The Hub Location and Pricing Problem

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Abstract

This paper introduces the joint problem of locating hubs on a network and determining transportation prices between the hubs. Two levels of decision makers are present in the problem acting non-cooperatively: hub transportation provider and customers. The objective of the hub transportation provider is to locate hubs and to set the prices (per unit of commodity) of crossing the hub arcs maximizing its profit, whereas the customers aim is to send their commodities, in the cheapest way, having the possibility of using the hub arcs at the price set by the hub transportation provider or using the existing network at a predefined tariff. The problem is modeled as a nonlinear bilevel programming formulation, which is in turn linearized, and strengthened through variable reductions as well as valid inequalities. The case in which the price of each hub arc is determined by applying a common discount factor to the predefined tariff in the existing network is also studied. Computational results of mixed integer programming models and a metaheuristic on instances adapted from the literature are presented.

Keywords: hub location, pricing, bilevel programming

1. Introduction

Hub and spoke topology is commonly used to model networks in air passenger and freight transportation, package delivery companies, postal service, telecommunications and rapid transit systems, among others. This topology provides connections between many origins and destinations via hub facilities that serve as switching, sorting, and consolidation points for flows of freight, passengers, or information. This transshipment role of the hubs facilities with the consequent consolidation typically provides a considerable reduction of the number of routes required to connect all origins and destinations in a region. The benefits of hub systems result from economies of scale because of consolidation flows on relatively few arcs, and as mentioned, from the reduced number of arcs required to connect all nodes. Hub Location Problems (HLPs) consider i) the design of hub networks by selecting a set of nodes to locate hubs, and by activating a set of links connecting origin/destination nodes to hubs and ii) the routing of commodities through the

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designed network optimizing an objective function (e.g. transportation costs). Regarding the economies of scale by the flow consolidation at hubs, most of the models in hub location apply a fixed discount factor to all inter-hub arcs (hub arcs) costs. HLPs have been studied extensively, with many variants based on the number of hubs to be located, hub capacity, and objective function type (minisum, minimax, covering, and ordered median function, among others). We refer the interested reader to the comprehensive reviews by Alumur and Kara (2008), Campbell and O'Kelly (2012), Farahani et al. (2013), Contreras (2015), Contreras and O'Kelly (2019), Alumur et al. (2021).

In this paper, we consider a multicommodity transportation problem, called the Hub Location and Pricing Problem (HLP-P), in which an existing directed network is used to deliver the commodities. A tariff per unit of transportation commodity is associated with every arc. A hub transportation provider wants to establish a new system of interlinked hubs that can be used in the origin-destination (O/D) routes of the commodities. Therefore, two levels of decision makers are present in the problem acting non-cooperatively (i.e., Stackelberg game). First, the hub transportation provider (the leader or the upper level) should decide: i) where to locate a fixed number of hubs and ii) the price per unit of commodity crossing the arcs connecting hubs to maximize its benefit taking into account travel costs of traversing hub arcs. Second, the customers (the follower or the lower level) should decide the routes for the transportation of each commodity The follower will choose the cheapest path to route each commodity taking into account the tariffs of the existing network and the prices set by the leader for using the hub arcs. This will determine which is the cheapest path for routing each commodity from O/D (i.e., using only the existing network or using both, if necessary, hub arcs and the existing network). If the follower chooses to route a commodity using hub arcs, this commodity will use at least one hub arc. Each commodity might be routed to the hubs or from the hub to the final destination, using the existing network. It is assumed that the follower will use the hub service for routing a commodity only if the overall amount to be paid for the corresponding O/D route is cheaper by a factor of γ than the the corresponding amount by using the direct connection of the existing network. We refer to this condition as the *saving threshold condition*.

An interpretation of the saving threshold condition comes from the fact that using the existing network allows for a direct delivery (using the same vehicle from the origin to destination) while using hub arcs typically entails transshipments at the access and exit node of the hubs system (transportation on hub arcs is often performed using a different mode, as, for example, big trucks, vessels, high speed trains, planes, etc...). Hence, γ could represent the minimum saving necessary for a follower to choose a route for a commodity with transshipments instead of a direct delivery. Actually, direct deliveries may have some advantages compared to using hub arcs; as for instance, they might be more comfortable (for passengers), safer (each transshipment of a cargo may damage the freight), sometimes even faster (for perishable freights this is an important risk factor), they might avoid possible delays (due to lack or errors of coordination in transshipments), etc. On the other hand, the main advantage of using hub arcs is the saving, therefore the follower decides to use the hub arcs for a commodity whenever the saving is large enough.

Regarding the applicability of this model, consider a railway company that wants to establish several railway terminals to transport new automobiles. Clients or users have two options: i) send the automobiles directly by trucks from the origin to the destination or ii) take them to one of the terminals using trucks, then the automobiles will be delivered by train to another terminal from where they will be transported to the destination by trucks. Clients will choose the O/D routes based on economical reasons whenever it is worth it (saving threshold condition). The railway company should set the delivery prices between each pair of its terminals to obtain the maximum benefit taking into account the travel costs.

High prices will mean less flow on hub arcs but more benefit per unit of commodity. On the contrary, low prices usually mean more flow in hub arcs but less profit per unit of commodity. Another application could be the case of a container shipping company that wishes to establish several international terminals that are connected by vessels. The goal of this company is to determine the location of these terminals and to set the transport prices of a TEU (Twenty-foot Equivalent Unit) between pairs of hubs to maximize its benefit.

In addition, although the new transportation system is characterised by the location of hubs, it is worth mentioning that our model differs from the classical hub location models. The goal of our model is not only to determine the location of hubs and the O/D routes of the commodities, but also to set the price of crossing every hub arc per unit of commodity to maximize the benefit of the leader (the tariffs for the arcs on the existing transportation network, the non-hub arcs, are known). In addition, direct connection routes through the arcs of the existing network remain available to the customers, i.e., the commodities do not necessarily have to be routed through a hub arc. Moreover, in classical hub location models a common discount factor (per unit of commodity) is applied for using hub arcs based on economy of scale reasons; in the proposed model the price for traversing each one of the hub arcs can have a different discount with respect to the tariff of using the corresponding arc in the existing network. Another difference with respect to the classical hub location models is the modelling of two decision makers: the leader (setting the hubs and the prices for using the hub arcs) and the follower (determining the O/D routes for each commodity).

Our contributions in this paper are threefold. First and foremost, we provide an exact nonlinear model for the HLP-P as well as its linearization, associated valid inequalities, and variable reductions to strengthen it. Secondly, we present a variant of the HLP-P, in which the decision maker is restricted to set the prices based on a common discount factor. Thirdly, we provide a hybrid metaheuristic that can solve both variants.

The rest of the paper is organized as follows. Section 2 reviews relevant literature related to HLP-P. In Section 3, we present the formal definition of the problem, and a proof of $\mathcal{NP}$-Hardness. In Section 4, we provide a nonlinear bilevel programming formulation for the HLP-P, and its linearization. Section 5 consists of properties of the HLP-P leading to variable reductions and valid inequalities for the linearized formulation. Section 6 provides the transformation of the formulation for the case of prices given by a common discount factor with respect to the direct connection tariffs, as well as a detailed description of a hybrid metaheuristic algorithm that we have developed for this variant. Section 7 contains the details of our computational experiments, and the managerial insights we derive from the experiments. Finally, we provide our conclusions in Section 8.

2. Literature review

HLPs concern the design of hub-and-spoke networks by locating a set of hub facilities and selecting a set of links to route flows between O/D pairs. Classical HLPs assume that hubs are fully interconnected and that direct connections between non-hub nodes are not allowed, i.e., O/D paths must contain at least one hub node, although many variants of this classical model have been analyzed since the seminal paper of O'Kelly (1986). The consolidation of flows on the inter hubs arcs allows us to use the economies of scale through a discount factor to the cost per unit of flow in these arcs.

O'Kelly and Miller (1994) proposed a classification of hub networks according to (i) whether non-hub nodes are singly allocated (non-hub nodes are assigned to exactly one hub) or multiply allocated (non-hub nodes may be assigned to more than one hub); (ii) whether hubs are fully or partially interconnected; and (iii) whether direct connections are allowed between pairs on non-hub nodes. Our model consider the location of q hubs, see, among others, the following references for different variants of hub location models considering a fixed number of hubs: Alumur et al. (2009), Aykin (1994), Bryan and O'Kelly (1999), Campbell (1994a,b, 1996), Ebery et al. (2000), Ernst and Krishnamoorthy (1996, 1998a,b, 1999), García et al. (2012), Kara and Tansel (2000), Puerto et al. (2016), Skorin-Kapov et al. (1996).

The model analyzed in this paper presents some features that considered simultaneously make the model novel, although some of them have already been considered in different variants of the classical hub location models in the literature. i) The case where direct connections are allowed between pairs on non-hub nodes was already analyzed in Aykin (1994), Nickel et al. (2001), Martins de Sá et al. (2015), Alibeyg et al. (2018). ii) As mentioned in the introduction, the objective function of HLP-P considers a profit driven model, Alibeyg et al. (2016, 2018) considers profit-oriented objectives in hubs network design problems that measure the tradeoff between the revenue due to served commodities and the overall network design and transportation costs. The main novelty of this model is that there is not a requirement of serving the demand of every commodity. iii) The transshipments are penalised in our model by the saving threshold condition to use hub-arcs (factor γ); Martins de Sá et al. (2015) considers that access to the first hub and the exit from the last hub are time consuming. iv) HLP-P model can be seen as the decision process of a transportation company (leader) aiming to attract flow (maximizing profit) through its inter hubs links competing against the existing network. In the literature we find other references considering competitive hub location models, see Marianov et al. (1999), Wagner (2008), Sasaki and Fukushima (2001), Sasaki (2005), Eiselt and Marianov (2009), Sasaki et al. (2014), Mahmutogullari and Kara (2016). The two main differences between HLP-P and the models in these references are: a) these models consider two hub transportation systems competing for the same market of origin-destination routes that can both adjust their prices and in our case, the hub transportation system competes with an existing transportation network with fixed prices for every arc, and b) every complete O/D route is covered by one of the competing transportation companies, and in our case, only the inter-hub connections are operated by the new company, the remaining parts of the O/D routes (from the origin node to the first hub and from the last hub to the destination node) will be performed using the existing transportation network.

