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Alan Weir

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Putnam, Gödel and Mathematical Realism Revisited

Alan Weir

Philosophy, University of Glasgow, Glasgow, UK

ABSTRACT

I revisit my 1993 paper on Putnam and mathematical realism focusing on the indispensability argument and how it has fared over the years. This argument starts from the claim that mathematics is an indispensable part of science and draws the conclusion, from holistic considerations about confirmation, that the ontology of science includes abstract objects as well as the physical entities science deals with.

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I am delighted to have been asked to contribute to the 30th anniversary edition of the *International Journal of Philosophical Studies* as one of the contributors to the first year of the revamped journal. My own contribution ‘Putnam, Gödel and Mathematical Realism’ was based on a talk I gave to a Royal Irish Academy conference on Hilary Putnam. I have only patchy memories of the conference and my talk – including that I was quite nervous giving the talk in front of Putnam and that he was very gracious. I don’t remember his comments on the talk, only him asking me afterwards to recite from Robert Burns’ ‘To a Louse’ in particular ‘wad some powr the giftie gie us to see oorsels as ithers see us’. In fact, the only thing I remember about the discussion period is the contribution from compatriot (at the time unknown to me) John Divers, complimenting me on my diction! Only myself and the other Glaswegian in the audience, Alex Miller, actually understood what John had said!

The paper does tackle the topics one would expect from the title: Putnam’s philosophy of mathematics and certain brands of mathematical realism, usually known as ‘platonism’ associated with Putnam and Gödel, respectively. But I used this as a springboard to present ideas that I was to develop further in later work. Here, I will focus largely on what is often known as the

CONTACT Alan Weir  alan.weir@glasgow.ac.uk

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Quine/Putnam indispensability argument in favour of mathematical platonism and on its fate over the last three decades. Moreover, I will mostly focus on the Quine part, not Putnam, even though in the *IJPS* article I focussed, for obvious reasons, on Putnam's take on it. But Quine, going back to my doctoral thesis on his philosophy language, and forward through the succeeding decades, has been much more of a focus of my attention than Putnam.

Indispensability

A rough version of the *indispensability* argument for mathematical platonism goes as follows. The ontology of a theory, an empirical scientific theory for example, is given by its existential consequences. However, successful scientific theories contain applied mathematics – substantial portions of pure and impure mathematics, including bridge principles linking the pure and the empirical. Thus, scientific theories entail not only the existence of electrons, black holes etc. but also whatever follows from their mathematical sub-theory. For many theories, and not just those in the physical sciences, this will entail an ontology including infinitely many natural numbers, uncountably many real and complex numbers, and so on. Since these objects are clearly not concrete entities, we have as much reason to believe in abstracta as in electrons.

Responses to the argument generally take two forms. One is a denial that scientific theories need to use mathematics for reasons other than convenience; this usually involves an attempt to fillet out the maths and generate a 'nominalised' but empirically equivalent version of the theory. This response is associated with Hartry Field (Field 1980). The second response, whilst conceding that substantive mathematical theories are an indispensable part of adequate empirical science, denies that scientists are 'ontologically committed' to abstracta notwithstanding the existential consequences such as that there are uncountably many reals etc.¹

A form of the argument is found in Putnam's *Philosophy of Logic* (Putnam 1971) and in more scattered fashion across Quine's corpus.² In this paper, the focus will be much more on Quine but I'll start with Putnam citing Quine:

The individual postulates of a theory generally have no (or very few) experiential consequences when we consider them in isolation from other statements of the theory As Quine puts it, sentences meet the test of experience 'as a corporate body and not one by one'. (Putnam 1988, 8–9)

Now add to this the claim that theories are confirmed or disconfirmed by their empirical consequences in an egalitarian fashion. For example, if T is confirmed then every $S \in T$ has the same degree of confirmation. The

‘ontological commitment’ of a theory is to the set of its existential consequences, committed to the same degree and the same way to each of those consequences. This is the doctrine of *confirmational holism*.

Any empirically adequate theory, any theory more confirmed than disconfirmed, will entail a host of mathematical existence claims. Hence, on this version of the indispensability argument, current science includes in its ontology not only electrons, gluons and black holes but also, and with equal force, infinitely many abstracta.

W.V. Quine and V. Quine

There is, however, some dispute about how exactly to interpret Quine on indispensability. And in the last three decades I have added my own two bawbees to contentious issues in the interpretation of Quine more generally, some of this bearing on indispensability. To simplify the discussion I’ll proceed by expounding the views of a fictional philosopher, I’ll call him Viennese Quine, or V. Quine for short, whose philosophical position is not one explicitly adopted at any stage of Quine’s career. I claim, however, that the position ascribed to this character makes the most coherent sense of doctrines and theses Quine has defended through most of his career though the real Quine’s position has some rough edges and is not entirely stable over the period. I’ll then ask how this position relates to indispensability.

V. Quine holds that sentences of an idealised regimentation of the languages used by scientists divide into two categories. Firstly observation sentences (observation categoricals in his later formulations) whose content is exhausted completely by their *empirical content*. And theoretical sentences whose meaning consists entirely in their logical connections with one another and observation sentences. Observation sentences are ‘directly keyed’ to sensory stimulation so that observation sentences do no more than record sensory input available to an observer on a given occasion. Apart from some atypical cases, a conjunction of the axioms of a finitely axiomatisable empirical theory for example, theoretical sentences have no empirical content and therefore no meaning at all to call their own.

