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Quantum Carroll/fracton particles

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Abstract: We classify and relate unitary irreducible representations (UIRs) of the Carroll and dipole groups, i.e., we define elementary quantum Carroll and fracton particles and establish a correspondence between them. Whenever possible, we express the UIRs in terms of fields on Carroll/Aristotle spacetime subject to their free field equations.

We emphasise that free massive (or "electric") Carroll and fracton quantum field theories are ultralocal field theories and highlight their peculiar and puzzling thermodynamic features. We also comment on subtle differences between massless and "magnetic" Carroll field theories and discuss the importance of Carroll and fractons symmetries for flat space holography.

Keywords: Global Symmetries, Space-Time Symmetries

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Contents

1 Introduction

In this work we classify and relate quantum Carroll and fracton particles, i.e., we classify unitary irreducible representations of the Carroll [1, 2] and dipole [3] groups and describe them, whenever possible, as fields in Carroll spacetime or the Aristotle spacetime underlying the fracton system. This is a continuation of our earlier paper [4], henceforth referred to as Part I, where we studied the classical elementary systems with Carroll and dipole (for us, "fracton"[1]) symmetries. In particular, in this paper we lift the Carroll/fracton correspondence proposed in Part I from the classical to the quantum realm.

A Lie group G is said to be a symmetry of a quantum mechanical model if the underlying Hilbert space of states admits a unitary representation[2] of G. In this sense it is typical to think of unitary representations as describing the symmetries of a quantum mechanical model, what we will somewhat loosely refer to as quantum symmetries in this paper. Indeed, some of these unitary representations may be constructed by geometrically quantising coadjoint orbits and, for some (but famously not all) Lie groups, all unitary representations arise in this way. In particular, if a group acts as symmetries of a given spacetime, we expect it to be realised as quantum symmetries of any natural quantum system defined on that spacetime. Conversely, it is often the case that unitary representations of a Lie group G can actually be realised on classical fields defined on a spacetime on which G acts by symmetries.

In any unitary representation of a Lie group, the elements of the Lie algebra give rise to hermitian operators which may be thought of as physical observables of the corresponding quantum system and in whose spectrum we might be interested. For example, the generator of time translations gives rise to the hamiltonian, which governs the energy spectrum of the theory.

Unitary representations need not be irreducible, but they typically decompose into irreducible components, which we may think of as the building blocks of the quantum symmetries of the given group. We call them the *elementary* quantum systems and they are the quantum counterpart of the coadjoint orbits describing the elementary classical

[1] We will not be able to give full justice to the broad field of "fractons" for which we refer the reader to [5–7] for reviews.

[2] This might be relaxed to only projective representations of G, but we can always restrict to honest representations by passing to a central extension of G. We shall assume that we have done this from now on. For the case of the Carroll and dipole groups there is no need to do this, since there are no nontrivial central extensions in $3 + 1$ dimensions and above.

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<table>
<tr><th>Carroll particles</th><th>Fractonic particles</th></tr>
<tr><td colspan="2">angular momentum $\boldsymbol{J}$</td></tr>
<tr><td colspan="2">momentum $\boldsymbol{P}$</td></tr>
<tr><td>center-of-mass $\boldsymbol{B}$</td><td>dipole moment $\boldsymbol{D}$</td></tr>
<tr><td>energy H</td><td>charge Q</td></tr>
<tr><td>—</td><td>energy H_F</td></tr>
</table>

Table 1. Conserved quantities in carrollian and fractonic theories [4].

systems. One of the earliest descriptions of elementary quantum systems is the Wigner classification [8] of unitary irreducible representations (UIRs, for short) of the Poincaré group, which describe the (free) particles we observe in nature, and forms a cornerstone of relativistic quantum field theory on Minkowski spacetime.

Thus if we wish to study quantum mechanical models with a certain symmetry group, it is useful to first understand the elementary quantum systems of that group. This motivates the study of UIRs of the Carroll and dipole groups and allows us to propose a definition of what we mean by elementary quantum Carroll and fracton particles.

We are able to treat carrollions and fractons simultaneously for the most part since free complex Carroll and fracton scalar theories and their symmetries essentially[3] coincide [9] (see also [10]). This led us in Part I to propose a correspondence, summarised in table 1, between all elementary carrollions and fractons, which we show in this work to persist at the quantum level.

Let us emphasise that out of these elementary ingredients one can of course build composite objects (which are then reducible), to which the above correspondence extends. The elementary dipoles should be contrasted with composite dipoles built out of two elementary monopoles, both of which have mobility, in contradistinction to isolated monopoles. They correspond to two Carroll particles of opposite energy [4]. Seen from this perspective it is even less surprising that composite Carroll particles with opposite mass can move [11] (see also [12, 13]). This is a manifestation of the fact that the dipole moment for nonzero charge depends on the choice of origin (see, e.g., [14]). Hence a monopole, which has nonzero charge ($q \neq 0$), cannot move without changing the dipole moment. As soon as the charge is zero, e.g., by adding another monopole with opposite charge, mobility in ways that do not change the total dipole moment is restored. This shows that it can be useful to think about Carroll and fracton systems from complementary perspectives.

Since carrollian physics is relevant for flat space holography at null [15], timelike [16] or spacelike [17] infinity it might be interesting to understand them from a fracton perspective. We will provide further comments concerning these interesting topics in section 7, but let us highlight that some of the structure of flat space holography can already be seen by using only Carroll symmetries, e.g., there is a radiative and non-radiative branch, related to the option of having vanishing or non-vanishing mass. In addition, the equations of

[3]The dipole group is a trivial central extension of the Carroll group.

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the non-radiative branch share similarities with the massless or magnetic carrollian field theory.

Another motivation comes from the successful and wide-ranging applications of tools that fall under the banner of "scattering amplitudes". See, e.g., [18] for a review. To apply this remarkable toolkit to carrollian or fractonic theories, one needs first of all an understanding of the possible quantum particles, i.e., the "in" and "out" states of scattering amplitudes. The UIRs classified in this paper provide the complete answer to this question and it could be interesting to employ this technology.

The construction of UIRs of the Carroll and dipole groups employed in this paper is essentially that pioneered by Wigner. The upshot of this method is that UIRs are carried by fields in momentum space, more precisely fields defined over orbits on momentum space of the "homogeneous" subgroup: e.g., Lorentz in the case of Poincaré. These fields transform under (unitary, irreducible) representations of the little group associated to these orbits. In the Poincaré case, from which we derive most of our intuition, the orbits are the mass shells and the little groups are the subgroups of the Lorentz group which preserve a given massive, massless or tachyonic momentum. There is however one crucial difference between the Poincaré and Carroll groups: in the latter, boosts commute and hence there are other choices of abelian subgroups from which we can induce. This is reflected in the existence of automorphisms of the Carroll group [4] which mix momenta and boosts. Although we will induce from characters of the translation group in this work, one could equally induce from characters of the group generated by boosts and time translations and in some cases, such as the centrons (in Carroll language) or elementary dipoles (in fracton language), it would perhaps be more natural to do that and hence express the corresponding UIRs as fields on centre-of-mass (or dipole-moment) space.

It is natural to wish to describe these representations in terms of classical fields defined over the relevant spacetime: Minkowski in the case of Poincaré, the eponymous spacetime in the case of Carroll or the Aristotle spacetime in the case of the fractons [9, 19]. Such fields transform according to representations of the homogeneous subgroup, which is the stabiliser of a chosen origin in the spacetime and hence in passing from the momentum space description to the spacetime description, there is always a choice to be made: we need to embed the representation of the little group (the so-called "inducing representation") into some representation of the homogeneous subgroup, a process known as "covariantisation" in the Physics literature. The embedding representation need not be unitary; although a typical consideration is that it should be as small as possible and, in any case, finite-dimensional, in order to arrive at spacetime fields with a finite number of components. It typically happens that the embedding representation is of larger dimension than the inducing representation and hence that the spacetime field has more degrees of freedom than the momentum space field. One way to cut down to the required number of degrees of freedom is by imposing field equations on the spacetime fields, so that the sought-after irreducible representation is carried not by all spacetime fields, but only by those which obey their field equations. There is a systematic way to arrive at the field equations, once the inducing representation has been covariantised, in terms of a group-theoretical generalisation of the Fourier transform. It is well known from the case of the Poincaré group, that this procedure is the origin of many of the familiar relativistic free field equations: Klein-Gordon, Dirac, Maxwell, Proca, linearised Einstein,... In this paper we will give similar descriptions for some of the UIRs of the Carroll and dipole groups. In particular, we will see that some of the massless low-helicity UIRs of the Carroll group are given by solutions of the three-dimensional euclidean Helmholtz, Dirac and topologically massive Maxwell equations. This is perhaps not surprising in that the massless helicity UIRs of the Carroll group are actually UIRs of the three-dimensional euclidean group, but what is perhaps novel is the interpretation of these well-known partial differential equations as irreducibility conditions for three-dimensional euclidean fields and, by extension, for massless carrollian fields (or neutral fractons).

It is worth highlighting the fact that the field equations we find do not necessarily agree with the Carroll-invariant field equations in the literature (see, e.g., [12, 20, 21]). The fundamental reason for this discrepancy is our different points of departure. Our principal aim in this paper is the classification of UIRs of the Carroll and dipole groups. These representations are given in terms of fields on momentum space and that suffices for the classification. Those fields have the precise number of degrees of freedom (roughly the dimension of the inducing representation) required to describe the UIRs. To express the UIRs in terms of spacetime fields, we must make a choice of how to covariantise the inducing representation and our approach has been to choose the simplest covariantisation: roughly, the one which adds the smallest number of extra degrees of freedom. For example, for massive Carroll UIRs (equivalently, charged fracton UIRs), the inducing representation is already covariant if we demand that the boosts (equivalently, the dipole generators) act trivially. This results in no additional field equations beyond the one coming from having fixed the energy. This may result in unfamiliar/surprising field transformation laws, since such an economical choice of covariantisation is not available for the Poincaré group. Indeed, what makes this possible is that the boosts commute in the homogeneous Carroll group, but do not in the Lorentz group, where the commutator of two boosts is a rotation.

By contrast, in the approach where one departs from building invariant actions for spacetime fields, it is perhaps not obvious (and indeed would have to be checked) that the resulting field equations project onto an irreducible subrepresentation of the representation carried by the off-shell spacetime fields.

This paper is organised as follows. In the remainder of this Introduction we provide a self-contained summary of the UIRs of the Carroll (section 1.1.1) and dipole (section 1.1.2) groups and whenever possible we summarise their description in terms of spacetime fields. Readers who are happy to skip some of the details could then continue with section 6 where we discuss Carroll and fracton quantum field theories and finish with the discussion in section 7, where we highlight, e.g., the relevance for flat space holography.

The more detailed treatment starts in section 2, where we review the basic results of Part I on the coadjoint orbits of the Carroll group, their structure and the action of automorphisms. The construction of the UIRs starts in section 3, based on the method of induced representations described in some detail in appendix A. As this topic is somewhat technical, we distill from the appendix a sort of algorithm to construct the UIRs and this is briefly recapped in section 3.1. Section 3.2 outlines the method and checks that the momentum space orbits admit invariant measures. Section 3.3 works out the inducing representations: the UIRs of the little groups of the momentum orbits. Two of the little groups are themselves semidirect products and require iterating the method of induced representations. In section 3.4 we put everything together and list the UIRs of the Carroll group: divided into those with nonzero energy (termed "massive" in Part I and treated in section 3.4.1) and those with zero energy (termed "massless" in Part I and treated in section 3.4.2). In section 3.4.3 we comment on a more unified description of the massless UIRs as induced representations from a larger subgroup of the Carroll group. We end the section with a conjectural correspondence between the UIRs and the (quantisable) coadjoint orbits. In section 4 we describe some of the UIRs found in the previous section in terms of classical fields on Carroll spacetime. This requires a continuation of the brief recap of the method of induced representations, describing the covariantisation procedure and the group-theoretical Fourier transform, and contained in section 4.1. We then do an example of a massive carrollian field (in section 4.2) and of a massless carrollian field with helicity (in section 4.3), which are the only two classes of UIRs which seem to admit a description in terms of finite-component carrollian fields. We work out the resulting field equations for the cases of helicities 0, $\frac{1}{2}$ and 1 and recover the three-dimensional Helmholtz, Dirac and topologically massive Maxwell equations, respectively. Section 5 is devoted to fractonic particles and fields. In section 5.1 we classify the UIRs of the dipole group by observing that there is a bijective correspondence between UIRs of the Carroll group and classes of UIRs of the dipole group distinguished solely by the fracton energy. We then describe some of these UIRs in terms of fields on Aristotle spacetime. We do the example of a charged monopole in section 5.2 and of what could be termed a neutral aristotelion in section 5.3. In section 6 we discuss Carroll and fracton quantum field theories in a second-quantisation language, some of their similarities, and highlight the relation to ultralocal field theories [22, 23]. Additionally, we comment on the intricate thermodynamic properties of these theories and show that there is a subtle difference between free massless Carroll theories and the magnetic Carroll theory. In section 7 we provide a discussion where we emphasise interesting connections to flat space holography and other intriguing topics for further exploration. The paper ends with two appendices: appendix A contains a short review of the method of induced representations, whereas appendix B contains some formulae in special coordinates for the 3-sphere adapted to the Hopf fibration, which we make use of in our discussion of massless UIRs of the Carroll group.

1.1 Summary

In this section we summarise the quantum Carroll and fracton particles and their field theories. In addition to the vacuum sector, UIRs fall into two broader classes outlined in table 2:

- II: massive Carroll particles $\hat{H} = E_0$ and charged monopoles $\hat{Q} = q$
- III − V: massless Carroll particle $\hat{H} = 0$ and neutral fractons $\hat{Q} = 0$

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		Carroll		Fractons, $E \in \mathbb{R}$	
I	$2s \in \mathbb{N}_0$	vacuum sector			
II	$2s \in \mathbb{N}_0$	$E_0 \neq 0$	massive spin s	$q \neq 0$	monopole spin s
III	$n \in \mathbb{Z}, p > 0$	massless helicity $\frac{n}{2} \simeq$ aristotelions			
III′	$n \in \mathbb{Z}$	$k > 0$	centrons	$d > 0$	elementary dipoles
IV$_\pm$	$n \in \mathbb{Z}, p > 0, \pm$	$k > 0$	(anti)parallel helicity $\frac{n}{2}$	$d > 0$	(anti)parallel dipoles
V$_\pm$	$\theta \in (0, \pi), p > 0$	$k > 0$	generic massless	$d > 0$	generic dipole

Table 2. Unitary irreducible representations of the Carroll and monopole/dipole group. By I to V we enumerate inequivalent UIRs of the Carroll and dipole groups. They broadly fall into two classes with different physical properties: II massive carrollions and charged monopoles versus III – V massless carrollions and neutral fractons.
In this table $\mathbb{N}_0$ denotes the non-negative natural numbers (i.e., including zero) and $\mathbb{Z}$ the integers. This shows that spin s and helicity $\frac{n}{2}$ are quantised. All other quantities are real. It is implicit that there is an additional, but mostly irrelevant, phase for the fractons. Further explanations are given in the summary section 1.1.

In each case the properties of the particles are quite distinct and many of the curious physical features of carrollian and fracton theories can be traced back to this fact.

We will present hermitian operators corresponding to our symmetries. They are related to the skew-hermitian Lie algebra generators via multiplication by i.

1.1.1 Quantum Carroll particles and fields

We are not the first to study the UIRs of the Carroll group, the massive UIRs were already constructed by Lévy-Leblond [1] and the massless sector was highlighted in [12] (see also appendix A in [20]), but this work provides the first classification. A similar feat has already been accomplished for the Poincaré [8] and, to a large extent, Galilei/Bargmann groups (we refer to the review [24] and references therein) and this work closes the final gap for quantum symmetries based on the maximally symmetric affine spacetimes [25].

We will now provide a summary of quantum Carroll particles, i.e., UIRs of the Carroll group, and discuss some of their properties. Further details are presented in section 3. Broadly speaking they fall into two classes, massive carrollions ($\hat{H} = E_0$) and massless carrollions ($\hat{H} = 0$), which have very distinct features.

We parametrise the Carroll group as

$$g = g(R, \boldsymbol{v}, \boldsymbol{a}, s) = e^{sH} e^{\boldsymbol{a}\cdot\boldsymbol{P}} e^{\boldsymbol{v}\cdot\boldsymbol{B}} R \tag{1.1}$$

where R is a rotation, $\boldsymbol{B}$ denote the Carroll boost generators, $\boldsymbol{P}$ the generators of spatial translations and H the generator of time translations. The Carroll Lie algebra is given by $(i, j, k = 1, 2, 3)$

$$[J_i, J_j] = \epsilon_{ijk} J_k \qquad [J_i, B_j] = \epsilon_{ijk} B_k \qquad [J_i, P_j] = \epsilon_{ijk} P_k \qquad [B_i, P_j] = \delta_{ij} H\,. \tag{1.2}$$

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The quantum Carroll particles, by which we mean UIRs of the (simply-connected) Carroll group, fall into several different classes listed below. The notation $\mathbb{N}_0$ denotes the non-negative integers and V_s stands for the complex spin-s irreducible representation of $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, of dimension $2s+1$.

I(s) vacuum sector with $2s \in \mathbb{N}_0$ with underlying Hilbert space V_s. When $s = 0$ this represents the vacuum, whereas for $s > 0$ these are spinning vacua. In this representation only the rotations act nontrivially and they do so via the spin-s irreducible representation.

II(s, E_0) massive spin s with $2s \in \mathbb{N}_0$ and $E_0 \in \mathbb{R} \setminus \{0\}$ [1]. This shows that the spin of massive quantum Carroll particles is indeed quantised. The underlying Hilbert space is given by square-integrable functions $\psi \in L^2(\mathbb{A}^3, V_s)$ and $\boldsymbol{p} \in \mathbb{A}^3$ parametrises the hyperplane in momentum space with $E = E_0$ (see figure 1).

The unitary action of G is given by[4]

$$(g \cdot \psi)(\boldsymbol{p}) = e^{i(E_0 s + \boldsymbol{p}\cdot\boldsymbol{a})} \rho(R) \psi(R^{-1}(\boldsymbol{p} + E_0 \boldsymbol{v})) \tag{1.3}$$

where $R \mapsto \rho(R)$ denotes the spin-s representation of $\mathrm{Spin}(3)$ and the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{A}^3} d^3p \, \langle \psi_1(\boldsymbol{p}), \psi_2(\boldsymbol{p}) \rangle_{V_s} , \tag{1.4}$$

where $\langle -, - \rangle_{V_s}$ is an SU(2)-invariant hermitian inner product on V_s. The hermitian operators that correspond to our conserved charges are in this basis given by

$$\hat{\boldsymbol{J}} = -i\boldsymbol{p} \times \frac{\partial}{\partial \boldsymbol{p}} + \hat{\boldsymbol{S}} \qquad \hat{\boldsymbol{B}} = -iE_0 \frac{\partial}{\partial \boldsymbol{p}} \qquad \hat{H} = E_0 \qquad \hat{\boldsymbol{P}} = \boldsymbol{p} , \tag{1.5}$$

where $\hat{\boldsymbol{S}}$ are the infinitesimal generators of the spin-s representation $\rho(R)$. Massive spin-s carrollions can then be labeled by

$$\hat{H} = E_0 \qquad \text{and} \qquad \hat{\boldsymbol{S}}^2 = s(s+1) , \tag{1.6}$$

which are multiples of the identity. For massive carrollions we can also define a position operator $\hat{\boldsymbol{X}}$

$$\hat{\boldsymbol{X}} = \frac{1}{E_0} \hat{\boldsymbol{B}} \tag{1.7}$$

which agrees with the intuition that the centre of mass of a massive Carroll particle is the energy multiplied by the position and satisfies the canonical commutation relations

$$[\hat{X}_i, \hat{P}_j] = -i\delta_{ij} . \tag{1.8}$$

We may alternatively diagonalise with respect to $\hat{\boldsymbol{B}}$, which is related to the above via a Fourier transform (see (3.30)) and express the representation in the "boost basis" as

$$(g \cdot \tilde{\psi})(\boldsymbol{k}) = e^{i(-E_0 s + \boldsymbol{k}\cdot\boldsymbol{v})} \rho(R) \tilde{\psi}(R^{-1}(\boldsymbol{k} - E_0 \boldsymbol{a})) , \tag{1.9}$$

[4] We could have added an additional label to our wavefunctions such that the specific $E = E_0$ hyperplane is explicit, e.g., we could have written $\psi_{E_0}(\boldsymbol{p})$. To reduce clutter we will leave the energy implicit.

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where the inner product is now given by

$$(\tilde{\psi}_1, \tilde{\psi}_2) = \int_{\mathbb{A}^3} d^3k \left\langle \tilde{\psi}_1(\boldsymbol{k}), \tilde{\psi}_2(\boldsymbol{k}) \right\rangle_{V_s} . \tag{1.10}$$

We provide further details for this UIR in section 3.4.1.

This representation can also be described using fields on Carroll spacetime, i.e., as massive Carroll field theories. These V_s-valued fields $\phi(t, \boldsymbol{x})$ are obtained from the V_s-valued momentum space fields $\psi(\boldsymbol{p})$ via a group-theoretical Fourier transform which, in this case, agrees with the classical Fourier transform:

$$\phi(t, \boldsymbol{x}) = e^{-iE_0 t} \int_{\mathbb{A}^3} d^3p e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \psi(\boldsymbol{p}) . \tag{1.11}$$

The field ϕ satisfies the obvious (and only) field equation

$$\frac{\partial \phi}{\partial t} = -iE_0 \phi. \tag{1.12}$$

The action of the Carroll group on the spacetime fields is given by

$$(g \cdot \phi)(t, \boldsymbol{x}) = \rho(R)\phi(t - s - \boldsymbol{v} \cdot (\boldsymbol{x} - \boldsymbol{a}), R^{-1}(\boldsymbol{x} - \boldsymbol{a})) \tag{1.13}$$

where we want to emphasise that the fields are scalars under boosts, since these act nontrivially only on the coordinates.

When we do not restrict to just one orbit and allow both energies $E = \pm E_0$ we are led to ultralocal (quantum) field theories [22, 23] or "electric Carroll field" theories [12, 21], as discussed in more detail in section 6.1.

$\mathrm{III}(n, p)$ massless helicity $\frac{n}{2}$ with real $p > 0$ and $n \in \mathbb{Z}$, so the helicity is now quantised. The underlying Hilbert space consists of complex-valued functions[5] on the complex plane, where the action of G is given by

$$(g \cdot \psi)(z) = e^{i\boldsymbol{a}\cdot\boldsymbol{\pi}(z)} \left(\frac{\eta + \bar{\xi} z}{|\eta + \bar{\xi} z|} \right)^{-n} \psi \left(\frac{\overline{\eta} z - \xi}{\eta + \bar{\xi} z} \right) . \tag{1.14}$$

In this case z is a stereographic coordinate on the sphere $\|\boldsymbol{p}\| = p$, the action of time translations and boosts is trivial and $\boldsymbol{\pi}(z)$, given in equation (3.37), satisfies $\|\boldsymbol{\pi}(z)\|^2 = p^2$. The rotation group SU(2) acts on z via linear fractional transformations:

$$R = \begin{pmatrix} \eta & \xi \\ -\bar{\xi} & \overline{\eta} \end{pmatrix} \in \mathrm{SU}(2) \quad \text{acts as} \quad z \mapsto \frac{\eta z + \xi}{\overline{\eta} - \bar{\xi} z} . \tag{1.15}$$

Consequently, $\hat{\boldsymbol{B}} = \hat{H} = 0$, however $\hat{\boldsymbol{P}} = \boldsymbol{\pi}(z)$ so that these representations are indeed specified by the helicity $\frac{n}{2}$ and $\hat{\boldsymbol{P}}^2 = p^2$. The inner product on the Hilbert space is given by

$$\langle \psi_1, \psi_2 \rangle := \int_{\mathbb{C}} \frac{2idz \wedge d\bar{z}}{(1 + |z|^2)^2} \overline{\psi_1(z)} \psi_2(z) . \tag{1.16}$$

[5] More precisely, square-integrable (relative to an SU(2)-invariant measure) sections of the line bundle $\mathscr{O}(-n)$ over the complex projective line, but the description in this summary suffices.