Finally, v) Pricing, it is worth mentioning that network pricing problems have attracted their fair share of attention. The problem of maximizing the toll revenue collected on a multicommodity transportation network, assuming that the user of the network are assigned to shortest paths with respect to the sum of tolls and initial costs, was considered, among others, by Heilporn et al. (2010), Brotcorne et al. (2011), Dewez et al. (2008). Bilevel programming is the tool of choice in these papers, since this technique models the customers behavior effectively. The tutorial by Labbé and Violin (2013) provides an excellent introduction to the use of bilevel programming for network pricing problems. On the other hand, the problem of jointly locating hubs and pricing arcs connecting the hubs has not been studied yet. The closest paper is by Brotcorne et al. (2008), in which the problem of Joint Design and Pricing on a Network (JDP) is considered. The aim is to select arcs of a given network to invest on and subsequently apply pricing to. The authors provide a bilevel programming formulation, a mixed-integer bilinear formulation, and a Lagrangian relaxation based solution method. HLP-P differs from JDP, since in HLP-P the arcs to be invested are given by the location of a given number of hubs, and the parameter γ provides a more general way of modelling customers behavior (JDP considers the case $\gamma = 1$).

Notably, Lüer-Villagra and Marianov (2013) have studied the Competitive Hub Location and Pricing Problem (CHLPP), where a new company sets its prices to compete with an existing one. The existing company utilizes a transportation network with a hub-and-spoke topology. The new company offers its services in the same market, using its own hub and spoke network and setting prices to maximize its profit. The customers behavior is assumed to conform with a logit model that determines the price-sensitivity of the customers decisions. The authors present a nonlinear formulation as well as a genetic algorithm. HLP-P differs from the CHLPP: i) in CHLPP two companies competes utilizing a transportation network with a hub and spoke topology (in our case, we just have the new company), ii) in CHLPP the commodities may pass through more than two hubs since the underlying transportation network is incomplete, whereas in HLP-P any route cannot cross more than two hubs (it is not an assumption in our model, but it results from the properties of the model), iii) the decision process of customers to choose the company and route is modeled using a logit model (in our case, each customer decides to use the alternative transportation system if the percentage of saving exceeds a threshold), and iv) the origin-destination flow could be split in different routes and companies (in our case, all the commodities with the same origin and destination are sent via the same route).

3. Problem definition

Consider a complete directed graph $G = (V, A)$ with vertex set V and arc set A, and a set of commodities K to be transported. Each arc $(i, j) \in A$ is associated with a fee per unit of commodity (known in advance) for traversing it, called *tariff*, $\bar{t}_{ij} \geq 0$. These are the prices in the existing network. We assume that these tariffs satisfy the triangle inequality as they represent the most competitive price in the existing network for a pair of origin and destination. We define T_k as the total amount of commodity $k \in K$ to be routed from its origin $o(k) \in V$ to its destination $d(k) \in V$. We denote the transportation costs per unit of commodity as $\bar{c}_{ij} \geq 0, \forall (i, j) \in A$, and assume that they satisfy the triangular inequality i.e. $\bar{c}_{ij} \leq \bar{c}_{ik} + \bar{c}_{kj}, \forall i, j, k \in V$. Note that the tariffs and costs need not be identical or even smoothly align to each other, since the former is a result of the existing competition between the transportation providers in the network, and the latter is a result of the infrastructure.

To maximize its profit the leader has to locate a fixed number of hubs and to set a fee per unit of commodity for traversing every hub arc, called price. In the decision process of the lower level, the follower chooses a route for each commodity using a hub arc only if the overall route fee (i.e., the sum of the prices or tariffs of the arcs in that route) is lower by a factor of $\gamma \in (0, 1]$ with respect to the tariff of direct connection in the existing network (saving threshold condition). HLP-P is the problem of finding the optimal locations of q hubs, and the prices p_{ij} of traversing the hub arcs to maximize the profit of the leader.

Before providing a proof of $\mathcal{NP}$-Hardness, we emphasize that defining HLP-P on a complete graph does not decrease the generality of the problem. To solve an HLP-P arising on an incomplete graph, as most real-world transportation networks are, it suffices to solve an All-Pairs Shortest (lowest tariff) Path problem to determine the values of $\bar{t}_{ij}$. The solution of the resulting HLP-P instance by using the lowest tariff paths will provide the tariffs of the arcs in this complete graph. Actually, to make his/her decision, the follower only needs to know the tariff between each pair of nodes and the price between hubs independently of the path defining these links in the underlying graph. From a practical point of view, the goal of the follower is to pay the least for transporting their commodities, regardless of the routes defined in the underlying graph. Note that using the same rationale, the assumption of tariffs obeying the triangle inequality does not decrease the problem generality.

Proposition 1. *HLP-P is $\mathcal{NP}$-Hard.*

Proof. By reduction from the *Max-Avg Facility Dispersion* (MAFD) problem, also known as the *Maxisum Dispersion Problem* (Kuby, 1987). We use the formal definition provided by Ravi et al. (1994): Given a set of nodes $V = \{v_1, v_2, ..., v_n\}$, a nonnegative distance $w(v_i, v_j)$ for each pair v_i, v_j of nodes, and an integer p such that $2 \leq p \leq n$, find a subset $P \subset V$ with $|P| = p$, such that the objective function $g(P) = \frac{2}{p(p-1)} \sum_{x,y \in P} w(x, y)$ is maximized.

To solve an instance of MAFD, we construct an instance of the HLP-P with n vertices and $q = p$ hubs. Starting with an empty commodity set K, we add a commodity k for each pair of vertices $v_i, v_j \in V : i < j$ with $o(k) = i, d(k) = j$, and $t_k = w(v_i, v_j)$. We set the tariffs as $\bar{t}_{ij} = 1, \forall (i, j) \in A$, and $\gamma = 0.5$. For this instance, no commodity k would use an existing network arc other than $(o(k), d(k))$, since its tariff is greater than $\gamma \cdot \bar{t}_{o(k),d(k)} = 0.5$. Consequently, the optimal solution of this instance of HLP-P only captures the demand of the commodities whose both origin and destination are hubs. To capture all such demand at maximum revenue, the optimal price of each inter-hub arc should be 0.5 (equal to the value of γ). Therefore, the optimal hub locations for the HLP-P instance match the optimal set of facilities for the MAFD, and the optimal objective value of the MAFD instance is equal to that of the HLP-P instance times $\frac{4}{p(p-1)}$.

□

4. Formulations

In addition to the notation introduced in Section 3, we define δ_i^+ as the set of outgoing arcs and δ_i^- as the set of incoming arcs for each $i \in V$. Moreover, we define parameters t_{ij}^k for $(i, j) \in A$, $k \in K$ to ease the modelling of the follower's choice of delivering routes, taking into account the saving threshold condition. This parameter is given by

$$t_{ij}^k := \begin{cases} \bar{t}_{ij}, & \text{if } (i, j) \in A \setminus \{(o(k), d(k))\} \\ \gamma \cdot \bar{t}_{ij}, & \text{if } (i, j) = (o(k), d(k)). \end{cases}$$

Finally, we define

$$b_i^k = \begin{cases} 1, & \text{if } i = o(k), \\ -1, & \text{if } i = d(k), \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For any $i \in V$, let x_i be equal to 1 if a hub is located at vertex i, and 0 otherwise. In addition, for any $(i, j) \in A$ and $k \in K$, let y_{ij}^k be equal to 1 if commodity k is transported on arc (i, j) without using the hub network, and 0 otherwise. Similarly, for any $(i, j) \in A$ and $k \in K$, let w_{ij}^k be equal to 1 if commodity k is transported on arc (i, j) using the hub network, and 0 otherwise. In the formulation to be presented below, the transportation variables are stated as $\hat{y}_{ij}^k$ and $\hat{w}_{ij}^k$ for the follower's perspectives. Finally, let p_{ij} be the price of using the hub network for traversing arc $(i, j) \in A$. A formulation for HLP-P can then be stated as: (HLP-P0)

$$
\begin{aligned}
\text{maximize} \quad & \sum_{k\in K}\sum_{(i,j)\in A} T_k(p_{ij}-\bar{c}_{ij})w_{ij}^k && (2)\\
\text{subject to} \quad & \sum_{i\in V} x_i = q && (3)\\
& p_{ij} \le \bar{t}_{ij} \quad \forall (i,j)\in A, && (4)\\
& \sum_{j\in\delta_i^+} w_{ij}^k \le x_i \quad \forall i\in V, k\in K, && (5)\\
& \sum_{i\in\delta_j^-} w_{ij}^k \le x_j \quad \forall j\in V, k\in K, && (6)\\
& \sum_{j\in\delta_i^+}(y_{ij}^k + w_{ij}^k) - \sum_{j\in\delta_i^-}(y_{ji}^k + w_{ji}^k) = b_i^k \quad \forall i\in V, \forall k\in K, && (7)\\
& p_{ij} \ge 0 \quad \forall (i,j)\in A, && (8)\\
& x_i \in \{0,1\} \quad \forall i\in V, && (9)\\
& y_{ij}^k, w_{ij}^k \in \{0,1\} \quad \forall (i,j)\in A, k\in K, && (10)\\
& \left.\begin{array}{l}
\sum_{(i,j)\in A}(t_{ij}^k y_{ij}^k + p_{ij}w_{ij}^k) = \\
\quad \text{minimize} \quad \sum_{(i,j)\in A}(t_{ij}^k \hat{y}_{ij}^k + p_{ij}\hat{w}_{ij}^k) \\
\quad \text{subject to} \ \sum_{j\in\delta_i^+}(\hat{y}_{ij}^k + \hat{w}_{ij}^k) - \sum_{j\in\delta_i^-}(\hat{y}_{ji}^k + \hat{w}_{ji}^k) = b_i^k \quad \forall i\in V, \\
\qquad\qquad\qquad \hat{y}_{ij}^k, \hat{w}_{ij}^k \ge 0 \qquad\qquad\qquad\qquad\qquad\qquad \forall (i,j)\in A,
\end{array}\right\} \forall k\in K. && (11)
\end{aligned}
$$

The objective function (2) aims to maximize the profit of the leader. Constraint (3) sets the maximum number of hubs to be located. Constraint set (4) requires the price of each arc to be upper bounded by its tariff in the existing transportation network, since it is suboptimal for a commodity to traverse a hub arc with a price that is higher than tariff of the underlying non-hub arc. Constraint sets (5) and (6) force a vertex to be a hub if at least one commodity is transported using the hub network on an arc emanating from or ending at it. Constraint set (7) states the flow conservation for the commodities. Constraint set (8) states that all prices are nonnegative. Constraint sets (9) and (10) require the hub and transportation decisions to be binary. Finally, constraint set (11) ensures the follower chooses the routes to pay the minimum amount of money for transporting all the commodities. The definition of t_{ij}^k ensures that commodity k uses the direct arc $y_{o(k),d(k)}^k$ unless there is a path with tariff less than or equal to $\gamma \cdot \bar{t}_{o(k),d(k)}$. Note that the lower level model can only use the links available for commodity transportation. Therefore, the families of constraints (5) and (6), considering x as fixed values by the upper level, should be in the formulation of the lower level model. However, using Proposition 2 in Brotcorne et al. (2008), those constraints can be moved to the upper level model. So, the resulting lower level model is a minimum cost flow problem where the binary nature of $(\hat{y},\hat{w})$-variables can be relaxed.