Now Viennese Quine is not a character who figures prominently in much recent discussion of the actual Quine who, many will claim, differs from V. Quine in a number of important philosophical respects. On observationality, for example, Quine deploys in his classic *Word and Object* (Quine, 1960) two completely distinct notions, on pp. 43 and 44, respectively (without seeming to notice this). On one conception, observationality is a matter of degree, sometimes this is the degree of intersubjective consensus on the sentence; on the other, observation sentences are ‘directly keyed’ to sensory

stimulation, which determines (penumbral indeterminacies aside) all their semantic content. Later, Quine combined both accounts (1990, p. 43). Moreover, Quine has a pronounced antipathy to dichotomies (e.g. that of analytic versus synthetic), so it is implausible, it is said, to interpret him as supporting an observation/theory dichotomy even one with vague boundaries. Theoreticity is a matter of degree, likewise observability.

My response³ is to say that while Quine does claim in places to believe in a spectrum of theoreticity, and so also observability, the concept of a special class of sentences directly connected with specific experiences or 'stimulations' is essential to his empiricism. Without such moorings, his position would threaten to collapse into a very post-modern 'it's language all the way up, down and across' position, with language unmoored from reality.

My reading of Quine's holism as leading to a form of nihilism about meaning for theoretical sentences and as a result deep scepticism about (theoretical) propositions, beliefs and related entities is also at odds with some major interpreters of Quine such as Peter Hylton (2007) and Gary Kemp (2006, 2010). I take V. Quine's argument to be that theoretical meaning cannot be explicated or 'reduced to' empirical meaning (initially characterised by him via his highly behaviouristic stimulus theory); therefore, there is no such thing. For Hylton and Kemp to identify Quine with Viennese Quine is to saddle the former with a major inconsistency given that his most famous article 'Two Dogmas' (Quine 1951) is an attack on reductionist versions of positivism on the basis of his holism⁴; charity in interpretation should rule out such a reading.

Well call me uncharitable but I find too much in the Quinean corpus which only makes sense if one views him as never giving up on a hard line verificationist reductionism on meaning, of never fully giving up the hostility to semantics found in the Vienna school before Tarski's rehabilitation of the notion of truth.

Indeed not only do I indict him as a reductionist, I challenge Quine's self-identification as a scientific realist. True there's always a grave danger in discussing such 'isms' of falling into mere terminological disputes but one can point, against Quine's claim to be a realist, to the 'argument from above' to indeterminacy of interpretation from underdetermination of theory by evidence: there can be pairwise inconsistent theories with exactly the same empirical consequences.⁵

Underdetermination, indeterminacy and Relativism

However, the key premiss of underdetermination is one which Quine came increasingly to doubt, or at least to understand in weaker and weaker forms.

His verificationism led him fairly inexorably to the view that two empirically equivalent theory formulations are actually formulations of one and the same theory, with at worst mere terminological differences and no genuine conflict (Weir 2006a, §3.).

But there is a different argument for indeterminacy which Quine sometimes employs and which is more relevant to the discussion of indispensability and confirmational holism, the argument from the semantic thesis of *holistic verificationism*.⁶ It goes as follows.⁷ Sentences in the observational sector of language have non-null, determinate empirical *meanings* (for a period identified by Quine with the stimulus meanings of his dispositionalist theory of *Word and Object*). Atypical cases aside, typical theoretical sentences have no empirical consequences on their own as per the Duhem/Quine thesis:

The physicist can never subject an isolated hypothesis to experimental test, but only a whole group of hypotheses; when the experiment is in disagreement with his predictions, what he learns is that at least one of the hypotheses constituting this group is unacceptable and ought to be modified; but the experiment does not designate which one should be changed. (Duhem 1906, 187)

(see also the earlier quotation from Putnam citing Quine). Hence, the typical theoretical sentence has no empirical meaning. As with the underdetermination argument, this leads to contradictory theoretical sentences having the same meaning and so, it is generally assumed, the same truth value even though:

countless native sentences admitting no independent check ... may be expected to receive radically unlike and incompatible renderings under the two systems. (Quine 1960, 72)

To preserve consistency (in the metatheory), we have to relativise truth. Relative to one 'conceptual scheme' the world starts with a Big Bang and ends a finite time later with a Big Crunch, relative to another it is infinite in time with infinitely many Bang:Crunch cycles, there is no absolute fact of the matter as to which is right; there are many other such examples.

I've considered a number of ways this argument can be filled out. Most simply (Weir 2005 §IV) via reinterpreting a language by perming one sentence with its negation. For example, let N be the conjunction of Newton's three laws of motion plus his inverse square gravity law. N entails no empirical hypotheses independently of further auxiliary hypotheses and boundary conditions; hence, it has no stimulus meaning and no empirical content. For the same reason, neither does its negation $\neg N$: without further hypotheses, there is no way we can derive predictions as to which objects are behaving in ways which falsify the laws. Both sentences then

trivially have the same (null) empirical meaning, and hence, it is standardly assumed, the same truth value, though $\vdash N \leftrightarrow \neg N$.

A more complex reading (Weir 2006a, §3; Weir 2013a, §6) of the Quinean argument allows that sentences with identical (null perhaps) empirical meaning can nonetheless differ in meaning, the difference lying in their different logical structure. Quine's verdict matrix theory gives him the resources to declare that two sentences (of a formalised regimentation of natural language) with isomorphic phrase structure trees are directly synonymous iff they match, that is if the respective terminal nodes are empirically equivalent observation sentences and the corresponding particles at the other nodes satisfy the same verdict matrices; they are directly asynonymous iff they do not match. One then appeals to the existence of a complex sentence P empirically equivalent with $\neg N$ but with a different phrase structure tree. We then have $\vdash N \leftrightarrow \neg P$, hence by soundness N and P have different truth values even though they, given the holistic verificationism, have the same meaning in the more fine-grained sense: empirically equivalent but not directly asynonymous. On the usual assumptions about the relationship between cognitive meaning and truth, this means they have the same truth value. Contradiction.