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This UIR can also be described using spacetime fields. One possibility is to covariantise the inducing representation of U(1) of weight n with boosts acting trivially into the spin-$|n/2|$, representation $V_{|n/2|}$ of SU(2) as the highest (if $n \geq 0$) or lowest (if $n \leq 0$) weight vectors in $V_{|n/2|}$. The $V_{|n/2|}$-valued spacetime fields $\phi(t, \boldsymbol{x})$ are given in terms of the momentum space fields $\psi(z)$ by

$$\phi(t, \boldsymbol{x}) = \int_{\mathbb{C}} \frac{2i dz \wedge d\bar{z}}{(1+|z|^2)^2} e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)} \rho(\sigma(z))\psi(z), \tag{1.17}$$

where $\sigma(z) \in \mathrm{SU}(2)$ is defined by

$$\sigma(z) = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} z & -1 \\ 1 & \bar{z} \end{pmatrix}. \tag{1.18}$$

Notice that the spacetime fields do not depend on t, all massless Carroll fields fulfil

$$\frac{\partial \phi}{\partial t} = 0\,, \tag{1.19}$$

so they are essentially euclidean three-dimensional fields. The action of G factors through the action of the three-dimensional euclidean group:

$$(g \cdot \phi)(\boldsymbol{x}) = \rho(R)\phi(R^{-1}(\boldsymbol{x} - \boldsymbol{a}))\,, \tag{1.20}$$

where we see that boosts act trivially. The additional field equations which project to the irreducible subrepresentation can be worked out for the lowest values of the helicity. For helicity 0 we obtain the Helmholtz equation

$$(\triangle + p^2)\phi(\boldsymbol{x}) = 0, \tag{1.21}$$

for a scalar field, where $\triangle$ is the laplacian acting on functions in three-dimensional euclidean space. For helicity 1/2, we obtain the three-dimensional euclidean Dirac equation

$$(\partial\!\!\!/ + ip)\,\phi(\boldsymbol{x}) = 0, \tag{1.22}$$

where now $\phi(\boldsymbol{x})$ is a 2-component field taking values in the spin-1/2 representation of SU(2). Finally, for helicity 1 we find the topologically massive Maxwell equation of [26, 27]:

$$\nabla \times \phi = p\phi, \tag{1.23}$$

where ϕ is now a three-dimensional vector field, which is metrically dual to the Hodge dual of the Maxwell field-strength.

III$'(n,k)$ centrons with real $k > 0$ and $n \in \mathbb{Z}$. They are in many ways analogous to the massless carrollions just described, so we will be brief. The underlying Hilbert space consists again of complex-valued functions on the complex plane and the action of G is given by

$$(g \cdot \psi)(z) = e^{i\boldsymbol{v}\cdot\boldsymbol{\pi}(z)} \left(\frac{\eta + \bar{\xi}z}{|\eta + \bar{\xi}z|}\right)^{-n} \psi\left(\frac{\bar{\eta}z - \xi}{\eta + \bar{\xi}z}\right), \tag{1.24}$$

where z is now a stereographic coordinate on the sphere $\|\boldsymbol{k}\| = k$ and is $\boldsymbol{\pi}(z)$ given in equation (3.37) with $p \mapsto k$. In this case the action of the time and spatial translations is trivial and the representations are uniquely specified by $\hat{\boldsymbol{P}} = \hat{H} = 0$, $\hat{\boldsymbol{B}}^2 = k^2$ and $n \in \mathbb{Z}$. The inner product on the Hilbert space is again given by (1.16).

We can also write down field theories for the centrons and they are mutatis mutandis the same as for the massless carrollions, with the interesting twist that they live naturally in "centre-of-mass space". In this sense they are more reminiscent of internal degrees of freedom, such as spin.

$\mathbb{IV}_\pm(n,p,k)$ (anti)parallel massless helicity $\frac{n}{2}$ with $n \in \mathbb{Z}$ and real $p, k > 0$. The underlying Hilbert space is again given by complex-valued functions on the complex plane and the action of G is given by

$$(g \cdot \psi)(z) = e^{i(\boldsymbol{a} \pm \frac{k}{p}\boldsymbol{v})\cdot\boldsymbol{\pi}(z)} \left(\frac{\eta + \bar{\xi} z}{|\eta + \bar{\xi} z|}\right)^{-n} \psi\left(\frac{\bar{\eta} z - \xi}{\eta + \bar{\xi} z}\right). \tag{1.25}$$

In this case z is a stereographic coordinate on the sphere $\|\boldsymbol{p}\| = p$. By inspection we see that $\mathbb{III}(n,p)$ is the limit of $\mathbb{IV}_\pm(n,p,k)$ as $k \to 0$, which results from formally putting $k = 0$ in the above expression for $g \cdot \psi$. The action of time translations is trivial, consequently $\hat{H} = 0$. For the momentum and centre-of-mass operators we obtain $\hat{\boldsymbol{P}} = \boldsymbol{\pi}(z)$ and $\hat{\boldsymbol{B}} = \pm\frac{k}{p}\boldsymbol{\pi}(z)$, respectively. Since $\boldsymbol{k} = \pm\frac{k}{p}\boldsymbol{p}$, the sign tells us whether we are in the parallel $(+)$ or antiparallel $(-)$ cases. In summary, we can characterise these UIRs by $\hat{\boldsymbol{P}}^2 = p^2$ and $\hat{\boldsymbol{B}}^2 = k^2$ and the sign of $\hat{\boldsymbol{P}} \cdot \hat{\boldsymbol{B}}$. The inner product on the Hilbert space is again given by (1.16).

$\mathbb{V}_\pm(p,k,\theta)$ generic massless with real $p, k > 0$ and $\theta \in (0, \pi)$. It is interesting to note that there are no discrete quantum numbers for the generic massless particles. The underlying Hilbert space is $L^2(S^3, \mathbb{C})$, which are the square-integrable functions on the round 3-sphere with values in a one-dimensional unitary representation of the nilpotent subgroup of G generated by boosts and translations. The unitary character of this representation is such that

$$\chi\left(e^{sH + \boldsymbol{a}\cdot\boldsymbol{P}} e^{\boldsymbol{v}\cdot\boldsymbol{B}}\right) = e^{i(\boldsymbol{a}\cdot\boldsymbol{p} + \boldsymbol{v}\cdot\boldsymbol{k})}, \tag{1.26}$$

where $\boldsymbol{p} = (0,0,p)$ and $\boldsymbol{k} = (k\sin\theta, 0, k\cos\theta)$. We identify S^3 with the SU(2) subgroup of G and we write the action of $g = g(R, \boldsymbol{a}, \boldsymbol{v}, s) \in G$ on $L^2(S^3, \mathbb{C})$ as

$$(g \cdot \Psi)(S) = e^{i(\boldsymbol{a}\cdot S\boldsymbol{p} + \boldsymbol{v}\cdot S\boldsymbol{k})} \Psi(R^{-1}S), \tag{1.27}$$

where $S \in \mathrm{SU}(2)$. The inner product is given by

$$\langle \Psi_1, \Psi_2 \rangle = \int_{S^3} d\mu(S) \overline{\Psi_1(S)} \Psi_2(S), \tag{1.28}$$

with $d\mu(S)$ the volume form of a round metric on S^3, or equivalently a bi-invariant Haar measure on SU(2). This representation breaks up as the orthogonal direct sum of two UIRs: $L^2_\pm(S^3,\mathbb{C})$, where $\Psi \in L^2_\pm(S^3,\mathbb{C})$ if and only if $\Psi(-S) = \pm\Psi(S)$ for all $S \in \mathrm{SU}(2)$. The sign labels two inequivalent quantisations of the same coadjoint orbit, a phenomenon typically associated to a disconnected stabiliser, which is indeed the underlying reason here too as discussed in section 3.4.3.

In summary, apart from that sign, the representation is uniquely specified by

$$\hat{H} = 0 \qquad \hat{\boldsymbol{P}}^2 = p^2 \qquad \hat{\boldsymbol{B}}^2 = k^2 \qquad \hat{\boldsymbol{P}} \cdot \hat{\boldsymbol{B}} = pk\cos\theta\,, \tag{1.29}$$

where $\hat{\boldsymbol{P}} = S\boldsymbol{p}$ and $\hat{\boldsymbol{B}} = S\boldsymbol{k}$.

From the point of view of path integral quantisation the subtleties in the quantisation of these orbits derives from the intricate constraint structure of the orbits, which are basically the classical analog of (1.29), as in section 3.5 of Part I.

1.1.2 Quantum fracton particles and fields

In this section we summarise the UIRs of the dipole group and some of their field-theoretic realisations on Aristotle spacetime. To the best of our knowledge there have been no attempts towards a classification of unitary irreducible representations of the dipole group. The dipole Lie algebra[6] is given by

$$[J_i, J_j] = \epsilon_{ijk} J_k \qquad [J_i, P_j] = \epsilon_{ijk} P_k \qquad [J_i, D_j] = \epsilon_{ijk} D_k \qquad [D_i, P_j] = \delta_{ij} Q\,, \tag{1.30}$$

with an additional generator H_F which is central and most notably the exchange of center-of-mass B_i with dipole moment D_i and Carroll energy H with charge Q, as shown in table 1.

Let us now discuss the quantum generalisation of the correspondence between Carroll and fracton particles [4]. As explained in section 5, UIRs of the dipole group are in bijective correspondence with the UIRs of the Carroll group, except that we extend them to a UIR of the dipole group by declaring that e^{sH_F} should act via the unitary character $\chi(e^{sH_F}) = e^{isE}$ for some $E \in \mathbb{R}$ which is to be interpreted as the fracton energy. We therefore use the same notation, but replacing the Carroll energy E_0 with the monopole charge q and the magnitude of the centre-of-mass k with the magnitude of the dipole moment d and adding a label E: hence the UIRs of the dipole group are $\mathrm{I}(s,E)$, $\mathrm{II}(s,q,E)$, $\mathrm{III}(n,p,E)$, $\mathrm{III}'(n,d,E)$, $\mathrm{IV}_\pm(n,p,d,E)$ and $\mathrm{V}_\pm(p,d,\theta,E)$, as can be seen in table 2.

Monopoles with charge q and spin s. For example, monopoles of charge $q \neq 0$ and spin s (where $2s$ is a non-negative integer) are given by the UIR $\mathrm{II}(s,q,E)$. The underlying Hilbert space is given by square-integrable functions $\psi \in L^2(\mathbb{A}^3, V_s)$ which means that they are functions on the hyperplane in momentum space $\boldsymbol{p} \in \mathbb{A}^3$ with fixed charge q and energy E (see figure 1 with $E \mapsto q$). They are valued in V_s, i.e., in the complex spin-s UIR of $\mathrm{SU}(2)$. The action of the dipole group is given as follows. If we let

$$g = g(R, \boldsymbol{m}, \boldsymbol{a}, \theta, s) = e^{sH_F + \theta Q + \boldsymbol{a}\cdot\boldsymbol{P}} e^{\boldsymbol{m}\cdot\boldsymbol{D}} R \tag{1.31}$$

[6] A better name might be monopole-dipole algebra, since the particles described by this algebra include monopoles as well as dipoles.

denote the generic element of the dipole group, we have, for ψ a V_s-valued field, that

$$(g \cdot \psi)(\boldsymbol{p}) = e^{i(q\theta + Es + \boldsymbol{p}\cdot\boldsymbol{a})}\rho(R)\psi(R^{-1}(\boldsymbol{p} + q\boldsymbol{m}))\,. \tag{1.32}$$

Let us emphasise that the dipole transformation acts as expected $\boldsymbol{p} \mapsto \boldsymbol{p} + q\boldsymbol{m}$ and $\rho(R)$ is a manifestation of the fact that these are spin s monopoles. We could have written $\psi_{q,E}(\boldsymbol{p})$ to emphasise that our functions are restricted to these specific charge q and energy E. The infinitesimal action of (1.32) is given by

$$\hat{\boldsymbol{J}} = -i\boldsymbol{p} \times \frac{\partial}{\partial \boldsymbol{p}} + \hat{\boldsymbol{S}} \qquad \hat{Q} = q \qquad \hat{\boldsymbol{D}} = -iq\frac{\partial}{\partial \boldsymbol{p}} \qquad \hat{H} = E \qquad \hat{\boldsymbol{P}} = \boldsymbol{p}\,, \tag{1.33}$$

which are hermitian operators with respect to the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{A}^3} d^3p\, \langle \psi_1(\boldsymbol{p}), \psi_2(\boldsymbol{p})\rangle_{V_s}\,. \tag{1.34}$$

The UIRs can then be uniquely labeled by

$$\hat{Q} = q \qquad\qquad \hat{H} = E \qquad\qquad \hat{\boldsymbol{S}}^2 = s(s+1)\,, \tag{1.35}$$

which are multiples of the identity. We can also define a position operator $\hat{\boldsymbol{X}}$

$$\hat{\boldsymbol{X}} = \frac{1}{q}\hat{\boldsymbol{D}} \tag{1.36}$$

which agrees with the intuition that the dipole moment is given by the charge times the position. The position operator satisfies the canonical commutation relation

$$[\hat{X}_i, \hat{P}_j] = -i\delta_{ij}\,. \tag{1.37}$$

We may alternatively diagonalise with respect to $\hat{\boldsymbol{D}}$, which is related to the above via a Fourier transform (see (3.30)) and express the representation in the "dipole basis"

$$(g \cdot \tilde{\psi})(\boldsymbol{d}) = e^{i(-Es + q\theta + \boldsymbol{d}\cdot\boldsymbol{m})}\rho(R)\tilde{\psi}(R^{-1}(\boldsymbol{d} - q\boldsymbol{a}))\,, \tag{1.38}$$

where the dipole moment is, as expected, shifted by the translations.[7] The inner product is then given by

$$(\tilde{\psi}_1, \tilde{\psi}_2) = \int_{\mathbb{A}^3} d^3d \left\langle \tilde{\psi}_1(\boldsymbol{d}), \tilde{\psi}_2(\boldsymbol{d})\right\rangle_{V_s}\,. \tag{1.39}$$

We may describe these UIRs in terms of fields on the Aristotle spacetime [9, 19] with coordinates $(t, \boldsymbol{x})$. The field $\phi(t, \boldsymbol{x})$ is obtained from $\psi(\boldsymbol{p})$ via a Fourier transform

$$\phi(t, \boldsymbol{x}) = e^{-iEt}\int_{\mathbb{A}^3} d^3p e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\psi(\boldsymbol{p})\,, \tag{1.40}$$

and the action of the generic element g of the dipole group in equation (1.31) is given by

$$(g \cdot \phi)(t, \boldsymbol{x}) = e^{iq(\theta + \boldsymbol{m}\cdot(\boldsymbol{x} - \boldsymbol{a}))}\rho(R)\phi(t - s, R^{-1}(\boldsymbol{x} - \boldsymbol{a}))\,, \tag{1.41}$$

[7]This choice of basis was also employed in appendix D in [28].

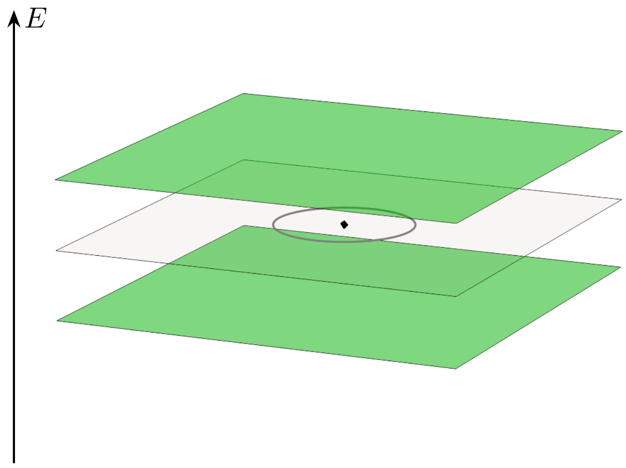

Figure 1. This figure shows the coadjoint orbits of Carroll group in momentum space $(E, \boldsymbol{p})$. Broadly they fall into two classes depending on vanishing or nonvanishing E.
When $E = E_0 \neq 0$, and since the energy is a Casimir, the orbits are given by three dimensional planes (depicted in green). When $E = 0$ the orbits are given by $\|\boldsymbol{p}\| = const.$ two spheres, one of which we have represented as a black circle. The whole $E = 0$ plane is foliated by such spheres, while the origin $\|\boldsymbol{p}\| = 0$ is the dot in the middle.
Let us emphasise that this figure only represents the $(E, \boldsymbol{p})$ part of the full dual space $(\boldsymbol{j}, \boldsymbol{v}, \boldsymbol{p}, E)$ and the complete structure of the orbits is more intricate and involves spin degrees of freedom.

with ρ the spin-s representation of SU(2). In particular, pure charge and dipole transformations act as expected via a phase

$$\phi(t, \boldsymbol{x}) \mapsto e^{iq(\theta + \boldsymbol{m}\cdot\boldsymbol{x})}\phi(t, \boldsymbol{x})\,. \tag{1.42}$$

The only field equation is

$$\frac{\partial \phi}{\partial t} = -iE\phi. \tag{1.43}$$

Readers who are happy to skip the details of the classification of the UIRs could continue with our discussion of Carroll and fracton quantum field theories in section 6.

2 Review of coadjoint orbits of the Carroll group

In Part I we classified the coadjoint orbits of the $(3+1)$-dimensional Carroll group. The Carroll Lie algebra $\mathfrak{g}$ is the ten-dimensional real Lie algebra spanned by J_i, B_i, P_i, H where $i = 1, 2, 3$ subject to the following non-zero brackets:

$$[J_i, J_j] = \epsilon_{ijk} J_k \qquad [J_i, B_j] = \epsilon_{ijk} B_k \qquad [J_i, P_j] = \epsilon_{ijk} P_k \qquad [B_i, P_j] = \delta_{ij} H\,, \tag{2.1}$$

where the Levi-Civita symbol ϵ_{ijk} is normalised so that $\epsilon_{123} = 1$. The connected Carroll group G is isomorphic to a semidirect product $G \cong K \ltimes T$, where $K \cong \mathrm{SO}(3) \ltimes \mathbb{R}^3$ and $T \cong \mathbb{R}^4$. The Lie algebra $\mathfrak{t}$ of T is spanned by P_i, H and the Lie algebra $\mathfrak{k}$ of K by J_i, B_i. The group K is isomorphic to the (connected) three-dimensional euclidean group.

#	Orbit representative $\alpha = (\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, E)$	$\dim \mathcal{O}_\alpha$	Equations for orbits
1	$(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, E_0)$	6	$E = E_0 \neq 0, E_0\boldsymbol{j} + \boldsymbol{p} \times \boldsymbol{k} = \boldsymbol{0}$
2	$(S\boldsymbol{u}, \boldsymbol{0}, \boldsymbol{0}, E_0)$	8	$E = E_0 \neq 0, \Vert\boldsymbol{j} + E_0^{-1}\boldsymbol{p} \times \boldsymbol{k}\Vert = S > 0$
3	$(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, 0)$	0	$E = 0, \boldsymbol{p} = \boldsymbol{0}, \boldsymbol{k} = \boldsymbol{0}, \boldsymbol{j} = \boldsymbol{0})$
4	$(j\boldsymbol{u}, \boldsymbol{0}, \boldsymbol{0}, 0)$	2	$E = 0, \boldsymbol{p} = \boldsymbol{0}, \boldsymbol{k} = \boldsymbol{0}, \Vert\boldsymbol{j}\Vert = j > 0$
5	$(h\boldsymbol{u}, k\boldsymbol{u}, \boldsymbol{0}, 0)$	4	$E = 0, \boldsymbol{p} = \boldsymbol{0}, \Vert\boldsymbol{k}\Vert = k > 0, \boldsymbol{j} \cdot \boldsymbol{k} = h\Vert\boldsymbol{k}\Vert \in \mathbb{R}$
6	$(h\boldsymbol{u}, \boldsymbol{0}, p\boldsymbol{u}, 0)$	4	$E = 0, \boldsymbol{k} = \boldsymbol{0}, \Vert\boldsymbol{p}\Vert = p > 0, \boldsymbol{j} \cdot \boldsymbol{p} = h\Vert\boldsymbol{p}\Vert \in \mathbb{R}$
$7_\pm$	$(h\boldsymbol{u}, \pm k\boldsymbol{u}, p\boldsymbol{u}, 0)$	4	$E = 0, \Vert\boldsymbol{p}\Vert = p > 0, \Vert\boldsymbol{k}\Vert = k > 0, \boldsymbol{p} \cdot \boldsymbol{k} = \pm pk, \boldsymbol{j} \cdot \boldsymbol{p} = h\Vert\boldsymbol{p}\Vert \in \mathbb{R}$
8	$(\boldsymbol{0}, k\cos\theta\boldsymbol{u} + k\sin\theta\boldsymbol{u}_\perp, p\boldsymbol{u}, 0)$	6	$E = 0, \Vert\boldsymbol{p}\Vert = p > 0, \Vert\boldsymbol{k}\Vert = k > 0, \boldsymbol{p} \cdot \boldsymbol{k} = pk\cos\theta, \theta \in (0, \pi)$

Table 3. Coadjoint orbits of the Carroll group.
This table provides an overview of the coadjoint orbits of the Carroll group. As indicated by the horizontal line they are separated into orbits with $E \neq 0$ and orbits with $E = 0$. The second column displays an orbit representative: the notation is such that $\boldsymbol{u} \in \mathbb{R}^3$ represents a fixed unit-norm vector and in the last row $\boldsymbol{u}_\perp \in \mathbb{R}^3$ is a second unit-norm vector perpendicular to $\boldsymbol{u}$. The third column is the dimension of the orbit and the last column provides the equations which define the orbits.

Elements $\alpha \in \mathfrak{g}^*$ in the dual of the Carroll Lie algebra are parametrised by the "momenta" of classical particles; that is, $\alpha = (\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, E)$ where $\boldsymbol{j} = \langle\alpha, \boldsymbol{J}\rangle$ is the angular momentum, $\boldsymbol{k} = \langle\alpha, \boldsymbol{B}\rangle$ is the centre of mass, $\boldsymbol{p} = \langle\alpha, \boldsymbol{P}\rangle$ is the linear momentum and $E = \langle\alpha, H\rangle$ is the energy. Coadjoint orbits belong to several classes distinguished in the first instance by the value of the Casimir elements H and W^2, which is the euclidean norm of

$$W_i := HJ_i + \epsilon_{ijk}P_jB_k\,. \tag{2.2}$$

On $\alpha = (\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, E)$,

$$H(\alpha) = E \qquad \text{and} \qquad W^2(\alpha) = \Vert E\boldsymbol{j} + \boldsymbol{p} \times \boldsymbol{k}\Vert^2. \tag{2.3}$$

Notice that since the energy is constant on each orbit, it is trivially bounded below, that being a typical physical requirement. See also figure 1.

In Part I, we arrived at the classification of coadjoint orbits displayed in table 3.

We also determined the structure of the coadjoint orbits as homogeneous fibre bundles over the K-orbits in $\mathfrak{t}^*$ and that plays an important rôle in the construction of induced representations. Briefly, we write $\alpha \in \mathfrak{g}^*$ as $(\kappa, \tau) \in \mathfrak{k}^* \oplus \mathfrak{t}^*$. Since K acts on T, it acts on $\mathfrak{t}$ and hence on $\mathfrak{t}^*$. We let $\mathcal{O}_\tau = K \cdot \tau$ denote the K-orbit of τ in $\mathfrak{t}^*$. Let K_τ denote the stabiliser of τ in K and let $\mathfrak{k}_\tau$ be its Lie algebra. We let $\kappa_\tau \in \mathfrak{k}_\tau^*$ denote the restriction of κ to $\mathfrak{k}_\tau$ and let $\mathcal{O}_{\kappa_\tau}$ denote its K_τ-coadjoint orbit. Then as explained, for example in [29]

#	$\alpha \in \mathfrak{g}^*$	$\tau \in \mathfrak{t}^*$	$\mathcal{O}_\tau$	K_τ	$\kappa \in \mathfrak{k}^*$	$\kappa_\tau \in \mathfrak{k}_\tau^*$	$\mathcal{O}_{\kappa_\tau}$	$\mathcal{O}_\alpha$
1	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, E_0 \neq 0)$	$(\mathbf{0}, E_0)$	$\mathbb{A}^3_{E=E_0}$	$\mathrm{SO}(3)$	$(\mathbf{0}, \mathbf{0})$	$\mathbf{0}$	$\{\mathbf{0}\}$	$T^*\mathbb{A}^3$
2	$(\boldsymbol{j} \neq \mathbf{0}, \mathbf{0}, \mathbf{0}, E_0 \neq 0)$	$(\mathbf{0}, E_0)$	$\mathbb{A}^3_{E=E_0}$	$\mathrm{SO}(3)$	$(\boldsymbol{j}, \mathbf{0})$	$\boldsymbol{j}$	$S^2_{\|j\|}$	$T^*\mathbb{A}^3 \times_{\mathbb{A}^3} (K \times_{K_\tau} S^2)$
3	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	$(\mathbf{0}, 0)$	$\{(\mathbf{0}, 0)\}$	K	$(\mathbf{0}, \mathbf{0})$	$(\mathbf{0}, \mathbf{0})$	$\{(\mathbf{0}, \mathbf{0})\}$	$\{0\}$
4	$(\boldsymbol{j} \neq \mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$	$(\mathbf{0}, 0)$	$\{(\mathbf{0}, 0)\}$	K	$(\boldsymbol{j}, \mathbf{0})$	$(\boldsymbol{j}, \mathbf{0})$	$S^2_{\|\boldsymbol{j}\|}$	S^2
5	$(\boldsymbol{j}, \boldsymbol{k} \neq \mathbf{0}, \mathbf{0}, 0)_{\boldsymbol{j}\times\boldsymbol{k}=\mathbf{0}}$	$(\mathbf{0}, 0)$	$\{(\mathbf{0}, 0)\}$	K	$(\boldsymbol{j}, \boldsymbol{k})$	$(\boldsymbol{j}, \boldsymbol{k})$	$T^*S^2_{\|\boldsymbol{k}\|}$	T^*S^2
6	$(\boldsymbol{j}, \mathbf{0}, \boldsymbol{p} \neq 0, 0)_{\boldsymbol{j}\times\boldsymbol{p}=\mathbf{0}}$	$(\boldsymbol{p}, 0)$	$S^2_{\|\boldsymbol{p}\|}$	$\mathrm{SO}(2) \ltimes \mathbb{R}^3$	$(\boldsymbol{j}, \mathbf{0})$	$(\boldsymbol{j}, \mathbf{0})$	$\{(\boldsymbol{j}, \mathbf{0})\}$	T^*S^2
$7_\pm$	$(\boldsymbol{j}, \boldsymbol{k} \neq \mathbf{0}, \boldsymbol{p} \neq \mathbf{0}, 0)_{\boldsymbol{k}\times\boldsymbol{p}=\boldsymbol{j}\times\boldsymbol{k}=\mathbf{0}}$	$(\boldsymbol{p}, 0)$	$S^2_{\|\boldsymbol{p}\|}$	$\mathrm{SO}(2) \ltimes \mathbb{R}^3$	$(\boldsymbol{j}, \boldsymbol{k})$	$(\boldsymbol{j}, \boldsymbol{k})$	$\{(\boldsymbol{j}, \boldsymbol{k})\}$	T^*S^2
8	$(\mathbf{0}, \boldsymbol{k}, \boldsymbol{p}, 0)_{\boldsymbol{k}\times\boldsymbol{p}\neq\mathbf{0}}$	$(\boldsymbol{p}, 0)$	$S^2_{\|\boldsymbol{p}\|}$	$\mathrm{SO}(2) \ltimes \mathbb{R}^3$	$(\mathbf{0}, \boldsymbol{k})$	$(\mathbf{0}, \boldsymbol{k})$	$T^*S^1_{\|\boldsymbol{k}\|}$	$T^*S^2 \times_{S^2} (K \times_{K_\tau} T^*S^1)$

Table 4. Deconstructing the coadjoint orbits.

