epsilon^{PP}=\bar{\epsilon}_{REE}$ if $\theta=1$.[19]

Now we show that our conditions recover Proposition 5 in AM. First, we have $\mu=log(r\pi_*)>0$ which implies $r^{-1}\leq\pi_*$ where r and π_* are the steady state gross real interest rate and inflation rate, respectively. Further, we set $\epsilon_2=0$ and $\epsilon_1=-\sigma\hat{M}_{t+1|t}=\sigma pr_L$. The critical threshold, ϵ^{PP} becomes: $-r_L\leq\mu\left(\frac{\theta}{\psi}+\frac{(\psi-p)}{p\psi}\right)$. Thus, a solution exists if and only if $\theta>1$ or $\theta\leq1$ and $-r_L\leq\mu\left(\frac{\theta}{\psi}+\frac{(\psi-p)}{p\psi}\right)$ as in AM.

A.2. *Proof of Proposition 2*

Define $z_t:=\pi_t+\mu/\psi$ and assume $\psi>0$, so that the positive interest rate regime arises when $z_t>0$ (equivalent to $\psi\pi_t>-\mu$), and the zero interest rate regime when $z_t\leq0$. Substituting out i_t, and $\pi_t=z_t-\mu/\psi$, equations (1)-(3) can be written as

$$x_t=x_t^e-\sigma\left(\psi z_t\mathbf{1}\{z_t>0\}-\mu-\pi_t^e\right)+\epsilon_t,$$
$$z_t=\mu/\psi+\lambda x_t+\beta\pi_t^e,$$

or, compactly, as

$$\begin{pmatrix}1 & \sigma\psi\mathbf{1}\{z_t>0\}\\ -\lambda & 1\end{pmatrix}\begin{pmatrix}x_t\\ z_t\end{pmatrix}=\begin{pmatrix}1 & \sigma\\ 0 & \beta\end{pmatrix}Y_t^e+\begin{pmatrix}\sigma\mu+\epsilon_t\\ \mu/\psi\end{pmatrix}, \tag{A.4}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function that takes the value 1 when its argument is true and zero otherwise. With $k=1$ in (7), the variable Y_t^e is predetermined. Coherence and completeness of (A.4) means that the model can be solved uniquely for x_t,z_t (equivalently x_t,π_t). Equation (A.4) is a piecewise-linear continuous simultaneous equations model for $(x_t,z_t)'$ whose coherence conditions (existence and uniqueness of equilibrium) are given by (Gourieroux et al., 1980, Theorem 1). Specifically,

$$\det\begin{pmatrix}1 & \sigma\psi\\ -\lambda & 1\end{pmatrix}\det\begin{pmatrix}1 & 0\\ -\lambda & 1\end{pmatrix}=1+\sigma\lambda\psi>0,$$

which always holds when $\sigma,\lambda,\psi>0$.

A.3. *Proof of Proposition 3*

The proof of Proposition 3 is a straightforward extension of the proof of Proposition 1. Define $\bar{q}:=Pr(\epsilon_t=2)=(1-p)/(2-p-q)$. The regime-specific levels of inflation in RPE i, $\hat{\pi}^i=(\pi_1^i,\pi_2^i)'$, are given by fixed point restrictions that have the same basic form as the REE fixed point restrictions except we replace q with $\bar{q}$ and p with $1-\bar{q}$. Therefore, RPE will exist if and only if

$$\epsilon_1\geq\bar{\epsilon}_{RPE}=\min\{\epsilon^{PP,RPE},\epsilon_2^{ZP,RPE}\}, \tag{A.5}$$

[19] The $\theta=1$ case arises if $a=\frac{(1-p)(1-\beta\rho)}{p}$ and $q=1$. To compute ϵ_2^{ZP}, set $a=\frac{(1-p)(1-\beta\rho)}{p}$ and compute $\lim_{q\to1}\epsilon_2^{ZP}$.

where $\epsilon^{PP,RPE}, \epsilon_2^{ZP,RPE}$ have the same form as $\epsilon^{PP}, \epsilon_2^{ZP}$ given in (A.1), (A.2) except we replace q and p with $\bar{q}$ and $1-\bar{q}$, respectively. In the special case $q=1$ (which implies $\bar{q}=1$), we have $\bar{\epsilon}_{RPE}=-\infty$, as the PP solution exists if and only if $\epsilon_1 > -\mu(1+\lambda\sigma\psi)(\lambda\psi)^{-1}+(1+\lambda\sigma)(\lambda\sigma(1-\psi))^{-1}\epsilon_2=\epsilon^{PP,RPE}$ and the ZP exists if and only if $\epsilon_1 \leq \epsilon^{PP,RPE}$. For $q<1$, one can show: $\epsilon^{PP}-\epsilon^{PP,RPE}=-\Xi_{PP}\rho$ and $\epsilon^{ZP}-\epsilon^{ZP,RPE}=-\Xi_{ZP}\rho$ where $\Xi_{PP} := \frac{a(1+a-\beta(\rho-1))(1-p)(\psi-1)(a\mu(\psi-1)+\lambda\epsilon_2\psi)}{\lambda\psi(a(1-\psi)(1-\rho)+(a+1)(q-1))((1-q)(1+a-\beta\rho)+a(\psi-1))} \leq 0$ and $\Xi_{ZP} := \frac{a(a\mu(\psi-1)+\lambda\epsilon_2\psi)(1+a-\beta(\rho-1))}{\lambda(a+1)(q-1)\psi(1+a-\beta\rho)} < 0$. Hence, $\bar{\epsilon}_{REE} \geq \bar{\epsilon}_{RPE}$ if and only if $p+q \geq 1$.

A.4. Proof of Proposition 4

Define $\delta := (M-1)(1-M_f\beta)+\lambda\sigma N$ and $Q := I_2-(M+M_f\beta+\lambda\sigma N)K+\beta M M_f K^2$.

Case $q<1$ Since $den^{PP,BR} := det\,(Q+\lambda\sigma\psi I_2) = ((1-M\rho)(1-M_f\beta\rho)+a(\psi-N\rho))((1-M)(1-M_f\beta)+a(\psi-N)) > 0$, the PP solution is given by:

$$\hat{\pi}^{PP,BR} = (Q+\lambda\sigma\psi I_2)^{-1}\begin{pmatrix}\lambda\epsilon_1\\ \lambda\epsilon_2\end{pmatrix}.$$

The PP solution exists if and only if $\psi\pi_j^{PP,BR} > -\mu$ for $j=1,2$. We have:

$$\frac{\partial\pi_1^{PP,BR}}{\partial\epsilon_1} = \frac{num_1^{PP,BR}}{den^{PP,BR}} > 0, \qquad \frac{\partial\pi_2^{PP,BR}}{\partial\epsilon_1} = \frac{num_2^{PP,BR}}{den^{PP,BR}} > 0,$$

where

$$\begin{aligned} num_1^{PP,BR} &:= \lambda(a\psi+\beta M M_f(q(p+q)-\rho)-Mq-q(\beta M_f+aN)+1) > 0,\\ num_2^{PP,BR} &:= \lambda(q-1)(\beta M_f(M(p+q)-1)-M-aN) > 0 \end{aligned}$$