Both those arguments have the defect that by re-interpreting only one sentence φ of the object language we get the result that if φ is mapped under the interpretation non-homophonically to an incompatible sentence ψ , then the occurrence of φ in $[\varphi \wedge \theta]$ means something different (namely φ) from its self-standing occurrence, where it means ψ . However, the interpretation can be made more general by taking it to be a map which permutes atomic open sentences with other open sentences also having null empirical content and distributes over the connectives. We get the same contradiction so long as some of the permuted contentless atomic sentences differ in truth value, e.g. the Newtonian gravitational law concerning the magnitude of a force between massive objects x and y expressed by

$$F_{xy} = g \frac{m_x m_y}{r^2}$$

versus its negated inequality.

$$F_{xy} \neq g \frac{m_x m_y}{r^2}.$$

Suppose, then, that Willard van Orman really is, or ought to be, Viennese Quine and should embrace, for the above reasons relativism about scientific truth, abandoning thereby scientific realism: what's the big deal for this concerning indispensability? The problem is that if Quine's indispensability argument assumes not just confirmational holism but holistic verificationism then Quine seems to end up not only as an anti-platonist but bereft even

of a physicalistic nominalistic ontology expunged of abstracta. He ends up with almost no ontology at all, an ontology solely of macroscopic ‘occasions’, the physicalist successor to the sense datum ontology favoured by Carnap against Neurath, but with no theoretical entities.

It might be countered that even if one embraces an anti-realism about truth, this does not entail anti-realism in ontology. Suppose, for example, one takes the opposite view to Quine, rejecting ontological relativity, relativity about reference (Quine 1969c) but openly adopting relativism about scientific truth. One might claim to believe in the existence of electrons, gluons, transfinite ordinals, countable ones anyway etc., but only in a ‘softer’ or ‘thinner’ sense than the more full-bodied existence the realist believes in.

Whatever the merits of the idea of ‘thin’ existence (very little in my view) this view is not, I think, common among current proponents of the indispensability argument. For if the anti-realist, instrumentalistic view is right they are committed to saying that both abstracta and concreta have the same demi-mondain existence: electrons and transfinite ordinals exist, according to one conceptual scheme, and do not exist according to another, with no absolute mind-independent fact of the matter as to which is right.⁸

An aside on Ontological Commitment

It will be seen I have been scare-quoting throughout ‘ontological commitment’ though the phrase is a classic piece of Quinean terminology. It is very unfortunate terminology however. For one thing, commitments are things which *people* undertake; it is not obvious why the concept should be extended to apply to *theories*. More importantly, the phraseology led him into obscure and unhelpful formulations:

entities of a given sort are assumed by a theory if and only if some of them must be counted among the values of the variables in order that the statements affirmed in the theory be true. (Quine 1953b, 103) [emphasis in original]

Counted among the values by whom, and for what purpose? What is this modal ‘must’ doing in a crucial Quinean criterion? What to make of ‘in order that the statements affirmed in the theory be true’ appended to ‘values of the variables’?

Famously Quine, always keen, I suspect, to find a striking phrase, often enlivened his exposition of the commitment doctrine with the slogan ‘to be is to be the value of a variable’.⁹ Many commentators also associate the doctrine with the expanded phrase ‘to be is to be the value of a *bound* variable’ [my emphasis] but as far as I can see this never occurs in Quine’s corpus. However, the emphasis on bound versus free variables which does occur frequently

So I have insisted down the years that to be is to be the value of a variable. More precisely, what one takes there to be are what one admits as values of one's bound variables. (Quine 1991, 26)

is important, because what Quine is really getting at here is the idea, familiar from model-theoretic semantics, of quantification over a domain.¹⁰ The idea then is that T is committed to α , or to φ s, if every domain in which the theory is true contains α (or the φ s). But this is a hopeless view because, if true, no consistent theory will have any ontological commitments, as is easy to show, at least if standard model theory is our guide.

Better, I suggest, to drop the notion of ontological commitment in favour of the idea of the *ontological content* of a theory explicated as follows. To say that φ s are part of the ontological content of T is to say more exactly that the proposition that φ s exist, equivalently, the proposition that there are φ s,¹¹ is a consequence of T. The ontological content of a theory is a set of existential propositions.¹²

So number theory entails that numbers exist, hence, we might more loosely say, *numbers* are part of its ontological content. Similarly any common sense account of the world entails that there are pianos so *pianos* are part of everyday ontology. But the looser way of talking has its dangers: if a paranormal theory entails that ghosts exist then on the loose way of talking *ghosts* are part of its ontology, but nothing can be literally part of something unless it exists. So in these cases best stick to the stricter formulation: the proposition that ghosts exist exists (pace Quine) and that is what is part of the ontological content of the paranormal theory.

Now Quine, it will be pointed out, often claims that gauging ontological content only makes sense relative to a paraphrase or translation (and there will be many pairwise conflicting ones on his account) into first-order predicate calculus (cf. Quine 1981b, 9). Quine actually swithers (vacillates) on this. Sometimes he says $\exists x$ is just a terminological variant of 'there exists something such that it . . .' (Quine 1969b, 97, 1991, 26–7) in which case there is no need to translate into formal logic; at other times, as noted (Quine 1981b, 9) he requires the formal symbol to be interpreted, semantically or in terms of its rules, as the singular existential quantifier of classical unbound logic. In which case, it is definitely not a terminological variant of a natural language locution but a conceptual innovation. But it begs too many meta-physical questions to insist on reading existence or being in those terms.