(see also [30]) the G-coadjoint orbit of $\alpha = (\kappa, \tau)$ is the fibred product

$$\begin{tikzcd} \mathcal{O}_\alpha \arrow[r] \arrow[d] & T^*\mathcal{O}_\tau \arrow[d] \\ K \times_{K_\tau} \mathcal{O}_{\kappa_\tau} \arrow[r] & \mathcal{O}_\tau \end{tikzcd} \tag{2.4}$$

over $\mathcal{O}_\tau$ of the cotangent bundle $T^*\mathcal{O}_\tau$ and the homogeneous fibre bundle over $\mathcal{O}_\tau$ whose fibre is the K_τ-coadjoint orbit of κ_τ. A more standard notation for that fibred product would be

$$\mathcal{O}_\alpha = T^*\mathcal{O}_\tau \times_{\mathcal{O}_\tau} (K \times_{K_\tau} \mathcal{O}_{\kappa_\tau}). \tag{2.5}$$

It is a symplectic manifold of dimension $2 \dim \mathcal{O}_\tau + \dim \mathcal{O}_{\kappa_\tau}$ and carries a Carroll-invariant symplectic structure. For every coadjoint orbit of the Carroll group, one can determine τ, $\mathcal{O}_\tau$, K_τ and the structure of the orbit. This was done in Part I, from where we borrow table 4.

In Part I we showed that automorphisms of G induce symplectomorphisms between coadjoint orbits (provided with their natural G-invariant Kirillov-Kostant-Souriau symplectic structure). Inner automorphisms preserve the coadjoint orbit, whereas outer automorphisms relate different coadjoint orbits. For instance, all the four-dimensional coadjoint orbits of the Carroll group (cases $5, 6, 7_\pm$) with the same value of $\|\boldsymbol{j}\|$ are related by automorphisms. In the same way, automorphisms also relate different representations of G. If $\rho : G \to \mathrm{U}(\mathscr{H})$ is a unitary representation of G on a Hilbert space $\mathscr{H}$ and $\varphi \in \mathrm{Aut}(G)$ is an automorphism, we may twist ρ by φ to arrive another representation ρ^φ defined simply by pre-composition: $\rho^\varphi(g) = \rho(\varphi(g))$ for all $g \in G$. Notice that by construction, ρ^φ is a representation on the same underlying Hilbert space. Again if φ is an inner automorphism, so $\varphi(g) = hgh^{-1}$ for some $h \in G$, then $\rho^\varphi(g) = \rho(h) \circ \varphi(g) \circ \rho(h)^{-1}$, so that the two representations are unitarily equivalent. However if φ is outer, then ρ and ρ^φ need not be equivalent and, indeed, often they are not.

The outer automorphisms of the Carroll group G were determined in Part I. They are given by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ acting on $\mathfrak{g}$ as follows:

$$J_i \mapsto J_i, \qquad B_i \mapsto \alpha B_i + \beta P_i, \qquad P_i \mapsto \gamma B_i + \delta P_i \qquad \text{and} \qquad H \mapsto \Delta H, \tag{2.6}$$

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where $\Delta = \alpha\delta - \beta\gamma \neq 0$ the determinant of the matrix. The dual action on $\mathfrak{g}^*$ is given as follows: $(\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, E) \mapsto (\boldsymbol{j}', \boldsymbol{k}', \boldsymbol{p}', E')$ with

$$\boldsymbol{j}' = \boldsymbol{j}, \qquad \boldsymbol{k}' = \frac{\delta \boldsymbol{k} - \beta \boldsymbol{p}}{\Delta}, \qquad \boldsymbol{p}' = \frac{\alpha \boldsymbol{p} - \gamma \boldsymbol{k}}{\Delta} \qquad \text{and} \qquad E' = \frac{E}{\Delta}. \tag{2.7}$$

We will find it convenient to also work out the action of automorphisms on the group G in our choice of parametrisation:

$$g(R, \boldsymbol{v}, \boldsymbol{a}, s) = e^{sH} e^{\boldsymbol{a}\cdot\boldsymbol{P}} e^{\boldsymbol{v}\cdot\boldsymbol{B}} R. \tag{2.8}$$

One finds after a short calculation (using the Baker-Campbell-Hausdorff formula) that

$$g(R, \boldsymbol{v}, \boldsymbol{a}, s) \mapsto g(R, \gamma\boldsymbol{a} + \alpha\boldsymbol{v}, \delta\boldsymbol{a} + \beta\boldsymbol{v}, s\Delta + \frac{1}{2}(\gamma\delta\|\boldsymbol{a}\|^2 + \alpha\beta\|\boldsymbol{v}\|^2) + \beta\gamma\boldsymbol{a}\cdot\boldsymbol{v}). \tag{2.9}$$

3 UIRs of the Carroll group

We now discuss UIRs of the Carroll group. Since the Carroll group is a (regular) semidirect product $K \ltimes T$, with T abelian, it follows from Mackey's Imprimitivity Theorem (see, e.g., [31, Ch. 17]) that all such representations are obtained via the method of induced representations, departing from a unitary one-dimensional representation of T and a unitary irreducible representation of its "little group". Furthermore, as shown by Rawnsley [29], these are precisely the representations arising via the geometric quantisation of the coadjoint orbits. Our approach is via induced representations, rather than the geometric quantisation of the coadjoint orbits, but the correspondence with coadjoint orbits provides a useful guide. Coadjoint orbits were described in section 2 and the method of induced representations is described in appendix A.

It is convenient to consider the universal cover of the Carroll group, which shares the coadjoint orbits with the Carroll group. From here onwards, we shall let G denote the universal cover of the Carroll group, whose maximal compact subgroup is $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, the universal cover of $\mathrm{SO}(3)$. Just as the Carroll group, its universal cover is a semi-direct product

$$G \cong (\mathrm{Spin}(3) \ltimes \mathbb{R}^3) \ltimes (\mathbb{R}^3 \oplus \mathbb{R}) = K \ltimes T, \tag{3.1}$$

where K now denotes the universal cover of the homogeneous Carroll group (isomorphic to the universal cover of the three-dimensional euclidean group) and $T \cong \mathbb{R}^4$ is the abelian normal subgroup of translations. We recall that the Casimir elements of the Carroll group are H (linear) and $\|H\boldsymbol{J} + \boldsymbol{P} \times \boldsymbol{B}\|^2$ (quartic).

3.1 Brief recap of the method of induced representations

Although more details are given in appendix A, we briefly recap the method of induced representations for a semidirect product $K \ltimes T$ with T abelian, emphasising the procedure, which we list as a sort of algorithm.

(1) Pick a complex one-dimensional unitary representation of T or, equivalently, an element $\tau \in \mathfrak{t}^*$ in the dual of its Lie algebra. Since T is abelian, all complex irreducible representations are one-dimensional and the unitary ones are given by characters

$$\chi_\tau(\exp X) = e^{i\langle \tau, X\rangle} \tag{3.2}$$

for all $X \in \mathfrak{t}$ and where $\tau \in \mathfrak{t}^*$. Therefore picking a one-dimensional unitary representation of T is equivalent to picking $\tau \in \mathfrak{t}^*$.

(2) Pick a complex unitary irreducible representation W of the stabiliser $K_\tau \subset K$ of $\tau \in \mathfrak{t}^*$. Let $\langle -, -\rangle_W$ denote a K-invariant hermitian inner product on W. The K-orbit $\mathcal{O}_\tau$ of τ is thus diffeomorphic to K/K_τ, but since T acts trivially on $\mathfrak{t}^*$, also diffeomorphic to G/H, with $H = K_\tau \ltimes T$. We will assume (and will check) that $\mathcal{O}_\tau$ admits a K-invariant measure. Although W is initially a representation of K_τ, it can be seen as a representation of H where $t \in T$ acts via the character χ_τ defined by τ.

(3) Pick a (possibly only locally defined) coset representative $\sigma : \mathcal{O}_\tau \to G$ for the orbit $\mathcal{O}_\tau \cong G/H$. Then for all $p \in \mathcal{O}_\tau$ (in the domain of σ) and all $g \in G$,

$$g^{-1}\sigma(p) = \sigma(g^{-1} \cdot p)h(g^{-1}, p), \tag{3.3}$$

which defines $h(g^{-1}, p) \in H$.

(4) Let $\psi : \mathcal{O}_\tau \to W$ and for $g \in G$, define

$$(g \cdot \psi)(p) = h(g^{-1}, p) \cdot \psi(g^{-1} \cdot p). \tag{3.4}$$

This defines a UIR of G on the Hilbert space of square-integrable functions $\mathcal{O}_\tau \to W$ relative to the inner product

$$(\psi_1, \psi_2) = \int_{\mathcal{O}_\tau} d\mu(p) \, \langle \psi_1(p), \psi_2(p)\rangle_W \,, \tag{3.5}$$

where $d\mu(p)$ is the invariant measure on $\mathcal{O}_\tau$.

We should remark that the above "algorithm" is an over-simplification and the reader is urged to read appendix A for a more detailed exposition, from where the above four points have been distilled. In particular, the "functions" $\psi : \mathcal{O}_\tau \to W$ are actually sections of a vector bundle $E_W = K \times_{K_\tau} W$ over $\mathcal{O}_\tau$ associated to the representation W of K_τ. We can also describe this vector bundle as $G \times_{K_\tau \ltimes T} W$ having extended the action of K_τ on W to the action of $K_\tau \ltimes T$ as discussed in point (2) above. In appendix A we also remind the reader that sections of E_W can be equivalently described as (Mackey) functions $K \to W$ which are equivariant under K_τ or, even, functions $G \to W$ which are equivariant under $K_\tau \ltimes T$. The representation of G carried by the sections of E_W is much more transparent when recast in the language of Mackey functions. It is in this language that the formula (3.4) (which is equation (A.9)) is arrived at, departing from equation (A.7), where $F : G \to W$ is the corresponding Mackey function.

3.2 Induced representations

We shall construct UIRs of G using the method of induced representations familiar from the case of the Poincaré group [8] and recalled above in section 3.1 and in more detail in appendix A.

Let $\tau = (\boldsymbol{p}, E) \in \mathfrak{t}^*$. This defines a unitary character χ_τ by

$$\chi_\tau(\boldsymbol{a}, s) = e^{i(\boldsymbol{p}\cdot\boldsymbol{a}+Es)}. \tag{3.6}$$

Let $K_\tau \subset K$ denote the stabiliser of τ. Even though K is the universal cover of the euclidean group, its action on $\mathfrak{t}^*$ factors through the action of the euclidean group and hence the K-orbit of τ is again the same $\mathcal{O}_\tau$ introduced in section 2. It is nevertheless K-equivariantly diffeomorphic to K/K_τ, even when the K-action is only locally effective.

We now choose a UIR W of K_τ and construct the homogeneous vector bundle

$$E_W := K \times_{K_\tau} W \to \mathcal{O}_\tau. \tag{3.7}$$

Sections of E_W are locally functions $\psi : \mathcal{O}_\tau \to W$ and they carry an action of G as in equation (3.4) which defines a UIR of G on the Hilbert space of square-integrable sections.

3.2.1 Invariant measures

The above of course depends on the existence of the invariant measure. In table 4 we see that there are three types of orbits $\mathcal{O}_\tau$: point-like orbits $\{(\boldsymbol{0}, 0)\}$, 2-spheres $S^2_{\|\boldsymbol{p}\|}$ and three-dimensional affine hyperplanes $\mathbb{A}^3_{E=E_0}$. Invariant measures are nowhere-vanishing top-rank forms on $\mathcal{O}_\tau$ which are K-invariant and, by Frobenius reciprocity, they are in bijective correspondence with K_τ-invariant elements in $\wedge^{\text{top}}(\mathfrak{k}/\mathfrak{k}_\tau)^* \cong \wedge^{\text{top}}\mathfrak{k}^0_\tau$, where $\mathfrak{k}^0_\tau \subset \mathfrak{k}^*$ is the annihilator of the Lie algebra $\mathfrak{k}_\tau$ of K_τ in the dual of the Lie algebra $\mathfrak{k}^*$ of K. We will use this to deduce that the orbits $\mathcal{O}_\tau$ in table 4 admit invariant measures.

Let J_i, B_j denote a basis for $\mathfrak{k}$ and λ^i, β^i the canonical dual basis for $\mathfrak{k}^*$. Then the coadjoint action is given by

$$\begin{aligned} \operatorname{ad}^*_{J_i}\beta^j &= \epsilon_{ijk}\beta^k \qquad & \operatorname{ad}^*_{B_i}\beta^j &= -\epsilon_{ijk}\lambda^k \\ \operatorname{ad}^*_{J_i}\lambda^j &= \epsilon_{ijk}\lambda^k \qquad & \operatorname{ad}^*_{B_i}\lambda^j &= 0. \end{aligned} \tag{3.8}$$

Ignoring the point-like orbit, we see that for the 2-sphere orbits $\mathfrak{k}_\tau$ is the span of J_3, B_i, whereas $\mathfrak{k}^0_\tau$ is the span of λ^1, λ^2 and one can check that $\lambda^1 \wedge \lambda^2 \in \wedge^2\mathfrak{k}^0_\tau$ is $\mathfrak{k}_\tau$-invariant and hence, since K_τ is connected, also K_τ-invariant. For the affine hyperplane orbits, $\mathfrak{k}_\tau$ is spanned by J_i and hence $\mathfrak{k}^0_\tau$ is spanned by β^i and it's not hard to see that $\beta^1\wedge\beta^2\wedge\beta^3 \in \wedge^3\mathfrak{k}^0_\tau$ is K_τ-invariant. We conclude that all orbits have invariant measures.

3.3 Inducing representations

We must now determine the UIRs W of K_τ, the so-called inducing representations. Depending on the orbit, as seen in figure 1, we have three possible isomorphism classes of stabilisers:

- K_τ: $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$ for the point-like orbits,

- $(\mathrm{Spin}(2) \ltimes \mathbb{R}^2) \times \mathbb{R}$ for the 2-spheres and
- $\mathrm{Spin}(3)$ for the affine hyperplanes.

The simplest case, which has already been discussed in [1], is that of the affine hyperplanes, since all irreducible representations of Spin(3) are well-known: they are finite-dimensional, unitary and isomorphic to the spin-s representation V_s for some $2s$ a non-negative integer, which is of dimension $2s+1$.

3.3.1 UIRs of $\mathrm{Spin}(3) \ltimes \mathbb{R}^3$

The three-dimensional euclidean group is again a semidirect product and hence we use again the method of induced representations. Now we let $\boldsymbol{k} \in \mathbb{R}^3$ and $\chi_{\boldsymbol{k}}$ be the unitary character defined by

$$\chi_{\boldsymbol{k}}(\boldsymbol{v}) := e^{i\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{3.9}$$

The group Spin(3) acts on such characters as $\chi_{\boldsymbol{k}} \mapsto \chi_{R\boldsymbol{k}}$, where in the expression $R\boldsymbol{k}$ we understand that $R \in \mathrm{Spin}(3)$ acts through its projection to SO(3). If $\boldsymbol{k} = \boldsymbol{0}$ (so that $\chi_{\boldsymbol{k}} \equiv 1$) the induced representation is then simply a UIR of Spin(3), which as mentioned above, is one of the spin-s representations V_s.

If $\boldsymbol{k} \neq \boldsymbol{0}$, the Spin(3)-orbit of $\chi_{\boldsymbol{k}}$ is a 2-sphere with typical stabiliser $\mathrm{U}(1) \subset \mathrm{SU}(2) \cong \mathrm{Spin}(3)$. The UIRs of U(1) are indexed by the integers. If $\lambda \in \mathrm{U}(1)$, or equivalently $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the representation indexed by $n \in \mathbb{Z}$ is the one-dimensional complex representation where λ acts by multiplication by λ^n. Let us call that representation $\mathbb{C}_n$.

We now define complex line bundles over the 2-sphere associated to such representations:

$$\mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbb{C}_n \to S^2. \tag{3.10}$$

We can identify these bundles as follows. First of all notice that we can identify SU(2) with the unit sphere in $\mathbb{C}^2$. Indeed if $(z_1, z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$, we form the special unitary matrix

$$g(z_1, z_2) := \begin{pmatrix} z_1 & \overline{z}_2 \\ -z_2 & \overline{z}_1 \end{pmatrix} \tag{3.11}$$

and every special unitary matrix is of this form. The 2-sphere is the complex projective line, which is the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We can restrict to the 3-sphere in $\mathbb{C}^2$ and quotient by the action of $\mathrm{U}(1) \subset \mathbb{C}$ given by right multiplication as follows:

$$g(z_1, z_2) \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} = \begin{pmatrix} z_1 & \overline{z}_2 \\ -z_2 & \overline{z}_1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} = \begin{pmatrix} \lambda z_1 & \overline{\lambda}\overline{z}_2 \\ -\lambda z_2 & \overline{\lambda}\overline{z}_1 \end{pmatrix} = g(\lambda z_1, \lambda z_2), \tag{3.12}$$

where $|\lambda| = 1$. Sections of the homogeneous line bundle $\mathrm{SU}(2) \times_{\mathrm{U}(1)} \mathbb{C}_n \to S^2$ are U(1)-equivariant functions $f : \mathrm{SU}(2) \to \mathbb{C}$ such that $f(gh) = h^{-1} \cdot f(g)$, or equivalently complex-valued functions of z_1, z_2 such that $f(\lambda z_1, \lambda z_2) = \lambda^{-n} f(z_1, z_2)$. These are the sections of the line bundle $\mathscr{O}(-n)$ over $\mathbb{CP}^1$. We may define an inner product on the space of sections by integrating the pointwise inner product on $\mathbb{C}_n$ against the SU(2)-invariant measure given by the volume form of a round metric on the 2-sphere. The resulting induced representation of the three-dimensional euclidean group is then carried by the square-integrable sections of $\mathscr{O}(-n)$ over $\mathbb{CP}^1$ for any $n \in \mathbb{Z}$. We will discuss them in more detail in section 3.4.2.

3.3.2 UIRs of $(\mathrm{Spin}(2) \ltimes \mathbb{R}^2) \times \mathbb{R}$

The stabiliser now is isomorphic to $(\mathrm{Spin}(2) \ltimes \mathbb{R}^2) \times \mathbb{R}$, where Spin(2) can be identified with the U(1) subgroup of SU(2) discussed in the previous section. Indeed, the action of SU(2) on $\mathbb{R}^3$ is the adjoint representation, which is self-dual and hence isomorphic to the coadjoint representation. Choosing $\boldsymbol{p} \in \mathbb{R}^3$ to correspond to the Lie algebra element

$$\begin{pmatrix} ip & 0 \\ 0 & -ip \end{pmatrix} \tag{3.13}$$

we see that the stabiliser of this element in SU(2) is

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \,\middle|\, |\lambda| = 1 \right\} \cong \mathrm{U}(1). \tag{3.14}$$

Irreducible representations of $(\mathrm{Spin}(2) \ltimes \mathbb{R}^2) \times \mathbb{R}$ are tensor products of irreducible representations of $\mathrm{Spin}(2) \ltimes \mathbb{R}^2$ and of $\mathbb{R}$. Complex irreducible representations of an abelian Lie group are one-dimensional. The unitary irreducible representations of $\mathbb{R}$ are complex one-dimensional and given by unitary characters labelled by a real number $w \in \mathbb{R}$, where $\chi_w(s) = \exp(iws)$ for all $s \in \mathbb{R}$. It is however more convenient notationally to consider $\mathrm{Spin}(2) \ltimes \mathbb{R}^3$ even when Spin(2) leaves invariant the third component of the vectors in $\mathbb{R}^3$.

Let us then determine the unitary irreducible representations of $\mathrm{Spin}(2) \ltimes \mathbb{R}^3$. Being also a semidirect product, we apply again the method of induced representations. Let again $\boldsymbol{k} \in \mathbb{R}^3$ and $\chi_{\boldsymbol{k}}$ be the unitary character:

$$\chi_{\boldsymbol{k}}(\boldsymbol{v}) := e^{i\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{3.15}$$

The group Spin(2) acts on such characters by restricting the adjoint action of SU(2). If we take $\boldsymbol{k} = (k_1, k_2, k_3)$, then $\lambda \cdot \boldsymbol{k} = \boldsymbol{k}'$ where $\boldsymbol{k}' = (k_1', k_2', k_3')$ with $k_3' = k_3$ and

$$k_1' + ik_2' = \lambda^2(k_1 + ik_2) \tag{3.16}$$

as shown by the conjugation, where we have used that $\overline{\lambda} = \lambda^{-1}$:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} k_3 & k_1 + ik_2 \\ k_1 - ik_2 & -k_3 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} k_3 & \lambda^2(k_1 + ik_2) \\ \lambda^{-2}(k_1 - ik_2) & -k_3 \end{pmatrix}. \tag{3.17}$$

Let us use the notation $\boldsymbol{k}^\perp = (k_1, k_2, 0)$ to denote the component of $\boldsymbol{k}$ orthogonal to $\boldsymbol{p}$. If $\boldsymbol{k}^\perp = \boldsymbol{0}$, then its Spin(2)-orbit is a single point. The stabiliser is Spin(2) itself and hence we choose a UIR $\mathbb{C}_n$ of Spin(2), as already discussed above.

If $\boldsymbol{k}^\perp \neq \boldsymbol{0}$, the Spin(2)-orbit of $\chi_{\boldsymbol{k}}$ is a circle and the stabiliser consists of all those $\lambda \in \mathrm{Spin}(2)$ with $\lambda^2 = 1$; that is the subgroup corresponding to ± 1, which is of course isomorphic to $\mathbb{Z}_2$. There are two irreducible representations of $\mathbb{Z}_2$, both unitary and one-dimensional, depending on whether -1 acts as 1 or as -1: they are call, respectively, the trivial and sign representations and we will denote them by $\mathbb{C}_\pm$, respectively. The associated homogeneous line bundles have an equivalent characterisation which may be more familiar. The group Spin(2) is the total space of the spin bundle over the circle

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$\mathrm{Spin}(2)/\mathbb{Z}_2$ and the homogeneous line bundles associated to the representations $\mathbb{C}_\pm$ are the corresponding spinor bundles $\Sigma_\pm \to S^1$. Sections of $\Sigma_\pm$ are typically known as Ramond spinors (for $+$) and Neveu-Schwarz spinors (for $-$) on the circle. The representation space of $\mathrm{Spin}(2) \ltimes \mathbb{R}^2$ is then the Hilbert space $L^2(S^1, \Sigma_\pm)$ of square-integrable spinor fields on the circle. But of course we are interested in $\mathrm{Spin}(2) \ltimes \mathbb{R}^3$. The third component acts via a character as explained above and hence we get a (trivial) line bundle $L_w \to S^1$ by which we may twist the spinors.

In summary, the UIRs of $(\mathrm{Spin}(2) \ltimes \mathbb{R}^2) \times \mathbb{R}$ come in several types:

- one-dimensional representations $\mathbb{C}_n \otimes \mathbb{C}_w$ where $n \in \mathbb{Z}$ and $w \in \mathbb{R}$; and
- infinite-dimensional representations $L^2(S^1, \Sigma_\pm \otimes L_w)$ for $w \in \mathbb{R}$, where L_w is a trivial line bundle over S^1.

We will discuss them in more detail in section 3.4.2.

3.4 UIRs of the Carroll group

We may summarise the above discussion by listing the UIRs of (the universal cover of) the Carroll group and in so doing we shall give them names.

