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Optimal Augmentation of a 2-Vertex-Connected Multigraph to an ℓ-Edge-Connected and 3-Vertex-Connected Multigraph [1]

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Abstract. Given an undirected multigraph $G = (V, E)$ and two positive integers ℓ and k, we consider the problem of augmenting G by the smallest number of new edges to obtain an ℓ-edge-connected and k-vertex-connected multigraph. In this paper, we show that the problem can be solved in $\tilde{O}(mn^2)$ time for any fixed ℓ and $k = 3$ if an input multigraph G is 2-vertex-connected, where $n = |V|$ and m is the number of pairs of adjacent vertices in G.

1. Introduction

The problem of augmenting a graph by adding the smallest number of new edges to meet edge-connectivity or vertex-connectivity requirement has been extensively studied as an important subject in network design, and many efficient algorithms have been developed so far. However, it was only very recent to have algorithms for augmenting both edge-connectivity and vertex-connectivity simultaneously (see [14, 15] for those results).

Let $G = (V, E)$ stand for an undirected multigraph with a set V of *vertices* and a set E of *edges*, where an edge with end vertices u and v is denoted by (u, v). A singleton set $\{x\}$ may be simply written as x, and " $\subset$ " implies proper inclusion while " $\subseteq$ " means " $\subset$ " or " $=$ ". For two disjoint subsets of vertices $X, Y \subset V$, we denote by $E_G(X, Y)$ the set of edges $e = (x, y)$ such that $x \in X$ and $y \in Y$, and also denote $|E_G(X, Y)|$ by $c_G(X, Y)$. In particular, $E_G(u, v)$ is the set of edges with end vertices u and v. We denote $|V|$ by n (or by $n(G)$). A multigraph G can be input as an integer edge-weighted simple graph, where we denote by m (or by $m(G)$) the number of edges in the weighted graph, i.e., the number of pairs of vertices which are adjacent in G. For a subset $V' \subseteq V$ in G, $G - V'$ denotes the subgraph induced by $V - V'$. A *partition* $X_1, \cdots, X_t$ of the vertex set V means a family of nonempty disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset V' of V. A *cut* is defined as a subset X of V with $\emptyset \neq X \neq V$, and the *size* of a cut X is defined by $c_G(X, V - X)$, which may also be written as $c_G(X)$. A cut with the minimum size is called a (*global*) *minimum cut*, and its size, denoted by $\lambda(G)$, is called the *edge-connectivity* of G. The *local edge-connectivity* $\lambda_G(x, y)$ for two vertices $x, y \in V$ is defined to be the minimum size of a cut in G that separates x and y (i.e., x and y belong to different sides of X and $V - X$), or equivalently the maximum number of edge-disjoint paths between x and y by Menger's theorem.

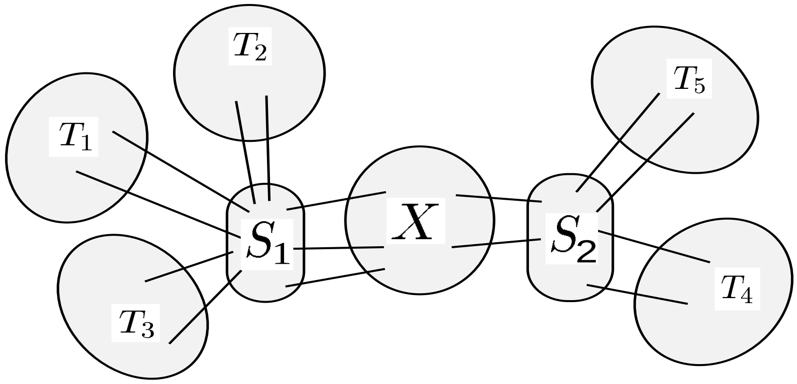

Figure 1. Illustration of a multigraph which has exactly two minimum disconnecting set S_1 and S_2. Each of cuts T_1, T_2, T_3 and $X \cup S_2 \cup T_4 \cup T_5$ is tight, since its neighbor set is the minimum disconnecting set S_1. Similarly with respect to the minimum disconnecting set S_2, each of cuts T_4, T_5 and $X \cup S_1 \cup T_1 \cup T_2 \cup T_3$ is tight. In particular, cuts T_i for $i = 1, \cdots, 5$ are minimal tight sets since no $T' \subset T_i$ is tight.

For a subset X of V, a vertex $v \in V - X$ is called a *neighbor* of X if it is adjacent to some vertex $u \in X$, and the set of all neighbors of X is denoted by $\Gamma_G(X)$. A maximal connected subgraph G' in a graph G is called a *component* of G (for notational convenience, the subset $X \subseteq V$ that induces G' is also called a component), and denote the number of components in G by $p(G)$.

A *disconnecting set* of G is defined as a cut S of V such that $p(G-S) > p(G)$ holds and no $S' \subset S$ has this property. Let $\bar{G}$ denote the simple graph obtained from G by replacing multiple edges in $E_G(u, v)$ by a single edge (u, v) for all $u, v \in V$. A component G' of G with $|V(G')| \geq 3$ always has a disconnecting set unless $\bar{G}$ is a complete graph K_n. If G is connected and contains a disconnecting set, then a disconnecting set of the minimum size is called a (*global*) *minimum disconnecting set*, and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G. On the other hand, we define $\kappa(G) = 0$ if G is not connected, and $\kappa(G) = n - 1$ if $\bar{G}$ is a complete graph K_n. The *local vertex-connectivity* $\kappa_G(x, y)$ for two vertices $x, y \in V$ is defined to be the number of internally-disjoint paths between x and y in $\bar{G}$. Again by Menger's theorem, $\kappa_G(x, y)$ for nonadjacent vertices x and y is equal to the minimum size of a disconnecting set that disconnects x and y.

A cut $T \subset V$ is called *tight* if $\Gamma_G(T)$ is a minimum disconnecting set in G (see Figure 1). A tight set T is called *minimal* if no proper subset T' of T is tight (hence, the induced subgraph $G[T]$ is connected).

In this paper, for a function $r_\lambda : \binom{V}{2} \to Z^+$ (resp., $r_\kappa : \binom{V}{2} \to Z^+$), where $\binom{V}{2}$ denotes the set of all pairs of vertices and Z^+ denotes the set of nonnegative integers, we say that $G = (V, E)$ is r_λ-*edge-connected* (resp., r_κ-*vertex-connected*) if $\lambda_G(x, y) \geq r_\lambda(x, y)$ (resp., $\kappa_G(x, y) \geq r_\kappa(x, y)$) holds for every $x, y \in V$. Then the *edge-connectivity augmentation problem* (resp., the *vertex-connectivity augmentation problem*) asks to augment G by the smallest number of new edges so that the resulting graph G' becomes r_λ-edge-connected (resp., r_κ-vertex-connected). When the requirement function r_λ (resp., r_κ) satisfies $r_\lambda(x,y) = \ell \in Z^+$ for all $x, y \in V$ (resp., $r_\kappa(x,y) = k \in Z^+$ for all $x, y \in V$), the problem is called the *uniform* ℓ-edge-connectivity augmentation problem (resp., the *uniform* k-vertex-connectivity augmentation problem). The problem of augmenting a multigraph by adding the smallest number of new edges to meet edge-connectivity or vertex-connectivity requirement has been extensively studied as important subjects in the network design problem, the data security problem [21] and the graph drawing problem [20] and others.

Watanabe and Nakamura [29] first proved that the uniform ℓ-edge-connectivity augmentation problem can be solved in polynomial time for any given integer ℓ. They observed the following lower bound $\lceil \alpha(G)/2 \rceil$ on the number of edges to be added to G to increase the edge-connectivity up to ℓ:

$$\alpha(G) = \max_{\text{subpartition } \mathcal{X} \text{ of } V} \left\{ \sum_{X \in \mathcal{X}} (\ell - c_G(X)) \right\}, \tag{1}$$

where the maximum is taken over all subpartitions $\mathcal{X}$ of V (note that each term $\ell - c_G(X)$ denote the "deficiency" of the cut X to have cut size at least ℓ, and adding one new edge reduces the deficiency of at most two cuts). By increasing the edge-connectivity one by one on the basis of the structural information of G, their algorithm obtains a set F of $\lceil \alpha(G)/2 \rceil$ new edges to make G ℓ-edge-connected. This F is an optimal solution since it attains the lower bound. Currently, an $O(e + \ell^2 n \log n)$ time algorithm due to Gabow [7], where $e = |E|$, is the fastest among existing algorithms of this type. Different from the approach by Watanabe and Nakamura, Cai and Sun [2] first pointed out that the uniform ℓ-edge-connectivity augmentation problem can be directly solved by applying the Lovász edge-splitting theorem. Based on this, Frank [6] gave an $O(n^5)$ time augmentation algorithm. Recently, an $O((m + n \log n) n \log n)$ time augmentation algorithm is proposed by Nagamochi and Ibaraki [27]. For a general requirement function r_λ, Frank [6] showed that the edge-connectivity augmentation problem can be solved in polynomial time by modifying the lower bound (1) and by using Mader's edge-splitting theorem [24]. The time complexity for this problem was recently improved by Gabow [8] to $O(n^3 m \log(n^2/m))$.

As to vertex-connectivity augmentation, several algorithms have been developed for the problem of augmenting a given $(k-1)$-vertex-connected graph by the minimum number of new edges to obtain a k-vertex-connected graph. By defining

$$\beta(G) = \max\{p(G - S) | S \text{ is a minimum disconnecting set in } G\}, \tag{2}$$

and

$$t(G) = \text{ the maximum number of pairwise disjoint minimal tight sets in } G, \tag{3}$$

We easily observe that

$$M(G) = \max\{\beta(G) - 1, \lceil t(G)/2 \rceil\}$$

provides a lower bound to this problem. Eswaran and Tarjan [3] proved that the problem for $k = 2$ can be solved by finding a set of $M(G)$ edges to make G 2-vertex-connected. Watanabe and Nakamura [30] stated the same result for $k = 3$. However, $M(G)$ can be smaller than the optimal value for general $k \geq 4$. Recently Jordán presented an $O(n^5)$ time approximation algorithm for this problem for a general k [18, 19]. The difference between the number of new edges added by his algorithm and the optimal value is at most $(k-3)/2$.

It is known that the uniform k-vertex-connectivity augmentation problem for $k \in \{2, 3, 4\}$ can be solved in polynomial time ([3, 12] for $k = 2$, [11, 30] for $k = 3$, and [13] for $k = 4$), where an input graph G may not be $(k-1)$-vertex-connected. However, whether there is a polynomial time algorithm for the uniform vertex-connectivity augmentation problem for an arbitrary k is still an open question (even if G is $(k-1)$-vertex-connected). For a general requirement function r_κ, the problem is shown to be NP-hard by Jordán [17].

In this paper, we consider the connectivity augmentation problem that includes the above two augmentation problems, as follows. The *edge-and-vertex-connectivity augmentation problem*, denoted by EVAP(r_λ, r_κ), is that of augmenting G by the smallest number of new edges so that the resulting graph G' becomes r_λ-edge-connected and r_κ-vertex-connected (hereafter we call such a multigraph (r_λ, r_κ)-connected). Without loss of generality, $r_\lambda(x,y) \geq r_\kappa(x,y)$ is assumed for all $x, y \in V$, since if a graph is r_κ-vertex-connected then it is r_κ-edge-connected. When the requirement function r_κ satisfies $r_\kappa(x,y) = k \in Z^+$ for all $x, y \in V$, this problem is also denoted as EVAP(r_λ, k). Recently, the authors proved that EVAP($r_\lambda, 2$) can be solved in $O(n^3 m \log(n^2/m))$ time [15].

As to the other problems of augmenting both edge-connectivity and vertex-connectivity, the following result was shown by Hsu and Kao [14]. Given a graph $G = (V, E)$ with two specified subsets V_1 and V_2 of V, there is a linear time algorithm for augmenting G by the smallest number of edges so that the resulting graph G' satisfies $\lambda_{G'}(x, x') \geq 2$ for all $x, x' \in V_1$ and $\kappa_{G'}(y, y') \geq 2$ for all $y, y' \in V_2$. In this problem, $V_1 \cap V_2 = \emptyset$ can be assumed without loss of generality (since $\kappa_{G'}(u,v) \geq 2$ implies $\lambda_{G'}(u,v) \geq 2$).

In a communication network, the edge-connectivity and vertex-connectivity are fundamental measurements of reliability in terms of link failures and node failures, respectively. These problems design a network which is invulnerable against both link and node failures.

In this paper, we consider problem EVAP($\ell, 3$); i.e., $r_\lambda(x,y) = \ell$ and $r_\kappa(x,y) = 3$ for all $x, y \in V$. We first present a lower bound on the number of edges that is necessary to make a given multigraph G $(\ell, 3)$-connected, and then show that the lower bound suffices or otherwise $O(\ell)$ edges suffices if the input graph is 2-vertex-connected. The task of constructing such set of new edges can be done in polynomial time for a fixed ℓ.

In Section 2, after introducing basic definitions and the concept of edge-splitting, we derive some lower bounds on EVAP($\ell, 3$). In Section 3, we outline our algorithm, named EV-AUGMENT3, that makes a given 2-vertex-connected graph G $(\ell, 3)$-connected by adding a new edge set whose size is equal to the lower bound or $O(\ell)$. The algorithm consists of five major steps. In Sections 4 – 8, we prove the correctness and running time of each step in our algorithm.

2. Preliminaries

2.1. *Definitions*

Given a multigraph $G = (V, E)$, its vertex set V and edge set E may be denoted by $V(G)$ and $E(G)$, respectively. For a subset $V' \subseteq V$ (resp., $E' \subseteq E$) in G, $G[V']$ (resp., $G[E']$) denotes the subgraph induced by V' (resp., $G[E'] = (V, E')$). For $V' \subset V$ (resp., $E' \subseteq E$), we denote subgraph $G[V - V']$ (resp., $G[E - E']$) by $G - V'$ (resp., $G - E'$). For an edge set F with $F \cap E = \emptyset$, we denote the augmented graph $G = (V, E \cup F)$ by $G + F$.