So, taking all the above into consideration, I suggest that the most fruitful way to think of 'ontological commitment' is in terms of the ontological content of theories and this should be glossed as the existential propositions entailed by the theory, propositions of the form 'there are φ s' or 'there exist φ s' with the correct formal paraphrase of these forms, if there is any in any current well-known system, a secondary matter.

Indispensability₁

Returning, then, to indispensability with this understanding of ontological content, how should the argument be developed by Viennese Quine? Field's fictionalism is usually taken to involve rejection of a specific form of indispensability, deductive indispensability.¹³ Call that *indispensability*₁: it states that substantial mathematics is ineluctably necessary in order to deduce (in the syntax version) the empirical consequences of current science (or in the semantic version for these consequences to follow logically). Field denies this, arguing that the mathematics is only needed for pragmatic reasons such as shortening the lengths of proofs.

If Field is wrong and confirmational holism is right, then modern science *seems* to have a pretty extensive array of mathematical entities as part of its ontological content. The qualification arises from the doubts raised by the Quinean holistic verificationism version of confirmational holism. Further exegetical issues with Quine arise, however. Was he really a confirmational holist, as characterised above? Apparently not very egalitarian in his politics, did he actually adopt an egalitarian attitude across all theoretical sentences in a confirmed theory? Or did he adopt a more differentiated attitude?

For Quine argues against taking seriously, in ontology, the very large cardinalities of 'higher set theory' and expresses deep reservations about affirming their existence¹⁴; (Quine 1991, pp. 94-5). Maddy criticises Quine for abandoning naturalism by failing to respect the autonomy of mathematics and its methods and instead subordinating it directly to the requirements of empirical science. She denies that this argument could be pressed in naturalistic service of the autonomy of astrology or theology (Maddy 1997, 184).

Slimming down his ontology to avoid exotic and unfathomably large sets might seem entirely reasonable to Viennese Quine. As an empiricist, he will want to eliminate 'theoretical fat' from his theories and espouse only the leanest, meanest theory which is empirically adequate. The strength of the mathematics needed for such theory is well short of that of higher set theory; it is generally considered, cf. Feferman (1993).

But the empiricist faces a grave danger here in the form of the slothfulness of the Absolute. If empirical adequacy is the sole criterion for the acceptability of a scientific theory, then if T is acceptable so is [T \wedge The Absolute is Lazy]. The obvious way to get rid of the Lazy Absolute is to add as a criterion for the goodness or acceptability of a theory a 'leanness' component – among two empirically equivalent theories prefer T₁ to T₂, *ceteris paribus*, if the ontological content of T₁ is a proper subset of that of T₂. But a theorem of Craig's, often adverted to by Quine (1975, 324-326) threatens to push such an empiricist to radical instrumentalism. Supposing the vocabulary of our language can be divided into two recursive exclusive and exhaustive sets, the

theoretic and the observable respectively, then if T_1 is a recursively enumerable theory then O_1 , the set of all its consequences with purely observational non-logical vocabulary, is recursively axiomatisable. So, it seems, we can replace theories of this type by programs which generate just the observation sentences.¹⁵ Empiricism, radical empiricism anyway, threatens to cast into the flames as sophistry and illusion not just empty obscurantist metaphysics, satirised by the figure of the indolent Absolute, but also all of theoretical science, including pure and applied mathematics, and with it their ontologies!

But is this radical empiricist position coherent? After all, Craig's result is itself an example of pure mathematics, as is the proof theory and model theory used in the investigation of formal theories. A move 'upstairs' to the metatheory can help shed light. As well as the ontological content of an object language theory T , we can inquire after the ontological content of an *explanatory* semantic metatheory MT for T . What is meant by 'explanatory semantic metatheory'? Well that in itself is a matter for philosophical debate; what I have in mind is not a formal model theory for the object language but theories which aim, at the most ambitious level, to explain what it is to understand it. Less ambitiously, to assign content to the sentences of the object language in a way which would usefully contribute to the most ambitious sort of theory.

Ontological Reduction

In Weir 1993, I tried out this idea with respect to projectivism in aesthetics and probability theory, distinguishing informational content, roughly Fregean sense, from what I then called 'explanatory' content, by Weir (2010) changing this to 'metaphysical content'. Starting with a rather Kaplanian example (Kaplan 1989), demonstratives such as 'that table', I said that there may be nothing more illuminating to say about the informational content of 'that table' than that it means 'that table'; but one might explain how a competent speaker grasps its use in a sentence such as 'that table was in the store yesterday' by giving the noun phrase content such as 'the most salient table'.¹⁶ And here informational content and explanatory content come apart, 'the most salient table was in the store yesterday' means something different.

The overall approach, a Sense/Circumstance/World (SCW) theory is (Weir 2010, Chapters 1 and 2), I think, not too contentious, though of course anathema to nihilists about systematic theories of meaning. The rather neo-Fregean idea is that utterances of declarative sentences in a given context (part of the 'circumstances') have context-independent senses – informational content – which, together with the given circumstance, yield propositions which are rendered true or false by 'The World'. Metaphysical content

specifies what in the world makes the proposition true or false in the circumstances. Sometimes, as in the demonstrative case, this amounts largely to filling out the content of the sense in such a way that it can confront reality truly or falsely. But in other cases the metaphysical content shows us that the utterances are not made true or false by a reality which corresponds, in the realist's sense, to that content.