The formulation presented above is nonlinear due to the objective function (2) as well as the bilevel programming constraints (11). To linearize this formulation, we define $z_{ij}^k = p_{ij} w_{ij}^k, \forall (i,j) \in A, k \in K$. We also define the dual variables u_i^k, associated with constraint $i \in V$ for commodity $k \in K$ within (11). The resulting linearized formulation is then:

(HLP-P1)

$$
\begin{aligned}
\text{maximize} \quad & \sum_{k \in K} \sum_{(i,j) \in A} T_k z_{ij}^k - \sum_{k \in K} \sum_{(i,j) \in A} T_k \bar{c}_{ij} w_{ij}^k && (12) \\
\text{subject to} \quad & (3)\text{–}(10), \\
& z_{ij}^k \leq p_{ij} \quad \forall (i,j) \in A, k \in K, && (13) \\
& z_{ij}^k \leq t_{ij}^k w_{ij}^k \quad \forall (i,j) \in A, k \in K, && (14) \\
& z_{ij}^k \geq p_{ij} - (1 - w_{ij}^k) \bar{t}_{ij} \quad \forall (i,j) \in A, k \in K, && (15) \\
& z_{ij}^k \geq 0 \quad \forall (i,j) \in A, k \in K, && (16) \\
& \sum_{(i,j) \in A} (t_{ij}^k y_{ij}^k + z_{ij}^k) = \sum_{i \in V} b_i^k u_i^k \quad \forall k \in K, && (17) \\
& u_i^k - u_j^k \leq t_{ij}^k \quad \forall (i,j) \in A, k \in K, && (18) \\
& u_i^k - u_j^k \leq p_{ij} \quad \forall (i,j) \in A, k \in K, && (19) \\
& u_i^k \text{ unrestricted} \quad \forall i \in V, k \in K. && (20)
\end{aligned}
$$

Constraint sets (13), (14), and (15) linearize the revenue made by the leader. Constraint set (16) requires all incomes to be nonnegative. Constraint set (17) replaces (11) to ensure that each commodity is transported in the cheapest way. Constraint sets (18), (19), and (20) stem from the duals of the follower's subproblems. Based on Farkas' lemma, necessary conditions of optimality for a linear program can be stated as simultaneous primal and dual feasibility, with equal objective function values. In this case, constraint set (7) ensures primal feasibility, constraint sets (18), (19), and (20) guarantee the dual feasibility, and (17) enforces the equality of the objective values. Hence, (17)-(20) is equivalent to (11).

We conclude this section with a remark on the possibility of aggregating commodities to decrease the problem size, which is a common and successful practice in the hub location literature. However, in the case of HLP-P, aggregation would require the linearization of the product of general integer variables (flows) and continuous variables (prices). This is not possible without defining additional variables and constraints that would result in a more compact yet weaker formulation. Hence, we opt to perform our analysis on the binary flow variables of model HLP-P1.

5. Problem Properties and Valid Inequalities

In this section, we first demonstrate a number of problem properties. Next, we provide variable reductions leading to a stronger and more compact model. Lastly, we present valid inequalities to strengthen the upper bound provided by the Linear Programming (LP) relaxation.

5.1. Problem properties

We start by showing that in any optimal solution for the HLP-P, the prices between the hubs obey the triangle inequality.

Lemma 1. *For any instance of HLP-P, there exists an optimal solution for which the prices between the hubs obey the triangle inequality, i.e.* $p_{il} \leq p_{ij} + p_{jl}\ \forall i, j, l \in V : x_i = x_j = x_l = 1$.

Proof. Assume that we have an optimal solution of an instance of HLP-P where $\exists i, j, l \in V : x_i = x_j = x_l = 1$ and $p_{il} > p_{ij} + p_{jl}$. If $\exists k \in K : w^k_{il} = 1$, then we can construct a solution for the follower's problem with $\hat{w}^k_{ab} = w^k_{ab}\, \forall (a, b) \in A \setminus \{(i, l), (i, j), (j, l))\}$ and $\hat{w}^k_{ij} = \hat{w}^k_{jl} = 1$, which has a lower objective value for that problem than the considered solution. This violates (11), which requires the objective value of the subproblem for the two solutions to be equal. On the other hand, if $w^k_{il} = 0\, \forall k \in K$, then the solution can be amended as $p'_{il} := p_{ij} + p_{jl}$. Since no commodities use this hub arc, the new solution has the same objective function value as the previous one. If this change of price results in a violation of the triangle inequality, i.e. $p'_{il} + p_{lm} < p_{im}$ for some $m \in V : x_m = 1$, then the argument can be re-applied to the variable p_{im}. Indeed, the argument can be recursively reapplied to obtain an optimal solution for which the prices between the hubs obey the triangle inequality. □

Lemma 1 enables us to restrict the search space, as we show next.

Lemma 2. *For any instance of HLP-P, the optimal path of each commodity* $k \in K$ *can be restricted to be only one of the following five types:*
i) $y^k_{o(k),d(k)} = 1$
ii) $w^k_{o(k),d(k)} = 1$
iii) $y^k_{o(k),i} = w^k_{i,d(k)} = 1$*, for some* $i \in V \setminus \{o(k), d(k)\}$
iv) $w^k_{o(k),i} = y^k_{i,d(k)} = 1$*, for some* $i \in V \setminus \{o(k), d(k)\}$
v) $y^k_{o(k),i} = w^k_{ij} = y^k_{j,d(k)} = 1$*, for some* $i, j \in V \setminus \{o(k), d(k)\}$.

Proof. The proof is based on three arguments:
a) Consider a path for commodity k, originating at $o(k)$ and ending at $d(k)$, consisting of y^k_{ij} and w^k_{ij} variables. Any sub-path of this path, consisting of only y^k_{ij} variables, can be reduced to a single arc starting at the origin of the first arc of the sub-path and ending at the destination of the last arc of the sub-path, since the tariffs satisfy the triangle inequality.
b) An argument similar to a) applies to the w^k_{ij} variables, since the prices between the hubs obey the triangle inequality (Lemma 1).
c) It is never suboptimal to convert a sub-path of three consecutive arcs in a path $w^k_{ab} = y^k_{bc} = w^k_{cd} = 1$

to $w^k_{ab} = w^k_{bc} = w^k_{cd} = 1$. Assume that we have a solution with $w^k_{ab} = y^k_{bc} = w^k_{cd} = 1$ for some $k \in K$. Firstly, since the follower is using the hub arc structure for commodity k, we have that the saving threshold condition is already fulfilled (the total price of the path from $o(k)$ to $d(k)$ is less than or equal to $\gamma \cdot t_{o(k),d(k)}$). Moreover, any path with a lower price or the same price will also satisfy this condition. Secondly, constraint set (4) sets the upper bound of p_{bc} to $\bar{t}_{bc}$. If $p_{bc} < \bar{t}_{bc}$, the follower would use the hub arc for commodity k, due to its lower price. For the case of $p_{bc} = \bar{t}_{bc}$, we will have a path with the same price. Hence, taking $w^k_{bc} = 1$ will does not increase the price of the path.

Applying arguments a), b), and c) repeatedly to any feasible solution results in one of the solution types described above. $\square$

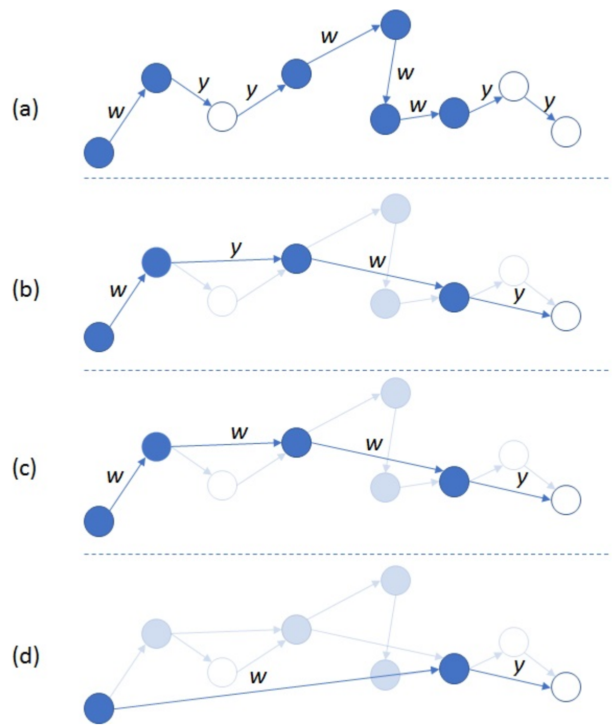

Figure 1: A demonstration of Lemma 2, where a given path is reduced to a path of type iv).

An application of Lemma 2 is depicted in Figure 1, where the shaded vertices correspond to the hubs as in Figure 1. The given path for the commodity comprises of 8 arcs, alternating between hub and existing network arcs, as shown in Figure 1a. Applying arguments a) and b) in the proof of Lemma 3 shrinks the consecutive hub and existing network arcs, resulting in Figure 1b. Argument c) in the proof of Lemma 3 allows the conversion of the second arc in Figure 1b to a hub arc, resulting in Figure 1c. A final application of argument b) brings us to Figure 1d, which depicts a solution of type iv).

A side result of Lemma 2 is that the values of the y^k_{ij} variables are implied by the values of the w^k_{ij} variables.

Lemma 3. *In any solution for the HLP-P, $w_{ij}^k = 1$ implies $y_{o(k),i} = 1$ if $o(k) \neq i$, and $y_{j,d(k)} = 1$ if $j \neq d(k)$.*

Proof. Follows from the solution types *iii)*, *iv)*, and *v)* described in Lemma 2. $\square$

Another implication of Lemma 2 is that an HLP-P instance can be solved as an MAFD instance for small enough values of γ, for which a compact formulation has been presented by Kuby (1987).