Thus, PP exists if and only if $\epsilon_1 > \epsilon^{PP,BR} = \max\{\epsilon_1^{PP,BR}, \epsilon_2^{PP,BR}\}$ where $\epsilon_1^{PP,BR}$ and $\epsilon_2^{PP,BR}$ solve $\psi\pi_1^{PP,BR}=-\mu$ and $\psi\pi_2^{PP,BR}=-\mu$, respectively. We have

$$\begin{aligned} \epsilon^{PP,BR} &= \frac{\eta_1\eta_2\eta_3}{\psi num_1^{PP,BR}}, \\ &\eta_1 := a(\psi-N)+(1-M)(1-M_f\beta) > 0,\\ &\eta_2 := (a(N+\psi)-(p+q)(aN+\beta M_f)+M\rho(\beta M_f\rho-1)+\beta M_f+1),\\ &\eta_3 := \frac{\lambda\epsilon_2(1-p)\psi(\beta M_f(M(p+q)-1)-aN-M)}{den^{PP,BR}}-\mu. \end{aligned} \tag{A.6}$$

From above, $(Q+\lambda\sigma\psi I_2)^{-1}((\lambda\epsilon_1,\lambda\epsilon_2)')$ is a ZP solution if $\epsilon_1=\epsilon^{PP,BR}$. If $det(Q+\lambda\sigma\psi e_2e_2') \neq 0$, then the ZP solution is given by

$$\hat{\pi}^{ZP,BR} = \left(Q+\lambda\sigma\psi e_2e_2'\right)^{-1}\begin{pmatrix}\lambda\epsilon_1+\lambda\sigma\mu\\ \lambda\epsilon_2\end{pmatrix}.$$

The ZP solution exists if and only if $\psi\pi_2^{ZP,BR} > -\mu \geq \psi\pi_1^{ZP,BR}$. We have:

$$\begin{aligned} \frac{\partial\pi_1^{ZP,BR}}{\partial\epsilon_1} &= \frac{\lambda(a(\psi-Nq)+\beta M_f(M(q(p+q)-\rho)-q)-Mq+1)}{den^{ZP,BR}},\\ \frac{\partial\pi_2^{ZP,BR}}{\partial\epsilon_1} &= \frac{\lambda(1-q)(aN+\beta M_f(1-M(p+q))+M)}{den^{ZP,BR}}. \end{aligned}$$

From the last equations, $\frac{\partial \pi_1^{ZP}}{\partial \epsilon_1} > 0$ and $\frac{\partial \pi_2^{ZP}}{\partial \epsilon_1} > 0$ if and only if $den^{ZP,BR} := \delta^2 - \delta(a\psi + (1-\rho)(M + aN + M_f\beta(1 - M(p+q)))) + (1-p)a\psi(M + aN + M_f\beta(1 - M(p+q))) > 0$. Solving for $\epsilon_1^{ZP,BR}$ and $\epsilon_2^{ZP,BR}$ such that $\psi\pi_1^{ZP,BR} = -\mu$ and $\psi\pi_2^{ZP,BR} = -\mu$, respectively, we have

$$\epsilon_1^{ZP,BR} - \epsilon_2^{ZP,BR} = \frac{(\mu((1-M)(1-M_f\beta) + a(\psi - N)) + \lambda\epsilon_2\psi)den^{ZP,BR}}{\lambda\epsilon_{\Delta ZP,BR}},$$

$$\epsilon_{\Delta ZP,BR} := (1-q)num_1^{ZP,BR}(M + aN + M_f\beta(1 - M(p+q))) > 0,$$

$$num_1^{ZP,BR} := \psi(a(\psi - Nq) + \beta M_f(M(q(p+q) - \rho) - q) - Mq + 1) > 0.$$

Therefore, if $den^{ZP,BR} > 0$ ($den^{ZP,BR} < 0$) then $\epsilon_2^{ZP,BR} < \epsilon_1 \le \epsilon_1^{ZP,BR}$ ($\epsilon_1^{ZP,BR} \le \epsilon_1 < \epsilon_2^{ZP,BR}$) is necessary and sufficient for existence of ZP. Further, we can show: $\epsilon_1^{ZP,BR} = \epsilon^{PP,BR}$ and

$$\epsilon_2^{ZP,BR} = \frac{\mu\eta_1(aN - (p+q)(aN + \beta M_f) + M\rho(\beta M_f\rho - 1) + \beta M_f + 1)}{\lambda(q-1)\psi(aN - \beta M M_f(p+q) + M + \beta M_f)} - \frac{\epsilon_2\lambda(aNp + \beta M_f(M(q - p\rho - 1) + p) + Mp - 1)}{\lambda(q-1)(aN - \beta M M_f(p+q) + M + \beta M_f)}. \tag{A.7}$$

Finally, if $det(Q + \lambda\sigma\psi e_2 e_2') = 0$ ($den^{ZP,BR} = 0$) then $\epsilon^{PP,BR} = \epsilon_2^{ZP,BR}$, and a continuum of ZP solutions exists if $\epsilon_1 = \epsilon^{PP,BR} = \epsilon_2^{ZP,BR}$ and no ZP solution exists if $det(Q + \lambda\sigma\psi e_2 e_2') = 0$ ($den^{ZP,BR} = 0$) and $\epsilon_1 \neq \epsilon^{PP,BR}$.

It is straightforward to show that the PZ solution does not exist if $det(Q + \lambda\sigma\psi e_1 e_1') = 0$. If $det(Q + \lambda\sigma\psi e_1 e_1') \neq 0$, the PZ solution is given by

$$\hat{\pi}^{PZ,BR} = \left(Q + \lambda\sigma\psi e_1 e_1'\right)^{-1} \begin{pmatrix} \lambda\epsilon_1 \\ \lambda\epsilon_2 + \lambda\sigma\mu \end{pmatrix}.$$

The PZ solution exists if and only if $\psi\pi_1^{PZ,BR} > -\mu \ge \psi\pi_2^{PZ,BR}$. One can show

$$\frac{\partial \pi_1^{PZ,BR}}{\partial \epsilon_1} = \frac{\lambda(1 - (M + aN)q + M_f(M + Mp(q-1) - q + M(q-1)q)\beta))}{den^{PZ,BR}} = \frac{\lambda num_1^{PZ,BR}}{den^{PZ,BR}},$$

$$\frac{\partial \pi_2^{PZ,BR}}{\partial \epsilon_1} = \frac{\lambda(1-q)(M + aN + M_f(1 - M(p+q))\beta)}{den^{PZ,BR}} = \frac{\lambda num_2^{PZ,BR}}{den^{PZ,BR}},$$

where $den^{PZ,BR} := det(Q + \lambda\sigma\psi e_1 e_1') = -M(aN\rho(\beta M_f(p+q) - 2) + a\psi(\beta M_f(p-1) - \beta M_f q\rho + q) + (\beta M_f - 1)(p+q)(\beta M_f\rho - 1)) + (aN + \beta M_f - 1)(aN\rho + \beta M_f\rho - 1) - a\psi(aNq + \beta M_f q - 1) + M^2(\beta M_f - 1)\rho(\beta M_f\rho - 1)$. Clearly $num_2^{PZ,BR} > 0$. Furthermore, it is straightforward to show that $num_1^{PZ,BR} \neq 0$ is necessary for existence of PZ solution. Solving for $\epsilon_1^{PZ,BR}$ and $\epsilon_2^{PZ,BR}$ such that $\psi\pi_1^{PZ,BR} = -\mu$ and $\psi\pi_2^{PZ,BR} = -\mu$, respectively, we have