In Weir 1993, I applied these ideas to a version of projectivism. An utterance of 'that passage is sublime' has as its informational content simply 'that passage is sublime' but we may assign as its explanatory/metaphysical content some fact about the conative attitudes of speakers, most crudely some sort of attitude of approval. This need not lead to radical subjectivism: speakers may be disposed to amend and revise their attitudes in the light of interaction with their peer group, so at least an intersubjective notion of correctness applies. The upshot is that aesthetic claims can be evaluated as true or false, at least by a disquotationalist who does not care to distinguish correctness from truth, thus side-stepping the Frege-Geach problem (Geach 1965). But the position is nonetheless quite clearly anti-realist as we see by ascending to the metalanguage. In the explanatory account of the content of aesthetic claims, there are no existential commitments to *sui generis* properties of beauty, sublimity and so on, just to conative attitudes. As Blackburn puts it, projectivism:

asks no more than this: a natural world and patterns of reaction to it.
(Blackburn 1984, 182)

Ontological reduction occurs if the ontological content of the explanatory metatheory is a proper subset (some details of syntax, the existence of sentence tokens and so on aside) of that of the object theory. A good account of how we grasp Shakespearean texts concerning ghosts, or a literary critic's claim that Fyodor Karamazov is a bigger villain than Stepan Oblonsky will not entail the existence of ghosts nor the existence of people corresponding to Dostoyevsky's Karamazov and Tolstoy's Oblonsky. It will only entail the existence of such things as tokens, verbal or written, of the texts of those authors and the evaluations, by knowledgeable readers of both texts, of counterfactuals concerning the ethical nature of any who would meet those descriptions.

Indispensability₂

What implications do the reflections on ontological content and ontological reduction have on the tenability of indispensability₁? Moderate empiricists of an anti-platonist bent have, in fact, moved the terms of the debate from indispensability₁, unrestricted indispensability, to a second type, let's call it indispensability₂, in which *explanation*, often causal explanation, is more to

the fore. If a theory posits entities of type *K* and *accords them a causal role*, however indirect, in the production of the phenomena recorded in the observational consequences of the theory, then *K*'s are part of its ontology. Those advocating this version of the indispensability argument are following Putnam more than Quine. For Putnam, despite many changes in his philosophical views over his career, has long emphasised explanation as a key goal of science and, contra Quine, rejected positivistic 'naturalistic' accounts of explanation – rejected indeed the whole fact/value distinction, e.g. Putnam (1987, p. 90), Putnam (2004). Post Putnam, philosophers such as Maddy (1997, 2005), Melia (2000, 2002), Leng (2002, 2010, 2012) and others have advocated or at least sympathetically presented a restricted, differentiated holism: some parts of theories, though playing a deductive role in generating empirical hypotheses, play no explanatory role. The ontological content of such sub-parts is no part of the overall scientific theory. If atoms, or infinitely deep water, or fluids as continuous substances, or most pointedly, natural numbers, play no explanatory role in empirically adequate science then the scientist is not committed to atoms, continuous fluids, natural numbers and so on.

Here the argument often takes a historical, quasi-anthropological turn, so allowing an empirical a posteriori element into philosophy, an entirely congenial move from Quine's perspective of course. Thus, Maddy investigates the debate concerning the existence of atoms (Maddy 1997, 143–151, Maddy 2005, 455–56). By 1860 atomic theory was able to compute stable atomic values; by the turn of the century, the addition of the kinetic theory improved the theory as evaluated by at least some of the scientific community. But it was not, she claims, until Perrin verified Einstein's 1905 Brownian motion calculations that a consensus in favour of the existence of atoms emerged. Thus she defends a more restricted, differential confirmational holism. The existential consequences which flow from some subtheories on their own need not be counted as part of the ontology of the theory. No need to adopt what Field called 'Heavy-Duty Platonism' (Field 1989, 185), no need to admit Avogadro's number, or 1.602×10^{-19} , the value of the electrostatic force in Coulombs, into one's ontology, for they play no causal role in the phenomena explained by the theories in which they, or rather terms referring to them, occur. Maddy indeed suggests that all of the mathematics might be treated as idealisation, the real numbers which model continuous fluids as well as the continuous fluids themselves, thus rejecting any form of platonism (2005, p. 457).

Clearly, a lot of weight is placed on the contested concept (or perhaps concepts) of explanation in these accounts of indispensability (as also in my account of ontological reduction which appeals to the notion of an explanatory theory of meaning and understanding). A lot of philosophical work then needs to be done on this weight-bearing part of the philosophical theory.

Complex considerations also arise when we look more deeply at a notion dealt with very summarily thus far, that of a theory being ‘held’ by some given person or community of people. What is it for a scientist to hold to the atomic theory whether or not she accepts an ontology of atoms. Is it to believe (or perhaps know) the theory? Or is it to accept the theory? What trust can we put in our translations from 19th scientific English or German into present day philosophical English?

Moreover, belief and acceptance are two different attitudes according to many philosophers, for example Van Fraassen (1980) and, deploying a different distinction resting on voluntariness not empirical content, L. J. Cohen (1989, 1992). Or are other attitudes important here? Whatever the attitude, is there a distinction in respect of it which differentiates those parts of the theory which bear the ontological burden and those which don’t? It doesn’t seem plausible that a chemist might have a lower degree of belief in the theorems of arithmetic than that nitric acid results from oxidation of ammonia. So if we suppose there is such a thing as degree of belief¹⁷ and that it interacts with preferences or some such conative state to generate actions (including speech acts) in ways idealised by decision theory, then it is not at all obvious how to extract differential “ontological” degrees of attitude from a person’s actions. For if one does not believe or accept the sub-theory how can one rationally derive the empirical consequences which are essential to the theories empirical success and applicability.

Maddy alludes to ‘direct detection’ of atoms¹⁸ as a factor in converting scientists such as Ostwald to belief in atoms:

These observations suggest, first, that holism is incorrect, that our best scientific theory is not simultaneously confirmed in all its parts, that at least in cases like atomic theory, some variety of ‘direct detection’ is required. (Maddy 2005, 455)

Certainly, in some non-mathematical cases, the difference in attitude is quite explicit. Ostwald affirms, sincerely, his belief in atoms where before he had similarly denied they existed. Unless a victim of self-deception the change in ontological content is unproblematic. However, there is no direct detection of numbers;¹⁹ therein lies the classic problem with mathematical platonism. But restricting ontology to the directly detectable looks like a very extreme form of empiricism.