Proposition 2. *Any HLP-P instance, where $\gamma < \min_{k \in K} \frac{\min_{i \in V \setminus \{o(k),d(k)\}} \{\bar{t}_{o(k),i}, \bar{t}_{i,d(k)}\}}{\bar{t}_{o(k),d(k)}}$, can be solved as an MAFD instance.*

Proof. For each $k \in K$, a lower bound for the price of solution types iii), iv), and v) of Lemma 2 can be computed as $\min_{i \in V \setminus \{o(k),d(k)\}} \{\bar{t}_{o(k),i}, \bar{t}_{i,d(k)}\}$. Dividing this lower bound by $\bar{t}_{o(k),d(k)}$ yields a lower bound for the values of γ for which the commodity would use one of these solution types. If $\gamma < \min_{k \in K} \frac{\min_{i \in V \setminus \{o(k),d(k)\}} \{\bar{t}_{o(k),i}, \bar{t}_{i,d(k)}\}}{\bar{t}_{o(k),d(k)}}$, then none of the commodities would use one of these solution types. Hence, the optimal hub locations can be determined by capturing solution type ii), through solving an MAFD on a complete undirected graph, with $w(v_i, v_j) = \sum_{\substack{k \in K: o(k)=i, d(k)=j \\ \text{or } o(k)=j, d(k)=i}} T_k$. The optimal price for each hub arc would be $p_{ij} = \gamma \cdot \bar{t}_{ij}$. $\square$

5.2. Variable reductions

We are now ready to state a transformation of HLP-P1 into a more compact and stronger formulation.

Proposition 3. *Replacing the constraint sets* (7) *and* (17) *in HLP-P1 with*

$$y_{o(k),d(k)} + \sum_{(i,j) \in A} w_{ij}^k = 1, \quad \forall k \in K, \tag{21}$$

and

$$t_{o(k),d(k)}^k y_{o(k),d(k)}^k + \sum_{(i,j) \in A} ((t_{o(k),i}^k + t_{j,d(k)}^k) w_{ij}^k) + \sum_{(i,j) \in A} z_{ij}^k = \sum_{i \in V} b_i^k u_i^k \quad \forall k \in K, \tag{22}$$

respectively, results in a valid formulation (HLP-P2), which dominates the original formulation.

Proof. The validity of both (21) and (22) follows from Lemmas 2 and 3. The resulting formulation dominates the original one since a feasible solution for the LP relaxation of HLP-P1 can be constructed from any feasible solution for the LP relaxation HLP-P2. This can be done by setting $y_{o(k),i}^k = \sum_{l \in \delta_i^+} w_{il}^k$ and $y_{j,d(k)}^k = \sum_{l \in \delta_j^-} w_{lj}^k$, $\forall k \in K; i, j \in V \setminus \{o(k), d(k)\}$. However, there exist solutions for the LP relaxation of HLP-P1 that contain multiple nonzero values for some $k \in K$, e.g. $w_{o(k),a}^k = y_{ab}^k = y_{bc}^k = w_{c,d(k)}^k = 1$, which cannot be translated to a feasible solution for the LP relaxation of HLP-P2. Hence we conclude that the space of feasible solutions is smaller for the LP relaxation of HLP-P2. $\square$

Note that the resulting formulation does not involve any y_{ij}^k variables other than $y_{o(k),d(k)}^k$, decreasing the number of variables by $|K| \times (|A| - 1)$. This reduction, however, cannot be directly reflected onto the follower's subproblems. An example is depicted in Figure 2, with three commodities and $\gamma = 0.95$ and $\bar{c}_{ij} = 0$ for any $i, j \in N$. Vertices 2, 3, and 5 are selected as the optimal set of hubs. An optimal solution is provided in Figure 2a, which provides services to commodities 1 and 2. For commodity 1, there exist two alternatives with the same price of 2.85, and the commodity follows the direct arc due to the saving threshold condition. Capturing commodity 3 would require a price of 0.85 for arc $(5, 3)$, effectively decreasing the price of arc $(2, 3)$ to 1.7 due to the triangle inequality, which is suboptimal. However, if the follower are restricted to use at most one hub arc for each commodity, the optimal solution becomes the one shown in Figure 2b, with the prices of $(2, 5), (5, 3)$ and $(2, 3)$ violating the triangle inequality. In this case, commodity 1 cannot use both arcs $(2, 5)$ and $(5, 3)$ at the indicated price due to the single arc restriction, and is still limited to a total price of 2.85.

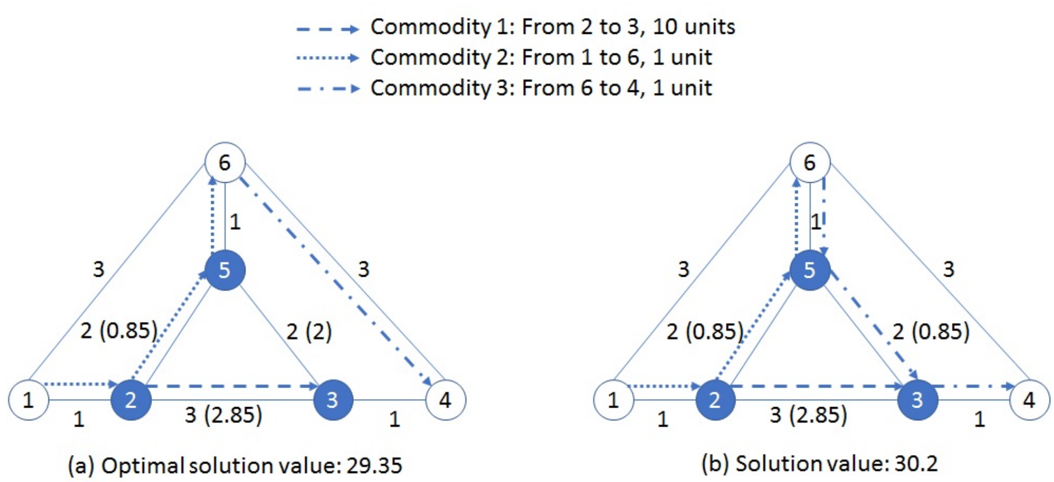

Figure 2: An example of the optimal prices, and the prices when the follower are restricted to use a single hub arc for each commodity.

We note that the triangle inequality cannot be enforced on all price variables due to Lemma 1, and enforcing prices that obey triangle inequality only between the hubs requires constraints involving big M terms. Next, we show that we can discard most members of the constraint set (18).

Proposition 4. *Constraint set* (19) *dominates* (18) $\forall k \in K, (i, j) \in A : i \neq o(k)$ *or* $j \neq d(k)$.

Proof. The proof follows from the fact that $p_{ij} \leq t_{ij}^k = \bar{t}_{ij}, \forall k \in K, (i, j) \in A : i \neq o(k)$ or $j \neq d(k)$ due the definition of t_{ij}^k and (4). $\square$

Proposition 4 decreases the number of constraints by $|K| \times (|A| - 1)$. In addition, a subset of the constraint set (19) as well as w_{ij}^k variables can also be discarded, as we now demonstrate.

Proposition 5. *For $k \in K, R_k = \{(i,j) \in A : i = d(k) \text{ or } j = o(k)\}$, the variables $w_{ij}^k, \hat{w}_{ij}^k : (i,j) \in R_k$ will never assume a positive value in any optimal solution for the HLP-P.*

Proof. In the absence of negative tariffs and prices, any optimal path for a commodity would be an acyclic path, so a commodity returning to its origin or departing from its destination would be suboptimal. □

We now focus on tightening the LP relaxation of HLP-P2.

Proposition 6. *The constraint sets*

$$\bar{c}_{ij} w_{ij}^k \le z_{ij}^k \le \max\{0, t_{o(k),d(k)}^k - t_{o(k),i}^k - t_{j,d(k)}^k\} w_{ij}^k \quad \forall (i,j) \in A, k \in K \tag{23}$$

is valid and dominates (14)*.*

Proof. The validity of (23) follows from the fact that the price the leader can set for a commodity k on arc (i,j) is bounded above by the tariff of direct connection times γ, minus the tariff of the arc from $o(k)$ to i and the tariff of the arc from j to $d(k)$. The domination property follows from the triangle inequality $\bar{t}_{o(k),d(k)} \le \bar{t}_{o(k),i} + \bar{t}_{ij} + \bar{t}_{j,d(k)}$ that yields $t_{o(k),d(k)}^k \le t_{o(k),i}^k + t_{ij}^k + t_{j,d(k)}^k$, which can be rewritten as $t_{o(k),d(k)}^k - t_{o(k),i}^k - t_{j,d(k)}^k \le t_{ij}^k$. Finally, the first inequality comes from the fact that the cost per unit of commodity of traversing a hub arc should be smaller than or equal to the price. □

A consequence of Proposition 6 is that we can eliminate w_{ij}^k and z_{ij}^k variables that will never assume a positive value. In the next proposition, we provide a general result for eliminating w_{ij}^k variables.

Proposition 7. *For $k \in K, S_k = \{(i,j) \in A : t_{o(k),d(k)}^k - t_{o(k),i}^k - t_{j,d(k)}^k \le \bar{c}_{ij}\} \cup \{(i,j) \in A : i,j \in V \setminus \{o(k), d(k)\} : t_{o(k),i}^k > t_{o(k),j}^k \text{ or } t_{i,d(k)}^k < t_{j,d(k)}^k\}$, the variables $w_{ij}^k, z_{ij}^k : (i,j) \in S_k$ can be discarded.*

Proof. S_k consists of two subsets, the first of which is a direct consequence of Proposition 6. To prove the validity of the second subset, consider the optimal path $\hat{y}_{o(k),i}^k = \hat{w}_{ij}^k = \hat{y}_{j,d(k)}^k = 1$ for commodity k, with total price $t_{o(k),i}^k + p_{ij} + t_{j,d(k)}^k \le t_{o(k),d(k)}^k$. The property $t_{o(k),i}^k + p_{ij} + t_{j,d(k)}^k \ge t_{o(k),i}^k + t_{j,d(k)}^k$ follows from the nonnegativity constraints (8). If $t_{o(k),i}^k > t_{o(k),j}^k$, then $t_{o(k),i}^k + t_{j,d(k)}^k > t_{o(k),j}^k + t_{j,d(k)}^k \ge t_{o(k),d(k)}^k$ by the triangle inequality, contradicting the assumption that the path is optimal. Similarly, if $t_{i,d(k)}^k < t_{j,d(k)}^k$, then $t_{o(k),i}^k + t_{j,d(k)}^k > t_{o(k),i}^k + t_{i,d(k)}^k \ge t_{o(k),d(k)}^k$. □

For the sake of brevity, let us define the set of feasible arcs for commodity k as $A_k = A \setminus (R_k \cup S_k)$, and the maximum income the leader can get from commodity k on arc (i,j) as $\hat{t}_{ij}^k := t_{o(k),d(k)}^k - t_{o(k),i}^k - t_{j,d(k)}^k$. Based on Propositions 5 and 7, constraint set (23) can be rewritten as:

$$\bar{c}_{ij} w_{ij}^k \le z_{ij}^k \le \hat{t}_{ij}^k w_{ij}^k \quad \forall k \in K, (i,j) \in A_k. \tag{24}$$

Note that Proposition 7 cannot be applied to $\hat{w}_{ij}^k$, due to Proposition 4 that modified the structure of the constraints corresponding to the dual of the follower's subproblems. We now provide the updated form of HLP-P2, for the sake of readability.