$$\epsilon_1^{PZ,BR} - \epsilon_2^{PZ,BR} = \frac{(\eta_1\mu + \psi\lambda\epsilon_2)den^{PZ,BR}}{\lambda(1-q)\psi(M + aN + M_f\beta(1 - M(p+q)))num_1^{PZ,BR}},$$

if $num^{PZ,BR} \neq 0$. There are three cases to consider. First, if $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} > 0$ then $\epsilon_1 > \epsilon_1^{PZ,BR} > \epsilon_2^{PZ,BR} \ge \epsilon_1$ is necessary for PZ existence, but not possible. Second, if $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} > 0$ then $\epsilon_1 < \epsilon_1^{PZ,BR} < \epsilon_2^{PZ,BR} \leq \epsilon_1$ is necessary for PZ existence, but not possible. In the third case, $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} < 0$, which implies $\epsilon_2^{PZ,BR} < \epsilon_1^{PZ,BR} < \epsilon_1$ is necessary and sufficient for PZ existence. Note that $den^{PZ,BR} > 0$ and $num_1^{PZ,BR} < 0$ cannot hold simultaneously because

$$den^{PZ,BR} = \delta(p-1)(M + aN + M_f\beta(1 - M(p+q))) + num_1^{PZ,BR}\eta_1 > 0,$$

requires $\delta < 0$ if $num_1^{PZ,BR} < 0$, but

$$num_1^{PZ,BR} = -\delta + (1-q)(M + aN + M_f\beta(1 - (p+q)M)) < 0,$$

requires $\delta > 0$. Hence, a PZ solution can only exist if $den^{PZ,BR} < 0$ and $num_1^{PZ,BR} < 0$ and $\epsilon_1 > \epsilon_1^{PZ,BR}$. One can show:

$$\epsilon_1^{PZ,BR} - \epsilon^{PP,BR} = \frac{\psi a(1-p)(aN + M + M_f\beta(1 - M(p+q)))(\lambda\psi\epsilon_2 + \mu\eta_1)}{-\lambda num_1^{ZP,BR} num_1^{PZ,BR}} \geq 0,$$

if PZ exists (since this requires $num_1^{PZ,BR} < 0$). Therefore, if the PZ exists then $\epsilon_1 \geq \epsilon^{PP,BR}$ and hence the PP or ZP solution also exists.

From above, $(Q + \lambda\sigma\psi e_2 e_2')^{-1}((\lambda\epsilon_1 + \lambda\sigma\mu, \lambda\epsilon_2)')$ is a ZZ solution if $\epsilon_1 = \epsilon_2^{ZP,BR}$ and $det(Q + \lambda\sigma\psi e_2 e_2') \neq 0$ ($den^{ZP,BR} \neq 0$). If $det(Q) \neq 0$ then the ZZ solution is given by

$$\hat{\pi}^{ZZ,BR} = (Q)^{-1}\begin{pmatrix}\lambda\epsilon_1 + \lambda\sigma\mu \\ \lambda\epsilon_2 + \lambda\sigma\mu\end{pmatrix}.$$

The ZZ solution exists if and only if $\psi\pi_j^{ZZ,BR} \leq -\mu$ for $j = 1, 2$. One can show that

$$\begin{aligned}
\frac{\partial\pi_1^{ZZ,BR}}{\partial\epsilon_1} &= \frac{\lambda((1 - (M + aN)q + M_f(M + Mp(q-1) - q + M(q-1)q)\beta))}{den^{ZZ,BR}} \\
&= \frac{\lambda num_1^{ZZ,BR}}{den^{ZZ,BR}}, \\
\frac{\partial\pi_2^{ZZ,BR}}{\partial\epsilon_1} &= \frac{\lambda(1-q)(M + aN + M_f\beta(1 - M(p+q)))}{den^{ZZ,BR}} = \frac{\lambda num_2^{ZZ,BR}}{den^{ZZ,BR}},
\end{aligned}$$

where $den^{ZZ,BR} := -\delta(-\delta + (1-\rho)(M + aN + M_f\beta(1 - (p+q)M))) = det(Q)$ and clearly $num_2^{ZZ,BR} > 0$. Solving for $\epsilon_1^{ZZ,BR}$ and $\epsilon_2^{ZZ,BR}$ such that $\psi\pi_1^{ZZ,BR} = -\mu$ and $\psi\pi_2^{ZZ,BR} = -\mu$, respectively, we have

$$\epsilon_1^{ZZ,BR} - \epsilon_2^{ZZ,BR} = \frac{den^{ZZ,BR}(\eta_1\mu + \lambda\psi\epsilon_2)}{\lambda(1-q)\psi(M + aN + M_f\beta(1 - M(p+q)))num_1^{ZZ,BR}},$$

if $num_1^{ZZ,BR} \neq 0$. There are the following cases to consider. First, if $den^{ZZ,BR} > 0$ and $num_1^{ZZ,BR} < 0$ then ZZ existence requires $\epsilon_2^{ZZ,BR} \geq \epsilon_1 \geq \epsilon_1^{ZZ,BR}$. Second, if $den^{ZZ,BR} < 0$ and $num_1^{ZZ,BR} > 0$ then ZZ existence requires $\epsilon_1 \geq \epsilon_2^{ZZ,BR} > \epsilon_1^{ZZ,BR}$. In the third case, $den^{ZZ,BR} < 0$ and $num_1^{ZZ,BR} < 0$ then ZZ existence requires $\epsilon_1^{ZZ,BR} \geq \epsilon_1 \geq \epsilon_2^{ZZ,BR}$. Now it can be shown that $\epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$ and

$$\epsilon_1^{ZZ,BR} - \epsilon^{PP,BR} = \frac{-a(1-p)(M + aN + M_f\beta(1 - M(p+q)))(\psi\lambda\epsilon_2 + \eta_1\mu)}{\lambda num_1^{ZZ,BR}\eta_4} \geq 0,$$

if $num_1^{ZZ,BR} < 0$, where $\eta_4 := (1-q)(aN + \beta M_f(1-M(p+q)) + M) + a(\psi - N) + (1-M)(1-\beta M_f) > 0$. Since $\epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$ and existence of ZZ in the first three cases only hinges on $\epsilon_1 \geq \epsilon_1^{ZZ,BR}$ if $num_1^{ZZ,BR} < 0$ it follows that the ZP or PP solution will exist if the ZZ solution exists in the first three cases and $\epsilon_1 > \max\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\}$.