3.4.1 UIRs with $E_0 \neq 0$

For representations with the value E_0 of H nonzero, the Hilbert space consists of the square-integrable functions $\mathbb{A}^3 \to V_s$, with $\mathbb{A}^3 \subset \mathfrak{t}^*$ the affine hyperplane with $E = E_0$ and V_s the complex spin-s representation of $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ of dimension $2s+1$, relative to the inner product

$$(\psi_1, \psi_2) = \int_{\mathbb{A}^3} d^3p \, \langle \psi_1(\boldsymbol{p}), \psi_2(\boldsymbol{p}) \rangle_{V_s} \,, \tag{3.18}$$

where $\langle -, - \rangle_{V_s}$ is an invariant hermitian inner product on V_s. These representations were already discussed in the original work of Lévy-Leblond [1]. They admit field-theoretic realisations on Carroll spacetime as we will review below in section 4.2. We shall call denote these UIRs by $\amalg(s, E_0)$ with the understanding that $E_0 \neq 0$.

Let us write the explicit action of the Carroll group G on these representations. The orbit $\mathcal{O}_\tau \subset \mathfrak{t}^*$ is the affine hyperplane $\mathbb{A}^3 \subset \mathfrak{t}^*$ consisting of points $(E_0, \boldsymbol{p})$ where $\boldsymbol{p} \in \mathbb{R}^3$. Let us choose the point $(E_0, \boldsymbol{0})$ as the origin and let us choose a coset representative $\sigma(\boldsymbol{p}) \in K$ so that $\sigma(\boldsymbol{p}) \cdot (E_0, \boldsymbol{0}) = (E_0, \boldsymbol{p})$. A quick calculation shows that $\sigma(\boldsymbol{p}) = \exp(-\frac{1}{E_0}\boldsymbol{p} \cdot \boldsymbol{B})$ works. In this paper and in contrast with Part I, we work with a different, more standard, parametrisation of the Carroll group. Let us factorise $g = tk$, with $t \in T$ and $k \in K$ as follows:

$$g(R, \boldsymbol{v}, \boldsymbol{a}, s) = \underbrace{e^{sH + \boldsymbol{a}\cdot\boldsymbol{P}}}_{t \in T} \underbrace{e^{\boldsymbol{v}\cdot\boldsymbol{B}} R}_{k \in K} . \tag{3.19}$$

We have that with such g,

$$(g \cdot \psi)(\boldsymbol{p}) = \chi_{(E_0, \boldsymbol{p})}(t) (k \cdot \psi)(\boldsymbol{p}), \tag{3.20}$$

where $\chi_{(E_0,\boldsymbol{p})}$ is the character given by

$$\chi_{(\boldsymbol{p},E_0)}(t) = e^{i(E_0 s + \boldsymbol{a}\cdot\boldsymbol{p})} \tag{3.21}$$

and we calculate

$$k^{-1}\sigma(\boldsymbol{p}) = \sigma(R^{-1}(\boldsymbol{p} + E_0\boldsymbol{v}))R^{-1}, \tag{3.22}$$

so that from equation (3.4) we arrive at the explicit expression for the action of the group element $g(R, \boldsymbol{v}, \boldsymbol{a}, s) \in G$ on $\psi \in L^2(\mathbb{A}^3, V_s)$:

$$(g \cdot \psi)(\boldsymbol{p}) = e^{i(E_0 s + \boldsymbol{p}\cdot\boldsymbol{a})}\rho(R)\psi(R^{-1}(\boldsymbol{p} + E_0\boldsymbol{v}))\,, \tag{3.23}$$

where $R \mapsto \rho(R)$ denotes the spin-s representation of Spin(3). It is understood that we are on the hyperplane where the energy is restricted to $E = E_0$.

Let us write down the hermitian operators corresponding to angular momentum $\hat{\boldsymbol{J}}$, energy $\hat{H}$, momentum $\hat{\boldsymbol{p}}$ and centre-of-mass $\hat{\boldsymbol{B}}$. They are related to the Lie algebra generators (2.1) via multiplication by i, explicitly $X_{\text{LieAlg}} = i\hat{X}$. For the representation at hand they are given by

$$\hat{\boldsymbol{J}} = -i\boldsymbol{p} \times \frac{\partial}{\partial \boldsymbol{p}} + \hat{\boldsymbol{S}} \qquad \hat{\boldsymbol{B}} = -iE_0\frac{\partial}{\partial \boldsymbol{p}} \qquad \hat{H} = E_0 \qquad \hat{\boldsymbol{P}} = \boldsymbol{p}\,. \tag{3.24}$$

Here $\hat{\boldsymbol{S}}$ are the infinitesimal generators of the spin-s representation $\rho(R)$. Together with $\hat{H}$ we can use them to uniquely label the $\mathbb{I}(s, E_0)$ representation since

$$\hat{H} = E_0 \qquad\qquad \hat{\boldsymbol{S}}^2 = s(s+1)\,, \tag{3.25}$$

which are multiples of the identity.

For massive carrollions we can also define a position operator $\hat{\boldsymbol{X}}$ [1]

$$\hat{\boldsymbol{X}} = \frac{1}{E_0}\hat{\boldsymbol{B}} \tag{3.26}$$

which transforms as expected under rotations and spatial translations. This definition also agrees with the intuition that the centre of mass of a massive Carroll particle is the energy multiplied by the position (classically written as $\boldsymbol{k} = E_0\boldsymbol{x}$ as in, e.g., Part I, section 3). When evaluated on the wavefunctions we recover the canonical commutation relations

$$[\hat{X}_i, \hat{P}_j] = -i\delta_{ij}\,. \tag{3.27}$$

We could have chosen to diagonalise not the momentum operator $\hat{\boldsymbol{P}}$, but with respect to the centre-of-mass $\hat{\boldsymbol{B}}$. In this case this leads us to "boost" or centre-of-mass wavefunctions on which the symmetries act as

$$(g \cdot \tilde{\psi})(\boldsymbol{k}) = e^{i(E_0 s + \boldsymbol{k}\cdot\boldsymbol{v})}\rho(R)\tilde{\psi}(R^{-1}(\boldsymbol{k} - E_0\boldsymbol{a}))\,. \tag{3.28}$$

with

$$\hat{\boldsymbol{J}} = -i\boldsymbol{k} \times \frac{\partial}{\partial \boldsymbol{k}} + \hat{\boldsymbol{S}} \qquad \hat{\boldsymbol{B}} = \boldsymbol{k} \qquad \hat{H} = E_0 \qquad \hat{\boldsymbol{P}} = iE_0\frac{\partial}{\partial \boldsymbol{k}}\,. \tag{3.29}$$

The momentum and boost eigenstates can be shown to be related via Fourier transforms

$$\tilde{\psi}(\boldsymbol{k}) = \int d^3p\, e^{-\frac{i}{E_0}\boldsymbol{k}\cdot\boldsymbol{p}}\psi(\boldsymbol{p}) \qquad \psi(\boldsymbol{p}) = \frac{1}{(2\pi|E_0|)^3}\int d^3k\, e^{\frac{i}{E_0}\boldsymbol{k}\cdot\boldsymbol{p}}\tilde{\psi}(\boldsymbol{k}) \tag{3.30}$$

and the inner product is given by

$$(\tilde{\psi}_1, \tilde{\psi}_2) = \int_{\mathbb{A}^3} d^3k \left\langle \tilde{\psi}_1(\boldsymbol{k}), \tilde{\psi}_2(\boldsymbol{k})\right\rangle_{V_s}. \tag{3.31}$$

Here $\mathbb{A}^3$ is now the $E = E_0$ hyperplane but in $(E, \boldsymbol{k})$ space. The relation between the momentum and boost eigenstates is analogous to the relation between momentum and position space eigenstates in nonrelativistic (galilean) quantum mechanics. This can be traced back to the commutation relation $[B_i, P_j] = \delta_{ij}H$ which mirrors the canonical commutation relation between position and momentum operators with the energy playing the rôle of $\hbar$. This also implies that the momentum and the centre-of-mass representations are Fourier transforms (in $\boldsymbol{p}$-$\boldsymbol{k}$ space) of each other.

Another way to see this relation between the momentum and boost basis is to look at the particle action, as in section 3.1 in Part I, for instance. The kinetic term in the canonical lagrangian can be written as being proportional to $\boldsymbol{p}\cdot\dot{\boldsymbol{k}}$. The analogous choice in the path integral quantisation approach is then the choice to calculate amplitudes either with regard to eigenstates of $\hat{\boldsymbol{P}}$ or $\hat{\boldsymbol{B}}$.

3.4.2 UIRs with $E_0 = 0$

There are four classes of UIRs with $E_0 = 0$:

(a) any finite-dimensional representation V_s of Spin(3) with all other generators acting trivially;

(b) the square integrable sections of $\mathscr{O}(-n)$ over the 2-sphere for any $n \in \mathbb{Z}$, with the translations of the Carroll group acting trivially;

(c) the square-integrable sections of a Hilbert bundle over the 2-sphere, whose fibres are the square-integrable spinors (with respect to either of the two spin structures) on the circle twisted by a trivial line bundle L_w: $L^2(S^1, \Sigma_\pm \otimes L_w)$;

(d) and the square-integrable sections of the line bundle over the 2-sphere associated to the one-dimensional representation $\mathbb{C}_n \otimes \mathbb{C}_w$ of $(\mathrm{Spin}(2) \ltimes \mathbb{R}^2) \times \mathbb{R}$.

We shall now discuss them in some detail and will discuss the possible field theoretical realisations of some of these representations in section 4.3.

Representations of class (a) require no further discussion. We shall denote them by $\mathrm{I}(s)$ with the understanding that $2s \in \mathbb{N}_0$.

Representations of classes (b) and (d). Let us first of all consider representations of class (d), under the assumption that $w = 0$, since as we shall see in section 4.3, these are the ones which can be realised as finite-component fields on Carroll spacetime. We take $\tau = (0, \boldsymbol{p})$ with $\boldsymbol{p} = (0, 0, p)$ and $p > 0$. The stabiliser K_τ consists of the boosts and U(1) subgroup of SU(2) consisting of diagonal matrices and $\mathcal{O}_\tau$ is the 2-sphere of radius p in $\mathbb{R}^3$. If we think of the sphere as the extended complex plane, we can effectively work in the complex plane. The round metric on the unit sphere pulls back to the Fubini-Study metric (up to a factor of 4):

$$g_{FS} = \frac{4dzd\overline{z}}{(1+|z|^2)^2} \tag{3.32}$$

whose associated volume form is

$$\omega = \frac{2idz \wedge d\overline{z}}{(1+|z|^2)^2}. \tag{3.33}$$

As a coset representative $\sigma : \mathbb{C} \to \mathrm{SU}(2)$ we may take[8]

$$\sigma(z) = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} z & -1 \\ 1 & \overline{z} \end{pmatrix}. \tag{3.34}$$

Notice that the map $\mathcal{O}_\tau \to \mathbb{C}$ is the stereographic projection from τ. Hence the complex plane $\mathbb{C}$ parametrises the orbit $\mathcal{O}_\tau$ excised of the actual point τ, which we only recover in the limit $z \to \infty$. (This may seem a little strange, but it is fine.) The action of $\sigma(z)$ on $\boldsymbol{p} = (0,0,p)$ results in $\boldsymbol{\pi}(z) = \sigma(z) \cdot \boldsymbol{p}$. To work out the expression for $\boldsymbol{\pi}(z)$, let us identify $\mathbb{R}^3$ with the space of hermitian traceless matrices in such a way that $\boldsymbol{p} = (p_1, p_2, p_3)$ is represented by the matrix

$$\begin{pmatrix} p_3 & p_1 + ip_2 \\ p_1 - ip_2 & -p_3 \end{pmatrix} \tag{3.35}$$

and the action of SU(2) on such hermitian matrices is via matrix conjugation. This is a linear action which preserves the trace (which is zero) and the determinant (which is $-\|\boldsymbol{p}\|^2$). For the chosen τ, we have $\boldsymbol{p} = (0,0,p)$, so that the matrix corresponding to $\boldsymbol{\pi}(z)$ is given by

$$\begin{pmatrix} \pi_3(z) & \pi_1(z) + i\pi_2(z) \\ \pi_1(z) - i\pi_2(z) & -\pi_3(z) \end{pmatrix} = \frac{1}{1+|z|^2} \begin{pmatrix} z & -1 \\ 1 & \overline{z} \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \begin{pmatrix} \overline{z} & 1 \\ -1 & z \end{pmatrix}, \tag{3.36}$$

resulting in

$$\pi_1(z) = \frac{2p\,\mathrm{Re}(z)}{1+|z|^2}, \qquad \pi_2(z) = \frac{2p\,\mathrm{Im}(z)}{1+|z|^2} \qquad \text{and} \qquad \pi_3(z) = \frac{(|z|^2-1)p}{1+|z|^2}. \tag{3.37}$$

As expected, it satisfies $\|\boldsymbol{\pi}(z)\|^2 = p^2$.

We consider functions $\psi : \mathbb{C} \to \mathbb{C}_n$, where $\mathbb{C}_n$ is a copy of the complex numbers with n reminding us how U(1) acts. We introduce the inner product

$$\langle \psi_1, \psi_2 \rangle := \int_{\mathbb{C}} \frac{2idz \wedge d\overline{z}}{(1+|z|^2)^2} \overline{\psi_1(z)} \psi_2(z) \tag{3.38}$$

[8]Here and in the sequel we think of z as a point in the complex plane and not as a holomorphic coordinate. Hence the notation $\sigma(z)$ or $\psi(z)$ is not meant to denote a holomorphic function, but simply a smooth function on the complex plane, as can be seen from the explicit form of $\sigma(z)$.

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and we let $\mathscr{H}$ denote the Hilbert space of square-integrable such functions. This space carries a UIR of the Carroll group which we now exhibit. Let $g = g(R, \boldsymbol{v}, \boldsymbol{a}, s) \in G$ be given as in equation (3.19). Then writing $g = tk$,

$$(g \cdot \psi)(z) = \chi_{(0,\boldsymbol{\pi}(z))}(t)(k \cdot \psi)(z). \tag{3.39}$$

Let us write $k = R\beta$ with $R \in \mathrm{SU}(2)$ and β a boost. Then

$$k^{-1}\sigma(z) = \beta^{-1}R^{-1}\sigma(z). \tag{3.40}$$

We will first work out $R^{-1}\sigma(z)$. Let's take

$$R = \begin{pmatrix} \eta & \xi \\ -\overline{\xi} & \overline{\eta} \end{pmatrix} \implies R^{-1} = \begin{pmatrix} \overline{\eta} & -\xi \\ \overline{\xi} & \eta \end{pmatrix} \qquad \text{with} \qquad |\eta|^2 + |\xi|^2 = 1. \tag{3.41}$$

A short calculation shows that

$$R^{-1}\sigma(z) = \sigma(w)\lambda(R, z), \tag{3.42}$$

where

$$w = \frac{\overline{\eta}z - \xi}{\eta + \overline{\xi}z} \qquad \text{and} \qquad \lambda(R, z) = \frac{\eta + \overline{\xi}z}{\left|\eta + \overline{\xi}z\right|}. \tag{3.43}$$

Notice that $z \mapsto w$ is the fractional linear transformation associated to R^{-1}, as expected. Therefore,

$$k^{-1}\sigma(z) = \beta^{-1}\sigma(w)\lambda(R, z) = \sigma(w)\underbrace{\sigma(w)^{-1}\beta^{-1}\sigma(w)}_{\beta'(w)}\lambda(R, z), \tag{3.44}$$

where $\beta'(w)$ is another boost. Since boosts act trivially on the inducing representation $\mathbb{C}_n$ and $\lambda \in \mathrm{U}(1)$ acts like λ^n, we arrive at the following action of $g = g(R, \boldsymbol{v}, \boldsymbol{a}, s)$ on $\psi : \mathbb{C} \to \mathbb{C}_n$:

$$(g \cdot \psi)(z) = e^{i\boldsymbol{a}\cdot\boldsymbol{\pi}(z)} \left(\frac{\eta + \overline{\xi}z}{|\eta + \overline{\xi}z|}\right)^{-n} \psi\left(\frac{\overline{\eta}z - \xi}{\eta + \overline{\xi}z}\right), \tag{3.45}$$

with $\boldsymbol{\pi}(z)$ given by equation (3.37) and $R \in \mathrm{SU}(2)$ is given by equation (3.41). Notice that time translations and carrollian boosts act trivially on momenta when the energy vanishes, so there is no s and $\boldsymbol{v}$ on the right-hand side. We denote these representations by $\mathrm{III}(n, p)$ with the understanding that $n \in \mathbb{Z}$ and $p > 0$.

By applying an automorphism $\varphi : G \to G$, we may obtain other representations from this one simply by precomposing: $\rho : G \to \mathrm{U}(\mathscr{H})$ changes to $\rho \circ \varphi : G \to \mathrm{U}(\mathscr{H})$. The outer automorphism $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$ acts on G as in equation (2.9) and hence from equation (3.45) we read off the following representation

$$(g \cdot \psi)(z) = e^{i(\delta\boldsymbol{a} + \beta\boldsymbol{v})\cdot\boldsymbol{\pi}(z)} \left(\frac{\eta + \overline{\xi}z}{|\eta + \overline{\xi}z|}\right)^{-n} \psi\left(\frac{\overline{\eta}z - \xi}{\eta + \overline{\xi}z}\right), \tag{3.46}$$

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By taking $\delta = 0$ (and hence $\beta \neq 0$) we obtain the representation of class (b) in the list at the start of this section, since in this case the translations act trivially. We denote those representations by $\mathbb{III}'(n,k)$, with the understanding that $n \in \mathbb{Z}$ and $k > 0$.

Similarly, taking δ and β both nonzero, we obtain the representation of class (d) with $w = \beta$. We denote these representations by $\mathbb{IV}_\pm(n,p,k)$ where the sign is the sign of w, and with the understanding that $n \in \mathbb{Z}$ and $p,k > 0$. It may seem a little odd that whereas twisting by an automorphism results in a representation with the same underlying vector space, our description of the representations of class (d) at the start of this section exhibits such representations in terms of sections of a different homogeneous line bundle. The conjectural resolution is that these representations are unitarily equivalent.

Representations of class (c). It remains to discuss representations in class (c). The description which follows by adhering to the method of induced representations seems a little exotic, since the representation is described as being carried by sections of an infinite-rank Hilbert bundle over the 2-sphere. We will show, however, that there is an equivalent description of these representations as honest functions on the round 3-sphere with values in a one-dimensional representation of the nilpotent subgroup of the Carroll group generated by boosts and translations.

To see this it is perhaps convenient to briefly recapitulate how one might arrive at such a description. We start by following the description of the representation as square-integrable sections of a Hilbert bundle over the 2-sphere. The fibre of the Hilbert bundle in an infinite-dimensional Hilbert space $\mathscr{H}^\pm_{k,\theta}$ which is a UIR of $\mathrm{Spin}(2) \ltimes \mathbb{R}^3$ and is described as follows. We pick a unitary character $\chi_{\boldsymbol{k}} : \mathbb{R}^3 \to \mathrm{U}(1)$ of $\mathbb{R}^3$ given by $\chi_{\boldsymbol{k}}\left(e^{\boldsymbol{v}\cdot\boldsymbol{B}}\right) = e^{i\boldsymbol{k}\cdot\boldsymbol{v}}$, with our chosen $\boldsymbol{k} = (k\sin\theta, 0, k\cos\theta)$ with $k > 0$ and $\theta \in (0,\pi)$. Remember that $\mathrm{Spin}(2)$ is the subgroup of $\mathrm{SU}(2)$ consisting of diagonal matrices, so they are labelled by a complex number ζ, say, of unit modulus. The orbit of the character $\chi_{\boldsymbol{k}}$ under $\mathrm{Spin}(2)$ is a circle of characters $\chi_{\boldsymbol{k}(\zeta)}$ where $\boldsymbol{k}(\zeta) = (k_1(\zeta), k_2(\zeta), k\cos\theta)$, where

$$k_1(\zeta) + ik_2(\zeta) = \zeta^2 k\sin\theta, \tag{3.47}$$

as was seen in equation (3.16). The stabiliser of $\boldsymbol{k}$ in $H := \mathrm{Spin}(2) \ltimes \mathbb{R}^3$ is the subgroup $H_{\boldsymbol{k}} := \mathbb{Z}_2 \times \mathbb{R}^3$ consisting of elements $e^{\boldsymbol{v}\cdot\boldsymbol{B}}(\pm\mathbb{1})$, with $\mathbb{1}$ the identity matrix. The UIRs of $H_{\boldsymbol{k}}$ are complex one-dimensional $\mathbb{C}_\pm \otimes \mathbb{C}_{\boldsymbol{k}}$, where $\mathbb{C}_\pm$ are the trivial and sign representations of $\mathbb{Z}_2$, respectively, and $\mathbb{C}_{\boldsymbol{k}}$ is the one-dimensional unitary representation of $\mathbb{R}^3$ given by with character $\chi_{\boldsymbol{k}}$. Let us choose a coset representative $\nu : S^1 \to \mathrm{Spin}(2)$, sending $\zeta \mapsto \nu(\zeta)$, so that $\nu(\zeta)\boldsymbol{k} = \boldsymbol{k}(\zeta)$. One such possibility is

$$\nu(\zeta) = \begin{pmatrix} \zeta^{1/2} & 0 \\ 0 & \zeta^{-1/2} \end{pmatrix}, \tag{3.48}$$

for some choice of square root function. Then we may describe $\psi \in \mathscr{H}^\pm_{k,\theta}$ as complex-valued functions $\psi(\zeta)$ on the circle with the inner product

$$\langle \psi_1, \psi_2 \rangle = \int_{S^1} \frac{d\zeta}{i\zeta} \overline{\psi_1(\zeta)} \psi_2(\zeta). \tag{3.49}$$

The unitary action of $h = e^{\boldsymbol{v}\cdot\boldsymbol{B}}\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix} \in \mathrm{Spin}(2) \ltimes \mathbb{R}^3$ on ψ is given by (in the case of Ramond spinors, for definiteness)

$$(h\cdot\psi)(\zeta) = e^{i\boldsymbol{v}\cdot\boldsymbol{k}(\zeta)}\psi(\lambda^{-2}\zeta). \tag{3.50}$$

The UIR of the Carroll group is carried by sections over the Hilbert bundle over the 2-sphere $S^2_p \in \mathbb{R}^3$ where $p = \|\boldsymbol{p}\| > 0$, associated to the UIR $\mathscr{H}^{\pm}_{k,\theta}$ of the stabiliser of $\boldsymbol{p} = (0,0,p)$. We denote these UIRs by $\mathbb{V}_{\pm}(p,k,\theta)$ with the understanding that $p,k>0$ and $\theta \in (0,\pi)$.

Locally, relative to a stereographic coordinate $z \in \mathbb{C}$ for the sphere, they are defined as functions $\psi : \mathbb{C} \to \mathscr{H}^{\pm}_{k,\theta}$. This means that $\psi(z) \in \mathscr{H}^{\pm}_{k,\theta}$ so it is itself a function $\zeta \mapsto \psi(z)(\zeta)$ as described above. Such a function is the result of currying a function $\Psi : \mathbb{C}\times S^1 \to \mathbb{C}$; that is, $\psi(z) = \Psi(z,-)$. In this description, translations and boosts are simultaneously diagonalised, which is possible for representations with $E = 0$, since in that case they commute. We therefore have that

$$\begin{aligned}\left(e^{\boldsymbol{a}\cdot\boldsymbol{P}}\cdot\Psi\right)(z,\zeta) &= e^{i\boldsymbol{a}\cdot\boldsymbol{\pi}(z)}\Psi(z,\zeta)\\ \left(e^{\boldsymbol{v}\cdot\boldsymbol{B}}\cdot\Psi\right)(z,\zeta) &= e^{i\boldsymbol{v}\cdot\boldsymbol{k}(z,\zeta)}\Psi(z,\zeta),\end{aligned} \tag{3.51}$$

where $\boldsymbol{\pi}(z) = \sigma(z)\boldsymbol{p}$ and $\boldsymbol{k}(z,\zeta) = \sigma(z)\boldsymbol{k}(\zeta)$. Notice that $\boldsymbol{\pi}(z)\cdot\boldsymbol{k}(z,\zeta) = \sigma(z)\boldsymbol{p}\cdot\sigma(z)\boldsymbol{k}(\zeta) = \boldsymbol{p}\cdot\boldsymbol{k}(\zeta) = pk\cos\theta$. Time translations act trivially, of course, on massless representations, so it remains to discuss the action of rotations. Let $R \in \mathrm{SU}(2)$ be given by equation (3.41). Then from equation (3.42) we have that

$$(R\cdot\Psi)(z,\zeta) = \lambda(R,z)^{-1}\Psi(w,\zeta), \tag{3.52}$$

where $\lambda(R,z)$ and w are given in equation (3.43). Using the action of Spin(2) on the circle S^1, we arrive at

$$(R\cdot\Psi)(z,\zeta) = \Psi\left(\frac{\overline{\eta}z-\xi}{\eta+\overline{\xi}z}, \left(\frac{\eta+\overline{\xi}z}{|\eta+\overline{\xi}z|}\right)^2\zeta\right). \tag{3.53}$$

Putting it all together we arrive at the action of $g = g(R,\boldsymbol{v},\boldsymbol{a},s)$ in equation (3.19) in this representation

$$(g\cdot\Psi)(z,\zeta) = e^{i(\boldsymbol{a}\cdot\boldsymbol{\pi}(z)+\boldsymbol{v}\cdot\boldsymbol{k}(z,\zeta))}\Psi\left(\frac{\overline{\eta}z-\xi}{\eta+\overline{\xi}z}, \left(\frac{\eta+\overline{\xi}z}{|\eta+\overline{\xi}z|}\right)^2\zeta\right). \tag{3.54}$$

Finally, this representation is unitary relative to the hermitian inner product given by

$$\langle\Psi_1,\Psi_2\rangle = \int_{\mathbb{C}}\frac{2dz\wedge d\overline{z}}{(1+|z|^2)^2}\int_{S^1}\frac{d\zeta}{\zeta}\overline{\Psi_1(z,\zeta)}\Psi_2(z,\zeta). \tag{3.55}$$

We will now describe these representations in a simpler fashion.