Related challenges also face my own account of ontological reduction. When the metatheorist seeks to give an explanatory theory for some object theory, say chemical theory in the late nineteenth century, she will, of course, do so from the standpoint of her own theory of the world or the relevant parts of it anyway. Thus, if she doesn’t believe in ghosts she must give a reductive metatheory for paranormal pseudo-science, if she doesn’t believe in atoms, a reductive account of atomic theory, if she doesn’t believe in ether, a reductive theory of ether-theories of electromagnetism. But if she does

believe in atoms, there is no incoherence in still pressing a reductive account of atomic theory. This might seem required in order to explain the thoughts of a sceptic about atoms, like the younger Ostwald. But here we encroach on the debates about externalism in philosophy of mind and language. Some will hold that Ostwald could not have had the thought that there are no atoms without 'atom' picking out or referring (plurally some will say) to atoms, even though he himself believed there were no such things.

We also approach modal and counterfactual territory which is uncomfortable for the Quinean. Does it make sense to conceive of a possible situation in which we, like all material beings, are Aristotelian substances, combinations of matter and form and so not constituted by whatever we actually are composed of (not that we've a very clear answer to that from the physicists!). If that scenario makes sense would Democritus be able to think the same thoughts as he actually does, in defending atomism? Quine would presumably reject such speculations out of hand, since modal talk should for him be confined to the market place, not serious theoretical matters. (Though it is not clear that defender of a dispositionalist theory of meaning is entitled to that view of modality.)

At any rate, there is plenty scope for further work to be done on reductionist accounts of theory using something like the SCW framework, whether in support of indispensability₂ or not.

Indispensability₃

In Weir 1991, 1993, I argue for a third type of indispensability of mathematics, which I'll unimagatively dub indispensability₃ and which I claim holds even if the first two, deductive and explanatory indispensability, do not. This is *conceptual* indispensability. Even if for every serious scientific theory there is a 'nominalised' empirical theory which can derive, and perhaps even explain, the same set of observational consequences as T, we still have to explain how we grasp the *meaning* of theoretical terms (unless we are very radical Quinean sceptics about meaning) explain what our understanding of those terms consist in. I argued that, unless we are to embrace the radical instrumentalism I have pinned on V. Quine, for terms of any significant degree of theoreticity, an *independent* grasp of the truths of a substantial body of mathematics is necessary:

if we quantify rough and ready empirical concepts of distance or whatever then, granted a knowledge of mathematics *independent of our general empirical theory*, we thereby extend our conceptual reach to magnitudes picked out via numbers which are too small or large to stand for observable entities but which are still within our mathematical comprehension. Though the physical magnitudes are non-mathematical, our only route to them must first use a mathematical ladder elevating or lowering us to a plane above or below ordinary observation. . . .

For mathematics to perform this role it is essential that its theorems have a determinate meaning which is independent of any empirical theory and this cannot be so if Putnam's indispensability picture is right. But nor can it be so on Field's view of mathematics as a body of falsehoods concerning a nonexistent realm, for where, then, do mathematical sentences acquire their content?. (Weir 1993, 260)

I did not take conceptual indispensability to be a conclusive reason for adopting platonism: it is utterly mysterious how we would grasp mathematical concepts on a platonistic view. Rather I applied a version of the informational content/metaphysical content to the mathematical case. The informational content of 'Every even number is the sum of two primes' is that every even number is the sum of two primes.²⁰ But its metaphysical content is that a concrete token which constitutes, or could constitute in an everyday sense of the term, a proof of that sentence exists. Hence, as with the treatment of aesthetics (subjective) probability or fictional discourse, one can with a good conscience embrace the theorems of mathematics without embracing a platonistic ontology; it is reduced away to an ontology of physical tokens.

Does this lead to strict finitism, since I restrict the entities which do form part of the (reductive) ontological content of maths to small, graspable concrete tokens? I answered no but did not expand more fully on this until the monograph *Truth through Proof* (Weir 2010, also Weir 1998b). In the Putnam article, I raised a worry posed by Gödel's first incompleteness theorem – any formal system whose theorems are recursively enumerable and which contains a fairly weak fragment of Peano Arithmetic, Robinson's arithmetic Q will do, is negation-incomplete, there is a sentence such that neither it nor its negation is provable. The problem there is that for *some* of the sentences known as 'Gödel sentences'²¹ it seems we can 'see', albeit not by dint of a proof from the axioms we started with, that those ones are true. A more difficult problem, I came to see, arises from a different Gödelian angle that posed by what I called 'concrete undecidables': short graspable sentences which various considerations, including Gödel's speed-up results,²² mean there will be no short graspable proof nor disproof. A plausible example from Neil Tennant is:

$(2^{2^{2^{2^2}}} + 1)$ is prime. (Tennant 1997, 152)

The response in the Putnam article, echoing Putnam (1975a), 77, was,

Gödel's result appeals to a very special notion of proof: roughly that it must be possible to programme a computer so that it churns out all and only theorems of the proof system, leaving aside what Russell called mere mechanical, or in this case hardware, limitations. This is generally thought to be a plausible constraint on proof, given the epistemic role it plays. I want to argue to the contrary. 1993, p. 265

and I expanded on this later (1998 and especially 2010, 2015) in the context of an outright formalist position. Far from embracing strict finitism, the formalism I eventually adopted rejects strict finitism as incoherent, in particular as incapable of accommodating the all-pervasive use of abbreviation in mathematics which means that our actual notations are never ‘downward closed’. For example we might introduce into system S tetration enabling us to bring ‘into reach’ numbers²³ such as $10 \uparrow 10000$ ²⁴ we could not name in S before; but the vast majority of smaller numbers will not be nameable in S and this point holds for any graspable system. This in turn renders inapplicable metamathematical techniques such as definition by recursion and proof by induction, without which metamathematics collapses.