In the fourth case, $den^{ZZ,BR} > 0$ and $num_1^{ZZ,BR} > 0$. One can show that:

$$\begin{aligned}
den^{ZZ,BR} &= -\delta(-\delta + (1-\rho)(M + aN + M_f\beta(1-(p+q)M))),\\
num_1^{ZZ,BR} &= -\delta^{-1}den^{ZZ,BR} + \eta_5\\
&= -\delta + (1-q)(M + aN + M_f\beta(1-(p+q)M)),\\
\eta_5 &:= (p-1)(M + aN + M_f\beta(1-(p+q)M)) \leq 0.
\end{aligned}$$

Therefore, $\delta < 0$ if and only if the fourth case ($num_1^{ZZ,BR} > 0$ and $den^{ZZ,BR} > 0$) applies. In the fourth case, ZZ existence requires $\epsilon_2^{ZP,BR} \geq \epsilon_1$. It is furthermore straightforward to show that if $num_1^{ZZ,BR} = 0$ and $det(Q) \neq 0$ then a ZZ exists if and only if $\epsilon_1 \geq \epsilon_2^{ZZ,BR} = \epsilon_2^{ZP,BR}$. Finally, if $det(Q) = 0$ ($den^{ZZ,BR} = 0$) and $\epsilon_1 = \epsilon_2^{ZP,BR}$ then a continuum of ZZ solutions exists and if $det(Q) = 0$ ($den^{ZZ,BR} = 0$) and $\epsilon_1 \neq \epsilon_2^{ZP,BR}$ then a ZZ solution does not exist.

From the analysis above, if a BRE exists then $\delta \geq 0$ and $\epsilon_1 \geq \min\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\}$ or $\delta < 0$. Further, if $\epsilon_1 \geq \epsilon^{PP,BR}$ then a PP or ZP exists because $det(Q + \lambda\sigma\psi I_2) > 0$. If $\epsilon^{PP,BR} > \epsilon_2^{ZP,BR}$, then $den^{ZP,BR} = det(Q + \lambda\sigma e_2 e_2') \neq 0$ and therefore a PP, ZP or ZZ solution exists if, in addition, $\epsilon_1 \geq \epsilon_2^{ZP,BR}$. If $\delta < 0$, then a ZZ exists for $\epsilon_1 \leq \epsilon_2^{ZP,BR}$. We conclude that a BRE exists if and only if

$$\epsilon_1 \geq \bar{\epsilon}_{BR} := \begin{cases} \min\left\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\right\}, & \text{if } \delta \geq 0 \\ -\infty, & \text{if } \delta < 0, \end{cases} \tag{A.8}$$

where $\epsilon^{PP,BR}$ and $\epsilon_2^{ZP,BR}$ are defined in (A.6) and (A.7), respectively.

Case $q = 1$ Note that $\epsilon_2^{ZP,BR}$ from (A.7) can be expressed as $\epsilon_2^{ZP,BR} = (q-1)^{-1}\chi_{BR}$ where, if $q = 1$, and $\chi^1 := -\delta + (1-p)(aN + M(1 - M_f\beta p) + M_f\beta(1-M)) \neq 0$:

$$\chi_{BR} := \frac{\chi^1(\psi\lambda\epsilon_2 + \mu((1-M)(1-M_f\beta) + a(\psi - N)))}{\lambda\psi(aN + M_f\beta(1-M) + M(1-M_f p\beta))}.$$

For the PP solution, we have $\pi_2^{PP,BR} = \frac{\lambda\epsilon_2}{(1-M)(1-M_f\beta)+a(\psi-N)} \geq 0$ and therefore $\psi\pi_2^{PP,BR} > -\mu$.[20] Further, $\partial\pi_1^{PP,BR}/\partial\epsilon_1 = \lambda/((1-Mp)(1-M_f\beta p) + a(\psi - Np)) > 0$ and $\psi\pi_1^{PP,BR} = -\mu$ if and only if $\epsilon_1 = \epsilon^{PP,BR}$ where $\epsilon^{PP,BR}$ is defined in (A.6) with $q = 1$. Therefore, PP exists if and only if $\epsilon_1 > \epsilon^{PP,BR}$, and a ZP solution always exists if $\epsilon_1 = \epsilon^{PP}$. For the ZP solution, we have $\pi_2^{ZP,BR} = \pi_2^{PP,BR}$ and therefore $\psi\pi_2^{ZP,BR} > -\mu$. If $\chi^1 \neq 0$, then: $\partial\pi_1^{ZP,BR}/\partial\epsilon_1 = \lambda/\chi^1$ and $\psi\pi_1^{ZP,BR} = -\mu$ if and only if $\epsilon_1 = \epsilon^{PP,BR}$ where $\epsilon^{PP,BR}$ is defined in (A.6) with $q = 1$. Therefore if $\chi^1 > 0$ then $\epsilon^{PP,BR} \geq \epsilon_1 > \epsilon_2^{ZP,BR} = -\infty$ is necessary and sufficient for existence of the ZP solution. Otherwise, if $\chi^1 < 0$ then $\epsilon_2^{ZP,BR} = +\infty$ and $\epsilon_1 \geq \epsilon^{PP,BR}$ is necessary

[20] It can be shown that $(Q + \lambda\sigma\psi I_2)^{-1}$ exists if $q = 1$.

and sufficient for existence of the ZP solution. Note that $\delta < 0$ implies $\chi^1 > 0$. Finally, $\chi^1 = 0$ implies[21]:

$$\epsilon_2^{ZP,BR} - \epsilon^{PP,BR} = \frac{(1-p)(1-MM_f\beta p)}{\lambda p\psi}\mu + \frac{(1-p)(1-MM_f\beta p)N\epsilon_2}{(1-M)(1-M_f\beta)p + (1-N)(1-p)(1-MM_f\beta p) + (1-Mp)(1-M_f p\beta)(\psi-1)} \geq 0,$$

and that a continuum of ZP solutions exists if $\epsilon_1 = \epsilon^{PP,BR}$, and no ZP solution exists if $\epsilon_1 \neq \epsilon^{PP,BR}$.

For the PZ solution, we have $\pi_2^{PZ,BR} = -\frac{\lambda\epsilon_2 + a\mu}{\delta}$. If $\delta < 0$, then $\pi_2^{PZ,BR} \geq 0$, and if $\delta > 0$, then $\psi\pi_2^{PZ,BR} = -\psi\frac{\lambda\epsilon_2 + a\mu}{\delta} \leq -\psi\mu < -\mu$, since $\delta \leq a$ and $\epsilon_2 \geq 0$. If $\delta = 0$ then a PZ solution does not exist. Therefore, $\psi\pi_2^{PZ,BR} + \mu < 0$ if and only if $\delta > 0$. Further, $\partial\pi_1^{PZ,BR}/\partial\epsilon_1 = \lambda/((1-Mp)(1-M_f\beta p) + a(\psi - Np)) > 0$ and $\psi\pi_1^{PZ,BR} = -\mu$ if and only if $\epsilon_1 = \epsilon_1^{PZ,BR}$ where

$$\epsilon_1^{PZ,BR} = \epsilon^{PP,BR} + \frac{a(1-p)(M(1-M_f\beta p) + aN + M_f\beta(1-M))(\lambda\psi\epsilon_2 + \mu\eta_1)}{\lambda\delta(a\psi - \delta)} \geq \epsilon^{PP,BR},$$

and $\epsilon^{PP,BR}$ is defined in (A.6) with $q = 1$. It follows that PZ exists if and only if $\delta > 0$ and $\epsilon_1 > \epsilon_1^{PZ,BR} \geq \epsilon^{PP,BR}$.