(HLP-P2)

$$\begin{aligned}
\text{maximize} \quad & \sum_{k\in K}\sum_{(i,j)\in A} T_k z_{ij}^k - \sum_{k\in K}\sum_{(i,j)\in A} T_k \bar{c}_{ij} w_{ij}^k && (25)\\
\text{subject to} \quad & \sum_{i\in V} x_i \le q && (26)\\
& p_{ij} \le \bar{t}_{ij} \quad \forall (i,j)\in A, && (27)\\
& \sum_{j\in\delta_i^+\cap A_k} w_{ij}^k \le x_i \quad \forall i\in V, k\in K, && (28)\\
& \sum_{i\in\delta_j^-\cap A_k} w_{ij}^k \le x_j \quad \forall j\in V, k\in K, && (29)\\
& y_{o(k),d(k)} + \sum_{(i,j)\in A_k} w_{ij}^k = 1, \quad \forall k\in K, && (30)\\
& z_{ij}^k \le p_{ij} \quad \forall k\in K, (i,j)\in A_k, && (31)\\
& z_{ij}^k \le \hat{t}_{ij}^k w_{ij}^k \quad \forall k\in K, (i,j)\in A_k, && (32)\\
& z_{ij}^k \ge p_{ij} - (1-w_{ij}^k)\bar{t}_{ij} \quad \forall k\in K, (i,j)\in A_k, && (33)\\
& t_{o(k),d(k)}^k y_{o(k),d(k)}^k + \sum_{(i,j)\in A_k} ((t_{o(k),i}^k + t_{j,d(k)}^k) w_{ij}^k) + \sum_{(i,j)\in A_k} z_{ij}^k \\
& \qquad = \sum_{i\in V} b_i^k u_i^k \quad \forall k\in K, && (34)\\
& u_{o(k)}^k - u_{d(k)}^k \le t_{o(k),d(k)}^k \quad \forall k\in K, && (35)\\
& u_i^k - u_j^k \le p_{ij} \quad \forall k\in K, (i,j)\in A\setminus R_k, && (36)\\
& u_i^k \text{ unrestricted} \quad \forall i\in V, k\in K, && (37)\\
& p_{ij} \ge 0 \quad \forall (i,j)\in A, && (38)\\
& x_i \in \{0,1\} \quad \forall i\in V, && (39)\\
& y_{o(k),d(k)}^k \in \{0,1\} \quad k\in K, && (40)\\
& w_{ij}^k \in \{0,1\} \quad \forall k\in K, (i,j)\in A_k, && (41)\\
& z_{ij}^k \ge \bar{c}_{ij} w_{ij}^k \quad \forall k\in K, (i,j)\in A_k. && (42)
\end{aligned}$$

5.3. Valid inequalities

We now present polynomial-sized valid inequalities for HLP-P1 and HLP-P2.

Proposition 8. *The following two family of constraints are valid for HLP-P1 and HLP-P2:*

$$p_{ij} \geq \bar{t}_{ij}(1 - x_i), \quad \forall (i; j) \in A; \tag{43}$$

$$p_{ij} \geq \bar{t}_{ij}(1 - x_j), \quad \forall (i; j) \in A. \tag{44}$$

Proof. Constraint sets (43) and (44) state that the price of an arc cannot be less than the tariff of direct transportation unless both ends of the arc are hubs. □

Our second valid inequality stems from (28) and (29).

Proposition 9. *The inequalities*

$$\sum_{j \in \delta_i^- \cap A_k} w_{ji}^k + \sum_{j \in \delta_i^+ \cap A_k} w_{ij}^k \leq x_i, \quad \forall i \in V, k \in K. \tag{45}$$

are valid for HLP-P2, and dominate (28) *and* (29)*.*

Proof. These valid inequalities imply that a location is a hub if a commodity flows into or out of it through a hub arc. The domination property follows from the fact that they are the result of lifting (28) and (29). □

We emphasize that we have presented (45) as a valid inequality rather than part of the formulation, since we have observed (28) and (29) to have a better computational performance. The next valid inequality provides upper bounds for the optimal prices.

Proposition 10. *The inequalities*

$$p_{ij} \leq \hat{t}_{ij}^k w_{ij}^k + \bar{t}_{ij}(1 - w_{ij}^k) \quad \forall k \in K, (i, j) \in A_k \tag{46}$$

are valid for HLP-P2.

Proof. Similar to (23), the validity of (46) follows from the fact that it is not profitable for commodity k to use the hub service on arc (i, j) if the sum of the tariff from $o(k)$ to i, the price of hub arc (i, j), and the tariff from j to $d(k)$ exceeds the direct connection tariff $t_{o(k),d(k)}^k$. Hence, commodity k using arc (i, j) implies that the price p_{ij} is less than or equal to $\hat{t}_{ij}^k$. □

The rest of the valid inequalities we present concerns pairs of commodities, k and k'.

Proposition 11. *The following inequalities are valid for HLP-P2:*

$$\sum_{j \in \delta_i^+ \cap A_{k'}} w_{ij}^{k'} \leq 1 - y_{o(k),d(k)}^k$$

$$\forall i \in V, k, k' \in K : d(k') = d(k), \bar{t}_{o(k),i} < \bar{t}_{o(k'),i}, \bar{t}_{o(k'),d(k')} \leq \bar{t}_{o(k),d(k)}, \tag{47}$$

$$\sum_{i \in \delta_j^- \cap A_{k'}} w_{ij}^{k'} \leq 1 - y_{o(k),d(k)}^k$$

$$\forall j \in V, k, k' \in K, o(k') = o(k), \bar{t}_{j,d(k)} < \bar{t}_{j,d(k')}, \bar{t}_{o(k'),d(k')} \leq \bar{t}_{o(k),d(k)}. \tag{48}$$

Proof. Let (x, y, w, z, p) be a solution of HLP-P2. To prove the validity of inequalities (47), assume that $w_{ij}^{k'} = 1$ for a given $(i, j) \in A \setminus (R_{k'} \cup S_{k'})$ and $k' \in K$. Then, $\bar{t}_{o(k'),i} + p_{ij} + \bar{t}_{j,d(k')} \leq t_{o(k'),d(k')}^{k'}$. For any $k \in K$ such that $\bar{t}_{o(k),i} < \bar{t}_{o(k'),i}$ and $d(k) = d(k')$, we have $\bar{t}_{o(k),i} + p_{ij} + \bar{t}_{j,d(k)} \leq \bar{t}_{o(k'),i} + p_{ij} + \bar{t}_{j,d(k')}$. If, in addition $\bar{t}_{o(k'),d(k')} \leq \bar{t}_{o(k),d(k)}$, then $\bar{t}_{o(k),i} + p_{ij} + \bar{t}_{j,d(k)} \leq t_{o(k),d(k)}^k$ and if the previous inequality is strict, it would imply that $y_{o(k),d(k)}^k = 0$, otherwise we can assume it without loss of generality, and the result follows. The validity of (48) can be proved in a similar way.

□

Proposition 12. *The following inequalities are valid for HLP-P2:*

$$w_{ij}^k \leq w_{ij}^{k'} \quad \forall k, k' \in K : o(k') = o(k), d(k') = j; (i, j) \in A_k \cap A_{k'}, \tag{49}$$

$$w_{ij}^k \leq w_{ij}^{k'} \quad \forall k, k' \in K : o(k') = i, d(k') = d(k); (i, j) \in A_k \cap A_{k'}. \tag{50}$$

Proof. Given a solution (x, y, w, z, p) for HLP-P2, consider $k, k' \in K$ be such that $o(k') = o(k)$ and $d(k') = j$, and $(i, j) \in A \setminus (R_k \cup S_k \cup R_{k'} \cup S_{k'})$, with $w_{ij}^k = 1$ and $w_{ij}^{k'} = 0$. Then, either $y_{o(k'),d(k')} = 1$ or $w_{i',j}^{k'} = 1$ for some $i' (\neq i) \in V$.

In the first case, it would imply that $(1 - \gamma)\bar{t}_{o(k'),d(k')} \leq \bar{t}_{o(k'),i} + p_{i,d(k')}$ and then, $(1 - \gamma)\bar{t}_{o(k'),d(k')} + \bar{t}_{d(k'),d(k)} \leq \bar{t}_{o(k),i} + p_{ij} + \bar{t}_{d(k'),d(k)}$. In addition $(1 - \gamma)\bar{t}_{o(k'),d(k')} + \bar{t}_{d(k'),d(k)} \geq (1 - \gamma)(\bar{t}_{o(k'),d(k')} + \bar{t}_{d(k'),d(k)}) \geq (1 - \gamma)\bar{t}_{o(k),d(k)}$. However, since $j = d(k')$ and $o(k) = o(k')$, if some of the previous inequalities are strict, it contradicts the fact that $w_{ij}^k = 1$, otherwise we can assume it without loss of generality, and the result follows.

If $w_{i',j}^{k'} = 1$, then $\bar{t}_{o(k),i'} + p_{i',j} < \bar{t}_{o(k),i} + p_{ij}$, but it contradicts the fact that $w_{ij}^k = 1$, because it is cheaper to get j from i' than from i.