3.4.3 A simpler description of the massless UIRs

As shown in appendix B, the domain of integration $\mathbb{C} \times S^1$ in equation (3.55) is a Hopf chart on the 3-sphere and the measure of integration is nothing but the volume form of a round metric on the 3-sphere, given in equation (B.10). This suggests very strongly an equivalent (and perhaps less exotic) characterisation of these UIRs, which we now describe.

Let $N \subset G$ denote the nilpotent subgroup of the Carroll group generated by boosts and translations and let $\chi : N \to \mathrm{U}(1)$ be defined by

$$\chi\left(e^{sH+\boldsymbol{a}\cdot\boldsymbol{P}}e^{\boldsymbol{v}\cdot\boldsymbol{B}}\right) = e^{i(\boldsymbol{a}\cdot\boldsymbol{p}+\boldsymbol{v}\cdot\boldsymbol{k})}, \tag{3.56}$$

where $\boldsymbol{p} = (0,0,p)$ and $\boldsymbol{k} = (k\sin\theta, 0, k\cos\theta)$, as above. Although N is not abelian, χ does define a one-dimensional representation since in any representation of N where H acts trivially, N acts effectively as an abelian group. Notice that $G/N \cong \mathrm{SU}(2)$, which is diffeomorphic to the 3-sphere. The character χ of N defines a homogeneous line bundle $L_\chi := G \times_N \mathbb{C}_\chi$ on S^3, where $\mathbb{C}_\chi$ is a copy of $\mathbb{C}$ where N acts via χ. Every smooth line bundle over S^3 is trivial,[9] and hence sections of L_χ are just smooth functions $S^3 \to \mathbb{C}_\chi$. Let $\mathscr{H} = L^2(S^3, \mathbb{C}_\chi)$ denote the square-integrable such functions relative to the measure coming from a round metric. We define a coset representative $\sigma : S^3 \to G$ to be the identification $S^3 \cong \mathrm{SU}(2) \subset G$. So we can actually write $\Psi(S)$ with $S \in \mathrm{SU}(2)$ and if we write $g = g(R, \boldsymbol{a}, \boldsymbol{v}, s)$ as in equation (2.8), its action on such functions is given, as described in appendix A (see, e.g., equation (A.4)), by

$$(g \cdot \Psi)(S) = e^{i(\boldsymbol{a}\cdot S\boldsymbol{p}+\boldsymbol{v}\cdot S\boldsymbol{k})}\Psi(R^{-1}S), \tag{3.57}$$

which is manifestly unitary relative to the inner product

$$\langle \Psi_1, \Psi_2 \rangle = \int_{S^3} d\mu(S)\overline{\Psi_1(S)}\Psi_2(S), \tag{3.58}$$

where $d\mu$ is a bi-invariant Haar measure on $\mathrm{SU}(2)$. Restricting to one of the Hopf charts $\mathbb{C}\times S^1$ as described in appendix B, we indeed recover the action of G given by equation (3.54) and the inner product in equation (3.55). There is one small detail: the representation just constructed is not actually irreducible. Why? Because of the action of the centre of $\mathrm{SU}(2)$. Given any function $\Psi : \mathrm{SU}(2) \to \mathbb{C}$ we can decompose it into a sum $\Psi_+ + \Psi_-$ of such functions where $\Psi_\pm(-S) = \pm\Psi_\pm(S)$; namely,

$$\Psi(S) = \underbrace{\frac{1}{2}(\Psi(S) + \Psi(-S))}_{=:\Psi_+(S)} + \underbrace{\frac{1}{2}(\Psi(S) - \Psi(-S))}_{=:\Psi_-(S)}. \tag{3.59}$$

This decomposes $L^2(S^3, \mathbb{C}_\chi)$ into the direct sum of two orthogonal[10] subspaces:

$$L^2(S^3, \mathbb{C}_\chi) = L^2_+(S^3, \mathbb{C}_\chi) \oplus L^2_-(S^3, \mathbb{C}_\chi), \tag{3.60}$$

[9] We may trivialise the bundle on each hemisphere, with transition function from the equatorial 2-sphere to the structure group. Homotopic transition functions define isomorphic bundles, but the second homotopy group of any Lie group is trivial, so we may extend the trivialisation to the whole S^3.

[10] This follows from the invariance of the Haar measure under multiplication by $-\mathbb{1}$: $d\mu(-S) = d\mu(S)$.

where $\Psi \in L^2_\pm(S^3, \mathbb{C}_\chi)$ if and only if $\Psi(-S) = \pm\Psi(S)$. This sign, of course, is the same sign distinguishing the two spin structures on the circle. The UIRs have underlying Hilbert spaces $L^2_\pm(S^3, \mathbb{C}_\chi)$ and they agree of course with the UIRs denoted $\mathbb{V}_\pm(p, k, \theta)$ above.

One could be forgiven for asking why we did not describe other massless representations of G in terms of functions on the 3-sphere. In fact, as we now explain, in a sense we have. Let $\boldsymbol{p}, \boldsymbol{k} \in \mathbb{R}^3$ be arbitrary and consider the unitary character of N defined by equation (3.56). What is the stabiliser $H_\chi \subset \mathrm{SU}(2)$ of χ? It clearly depends on $\boldsymbol{p}$ and $\boldsymbol{k}$. The stabiliser always includes the centre $\mathbb{Z}_2$, since the action of SU(2) on $\mathbb{R}^3$ is the adjoint representation and the centre lies in its kernel, but it may be larger. In all cases, Mackey's method would instruct us to induce a representation of G as square-integrable sections of a homogeneous vector bundle over $\mathrm{SU}(2)/H_\chi = G/(H_\chi \ltimes N)$ associated to a UIR of $H_\chi \ltimes N$, obtained from a complex UIR of H_χ by having N act via the character χ. Let us now go through the different possibilities.

- If $\boldsymbol{p} = \boldsymbol{k} = \boldsymbol{0}$, then the character $\chi \equiv 1$ and $H_\chi = \mathrm{SU}(2)$. Thus we induce a representation of G from a UIR of SU(2) (since N acts trivially) which is carried by sections of the corresponding homogeneous vector bundle over $\mathrm{SU}(2)/\mathrm{SU}(2)$, which is a point, over which a vector bundle is just a vector space. In other words, the representation is simply an UIR of SU(2) with N acting trivially. These are the Carroll UIRs of class $\mathbb{I}(s)$.

- If $\boldsymbol{p}, \boldsymbol{k}$ are collinear (but not both zero), χ is stabilised by the Spin(2) subgroup of SU(2) which fixes the direction singled out by $\boldsymbol{p}, \boldsymbol{k}$. Thus we induce a UIR of G from a UIR of $\mathrm{Spin}(2) \ltimes N$ which is carried by sections of the corresponding homogeneous vector bundle over $\mathrm{SU}(2)/\mathrm{Spin}(2)$, which is the 2-sphere. Since N acts via χ and Spin(2) stabilises χ, the inducing representation is a representation of $\mathrm{Spin}(2) \times N$, hence a tensor product of a UIR of Spin(2), which is $\mathbb{C}_n$ for some $n \in \mathbb{Z}$, and the UIR of N defined by χ. The fibration $\mathrm{SU}(2) \to S^2$ is the Hopf fibration discussed in appendix B. If $\boldsymbol{p} = \boldsymbol{0}$ these are the Carroll UIRs of class $\mathbb{III}'(n, k)$, whereas if $\boldsymbol{p} \neq \boldsymbol{0}$ they are the Carroll UIRs of class $\mathbb{III}(n, p)$ if $k = 0$ or of class $\mathbb{IV}_\pm(n, p, k)$ where $\boldsymbol{k} = \pm\frac{k}{p}\boldsymbol{p}$.

- Finally if $\boldsymbol{p}$ and $\boldsymbol{k}$ are not collinear, then they span a plane and $H_\chi = \mathbb{Z}_2$. The UIRs are square-integrable sections of a homogeneous vector bundle over $\mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(3)$ associated to either the trivial or the sign representation of $\mathbb{Z}_2$, or we could just remain on SU(2) as described earlier in this section and project to functions which are either odd or even under the action of the centre. These are of course the Carroll UIRs of class $\mathbb{V}_\pm(p, k, \theta)$. The fact that H_χ is not connected is responsible for the sign, which labels two inequivalent quantisations of the same coadjoint orbit.

Since, as explained in appendix A, sections of homogeneous vector bundles over G/H lift to H-equivariant functions on G (the so-called Mackey functions), all the massless UIRs of the Carroll group can indeed be described in terms of functions on SU(2), except that the functions are equivariant under the action of SU(2), Spin(2) or $\mathbb{Z}_2$ with values in UIRs

	#	$\alpha \in \mathfrak{g}^*$	$\mathscr{O}_\tau$	K_τ	inducing representation of K_τ	UIR of G
$\mathbb{II}(s=0,E_0)$	1	$(\mathbf{0},\mathbf{0},\mathbf{0},E_0\neq 0)$	$\mathbb{A}^3_{E=E_0}$	$\mathrm{Spin}(3)$	$\mathbb{C}$	$L^2(\mathbb{A}^3)$
$\mathbb{II}(s\neq 0,E_0)$	2	$(\boldsymbol{j}\neq\mathbf{0},\mathbf{0},\mathbf{0},E_0\neq 0)$	$\mathbb{A}^3_{E=E_0}$	$\mathrm{Spin}(3)$	$V_{s\neq 0}$	$L^2(\mathbb{A}^3,V_s)$
$\mathbb{I}(s=0)$	3	$(\mathbf{0},\mathbf{0},\mathbf{0},0)$	$\{(\mathbf{0},0)\}$	K	$\mathbb{C}$	$\mathbb{C}$
$\mathbb{I}(s\neq 0)$	4	$(\boldsymbol{j}\neq\mathbf{0},\mathbf{0},\mathbf{0},0)$	$\{(\mathbf{0},0)\}$	K	$V_{s\neq 0}$	V_s
$\mathbb{III}'(n,k)$	5	$(\boldsymbol{j},\boldsymbol{k}\neq\mathbf{0},\mathbf{0},0)_{\boldsymbol{j}\times\boldsymbol{k}=\mathbf{0}}$	$\{(\mathbf{0},0)\}$	K	$L^2(S^2,\mathscr{O}(-n))$	$L^2(S^2,\mathscr{O}(-n))$
$\mathbb{III}(n,p)$	6	$(\boldsymbol{j},\mathbf{0},\boldsymbol{p}\neq\mathbf{0},0)_{\boldsymbol{j}\times\boldsymbol{p}=\mathbf{0}}$	S^2_p	$\mathrm{Spin}(2)\ltimes\mathbb{R}^3$	$\mathbb{C}_n$	$L^2(S^2,\mathscr{O}(-n))$
$\mathbb{IV}_\pm(n,p,k)$	7	$(\boldsymbol{j},\boldsymbol{k}\neq\mathbf{0},\boldsymbol{p}\neq\mathbf{0},0)_{\boldsymbol{k}\times\boldsymbol{p}=\boldsymbol{j}\times\boldsymbol{k}=\mathbf{0}}$	S^2_p	$\mathrm{Spin}(2)\ltimes\mathbb{R}^3$	$\mathbb{C}_n\otimes\mathbb{C}_k$	$L^2(S^2,\mathscr{O}(-n))$
$\mathbb{V}_\pm(p,k,\theta)$	8	$(\mathbf{0},\boldsymbol{k},\boldsymbol{p},0)_{\boldsymbol{k}\times\boldsymbol{p}\neq\mathbf{0}}$	S^2_p	$\mathrm{Spin}(2)\ltimes\mathbb{R}^3$	$\mathscr{H}^\pm_{k,\theta}:=L^2(S^1_{k\sin\theta},\Sigma_\pm\otimes L_{k\cos\theta})$	$L^2_\pm(S^3,\mathbb{C}_{p,k,\theta})$

Table 5. Coadjoint orbits and unitary irreducible representations of the Carroll group. The table lists a representative α of each class of coadjoint orbit $\mathcal{O}_\alpha$, the base $\mathcal{O}_\tau$ of the fibration which describes $\mathcal{O}_\alpha$ and the little groups $K_\tau \subset K$ from which we induce the unitary irreducible representations of the Carroll group. In each row we also list the vector space of the inducing representation of K_τ as well as that of the induced representation. The notation $L^2(X,V)$ means either L^2 functions $X \to V$, when V is a vector space, or L^2 sections of a vector bundle V over X. The bundles $\Sigma_\pm$ are the Ramond $(+)$ and Neveu-Schwarz $(-)$ spinor bundles over the circle and $L_{k\cos\theta}$ is a trivial line bundle over the circle associated to a unitary character of $\mathbb{R}$, thought of as a one-dimensional abelian group. As explained in the bulk of the paper, the resulting description of this UIR in terms of square-integrable sections of a homogeneous Hilbert bundle over the 2-sphere can be simplified to square-integrable functions on the 3-sphere with values in a one-dimensional complex unitary representation of the subgroup of G generated by boosts and translations labelled by (p,k,θ) which are either odd or even under the action of the centre $\mathbb{Z}_2$.

of SU(2), Spin(2) or $\mathbb{Z}_2$, respectively. That is precisely the description of the UIRs detailed above.

3.4.4 Relation with coadjoint orbits

Following Rawnsley [29], these induced representations are obtained by geometric quantisation of the coadjoint orbits of the Carroll group, whose classification was recalled in section 2. This suggests a sort of correspondence between coadjoint orbits of the Carroll group and the induced representations we have determined, which is described in table 5.

Any such correspondence between coadjoint orbits and UIRs needs to be qualified. Firstly, not every orbit is quantisable: indeed it is clear from the table that not all orbits in classes $\#2,4,5,6,7_\pm$ are quantisable, since the angular momentum is quantised. On the other hand, the orbits of type 8 admit two different quantisations: corresponding to the two different spin structures on the circle. This is due to the fact that the stabiliser of the character is disconnected. Both of these phenomena are well established, as explained, for example, in the Introduction to [32].

Short of actually geometrically quantising the coadjoint orbits, the correspondence in the table must remain largely conjectural, but we are fairly confident in its validity, with the above caveats.

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4 Carrollian fields

In this section we describe some of the UIRs of the Carroll group in terms of fields in Carroll spacetime. We will do one example with $E \neq 0$ and one with $E = 0$, which roughly correspond to electric and magnetic Carroll fields, respectively. Before doing so, we will briefly recap the method, which we explain in more detail in appendix A.

Carroll spacetime is the homogeneous space G/K and fields on Carroll spacetime which transform in some representation of the Carroll group are sections of homogeneous vector bundles associated to representations of K. In the previous section we exhibited the UIRs of the Carroll group as sections of homogeneous vector bundles over the orbits $\mathcal{O}_\tau$ in momentum space associated to a UIR W of K_τ. To view such representations as fields in Carroll spacetime requires us firstly to make a choice of representation V of K which, when restricted to K_τ, contains a subrepresentation isomorphic to W. This first step is known as "covariantisation" in the Physics literature. We then apply a group-theoretical Fourier transform to go from sections of homogeneous vector bundles over $G/(K_\tau \ltimes T)$ to sections of homogeneous vector bundles over G/K.

We may contrast this with the case of massive representations of the Poincaré group. In both cases, the little group $K_\tau \cong \mathrm{SU}(2)$ and W is one of the complex spin-s representations. Covariantisation differs: in the Poincaré case we need to embed W in a finite-dimensional (and hence non-unitary) representation of Lorentz group, whereas in the Carroll case we need to embed W in a finite-dimensional representation of the homogeneous Carroll group, which is isomorphic to the three-dimensional euclidean group. The main difference, which will have very visible repercussions, is that in the Carroll case the representation W can be made itself covariant by declaring the boosts, which form an ideal, to act trivially; whereas in the Lorentz case this is not possible for any positive spin.

4.1 Brief recap of the method of induced representations (continued)

We retake the algorithm started in section 3.1 with the procedure of covariantising the induced representations in terms of fields on Carroll spacetime $M = G/K$.

(5) Pick a representation V of K whose restriction to K_τ contains a subrepresentation isomorphic to W. This representation need not be unitary. We typically take it to be irreducible, so as to minimise the number of extra degrees of freedom we are introducing. We extend V to a representation of G by declaring $t \in T$ to act via the character χ_τ, just in the same way we extended W to a representation of $H = K_\tau \ltimes T$.

(6) Let $\zeta : M \to T$ be the coset representative for Carroll spacetime sending $x = (t, \boldsymbol{x}) \mapsto \zeta(x) = \exp(tH + \boldsymbol{x} \cdot \boldsymbol{P})$ and define $\phi : M \to V$ by

$$\phi(x) = \int_{\mathcal{O}_\tau} d\mu(p) \chi_{\sigma(p)\cdot\tau}(\zeta(x)^{-1})\sigma(p) \cdot \psi(p), \tag{4.1}$$

where $\psi : \mathcal{O}_\tau \to V$.

(7) Let $g \in G$ and define $k(g^{-1}, x) \in K$ by

$$g^{-1}\zeta(x) = \zeta(g^{-1} \cdot x)k(g^{-1}, x). \tag{4.2}$$

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Then the action of $g \in G$ on the carrollian field ϕ is given by

$$(g \cdot \phi)(x) = k(g^{-1}, x) \cdot \phi(g^{-1} \cdot x). \tag{4.3}$$

(8) This representation of G need neither be unitary nor irreducible. To remedy this we must impose field equations which say that ϕ lies in the kernel of a (pseudo-)differential operator which is obtained via the Fourier transform (4.1) from the projector from V to W.

Again, these steps are somewhat over-simplified and the more detailed story is contained in appendix A. In particular, the "functions" $\phi : M \to V$ are actually sections of a homogeneous vector bundle $E_V = G \times_K V$ over M. These sections are obtained via a group-theoretical version of the Fourier transform at the level of the Mackey functions. Indeed, equation (4.1), which is equation (A.19), is obtained by evaluating the Fourier-transformed Mackey function $\widehat{F}$ defined by equation (A.10) at the coset representative. Similarly, the expression (4.3), which agrees with equation (A.25) is obtained from the more natural transformation law (A.22) of the Fourier-transformed Mackey function $\widehat{F}$.

4.2 Representations with $E \neq 0$ in terms of carrollian fields

Here we illustrate the method by working out an explicit example of unitary irreducible representation of (the universal cover of) the Carroll group with $E_0 \neq 0$ as fields on Carroll spacetime. The group element $g(R, \boldsymbol{v}, \boldsymbol{a}, s)$ is given by equation (3.19), where we assume that G stands for the universal cover of the Carroll group and K for the universal cover of the euclidean group. In particular, R in equation (3.19) belongs to $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$.

As discussed above, this representation is carried by the Hilbert space of functions $\mathbb{A}^3 \to V_s$ where $\mathbb{A}^3 \subset \mathfrak{t}^*$ is the affine plane with $E = E_0$ and V_s is the complex spin-s representation of Spin(3), which are square-integrable relative to the inner product defined by integrating the pointwise invariant hermitian inner product on V_s over $\mathbb{A}^3$ relative to the euclidean measure, as in equation (3.18). The action of the Carroll group G on this representation was given above in equation (3.23).

We are ultimately interested in fields defined on Carroll spacetime, so we may now covariantise this representation by first of all embedding the representation V_s of K_τ into a representation of K. One particularly economical choice is to declare that the boosts act trivially. As described in the introduction this might seem unconventional from the point of view of Lorentz invariant theories, since this is impossible for the Lorentz group unless $s = 0$. Indeed, the commutator of two boosts is a rotation and hence if the boosts were to act trivially, so would the rotations. As explained in appendix A and briefly recapped above, we then need to Fourier transform to arrive at sections of a homogeneous vector bundle over Carroll spacetime.

Let $\zeta : G/K \to T$ be a coset representative sending[11] $x = (t, x^a) \mapsto \exp(tH + x^a P_a) \in T$. Recall that for $\boldsymbol{p} \in \mathbb{A}^3$, $\sigma(\boldsymbol{p}) = \exp(-\frac{1}{E_0}\boldsymbol{p} \cdot \boldsymbol{B})$ so that

$$\sigma(\boldsymbol{p})^{-1}\zeta(t, \boldsymbol{x})\sigma(\boldsymbol{p}) = \zeta(t + E_0^{-1}\boldsymbol{x} \cdot \boldsymbol{p}, \boldsymbol{x}), \tag{4.4}$$

[11]Now t is the time coordinate and no longer a generic element of the translation subgroup T.

so that

$$\chi_\tau(\sigma(\boldsymbol{p})^{-1}\zeta(t,\boldsymbol{x})^{-1}\sigma(\boldsymbol{p})) = e^{-iE_0 t - i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{4.5}$$

and hence $\phi(t,\boldsymbol{x})$, which we remind the reader takes values in the spin-s representation of SU(2), is given by

$$\phi(t,\boldsymbol{x}) = e^{-iE_0 t}\int_{\mathbb{A}^3} d^3p e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\psi(\boldsymbol{p})\,, \tag{4.6}$$

which is, up to the factor $e^{-iE_0 t}$ and normalisation, the spatial Fourier transform of $\psi(\boldsymbol{p})$. Observe that these fields obey

$$\frac{\partial\phi}{\partial t} = -iE_0\phi, \tag{4.7}$$

suggesting that they are "electric" Carroll fields.

The action of the Carroll group on such fields is easily worked out. Let $g = g(R,\boldsymbol{v},\boldsymbol{a},s) = g(0,\boldsymbol{0},\boldsymbol{a},s)\exp(\boldsymbol{v}\cdot\boldsymbol{B})R$, with $R \in \mathrm{Spin}(3)$. Then

$$g^{-1}\zeta(x) = R^{-1}e^{-\boldsymbol{v}\cdot\boldsymbol{B}}t^{-1}\zeta(x) = R^{-1}e^{-\boldsymbol{v}\cdot\boldsymbol{B}}\zeta(x') \tag{4.8}$$

where if $x = (t,\boldsymbol{x})$ then

$$x' = (t-s, \boldsymbol{x}-\boldsymbol{a}). \tag{4.9}$$

Continuing with the calculation,

$$R^{-1}e^{-\boldsymbol{v}\cdot\boldsymbol{B}}\zeta(x') = \zeta(x'')R^{-1}e^{-\boldsymbol{v}\cdot\boldsymbol{B}}, \tag{4.10}$$

where

$$x'' = (t-s-\boldsymbol{v}\cdot(\boldsymbol{x}-\boldsymbol{a}), R^{-1}(\boldsymbol{x}-\boldsymbol{a})), \tag{4.11}$$

resulting in

$$(g\cdot\phi)(t,\boldsymbol{x}) = \rho(R)\phi(t-s-\boldsymbol{v}\cdot(\boldsymbol{x}-\boldsymbol{a}), R^{-1}(\boldsymbol{x}-\boldsymbol{a})), \tag{4.12}$$

with ρ the spin-s representation of SU(2).

Notice that under the different kinds of Carroll transformations, the field ϕ behaves as follows:

- under rotations with $R \in \mathrm{Spin}(3)$,

$$(R\cdot\phi)(t,\boldsymbol{x}) = \rho(R)\phi(t,R^{-1}\boldsymbol{x}); \tag{4.13}$$

- under boosts,

$$(e^{\boldsymbol{v}\cdot\boldsymbol{B}}\cdot\phi)(t,\boldsymbol{x}) = \phi(t-\boldsymbol{v}\cdot\boldsymbol{x},\boldsymbol{x}); \tag{4.14}$$

- and under translations,

$$(e^{sH+\boldsymbol{a}\cdot\boldsymbol{P}}\cdot\phi)(t,\boldsymbol{x}) = \phi(t-s,\boldsymbol{x}-\boldsymbol{a}). \tag{4.15}$$

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It is interesting to note that the boosts have no action on the field ϕ, but only on coordinates, which is quite distinct to Poincaré where Lorentz boosts also transform the fields. This is easy to explain: since the commutator of two Lorentz boosts is a rotation, we cannot covariantise the inducing representation in the Poincaré case by simply declaring the boosts to act trivially.