For the ZZ solution, we have $\pi_2^{ZZ,BR} = \pi_2^{PZ,BR}$, and therefore $\psi\pi_2^{ZZ,BR} + \mu \leq 0$ if and only if $\delta > 0$. Furthermore, if $\chi^1 \neq 0$ then $\partial\pi_1^{ZZ,BR}/\partial\epsilon_1 = \lambda/\chi^1$ and $\psi\pi_1^{ZZ,BR} = -\mu$ if and only if $\epsilon_1 = \epsilon_1^{ZZ,BR} = \epsilon_1^{PZ,BR} \geq \epsilon^{PP,BR}$ where $\epsilon^{PP,BR}$ is defined in (A.6) with $q = 1$. Therefore if $\chi^1 > 0$ and $\delta > 0$ then $\epsilon_1^{ZZ,BR} \geq \epsilon_1 > \epsilon_2^{ZP,BR} = -\infty$ is necessary and sufficient for existence of the ZZ solution. Otherwise, if $\chi^1 < 0$ then $\epsilon_2^{ZP,BR} = +\infty$ and $\epsilon_1 \geq \epsilon_1^{ZZ,BR} \geq \epsilon^{PP,BR}$ is necessary and sufficient for existence of the ZZ solution. If $\chi^1 = 0$ and $\delta > 0$ then $\epsilon_2^{ZP,BR} - \epsilon^{PP,BR} \geq 0$ as shown above and a continuum of ZZ solutions exist if and only if

$$\epsilon_1 = \epsilon^{PP,BR} + \epsilon_2 + \frac{(1-Mp)(1-M_f p\beta)\mu}{\lambda Np} + \frac{\epsilon_2(1-p)N(1-M_f Mp\beta)}{(1-p)(1-MM_f p\beta)(\psi - N) + (1-M)p(1-M_f\beta)\psi} \geq \epsilon^{PP,BR}.$$

We conclude that a BRE exists if and only if

$$\epsilon_1 \geq \bar{\epsilon}_{BR} := \begin{cases} \min\left\{\epsilon^{PP,BR}, \epsilon_2^{ZP,BR}\right\}, & \text{if } \delta \geq 0 \\ -\infty, & \text{if } \delta < 0, \end{cases} \tag{A.9}$$

where $\epsilon^{PP,BR}$ and $\epsilon_2^{ZP,BR}$ are defined in (A.6) and (A.7), respectively, with $q = 1$.

[21] The $\chi^1 = 0$ case arises if $a = \frac{(1-Mp)(1-M_f\beta\rho) + M_f(q-1)(1-M)\beta}{Np}$ and $q = 1$. To compute $\epsilon_2^{ZP,BR}$, set $a = \frac{(1-Mp)(1-M_f\beta\rho) + M_f(q-1)(1-M)\beta}{Np}$ and compute $\lim_{q\to 1}\epsilon_2^{ZP,BR}$.

A.5. Proof of Proposition 5

Suppose $\delta = (M-1)(1-M_f\beta)+aN<0$, which implies $det(Q) = den^{ZZ,BR} = -\delta(-\delta+(1-\rho)(M+aN+M_f\beta(1-(p+q)M)))>0$, $det(Q+\lambda\sigma\psi e_2e_2') = den^{ZP,BR} = \delta^2-\delta(a\psi+(1-\rho)(M+aN+M_f\beta(1-M(p+q))))+(1-p)a\psi(M+aN+M_f\beta(1-M(p+q)))>0$, and $num_1^{PZ,BR} = ((1-q)(M+aN+M_f\beta(1-M(p+q)))-\delta)>0$, from Proposition 4. Also by Proposition 4: $num_1^{PZ,BR}>0$ implies no PZ; ZZ exists under $\delta<0$ if and only if $q<1$ and $\epsilon_1 \le \epsilon_2^{ZP,BR}$; $den^{ZP,BR}>0$ implies $\epsilon^{PP,BR}>\epsilon_2^{ZP,BR}$, $\epsilon_2^{ZP,BR}=-\infty$ if $q=1$, and ZP exists if and only if $\epsilon^{PP,BR}\ge\epsilon_1>\epsilon_2^{ZP,BR}$. Define $\epsilon^{ZP,BR} := \epsilon_2^{ZP,BR}$. We conclude that the PP solution is the unique BRE when $\epsilon_1>\epsilon^{PP,BR}$, the ZP solution is the unique BRE when $\epsilon^{PP,BR}\ge\epsilon_1>\epsilon^{ZP,BR}$. Otherwise, the ZZ solution is the unique solution if $q<1$ and $\epsilon_1\le\epsilon^{ZP,BR}$. If $\delta\ge 0$ then by Proposition 4 there exist p, q, ϵ_1 and $\epsilon_2 \ge 0$ for which there are no solutions or multiple solutions.[22]

A.6. Proof of Proposition 6

The proof of Proposition 6 is a straightforward extension of the proof of Proposition 4. Define $\bar{q} := Pr(\epsilon_t = 2) = (1-p)/(2-p-q)$. The regime-specific levels of inflation in BR-RPE i, $\hat{\pi}^i = (\pi_1^i, \pi_2^i)'$, are given by fixed point restrictions that have the same basic form as the BRE fixed point restrictions except we replace q with $\bar{q}$ and p with $1-\bar{q}$. Therefore, BR-RPE will exist if and only if

$$\epsilon_1 \ge \bar{\epsilon}_{BR,RPE} := \begin{cases} \min\left\{\epsilon^{PP,BR,RPE}, \epsilon_2^{ZP,BR,RPE}\right\}, & \text{if } \delta \ge 0 \\ -\infty, & \text{if } \delta < 0, \end{cases} \tag{A.10}$$

where $\delta = (M-1)(1-M_f\beta)+\lambda\sigma N$, and $\epsilon^{PP,BR,RPE}$ and $\epsilon_2^{ZP,BR,RPE}$ are defined in (A.6) and (A.7), respectively, assuming $p = 1-\bar{q}$, and $q=\bar{q}$. In the special case $q=1$ (which implies $\bar{q}=1$), we have $\bar{\epsilon}_{BR,RPE} = -\infty$ for any δ, as the PP solution exists if and only if $\epsilon_1 > -\mu(1+\lambda\sigma\psi)(\lambda\psi)^{-1} + (M(1-M_f\beta)+M_f\beta+\lambda\sigma N)((M-1)(1-M_f\beta)+\lambda\sigma(N-\psi))^{-1}\epsilon_2 = \epsilon^{PP,BR,RPE}$ and the ZP exists if and only if $\epsilon_1 \le \epsilon^{PP,BR,RPE}$. For $q<1$, one can show: $\epsilon^{PP,BR} - \epsilon^{PP,BR,RPE} = -\Xi_{PP}^B\rho$ and $\epsilon^{ZP,BR} - \epsilon^{ZP,BR,RPE} = -\Xi_{ZP}^B\rho$ where