We say that a cut X *separates* two disjoint subsets Y and Y' of V if $Y \subseteq X$ and $Y' \subseteq V - X$ (or $Y \subseteq V - X$ and $Y' \subseteq X$) hold. In particular, a cut X separates vertices x and y if $x \in X$ and $y \in V - X$ (or $x \in V - X$ and $y \in X$) hold. A cut X *intersects* another cut Y if none of subsets $X \cap Y$, $X - Y$ and $Y - X$ is empty. A family $\mathcal{X}$ of subsets $X_1, \cdots, X_p$ is called *laminar* if no two subsets in $\mathcal{X}$ intersect each other (possibly $X_i \subseteq X_j$ for some $X_i, X_j \in \mathcal{X}$). We say that a cut X *crosses* another cut Y if they intersect each other and in addition $V - (X \cup Y) \neq \emptyset$ holds. If two cuts X and Y cross each other in $G = (V, E)$, the following properties hold:

$$\begin{array}{l} c_G(X) + c_G(Y) = c_G(X \cap Y) + c_G(X \cup Y) + 2c_G(X - Y, Y - X), \\ c_G(X) + c_G(Y) = c_G(X - Y) + c_G(Y - X) + 2c_G(X \cap Y, V - (X \cup Y)), \end{array} \tag{4}$$

$$|\Gamma_G(X)| + |\Gamma_G(Y)| \geq |\Gamma_G(X \cap Y)| + |\Gamma_G(X \cup Y)|. \tag{5}$$

The next property can be easily obtained by (4).

LEMMA 1 *Let X, Y be two crossing cuts which satisfy $c_G(X) \leq \ell + 1$ and $c_G(Y) \leq \ell + 1$ in an ℓ-edge-connected graph $G = (V, E)$. If $c_G(X \cap Y, V - (X \cup Y)) \geq 1$ and $c_G(X - Y, Y - X) \geq 1$, then the following statements hold:*
(1) $c_G(X \cup Y) = c_G(X \cap Y) = c_G(X - Y) = c_G(Y - X) = \ell$.
(2) $c_G(X \cap Y, V - (X \cup Y)) = c_G(X - Y, Y - X) = 1$. □

We say that a disconnecting set $S \subset V$ *disconnects* two disjoint subsets Y and Y' of $V - S$ if no two vertices $x \in Y$ and $y \in Y'$ are connected in $G - S$. In particular, a disconnecting set S disconnects vertices x and y in $V - S$ if x and y are contained in different components of $G - S$. A vertex v is called a *disconnecting vertex* in G if $S = \{v\}$ is a disconnecting set in G. A pair $\{v_1, v_2\}$ of vertices is called a *disconnecting pair* in G if $S = \{v_1, v_2\}$ is a disconnecting set in G. For a disconnecting pair S, there is the component X of G such that $X \supseteq S$, and we call the components in $G[X]-S$ *the S-components*. Note that the vertex pair S is a disconnecting pair in G if and only if there are at least two S-components. If $\kappa(G)=2$, then any S-component of a disconnecting pair S is also a tight set. We easily obtain the next lemmas.

LEMMA 2 *Let G be ℓ-edge-connected, and X be a cut in G. If $G[X]$ is not connected, then $c_G(X)\geq 2\ell$.* □

LEMMA 3 *Let $S=\{v_1,v_2\}$ be a disconnecting pair and T be an S-component in a 2-vertex-connected graph $G=(V,E)$. If G has an edge (v_1,v_2) in E, then the induced subgraph $G[T\cup S]$ is also 2-vertex-connected.* □

LEMMA 4 *If $\kappa(G)=2$, then any two minimal tight sets X and Y in G are pairwise disjoint.* □

We denote the set of minimal tight sets in a 2-vertex-connected graph G by $\mathcal{T}(G)$.

2.2. Edge-Splitting

In this subsection, we review an operation of *edge-splitting*. Given a multigraph $G=(V,E)$, a designated vertex $s\in V$, vertices $u,v\in\Gamma_G(s)$ (possibly $u=v$) and a nonnegative integer $\delta\leq\min\{c_G(s,u),c_G(s,v)\}$, we construct graph $G'=(V,E')$ by deleting δ edges from $E_G(s,u)$ and $E_G(s,v)$, respectively, and adding new δ edges to $E_G(u,v)$: $c_{G'}(s,u):=c_G(s,u)-\delta$, $c_{G'}(s,v):=c_G(s,v)-\delta$, $c_{G'}(u,v):=c_G(u,v)+\delta$, and $c_{G'}(x,y):=c_G(x,y)$ for all other pairs $x,y\in V$. In the case $u=v$, we interpret that $c_{G'}(s,u):=c_G(s,u)-2\delta, c_{G'}(u,u):=c_G(u,u)+\delta$, and $c_{G'}(x,y):=c_G(x,y)$ for all other pairs $x,y\in V$, where an integer δ is chosen so as to satisfy $0\leq\delta\leq\frac{1}{2}c_G(s,u)$. We say that G' is obtained from G by *splitting* δ pair of edges (s,u) and (s,v) (or by splitting (s,u) and (s,v) by size δ). A sequence of splittings is *complete* if the resulting graph G' does not have any neighbor of s.

The following theorem is proven by Lovász [23].

THEOREM 1 [6, 23] *Let $G=(V,E)$ be a multigraph with a designated vertex $s\in V$ with even $c_G(s)$, and $\ell\geq 2$ be an integer such that $\lambda_G(x,y)\geq\ell$ for all pairs $x,y\in V-s$. Then for each $u\in\Gamma_G(s)$ there is a vertex $v\in\Gamma_G(s)$ such that the graph G' obtained by splitting one pair of edges (u,s) and (s,v) satisfies $\lambda_{G'}(x,y)\geq\ell$ for all pairs $x,y\in V-s$.* □

Repeating the splitting in this theorem, we see that, if $c_G(s)$ is even, there always exists a complete splitting at s such that the resulting graph G' satisfies $\lambda_{G'-s}(x,y)\geq\ell$ for every pair of $x,y\in V-s$. It is shown in [1, 27, 28] that such a complete splitting at s can be computed in $O((m+n\log n)n\log n)$ time and the number of new pairs of vertices which became adjacent after the complete splitting is $O(n)$.

2.3. *Lower Bound*

For a multigraph G and a fixed integer $\ell \geq 3$, let $opt(G)$ denote the optimal value of EVAP$(\ell, 3)$ in G, i.e., the minimum size $|F|$ of a set F of new edges to obtain an $(\ell, 3)$-connected graph $G+F$. In this section, we derive two types of lower bounds, $\alpha(G)$ and $\beta(G)$, on $opt(G)$.

Let X be a cut in G. To make G ℓ-edge-connected and 3-vertex-connected, it is necessary to add

(a) at least $\max\{\ell - c_G(X), 0\}$ edges between X and $V-X$ (see Figure 2(a)), or

(b) at least $\max\{3 - |\Gamma_G(X)|, 0\}$ edges between X and $V-X-\Gamma_G(X)$ if $V-X-\Gamma_G(X) \neq \emptyset$ (see Figure 2(b)).

Given a subpartition $\mathcal{X} = \{X_1, \cdots, X_p, X_{p+1}, \cdots, X_q\}$ of V, where $V - X_i - \Gamma_G(X_i) \neq \emptyset$ holds for $i = p+1, \cdots, q$, we can sum up "deficiencies" $\max\{\ell - c_G(X_i), 0\}$, $i = 1, \cdots, p$, and $\max\{3 - |\Gamma_G(X_i)|, 0\}$, $i = p+1, \cdots, q$. As adding one edge to G contributes to the deficiency of at most two cuts in $\mathcal{X}$, we need at least $\lceil \alpha(G)/2 \rceil$ new edges to make G $(\ell, 3)$-connected, where

$$\alpha(G) = \max_{\text{all subpartitions } \mathcal{X}} \left\{ 0,\ \sum_{i=1}^{p} (\ell - c_G(X_i)) + \sum_{i=p+1}^{q} (3 - |\Gamma_G(X_i)|) \right\}, \quad (6)$$

and the maximum is taken over all subpartitions $\mathcal{X} = \{X_1, \cdots, X_p, X_{p+1}, \cdots, X_q\}$ of V with $V - X_i - \Gamma_G(X_i) \neq \emptyset$, $i = p+1, \cdots, q$. This lower bound is a generalization of the lower bound (1) for the edge-connectivity augmentation problem [2] and the lower bound (2) for the vertex-connectivity augmentation problem [3]. Note that in this paper, $|\Gamma_G(X_i)| \geq 2$ holds for $i = p+1, \cdots, q$ in (6) and $|\Gamma_G(X_i)|$ is exactly 2 whenever $3 - |\Gamma_G(X_i)| > 0$, since an input graph is always 2-vertex-connected.

We now consider another case in which different type of new edges become necessary. For a pair of vertices $S = \{v_1, v_2\}$ of G, let $T_1, \cdots, T_q$ denote all the components in $G - S$, where $q = p(G - S)$. To make G 3-vertex-connected, a new edge set F must be added to G so that all T_i form a single connected component in $(G+F) - S$. For this, it is necessary to add

(c) at least $p(G - S) - 1$ edges to connect all components in $G - S$.

This is illustrated in Figure 2(c), for which $T_1, T_2, \cdots, T_5$ denote all the components in $G - S$. Moreover, if $\ell > c_G(v_i)$ holds for a $v_i \in S$, then it is necessary to add at least $\ell - c_G(v_i)$ edges for $i = 1, 2$ in order to make G ℓ-edge-connected. Since adding an edge between v_1 and v_2 contribute to the requirement of both v_1 and v_2, we require

(d) at least $\max\{\ell - c_G(v_1), \ell - c_G(v_2), 0\}$ edges for $S = \{v_1, v_2\}$.

In the above discussion, all augmented edges in (c) are incident to neither v_1 nor v_2, and all augmented edges in (d) are incident to v_1 or v_2; hence there is no edge that belong to both (c) and (d). Therefore, it is necessary to add

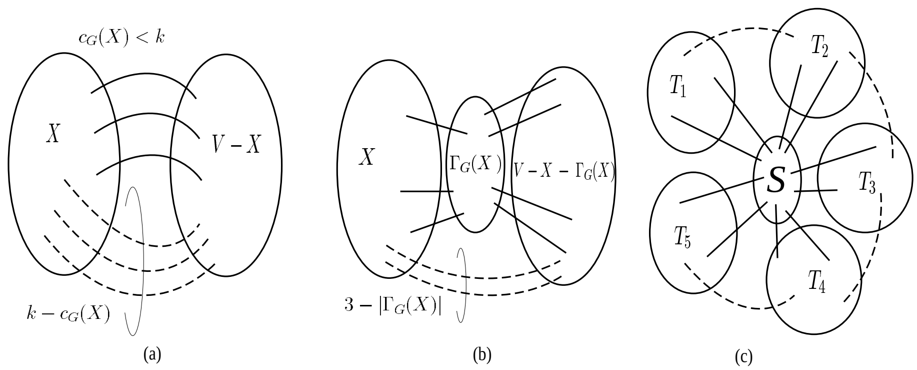

Figure 2. Illustrations of edge augmentations (broken edges are those augmented).

(e) at least $p(G-S)-1+\max\{\ell-c_G(v_1), \ell-c_G(v_2), 0\}$ edges for $S = \{v_1, v_2\}$ of G.

Define

$$\beta(G) = \max_{\substack{\text{all vertex pairs} \\ S=\{v,v'\} \text{ in } G}} \left[p(G-S) - 1 + \max\{\ell - c_G(v), \ell - c_G(v'), 0\} \right]. \quad (7)$$

Thus at least $\beta(G)$ new edges are necessary to make G $(\ell, 3)$-connected. The next lemma combines the above two lower bounds.

Lemma 5 **(Lower Bound)** *For a given multigraph G, let*

$$\gamma(G) = \max\{\lceil \alpha(G)/2 \rceil, \beta(G)\}.$$

Then $\gamma(G) \le opt(G)$ holds, where $opt(G)$ denotes the minimum number of edges augmented to make G $(\ell, 3)$-connected. □

In this paper, we prove the next result.

Theorem 2 *Let G be a 2-vertex-connected multigraph G with n vertices and m adjacent vertex pairs. For any integer $\ell \ge 3$,*

$$\gamma(G) \le opt(G) \le \max\{\gamma(G), 2\ell - 3\}$$

holds and an optimal solution of EVAP(ℓ,3) *can be found in $O((32\ell^4)^{2\ell-3}\ell n^3 + n^2 m + n^3 \log n)$ time. Furthermore, if $c_G(v) \ge \ell$ for all $v \in V$ or $\gamma(G) \ge 2\ell - 3$, then*

$$opt(G) = \gamma(G)$$

holds and an optimal solution F of EVAP(ℓ,3) *with $m(G+F) = O(m+n)$ can be found in $O(n^2m + n^3 \log n)$ time. In addition, in case of $\gamma(G) \le 2\ell - 3$, a feasible solution F' of* EVAP(ℓ,3) *such that*

$$|F'| \le \min\{2opt(G) - 1, 2\ell - 3, opt(G) + (\ell+1)/2\}$$

can be found in $O(n^2m + n^3 \log n)$ time. □

3. A Polynomial Time Algorithm for EVAP(ℓ,3)

In this section, we present a polynomial time algorithm, called EV-AUGMENT3, for solving EVAP(ℓ, 3) for a given 2-vertex-connected graph and a fixed integer ℓ. For a graph $G = (V, E)$, let $\mathcal{P}_3(G)$ denote the set of all vertex pairs x, y such that $\kappa_G(x, y) \ge 3$. Thus $\mathcal{P}_3(G) = \binom{V}{2}$ if $\kappa(G) \ge 3$. An operation of removing a subset $F' \subseteq E$ and adding a set F'' of new edges with $|F''| = |F'|$ is called a *shifting*, and denoted by F''/F'. In particular, a shifting F''/F' is called a *switching* if it does not change degree $c_G(v)$ of any vertex v in G. Given an ℓ-edge-connected graph G, a sequence of switchings or shiftings of edges is called *improving* in G if the resulting graph G' remains ℓ-edge-connected and $\mathcal{P}_3(G') \supseteq \mathcal{P}_3(G)$ holds. An edge $e = (u, w)$ is called *switching-admissible* (abbreviated to *s-admissible* in the subsequent discussion) with respect to a disconnecting pair S if $S \cap \{u, w\} = \emptyset$ and $p(G - S) = p((G - S) - e)$ hold.

The algorithm EV-AUGMENT3 consists of the following five major steps. In each step, we also describe some properties which verify its correctness. The proof for these properties will be given in the subsequent sections. Figure 3 illustrates the process of these five steps for an example multigraph.