We may calculate the value of the casimirs H and W^2 on such fields. Let $g(\lambda) = (R(\lambda), \boldsymbol{v}(\lambda), \boldsymbol{a}(\lambda), s(\lambda))$ denote a curve through the identity on G; that is, $R(0) = \mathbb{1}$, $\boldsymbol{v}(0) = \boldsymbol{a}(0) = \boldsymbol{0}$ and $s(0) = 0$. Let us now act with $g(\lambda)$ on a field ϕ according to equation (4.12) to obtain a curve in the space of fields. Its derivative $\frac{d}{d\lambda}\Big|_{\lambda=0}$ gives

$$\rho(R'(0))\phi + \frac{\partial \phi}{\partial t}(-s'(0) - \boldsymbol{v}'(0)\cdot \boldsymbol{x}) - (R'(0)\boldsymbol{x})\cdot \boldsymbol{\partial}\phi - \boldsymbol{a}'(0)\cdot \boldsymbol{\partial}\phi, \tag{4.16}$$

where we also use the notation ρ for the representation of $\mathfrak{so}(3)$ on V_s. We can then read off the action of the generators of $\mathfrak{g}$ on ϕ:

$$\begin{aligned} H\phi &= -\frac{\partial \phi}{\partial t} \\ P_i\phi &= -\frac{\partial \phi}{\partial x^i} \\ B_i\phi &= -x_i\frac{\partial \phi}{\partial t} \\ J_i\phi &= \rho(J_i)\phi + \epsilon_{ijk}x_k\frac{\partial \phi}{\partial x^j}. \end{aligned} \tag{4.17}$$

From this it follows that $H\phi = iE_0\phi$ and that $W_i := HJ_i + \epsilon_{ijk}P_jB_k$ acts simply as

$$\begin{aligned} W_i\phi &= \underbrace{\rho(J_i)\frac{\partial \phi}{\partial t} + \epsilon_{ijk}x_k\frac{\partial^2\phi}{\partial t\partial x^j}}_{HJ_i\phi} + \underbrace{\left(-\epsilon_{ijk}x_k\frac{\partial^2\phi}{\partial t\partial x^j}\right)}_{\epsilon_{ijk}P_jB_k\phi} \\ &= \rho(J_i)\frac{\partial\phi}{\partial t}, \end{aligned} \tag{4.18}$$

and hence

$$W^2\phi = \rho(J^2)\frac{\partial^2\phi}{\partial t^2} = E_0^2 s(s+1)\phi, \tag{4.19}$$

where we have used that $\rho(J^2)$ acts by scalar multiplication by $-s(s+1)$ on V_s.

4.3 A class of $\boldsymbol{E = 0}$ representations in terms of carrollian fields

Let us consider the other class of representations of the Carroll group which can be carried by (finite-component) fields on Carroll spacetime: namely, the ones carried by the square-integrable sections of the line bundle $\mathscr{O}(-n)$ over $\mathbb{CP}^1$, equivalently the vector bundle over the sphere associated to the one-dimensional representation $\mathbb{C}_n$ of $\mathrm{U}(1) \subset \mathrm{SU}(2)$, with boosts acting trivially.

Let us first see that relaxing this condition by inducing from $\mathbb{C}_n \otimes \mathbb{C}_w$ with $w \neq 0$ results in infinite-component fields. Indeed, as discussed in appendix A and recapped briefly in section 4.1, the first step is to embed the representation $\mathbb{C}_n \otimes \mathbb{C}_w$ of $(\mathrm{U}(1) \ltimes \mathbb{R}^2) \times \mathbb{R}$ into

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a representation of $K \cong \mathrm{SU}(2) \ltimes \mathbb{R}^3$. It is not difficult to see, however, that any such representation would have to be infinite-dimensional unless $w = 0$. Indeed, suppose that there is a vector ψ in that representation with $e^{\boldsymbol{v}\cdot\boldsymbol{B}}\psi = e^{i\chi(\boldsymbol{v})}\psi$, with χ the character $\chi(v_1, v_2, v_3) = wv_3$. Let $R \in \mathrm{Spin}(3)$ and consider the vector $R\psi$. Then

$$e^{\boldsymbol{v}\cdot\boldsymbol{B}}R\psi = RR^{-1}e^{\boldsymbol{v}\cdot\boldsymbol{B}}R\psi = Re^{R^{-1}\boldsymbol{v}\cdot\boldsymbol{B}}\psi = e^{i\chi(R^{-1}\boldsymbol{v})}R\psi = e^{i(R\chi)(\boldsymbol{v})}R\psi, \tag{4.20}$$

so that the representations with characters $R\chi$ in the SO(3)-orbit of χ also appear. These are labelled by the sphere of radius $|w|$ in the dual of $\mathbb{R}^3$, so that they are uncountable unless $w = 0$.

Therefore let us take $w = 0$ from now on and let us first of all embed the representation $\mathbb{C}_n$ of U(1) inside an irreducible finite-dimensional representation V of SU(2). Every such V is isomorphic to V_s for some spin s and $\mathbb{C}_n$ embeds in V_s provided that $|n| \leq 2s$. The smallest irreducible representation containing $\mathbb{C}_n$ is therefore V_s with $s = |\frac{n}{2}|$, where it appears as either the subspace of highest (if $n > 0$) or lowest (if $n < 0$) weight vectors in V_s. Let $V = V_{|\frac{n}{2}|}$ from now on; although of course other representations are possible.[12]

Now define $\zeta : G/K \to T$ by $\zeta(x) = \exp(tH + \boldsymbol{x}\cdot\boldsymbol{P})$ as a coset representative for Carroll spacetime M and we obtain a field ϕ on M taking values in V given by

$$\phi(t, \boldsymbol{x}) = \int_{\mathbb{C}} \frac{2idz \wedge d\bar{z}}{(1+|z|^2)^2} e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)}\rho(\sigma(z))\psi(z), \tag{4.21}$$

with ρ the spin-s representation of SU(2). Notice that such fields do not depend on the time coordinates in Carroll spacetime: $\frac{\partial\phi}{\partial t} = 0$, so we may simply write $\phi(\boldsymbol{x})$.

The action of the Carroll group on ϕ is easy to work out. Let $g = \exp(sH + \boldsymbol{a}\cdot P)e^{\boldsymbol{v}\cdot\boldsymbol{B}}R \in G$, where $R \in \mathrm{SU}(2)$. For this choice of g,

$$g^{-1}\zeta(x) = R^{-1}e^{-\boldsymbol{v}\cdot\boldsymbol{B}}\zeta(x') \quad \text{where} \quad x' = (t - s, \boldsymbol{x} - \boldsymbol{a}). \tag{4.22}$$

Continuing with the calculation,

$$R^{-1}e^{-\boldsymbol{v}\cdot\boldsymbol{B}}\zeta(x') = \zeta(x'')R^{-1}e^{-\boldsymbol{v}\cdot\boldsymbol{B}} \quad \text{where} \quad x'' = \Big(t - s - \boldsymbol{v}\cdot(\boldsymbol{x}-\boldsymbol{a}), R^{-1}(\boldsymbol{x}-\boldsymbol{a})\Big). \tag{4.23}$$

Since ϕ does not depend on time (t), we arrive at

$$(g\cdot\phi)(\boldsymbol{x}) = \rho(R)\phi(R^{-1}(\boldsymbol{x}-\boldsymbol{a})), \tag{4.24}$$

showing that both the boosts and the time translations act trivially. Both casimirs H and W^2 act trivially on such fields.

Since the fields are defined on the mass shell where $\|\boldsymbol{p}\|^2 = p^2$ is a constant, the field ϕ satisfies the Helmholtz equation

$$(\triangle + p^2)\phi(\boldsymbol{x}) = 0, \tag{4.25}$$

[12] This ought to be familiar from relativistic fields, where, for instance, the representation describing a massive scalar field can be embedded as either a Lorentz scalar subject to the Klein-Gordon equation or as a massive vector field subject to the equation $\partial_\mu\partial\cdot V = m^2V_\mu$, which is the complementary equation to the Proca equation on the massive vector field. In this case the only degree of freedom is precisely the divergence $\partial\cdot V$ which is a scalar field obeying the Klein-Gordon equation.

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with $\triangle$ the laplacian in $\mathbb{R}^3$. This is easily derived by simply inserting 0 in the form $\|\boldsymbol{\pi}(z)\|^2 - p^2$ in the integral (4.21) and then noticing that $\|\boldsymbol{\pi}(z)\|^2$ is (minus) the laplacian acting on $e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)}$.

The irreducible representation is carried by those fields ϕ for which the integral in equation (4.21) exists and for which ψ is either a lowest or highest weight vector in V. For definiteness, let us assume that ψ is a highest weight vector so that it lies in the kernel of $\rho(J_+)$, with $J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. As explained at the end of appendix A, we may derive an equation for such ϕ as follows. We notice that since $\rho(J_+)\psi(z) = 0$, we have that

$$\int_{\mathbb{C}} \frac{2idz \wedge d\overline{z}}{(1+|z|^2)^2} e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)} \rho(\sigma(z))\rho(J_+)\psi(z) = 0. \tag{4.26}$$

Let us define $J_+(z) := \sigma(z) J_+ \sigma(z)^{-1}$, so that the above equation becomes

$$\int_{\mathbb{C}} \frac{2idz \wedge d\overline{z}}{(1+|z|^2)^2} e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)} \rho(J_+(z))\rho(\sigma(z))\psi(z) = 0. \tag{4.27}$$

We work out $J_+(z)$ explicitly to be

$$\begin{aligned} J_+(z) = \sigma(z) J_+ \sigma(z)^{-1} &= \frac{1}{1+|z|^2} \begin{pmatrix} z & -1 \\ 1 & \overline{z} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{z} & 1 \\ -1 & z \end{pmatrix} \\ &= \frac{1}{1+|z|^2} \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}. \end{aligned} \tag{4.28}$$

To recognise the (pseudo-)differential operator defined by $\rho(J_+(z))$, let us now calculate the spatial derivatives of $\phi(\boldsymbol{x})$ by differentiating under the integral sign:

$$\frac{\partial}{\partial x^j}\phi(\boldsymbol{x}) = \int_{\mathbb{C}} \frac{2dz \wedge d\overline{z}}{(1+|z|^2)^2} \pi_j(z) e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)} \rho(\sigma(z))\psi(z), \tag{4.29}$$

so that, using the components of $\boldsymbol{\pi}(z)$ in equation (3.37),

$$\begin{aligned} \frac{\partial}{\partial x^1}\phi(\boldsymbol{x}) &= \int_{\mathbb{C}} \frac{4pdz \wedge d\overline{z}}{(1+|z|^2)^3} \operatorname{Re}(z) e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)} \rho(\sigma(z))\psi(z) \\ \frac{\partial}{\partial x^2}\phi(\boldsymbol{x}) &= \int_{\mathbb{C}} \frac{4pdz \wedge d\overline{z}}{(1+|z|^2)^3} \operatorname{Im}(z) e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)} \rho(\sigma(z))\psi(z) \\ \frac{\partial}{\partial x^3}\phi(\boldsymbol{x}) &= \int_{\mathbb{C}} \frac{2pdz \wedge d\overline{z}}{(1+|z|^2)^3} (|z|^2 - 1) e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)} \rho(\sigma(z))\psi(z). \end{aligned} \tag{4.30}$$

This sets up the following dictionary:

$$\begin{aligned} \frac{z}{1+|z|^2} &\rightsquigarrow \frac{i}{2p}(\partial_1 + i\partial_2) \\ \frac{1}{1+|z|^2} &\rightsquigarrow -\frac{i}{2p}(\partial_3 + ip) \\ \frac{z^2}{1+|z|^2} &\rightsquigarrow -\frac{i}{2p}\frac{(\partial_1 + i\partial_2)}{\partial_3 + ip}. \end{aligned} \tag{4.31}$$

It is then a relatively simple matter to take the $\rho(J_+(z))$ in the integrand and write it as a (pseudo-)differential operator acting on the integral, obtaining the somewhat formal expression

$$\rho\left(\begin{pmatrix} \partial_1 + i\partial_2 & \frac{(\partial_1+i\partial_2)^2}{\partial_3+ip} \\ -(\partial_3 + ip) & -(\partial_1 + i\partial_2) \end{pmatrix}\right)\phi(\boldsymbol{x}) = 0, \tag{4.32}$$

where we have omitted an inconsequential overall factor of $1/(2ip)$ in the left-hand side.

A similar calculation starting with $J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, whose kernel consists of the lowest weight vectors, results in

$$\rho\left(\begin{pmatrix} -(\partial_1 - i\partial_2) & -(\partial_3 + ip) \\ \frac{(\partial_1-i\partial_2)^2}{\partial_3+ip} & \partial_1 - i\partial_2 \end{pmatrix}\right)\phi(\boldsymbol{x}) = 0, \tag{4.33}$$

which is the relevant equation in the case of negative helicity.

These equations look non-local due to the presence of the resolvent $(\partial_3 + ip)^{-1}$, but we can try to make sense of them. Let us discuss some explicit cases.

4.3.1 Helicity 0

This case needs no discussion, since there ρ is the trivial representation and the only condition on the scalar field is the Helmholtz equation (4.25).

4.3.2 Helicity $\frac{1}{2}$

In this case, ρ is the defining representation of SU(2), so the identity map. Here $\phi(\boldsymbol{x})$ is a 2-component field and once we take into account the Helmholtz equation (4.25), the solutions to equation (4.32) are also the solutions to the Dirac-like equation

$$(\partial\!\!\!/ + ip)\phi = 0, \tag{4.34}$$

where $\partial\!\!\!/ = \gamma^i\partial_i$ with γ^i the representation of $C\ell(0,3)$ given by

$$\gamma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.35}$$

4.3.3 Helicity 1

The representation ρ now is the adjoint representation of SU(2). Let $R \in \mathrm{SU}(2)$ be given by equation (3.41). Then the adjoint representation of R is given by

$$\rho(R) = \begin{pmatrix} \mathrm{Re}(\eta^2 - \xi^2) & -\mathrm{Im}(\eta^2 + \xi^2) & -2\,\mathrm{Re}(\eta\xi) \\ \mathrm{Im}(\eta^2 - \xi^2) & \mathrm{Re}(\eta^2 + \xi^2) & -2\,\mathrm{Im}(\eta\xi) \\ 2\,\mathrm{Re}(\eta\bar{\xi}) & -2\,\mathrm{Im}(\eta\bar{\xi}) & |\eta|^2 - |\xi|^2 \end{pmatrix}, \tag{4.36}$$

which can be checked to be a matrix in SO(3). The matrix $J_+(z)$ belongs to $\mathfrak{sl}(2,\mathbb{C})$, the complexification of $\mathfrak{su}(2)$. The way we calculate $\rho(J_+(z))$ is as follows. We consider $X \in \mathfrak{su}(2)$ and exponentiate $\exp(tX) \in \mathrm{SU}(2)$ and differentiate $\rho(\exp(tX))$ at $t = 0$. This

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gives us a map (also denoted) $\rho : \mathfrak{su}(2) \to \mathfrak{so}(3)$, which we extend complex-linearly to $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{so}(3, \mathbb{C})$, which can then be evaluated at $J_+(z)$.

Let $X \in \mathfrak{su}(2)$ be given by

$$X = \begin{pmatrix} i\gamma & \alpha + i\beta \\ -\alpha + i\beta & -i\gamma \end{pmatrix} \tag{4.37}$$

and let $\|X\| := \sqrt{\alpha^2 + \beta^2 + \gamma^2}$. Then $\exp(tX) \in \mathrm{SU}(2)$ is given by a matrix of the form in equation (3.41) with

$$\begin{aligned} \eta(t) &= \cos\left(t\|X\|\right) + i\gamma \frac{\sin\left(t\|X\|\right)}{\|X\|} \\ \xi(t) &= (\alpha + i\beta) \frac{\sin\left(t\|X\|\right)}{\|X\|}. \end{aligned} \tag{4.38}$$

We insert this into equation (4.36) and differentiate with respect to t and evaluate at $t = 0$ to obtain

$$\rho(X) = \begin{pmatrix} 0 & -2\gamma & -2\alpha \\ 2\gamma & 0 & -2\beta \\ 2\alpha & 2\beta & 0 \end{pmatrix}, \tag{4.39}$$

which we extend to $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{so}(3, \mathbb{C})$ simply by allowing $\alpha, \beta, \gamma \in \mathbb{C}$. For $X = J_+(z)$, we have

$$2\alpha = \frac{1 + z^2}{1 + |z|^2}, \qquad 2\beta = \frac{i(1 - z^2)}{1 + |z|^2} \qquad \text{and} \qquad \gamma = \frac{iz}{1 + |z|^2}, \tag{4.40}$$

so that

$$\rho(J_+(z)) = \frac{1}{1 + |z|^2} \begin{pmatrix} 0 & -2iz & -(1 + z^2) \\ 2iz & 0 & i(z^2 - 1) \\ 1 + z^2 & i(1 - z^2) & 0 \end{pmatrix}. \tag{4.41}$$

Using the dictionary in equation (4.31), we may write the pseudo-differential equation for helicity-1 fields as

$$\begin{pmatrix} 0 & \partial_1 + i\partial_2 & \frac{i}{2}\left(\partial_3 + ip + \frac{(\partial_1 + i\partial_2)^2}{\partial_3 + ip}\right) \\ -(\partial_1 + i\partial_2) & 0 & -\frac{1}{2}\left(\partial_3 + ip - \frac{(\partial_1 + i\partial_2)^2}{\partial_3 + ip}\right) \\ -\frac{i}{2}\left(\partial_3 + ip + \frac{(\partial_1 + i\partial_2)^2}{\partial_3 + ip}\right) & \frac{1}{2}\left(\partial_3 + ip - \frac{(\partial_1 + i\partial_2)^2}{\partial_3 + ip}\right) & 0 \end{pmatrix} \phi(\boldsymbol{x}) = 0, \tag{4.42}$$

where $\phi(\boldsymbol{x})$ is now a 3-component field subject to the Helmholtz equation (4.25). Let us write $\phi = (\phi_1, \phi_2, \phi_3)$. Then the solutions of the equation (4.42) agree with the solutions of the following differential equations:

$$\begin{aligned} (\partial_1 + i\partial_2)(\phi_1 - i\phi_2) &= -(\partial_3 + ip)\phi_3 \\ (\partial_1 + i\partial_2)\phi_3 &= (\partial_3 + ip)(\phi_1 + i\phi_2). \end{aligned} \tag{4.43}$$

Under the assumption that ϕ_i are real fields, these equations imply the Helmholtz equations $\left(\partial_1^2 + \partial_2^2 + \partial_3^2\right)\phi_i = -p^2\phi_i$ for $i = 1, 2, 3$, and in fact, breaking the equations up into real and imaginary parts, we find that they are equivalent to a much simpler equation:

$$p\phi_i = \epsilon_{ijk}\partial_j\phi_k, \tag{4.44}$$

or, equivalently, $d\phi = p \star \phi$, thinking of $\phi \in \Omega^1(\mathbb{R}^3)$. We recognise this equation as the field equation of euclidean topologically massive Maxwell theory [26, 27], with ϕ playing the rôle of the Hodge dual of the Maxwell field-strength.

4.3.4 Remarks

The massless UIRs with helicity are such that time translations and boosts act trivially. Hence they are actually UIRs of the three-dimensional euclidean group and, presumably, any euclidean three-dimensional field theory should serve as a starting point to constructing massless carrollian field theories with vanishing centre-of-mass charge. Since the centre of mass vanishes for these theories, the symmetries and consequently the particles and theories are basically represented by aristotelian symmetries where the boosts play no rôle (see, e.g., [25]).

In massless relativistic theories, the field equations typically contain solutions with both signs of the helicity. This is not the case here. Indeed, performing a similar calculation for helicity $-\frac{1}{2}$ results in the "opposite" Dirac equation:

$$(\not{\partial} - ip)\phi = 0. \tag{4.45}$$

Clearly only the only field obeying this and equation (4.34) simultaneously is $\phi = 0$.

A final remark is that the above equations for helicity $\frac{1}{2}$ and helicity 1 are first-order partial differential equations which imply the Helmholtz equation, which is a second-order partial differential equation. This is typically one of the (mathematical) signatures of supersymmetry and it would be interesting to explore this further.

5 Fractonic particles and fields

The correspondence between Carroll and fracton particles established in Part I persists upon quantisation. As recalled in Part I, the dipole algebra corresponding to our notion of fracton is isomorphic to a trivial one-dimensional central extension of the Carroll algebra. The additional generator is the time translation generator of the Aristotle spacetime underlying the fracton dynamics. The time translation generator of the Carroll spacetime is the electric charge generator of the dipole algebra, whereas the Carroll boost generators are the dipole generators. There are, of course, no boosts in an Aristotle spacetime. A UIR of the dipole group is the tensor product of a UIR of the Carroll group and a one-dimensional UIR of the one-dimensional Lie group generated by the additional central generator. Any one-dimensional UIR is characterised by a character, which in the absence of any criterion which would imply the quantisation of energy of the quantum fracton, is simply given by a real number. On the other hand, if we do demand (as might seem reasonable) that the electric charge be quantised, then this would restrict the Carroll UIRs with a fractonic interpretation to those where E_0 is quantised in units of an elementary fracton charge.

5.1 Unitary irreducible representations of the dipole group

Let $\widetilde{G} = G \times \mathbb{R}$ denote the dipole group. The Aristotle spacetime A underlying the fractonic theory is a homogeneous space of $\widetilde{G}$ with stabiliser $\widetilde{K} = K \times \mathbb{R}$, where $\widetilde{K}$ is the group generated by rotations, boosts and time translations (in the Carroll language) or by rotations, dipole and electric charge generators (in the fracton language). In other words, we have a $\widetilde{G}$-equivariant diffeomorphism

$$A \cong \widetilde{G}/\widetilde{K} \cong (G/(K \times \mathbb{R})) \times \mathbb{R}, \tag{5.1}$$

where the copy of $\mathbb{R}$ in the "denominator" is the Carroll time translation subgroup, whereas that in the "numerator" is the fracton time translation subgroup. In this section we will let t denote the aristotelian time coordinate. We will introduce H_F as the aristotelian time translation generator and we shall relabel the Carroll generators $H, \boldsymbol{B}$ to $Q, \boldsymbol{D}$, with brackets $[D_i, P_j] = \delta_{ij} Q$. We will also introduce the notation $\widetilde{T} = T \times \mathbb{R}$ and $\widetilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{R} H_F$. A typical element $\widetilde{\tau} \in \widetilde{\mathfrak{t}}^*$ is denoted now $(q, \boldsymbol{p}, E)$ corresponding to the electric charge, the spatial momentum and the energy, respectively: in other words, $\langle \widetilde{\tau}, Q\rangle = q$, $\langle \widetilde{\tau}, \boldsymbol{P}\rangle = \boldsymbol{p}$ and $\langle \widetilde{\tau}, H_F\rangle = E$.

UIRs of the dipole group have a constant (fracton) energy. Letting E now denote the fracton energy, we have the following UIRs of the dipole group, where $\mathbb{C}_E$ denotes the one-dimensional representation of the aristotelian time translation subgroup associated with the character $\chi(e^{sH_F}) = e^{isE}$:

$\mathbf{I}(\boldsymbol{s}, \boldsymbol{E})$ $\cong \mathrm{I}(s) \otimes \mathbb{C}_E$, with $2s \in \mathbb{N}_0$ and $E \in \mathbb{R}$;

$\mathbf{II}(\boldsymbol{s}, \boldsymbol{q}, \boldsymbol{E})$ $\cong \mathrm{II}(s, q) \otimes \mathbb{C}_E$, with $2s \in \mathbb{N}_0$, $q \in \mathbb{R}$ (or $\mathbb{Z}$ if quantised) and $E \in \mathbb{R}$;

$\mathbf{III}(\boldsymbol{n}, \boldsymbol{p}, \boldsymbol{E})$ $\cong \mathrm{III}(n, p) \otimes \mathbb{C}_E$, with $n \in \mathbb{Z}$, $p > 0$ and $E \in \mathbb{R}$;

$\mathbf{III}'(\boldsymbol{n}, \boldsymbol{d}, \boldsymbol{E})$ $\cong \mathrm{III}'(n, k) \otimes \mathbb{C}_E$, with $n \in \mathbb{Z}$, $d > 0$ and $E \in \mathbb{R}$;

$\mathbf{IV}_{\pm}(\boldsymbol{n}, \boldsymbol{p}, \boldsymbol{d}, \boldsymbol{E})$ $\cong \mathrm{IV}_{\pm}(n, p, k) \otimes \mathbb{C}_E$, with $n \in \mathbb{Z}$, $p > 0$, $d > 0$ and $E \in \mathbb{R}$;

$\mathbf{V}_{\pm}(\boldsymbol{p}, \boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{E})$ $\cong \mathrm{V}_{\pm}(p, d, \theta) \otimes \mathbb{C}_E$, with $p, d > 0$, $\theta \in (0, \pi)$ and $E \in \mathbb{R}$.