$$\Xi_{PP}^B := \frac{(p-1)(\eta_6+MM_f\beta)\eta_7(\lambda\epsilon_2\psi+\mu\eta_7)}{\lambda\psi(\eta_7+(1-q)\eta_6)((1-p)(\lambda\sigma-\delta)+((1+\lambda\sigma)(1-q)+\lambda\sigma(1-\rho)(\psi-1)))},$$

$$\Xi_{ZP}^B := \frac{\delta(\eta_6+MM_f\beta)(\lambda\epsilon_2\psi+\mu\eta_7)}{\lambda(q-1)(M+\lambda\sigma N+M_f\beta(1-M))\eta_6\psi},$$

and $\eta_6 := M(1-M_f\beta p)+\lambda\sigma N+M_f\beta(1-qM)>0$, and $\eta_7 := (a(\psi-N)+(1-M)(1-M_f\beta))>0$. Since $\delta\le\lambda\sigma$, it is straightforward to show that $\Xi_{PP}^B\le 0$. Further, if $\delta\ge 0$ then $\Xi_{ZP}^B \le 0$. It follows that $\bar{\epsilon}_{BR} \ge \bar{\epsilon}_{BR,RPE}$ if $\delta \ge 0$ and $p+q-1\ge 0$ or $\delta<0$.

[22] Alternatively, one can show that $(M-1)(1-M_f\beta)+\lambda\sigma N<0$ ensures completeness and coherence using techniques developed by AM. Results available on request.

A.7. Proof of Proposition 7

Consider Proposition 7. To assess E-stability of a REE, we express $Y^i = (Y_1^{i'}, Y_2^{i'})'$ as a function of agents' expectations, $\tilde{Y}^e = (Y_1^{e'}, Y_2^{e'})'$:

$$Y^{PP}(\tilde{Y}^e) := \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_P & qA_P \end{pmatrix} \tilde{Y}^e + \Gamma^{PP},$$
$$Y^{ZP}(\tilde{Y}^e) := \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_P & qA_P \end{pmatrix} \tilde{Y}^e + \Gamma^{ZP},$$
$$Y^{PZ}(\tilde{Y}^e) := \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_Z & qA_Z \end{pmatrix} \tilde{Y}^e + \Gamma^{PZ},$$
$$Y^{PP}(\tilde{Y}^e) := \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_Z & qA_Z \end{pmatrix} \tilde{Y}^e + \Gamma^{ZZ},$$

where Γ^i collect terms that do not depend on beliefs, $\tilde{Y}^e$. It immediately follows that

$$DT_{Y^{PP}} = K \otimes A_P - I, \qquad DT_{Y^{ZP}} = \begin{pmatrix} pA_Z & (1-p)A_Z \\ (1-q)A_P & qA_P \end{pmatrix} - I,$$
$$DT_{Y^{ZZ}} = K \otimes A_Z - I, \qquad DT_{Y^{PZ}} = \begin{pmatrix} pA_P & (1-p)A_P \\ (1-q)A_Z & qA_Z \end{pmatrix} - I.$$

REE i is E-stable if the real parts of the eigenvalues of DT_{Y^i} are negative. Since the real parts of the eigenvalues of $DT_{Y^{PP}}$ are negative and the real part of an eigenvalue of $DT_{Y^{ZZ}}$ is positive, the PP (ZZ) solution is always (never) E-stable. The following condition is necessary for E-stability of the ZP solution: $Det(DT_{Y^{ZP}}) = \frac{a}{1+a\psi} den^{ZP} > 0$, where den^{ZP} is defined in the proof of Proposition 1. By Proposition 1, $den^{ZP} > 0$ implies $\epsilon^{PP} > \epsilon_2^{ZP}$, where $\epsilon^{PP}, \epsilon_2^{ZP}$ are defined in the proof of Proposition 1, and hence $\epsilon_1 > \epsilon^{PP}$ is necessary for existence of PP and $\epsilon_1 \leq \epsilon^{PP}$ is necessary for existence of ZP. It follows that the E-stability and existence of the ZP solution precludes existence of the PP solution. The following condition is necessary for E-stability of the PZ solution: $Det(DT_{Y^{PZ}}) = \frac{1}{1+a\psi} den^{PZ} > 0$, where den^{PZ} is defined in the proof of Proposition 1. By Proposition 1, $den^{PZ} < 0$ is necessary for PZ existence. We conclude that the PZ solution can never be E-stable.[23]

In sum, if the PP solution exists it is E-stable. If the ZP solution exists and is E-stable then the PP solution does not exist. The ZZ and PZ solutions are never E-stable.

A.8. Proof of Proposition 8

To assess E-stability of each RPE, we express the RPE unconditional mean of inflation and output as a function of agents' expectations, Y^e:

$$\bar{Y}^{PP}(Y^e) := A_P Y^e + \bar{\Gamma}^{PP}, \qquad \bar{Y}^{ZP}(Y^e) := (\bar{q}A_P + (1-\bar{q})A_Z) Y^e + \bar{\Gamma}^{ZP},$$
$$\bar{Y}^{ZZ}(Y^e) := A_Z Y^e + \bar{\Gamma}^{ZZ} \qquad \bar{Y}^{PZ}(Y^e) := ((1-\bar{q})A_P + \bar{q}A_Z) Y^e + \bar{\Gamma}^{PZ},$$

where $\bar{\Gamma}^i$ collect terms that do not depend on beliefs, Y^e. It immediately follows that

[23] If $q = 1$, the PP exists and is E-stable if and only if $\epsilon_1 > \epsilon^{PP}$ and if $Det(DT_{Y_{ZP}}) > 0$ then $\theta > 1$, such that ZP exists if and only if $\epsilon_1 \leq \epsilon^{PP}$ by Proposition 1. The ZZ and PZ solutions cannot be E-stable.

$$DT_{\bar{Y}PP} = A_P - I, \qquad DT_{\bar{Y}ZP} = \bar{q}A_P + (1-\bar{q})A_Z - I,$$
$$DT_{\bar{Y}ZZ} = A_Z - I, \qquad DT_{\bar{Y}PZ} = (1-\bar{q})A_P + \bar{q}A_Z - I.$$

It is straightforward to show that the real parts of the eigenvalues of $DT_{\bar{Y}PP}$ are negative and the real part of an eigenvalue of $DT_{\bar{Y}ZZ}$ is positive. Therefore, the PP (ZZ) RPE is always (never) E-stable. The ZP RPE is E-stable if and only if

$$tr(DT_{\bar{Y}ZP}) = \beta + a - \frac{a\bar{q}\psi(\beta + a + 1)}{a\psi + 1} - 1 < 0,$$
$$Det(DT_{\bar{Y}ZP}) = \frac{\bar{q}a(a\psi + \psi)}{a\psi + 1} - a > 0,$$

where $tr(B)$ denotes the trace of matrix B. We have $tr(DT_{\bar{Y}ZP}) < 0 < Det(DT_{\bar{Y}ZP})$ if and only if $\bar{q}(1+a)\psi - 1 - a\psi > 0$. From the proofs of Propositions 1 and 3:

$$\epsilon^{PP,RPE} - \epsilon_2^{ZP,RPE} = v(\bar{q}(1+a)\psi - 1 - a\psi)),$$
$$v := \frac{a(\lambda\epsilon_2\psi + a\mu(\psi - 1))}{\lambda(1-\bar{q})\psi(a+1)(a(\psi - \bar{q}) + 1 - \bar{q})} > 0.$$