□

Proposition 13. *The following inequalities are valid for HLP-P2:*

$$z_{ij}^k \geq p_{ij} - (1 - w_{ij}^{k'})\bar{t}_{ij} - \hat{t}_{ij}^{k'}(w_{ij}^{k'} - w_{ij}^k), \tag{51}$$

$$\forall k, k' \in K, (i, j) \in A_k \cap A_{k'},$$

$$z_{ij}^{k'} - z_{ij}^k \leq \hat{t}_{o(k'),d(k')}^{k'}(w_{ij}^{k'} - w_{ij}^k) : \forall k, k' \in K,$$

$$(i, j) \in A_k \cap A_{k'} : o(k') = o(k), d(k') = j \text{ or } o(k') = i, d(k') = d(k). \tag{52}$$

Proof. Let (x, y, w, z, p) be a feasible solution for HLP-P2. To prove the validity of (51), if $w_{ij}^k = w_{ij}^{k'} = 1$ the inequality becomes $z_{ij}^k \geq p_{ij}$, which is valid since in this case $z_{ij}^k = p_{ij}$. Analogously, if $w_{ij}^k = w_{ij}^{k'} = 0$ the inequality becomes $z_{ij}^k \geq p_{ij} - \bar{t}_{ij}$ which is valid because in this case $z_{ij}^k = 0$ and $p_{ij} \leq \bar{t}_{ij}$. If $w_{ij}^k = 1$ and $w_{ij}^{k'} = 0$, the inequality becomes $z_{ij}^k \geq p_{ij} - \bar{t}_{ij} + t_{o(k'),d(k')}^{k'} - t_{o(k'),i}^{k'} - t_{j,d(k')}^{k'}$, since $t_{o(k'),d(k')}^{k'} - t_{o(k'),i}^{k'} - t_{j,d(k')}^{k'} \leq \bar{t}_{ij}$ and $z_{ij}^k = p_{ij}$ the inequality is valid. Finally, if $w_{ij}^k = 0$ and $w_{ij}^{k'} = 1$, the inequality becomes $z_{ij}^k \geq p_{ij} - (t_{o(k'),d(k')}^{k'} - t_{o(k'),i}^{k'} - t_{j,d(k')}^{k'})$ which is valid by (46).

To prove the validity of (52), observe that if $z_{ij}^{k'} - z_{ij}^k = 0$ then $w_{ij}^{k'} - w_{ij}^k = 0$ and the inequality holds. Then assuming that $z_{ij}^{k'} - z_{ij}^k \neq 0$, by (49) and (50), $w_{ij}^{k'} - w_{ij}^k$ is equal to 1. Hence, $w_{ij}^k = 0$ and $w_{ij}^{k'} = 1$ and inequality (52), becomes $z_{ij}^{k'} \leq \hat{t}_{o(k'),d(k')}^{k'}$ which is valid by (46), which proves the validity of the inequality. □

6. A common inter-hub discount factor HLP-P model

Most of the literature on hub location problems applies economies of scale by using a discount factor, $\alpha \in [0, 1]$, to the cost of of traversing any arc connecting the hubs. In the case of HLP-P, using discount factor decreases the search space, since all prices between the hubs are determined by multiplying the tariffs of the existing network a common discount factor. Regarding the formulation of this problem, the follower's subproblems require the price for each arc, and the leader can set the prices only for the arcs between the selected hub locations. Consequently, the formulation for a common discount factor HLP-P model uses all the variables of HLP-P2 and the additional decision variable $\alpha \in [0, \gamma]$ that denotes the common discount factor. Accordingly, the following constraints must be added to HLP-P2:

$$p_{ij} \geq \alpha \cdot \bar{t}_{ij} \quad \forall (i, j) \in A, \tag{53}$$

$$p_{ij} \leq \alpha \cdot \bar{t}_{ij} + (2 - x_i - x_j)\bar{t}_{ij} \quad \forall (i, j) \in A. \tag{54}$$

Constraint set (53) establishes the lower bound for the prices based on α. Constraint set (54) requires the upper bound to be equal to the lower bound for the arcs between the hubs. In addition, the following inequalities are valid for a common inter-hub discount factor case, and can replace (13) by

$$z_{ij}^k \leq \alpha \cdot \bar{t}_{ij} \quad \forall k \in K, (i, j) \in A_k. \tag{55}$$

We name the the resulting formulation for a common inter-hub discount factor HLP-P as HLP-P2', and emphasize that all results of Sections 2 and 3 including the proof of $\mathcal{NP}$-Hardness also apply to HLP-P2'.

Although the use of a common discount factor does not offer a simplification regarding the formulation, the smaller search space allows us to design an efficient heuristic algorithm. The objective function value of a given solution (α and hub locations) can be computed in polynomial time, using the well-known method of determining the cheapest path for each commodity, identifying the hub arcs that have been used, and accounting the profit of the decision maker accordingly (O'Kelly et al., 2015). The shortest paths between all pairs of vertices can be determined at a complexity of $O(|V|^3)$ using the Floyd-Warshall algorithm, with a modification to ensure that the paths verify the saving threshold condition. However, Lemma 2 allows us to decrease the search space to single hub arcs, and consequently to compute the objective function value in $O(|K|) = O(|V|^2)$ time. The pseudocode to compute the objective value is provided in Algorithm 1. Within the algorithm, r_k denotes the cost of the shortest path for commodity k, whereas s_k represents the leader's profit from this path.

Algorithm 1 Objective value computation

ObjectiveValue (scaling factor α, hub set $H \subset V$)
For $k \in K$
  Initialize $r_k = t^k_{o(k),d(k)}$ and $s_k = 0$
  For $i \in H$
    For $j \in H \setminus \{i\}$
      If $(r_k > t^k_{o(k),i} + \alpha \cdot \bar{t}_{ij} + t^k_{j,d(k)}$ and $\alpha \bar{t}_{ij} \geq \bar{c}_{ij})$ **or** $(r_k = t^k_{o(k),i} + \alpha \cdot \bar{t}_{ij} + t^k_{j,d(k)}$ **and** $s_k < \alpha \cdot \bar{t}_{ij} - \bar{c}_{ij})$. **Then**
        $r_k = t^k_{o(k),i} + \alpha \cdot \bar{t}_{ij} + t^k_{j,d(k)}$
        $s_k = \alpha \cdot \bar{t}_{ij} - \bar{c}_{ij}$ and $\mathcal{T} := \mathcal{T} \cup \{(i,j)\}$
      End If
    End For
  End For
End For
Return $\sum_{k \in K} T_k s_k$

Using Algorithm 1 to compute the objective value enables us to restrict the search to the scaling factor α and the set of hubs H. A straightforward algorithm for single dimensional nonlinear optimization is sampling points with uniform intervals within the lower and upper bounds of the decision variable. Although there exist sophisticated direct search methods (e.g. Pattern Search), we use the direct sampling approach with a small search interval due to its simplicity. Based on this idea, we present our hybrid metaheuristic as Algorithm 2, which combines sampling and Tabu Search. Let us define the tabu list as T and iteration limit as κ_{max}. In addition, let us denote the objective function value of a solution by θ, and write θ' for a candidate solution, θ_{max} for the best candidate solution within the local search, and θ^* for the best known solution. Finally, we will denote the best known scaling factor as α^*.

The outer loop of Algorithm 2 (S-TS) samples α with intervals of ϵ (lines 3-25), starting from the maximum feasible value of γ down to ϵ. Tabu Search forms the inner loop (lines 6-23), searching for the best set of hubs by relocating hubs one at a time. The tabu list aims to diversify the search by forcing the vertices in the tabu list to remain as a hub, or forbidding them to become hubs (line 14). The *aspiration* criterion (line 11) overrides the tabu condition to update the best-known solution. The last step (line 27) consists of solving the linear programming problem of pricing by fixing the variables corresponding to the

Algorithm 2 S-TS

Initialize $\alpha = \gamma$, and H as the q-median solution for $G = (V, A)$
$\alpha^* = \alpha$, $H^* = H$, $\theta^* = \text{ObjectiveValue}(\alpha^*, H^*)$
While $\alpha \geq \epsilon$
  $H = H^*$
  Initialize tabu list $T = \emptyset$
  For $\kappa = 1$ **to** κ_{max}
    $\theta_{max} = 0$
    For $i \in H$
      For $j \in V \setminus H$
        $\theta' = \text{ObjectiveValue}(\alpha, (H \setminus \{i\}) \cup \{j\})$
        If $\theta' > \theta^*$ **Then**
          $\theta^* = \theta'$, $\alpha^* = \alpha$, $H^* = (H \setminus \{i\}) \cup \{j\}$
          $\theta_{max} = \theta'$, $i_{max} = i$, $j_{max} = j$
        Else If $i \notin T$ **and** $j \notin T$ **and** $\theta' > \theta_{max}$ **Then**
          $\theta_{max} = \theta'$, $i_{max} = i$, $j_{max} = j$
        End If
      End For
    End For
    $H = (H \setminus \{i_{max}\}) \cup \{j_{max}\}$
    $T = T \cup \{i_{max}\} \cup \{j_{max}\}$
    Increment the tabu tenure of the vertices in T
    Remove the vertices in T that have reached the tabu tenure limit
  End For
  $\alpha = \alpha - \epsilon$
End While
Use Algorithm 1 to compute the optimal routes for α^* and H^*
Solve the pricing problem by fixing x as H^* and y, w as the optimal routes for α^* and H^*
Return (α^*, H^*)

hubs in H^* and the corresponding route decisions computed using Algorithm 1.

7. Computational Experiments

We have implemented HLP-P2 and HLP-P2' using the callable library of CPLEX 12.10 (64-bits) and C++. We have conducted experiments on the nodes of computing cluster *Balena* hosted at the University of Bath, each with Intel Xeon Platinum CPUs running at 2.70 GHz. Algorithm S-TS was utilized for generating initial solutions. Even for the largest instances, the heuristic did not take more than 10 minutes to complete. Our first experiments aim to test the effect of the variable reductions and the valid inequalities, within a wall clock time limit of 4 hours.

Motivated by the computational success of Contreras et al. (2011) and the work of O'Kelly et al. (2015), we have opted to use Benders decomposition for our models. We refer the interested reader to the aforementioned works for the details how Benders decomposition can be applied to hub location problems. In addition to the traditional branch-and-cut algorithm, we have used the automated Benders decomposition feature of CPLEX for different levels of granularity of the slave problems. Our initial computational experiments have shown the best choice of Benders decomposition to be with K slave problems, each containing the u_i^k variables and the z_{ij}^k variables for a commodity, encompassing both routing and linearization decisions. We have observed this setting to outperform our implementation for the original formulation and we have used it to solve the instances that we have generated.

To test the computational performance of our models and algorithms, we have adapted the Civil Aviation Board data that is available on the ORLIB website (Beasley, 2005). We have computed the tariffs using the formula $\bar{t}_{ij} = d_{ij} \, \forall (i,j) \in A$. We have determined the variable arc costs as $\bar{c}_{ij} = \lambda \times d_{ij} \, \forall (i,j) \in A$, where λ is a parameter we have used for experimentation. We have generated instances for each value of $|V| \in \{10, 15, 20, 25\}$, $q \in \{3, ..., \lfloor |V|/3 \rfloor\}$, $\gamma \in \{0.9, 0.925, 0.95, 0.975, 1\}$, and $\lambda \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$. For instances with $|V| < 25$, we have deleted data that pertain to the nodes with indices higher than $|V|$. The instances are available upon request from the first author.