These representations have the same underlying Hilbert space as the corresponding representations of the Carroll group and the group element $(g, s) \in G \times \mathbb{R}$ acts as

$$(g, s) \cdot \psi = e^{iEs} g \cdot \psi. \tag{5.2}$$

Let $\widetilde{\tau} = (q, \boldsymbol{p}, E) \in \widetilde{\mathfrak{t}}^*$ and let $\chi_{\widetilde{\tau}}$ denote the unitary character defined by

$$\chi_{\widetilde{\tau}}\left(e^{\theta Q + \boldsymbol{a}\cdot\boldsymbol{P} + sH_F}\right) = e^{i(\theta q + \boldsymbol{a}\cdot\boldsymbol{p} + sE)}. \tag{5.3}$$

The first step in deriving the expressions for the aristotelian fields is to extend the K_τ-representation W first to a representation $W \otimes \mathbb{C}_E$ of $K_\tau \times \mathbb{R}$, and then to a representation $V \otimes \mathbb{C}_E$ of $K \times \mathbb{R}$. As we did earlier with the Carroll group, we may extend $V \otimes \mathbb{C}_E$ to a representation of the whole dipole group via the character $\chi_{\widetilde{\tau}}$ above. These unitary representations are described as sections of homogeneous vector bundles over $\mathcal{O}_{\widetilde{\tau}} \cong \mathcal{O}_\tau$. Let $\zeta : A \to \widetilde{T}$ send the aristotelian coordinates $(t, \boldsymbol{x}) \mapsto \exp(\boldsymbol{x} \cdot \boldsymbol{P} + tH_F)$.

5.2 Charged aristotelian fields

Let us consider the case of UIRs with nonzero electric charge, so of class $\mathbb{II}(s, q, E)$ with $q \neq 0$. They correspond to $\widetilde{\tau} = (q, \mathbf{0}, E)$. The coset representative $\sigma : \mathcal{O}_{\widetilde{\tau}} \to K$ is given as before by $\sigma(\boldsymbol{p}) = \exp(-\frac{1}{q}\boldsymbol{p} \cdot \boldsymbol{D})$. Then a short calculation gives

$$\zeta(t, \boldsymbol{x})\sigma(\boldsymbol{p}) = \sigma(\boldsymbol{p})\zeta(t, \boldsymbol{x}) \exp\left(\frac{1}{q}\boldsymbol{p} \cdot \boldsymbol{x}Q\right), \tag{5.4}$$

which results in the aristotelian field

$$\phi(t, \boldsymbol{x}) = e^{-iEt} \int_{\mathbb{A}^3} d^3p e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \psi(\boldsymbol{p}), \tag{5.5}$$

taking values in the spin-s representation of SU(2). To work out the action of $\widetilde{G}$ on such fields, we first consider the general group element

$$g = g(R, \boldsymbol{m}, \boldsymbol{a}, \theta, s) = e^{sH_F + \theta Q + \boldsymbol{a}\cdot\boldsymbol{P}} e^{\boldsymbol{m}\cdot\boldsymbol{D}} R, \tag{5.6}$$

with $R \in \mathrm{SU}(2)$ and calculate

$$g^{-1}\zeta(t, \boldsymbol{x}) = \zeta(t - s, R^{-1}(\boldsymbol{x} - \boldsymbol{a}))e^{-\theta - \boldsymbol{m}\cdot(\boldsymbol{x}-\boldsymbol{a})Q} R^{-1} e^{-\boldsymbol{m}\cdot\boldsymbol{D}}, \tag{5.7}$$

from where we deduce (as in appendix A) that

$$(g \cdot \phi)(t, \boldsymbol{x}) = e^{iq(\theta + \boldsymbol{m}\cdot(\boldsymbol{x}-\boldsymbol{a}))} \rho(R)\phi(t - s, R^{-1}(\boldsymbol{x} - \boldsymbol{a})), \tag{5.8}$$

with ρ the spin-s representation of SU(2). Let us emphasise that, as expected, there is no action of the dipole transformations on the coordinates, cf. (4.6), and they indeed transform as expected under charge and dipole transformations

$$\left(e^{\theta Q + \boldsymbol{m}\cdot\boldsymbol{D}} \cdot \phi\right)(t, \boldsymbol{x}) = e^{iq(\theta + \boldsymbol{m}\cdot\boldsymbol{x})} \phi(t, \boldsymbol{x}). \tag{5.9}$$

5.3 Neutral aristotelian fields

These provide field theoretical realisations of UIRs of class $\mathbb{III}(n, p, E)$. Here $\widetilde{\tau} = (0, \boldsymbol{p}, E)$ and $\mathcal{O}_{\widetilde{\tau}}$ is the 2-sphere of radius $\|\boldsymbol{p}\|$. We think of the sphere as the extended complex plane as we did when discussing carrollian fields, with coset representative $\sigma(z)$ given by (3.34) and $\zeta(t, \boldsymbol{x}) = \exp(\boldsymbol{x} \cdot \boldsymbol{P} + tH_F)$ as above. The expression for the neutral field is now

$$\phi(t, \boldsymbol{x}) = e^{-iEt} \int_{\mathbb{C}} \frac{2idz \wedge d\bar{z}}{(1 + |z|^2)^2} e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)} \rho(\sigma(z))\psi(z), \tag{5.10}$$

where $\boldsymbol{\pi}(z)$ is given by equation (3.37). To work out the $\widetilde{G}$-action, we consider $g = g(R, \boldsymbol{m}, \boldsymbol{a}, \theta, s)$ as above and (via equation (A.10)) work out that

$$(g \cdot \phi)(t, \boldsymbol{x}) = \rho(R)\phi(t - s, R^{-1}(\boldsymbol{x} - \boldsymbol{a})), \tag{5.11}$$

since $q = 0$ now. Again the irreducible representations are those where ϕ is either a highest (if $n > 0$) or lowest (if $n < 0$) weight vector of the spin-$|\frac{n}{2}|$ representation ρ of SU(2). The discussion in section 4.3 applies mutatis mutandis. The field equations in that section for the cases of $s = 0, \frac{1}{2}, 1$ are to be supplemented by the condition $\frac{\partial\phi}{\partial t} = -iE\phi$.

6 Quantum field theory

The unitary irreducible representations of the Carroll and dipole groups can be used as a starting point for the development of carrollian[13] and fractonic quantum field theories, via the process known as second quantisation. Multiparticle states formed from the quantum states of the unitary irreducible representations presented in section 1.1 will then span the corresponding Fock spaces.

The following discussion centers around non-interacting quantum field theories, with a particular focus on scalar representations corresponding to massive carrollion/fractonic monopoles $\mathbb{II}(s=0)$ and massless carrollions/aristotelions $\mathbb{III}(n=0,p)$.

6.1 Massive carrollions/fractonic monopole

While most of what we are saying can be generalised to generic spin we will in the following restrict to spin $s=0$. This means the we focus on massive carrollions and fractonic monopoles, corresponding to UIRs of class $\mathbb{II}(s=0,E_0)$ and $\mathbb{II}(s=0,q,E)$, respectively.

The quantum field theory of an interacting real massive scalar field with action

$$I=\int dt d^3x\left(\frac{1}{2}\dot{\phi}^2-\frac{1}{2}E_0^2\phi^2-V(\phi)\right), \tag{6.1}$$

was studied by Klauder in the early 70s [22, 23] (see also the references therein and chapter 10 in [34] for an useful overview). Due to the absence of gradient terms $(\partial\phi)^2$ and the resulting independence of the evolution of the dynamics at each point in space, he called this scalar field theory "ultralocal." In recent times this theory has reemerged as "electric" Carroll scalar field theory [12, 21]. The space of solutions of the free theory, where $V(\phi)=0$, coincides with that of the free field equation for the massive carrollion in (4.7) when we consider simultaneously solutions with positive and negative energies. Therefore, in the context of second quantisation, the Fock space of this model will be spanned by multiparticle states formed from the vectors of the unitary irreducible representation $\mathbb{II}(s=0,\pm E_0)$. In particular, when considering configurations involving both positive and negative energies, according to (4.6) the wave function takes the form

$$\phi\left(t,\boldsymbol{x}\right)=\int_{\mathbb{A}^3}d^3p\left(e^{-i(E_0t+\boldsymbol{p}\cdot\boldsymbol{x})}\psi\left(\boldsymbol{p}\right)+e^{i(E_0t+\boldsymbol{p}\cdot\boldsymbol{x})}\psi^\dagger\left(\boldsymbol{p}\right)\right). \tag{6.2}$$

The operators $\psi^\dagger(\boldsymbol{p})$ and $\psi(\boldsymbol{p})$ are interpreted as creation and annihilation operators. Thus, if the vacuum state is denoted by $|0\rangle$, then one-particle states belonging to the representation $\mathbb{II}(s=0,E_0)$ are given by

$$|\boldsymbol{p}\rangle=\psi^\dagger\left(\boldsymbol{p}\right)|0\rangle \tag{6.3}$$

or $|E_0,\boldsymbol{p}\rangle$ if we wish to make the specific energy explicit.

This theory can be derived from an ultrarelativistic limit $(c\to 0)$ of a real Klein-Gordon field. In this limit, all the frequencies of the relativistic scalar field $\omega=\sqrt{E_0^2+c^2\boldsymbol{p}^2}$

[13] We are aware of a forthcoming work which also discusses Carroll quantum field theories [33].

collapse to a fixed value $\omega = E_0$. Geometrically, this is the limit in which a hyperboloid in momentum space tends to the plane, pictured in figure 1 (the full limit is given in Part I, figure 2). Therefore, every one-particle state (6.3) possesses exactly the same energy E_0, irrespective of its momentum. Consequently, there exist an infinite degeneracy of states with identical energy, which follows from the fact that the Hamiltonian is a central element of the Carroll algebra.

The main points discussed in the carrollian context are also valid for fractons, albeit with a different physical interpretation. For example, the free part of the action that describes the complex scalar field model of fractons introduced by Pretko [35] is given by

$$I = \int dt d^3x \left[\dot{\phi}^* \dot{\phi} - E^2 \phi^* \phi\right] , \tag{6.4}$$

This theory is described by wave functions that belong to the unitary irreducible representation $\mathbb{II}(s = 0, q, E)$, which corresponds to the charged aristotelian fields discussed in section 5.2.

Similar to the Carroll case, there is also an infinite degeneracy of eigenstates having the same energy. In the context of fractons, the degeneracy also extends to the vacuum because the representations of class $\mathbb{II}$ allow for a vanishing value of the energy. For example we are free to create a large number of monopoles with zero energy and arbitrary momentum. The degeneracy in the energy makes the study of statistical mechanics of fractons with dipole symmetries a non-trivial task. A possibility explored earlier in the literature (see footnote 1 and [28] and references therein) is to describe these models on a finite lattice, that acts as a UV regulator for the momentum. However, the continuum limit of quantities such as the partition function in the canonical ensemble, or the entropy in the microcanonical ensemble, is not clearly understood. This can be attributed to the fact that the Gibbs operator $e^{-\beta \hat{H}}$ is not trace class.[14] One potential approach to address this problem could be to add additional operators to the partition function in order to lift the infinite degeneracy, akin to what is usually done in the representation theory of infinite-dimensional algebras (e.g., W-algebras) in order to compute characters. Therefore, the thermodynamics for carrollian or fracton theories of this type is subtle.

The previous discussions were mainly centred on free field theories. However, it is natural to explore the implications of incorporating interactions. A potential that is invariant under dipole transformations and is of quartic order in the field ϕ is given by [35]

$$V = \lambda \left|\phi \partial_i \partial_j \phi - \partial_i \phi \partial_j \phi\right|^2 + \lambda' \phi^{*2} \left(\phi \partial_i \partial^i \phi - \partial_i \phi \partial^i \phi\right) , \tag{6.5}$$

[14] For a single massive Carroll particle we compute the character for a pure imaginary time translations, or equivalently, the canonical partition function. Formally this is given by $\mathrm{Tr}(e^{-\beta \hat{H}}) \propto \int d^3p e^{-\beta E_0} \delta^3(0)$ where E_0 is independent of $\boldsymbol{p}$. We see two divergences, the first $\delta^3(0)$ is due to the infinite volume in space and therefore an infrared divergence. It is also present for massive Poincaré particles (see, e.g., [30]) and one way to regulate it is by putting the particle in a box. The second divergence is due to $\int d^3p$ the infinite volume integral in momentum space (see figure 1) and it is therefore an ultraviolet divergence. This is a carrollian feature due to the fact that the energy is independent of the momentum and can potentially be regulated by putting the particle on a lattice. Similar remarks apply to the monopoles.

where λ and λ' are coupling constants. The unitary irreducible representations of class $\mathbb{II}$ can be used to describe asymptotic states for this class of interacting quantum field theories.

Note that in theories with conserved dipole charges, the notion of a scattering process between fractonic monopoles is subtle. If one assumes that the initial and final states are separated by large enough distances such that the interaction between them is negligible, then the free monopoles will not move and the scattering process will never take place. For a scattering process to exhibit non-trivial behaviour, it is necessary for at least some of the interaction to influence the asymptotic monopole states. Alternatively, due to their unrestricted mobility, the scattering between dipoles emerges as a natural physical process. A composite dipole can arise from the binding of two monopole states possessing charges of equal magnitude but opposite signs. Conversely, fundamental dipoles correspond to the unitary irreducible representations of classes $\mathbb{III}'$, $\mathbb{IV}_{\pm}$ and $\mathbb{V}_{\pm}$.

6.2 Massless carrollions/aristotelions

Massless carrollions/aristotelions are built on UIRs of class $\mathbb{III}(n = 0, p)$ where we again focus on vanishing helicity, hence $n = 0$.

As discussed in section 4.3, the wave function for scalar massless carrollions/aristotelions is, basically by construction, time-independent:

$$\partial_t \phi = 0, \tag{6.6}$$

and satisfies the Helmholtz equation

$$(\triangle + p^2)\phi(\boldsymbol{x}) = 0. \tag{6.7}$$

If the wave function $\phi(\boldsymbol{x})$ is real, it will be promoted to a hermitian operator in the second quantisation. Following (4.21) the general solution to the Helmholtz equation can be written as

$$\phi(\boldsymbol{x}) = \int_{\mathbb{C}} \frac{2idz \wedge d\bar{z}}{(1+|z|^2)^2} \left(e^{-i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)}\psi(z) + e^{i\boldsymbol{x}\cdot\boldsymbol{\pi}(z)}\psi^\dagger(z) \right). \tag{6.8}$$

The functions $\psi^\dagger(z)$ and $\psi(z)$ can be promoted to creation and annihilation operators, respectively. Therefore, one-particle states belonging to the representation of class $\mathbb{III}(n = 0, p)$ can be obtained by acting on the vacuum $|0\rangle$ according to

$$|z\rangle = \psi^\dagger(z)\,|0\rangle\ , \tag{6.9}$$

where z denotes a point on the 2-sphere defined by the constraint $\|\boldsymbol{p}\|^2 = p^2$. All states associated with massless carrollions/aristotelions have zero energy hence standard thermodynamics and statistical mechanics for massless carrollian quantum field theories is trivial. This observation is in line with the time-independence of the field $\phi(\boldsymbol{x})$ and the absence of a notion of ergodicity.

It is worth mentioning that whilst this theory shares similarities with the scalar field theory referred to as the “magnetic” Carroll scalar field theory proposed in [12, 21], the theories are not equivalent. The field equation of the magnetic Carroll scalar field is described by the Helmholtz equation supplemented with a source term

$$(\triangle + p^2)\phi(\boldsymbol{x}) = \dot{\pi}\,(t, \boldsymbol{x})\,, \tag{6.10}$$

where $\pi\,(t, \boldsymbol{x})$ is the canonical momentum conjugate to $\phi(\boldsymbol{x})$. Thus, does not describe the free theory corresponding to a UIR of the Carroll group. To characterise a UIR of the Carroll group, additional conditions corresponding to $\dot{\pi} = 0$ need to be taken into account. In [4] we proposed an action of the form

$$I[\phi, \pi, u] = \int dt d^3x \left(\pi\dot{\phi} - u(\triangle + p^2)\phi\right). \tag{6.11}$$

As the magnetic theory, characterized by eq. (6.10), does not describe a UIR, it opens the possibility for its statistical mechanics to be non-trivial. We leave this question for future investigations.

7 Discussion

We classified and related unitary irreducible representations of the Carroll and dipole groups, i.e., we defined quantum Carroll and fracton particles, which are summarised in section 1.1 and table 1. This lifts the correspondence between elementary Carroll and fracton particles [4] to the quantum world.

The UIRs have distinctive features depending on whether the Carroll energy/fracton charge vanishes or not. Isolated massive carrollions and monopoles have quantised spin and a position operator and, when isolated, do not move. The can be represented either as wave functions of momenta or in centre-of-mass/dipole-moment space, which are related via a Fourier transform.

On the other hand for vanishing energy (or fracton charge) we find massless carrollions or neutral fractons. When the centre-of-mass or dipole moment vanishes, massless carrollions or aristotelions have quantised helicity and are intrinsically very similar.

We also constructed field theories in Carroll or Aristotle spacetime which we supplemented by free field equations projecting to the UIR. Alternatively, the wave equations for scalar fields could be derived by applying Dirac quantisation to the constraints that define the classical particle models[15] described in Part I.

Finally, we commented on the Carroll and fracton quantum field theories for which the particles can be thought of as elementary excitations. We highlighted the relation of massive Carroll and charged fracton theories to ultralocal theories [22, 23] and emphasised that massive Carroll and (charged) fracton theories share similarities, like puzzling thermodynamic behaviour related to their infinite degeneracy with regard to energy and charge. We also pointed out that the magnetic scalar theory has a different number of degrees of freedom with respect to the free theory and presented an alternative action (6.11) which does.

[15] A different class of classical fracton particle models with subsystem symmetries were constructed in [36].

There are various interesting points for further exploration:

Interacting theories. Even though most of the discussion has concentrated on free theories, it is worth exploring the potential application of our analysis to interacting theories. Some carrollian/fractonic models with non-trivial interactions include self-interacting fracton scalar field theories and their coupling to fracton electrodynamics [28, 35, 37, 38], Carroll scalar fields and their coupling to Carroll electrodynamics [22, 39–41], theories of Yang-Mills type [42, 43], Carrollian gravity [21, 44–54], lower dimensional Chern-Simons theories [55–57] or extensions thereof [58–60], Carrollian JT or dilaton gravity [61, 62] or their supergravity versions [63], and theories with spacetime subsystem symmetries [64, 65]. An extensive list of references can be found in the reviews [5, 6, 66]. It would be interesting to explore whether these theories can be regarded as descriptions of interacting particles that belong to the UIRs of the Carroll and/or dipole groups, and to investigate the role played by the elementary particles in the development of perturbation theory.

It might be interesting to contrast this with the existing techniques used for ultralocal quantum field theories (e.g., [34] section 10).

Relation to timelike symmetries. In [38], "timelike" higher-form global symmetries were used to describe fractons. It might be interesting to understand the relation between these generalised symmetries and the definition of fractons as UIRs.

Lattice field theory. In this work we have focused on continuous symmetries. It might be interesting to understand the description of these symmetries on the lattice. See [38] and [28, Appendix D] for interesting comments.

Generic massless and dipole particles. The generic massless or generic dipole representations have quite exotic properties, but let us highlight some of their intricate features.

Similar to the massless continuous- or infinite-spin representations of the Poincaré group, they are actually the generic case in the massless/neutral sector (e.g., [67] provides a review of the Poincaré case). Continuous-spin particles share some similarities with this case and it would be interesting to further contrast their interesting properties. Continuous-spin UIRs are often discarded by causality [68, 69] arguments, but it is not clear if this applies to the case at hand, after all the theories are not Poincaré invariant. Continuous-spin particles too cannot be localised to a point (see, e.g., [70] and references therein) and again maybe this is a feature, rather than a bug, for carrollions or dipoles?

Carroll/fractons and flat space holography. Carrollian physics naturally emerges in the study of the asymptotic structure of spacetime in the absence of a cosmological constant (see, e.g., [16, 17, 71–77]), due to the underlying equivalence between BMS and conformal Carroll algebras [71]. Although our work does not focus on the con-

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formal extension, some of the structure of flat space holography is already dictated by Carroll symmetries alone:[16]

(i) As shown, e.g., in [78, 79] the asymptotic behaviour of a massless scalar field on a flat spacetime at null infinity exhibits two alternative sectors: a radiative and a non-radiative one. From the perspective of (conformal) carrollian symmetries these two branches can be traced back to the imposition of Carroll symmetries and the choice of vanishing or non-vanishing energy, as we have also presented in this work and in Part I.

One way to see this is by looking at the two-point functions of Carroll field theories which have two branches by only imposing Carroll symmetries [12, 73, 80], with further refinements once the extended symmetries are taken into account.

(ii) If we focus on the non-radiative sector and we denote the leading and subleading terms of the scalar field by ϕ_0 and ϕ_1, respectively, then they satisfy the following equations:

$$\dot{\phi}_0 = 0\,, \qquad (\Delta_{S^2} + h\,(h-1))\,\phi_0 = -2h\dot{\phi}_1, \tag{7.1}$$

where h is a real number and Δ_{S^2} is the Laplacian on the round two-sphere. Note that these equations coincide with that of the magnetic Carroll scalar field (6.10), with $p^2 = h\,(h-1)$ and $\pi = -2h\phi_1$. However, equation (7.1) is defined on the 2-sphere as a consequence of the topology of future (or past) null infinity. It could be interesting to explore this relation and the possible role of the UIR $\mathrm{III}(n,p)$ for flat space holography.

(iii) Massless carrollions (representations $\mathrm{III}(n,p)$) are zero-energy states naturally defined on the "celestial sphere." It might be worth to explore the potential connection with soft degrees of freedom.

Timelike infinity and fractons on hyperbolic space. We also proposed to generalise this correspondence to curved space [4]. This means to add

$$[P_a, P_b] = -\Lambda J_{ab} \qquad\qquad [P_a, H] = \Lambda B_a \tag{7.2}$$

to the Carroll algebra (2.1), leading to AdS Carroll [25, 81] and

$$[P_a, P_b] = -\Lambda J_{ab} \qquad\qquad [P_a, Q] = \Lambda D_a \tag{7.3}$$

to the dipole algebra (1.30). This means that the underlying space geometry is now not flat but given by three-dimensional hyperbolic space or the 3-sphere, for $\Lambda < 0$ and $\Lambda > 0$, respectively. The relation to timelike infinity is given by the fact

[16] Some of the relevance of Carroll symmetries derives from the conformal extension of $2+1$ dimensional Carroll symmetries. Even though the Carroll algebra in $2+1$ dimensions allows for nontrivial central extensions, the conformal extension does not (basically because it is isomorphic to the Poincaré algebra). Therefore most of our results apply to this case.

that AdS Carroll is the homogeneous model for the blow up of timelike infinity of asymptotically flat spacetimes [16].

For the quantum version of the correspondence on curved space it is interesting to note that AdS Carroll is a homogeneous space of the Poincaré group. Consequently, the quantum particles of the algebras described above are the same as the ones classified by Wigner [8]. In this sense AdS Carroll, fractons on curved space and flat space (holography) are indeed connected. (In this context see also [9, 19, 82, 83]. It could be interesting to understand if there is any relation to the models discussed in [84, 85]).

The tools used in this work are not restricted to the Carroll and dipole groups and we will discuss other particles with restricted mobility in a future work [86].

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A The method of induced representations

In this appendix we will summarise the salient points of the method of induced representations to construct unitary irreducible representations of a group with an abelian normal subgroup and their description as spacetime fields subject to the free field equations. This was pioneered by Wigner [8] for the case of the Poincaré group and developed into a mathematical theory by Mackey. Most of this material is standard. See, for example, [31] or [87–90].

A.1 Induced representations à la Mackey

Let $G = K \ltimes T$ be a connected Lie group with T an abelian normal subgroup which we will assume to be simply connected, so isomorphic to $\mathbb{R}^n$ for some n. We shall let $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{t}$ denote their respective Lie algebras. Due to the semidirect product structure, K acts on T and hence on $\mathfrak{t}^*$. Let $\tau \in \mathfrak{t}^*$ and let $\mathcal{O}_\tau = K \cdot \tau$ denote its K-orbit. Let $K_\tau \subset K$ denote the stabiliser subgroup, so that $\mathcal{O}_\tau$ is K-equivariantly diffeomorphic to K/K_τ. We will use in the sequel an equivalent description of $\mathcal{O}_\tau$ as $G/(K_\tau \ltimes T)$; although G does not act effectively, since T acts trivially on $\mathcal{O}_\tau$. Every $\tau \in \mathfrak{t}^*$ defines a character χ_τ of T and hence a one-dimensional unitary representation: if $t = \exp(X) \in T$, for some $X \in \mathfrak{t}$, then $\chi_\tau(t) = e^{i\langle \tau, X\rangle}$ with $\langle -, - \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ denoting the dual pairing. We will let W denote a complex unitary irreducible representation of K_τ. It is a representation of $K_\tau \ltimes T$ via

$$(th) \cdot w = \chi_\tau(t) h \cdot w \tag{A.1}$$

for all $t \in T$, $h \in K_\tau$ and $w \in W$. In this representation, K_τ and T commute. Hence th and ht act in the same way.