Therefore, if the ZP RPE is E-stable then $\epsilon^{PP,RPE} > \epsilon_2^{ZP,RPE}$ and the condition for PP existence becomes $\epsilon_1 > \epsilon^{PP,RPE}$ and the condition for ZP existence becomes $\epsilon^{PP,RPE} \geq \epsilon_1 > \epsilon_2^{ZP,RPE}$ as demonstrated in the proofs of Propositions 1 and 3.[24] Hence, if the ZP RPE exists and is E-stable then the PP solution does not exist. The PZ solution is E-stable if and only if

$$tr(DT_{\bar{Y}PZ}) = \frac{\beta - 2a\psi + a - 1}{a\psi + 1} + \frac{\bar{q}\left(\beta a\psi + a^2\psi + a\psi\right)}{a\psi + 1} < 0,$$
$$Det(DT_{\bar{Y}PZ}) = -\frac{a(1-\psi)}{a\psi + 1} - \frac{a\bar{q}(a\psi + \psi)}{a\psi + 1} > 0,$$

which holds if and only if $0 < \psi - 1 - \bar{q}\psi(1+a) = den^{PZ,RPE}a^{-1}$ where $den^{PZ,RPE}$ is equal to den^{PZ} defined in the Proposition 1 proof when $q = \bar{q}$ and $p = 1 - \bar{q}$. From the proof of Proposition 3, the PZ RPE only exists if $den^{PZ,RPE} < 0$. Hence the PZ RPE is never E-stable.

Therefore, the PP RPE is the only E-stable RPE solution when $\epsilon_1 > \epsilon^{PP,RPE}$, and the ZP RPE is the only E-stable RPE solution when $\epsilon^{PP,RPE} \geq \epsilon_1 > \epsilon_2^{ZP,RPE}$. It follows that a unique E-stable RPE solution exists when $\epsilon_1 > \bar{\epsilon}_{RPE}$.

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jet.2023.105745.

References

Adam, K., 2007. Experimental evidence on the persistence of output and inflation. Econ. J. 117, 603–636.
Angeletos, George-Marios, Lian, Chen, 2018. Forward guidance without common knowledge. Am. Econ. Rev. 108 (9), 2477–2512.

[24] If $\bar{q} = 1$, the PP exists and is E-stable if and only if $\epsilon_1 > \epsilon^{PP,RPE}$ and the ZP exists and is E-stable if and only if $\epsilon_1 \leq \epsilon^{PP,RPE}$. The ZZ and PZ solutions cannot be E-stable.

Angeletos, George-Marios, Lian, Chen, 2023. Determinacy without the Taylor Principle. J. Polit. Econ. 131 (8).
Armenter, Roc, 2018. The perils of nominal targets. Rev. Econ. Stud. 85, 50–86.
Aruoba, S. Boragan, Cuba-Borda, Pablo, Schorfheide, Frank, 2018. Macroeconomic dynamics near the ZLB: a tale of two countries. Rev. Econ. Stud. 85 (1), 87–118.
Aruoba, S. Borağan, Cuba-Borda, Pablo, Higa-Flores, Kenji, Schorfheide, Frank, Villalvazo, Sergio, 2021. Piecewise-linear approximations and filtering for DSGE models with occasionally binding constraints. Rev. Econ. Dyn. 41, 96–120.
Ascari, G., Mavroeidis, S., 2022. The unbearable lightness of equilibria in a low interest rate environment. J. Monet. Econ. 127, 1–17.
Assenza, T., Heemeijer, P., Hommes, C., Massaro, D., 2021. Managing self-organization of expectations through monetary policy: a macro experiment. J. Monet. Econ. 117, 170–186.
Audzei, V., Slobodyan, S., 2022. Sparse restricted perceptions equilibrium. J. Econ. Dyn. Control, 139.
Benhabib, J., Evans, G., Honkapohja, S., 2014. Liquidity traps and expectation dynamics: fiscal stimulus or fiscal austerity? J. Econ. Dyn. Control 45, 220–238.
Benhabib, J., Schmitt-Grohé, S., Uribe, M., 2001. The perils of Taylor rules. J. Econ. Theory 96, 40–69.
Beshears, J., Choi, J., Fuster, A., Laibson, D., Madrian, B.C., 2013. What goes up must come down? Experimental evidence on intuitive forecasting. Am. Econ. Rev. 103 (3), 570–574.
Bianchi, F., Melosi, L., Rottner, M., 2021. Hitting the elusive inflation target. J. Monet. Econ. 124, 107–122.
Bilbiie, F.O., 2022. Neo-fisherian policies and liquidity traps. Am. Econ. J. Macroecon. 14 (4), 378–403.
Boneva, Lena Mareen, Braun, R. Anton, Waki, Yuichiro, 2016. Some unpleasant properties of loglinearized solutions when the nominal rate is zero. J. Monet. Econ. 84 (C), 216–232.
Branch, B., Davig, T., McGough, B., 2013. Adaptive learning in regime-switching models. Macroecon. Dyn. 17 (5), 998–1022.
Branch, W.A., Evans, G.W., 2006a. Intrinsic heterogeneity in expectation formation. J. Econ. Theory, 264–295.
Branch, W.A., Evans, G.W., 2006b. A simple recursive forecasting model. Econ. Lett., 158–166.
Branch, William A., 2006. Restricted perceptions equilibria and learning in macroeconomics. In: Post Walrasian Macroeconomics: Beyond the Dynamic Stochastic General Equilibrium Model. Cambridge University Press, New York.
Branch, William A., 2022. Misspecification and the Restricted Perceptions Approach. Technical Report.
Branch, William A., Gasteiger, Emanuel, 2018. Endogenously (Non-)Ricardian Beliefs. Technical Report.
Bullard, J., Eusepi, S., 2014. When does determinacy imply expectational stability? Int. Econ. Rev. 55 (1).
Bullard, J., Evans, G.W., Honkapoha, S., 2008. Monetary policy, judgment and near-rational exuberance. Am. Econ. Rev. 98, 1163–1177.
Christiano, Lawrence, Eichenbaum, Martin S., Johannsen, Benjamin K., 2018. Does the New Keynesian Model Have a Uniqueness Problem? Working Paper. National Bureau of Economic Research.
Cole, S., 2021. Learning and the effectiveness of central bank forward guidance. J. Money Credit Bank. 53, 157–200.
Eggertsson, G., Woodford, M., 2003. The zero bound on interest rates and optimal monetary policy. Brookings Pap. Econ. Act. 34, 139–235.
Eggertsson, Gauti B., 2011. What fiscal policy is effective at zero interest rates? In: NBER Macroeconomics Annual, vol. 25. National Bureau of Economic Research, pp. 59–112.
Eggertsson, Gauti B., Egiev, Sergey K., Lin, Alessandro, Platzer, Josef, Riva, Luca, 2021. A toolkit for solving models with a lower bound on interest rates of stochastic duration. Rev. Econ. Dyn.
Eusepi, S., Gibbs, C., Preston, B., 2021. Forward Guidance with Unanchored Expectations. Technical Report, Bank of Finland Research Discussion Papers 11/2021.
Evans, D., Evans, G., McGough, B., 2022a. The RPEs of RBCs and other DSGEs. J. Econ. Dyn. Control 143.
Evans, G., McGough, B., 2018. Interest-rate pegs in new Keynesian models. J. Money Credit Bank. 50, 939–965.
Evans, G., McGough, B., 2020. Stable near-rational sunspot equilibria. J. Econ. Theory, 186.
Evans, G., Guse, E., Honkapohja, S., 2008. Liquidity traps, learning and stagnation. Eur. Econ. Rev. 52 (8), 1438–1463.
Evans, G., Honkapohja, S., 1994. On the local stability of sunspot equilibria under adaptive learning rules. J. Econ. Theory 64, 142–161.
Evans, G., Honkapohja, S., 1998. Convergence of learning algorithms without a projection facility. J. Math. Econ. 30, 59–86.
Evans, G., Honkapohja, S., 2001. Learning and Expectations in Macroeconomics. Princeton University Press.
Evans, G., Honkapohja, S., Mitra, K., 2022b. Expectations, stagnation and fiscal policy: a nonlinear analysis. Int. Econ. Rev. 63 (3), 1397–1425.
Evans, G., Honkapohja, S., Sargent, T.J., 1993. Deterministic cycles when some agents perceive them to be random fluctuations. J. Econ. Dyn. Control, 705–721.