So idealisation is necessary but, inveighing against ‘supernaturalised epistemology’ – appeal to superhuman figures, angels, genies and the like – I argue that idealisations such as countably infinite languages, uncountably infinite languages with infinitary logic, decidability results, completeness results, these are all licit so long as there is a *concrete* proof (or at least proof sketch) of the existence claims or important results, such as the negation-completeness of Peano arithmetic augmented with the ω rule. The idea that there is some epistemic, methodological or ontologically important distinction between finite structures however large – in standard first-order logic, most proofs have more than $10 \uparrow 10000$ steps – and infinitary structures is I conjecture, a hangover from finitism and constructivism.

Formalism and Proof

Later sections of the 1993 paper explore the non-Gödelian notion of proof, using an unorthodox proof system, a bit like Jeffrey’s ‘coupled tree’ system (1967, pp. 93ff.) but more complex, and including an infinitary cut rule (1993, p. 278). I remember working on this system, but my notes on it are no longer retrievable!

I identified truth in mathematics with proof but proof from what? I answered: from analytically true axioms, citing Hume’s Principle (HP), that the number of ϕ s = the number of ψ s just in case there is a one:one correspondence between the ϕ s and the ψ s. I still reject Quine’s view that the only scientific notion of sentence meaning is a very thin empirical one attaching to only a few sentences but have now come to believe almost the opposite view – that meaning is a very rich, messy notion, indeed this applies to the meaning of ‘meaning’ itself. I no longer appeal to analytic principles as fundamental to mathematical truth. As a formalist, I hold that any consistent system²⁵ generates truths though some are more interesting than others. A growing scepticism about the neo-logicist programme of founding mathematics on analytic or meaning-constitutive proofs such as HP was partly

responsible for this change.²⁶ Another was increasing scepticism about the conventional responses to the semantic and set-theoretic paradoxes. This surfaced in 1993 in approval of a similar scepticism in Putnam.

The paradoxes themselves, however, are hardly less paradoxical than the solutions to which the logical community has been driven. (Putnam 1990, 15)

Graham Priest's argument that non-naïve solutions to the paradoxes only run into 'revenge paradoxes' themselves, stated in terms of the new notions introduced to resolve the original paradoxes, also convinced me, though I never accepted the dialetheist thesis that there are true contradictions (cf. 'There are no true contradictions', Weir 2004).

I argued in 1993 that the naïve rules for truth and class, e.g. for truth the interderivability of $\langle s \text{ is true} \rangle$ with ϕ , where substituends for s name substituends for ϕ , are as analytic as HP or indeed conjunction elimination. If we add classical structural rules to these analytic rules, then we get contradiction and indeed triviality, but the response I argued then – and now – is to weaken the structural rules in a rather pragmatic fashion, in order to avoid inconsistency whilst retaining the most powerful logic possible which still retains the naïve semantic and set-theoretic principles. Such, however, is the grip of the hierarchical picture that my attempt to go naïve there really just pushed the hierarchy up into the transfinite!²⁷ I still remain convinced that naïve set theory and semantics are consistent in something like the logical framework I presented but have to confess my attempts to demonstrate this have yet to meet with success.²⁸

I adopted a very radical revision of the structural rules which required introducing a semantic element into proofhood, a proscription of sentences which were not *actually* determinately true nor determinately false from occurring at certain stages in proofs. There are two nice phrases in Belfast, where I taught at that time, indeed for over thirty years, which come in useful here. If one utters something preposterous one will be told to 'catch yourself on' and 'wise up'. I quickly wised up and abandoned this semantic idea for a more conventional notion of proofhood as purely syntactic. Still heterodox to be sure, in allowing not only infinitary proofs but adding constraints on classical structural rules in terms of the determinacy of assumptions in certain operational rules, $\langle \phi \text{ is indeterminate} \rangle \equiv \text{df. } \phi \leftrightarrow \neg\phi$, the biconditional Lukasiewiczian (Weir 2013b). The result is a proof system in which transitivity of entailment is restricted though of course only in a limited fashion so as to allow the chaining of lemma piled on lemma so necessary in mathematics. This is still a heterodox view – I was baffled when my former teacher Neil Tennant introduced me to his version of non-transitivity – but is now being treated seriously by many excellent young logicians working in the thriving area of 'substructural logic'²⁹ though a rather different way of effecting a similar result, restrictions on the structural principle of contraction, is probably a more popular approach.

Conclusion

I ended the paper somewhat lyrically:

the body of mathematical truths is, in Adam Ferguson's words (on a different subject), 'the result of human action but not . . . of any human design' (Ferguson 1767, 187) ... like many human products – language, law, economics – it is to a large extent autonomous of its creators, a fact which is surely responsible, in part, for its great beauty and power. (Weir 1993, 283)

and it impressed me, if no one else, enough that I copied it over for the ending of the book! In the interim, the fortunes of formalism have perhaps improved a little. My Stanford Encyclopedia article on formalism (Weir 2023) notes some of the developments, but a variant of Field's fictionalism is probably a more popular form of anti-platonism – I compare the two programmes in that piece. Of the other issues discussed, sub-structural logics, non-hierarchical solutions to the paradoxes, the ins and outs of indispensability, these are still the subject of lively discussion and excellent research. Putnam, too, is still a focus of research as are the other major figures of my earlier years in the profession – Quine of course, Davidson, Kripke, Lewis. Who will emerge as the most important in the long term? My money would be on Quine being thus recognised in thirty years, certainly fifty years time: if the conditions still allow people to conduct philosophical or other highly theoretical research that is.