We will let $E_W := K \times_{K_\tau} W$ denote the homogeneous vector bundle over $\mathcal{O}_\tau$ associated to W. We will let $\Gamma(E_W)$ denote the space of smooth sections of $E_W \to \mathcal{O}_\tau$. Such sections admit a different characterisation in terms of so-called Mackey functions: smooth functions $f : K \to W$ which are K_τ-equivariant: that is, for all $k \in K$ and $h \in K_\tau$,

$$f(kh) = h^{-1} \cdot f(k), \tag{A.2}$$

where $\cdot$ stands for the linear K_τ-action on W. We will let $C^\infty_{K_\tau}(K, W)$ denote the vector space of Mackey functions. Functions on K_τ pull back to K_τ-invariant functions on K and in this way, $C^\infty_{K_\tau}(K, W)$ becomes a $C^\infty(\mathcal{O}_\tau)$-module.

Lemma 1. *$\Gamma(K \times_{K_\tau} W)$ and $C^\infty_{K_\tau}(K, W)$ are isomorphic as $C^\infty(\mathcal{O}_\tau)$-modules.*

Proof (sketch). Let $\sigma : \mathcal{O}_\tau \to K$ be a coset representative. This may only be partially defined, but we will assume that it is defined in an open dense subset of $\mathcal{O}_\tau$ which is of measure zero relative a K-invariant measure on $\mathcal{O}_\tau$, which we will also assume exists.[17]

If $f \in C^\infty_{K_\tau}(K, W)$, we define $\psi \in \Gamma(E_W)$ by $\psi(p) = f(\sigma(p))$ for all $p \in \mathcal{O}_\tau$. Conversely, if $\psi \in \Gamma(E_W)$, we define $f \in C^\infty_{K_\tau}(K, W)$ as follows: $f(\sigma(p)h) = h^{-1} \cdot \psi(p)$, where $h \in K_\tau$. This results in a K_τ-equivariant function by construction and it is not hard to show that it is independent of the choice of coset representative. $\square$

The vector bundle E_W can also be described as a homogeneous vector bundle $G \times_{K_\tau \ltimes T} W$, with the action of $K_\tau \ltimes T$ on W given by (A.1) and its sections can therefore be described equivalently as $(K_\tau \ltimes T)$-equivariant functions $G \to W$. In other words, as in Lemma 1, we have an isomorphism $C^\infty_{K_\tau}(K, W) \cong C^\infty_{K_\tau \ltimes T}(G, W)$ of $C^\infty(\mathcal{O}_\tau)$-modules, where we lift $C^\infty(\mathcal{O}_\tau)$ to G as the $(K_\tau \ltimes T)$-invariant functions. Under this isomorphism, a given $f \in C^\infty_{K_\tau}(K, W)$ is sent to $F : G \to W$, defined by

$$F(tk) := \chi_{k\cdot\tau}(t^{-1}) f(k) \tag{A.3}$$

for all $t \in T$ and $k \in K$. The function F is $(K_\tau \ltimes T)$-equivariant by construction.

The virtue of the Mackey functions $C^\infty_{K_\tau \ltimes T}(G, W)$ is that they admit a natural action of G which is very easy to describe. For $F \in C^\infty_{K_\tau \ltimes T}(G, W)$ and all $g, g' \in G$ we have

$$(g \cdot F)(g') = F(g^{-1} g'). \tag{A.4}$$

Let $g = th$ with $t \in T$ and $h \in K$ and choose $g' = k \in K$. Then,

$$\begin{aligned}
(th \cdot F)(k) &= F(h^{-1} t^{-1} k) \\
&= F(h^{-1} k k^{-1} t^{-1} k) \\
&= \chi_\tau(k^{-1} t k) F(h^{-1} k) \\
&= \chi_{k\cdot\tau}(t) F(h^{-1} k).
\end{aligned}$$

[17] By considering multiplier representations, we need only assume that the measure is quasi-invariant, but in the examples we have in mind, we will always have K-invariant measures.

Therefore we conclude that $g = th$ acts on a Mackey function $f : K \to W$ as

$$(g \cdot f)(k) = \chi_{k\cdot\tau}(t)\, f(h^{-1}k) \tag{A.5}$$

where $k \in K$.

Lemma 2. *The transformed function $g \cdot f$ is K_τ-equivariant.*

Proof. This follows from the fact that left- and right-multiplications commute and $h\cdot\tau = \tau$ for $h \in K_\tau$, so that

$$\begin{aligned}(g \cdot f)(kh') &= \chi_{kh'\cdot\tau}(t) f(h^{-1}kh') \\ &= \chi_{k\cdot\tau}(t)(h')^{-1} \cdot f(h^{-1}k) \\ &= (h')^{-1} \cdot \left(\chi_{k\cdot\tau}(t) f(h^{-1}k)\right) \\ &= (h')^{-1} \cdot (g \cdot f)(k).\end{aligned}$$

□

Therefore we get a representation of G on $C^\infty_{K_\tau}(K, W)$.

Proposition 3. *The representation of G on $C^\infty_{K_\tau}(K, W)$ just described is unitary relative to the hermitian inner product*

$$(f_1, f_2) = \int_{\mathcal{O}_\tau} d\mu(k \cdot \tau)\, \langle f_1, f_2\rangle_W\, (k \cdot \tau), \tag{A.6}$$

where $d\mu$ is a K-invariant measure on $\mathcal{O}_\tau$ and $\langle f_1, f_2\rangle_W$ is function on $\mathcal{O}_\tau$ which pulls back, via the orbit map $K \to \mathcal{O}_\tau$ sending $k \mapsto k \cdot \tau$, to the function $k \mapsto \langle f_1(k), f_2(k)\rangle_W$.

Proof. By assumption, W is a unitary representation of K_τ with hermitian inner product $\langle -, -\rangle_W$ and hence for all $h \in K_\tau$ and $k \in K$,

$$\begin{aligned}\langle f_1(kh), f_2(kh)\rangle_W &= \left\langle h^{-1} \cdot f_1(k), h^{-1} \cdot f_2(k)\right\rangle_W && \text{(since } f_1, f_2 \text{ are equivariant)} \\ &= \langle f_1(k), f_2(k)\rangle_W\,, && \text{(since } \langle -, -\rangle_W \text{ is } K_\tau\text{-invariant)}\end{aligned}$$

hence the function is the pull-back of a unique function on $\mathcal{O}_\tau$ and it is that function that we integrate against the invariant measure.

Unitarity of the G-representation now follows because with $g = th$

$$\begin{aligned}(g \cdot f_1, g \cdot f_2) &= \int_{\mathcal{O}_\tau} d\mu(k \cdot \tau)\, \langle g \cdot f_1, g \cdot f_2\rangle_W\, (k \cdot \tau) \\ &= \int_{\mathcal{O}_\tau} d\mu(k \cdot \tau) \left\langle \chi_{k\cdot\tau}(t) f_1(h^{-1}k), \chi_{k\cdot\tau}(t) f_2(h^{-1}k)\right\rangle_W \\ &= \int_{\mathcal{O}_\tau} d\mu(k \cdot \tau) \left\langle f_1(h^{-1}k), f_2(h^{-1}k)\right\rangle_W && \text{(since } \langle -, -\rangle_W \text{ is hermitian)} \\ &= \int_{\mathcal{O}_\tau} d\mu(k \cdot \tau)\, \langle f_1, f_2\rangle_W\, (h^{-1}k \cdot \tau) \\ &= \int_{\mathcal{O}_\tau} d\mu(hk' \cdot \tau))\, \langle f_1, f_2\rangle_W\, (k' \cdot \tau) && \text{(changing variables to } k' = h^{-1}k) \\ &= \int_{\mathcal{O}_\tau} d\mu(k' \cdot \tau)\, \langle f_1, f_2\rangle_W\, (k' \cdot \tau) && \text{(invariance of the measure)} \\ &= (f_1, f_2).\end{aligned}$$

□

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We can transport this unitary representation on $C^\infty_{K_\tau}(K,W)$ to a unitary representation of G on sections of E_W. Explicitly, if $\psi \in \Gamma(E_W)$ and $g \in G$,

$$(g \cdot \psi)(p) := (g \cdot F)(\sigma(p)) = F(g^{-1}\sigma(p)). \tag{A.7}$$

Using the product

$$g^{-1}\sigma(p) = \sigma(g^{-1} \cdot p)h(g^{-1}, p), \tag{A.8}$$

where $h(g^{-1}, p) \in K_\tau \ltimes T$, we can rewrite equation (A.7) as

$$(g \cdot \psi)(p) = h(g^{-1}, p)^{-1} \cdot \psi(g^{-1} \cdot p). \tag{A.9}$$

The function $h : G \times \mathcal{O}_\tau \to K_\tau \ltimes T$ satisfies some cocycle properties which guarantee that the above is indeed a representation of G.

It is a fundamental result in this subject that if W is both unitary and irreducible as a representation of K_τ, then so is (the Hilbert space completion of the square-integrable sections in) $\Gamma(K \times_{K_\tau} W)$ as a representation of G, which we denote by $L^2(\mathcal{O}_\tau, K \times_{K_\tau} W)$. We proved unitarity in Proposition 3. Irreducibility follows from a straightforward application of the SNAG theorem, as we now briefly sketch.

Proposition 4. *Let $\mathscr{H} = L^2(\mathcal{O}_\tau, K \times_{K_\tau} W)$ and let $U : G \to \mathrm{U}(\mathscr{H})$ be the unitary induced representation constructed above. Then if W is an irreducible representation of K_τ, $\mathscr{H}$ is an irreducible representation of G.*

Proof (sketch). Let $\mathscr{H}' \subset \mathscr{H}$ be a G-invariant subspace of $\mathscr{H}$. Then the orthogonal projection onto $\mathscr{H}'$ is a continuous operator on $\mathscr{H}$ which commutes with the action of G. We are done if we show that any such operator is necessarily a multiple of the identity, so that $\mathscr{H}'$ is not then a proper subspace.

Let A be a continuous operator on $\mathscr{H}$ which commutes with the action of G. In particular it commutes with the action of the translation subgroup $T \subset G$. By the SNAG theorem (see, e.g., [31, section 6.2]), A acts pointwise $(A \cdot \psi)(p) = A(p) \cdot \psi(p)$ for all $p = \sigma(p) \cdot \tau \in \mathcal{O}_\tau$ and since A commutes with the action of $\sigma(p)$, it follows that

$$\begin{aligned}
A(p) \cdot \psi(p) &= (A \cdot \psi)(p) \\
&= \left(\mathrm{U}(\sigma(p)^{-1}) \cdot A \cdot \psi\right)(\tau) \\
&= (A \cdot \mathrm{U}(\sigma(p)^{-1}) \cdot \psi)(\tau) \\
&= A(\tau) \cdot (\mathrm{U}(\sigma(p)^{-1}) \cdot \psi)(\tau) \\
&= A(\tau) \cdot \psi(p),
\end{aligned}$$

so that $A(p) = A(\tau)$ for all $p \in \mathcal{O}_\tau$. But $A(\tau) \in \operatorname{End} W$ commutes with the action of K_τ and since by hypothesis W is a complex irreducible representation of K_τ, Schur's Lemma guarantees that $A(\tau)$ is a multiple of the identity, and hence so is A. □

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A.2 Induced representations as free field theories

Mackey theory exhibits UIRs of G as (square-integrable) sections of bundles over $\mathcal{O}_\tau$, which in the case of kinematical groups such as the Poincaré, Galilei or Carroll groups, is an orbit in momentum space. It is however often desirable to exhibit the representation as fields on the kinematical spacetime: Minkowski, Galilei or Carroll, say. Such a spacetime is a homogeneous space of G which is G-equivariantly diffeomorphic to G/K. Such fields are therefore sections of homogeneous vector bundles over G/K associated to representations of K. Since we only have W, which is a representation of K_τ, this entails a *choice*: namely, that of a representation V of K which, when restricted to K_τ, contains a subrepresentation isomorphic to W. The representation V need not be unitary, of course.

Given $f \in C^\infty_{K_\tau}(K, V)$ we get $F \in C^\infty_{K_\tau \ltimes T}(G, V)$ as before, by having T act via the character χ_τ. We now define $\widehat{F} : G \to V$ by

$$\widehat{F}(g) := \int_{\mathcal{O}_\tau} d\mu(p)\sigma(p) \cdot F(g\sigma(p)) \tag{A.10}$$

where $d\mu$ is a K-invariant measure on $\mathcal{O}_\tau$ and we are assuming that the coset representative σ is defined in the complement of a set of measure zero. We now show that $\widehat{F}$ is K-equivariant.

Proposition 5. $\widehat{F} \in C^\infty_K(G, V)$.

Proof. For all $k \in K$ and $g \in G$,

$$\widehat{F}(gk) = \int_{\mathcal{O}_\tau} d\mu(p)\sigma(p)F(gk\sigma(p)). \tag{A.11}$$

We now have

$$k\sigma(p) = \sigma(k \cdot p)h(k, p) \tag{A.12}$$

for some $h(k, p) \in K_\tau$. Since F is in particular K_τ-equivariant,

$$F(gk\sigma(p)) = F(g\sigma(k \cdot p)h(k, p)) = h(k, p)^{-1} \cdot F(g\sigma(k \cdot p)). \tag{A.13}$$

But now notice that $\sigma(p)h(k, p)^{-1} = k^{-1}\sigma(k \cdot p)$, so that

$$\widehat{F}(gk) = \int_{\mathcal{O}_\tau} d\mu(p)k^{-1}\sigma(k \cdot p)F(g\sigma(k \cdot p)). \tag{A.14}$$

Let $k \cdot p = p'$. By the invariance of the measure, $d\mu(p) = d\mu(p')$ and hence

$$\widehat{F}(gk) = \int_{\mathcal{O}_\tau} d\mu(p')k^{-1}\sigma(p')F(g\sigma(p')) = k^{-1} \cdot \widehat{F}(g). \tag{A.15}$$

$\square$

It follows that $\widehat{F}$ defines a section of the homogeneous vector bundle $G \times_K V$ over G/K, which we can describe explicitly as follows. Let $\zeta : G/K \to T$ be a coset representative. Again this may only be locally defined, but we assume it is defined on the complement of a set of measure zero on G/K relative to a G-invariant measure. We define a section ϕ of $G \times_K V$ by

$$\phi(x) := \widehat{F}(\zeta(x)) = \int_{\mathcal{O}_\tau} d\mu(p)\sigma(p) \cdot F(\zeta(x)\sigma(p)). \tag{A.16}$$

Since T is a normal subgroup, we can write $\zeta(x)\sigma(p) = \sigma(p)\zeta(x')$, where

$$\zeta(x') = \sigma^{-1}(p)\zeta(x)\sigma(p), \tag{A.17}$$

and moreover

$$\chi_\tau(\zeta(x')^{-1}) = \chi_{\sigma(p)\cdot\tau}(\zeta(x)^{-1}), \tag{A.18}$$

so that

$$\phi(x) = \int_{\mathcal{O}_\tau} d\mu(p)\chi_{\sigma(p)\cdot\tau}(\zeta(x)^{-1})\sigma(p) \cdot \psi(p), \tag{A.19}$$

where we have used that $F(\sigma(p)) = \psi(p)$. If $\zeta(x) = \exp(X)$, then

$$\chi_{\sigma(p)\cdot\tau}(\zeta(x)^{-1}) = e^{-i\langle\sigma(p)\cdot\tau,X\rangle} \tag{A.20}$$

so that

$$\phi(x) = \int_{\mathcal{O}_\tau} d\mu(p)e^{-i\langle\sigma(p)\cdot\tau,X\rangle}\sigma(p) \cdot \psi(p) \tag{A.21}$$

is seen to be a group-theoretical generalisation of the Fourier transform: it relates a section ψ of $K \times_{K_\tau} W$ over $\mathcal{O}_\tau$ to a section ϕ of $G \times_K V$ over G/K. It bears reminding that we have made a choice of representation V. Other choices (such as coset representatives) are immaterial.

Finally, the G-action on ϕ is given by

$$(g \cdot \phi)(x) := \widehat{F}(g^{-1}\zeta(x)). \tag{A.22}$$

We may expand this using

$$g^{-1}\zeta(x) = \zeta(g^{-1} \cdot x)k(g^{-1}, x) \tag{A.23}$$

where $k : G \times G/K \to K$ is thus defined. Then

$$\widehat{F}(g^{-1}\zeta(x)) = \widehat{F}(\zeta(g^{-1} \cdot x)k(g^{-1}, x)) = k(g^{-1}, x)^{-1} \cdot \widehat{F}(\zeta(g^{-1} \cdot x)), \tag{A.24}$$

or, finally,

$$(g \cdot \phi)(x) = k(g^{-1}, x)^{-1} \cdot \phi(g^{-1} \cdot x). \tag{A.25}$$

In those cases where V properly contains W, the representation of G on sections of $G \times_K V$ is not irreducible. To restore irreducibility we need to somehow project our fields to W. This is an algebraic operation on the fibre of $K \times_{K_\tau} V$ at the identity coset in $\mathcal{O}_\tau$. We can then extend it to a point-dependent projector to the sub-bundle $K \times_{K_\tau} W \subset K \times_{K_\tau} V$ and via the generalised Fourier transform they become (pseudo-)differential operators which ought to be interpreted as free field equations for sections of $G \times_K V$.

To see how this goes about, let us assume that $W = \ker \Phi$ for some $\Phi \in \operatorname{End} V$. (This does not mean that Φ is K_τ-equivariant, by the way.) Let $F \in C^\infty_{K_\tau \ltimes T}(G, V)$ actually take values in W, so that it is in the image of the natural embedding $C^\infty_{K_\tau \ltimes T}(G, W) \subset C^\infty_{K_\tau \ltimes T}(G, V)$. Then for all $g \in G$, $\Phi F(g) = 0$ and hence integrating,

$$\int_{\mathcal{O}_\tau} d\mu(p)\sigma(p)\Phi F(g\sigma(p)) = 0. \tag{A.26}$$

In particular this is true for $g = \zeta(x)$. We can rewrite this as

$$0 = \int_{\mathcal{O}_\tau} d\mu(p) \underbrace{\sigma(p)\Phi\sigma(p)^{-1}}_{=:\Phi_p} \sigma(p)F(\zeta(x)\sigma(p)). \tag{A.27}$$

As above, we have that $\zeta(x)\sigma(p) = \sigma(p)\zeta(x')$ and hence

$$\begin{aligned} 0 &= \int_{\mathcal{O}_\tau} d\mu(p)\Phi_p\sigma(p)F(\sigma(p)\zeta(x')) \\ &= \int_{\mathcal{O}_\tau} d\mu(p)\chi_\tau(\zeta(x')^{-1})\Phi_p\sigma(p)F(\sigma(p)) && \text{(using the equivariance of } F\text{)} \\ &= \int_{\mathcal{O}_\tau} d\mu(p)\chi_{\sigma(p)\cdot\tau}(\zeta(x))^{-1}\Phi_p\sigma(p)\psi(p). \end{aligned}$$

The above integral defines the action of a pseudo-differential operator $\widehat{\Phi}_x$ on the field $\phi(x)$ given by

$$\begin{aligned} \widehat{\Phi}_x\phi(x) &= \widehat{\Phi}_x \int_{\mathcal{O}_\tau} d\mu(p)\chi_{\sigma(p)\cdot\tau}(\zeta(x)^{-1})\sigma(p)\cdot\psi(p) && \text{(by equation (A.19))} \\ &= \int_{\mathcal{O}_\tau} d\mu(p)\chi_{\sigma(p)\cdot\tau}(\zeta(x))^{-1}\Phi_p\sigma(p)\psi(p). \end{aligned}$$

In some cases, depending on the form of the Fourier-like kernel $\chi_{\sigma(p)\cdot\tau}(\zeta(x))^{-1}$ and the form of Φ_p, $\widehat{\Phi}_x$ is an honest differential operator; but in any case, we obtain a field equation (which may be non-local): $\widehat{\Phi}_x\phi(x) = 0$.

B Hopf charts on SU(2)

In this appendix we record some useful charts on SU(2) adapted to the Hopf fibration $\mathrm{SU}(2) \to S^2$ which play a rôle in our description of UIRs of the Carroll group of class $\underline{\mathrm{V}}$.

The Lie group SU(2) is diffeomorphic to the 3-sphere and this is made transparent by embedding the 3-sphere in $\mathbb{C}^2$ as the unit sphere: $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ and then identifying $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$ with the special unitary matrix $\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$. Let us define the following open subsets of S^3:

$$\begin{aligned} U_1 &= \left\{(z_1, z_2) \in S^3 \;\middle|\; z_1 \neq 0\right\} \\ U_2 &= \left\{(z_1, z_2) \in S^3 \;\middle|\; z_2 \neq 0\right\}. \end{aligned} \tag{B.1}$$

Clearly, $S^3 = U_1 \cup U_2$. We define surjective maps $\pi_1 : U_1 \to \mathbb{C}$ and $\pi_2 : U_2 \to \mathbb{C}$ by $\pi_1(z_1, z_2) = z_2/z_1$ and $\pi_2(z_1, z_2) = z_1/z_2$, which are nothing but the restriction of the Hopf fibration $\mathrm{SU}(2) \to S^2$ to each of U_1 and U_2 composed with stereographic projection from (the complement of a point in) S^2 to $\mathbb{C}$.

For $z \in \mathbb{C}$, the fibre $\pi_1^{-1}(z)$ consists of those (z_1, z_2) such that $z_2/z_1 = z$ and $|z_1|^2 + |z_2|^2 = 1$; that is,

$$\pi_1^{-1}(z) = \left\{ \left(\frac{\zeta}{\sqrt{1+|z|^2}}, \frac{z\zeta}{\sqrt{1+|z|^2}} \right) \;\middle|\; |\zeta| = 1 \right\}, \tag{B.2}$$

which is diffeomorphic to a circle. Similarly,

$$\pi_2^{-1}(z) = \left\{ \left(\frac{z\zeta}{\sqrt{1+|z|^2}}, \frac{\zeta}{\sqrt{1+|z|^2}} \right) \;\middle|\; |\zeta| = 1 \right\}. \tag{B.3}$$

This allows us to establish charts[18] $\varphi_1 : U_1 \to \mathbb{C} \times S^1$ and $\varphi_2 : U_2 \to \mathbb{C} \times S^1$ by

$$\varphi_1(z_1, z_2) = \left(\frac{z_2}{z_1}, \frac{z_1}{|z_1|} \right) \qquad \text{and} \qquad \varphi_2(z_1, z_2) = \left(\frac{z_1}{z_2}, \frac{z_2}{|z_2|} \right), \tag{B.4}$$

whose inverses give parametrisations of U_1 and U_2 in terms of $\mathbb{C} \times S^1$:

$$\varphi_1^{-1}(z,\zeta) = \left(\frac{\zeta}{\sqrt{1+|z|^2}}, \frac{z\zeta}{\sqrt{1+|z|^2}} \right) \qquad \text{and} \qquad \varphi_2^{-1}(z,\zeta) = \left(\frac{z\zeta}{\sqrt{1+|z|^2}}, \frac{\zeta}{\sqrt{1+|z|^2}} \right). \tag{B.5}$$

On the overlap $U_1 \cap U_2$, we have that the transition functions are given by

$$\varphi_1 \circ \varphi_2^{-1} : (z, \zeta) \mapsto \left(\frac{1}{z}, \frac{z}{|z|}\zeta \right) \tag{B.6}$$

and a formally identical expression for $\varphi_2 \circ \varphi_1^{-1}$.

Let $g_1 : \mathbb{C} \times S^1 \to \mathrm{SU}(2)$ and $g_2 : \mathbb{C} \times S^1 \to \mathrm{SU}(2)$ be the compositions of the parametrisations with the identification between $S^3 \subset \mathbb{C}^2$ and $\mathrm{SU}(2)$. Explicitly,

$$g_1(z,\zeta) = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} \zeta & z\zeta \\ -\overline{z}\zeta^{-1} & \zeta^{-1} \end{pmatrix} \tag{B.7}$$

and

$$g_2(z,\zeta) = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} z\zeta & \zeta \\ -\zeta^{-1} & \zeta^{-1}\overline{z} \end{pmatrix}. \tag{B.8}$$

We may use these maps to pull-back to $\mathbb{C} \times S^1$ the left-invariant (say) Maurer-Cartan one-form on $\mathrm{SU}(2)$. Let us work with g_2 for definiteness:

$$g_2^{-1} dg_2 = \frac{1}{1+|z|^2} \begin{pmatrix} \frac{1}{2}(\overline{z}dz - zd\overline{z}) + (|z|^2 - 1)\frac{d\zeta}{\zeta} & -d\overline{z} + 2\overline{z}\frac{d\zeta}{\zeta} \\ dz + 2z\frac{d\zeta}{\zeta} & -\frac{1}{2}(\overline{z}dz - zd\overline{z}) - (|z|^2 - 1)\frac{d\zeta}{\zeta} \end{pmatrix} \tag{B.9}$$

The round metric on S^3 agrees with the natural bi-invariant metric on $\mathrm{SU}(2)$ (both defined up to scale), whose volume form is given by

$$\mathrm{dvol} = -\frac{1}{3} \operatorname{Tr} \left(g_2^{-1} dg_2 \right)^3 = \frac{2dz \wedge d\overline{z}}{(1+|z|^2)^2} \frac{d\zeta}{\zeta}, \tag{B.10}$$

which agrees with the invariant measure in the inner product of the Carroll UIRs of class $\underline{\mathrm{V}}$.

[18] We use the word loosely, since $\mathbb{C} \times S^1$ is not diffeomorphic to an open subset of $\mathbb{R}^3$.

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