Algorithm EV-AUGMENT3

Input: An undirected multigraph $G = (V, E)$ with $\kappa(G) \ge 2$, and an integer $\ell \ge 3$.

Output: A set of new edges F such that $|F| = opt(G)$ and $G^* = G + F$ satisfies $\lambda(G^*) \ge \ell$ and $\kappa(G^*) \ge 3$.

Step I (Addition of vertex s and associated edges): Add a new vertex s together with a set F_1 of edges between s and V so that the resulting graph $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfies

$$c_{G_1}(X) \ge \ell \quad \text{for all cuts } X \subset V, \tag{8}$$
$$|\Gamma_{G_1}(X)| \ge 3 \quad \text{for all cuts } X \subset V \text{ with } V - X - \Gamma_{G_1}(X) \ne \emptyset, \tag{9}$$

where $|F_1|$ is *minimum* subject to (8) and (9).

PROPERTY 1 *The above set of edges F_1 satisfies $|F_1| = \alpha(G)$.* □

Remark 3.1: Condition (8) implies that $\lambda_{G_1}(x, y) \ge \ell$ holds for all pairs $x, y \in V$. Condition (9) implies that, for any disconnecting pair S in G, each S-component

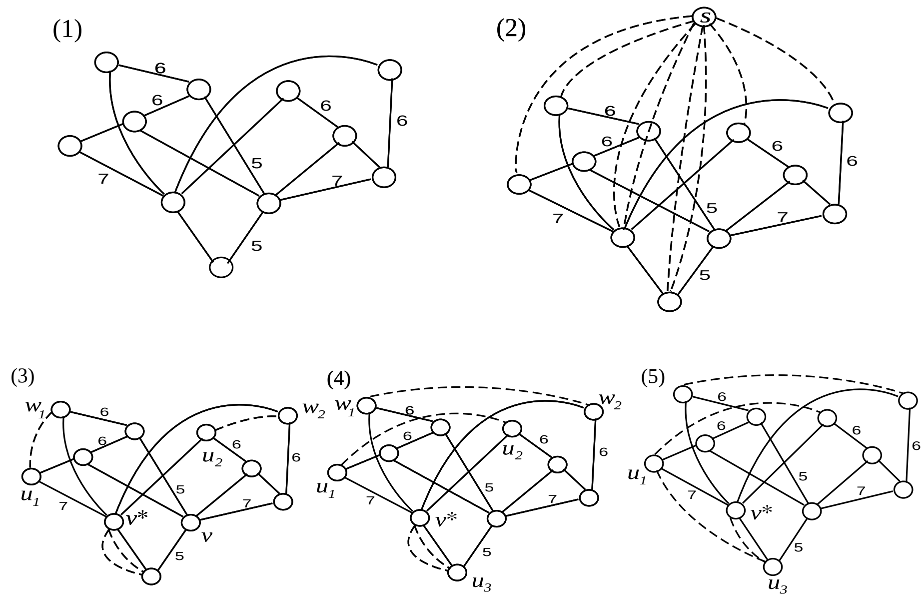

Figure 3. Computational process of algorithm EV-AUGMENT3 for $\ell = 8$. (1) An input graph $G = (V, E)$ with $\lambda(G) = 6$ and $\kappa(G) = 2$, where the number beside each edge is the multiplicity of the edge (the numbers for multiplicity 1 are omitted). The two lower bounds in Section 2 are $\lceil \frac{\alpha(G)}{2} \rceil = \frac{8}{2} = 4$ and $\beta(G) = 2$. (2) $G_1 = (V \cup \{s\}, E \cup F_1)$ obtained by Step I. Edges in F_1 are drawn as broken lines. Now $\lambda(x, y) \geq 8$ and $\kappa(x, y) \geq 3$ for all $x, y \in V$. (3) $G_2 = (V, E \cup F_2)$ obtained from G_1 in Step II. The G_2 satisfies $\lambda(G_2) \geq 8$ but has a disconnecting pair $S = \{v^*, v\}$. Edges $(u_i, w_i) \in F_2$ $(i = 1, 2)$ are s-admissible with respect to the disconnecting pair S. (4) $G_2' = (V, E \cup F_2')$ obtained from G_2 by switching $\{(u_1, u_2), (w_1, w_2)\}/\{(u_1, w_1), (u_2, w_2)\}$ in Step III. Then $\lambda(G_2') \geq 3$ holds and the number of S-components is decreased by one. Moreover, no switching is improving in G_2' (that is, $G_2' = G_3$ and $F_2' = F_3$). (5) $G_4 = (V, E \cup F_4)$ obtained by the shifting $\{(u_1, u_3)\}/\{(v^*, u_3)\}$ in Step IV. This G_4 is $(8, 3)$-connected.

T contains a vertex adjacent to s in G_1. Thus, $\kappa_{G_1}(x, y) \geq 3$ holds for all pairs $x, y \in V$.

Remark 3.2: A minimal set F_1 (i.e., any proper subset of F_1 violates (8) or (9)) can be obtained by first adding a sufficiently large number (say, ℓ) of edges between s and each vertex $v \in V$, and then discarding these new edges one by one as far as none of (8) and (9) is violated. However, this may not minimize $|F_1|$ among such sets F_1. For this, we need to modify set F_1 according to some algorithm until $|F_1|$ becomes the minimum. The details will be given in Section 4.

Step II (Edge-splitting): If $c_{G_1}(s)$ is odd, then add one edge $\hat{e} = (s, w)$ to F_1 for a vertex w arbitrarily chosen from V. Denote the resulting graph again by G_1.

Then find a complete edge-splitting at s in $G_1 = (V \cup \{s\}, E \cup F_1)$ to obtain $G_2 = (V, E \cup F_2)$ (ignoring the isolated vertex s). G_2 preserves the ℓ-edge-connectivity, i.e., $\lambda_{G_2}(x, y) \geq \ell$ holds for all pairs $x, y \in V$. By Theorem 1, there always exists such a complete edge-splitting, and it can be computed in polynomial time, as noted Section 2.2.

If $\kappa(G_2) \geq 3$, then we are done, because $|F_2| = |F_1|/2 = \lceil \alpha(G)/2 \rceil$ attains the lower bound of Lemma 5. Otherwise, we can observe from (9) that

$$T \cap V[F_2] \neq \emptyset \text{ for all } T \in \mathcal{T}(G_2), \tag{10}$$

where $\mathcal{T}(G_2)$ denotes a family of all minimal tight sets in G_2. (If $T \cap V[F_2] = \emptyset$ holds for some $T \in \mathcal{T}(G_2)$, then T contained no end vertex of an edge in F_1 of Step I, i.e., $|\Gamma_{G_1}(T)| = 2$, contradicting (9).)

Go to Step III.

Step III (Switching edges): Now $G_2 = (V, E \cup F_2)$ satisfies

$$\lambda(G_2) \geq \ell \text{ and } (10). \tag{11}$$

Property (11) is maintained in this step during which we try to regain 3-vertex-connectivity of G_2 by switching some edges in F_2. Based on (11) we can derive the following sufficient conditions which admit an improving switching such that at least one new pair of vertices in G_2 becomes 3-vertex-connected.

PROPERTY 2 *If G_2 has a disconnecting pair $S = \{v_1, v_2\}$ and two edges $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2) \in F_2$ which satisfy one of the following* (a)–(d) *(see Figure 4), then one of the switchings*

$$\{(u_1, u_2), (w_1, w_2)\}/\{e_1, e_2\} \text{ and } \{(u_1, w_2), (w_1, u_2)\}/\{e_1, e_2\}$$

is improving in G_2 and the number of S-components (recall that an S-component is also a tight set by assumption $\kappa(G_2) \geq 2$) in G_2 decreases at least by one after the switching.

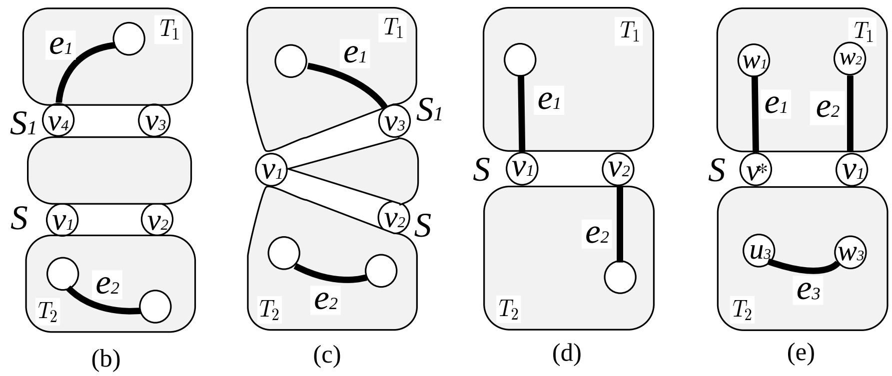

Figure 4. Illustrations of the graph G_2 satisfying conditions (b) – (d) in Property 2 and (e) in Property 3.

(a) *$V[e_1] \subseteq T_1$ and $V[e_2] \subseteq T_2$ hold for two distinct S-components T_1 and T_2 in G_2, and e_1 or e_2 (say, e_1) is s-admissible with respect to S.*

(b) *G_2 has another disconnecting pair S_1 with $S \cap S_1 = \emptyset$, and an S_1-component T_1 and an S-component T_2 such that $T_1 \cap T_2 = \emptyset$, $V[e_1] \subseteq T_1 \cup S_1$, $V[e_2] \subseteq T_2 \cup S$ and $V[e_i] \cap T_i \neq \emptyset$ $(i = 1, 2)$ hold.*

(c) *G_2 has another disconnecting pair $S_1 = \{v_1, v_3\}$ with $v_3 \neq v_2$, and an S_1-component T_1 and an S-component T_2 such that $T_1 \cap T_2 = \emptyset$, $V[e_1] \subseteq T_1 \cup \{v_3\}$ and $V[e_2] \subseteq T_2 \cup \{v_2\}$ hold.*

(d) *$v_1 \in V[e_1] \subseteq T_1 \cup \{v_1\}$ and $v_2 \in V[e_2] \subseteq T_2 \cup \{v_2\}$ hold for two distinct S-components T_1 and T_2.* □

Note that an improving switching preserves the property (11) of G_2, and we can continue switching pairs of edges in F_2 until none of the above conditions (a)–(d) holds in G_2. In particular, if G_2 has two minimal tight sets $T_1, T_2 \in \mathcal{T}(G_2)$ such that $\Gamma_{G_2}(T_1) \cap \Gamma_{G_2}(T_2) = \emptyset$, then we can immediately apply condition (b) to $S_1 = \Gamma_{G_2}(T_1)$ and $S = \Gamma_{G_2}(T_2)$, because such T_i has at least one edge $e_i \in F_2$ with $T_i \cap V[e_i] \neq \emptyset$ by (10). Therefore, we can reduce the number of minimal tight sets in G_2 until there are no disconnecting pairs S and S' such that $S \cap S' = \emptyset$. If the resulting graph G_2 has exactly three disconnecting pairs $\{v_a, v_b\}, \{v_b, v_c\}$ and $\{v_c, v_a\}$, then it is not difficult to see that one of the conditions (a),(c) and (d) always holds in G_2, and we can reduce the number of disconnecting pairs at least by one by a sequence of improving switchings. Therefore, we can assume that any disconnecting pair in the resulting graph G_2 contains one common vertex, say v^*. Let $S_i = \{v^*, v_i\}, i = 1, \cdots, q$ denote all disconnecting pairs in the G_2. (Note that the number of all S_i-components over all S_i is $O(n)$, since no two S_i- and S_j-components intersect each other.) We now repeat switching edges in F_2 which satisfy one of the above (a)-(d) or the following new condition (e).

PROPERTY 3 *Let* $S_1 = \{v^*, v_1\}$ *be a disconnecting pair in* G_2*. If* G_2 *has three edges* $e_k = (u_k, w_k) \in F_2$ $(k = 1, 2, 3)$ *that satisfy the following condition* (e) *(see Figure 4), then at least one of switchings* $\{(v^*, v_1), (w_1, u_3), (w_2, w_3)\}/\{e_1, e_2, e_3\}$ *and* $\{(v^*, v_1), (w_1, w_3), (w_2, u_3)\}/ \{e_1, e_2, e_3\}$ *is improving in* G_2 *and decreases the number of* S_1*-components in* G_2 *at least by one.*

(e) *For two distinct* S_1*-components* T_1 *and* T_2*,* $u_1 = v^*$*,* $u_2 = v_1$*,* $\{w_1, w_2\} \subseteq T_1$ *and* $V[e_3] \subseteq T_2$ *hold.* □

Let $G_3 = (V, E \cup F_3)$ denote the resulting graph obtained from $G_2 = (V, E \cup F_2)$ by the above switchings, where G_3 satisfies none of the conditions (a)-(e). Then G_3 has the following properties (See an example of G_3 in Figure A.1 in the appendix).

PROPERTY 4 *Assume that none of the conditions* (a)–(e) *holds in* G_3*. For each disconnecting pair* $S_i = \{v^*, v_i\}$*,* $i = 1, \cdots, q$*, any edge in* F_3 *incident to* v_i *is also incident to* v^**. If* $q \geq 2$*, then there is an edge in* F_3 *incident to* v^**. Moreover,* G_3 *satisfies exactly one of the following conditions* (i) – (iii) :

(i) $q = 1$ *holds and none of edges in* F_3 *is s-admissible with respect to* S_1*.*

(ii) $q = 1$ *holds and there is an edge in* F_3 *which is s-admissible with respect to* S_1*. Exactly one* S_1*-component contains edges in* F_3 *that are s-admissible with respect to* S_1*, and any edge in* F_3*, not contained in the* S_1*-component, is incident to* v^**.*

(iii) $q \geq 2$ *holds. For each disconnecting pair* S_i*,* $i = 1, \cdots, q$*, if there is an edge in* F_3 *which is s-admissible with respect to* S_i*, then exactly one* S_i*-component contains edges in* F_3 *that are s-admissible with respect to* S_i*, and any edge in* F_3*, not contained in the* S_i*-component, is incident to* v^**.* □

If G_3 has no disconnecting pair (i.e., $\kappa(G_3) \geq 3$), then halt by outputting F_3 as an optimal solution with $|F_3| = \lceil \alpha(G)/2 \rceil$, which attains our lower bound $\gamma(G)$. Otherwise, go to Step IV.