Fernández-Villaverde, Jesús, Gordon, Grey, Guerrón-Quintana, Pablo, Rubio-Ramirez, Juan F., 2015. Nonlinear adventures at the zero lower bound. J. Econ. Dyn. Control, 57.

Gabaix, X., 2020. A behavioral new Keynesian model. Am. Econ. Rev. 110 (8), 2271–2327.

Gibbs, C., McClung, N., 2023. Does my model predict a forward guidance puzzle? Rev. Econ. Dyn. https://doi.org/10.1016/j.red.2023.03.001.

Gourieroux, C., Laffont, J.J., Monfort, A., 1980. Coherency conditions in simultaneous linear equation models with endogenous switching regimes. Econometrica, 675–695.

Guerrieri, Luca, Iacoviello, Matteo, 2015. OccBin: a toolkit for solving dynamic models with occasionally binding constraints easily. J. Monet. Econ. 70, 22–38.

Gust, Christopher, Herbst, Edward, López-Salido, David, Smith, Matthew E., 2017. The empirical implications of the interest-rate lower bound. Am. Econ. Rev. 107 (7), 1971–2006.

Hajdini, I., 2022. Mis-specified Forecasts and Myopia in an Estimated New Keynesian Model. Technical Report, Federal Reserve Bank of Cleveland WP 22-03.

Hommes, C., Mavromatis, K., Ozden, T., Zhu, M., forth.. Behavioral Learning Equilibria in the New Keynesian model. Quant. Econ.

Hommes, C., Sorger, G., 1997. Consistent expectations equilibria. Macroecon. Dyn. 2, 287–321.

Hommes, C., Zhu, M., 2014. Behavioral learning equilibria. J. Econ. Theory 150, 778–814.

Honkapohja, S., Mitra, K., 2020. Price level targeting and evolving credibility. J. Monet. Econ. 116, 88–103.

Honkapohja, S., McClung, N., 2021. On Robustness of Average Inflation Targeting. Technical Report, CEPR Discussion Paper 16001.

Marcet, A., Nicolini, J.P., 2003. Recurrent hyperinflations and learning. Am. Econ. Rev. 93 (5), 1476–1498.

Marcet, A., Sargent, T.J., 1989. Convergence of least-squares learning in environments with hidden state variables and private information. J. Polit. Econ., 1306–1322.

McClung, N., 2020. E-stability vis-a-vis determinacy in regime-switching models. J. Econ. Dyn. Control, 121.

McKay, Alisdair, Nakamura, Emi, Steinsson, Jón, 2016a. The Discounted Euler Equation: A Note. Technical Report. National Bureau of Economic Research.

McKay, Alisdair, Nakamura, Emi, Steinsson, Jón, 2016b. The power of forward guidance revisited. Am. Econ. Rev. 106 (10), 3133–3158.

Meggiorini, G., Milani, F., 2021. Behavioral New Keynesian Models: Learning vs. Cognitive Discounting. Technical Report, CESifo Working Paper No. 9039.

Mertens, K., Ravn, M., 2014. Fiscal policy in an expectations-driven liquidity trap. Rev. Econ. Stud. 81, 1637–1667.

Moberly, J., 2022. Implications of Endogenous Cognitive Discounting. Technical Report.

Nakata, T., Schmidt, S., 2019. Conservatism and liquidity traps. J. Monet. Econ. 104, 37–47.

Nakata, T., Schmidt, S., 2022. Expectations-driven Liquidity Traps: Implications for Monetary and Fiscal Policy. Am. Econ. J. Macroecon. 14 (4).

Nakata, Taisuke, 2018. Reputation and liquidity traps. Rev. Econ. Dyn. 28, 252–268.

Del Negro, Marco, Giannoni, Marc P., Patterson, Christina, 2023. The forward guidance puzzle. J. Polit. Econ. Macroecon. 1 (1), 43–79.

Ormeno, A., Molnár, K., 2015. Using survey data of inflation expectations in the estimation of learning and rational expectations models. J. Money Credit Bank. 47 (4), 673–699.

Preston, B., 2005. Learning about monetary policy rules when long-horizon expectations matter. Int. J. Cent. Bank. 1 (2).

Slobodyan, S., Wouters, R., 2012. Learning in a medium-scale DSGE model with expectations based on small forecasting models. Am. Econ. J. Macroecon. 4 (2), 65–101.

Woodford, M., 1990. Learning to believe in sunspots. Econometrica 58 (2), 277–307.

Woodford, M., 2003. Interest and Prices: Foundation of a Theory of Monetary Policy. Princeton University Press.

Woodford, M., Xie, Y., 2022. Fiscal and monetary stabilization policy at the zero lower bound: consequences of limited foresight. J. Monet. Econ. 125, 18–35.

References from supplementary material

Airaudo, Marco, Hajdini, Ina, 2021. Consistent expectations equilibria in Markov regime switching models and inflation dynamics. Int. Econ. Rev. 62 (4), 1401–1430.

Diba, B., Loisel, O., 2021. Pegging the interest rate on bank reserves: a resolution of new Keynesian puzzles and paradoxes. J. Monet. Econ. 118, 230–244.

Evans, G., McGough, B., 2018a. Equilibrium selection, observability and backward-stable solutions. J. Monet. Econ. 98, 1–10.

Farhi, E., Werning, I., 2019. Monetary policy, bounded rationality, and incomplete markets. Am. Econ. Rev. 109 (11), 3887–3928.

Jorgensen, P., Lansing, K., 2021. Anchored Inflation Expectations and the Slope of the Phillips Curve. Technical Report, Federal Reserve Bank of San Francisco Working Paper 2019-27.

Mendes, R.R., 2011. Uncertainty and the Zero Lower Bound: A Theoretical Analysis. Technical Report, 14 MPRA Paper 59218. University Library of Munich, Germany.