Alan Weir,
 Roinn na Feallsanachd/Philosophy
 Sgoil nan Daonnachdan/School of Humanities
 Oilthigh Ghlaschu/University of Glasgow
 GLASGOW G12 8QQ

Notes

1. For an overall view and references to the extensive literature, see Colyvan in the Stanford Encyclopedia Colyvan 2023; see also his monograph Colyvan (2001).
2. Quine (1948, 1951, 1976b, 1981b, 1981c); for Putnam's much less positivistic version see especially Putnam 1975b.
3. For more detail see Weir, (2013a) §7.
4. It should be noted, however, that such holism was already to be found in Carnap and especially Neurath.
5. For discussion see Weir, (2006a) §3 and 2013a §§6–7.
6. It is sometimes said that Quine derives, via his verificationism, a form of *semantic holism*; I am calling this *holistic verificationism* but note that, on my account, it does not apply to observation sentences, not in a fully coherent version of Quine's position anyway.

7. The argument is to be found in (Quine 1969a, 80–81), but the argument is already pre-figured in section V of ‘Two Dogmas’, Quine (1951). Føllesdal (1973, 290–1) seems to give the same reading. (Quine 1986, 155–6) endorses Føllesdal’s account.
8. Quine, trying to hold on to vestiges of realism, at one point proposes adopting a ‘sectarian’ line on such issues, see (Weir 2006a, 246).
9. The earliest occurrence appears to be in ‘Designation and Existence, Quine (1939), 708, a paper containing the ‘bulk’, Quine says, of a paper read at Fifth International Congress for the Unity of Science, Cambridge, Mass., September 9, 1939, under the title ‘A Logistical Approach to the Ontological Problem.’ The conference proceedings never appeared, owing to the German invasion of the Netherlands. Quine published it Quine (1939/76) in *Ways of Paradox*, in 1976a although there is very little literal overlap with ‘Designation and Existence’.
10. Which could be the universal set, supposing there is such a thing, as Quine did in his NF set theory. Lacking the power set axiom, however, NF cannot form a basis, even if it is consistent, for standard semantics. Some philosophers reject ‘domain semantics’ route and branch, for some discussion of this and the related notion of ‘absolute generality’, see Weir (2006b).
11. Here I am in agreement with (Quine 1969b, 100), that there is (there exists) no important metaphysical difference between Being and Existence.
12. A sceptic about propositions, actual Quine as well as Viennese Quine for example, would need to re-fashion this account in terms of interpreted sentences.
13. For a critique, see Weir (1991).
14. Maddy (2005), 445–457, and for comments Weir (2005) §III.
15. Although theory has practical benefits. We amend and change theories in the light of experimental evidence and new ideas, often guided by a model suggested by the theory; this we cannot do with the program. Still, this is surely too pragmatic a justification for theory to satisfy a scientific realist.
16. As a first attempt at explanatory theory; more complex and decontextualised contents will be more explanatory. The idea is analogous to that in two-dimensional semantics but not the same. There is no appeal, even as a *façon de parler*, to worlds as points of evaluation for example.
17. Or of acceptance or whatever cognitive attitude play the role of determining what theories a person holds.
18. Perhaps we can give more content to ‘direct detection’ by using Hacking’s idea, in slogan form, that ‘if you can spray it, it’s real’, i.e. causal interaction such as the spraying of a niobium ball using an electron gun, is required before scientists come to believe in the existence of newly proposed entities, such as (polarized) electrons and niobium atoms (Hacking 1983, 274.).
19. Most believe. But not, it seems, Gödel. I have a brief discussion of his apparent belief in something like perception of numbers in 1993, p. 256.
20. Some unpacking of the informational content here might be reasonable given how people understand concepts such as ‘even’ and ‘prime’ but it is still clear that it is not part of the Fregean sense that there exists a concrete proof token of the sentence, the Fregean cognitive test: one can believe(disbelieve) the sentence but not one expressing its metaphysical content, or vice versa, shows that.

21. Only some, various conditions have to apply. See Milne (2007).
22. Roughly, there is, for any standard formal system S , no fixed polynomial function f such that for every theorem, its shortest proof is no more than $f(x)$ long where x is the length of the theorem.
23. There is no inconsistency in the ‘neo-formalist’ as I baptised my version of formalism, talking freely about numbers since the neo-formalist believes standard mathematics is true; the anti-platonism consists in giving an anti-realist, ontologically reductionist, at the metatheoretic level, account of mathematical truth.
24. Here ‘ \uparrow ’ represents tetration or ‘superexponentiation’, so that’s a very large number: $n \uparrow 0 = 1$, $n \uparrow k+1 = n^{n \uparrow k}$ so it’s an exponential stack of 10,000 tens.
25. Actually I require a little more, that certain ‘primality’ properties hold of the system. See Weir (2010), 10 and *passim*.
26. To be found in papers on neo-logicism such as Weir (2003) and (Shapiro and Weir 1999, 2000).
27. With an appeal in effect to free logic to defuse the paradoxical reasoning and give the appearance of fully naïve comprehension.
28. See (Weir 1998a) for my argument that naïve comprehension ‘trumps’ classical structural rules.
29. For a general account see Restall (2000); for some among many papers in the expanding area of non-transitive and non-contractive logic see Ripley (2012), Zardini (2013, 2019), Petrukhin and Shangin (2023).

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