Step IV (Shifting edges): For the common vertex v^* in all disconnecting pairs $S_1, \ldots, S_q$ in G_3, we apply the following shifting operation in F_3.

PROPERTY 5 *Assume that none of the conditions* (a)–(e) *of Properties 2 and 3 holds in* G_3*, and that* $c_{G_3}(v^*) > \ell$ *and* $E_{G_3}(v^*) \cap F_3 \neq \emptyset$ *hold. Then at least one of the following properties* (i)–(iii) *holds.*

(i) *G_3 has an edge $e_1 = (v^*, w_1) \in F_3$ and a vertex $u_1 \in V - v^*$ such that shifting $\{(u_1, w_1)\}/\ \{(v^*, w_1)\}$ is improving in G_3, and decreases the number of S-components for some disconnecting pair S in G_3.*

(ii) *G_3 has edges $e_1 = (v^*, w_1), e_2 = (u_2, w_2) \in F_3$ and a $u_1 \in V - v^*$ such that shifting $\{(u_1, u_2),\ (w_1, w_2)\}/\ \{(v^*, w_1), (u_2, w_2)\}$ is improving in G_3 and decreases the number of S-components for some disconnecting pair S in G_3.*

(iii) *G_3 has edges $e_1 = (v^*, w_1), e_2 = (u_2, w_2) \in F_3$ such that shifting $\{(v^*, u_2), (w_1, w_2)\}/\ \{(v^*, w_1), (u_2, w_2)\}$ is improving in G_3 and decreases the number of S-components for some disconnecting pair S in G_3.* □

Note that shifting in the above (i) and (ii) reduces $|F_3 \cap c_{G_3}(v^*)|$ at least by one. After such a shifting, we check if the new F_3 satisfies one of the conditions (a)-(e) (if some of (a)-(e) holds, we switch those edges to reduce the number of S_i-components of an S_i). With Property 5, we can find a sequence of improving shiftings of edges in F_3 until $c_{G_3}(v^*) \leq \ell$ or $E_{G_3}(v^*) \cap F_3 = \emptyset$ holds in the resulting graph G_3.

Let $G_4 = (V, E \cup F_4)$ denote the resulting graph G_3. If $\kappa(G_4) \geq 3$, then halt by outputting F_4 as an optimal solution. Otherwise, we see that G_4 has the following property.

Property 6 *In addition to the property mentioned in Property 4, we have $c_{G_4}(v^*) = \ell$ in G_4 whenever $E_{G_4}(v^*) \cap F_4 \neq \emptyset$; the number of edges in F_4 incident to S_i is equal to* $\max\{\ell - c_G(v^*), \ell - c_G(v_i), 0\}$ *for any disconnecting pair $S_i = \{v^*, v_i\}$.* □

Then we consider the following property.

Property 7 *Assume that G_4 has an edge in F_4 s-admissible with respect to S for each disconnecting pair S in G_4. Let T_1 be an S-component in G_4 containing an edge in F_4 s-admissible with respect to S, such that no S'-component $T' \subset T_1$ with $S' \neq S$ contains an edge in F_4 s-admissible with respect to S', where S and S' are disconnecting pairs in G_4. Let $S = \{v^*, v_1\}$ and T_i denote the other S-components for $i = 2, \cdots, t(\geq 2)$. (Recall that Property 4(iii) says that all edges in F_4 that are contained in $T_i \neq T_1$ are incident to v^*.) If T_1 has at least* $\min\{\ell - 1, n - 2\}$ *edges in F_4 s-admissible with respect to S, then there is a sequence of improving switchings of edges in F_4 which decreases the number of the S-components by one.* □

We now repeat switching and shifting edges in F_4 as long as one of the conditions (a)-(e) or Property 7 holds. Let $G_5 = (V, E \cup F_5)$ denote the resulting graph. If G_5 has no disconnecting pair, then we are done, since $|F_5| = \lceil \alpha(G)/2 \rceil$ implies by Lemma 5 that G_5 is optimally augmented. Otherwise, go to Step V. Such G_5 is characterized as follows.

Property 8 *Let $S_i = \{v^*, v_i\}$ $(i = 1, \cdots, q)$ denote all the disconnecting pairs in G_4, where $|E_{G_5}(v^*) \cap F_5| \geq |E_{G_5}(v_1) \cap F_5|$ is assumed if $q = 1$. Then any edge in F_5 incident to v_i is also incident to v^*. Moreover, G_5 satisfies one of the following properties:*

(i) *$q = 1$ and $|E_{G_5}(v^*) \cap F_5| = \max\{0, \ell - c_G(v^*)\}$ hold and none of the edges in F_5 is s-admissible with respect to S_1.*

(ii) *$c_{G_5}(v^*) = \ell$ and $|E_{G_5}(v^*) \cap F_5| \geq 1$ hold, and each disconnecting pair S_i satisfies the following: All edges in F_5 which are not incident to v^* are contained in exactly one S_i-component T. Furthermore, there is a disconnecting pair S such that at most $\ell - 2$ edges in F_5 are s-admissible with respect to S.* □

Step V (Edge augmentation):

Property 9 *If G_5 satisfies (i) of Property 8, then $p(G_5 - S_1) = p(G - S_1) + \max\{\ell - c_G(v^*), \ell - c_G(v_1), 0\} - |F_5|$, where $S_1 = \{v^*, v_1\}$ is a disconnecting pair in G_5.* □

From this and the definition of $\beta(G)$ (see (7)), we obtain $p(G_5 - S_1) - 1 = p(G - S_1) - 1 + \max\{\ell - c_G(v^*), \ell - c_G(v_1), 0\} - |F_5| \leq \beta(G) - |F_5|$. Let $T_1, \cdots, T_r$ be all the S_1-components in G_5, where $r = p(G_5 - S_1)$. Add to G_5 a set of $r - 1$ new edges $F_6 = \{(x_i, x_{i+1}) \mid i = 1, \cdots, r-1\}$, where x_i is an arbitrary vertex in T_i. This makes G_5 3-vertex-connected. Note that $|F_5| + |F_6| = |F_5| + p(G_5 - S_1) - 1 \leq \beta(G)$ implies that $F_5 \cup F_6$ is an optimal solution. Clearly, such F_6 can be obtained in linear time by computing all S_1-components in G_5 by a triconnected component algorithm [10].

Now we consider the case in which G_5 satisfies (ii) of Property 8 (note that if $c_G(v) \geq \ell$ holds for all $v \in V$ originally, then case (ii) in Property 8 cannot occur). In this case, $\gamma(G) < opt(G)$ may hold as examplified in Figure 5. Then an upper bound on $opt(G)$ is given by

$$opt(G) \leq |F_5| + |\mathcal{T}(G_5)| - 1, \tag{12}$$

because G_5 (which is already ℓ-edge-connected) can be augmented to a 3-vertex-connected graph by another set of $|\mathcal{T}(G_5)| - 1$ new edges.

By (10), any minimal tight set in G_5 contains an end vertex of an edge in F_5. As any two minimal tight sets in $\mathcal{T}(G_5)$ are pairwise disjoint in G (by Lemma 4), we have $|\mathcal{T}(G_5)| \leq |F_5| = \lceil \alpha(G)/2 \rceil$ and hence $opt(G) \leq 2\lceil \alpha(G)/2 \rceil - 1$ by (12). Furthermore we can show that $|F_5| = O(\ell)$ holds.

Property 10 *If G_5 satisfies* (ii) *of Property 8, then we have $c_G(v^*) < \ell$, $|F_5| = \lceil \alpha(G)/2 \rceil \leq 2\ell - 4$, $|F_5| + |\mathcal{T}(G_5)| - 1 \leq 2\ell - 3$, $|\mathcal{T}(G_5)| \leq (\ell + 1)/2$ and $opt(G) \leq \min\{2\ell - 3, 2\lceil \alpha(G)/2 \rceil - 1\}$.* □

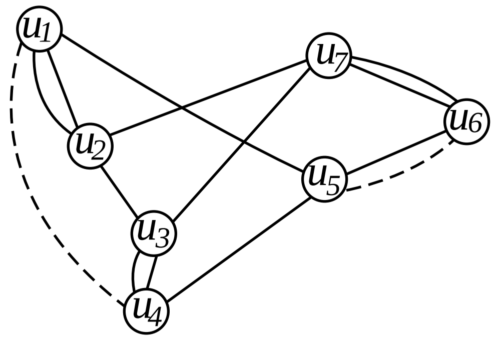

Figure 5. A graph which cannot be made $(4,3)$-connected by adding $\gamma(G)$ edges, where $\lceil \frac{\alpha(G)}{2} \rceil = \frac{4}{2} = 2$ and $\beta(G) = 2$, respectively. Broken lines illustrate edges $F_4 = \{(u_1, u_4), (u_5, u_6)\}$ obtained after Step IV of algorithm EV-AUGMENT3. The instance cannot be made $(4,3)$-connected by adding any 2 edges (but 3 edges $F_4 \cup \{(u_4, u_6)\}$ suffice).

We can first show that adding a set F_6' of new edges to make G_5 3-vertex-connected, where F_6' satisfies $|F_6'| \leq \min\{\lceil \alpha(G)/2 \rceil - 1, (\ell+1)/2 - 1\}$, we obtain an approximate solution $F' = F_5 \cup F_6'$ with $|F'| \leq \min\{2opt(G) - 1, 2\ell - 3\}$ and $|F'| - opt(G) \leq (\ell+1)/2$. An optimal solution F can then be found by inspecting all possible choices of F with $|F| \leq 2\ell - 3$. □

Finally we close this section by outlining all steps of algorithm EV-AUGMENT3.

Input: An undirected multigraph $G = (V, E)$ with $\kappa(G) \geq 2$, and an integer $\ell \geq 3$.

Output: A set of new edges F such that $|F| = opt(G)$ and $G^* = G + F$ satisfies $\lambda(G^*) \geq \ell$ and $\kappa(G^*) \geq 3$.

Step I Adding a new vertex s and a minimum set F_1 of edges between s and V to obtain a graph $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfying (8) and (9), where $|F_1|$ attains $\alpha(G)$.

Step II Add one arbitrary edge between s and V in G_1 if $c_{G_1}(s)$ is odd. Find a complete edge-splitting at s in G_1 to obtain an $(\ell, 2)$-connected graph $G_2 = (V, E \cup F_2)$ (ignoring the isolated vertex s).

Step III Continue switching two or three edges in F_2 to obtain a graph $G_3 = (V, E \cup F_3)$ such that every disconnecting pair S_i has one common vertex, say v^*, and the number of S_i-components containing edges in F_3 s-admissible with respect to S_i is at most one.

Step IV Continue switching or shifting some edges in F_3 to obtain $G_4 = (V, E \cup F_4)$ such that, in addition to the property of G_3, the number of edges in F_4

incident to S_i is equal to $\max\{\ell - c_G(v^*), \ell - c_G(v_i), 0\}$ for any disconnecting pair $S_i = \{v^*, v_i\}$.

Next continue switching some edges in F_4 until we obtain a graph $G_5 = (V, E \cup F_5)$ such that, in addition to the property of G_4, (i) there exists exactly one disconnecting pair S_1 and none of the edges in F_5 is s-admissible with respect to S_1, or (ii) $|E_{G_5}(v^*) \cap F_5| \geq 1$ holds and at most $\ell - 2$ edges in F_5 are s-admissible with respect to S_i for some S_i.

Step V In the case of (i), add a set F_6 of new edges to G_5 with $|F_6| = \beta(G) - \lceil \alpha(G)/2 \rceil - 1$ to obtain an $(\ell, 3)$-connected multigraph.

In the case of (ii), find an optimal solution F by inspecting all possible choices of F with $|F| \leq 2\ell - 3$. ☐

4. Justification of Step I

This section shows how to find a minimum size of set F_1 that satisfies (8) and (9) in Step I. An algorithm for finding such F_1 is described as follows.

Algorithm ADD-EDGE

Input: An undirected multigraph $G = (V, E)$ with $\kappa(G) \geq 2$, and an integer $\ell \geq 3$.

Output: An undirected multigraph $G_1 = (V \cup \{s\}, E \cup F_1)$ such that G_1 satisfies (8), (9) and $|F_1| = \alpha(G)$.

1. Let $\mathcal{T}(G) = \{T_1, ..., T_r\}$ be the set of all minimal tight sets in G (any two T_i and T_j are mutually disjoint by Lemma 4). From each T_i, we arbitrarily choose a vertex $t_i \in T_i$. Add to G a new vertex s and a set $F' = \{(s, t_i) \mid i = 1, ..., r\}$ of new edges. Clearly, the resulting graph $G_1' = (V \cup \{s\}, E \cup F')$ satisfies $\kappa_{G_1'}(x, y) \geq 3$ for all $x, y \in V$.

2. Add to G_1' a set F'' of new edges between s and V such that the resulting graph $G_1'' = (V \cup \{s\}, E \cup F' \cup F'')$ satisfies $\lambda_{G_1''}(x, y) \geq \ell$ for all $x, y \in V$, and F'' is minimal subject to this property. (This can be done by first adding a sufficiently large number (say, ℓ) of edges between s and each vertex $v \in V$, and discarding them one by one as long as (8) or (9) is not violated. We show later in Lemma 7 that this can be done efficiently.)

An edge $e \in F' \cup F''$ is called *λ-critical* if removal of e violates the ℓ-edge-connectivity between some $x, y \in V$. Let $\hat{F}$ be the set of λ-critical edges (hence $F'' \subseteq \hat{F}$). Thus, for each edge $(s, u) \in \hat{F}$, there is the *unique* cut $X_u \subset V$ such that

$$u \in X_u,\ c_{G_1''}(X_u) = \ell, \text{ and } c_{G_1''}(X) > \ell \text{ for all cuts } X \text{ with } u \in X \subset X_u \tag{13}$$

(possibly $X_u = X_v$ for distinct $u, v \in V[\hat{F}] - s$). Such a cut X_u is called *λ-critical* (with respect to u). It is not difficut to verify the following property.