7.1. Results for arc pricing

The computational results of HLP-P2 for $|V| \in \{10, 15, 20\}$ are provided in Table 1. We have observed the value of γ to have no significant effect on the optimality gap and the CPU time requirement of an instance, so we opt to display our results for each value $|V|, q$ and λ, averaged over the listed values of γ. The results clearly show the necessity for the valid inequality sets (43) and (44), since their presence improves the average CPU time requirement in most of the cases. There also is a clear trend of decreasing CPU time requirement as λ increases. This is expected since an increase in λ means an increase in the variable arc costs $\bar{c}_{ij}$, which results in many of the arcs becoming disadvantageous for the hub transportation provider, thereby decreasing the search space. The formulation HLP-P2 with the valid inequalities (43) and (44) has successfully solved all instances $|V| \in \{10, 15, 20\}$.

For the larger instances with $|V| = 25$, our results are presented in Table 2. A significantly smaller number of instances were solved to optimality. Based on the results of Table 1, we believe that the final incumbent solutions of all the instances are of high quality, which are not reflected onto the table due to the weaker upper bounds provided by HLP-P2. We conclude that the computational reach of HLP-P2 is limited to $|V| = 25$.

7.2. Results for a common discount factor

We have replicated the tests described in the previous subsection for HLP-P2'. The results are presented in Tables 3 and 4. The results of HLP-P2' can be observed to be better than that of HLP-P2. We attribute this to both the reduced search space for the prices due to the common discount factor, and the affinity of Algorithm S-TS to the common discount factor by its design that searches over the possible values of α. Similar to HLP-P2, we conclude that the computational reach of HLP-P2' is limited to $|V| = 25$, albeit with a much better performance for the instances with small q values.

Table 1: Arc Pricing results for $|V| \in \{10, 15, 20\}$

			Without (43) and (44)		With (43) and (44)	
$\lvert V\rvert$	q	λ	Average gap	Average CPU time (sec.)	Average gap	Average CPU time (sec.)
10	3	0	**0.00%**	23.56	**0.00%**	18.61
		0.1	**0.00%**	22.86	**0.00%**	17.69
		0.2	**0.00%**	21.68	**0.00%**	18.47
		0.3	**0.00%**	21.36	**0.00%**	17.75
		0.4	**0.00%**	17.73	**0.00%**	13.72
		0.5	**0.00%**	11.37	**0.00%**	9.12
15	3	0	**0.00%**	93.73	**0.00%**	79.14
		0.1	**0.00%**	92.77	**0.00%**	71.53
		0.2	**0.00%**	82.02	**0.00%**	67.87
		0.3	**0.00%**	76.30	**0.00%**	57.00
		0.4	**0.00%**	68.83	**0.00%**	51.57
		0.5	**0.00%**	57.07	**0.00%**	48.87
	4	0	**0.00%**	88.37	**0.00%**	79.88
		0.1	**0.00%**	87.97	**0.00%**	72.59
		0.2	**0.00%**	73.91	**0.00%**	70.56
		0.3	**0.00%**	70.43	**0.00%**	72.35
		0.4	**0.00%**	66.60	**0.00%**	53.79
		0.5	**0.00%**	56.47	**0.00%**	47.45
	5	0	**0.00%**	104.45	**0.00%**	100.78
		0.1	**0.00%**	90.31	**0.00%**	76.61
		0.2	**0.00%**	80.03	**0.00%**	75.73
		0.3	**0.00%**	71.88	**0.00%**	64.02
		0.4	**0.00%**	60.92	**0.00%**	53.00
		0.5	**0.00%**	64.59	**0.00%**	51.51
20	3	0	**0.00%**	873.33	**0.00%**	628.84
		0.1	**0.00%**	766.71	**0.00%**	450.14
		0.2	**0.00%**	851.37	**0.00%**	409.74
		0.3	**0.00%**	578.69	**0.00%**	385.69
		0.4	1.62%	3476.67	**0.00%**	282.41
		0.5	**0.00%**	494.68	**0.00%**	258.84
	4	0	**0.00%**	1890.98	**0.00%**	825.70
		0.1	**0.00%**	989.93	**0.00%**	583.07
		0.2	**0.00%**	676.10	**0.00%**	320.08
		0.3	**0.00%**	776.36	**0.00%**	273.43
		0.4	**0.00%**	514.22	**0.00%**	243.60
		0.5	**0.00%**	362.05	**0.00%**	189.67
	5	0	0.67%	11705.25	0.13%	7634.34
		0.1	0.38%	8103.68	**0.00%**	3240.06
		0.2	0.04%	3959.85	**0.00%**	1411.87
		0.3	**0.00%**	872.88	**0.00%**	697.06
		0.4	**0.00%**	645.29	**0.00%**	356.83
		0.5	1.06%	3942.78	**0.00%**	210.45
	6	0	4.85%	14400.00	3.92%	14400.00
		0.1	4.09%	14400.00	3.90%	14400.00
		0.2	1.52%	14400.00	0.84%	10692.68
		0.3	0.34%	5923.45	**0.00%**	6063.56
		0.4	**0.00%**	2560.81	**0.00%**	854.68
		0.5	1.38%	6063.96	**0.00%**	366.29

Table 2: Arc pricing results for $|V| = 25$

q	λ	Average gap	Average CPU time (sec.)
3	0	5.11%	14440.00
	0.1	2.26%	14440.00
	0.2	0.35%	8524.02
	0.3	0.07%	4900.43
	0.4	0.04%	3750.18
	0.5	0.43%	3529.07
4	0	11.73%	14440.00
	0.1	8.13%	14440.00
	0.2	5.91%	14440.00
	0.3	1.50%	14440.00
	0.4	0.23%	6381.52
	0.5	0.00%	1433.74
5	0	13.06%	14440.00
	0.1	12.13%	14440.00
	0.2	6.90%	14440.00
	0.3	4.64%	14440.00
	0.4	1.74%	11916.22
	0.5	0.12%	4038.04
6	0	16.32%	14440.00
	0.1	12.59%	14440.00
	0.2	9.07%	14440.00
	0.3	4.83%	14440.00
	0.4	2.96%	14440.00
	0.5	1.32%	9778.79
7	0	16.20%	14440.00
	0.1	15.08%	14440.00
	0.2	10.79%	14440.00
	0.3	7.26%	14440.00
	0.4	3.80%	14440.00
	0.5	2.24%	14440.00

Table 3: Common discount factor results for $|V| \in \{10, 15, 20\}$

			Without (43) and (44)		With (43) and (44)	
$\|V\|$	q	λ	Average gap	Average CPU time (sec.)	Average gap	Average CPU time (sec.)
10	3	0	**0.00%**	16.74	**0.00%**	17.87
		0.1	**0.00%**	15.16	**0.00%**	16.51
		0.2	**0.00%**	15.00	**0.00%**	14.84
		0.3	**0.00%**	12.06	**0.00%**	12.73
		0.4	**0.00%**	12.23	**0.00%**	10.55
		0.5	**0.00%**	8.76	**0.00%**	8.98
15	3	0	**0.00%**	82.52	**0.00%**	77.86
		0.1	**0.00%**	72.10	**0.00%**	71.12
		0.2	**0.00%**	71.69	**0.00%**	84.19
		0.3	**0.00%**	66.18	**0.00%**	64.08
		0.4	**0.00%**	58.73	**0.00%**	64.62
		0.5	**0.00%**	46.55	**0.00%**	55.51
	4	0	**0.00%**	78.04	**0.00%**	72.83
		0.1	**0.00%**	76.83	**0.00%**	83.05
		0.2	**0.00%**	70.30	**0.00%**	72.03
		0.3	**0.00%**	65.55	**0.00%**	62.60
		0.4	**0.00%**	57.39	**0.00%**	60.92
		0.5	**0.00%**	47.31	**0.00%**	54.48
	5	0	**0.00%**	77.46	**0.00%**	80.35
		0.1	**0.00%**	70.57	**0.00%**	72.25
		0.2	**0.00%**	72.24	**0.00%**	74.71
		0.3	**0.00%**	67.58	**0.00%**	72.27
		0.4	**0.00%**	52.61	**0.00%**	56.43
		0.5	**0.00%**	49.30	**0.00%**	56.76
20	3	0	**0.00%**	589.77	**0.00%**	511.87
		0.1	**0.00%**	565.69	**0.00%**	385.74
		0.2	**0.00%**	567.57	**0.00%**	369.97
		0.3	**0.00%**	537.39	**0.00%**	297.65
		0.4	**0.00%**	393.20	**0.00%**	258.63
		0.5	**0.00%**	730.25	**0.00%**	235.33
	4	0	**0.00%**	925.28	**0.00%**	562.83
		0.1	**0.00%**	623.88	**0.00%**	450.22
		0.2	**0.00%**	476.91	**0.00%**	343.32
		0.3	0.50%	3224.45	**0.00%**	297.92
		0.4	**0.00%**	414.56	**0.00%**	262.19
		0.5	**0.00%**	2569.09	**0.00%**	186.53
	5	0	**0.00%**	1321.31	**0.00%**	1139.61
		0.1	**0.00%**	865.31	**0.00%**	538.82
		0.2	**0.00%**	598.33	**0.00%**	431.70
		0.3	**0.00%**	906.81	**0.00%**	324.18
		0.4	**0.00%**	681.39	**0.00%**	285.53
		0.5	**0.00%**	3892.30	**0.00%**	204.25
	6	0	**0.00%**	3342.60	**0.00%**	2274.24
		0.1	**0.00%**	2046.16	**0.00%**	1879.77
		0.2	**0.00%**	1570.62	**0.00%**	1264.79
		0.3	**0.00%**	738.27	**0.00%**	821.97
		0.4	**0.00%**	646.08	**0.00%**	570.65
		0.5	**0.00%**	664.80	**0.00%**	541.33

Table 4: Common discount factor results for $|V| = 25$

q	λ	Average gap	Average CPU time (sec.)
3	0	0.06%	7316.26
	0.1	**0.00%**	3398.63
	0.2	**0.00%**	1527.35
	0.3	**0.00%**	1156.08
	0.4	**0.00%**	983.12
	0.5	**0.00%**	772.50
4	0	1.29%	13172.81
	0.1	0.18%	7697.50
	0.2	**0.00%**	3444.30
	0.3	**0.00%**	2018.72
	0.4	**0.00%**	1702.84
	0.5	**0.00%**	970.00
5	0	16.09%	14400.00
	0.1	9.67%	14400.00
	0.2	3.11%	12329.40
	0.3	0.46%	7604.84
	0.4	**0.00%**	3039.48
	0.5	**0.00%**	2970.20
6	0	15.51%	14400.00
	0.1	12.50%	14400.00
	0.2	10.11%	14400.00
	0.3	8.07%	14400.00
	0.4	1.21%	7657.66
	0.5	**0.00%**	3741.38
7	0	17.02%	14400.00
	0.1	14.84%	14400.00
	0.2	13.95%	14400.00
	0.3	12.27%	14400.00
	0.4	7.22%	14400.00
	0.5	2.70%	7931.19

7.3. Results for the S-TS algorithm

The performance of the S-TS algorithm was instrumental to the successful solution of many instances, which we present below in Table 5. The results we provide are based for the instances with $|V| \in \{10, 15, 20\}$, for which the optimality gaps are small. We analyze the performance of the algorithm based on its result before and after the solution of the pricing algorithm in step 27, which we refer to as "Stage 1" and "Stage 2", respectively. The results are aggregated on instances with the same $|V|, q$, and λ as in the preceding subsections.