Property 11 *For each λ-critical cut $X_u \subset V$, every cut $Y \subset V$ with $c_{G_1''}(Y) = \ell$ and $Y \cap X_u \neq \emptyset$ satisfies $Y \subseteq X_u$ or $X_u \subseteq Y$.* □

Property 12 *The family $\mathcal{X}_1$ of all λ-critical cuts X_u, $u \in V[\hat{F}] - s$, is a laminar family.* □

By definition, for any cut $X \in \mathcal{X}_1$, the set of vertices $u \in \Gamma_{G_1''}(s)$ such that the X is λ-critical with respect to u is given by $(X - \cup_{Y \in \mathcal{X}_1,\ Y \subset X} Y) \cap \Gamma_{G_1''}(s)$. Those vertices are called *free vertices* of X in G_1''. Thus,

for any free vertex u of $X \in \mathcal{X}_1$ and a vertex $v \in X$, shifting $\{(s,v)\}/\{e\}$ preserves property (8), where $e = (s,u)$.

This is because otherwise G_1'' would have a cut Y with $u \in Y$, $v \notin Y$, and $c_{G_1''}(Y) = \ell$, contradicting Property 11. In particular, for a cut $X' \in \mathcal{X}_1$ with $X' \supset X$ and the set W' of free vertices of X', all vertices in W' remain to be free vertices of X' in the graph resulting from such shiftings of edges within X.

3. While F' contains a non λ-critical edge $e = (s,t_i)$ such that removal of e preserves (9), we repeat setting $F' := F' - e$, $G_1'' := G_1'' - e$ and recomputing the set of λ-critical edges in $F' \cup F''$ and its laminar family $\mathcal{X}_1$ of λ-critical cuts in the resulting graph G_1''. This G_1'' now has the following property.

Property 13 *The above graph G_1'' satisfies* (8) *and* (9). □

4. A minimal tight set $T_i \in \mathcal{T}(G)$ is called *κ-critical* if $c_{G_1''}(s,T_i) = 1$, and the vertex in $T_i \cap \Gamma_{G_1''}(s)$ is called a *κ-critical* neighbor. We say that a κ-critical minimal tight set $T_i \in \mathcal{T}(G)$ *violates* a cut $X \in \mathcal{X}_1$ if

$$T_i \cap X \neq \emptyset \text{ and } (T_i - X) \cap \Gamma_{G_1''}(s) \neq \emptyset.$$

Let $\mathcal{T}^{(1)}$ be the family of κ-critical minimal tight sets $T_i \in \mathcal{T}(G)$ such that their κ-critical neighbors t_i belong to $V - \cup_{X \in \mathcal{X}_1} X$. Let $\mathcal{X}^{(1)}$ be the family of all cuts X' in $\mathcal{X}_1$ that are violated by some $T \in \mathcal{T}^{(1)}$. If $\mathcal{X}^{(1)} = \emptyset$, then halt, outputting the current $F_1 = F' \cup F''$ and $\mathcal{X}_1$.

Assume that $\mathcal{X}^{(1)} \neq \emptyset$. If there is a free vertex u^* of some $X' \in \mathcal{X}^{(1)}$ such that u^* is not κ-critical, then we choose a vertex $u_i \in X' \cap T_i$ with $T_i \in \mathcal{T}^{(1)}$, and execute shifting $\{(s,u_i)\}/\{(s,u^*)\}$ in $F' \cup F''$. Then remove edge $(s,t_i) \in E_{G_1''}(s,T_i)$ from F'. In this case, we recompute the family of λ-critical cuts in the resulting graph G_1'', which clearly satisfies (8) and (9), and apply the same procedure of step 4 after recomputing the new family $\mathcal{X}_1$ of λ-critical cuts.

Otherwise (i.e., if all free vertices of cuts in $\mathcal{X}^{(1)}$ are κ-critical), then let $\mathcal{T}^{(2)}$ be the family of all κ-critical minimal tight sets T corresponding to those κ-critical neighbors, and $\mathcal{X}^{(2)}$ be the family of all cuts X' in $\mathcal{X}_1 - \mathcal{X}^{(1)}$ that are violated by some $T \in \mathcal{T}^{(2)}$. If $\mathcal{X}^{(2)} = \emptyset$, then $\mathcal{X}_1 := \mathcal{X}_1 - \mathcal{X}^{(1)}$ and halt, outputting the current $F_1 = F' \cup F''$ and $\mathcal{X}_1$.

If not, we define $\mathcal{T}^{(j)}$ and $\mathcal{X}^{(j)}$, $j = 3, 4, \ldots$, in a similar way until $\mathcal{X}^{(j')}$ becomes empty or a cut $X_{j'} \in \mathcal{X}^{(j')}$ has a non κ-critical free vertex u^*. In the former case, let $\mathcal{X}_1 := \mathcal{X}_1 - \cup_{1 \le j < j'} \mathcal{X}^{(j)}$ and halt, outputting the current $F_1 = F' \cup F''$ and $\mathcal{X}_1$.

In the latter case, we can find a sequence of tight sets $T^j \in \mathcal{T}^{(j)}$ and $X_j \in \mathcal{X}^{(j)}$, $1 \le j \le j'$ such that κ-critical minimal tight set T^j violates X_j. For $u^j \in T^j \cap X_j$, $j = 1, 2, \cdots, j'$, we execute shiftings $(s, u^j)/(s, t^{j+1})$, where $t^{j'+1} = u^*$ and t^j $(1 \le j \le j')$ denote the κ-critical neighbor in each T^j, and then remove the edge (s, t^1) from the resulting graph.

Property 14 *The resulting graph satisfies* (8) *and* (9). □

Then we restart the same procedure of step 4 after recomputing the new family $\mathcal{X}_1$ of λ-critical cuts. □

Remark 4.1: The number of recomputing laminar families $\mathcal{X}_1$ during ADD-EDGE is at most n, since the number of neighbors of s that are κ-critical and non λ-critical decreases by one after each execution of a sequence of shiftings.

Proof of Property 14: Let G_1^* be the resulting graph after step 4, and G_1'' be the graph immediately before this iteration of step 4. G_1^* satisfies (9) since every T_i satisfies $c_{G_1^*}(s, T_i) > 0$.

Next we show that G_1^* satisfies (8). Since (s, t_1) is not λ-critical, removal of (s, t_1) preserves (8). Assume that we execute shiftings $(s, u^j)/(s, t^{j+1})$, $j \in \{1, \cdots, j'\}$ one by one, in the order of $j = i_1, i_2, \ldots, i_{j'}$ such that $X_{i_h} \not\subset X_{i_{h+1}}$ for $1 \le h < j'$. Assume that, after shifting $(s, u^{i_k})/(s, t^{i_k+1})$, the resulting graph G^{i_k} violates (8) for the first time. Let $Y \subset V$ be a cut with $c_{G^{i_k}}(Y) = \ell - 1$. Then $c_{G^{i_{k-1}}}(Y) = \ell$, $t^{i_k+1} \in Y$, and $u^{i_k} \notin Y$ hold, where $G^{i_{k-1}}$ denotes the graph immediately before shifting $(s, u^{i_k})/(s, t^{i_k+1})$ is applied. Since $\{u^{i_k}, t^{i_k+1}\} \subseteq X_{i_k} \in \mathcal{X}_1$, it follows $X_{i_k} - Y \neq \emptyset$ and $X_{i_k} \cap Y \neq \emptyset$. By $s \in (V \cup s) - (X_{i_k} \cup Y)$, if $Y - X_{i_k} \neq \emptyset$ holds, then Y and X_{i_k} cross each other. Then, by (4), $c_{G^{i_{k-1}}}(X_{i_k}) + c_{G^{i_{k-1}}}(Y) = c_{G^{i_{k-1}}}(X_{i_k} - Y) + c_{G^{i_{k-1}}}(Y - X_{i_k}) + 2c_{G^{i_{k-1}}}(X_{i_k} \cap Y, (V \cup s) - (X_i \cup Y))$ holds. Since $G^{i_{k-1}}$ satisfies (8) and $c_{G^{i_{k-1}}}(X_{i_k}) = c_{G^{i_{k-1}}}(Y) = \ell$, $c_{G^{i_{k-1}}}(X_{i_k} \cap Y, (V \cup s) - (X_{i_k} \cup Y)) = 0$ holds, contradicting $(s, t^{i_k+1}) \in E_{G^{i_{k-1}}}(s, X_{i_k} \cap Y)$. Thus $Y - X_{i_k} = \emptyset$. Now $t^{i_k+1} \in Y$, $c_{G^{i_{k-1}}}(Y) = \ell$ and $Y \subset X_{i_k}$ hold. Since t^{i_k+1} was originally a free vertex of X_{i_k} in G_1'', we have $c_{G_1''}(Y) > \ell$. Hence for some $r < k$, there is a cut $X_{i_r} \in \mathcal{X}_1$ such that $t^{i_r+1} \in Y$, $u^{i_r} \in X_{i_r} - Y$ and $c_{G^{i_{r-1}}}(Y) = \ell + 1$. Since t^{i_r+1} is a free vertex of X_{i_r}, we have $X_{i_r} \subset X_{i_k}$ by the definition of free vertices. This, however, contradicts the order of $i_1, i_2, \ldots, i_{j'}$. ■

The following lemma implies Property 1.

Lemma 6 *ADD-EDGE finds a set* F_1 *of new edges between* s *and* V *such that* $|F_1| = \alpha(G)$ *and* $G_1 = (V \cup \{s\}, E \cup F_1)$ *satisfies* (8) *and* (9).

Proof: Property 14 implies that G_1 satisfies (8) and (9). We prove $|F_1| \le \alpha(G)$ in order to show $|F_1| = \alpha(G)$, since $|F_1| \ge \alpha(G)$ is easily seen. Let $\mathcal{X}_1$ be the family of λ-critical cuts in the graph $G_1 = (V \cup \{s\}, E \cup F_1)$ obtained by ADD-EDGE. Now for $\mathcal{X}_1' = \{X_1, \cdots, X_p\}$ and $\mathcal{X}_2 = \{X_{p+1}, \cdots, X_q\}$, where $\mathcal{X}_1'$ denotes the family of all cuts $X \in \mathcal{X}_1$ such that no other cut $X' \in \mathcal{X}_1$ contains X, and $\mathcal{X}_2$ denotes the family of κ-critical minimal tight sets $T_i \in \mathcal{T}(G)$ such that their κ-critical neighbors t_i belong to $V - \cup_{X \in \mathcal{X}_1} X$, it holds

$$|F_1| = \sum_{i=1}^{p} (\ell - c_G(X_i)) + \sum_{i=p+1}^{q} (3 - |\Gamma_G(X_i)|).$$

From definition of $\alpha(G)$, we have $|F_1| \le \alpha(G)$. ■

4.1. Complexity of Step I

Here we analyze the complexity of Algorithm ADD-EDGE.

LEMMA 7 *Algorithm ADD-EDGE can be implemented to run in $O(n^2(m+n\log n))$ time.*

Proof: All minimal tight sets in $\mathcal{T}(G)$ can be computed in linear time [10]. We shall show below that the set F'' of λ-critical edges and the laminar family $\mathcal{X}_1$ of λ-critical cuts can be computed in $O(nm + n^2 \log n)$ time. To attain this time bound, we use the algorithm AUGMENT in [27] with a slight modification, which we describe as algorithm LAMINAR. Given a multigraph $G' = (V + s, E')$ (where possibly $E' \cap E_{G'}(s) \ne \emptyset$) and a set E_1 of edges between s and V such that $G' + E_1$ satisfies (8), algorithm LAMINAR computes an edge set $F \subset E_1$ such that $G' + F$ satisfies (8) and F is minimal subject to this property, i.e., every edge in F is λ-critical in the resulting graph $G' + F$ (where some edge in E' may be λ-critical in $G' + F$). This algorithm also finds a laminar family $\mathcal{X}$ of λ-critical cuts X_u for all $u \in V[F] - s$ ($\mathcal{X}$ may not contain a λ-critical cut X_v for some λ-critical edge $(s, v) \in E'$).

We first prove the correctness of algorithm LAMINAR in Lemma 8, and later show how to use algorithm LAMINAR in ADD-EDGE.

Algorithm LAMINAR$(G', E_1; F, \mathcal{X})$
Input: a multigraph $G' = (V + s, E')$ and a set E_1 of edges between s and V such that $G' + E_1$ satisfies (8).
Output: a minimal edge set $F \subset E_1$ such that $G' + F$ satisfies (8) and F is minimal subject to this property, and a laminar family $\mathcal{X}$ of λ-critical cuts X_u for all $u \in V[F] - s$.

$F := \emptyset$;
Let $U = \{u_1, u_2, \cdots, u_r\}$ be the set of vertices $u \in V$ with $c_{G'}(u) < \ell$;
For each $u_i \in U$ in G', add to F $\ell - c_{G'}(u_i)$ edges between s and u_i;
$\mathcal{X} := \{\{u_1\}, \{u_2\}, \cdots, \{u_r\}\}$

5 $H := G' + F$;
6 **while** $|V(H)| \geq 4$ **do** /* $c_H(u) \geq \ell$ holds for all vertices $u \in V(H) - s$ */
7 Find two vertices $v, w \in V(H) - s$ with $\lambda_H(v, w) \geq \ell$;
8 Contract v and w into a single vertex x^*, and let H be the resulting graph;
9 **if** $c_H(x^*) < \ell$ **then**
10 Let $X^* \subseteq V$ be the set of vertices which have been contracted so far into x^*;
11 Choose an arbitrary set F_{X^*} of $\ell - c_H(x^*)$ edges from $E_1 \cap E_{G'+E_1}(s, X^*)$;
/* $|E_1 \cap E_{G'+E_1}(s, X^*)| \geq \ell - c_{G'+E_1}(X^*)$ holds since $G' + E_1$ satisfies (8). */
12 $F := F \cup F_{X^*}$;
13 Let H denote the graph H augmented with new $\ell - c_H(x^*)$ edges between s and x^*;
/* Note that these $\ell - c_H(X^*)$ edges correspond to the edge set F_{X^*} in G'. */
14 Discard from $\mathcal{X}$ all cuts X such that $X \cap V[F_{X^*}] \neq \emptyset$, and then $\mathcal{X} := \mathcal{X} \cup \{X^*\}$
15 **end** /* if */
16 **end** /* while */

LEMMA 8 *Given a multigraph $G' = (V + s, E')$ and a set E_1 of edges between s and V such that $G' + E_1$ satisfies* (8), LAMINAR *computes in $O(n(m + n \log n))$ time a subset F of all λ-critical edges in $G' + F$, and a laminar family $\mathcal{X}$ such that each cut X in $\mathcal{X}$ is a λ-critical cut for all vertices $u \in X \cap V[F]$, where $n = |V|$ and m is the number of pairs of adjacent vertices in G'.*

Proof: (Sketch) The running time is derived from a similar argument for algorithm AUGMENT in [27]. Since two vertices v, w in line 7 are contracted only when $\lambda_H(v, w) \geq \ell$, it is easy to see that the final graph $G' + F$ satisfies (8). Clearly $\mathcal{X}$ is always a laminar family of cuts X with size ℓ during execution of LAMINAR, since the contraction in line 8 keeps $\mathcal{X}$ laminar, and the discard in line 14 makes the sizes of all cuts in $\mathcal{X}$ ℓ. Some cut X may be discarded in line 14, but in this case, a new cut $X^* \supset X$ is added to $\mathcal{X}$ (note that every discarded cut X has been added to $\mathcal{X}$ and been contracted to one vertex in H). This implies that for each vertex $u \in V[F] - s$, $\mathcal{X}$ contains a cut X with $u \in X$. Thus, all edges in F are λ-critical in $G' + F$. Finally we show that every cut X in the final $\mathcal{X}$ is a λ-critical cut for all $u \in X \cap V[F]$. As is shown in the correctness of algorithm AUGMENT in [27], we see that the cut X^* in line 10 satisfies $c_{G'+F}(X^*) < \ell$ and $c_{G'+F}(X') \geq \ell$ for all cuts $X' \subset X^*$ for the current F. Thus, for every edge (s, u) in the set $F_{X^*} \subseteq E_1 \cap E_{G'+E_1}(s, X^*)$ in line 11, it holds $c_{G'+F+F_{X^*}}(X') > \ell$ for all cuts $X' \subset X^*$; i.e., X^* is a λ-critical cut of u in $G' + F$ with new $F := F + F_{X^*}$. Also, for each edge $(s, v) \in F$ with $v \in X \subseteq X^*$ for some X in the current $\mathcal{X}$, X^* will be a new λ-critical cut of v in $G' + F$ with new $F := F \cup F_{X^*}$. Therefore such

X can be discarded when X^* is added to $\mathcal{X}$ in line 14. This proves the lemma. ■

Now we show how to execute ADD-EDGE by using LAMINAR. First we show how to compute edge sets F'' and $\hat{F}$ and a laminar family $\mathcal{X}_1$ by applying LAMINAR in Step 2 of ADD-EDGE. Firstly letting $G_a := (V+s, E \cup F')$ and $E_1 := \{\ell$ multiple edges $(s,v) \mid v \in V\}$, we execute LAMINAR$(G_a, E_1; F_a, \mathcal{X}_a)$ to compute a set F'' of λ-critical edges in E_1 such that $G_a + F_a$ satisfies (8). Next we consider the graph $G_b = (V+s, E \cup F_a)$ obtained from $(V+s, E \cup F' \cup F_a)$ by removing edges in F', and execute LAMINAR$(G_b, F'; F_b, \mathcal{X}_b)$ to obtain a minimal subset F_b of F' such that $G_b + F_b$ satisfies (8), where clearly every edge in $F_a \cup F_b$ is λ-critical in $G_b + F_b = (V+s, E \cup F_a \cup F_b)$. This F_a gives F''. Then let $G_c = (V+s, E)$ and execute LAMINAR$(G_c, F_a \cup F_b; F_c, \mathcal{X}_c)$ to obtain a laminar family $\mathcal{X}_c$ of λ-critical cuts for all vertices $u \in V[F_c] - s$, where $F_c = F_a \cup F_b$ holds. Finally, we put back all edges in $F' - F_b$ to $(V+s, E \cup F_b \cup F_a)$, and discard from $\mathcal{X}_c$ all cuts X such that there is an edge $(s,u) \in F' - F_b$ with $u \in X$, since such cut X is no longer λ-critical in $(V+s, E \cup F' \cup F_a)$. Then the resulting family gives $\mathcal{X}_1$, and a set $\hat{F}$ of all λ-critical edges in $(V+s, E \cup F' \cup F_a)$ is obtained by $\{(s,u) \in F_a \cup F_b \mid u \in X \in \mathcal{X}_1\}$, where $F_a \subseteq \hat{F}$. By Lemma 8, those F'', $\mathcal{X}_1$ and $\hat{F}$ can be computed in $O(n(m + n \log n))$ time.

Recomputing $\mathcal{X}_1$ in Steps 3 and 4 of ADD-EDGE can be done in a similar way. Now two edge sets F' and F'' are given, where F' is a set of edges (s, t_i) with a κ-critical neighbor t_i and each edge in F'' is λ-critical in $(V+s, E \cup F' \cup F'')$. Instead of setting E_1 by $\{\ell$ multiple edges $(s,v) \mid v \in V\}$, we set $E_1 := F''$ and execute the same $O(n(m + n \log n))$ time procedure as above, to obtain a set $\hat{F}$ of λ-critical edges and a laminar family $\mathcal{X}_1$ in $(V+s, E \cup F' \cup F'')$ (and hence we can find a κ-critical minimal tight set $T_i \in \mathcal{T}(G)$ that violates some cuts $X \in \mathcal{X}_1$).

As mentioned in Remark 4.1, the number of recomputing $\mathcal{X}_1$ in Steps 3 or 4 of ADD-EDGE is $O(n)$. Thus, Step I can be executed in $O(n^2 m + n^3 \log n)$ time, proving Lemma 7. ■

Before concluding this section, we show the following property, which will be used in Section 8.1 to design an algorithm to solve EVAP$(\ell, 3)$ by an enumerative approach. This is about the property of each cut X_i which corresponds to the vertex x^* in H when $c_H(x^*) < \ell$ and $E_H(s, x^*) = \emptyset$ hold in lines 8, 9 in LAMINAR; i.e., no edge has been added between s and X_i so far but some edges are now added for the first time.

LEMMA 9 *Given a multigraph $G' = (V+s, E)$ with no edge between s and V, we can find in $O(n(m + n \log n))$ time a subpartition $\mathcal{Z} = \{X_1, X_2, \ldots, X_r\}$ of V and $Z_1 = \{x_i \in X_i \mid i = 1, 2, \ldots, r\}$ which satisfy*

(i) *For each $X_i \in \mathcal{Z}$, $c_{G'}(X_i) < \ell$ and $c_{G'}(X') \geq \ell$ for all cuts $X' \subset X_i$.*

(ii) *For any cut Y with $c_{G'}(Y) < \ell$, there is a vertex in $Y \cap Z_1$.*

(iii) *If a cut Y with $c_{G'}(Y) < \ell$ crosses some cut $X_i \in \mathcal{Z}$, then $Y - X_i$ contains at least one vertex $x_j \in V[F] - s$.*

Proof: By setting $E_1 := \{\ell$ multiple edges $(s, v) \mid v \in V\}$, we execute LAMINAR(G, $E_1; F, \mathcal{X}$), with the following slight restriction on choosing an edge set F_{X^*} in line 11. We first choose a vertex $v \in X^*$ which is already incident to s by an edge in the current F, or an arbitrary vertex $u \in X^*$ if no vertex in X is incident to s. Then we choose F_{X^*} as a set of $\ell - c_H(x^*)$ multiple edges (s, u). Obviously, LAMINAR correctly computes a minimal edge set F of λ-critical edges and a laminar family $\mathcal{X}$ for those edges, since F_{X^*} can be arbitrarily chosen. Now consider the set $\mathcal{X}^*$ of all cuts X^* obtained in line 10. Clearly, $\mathcal{X} \subseteq \mathcal{X}^*$ holds, and $\mathcal{X}^* - \mathcal{X}$ is the family of all cuts discarded from $\mathcal{X}$ during LAMINAR. Let $Z_1 = V[F] - s$ and $\mathcal{Z}$ be the family of minimal cuts in $\mathcal{X}^*$, i.e., the family of cuts $X \in \mathcal{X}^*$ such that no proper subset X' of X belongs to $\mathcal{X}^*$. By construction of F, we see that $|Z_1| = |\mathcal{Z}|$ and for each cut $X \in \mathcal{Z}$, $|Z_1 \cap X| = 1$ holds. As observed in the proof of Lemma 8, when a cut $X \in \mathcal{Z}$ is chosen in line 10 of LAMINAR, it satisfies (i) in the current graph (and hence it satisfies (i) in G' since there has been no edge between s and X). Next we show (ii). Since $G' + F$ satisfies (8), any cut Y with $c_{G'}(Y) < \ell$ must contain a vertex in $Z_1 = V[F] - s$ (otherwise $c_{G'+F}(Y) = c_{G'}(Y) < \ell$ would contradict (8)). Now let us prove (iii). Consider a cut Y with $c_{G'}(Y)$ crossing a cut $X \in \mathcal{Z}$, and assume that $(Y - X) \cap Z_1 = \emptyset$. By (ii), $Y \cap Z_1 \neq \emptyset$, but by $(Y - X) \cap Z_1 = \emptyset$, we have $Y \cap Z_1 \subseteq X \cap Y$. Thus $|X \cap Z_1| = 1$ implies that $Y \cap Z_1$ contains a unique vertex, which belongs to X. Hence $(X - Y) \cap Z_1 = \emptyset$. Since $G' + F$ satisfies (8), $(Y - X) \cap Z_1 = (X - Y) \cap Z_1 = \emptyset$ implies $c_{G'}(Y - X) = c_{G'+F}(Y - X) \geq \ell$ and $c_{G'}(X - Y) = c_{G'+F}(X - Y) \geq \ell$. From (4), we have $2\ell > c_{G'}(X) + c_{G'}(Y) \geq c_{G'}(X - Y) + c_{G'}(Y - X) \geq 2\ell$, a contradiction. The time complexity is immediate from Lemma 8. ■

5. Justification of Step II

Let $G_1 = (V \cup \{s\}, E \cup F_1)$ be the graph obtained from a given graph $G = (V, E)$ after Step I. In Step II, a graph $G_2 = (V, E \cup F_2)$ is constructed from G_1 by applying a complete edge-splitting at s. Then the correctness of Step II is immediate from Lovász's theorem (Theorem 1).

It is known in [27, 1] that such a complete splitting can be done in $O(n(m + n \log n) \log n)$ time, which is the time required for Step II. Also, the resulting set F_2 of edges satisfies $m(G + F_2) = m + O(n)$ [1]; that is, the number of pairs of vertices which newly became adjacent in G_2 is $O(n)$.

6. Justification of Step III

Let $G_2 = (V, E \cup F_2)$ be the graph obtained after Step II. Now G_2 is ℓ-edge-connected but has disconnecting pairs. To show that Step III works correctly, we need to prove Properties 2, 3 and 4 in Step III. For this, we first prove the following three claims.

Claim 1 *Let $G = (V, E)$ be an $(\ell, 2)$-connected graph for $\ell \geq 2$, and let $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ be two edges in E with $V(e_1) \cap V(e_2) = \emptyset$. Assume that $G - \{e_1, e_2\}$ is 2-vertex-connected and that one of the induced subgraphs $G[X]$ and $G[V - X]$ is not connected for any cut X that separates $\{u_1, u_2\}$ and $\{w_1, w_2\}$. Then switching $\{(u_1, u_2), (w_1, w_2)\}/\{e_1, e_2\}$ is improving in G.*

Proof: Let G' denote the graph resulting from G by switching $\{(u_1, u_2), (w_1, w_2)\}/ \{e_1, e_2\}$; i.e., $G' = G + \{(u_1, u_2), (w_1, w_2)\} - \{e_1, e_2\}$.

We first show that G' remains ℓ-edge-connected. Let X be a cut in G'. Since if X does not separate $\{u_1, u_2\}$ and $\{w_1, w_2\}$, then $c_{G'}(X) \geq c_G(X) \geq \ell$, we assume that X separates $\{u_1, u_2\}$ and $\{w_1, w_2\}$. Hence $c_{G'}(X) = c_G(X) - 2$. Since at least one of $G[X]$ and $G[V - X]$ is not connected by assumption, we have $c_G(X) = c_G(V - X) \geq 2\ell \geq \ell + 2$ by Lemma 2. Therefore $c_{G'}(X) \geq \ell$ holds for all cuts X.

Next we show that $\mathcal{P}_3(G') \supseteq \mathcal{P}_3(G)$ holds. Assume that there is a pair $x, y \in V$ such that $\kappa_G(x, y) > \kappa_{G'}(x, y) = 2$. Let $S = \{v_1, v_2\}$ be a disconnecting pair in G' that disconnects the x and y, and let $T_1, \ldots, T_r$ be the S-components in G'; we assume $x \in T_1$ and $y \in T_2$. Let $W_1 = T_1$ and $W_2 = T_2 \cup \cdots \cup T_r$. Clearly, $E_G(W_1, W_2) - \{e_1, e_2\} = E_{G'}(W_1, W_2) = \emptyset$. Since S does not disconnect x and y in G by $\kappa_G(x, y) > 2$, we have $E_G(W_1, W_2) \cap \{e_1, e_2\} \neq \emptyset$. Assume without loss of generality that $e_1 = (u_1, w_1) \in E_G(W_1, W_2)$, $u_1 \in W_1$ and $w_1 \in W_2$. Thus $u_2 \in W_1 \cup S$ and $w_2 \in W_2 \cup S$ must hold by $E_{G'}(W_1, W_2) = \emptyset$. From $u_2 \neq w_2$, we can assume without loss of generality that $u_2 \in W_1 \cup \{v_1\}$ and $w_2 \in W_2 \cup \{v_2\}$. From the 2-vertex-connectivity of $G - \{e_1, e_2\}$, we see that each of the subgraphs $G[W_1 \cup \{v_1\}]$ and $G[W_2 \cup \{v_2\}]$ is connected. This means that cut $X = W_1 \cup \{v_1\}$ separates $\{u_1, u_2\}$ and $\{w_1, w_2\}$, and both induced subgraphs $G[X]$ and $G[V - X]$ are connected, contradicting the assumption. ∎

Claim 2 *Let $G = (V, E)$ be an $(\ell, 2)$-connected graph for $\ell \geq 2$ which has a disconnecting pair $S = \{v_1, v_2\}$. Let $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ be two edges in E with $V(e_1) \cap V(e_2) = \emptyset$, such that $\{u_1, w_1\} \subseteq T_1 \cup \{v_1\}$ and $\{u_2, w_2\} \subseteq T_2 \cup \{v_2\}$ hold for two S-components T_1 and T_2 in G. Assume that $G - \{e_1, e_2\}$ is 2-vertex-connected. Then switching $\{(u_1, u_2), (w_1, w_2)\}/\{e_1, e_2\}$ or $\{(u_1, w_2), (u_2, w_1)\}/\{e_1, e_2\}$ preserves the ℓ-edge-connectivity of G.*

Proof: Assume that both switchings in the claim statement violate the ℓ-edge-connectivity in G: i.e., G has two cuts X and Y such that X (resp., Y) satisfies $\{u_1, u_2\} \subseteq X$ and $\{w_1, w_2\} \subseteq V - X$ (resp., $\{u_2, w_1\} \subseteq Y$ and $\{u_1, w_2\} \subseteq V - Y$) and $c_G(X) \leq \ell + 1$ (resp., $c_G(Y) \leq \ell + 1$). Without loss of generality, let $u_1 \in T_1$ and $u_2 \in T_2$. By Lemma 2, each of subgraphs $G[X]$, $G[V - X]$, $G[Y]$, and $G[V - Y]$ is connected. Thus, $|S \cap X| = |S \cap Y| = 1$ must hold. Now X and Y cross each other since $u_1 \in X - Y$, $w_1 \in Y - X$, $u_2 \in X \cap Y$, and $w_2 \in V - (X \cup Y)$. By Lemmas 1(1) and 2, we see that each of the subgraphs $G[X \cap Y]$, $G[V - (X \cup Y)]$, $G[X - Y]$, and $G[Y - X]$ is connected. Also, by Lemma 1 (2), $|E_G(X \cap Y, V - (X \cup Y))| = |E_G(X - Y, Y - X)| = 1$ holds, from which we have $E_G(X \cap Y, V - (X \cup Y)) = \{(u_2, w_2)\}$

and $E_G(X-Y, Y-X) = \{(u_1, w_1)\}$. In what follows, we assume without loss of generality that $v_1 \in X$ and $v_2 \notin X$.