For the arc pricing problem, Stage 1 results have an overall average deviation of 7.77%, and we observe the average deviation to increase as $|V|$ and q increase. This relatively low performance can be attributed to the design of Algorithm 1 that computes the objective value based on the common discount factor method. However, the pricing problem step remedies this behaviour, and successfully decreases the average gap after Stage 2 to 3.87%.

For the common discount factor case, Stage 1 performance of S-TS is significantly better, with an overall average deviation of 0.40%. The subsequent pricing problem successfully finds a near-optimal solution for every instance we have attempted to solve. The overall average deviation of Stage 2 is consequently 0.04%. We conclude that S-TS performs well for for the arc pricing case, and remarkably well for the common discount factor case.

7.4. Managerial insights

Our analysis and computational result reveal two major managerial insights. The first one is the performance gap between arc pricing and mill pricing. Indeed, comparison of the results for HLP-P2 and HLP-P2' show that HLP-P2 may result in an objective function value that is 7.5% higher than the HLP-P2' on the average. The minimum and maximum performance differences were observed to be 3% and 15.5%, respectively. Decisions makers should therefore be aware that the simplicity of a common discount factor implies higher operational costs. Assessing the economic implications of the arc strategy is therefore an important task.

We have also performed an analysis of the effect of the pricing model on the location decisions on randomly generated instances in the Euclidean plane. HLP-P2' aims to compensate for the price restriction by relocating a number of hubs. The hub decisions are identical for approximately 20% of the instances, and the difference is a single hub for approximately 30% of the instances. Consequently, the solutions differ by 2 or more hubs for 50% of the solutions, a significant difference given that $q \in \{3, ..., 7\}$.

As an example, the instance depicted in Figure 3 with $|V| = 20, q = 3, \lambda = 0, \sigma = 0.5, \gamma = 0.925$ has two drastically different optimal solutions based on the type of pricing. The values associated with the arcs in Figure 3 are the ratios of the optimal prices to the costs, for ease of comparison. It can be observed

Table 5: Performance of the S-TS algorithm for $|V| \in \{10, 15, 20\}$

$\lvert V\rvert$	q	λ	Arc pricing Stage 1 gap	Arc pricing Stage 2 gap	Common discount factor Stage 1 gap	Common discount factor Stage 2 gap
10	3	0	7.93%	3.67%	0.34%	**0.00%**
		0.1	7.48%	3.16%	0.40%	0.01%
		0.2	7.67%	3.31%	0.51%	**0.00%**
		0.3	8.00%	3.34%	0.45%	0.09%
		0.4	9.06%	4.75%	0.44%	0.08%
		0.5	8.15%	4.96%	0.71%	0.03%
15	3	0	6.38%	2.56%	0.18%	**0.00%**
		0.1	6.33%	3.27%	0.26%	0.05%
		0.2	6.59%	3.06%	0.40%	0.08%
		0.3	6.97%	4.04%	0.46%	**0.00%**
		0.4	7.66%	4.64%	0.41%	**0.00%**
		0.5	7.24%	4.82%	0.65%	**0.00%**
	4	0	5.84%	3.30%	0.28%	0.06%
		0.1	5.76%	2.88%	0.34%	**0.00%**
		0.2	6.06%	3.00%	0.39%	**0.00%**
		0.3	6.46%	2.87%	0.41%	0.02%
		0.4	7.21%	2.87%	0.50%	0.03%
		0.5	7.33%	3.12%	0.74%	**0.00%**
	5	0	8.52%	5.04%	0.30%	**0.00%**
		0.1	8.00%	4.40%	0.34%	**0.00%**
		0.2	7.68%	3.51%	0.34%	0.01%
		0.3	7.67%	3.39%	0.41%	**0.00%**
		0.4	8.31%	3.94%	0.50%	**0.00%**
		0.5	9.66%	4.73%	0.65%	0.07%
20	3	0	4.98%	2.89%	0.25%	**0.00%**
		0.1	4.77%	2.39%	0.28%	**0.00%**
		0.2	4.73%	1.99%	0.32%	**0.00%**
		0.3	5.28%	2.04%	0.39%	0.09%
		0.4	6.12%	2.79%	0.41%	0.04%
		0.5	6.76%	3.48%	0.44%	0.09%
	4	0	7.20%	4.28%	0.27%	0.07%
		0.1	6.75%	3.43%	0.28%	0.05%
		0.2	6.54%	2.71%	0.29%	0.03%
		0.3	6.53%	3.23%	0.33%	**0.00%**
		0.4	6.56%	3.26%	0.42%	0.05%
		0.5	6.31%	2.89%	0.59%	0.12%
	5	0	8.72%	2.81%	0.22%	0.01%
		0.1	9.26%	2.55%	0.24%	**0.00%**
		0.2	10.07%	2.03%	0.29%	0.04%
		0.3	10.66%	5.44%	0.34%	0.03%
		0.4	10.93%	7.12%	0.44%	0.01%
		0.5	10.48%	6.96%	0.92%	0.27%
	6	0	9.41%	4.66%	0.19%	0.10%
		0.1	9.79%	4.17%	0.23%	0.13%
		0.2	9.61%	4.76%	0.42%	0.25%
		0.3	10.28%	6.08%	0.39%	0.06%
		0.4	11.31%	7.47%	0.31%	0.02%
		0.5	12.02%	7.55%	0.55%	0.03%

that the arc pricing model deviates from the optimal α value of 0.445 for every hub arc, and also utilizes asymmetrical prices to maximize the overall profit. The performance gap between these two solutions is 5.54%. We conclude that the hub location and pricing decisions are tightly coupled, and the choice of the pricing model is crucial to the long-term performance of the system, since the location decisions are usually long term.

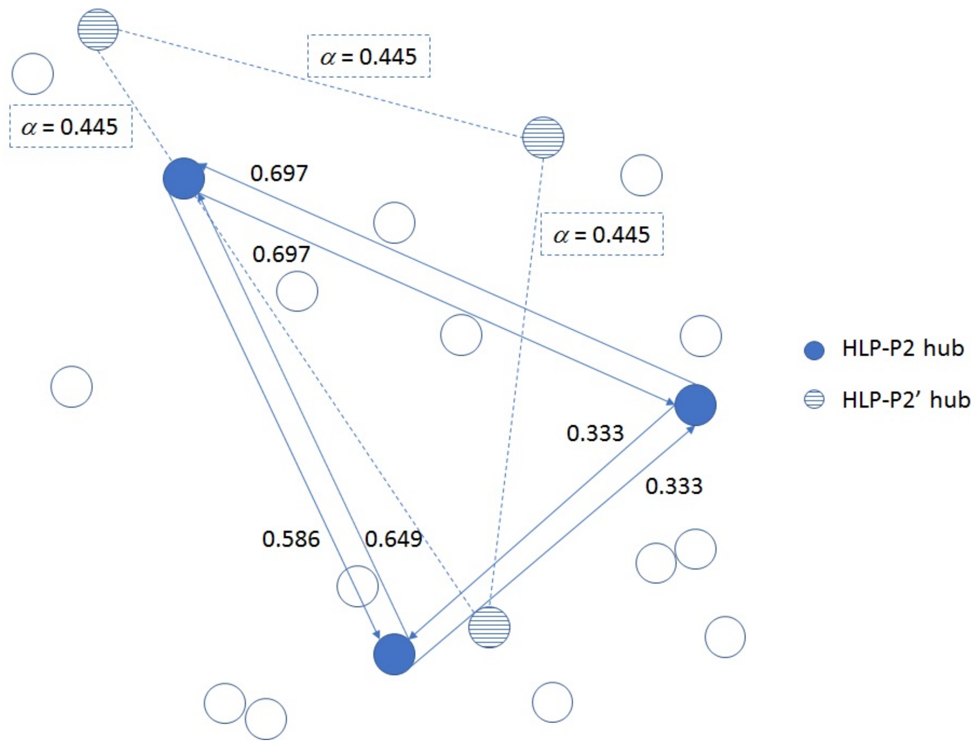

Figure 3: Comparison of the optimal solutions of HLP-P2 and HLP-P2', for an instance with $|V| = 20, q = 3, \sigma = 0.5, \gamma = 0.925$.

8. Conclusion

In this paper, we have introduced the HLP-P, the joint problem of determining hub locations and the price of service between each pair of hubs. We have provided a proof of $\mathcal{NP}$-Hardness and a nonlinear model. We have presented a linearized model and structural properties of its optimal solutions that have led to variable reductions and valid inequalities. We have analyzed both arc pricing and a common inter hub-hub discount factor HLP-P models. We have also developed a metaheuristic, and tested our formulations and metaheuristic on instances adapted from the literature.

The computational reach of the formulations is observed to be $|V| = 25$. The metaheuristic consistently returns optimal or near-optimal solutions, with an average deviation of 0.40 from the optimal solution or the best known bound for the common discount factor model. The average deviation for the arc pricing model is 3.87%. We hope to improve upon these results in subsequent work, possibly through solving the pricing problem (by fixing the hub decisions and the follower's routes and) at regular intervals within the metaheuristic.

Our problem definition and the resulting models depend on the assumption of completely rational behaviour for the followers, i.e. always using the cheapest path. However, it is well-known that the customers may not opt for shortest path every time they use the system, due to temporal disruptions of the transportation network, or other external conditions. We hope to remedy this limitation by using a logit model to describe the follower behaviour, as in the paper by Lüer-Villagra and Marianov (2013).

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