Case-1: $v_1 \in Y$ and $v_2 \notin Y$. Now $\{v_2\} \subseteq V - (X \cup Y)$ implies $w_1 \in T_1$ and $u_2 \in T_2 \cap (X \cap Y)$. Note that $S \cap (X-Y) = S \cap (Y-X) = \emptyset$. Since $G[X-Y]$ is connected, $(X-Y) \cap T_2 = \emptyset$ must hold (since otherwise $(X-Y) \cap T_2$ and $u_1 \in (X-Y) \cap T_1$ would be disconnected by the disconnecting pair S in G). Similarly, the connectivity of $G[Y-X]$ and $w_1 \in T_1$ imply $(Y-X) \cap T_2 = \emptyset$. Thus, $T_2 \cap (X \cap Y) = T_2 \cap (X \cup Y)$. Hence, by $u_2 \in T_2 \cap (X \cup Y)$ and $E_G(X \cap Y, V - (X \cup Y)) = \{(u_2, w_2)\}$ we have $E_G(T_2 \cap (X \cup Y), (T_2 \cup \{v_2\}) - (X \cup Y)) = \{(u_2, w_2)\}$. However, this means $\Gamma_{G-\{e_1,e_2\}}(T_2 \cap (X \cup Y)) \subseteq \{v_1\}$, contradicting $\kappa(G - \{e_1, e_2\}) \geq 2$.

Case-2: $v_2 \in Y$ and $v_1 \notin Y$. Now $w_2 \in V - (X \cup Y)$ implies $w_2 \in T_2$. Note that $S \cap (X \cap Y) = S \cap (V - (X \cup Y)) = \emptyset$ holds. Similarly to Case-1, we can see $T_1 \subseteq (X-Y) \cup (Y-X)$ from the connectedness of $G[V - (X \cup Y)]$. However, $E_G(X-Y, Y-X) = \{(u_1, w_1)\}$ means $\Gamma_{G-\{e_1,e_2\}}(T_1 \cap Y) \subseteq \{v_2\}$, contradicting $\kappa(G - \{e_1, e_2\}) \geq 2$. ■

Claim 3 *Let S be a disconnecting pair in a 2-vertex-connected graph G. Assume that an S-component T contains an s-admissible edge $e = (u, w)$ with respect to a disconnecting pair S. Then $G[T] - e$ contains a path between u and w.*

Proof: From the definition $p((G-e) - S) = p(G - S)$, T is still a connected component in $(G-e) - S$. So there is a path between u and w in $G[T] - e$ (see an example for s-admissible edges in Figure A.2 in the appendix). ■

Proof of Property 2: Case-(d): We assume without loss of generality $u_1 \in T_1$, $w_1 = v_1$, $u_2 \in T_2$ and $w_2 = v_2$, and show that $\{(v_1, v_2), (u_1, u_2)\}/\{e_1, e_2\}$ is improving in G_2. Let $G_2' := G_2 + \{(v_1, v_2), (u_1, u_2)\} - \{e_1, e_2\}$. For every cut X that separates $\{v_1, v_2\}$ and $\{u_1, u_2\}$, at least one of $G_2[X]$ and $G_2[V - X]$ is not connected, since $\{v_1, v_2\}$ is a disconnecting pair in G_2. Thus Claim 1 implies that $\{(v_1, v_2), (u_1, u_2)\}/\{e_1, e_2\}$ is improving in G_2. Each T_i, $i = 1, 2$ remains to be an S-component in $G_2 - \{e_1, e_2\}$, since e_i is incident to S and $\kappa(G_2 - \{e_1, e_2\}) \geq 2$. Hence $(u_1, u_2) \in E_{G_2'}(T_1, T_2)$ implies that $G_2'[T_1 \cup T_2]$ is connected. Therefore $p(G_2' - S) < p(G_2 - S)$ holds.

Cases-(a),(b),(c): In cases (b) and (c), it is not difficult to see from $\kappa(G_2 - \{e_1, e_2\}) \geq 2$ that e_1 (resp., e_2) is s-admissible to S (resp., S_1) in G_2 (detailed proofs in [16]). Then we can assume that e_1 (resp., e_2) is incident to S (resp., S_1), since otherwise e_1, e_2 and S (or S_1) satisfy condition (a). Thus, in cases (b) and (c), we assume that exactly one of the end vertices of $e_1 = (u_1, w_1)$ (resp., $e_2 = (u_2, w_2)$) belongs to $S_1 = \{v_3, v_4\}$ (resp., $S = \{v_1, v_2\}$). Now we consider cases (a)(b)(c).

(i) We first show $p(G_2' - S) < p(G_2 - S)$ holds in the graph G_2' obtained from G_2 by any of the two switchings $\{(u_1, u_2), (w_1, w_2)\}/\{e_1, e_2\}$ and $\{(u_1, w_2), (u_2, w_1)\}/\{e_1, e_2\}$. This follows because the s-admissibility of e_1 implies that $G_2'[T_1' \cup T_2]$ is connected, where T_1' denotes the S-component with $V[e_1] \subseteq T_1'$ (where $T_1' = T_1$ in case (a)).

(ii) From the assumption on G_2, edges e_1 and e_2 satisfy the statements of Claim 2 in all cases (a) – (c). Therefore the resulting graph obtained from at least one of switchings $\{(u_1,u_2),(w_1,w_2)\}$ $/\{e_1,e_2\}$ or $\{(u_1,w_2),(u_2,w_1)\}$ $/\{e_1,e_2\}$ is also ℓ-edge-connected.

(iii) We show that at least one of the switchings $\{(u_1,u_2),(w_1,w_2)\}$ $/\{e_1,e_2\}$ or $\{(u_1,w_2),(u_2, w_1)\}$ $/\{e_1,e_2\}$ is improving. Without loss of generality, assume that the graph G_2^* obtained from G_2 by switching $\{(u_1,u_2),(w_1,w_2)\}/\{e_1,e_2\}$ preserves the ℓ-edge-connectivity. We show that if there is a pair of vertices $x,y \in V$ in G_2^* such that $\kappa_{G_2}(x,y) > \kappa_{G_2^*}(x,y) = 2$, then the other switching $\{(u_1,w_2),(w_1,u_2)\}/\{e_1,e_2\}$ is improving in G_2.

Assume that G_2^* has a pair $x,y \in V$ such that $\kappa_{G_2}(x,y) > \kappa_{G_2^*}(x,y) = 2$. Let $S' = \{v_1', v_2'\}$ denote a disconnecting pair in G_2^* that disconnects x and y. Clearly, $S' \neq S$ (because $S = S'$ would imply $\kappa_{G_2}(x,y) = 2$). Let $W_1, W_2, \cdots, W_q$ ($q \geq 2$) be the S'-components of G_2^*, where $x \in W_1$ and $y \in W_2$. Since the S' does not disconnect x and y in G_2, $E_{G_2}(W_1,W_2)$ contains e_1 or e_2. Also no edge other than e_1 and e_2 can belong to $E_{G_2}(W_1,W_2)$. It is easy to see $W_1 \cup W_2 \cup S' \supseteq \{u_1,w_1,u_2,w_2\}$. Also note that $u_i,w_i \in W_j$ cannot hold for any $i \neq j$ with $i \in \{1,2\}$ and $j \in \{1,2\}$ (otherwise (u_1,u_2) or (w_1,w_2) would belong to $E_{G_2^*}(W_1,W_2)$). Therefore, for each $i = 1,2$, we have $e_i = (u_i,w_i) \in E_{G_2}(W_1,W_2)$ or $\{u_i,w_i\} \cap S' \neq \emptyset$.

Case-1: We first consider the case $e_1 \in E_{G_2}(W_1,W_2)$. Since $G_2[T_1'] - e_1$ is connected by Claim 3, we can assume without loss of generality that $v_1' \in T_1'$, $u_1 \in W_1$ and $w_1 \in W_2$ hold, from which $u_2 \in W_1 \cup S'$ and $w_2 \in W_2 \cup S'$ follow.

It is not difficult to see that $v_2' \notin T_1'$ holds by $\kappa(G_2 - \{e_1,e_2\}) \geq 2$. Then, we see that $S \not\subset W_1$ holds, because otherwise $\kappa(G_2 - \{e_1,e_2\}) \geq 2$ implies $|\Gamma_{G_2-\{e_1,e_2\}}(T_1' \cap W_2)| \geq 2$ and $v_2' \in T_1'$. Similarly, we obtain $S \not\subset W_2$. In what follows, we consider two cases (α) $S \cap W_1 \neq \emptyset \neq S \cap W_2$ (hence $v_2' \notin S$) and (β) $v_2' \in S$ and $S \cap W_1 \neq \emptyset$, or $v_2' \in S$ and $S \cap W_2 \neq \emptyset$.

(α) $S \cap W_1 \neq \emptyset \neq S \cap W_2$ and $v_2' \notin S$. Without loss of generality, let $v_1 \in W_1$ and $v_2 \in W_2$. From $v_2' \notin T_1'$, we have $v_2' \in T_2$. Note that $G_2[V - (W_1 \cup W_2 \cup S')]$ is a collection of S'-components in G_2 if $V - (W_1 \cup W_2 \cup S') \neq \emptyset$. But since S is a disconnecting pair in G_2, and $v_1' \in T_1'$ and $v_2' \in T_2$ hold, it follows $G_2 - G_2[W_1 \cup W_2 \cup S'] = \emptyset$. Hence $G_2 = G_2[W_1 \cup W_2 \cup S']$.

We claim that, for any cut X that separates $\{u_1,w_2\}$ and $\{u_2,w_1\}$, at least one of $G_2[X]$ and $G_2[V - X]$ is not connected. If this claim holds, then Claim 1 implies that switching $\{(u_1,w_2),(u_2,w_1)\}/\{e_1,e_2\}$ is improving in G_2. For this we can consider the graph $G_2 - \{e_1,e_2\}$, because neither $V[e_1]$ nor $V[e_2]$ is contained in X or $V - X$.

If $T_2 \cap W_1 \neq \emptyset \neq T_2 \cap W_2$ holds, then any two vertices in $\{v_1,v_2,v_1',v_2'\}$ forms a disconnecting pair in $G_2[W_1 \cup W_2 \cup S'] - \{e_1,e_2\}$, since $G_2[W_1 \cup W_2 \cup S']$ has only edges $\{e_1,e_2\}$ or $\{e_1\}$ between W_1 and W_2, and there is no edge between T_1' and T_2 (see Figure 6(i)). Note that this implies the claim. The case of $T_2 \cap W_1 = \emptyset$ or $T_2 \cap W_2 = \emptyset$ can be similarly handled (see [16] for details).

(β) The case of $v_2' \in S$. In this case, it holds either $v_1 \in W_1$ and $v_2 = v_2'$, or $v_1 = v_2'$ and $v_2 \in W_2$. We consider only the case $v_1 \in W_1$ and $v_2 = v_2'$ (since the other case can be treated symmetrically). From $v_2 = v_2'$ and $v_1 \in W_1$,

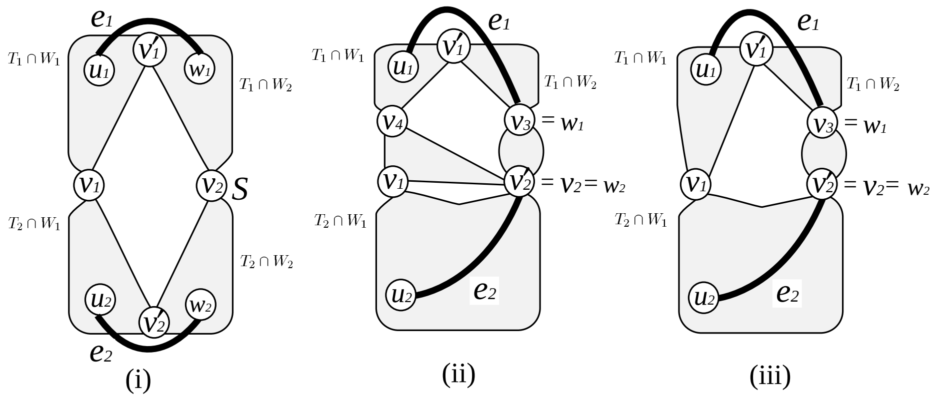

Figure 6. Illustrations of the graph G_2 such that $\mathcal{P}_3(G_2) \not\subseteq \mathcal{P}_3(G_2^*)$ holds, where G_2^* denotes the graph obtained from G_2 (which satisfies conditions (a), (b), or (c) in Property 2) by switching $\{(u_1, u_2),(w_1, w_2)\}/\{e_1, e_2\}$. (i) G_2 satisfies Case-1(α) and $T_2 \cap W_1 \neq \emptyset \neq T_2 \cap W_2$. (ii) G_2 satisfies Case-1(β) and the condition (b) in Property 2. (iii) G_2 satisfies Case-1(β) and the condition (c) in Property 2.

we have $T_2 \cap W_2 = \emptyset$, since otherwise $\Gamma_{G_2 - \{e_1, e_2\}}(T_2 \cap W_2) \subseteq \{v_2\}$ contradicts $\kappa(G_2 - \{e_1, e_2\}) \geq 2$. This implies that $v_2 = v_2'$ is one of the end vertices of e_2, since $\{u_2, w_2\} \subset W_1$ cannot hold. Note that this case cannot occur in case (a) of Property 2, where e_2 is not incident to S. Therefore we only consider cases (b) and (c) of Property 2. As described above, it suffices to consider the case of $w_1 = v_3$ and $w_2 = v_2$. By $w_2 \in W_2 \cup S'$, let $w_2 = v_2 = v_2'$. We claim that, for any cut X that separates $\{u_1, w_2\}$ and $\{u_2, w_1\}$, at least one of $G_2[X]$ and $G_2[V - X]$ is not connected. To prove the claim, first note that, in case (b) (resp., case (c)), any two vertices in $\{v_1', v_2', v_4\}$ (resp., $\{v_1', v_2', v_4 = v_1\}$) is a disconnecting pair in $G_2 - e_1$ (as illustrated in Figure 6(ii) (resp., (iii))). The proof of the claim uses an argument similar to the case (α) and the structure of disconnecting pairs S and S_1, but is omitted here (see [16] for details).

Case-2: $e_1 \notin E_{G_2}(W_1, W_2)$. In this case, e_1 is incident to a vertex in S' (say, v_1'). Hence $e_2 \in E_{G_2}(W_1, W_2)$ holds. Moreover, $S \cap W_1 \neq \emptyset$, $S \cap W_2 \neq \emptyset$ and $v_2' \in T_2$ hold, as shown in Case-1, and hence $G_2[T_2] - \{e_2\}$ has a path between u_2 and w_2 via v_2'. Therefore e_2 is s-admissible with respect to S in case of (a). In cases (b) and (c), e_2 is s-admissible with respect to S_1, as shown similarly to the proof of s-admissibility of e_1 with respect to S, in the first part of this proof. Therefore it can be proved by reversing the roles of e_1 and e_2, and those of S_1 and S, respectively.

This completes the proof of Property 2. ∎

Proof of Property 3: We give the proof in three stages (1)(2) and (3).

(1) It is easy to see that $p(G'_2 - S_1) < p(G_2 - S_1)$ holds for the graph G'_2 if G'_2 is obtained from G_2 by switching $\{(v^*, v_1), (w_1, u_3), (w_2, w_3)\}/\ \{e_1, e_2, e_3\}$, since $G'_2[T_1 \cup T_2]$ is connected. By exchanging u_3 and w_3, we see that $p(G'_2 - S_1) < p(G_2 - S_1)$ also holds if G'_2 is obtained from G_2 by switching $\{(v^*, v_1), (w_1, w_3), (w_2, u_3)\}/\ \{e_1, e_2, e_3\}$.

(2) We show that at least one of the switchings $\{(v^*, v_1), (w_1, u_3), (w_2, w_3)\}/\{e_1, e_2, e_3\}$ and $\{(v^*, v_1), (w_1, w_3), (w_2, u_3)\}/\{e_1, e_2, e_3\}$ preserves ℓ-edge-connectivity of G_2. First, by applying Claim 2 to e_1 and e_3, the graph G'_2 resulting from at least one of the switchings $\{(v^*, u_3), (w_1, w_3)\}/\ \{e_1, e_3\}$ and $\{(v^*, w_3), (w_1, u_3)\}/\ \{e_1, e_3\}$ is ℓ-edge-connected. Next we show that switching $\{(u_3, w_2), (v^*, v_1)\}/\{(v^*, u_3), e_2\}$ (resp., $\{(w_3, w_2), (v^*, v_1)\}/\{(v^*, w_3), e_2\}$) preserves the ℓ-edge-connectivity in the former case (resp., in the latter case). We prove only the former case, since the latter is similar. If a cut X does not separate $\{v^*, v_1\}$ and $\{u_3, w_2\}$, then $c_{G''_2}(X) \geq c_{G'_2}(X) \geq \ell$ holds, where G''_2 denotes the graph obtained from G'_2 by switching $\{(u_3, w_2),\ (v^*, v_1)\}/\{(v^*, u_3), e_2\}$. Therefore we assume that X separates $\{v^*, v_1\}$ and $\{u_3, w_2\}$. We consider a cut X with $\{v^*, v_1\} \subseteq X$ and $\{u_3, w_2\} \subseteq V - X$. Note that $c_{G''_2}(X) = c_{G'_2}(X) - 2$ holds. If $w_1 \in X$ or $w_3 \in X$ holds, then $c_{G'_2}(X) = c_{G'_2}(V - X) \geq 2\ell \geq \ell + 2$ by Lemma 2 since $G'_2[V - X]$ is not connected. Hence $c_{G''_2}(X) = c_{G'_2}(X) - 2 \geq \ell$ holds. If $w_1 \notin X$ and $w_3 \notin X$ hold, then $c_{G'_2}(X) = c_{G_2}(X)$ holds by construction of G'_2. Now, since $G_2[V - X]$ is not connected, $c_{G_2}(X) = c_{G_2}(V - X) \geq 2\ell \geq \ell + 2$ by Lemma 2, and hence $c_{G''_2}(X) = c_{G'_2}(X) - 2 = c_{G_2}(X) - 2 \geq \ell$ holds. Therefore $c_{G''_2}(X) \geq \ell$ holds for all cuts X.

(3) Without loss of generality, assume that $G'_2 := G_2 - \{e_1, e_2, e_3\} \cup \{(v^*, v_1), (w_1, u_3), (w_2, w_3)\}$ is ℓ-edge-connected. We show $\mathcal{P}_3(G'_2) \supseteq \mathcal{P}_3(G_2)$.

Clearly, $\mathcal{P}_3(G'_2 \cup \{e_1, e_2, e_3\}) \supseteq \mathcal{P}_3(G_2)$. First we show $\mathcal{P}_3(G'_2 \cup \{e_2, e_3\}) \supseteq \mathcal{P}_3(G_2)$. If $G'_2 \cup \{e_2, e_3\}$ has a pair of vertices $x, y \in V$ such that $\kappa_{G'_2 \cup \{e_1, e_2, e_3\}}(x, y) \geq 3$ but $\kappa_{G'_2 \cup \{e_1, e_2\}}(x, y) = 2$, then there is a disconnecting pair that disconnects v^* and w_1 in $G'_2 \cup \{e_2, e_3\}$. Hence it is sufficient to show $\kappa_{G'_2 \cup \{e_2, e_3\}}(v^*, w_1) \geq 3$. Since G'_2 has an edge (v^*, v_1), $G'_2[T_1 \cup S_1]$ has two internally disjoint paths P_1, P_2 between v^* and w_1 by Lemma 3. There also exists a path P_3 between v^* and u_3 in $G'_2 \cup \{e_2, e_3\} - (T_1 \cup \{v_1\})$ since T_2 is an S_1-component. By construction, all of P_1, P_2 and $P_3 \cup (u_3, w_1)$ are paths between v^* and w_1, which are internally disjoint. Therefore we have $\kappa_{G'_2 \cup \{e_2, e_3\}}(v^*, w_1) \geq 3$.

Next we prove $\mathcal{P}_3(G'_2 \cup \{e_3\}) \supseteq \mathcal{P}_3(G_2)$. This can be proved by property $\kappa_{G'_2 \cup \{e_3\}}(v_1, w_2) \geq 3$. But we omit the proof since it is similar to the proof of $\mathcal{P}_3(G'_2 \cup \{e_2, e_3\}) \supseteq \mathcal{P}_3(G_2)$.

Finally, we show $\kappa_{G'_2}(u_3, w_3) \geq 3$ in order to prove $\mathcal{P}_3(G'_2) \supseteq \mathcal{P}_3(G_2)$. By Lemma 3, $G'_2[T_2 \cup S_1]$ has two internally disjoint paths P_1, P_2 between u_3 and w_3. Note that $G'_2[T_1]$ has a path P_3 between w_1 and w_2 since $G'_2[T_1]$ $(= G_2[T_1] - \{e_1, e_2\})$ is still connected. From construction, all of P_1, P_2, and $(u_3, w_1) \cup P_3 \cup (w_2, w_3)$ are paths between u_3 and w_3, and internally disjoint. Therefore $\kappa_{G'_2}(u_3, w_3) \geq 3$ holds.

■

Proof of Property 4: Let $S_i = \{v^*, v_i\}$ denote all the disconnecting pairs in G_3 for $i = 1, \cdots, q$; we choose v^* so that $|E_{G_3}(v^*) \cap F_3| \geq |E_{G_3}(v_1) \cap F_3|$ in the case $q = 1$. The proof is done in two steps (1) and (2).

(1) We first show that any edge in F_3 incident to v_i is also incident to v^*. Without loss of generality, assume that there is an edge $e = (v_1, u) \in F_3$ with $u \neq v^*$. Clearly, u is contained in an S_1-component T. Then G_3 has an edge $(v^*, w) \in F_3$ with $w \neq v_1$, by the definition of v^*. If w is in an S_1-component $T' \neq T$, then edges e and (v^*, w) satisfy condition (d) of Property 2, contradicting the assumption on G_3. Thus all edges $(v^*, w) \in F_3$ incident to v^* satisfy $w \in T$. Then G_3 has another S_1-component $T' \neq T$ that contains $e' \in F_3 \cap E(G_3[T' \cup \{v_1\}])$, since S_1 is a disconnecting pair in G_3 satisfying (10). If e' is not incident to S_1, then these edges $e, (v^*, w)$ and e' satisfy condition (e) of Property 3, contradicting the assumption on G_3. If e' is incident to S_1, then edges e' and (v^*, w) satisfy condition (d) of Property 2, contradicting the assumption on G_3.

(2) We first show that if G_3 has two disconnecting pairs $S_1 = \{v^*, v_1\}$ and $S_2 = \{v^*, v_2\}$, then there is an edge in F_3 incident to v^*. If there is no edge in F_3 incident to v^*, then G_3 has another S_1-component T_1' ($\neq T_1$) and an S_2-component $T_2 \subset T_1$ for some S_1-component T_1. Then (10) says that $T_2 \cup S_2$ (resp., $T_1' \cup S_1$) has an edge $e_2 \in F_3$ (resp., $e_1' \in F_3$). Since neither e_2 nor e_1' is incident to S_1, the edges e_2 and e_1' would satisfy (c), a contradiction.

We finally show that exactly one of (i)–(iii) holds in G_3. For this, we assume that (i) does not hold in G_3. Hence in case of $q = 1$, there is an edge $f_1 \in F_3$ which is s-admissible with respect to S_1. Then every S_1-component T not containing the edge f_1 contains no edge in F_3 in its induced graph $G_3[T]$ (since otherwise condition (a) would hold). This implies (ii). Similarly, we can see that the case of $q \geq 2$ implies (iii). ■

Step III is executed in $O(n^2 m + n^3 \log n)$ time as follows. The ℓ-edge-connectivity in a given graph can be checked in $O(nm + n^2 \log n)$ time [26]. For two vertices $x, y \in V$, testing whether $\kappa_G(x, y) \geq 3$ can be done in $O(n^{1/2} m)$ time using a maximum flow technique [4]. Therefore, for a subset F' where $|F'|$ is a constant, we can test whether a shifting F''/F' is improving in a graph $G' = (V, E')$ or not in $O(nm + n^2 \log n)$ time. As observed after Property 2 in Section 3, we can apply case (b) of Property 2 as long as G_2 has two disjoint minimal tight sets. Since G_2 has at most $|\mathcal{T}(G_2)| < n$ minimal tight sets, we need $O(n)$ switching operations until any two disconnecting pairs S and S' have a common vertex. Hence it takes $O(n(nm + n^2 \log n))$ time to eliminate disjoint disconnecting pairs in G_2.

Note that if there is no disjoint disconnecting pair, then the total number of S-components over all disconnecting pairs is $O(n)$. For each S-component, we can identify all s-admissible edges with respect to S in $O(m)$ time by applying a triconnected component algorithm [10] to $G_2 - F_2$. From this observation, an improving switching which satisfies one of (a)-(e) can be found in $O(nm)$ time. Since the number of S-components decreases after each improving switching, it takes $O(n^2 m + n^3 \log n)$ time to obtain a graph $G_3 = G_2$ for which none of (a)-

(e) holds. Since the total number of switching operations in Step III is $O(n)$, $m(G+F_3) = m(G_3) = O(m+n)$ still holds.

7. Justification of Step IV

Let $G_3 = (V, E \cup F_3)$ be the graph obtained in Step III. Now G_3 is ℓ-edge-connected but has disconnecting pairs. Moreover, any two disconnecting pairs S_1 and S_2 satisfy $S_1 \cap S_2 = \{v^*\}$. Let $S_i = \{v^*, v_i\}$ denote all disconnecting pairs in G_3, $i = 1, \cdots, q$. If G_3 has only one disconnecting pair S_1, then we define that v^* satisfies $|E_{G_3}(v^*) \cap F_3| \geq |E_{G_3}(v_1) \cap F_3|$.

To show that Step IV works correctly, we prove Properties 5 – 8 in Step IV, but start with the following claim.

Claim 4 *Let $G' = (V, E')$ be an ℓ-edge-connected and 2-vertex-connected graph which has a vertex v such that $c_{G'}(v) > \ell$ and $|\Gamma_{G'}(v)| \geq 2$, and $e = (v, w)$ be an edge in $E_{G'}(v)$ such that $G' - e$ remains 2-vertex-connected. Let X be a cut X such that $v \in X$, $w \in V - X$, $c_{G'}(X) = \ell$ and $c_{G'}(X') > \ell$ hold for all cuts $X' \subset X$ with $v \in X'$ if $\lambda_{G'}(w, v) = \ell$; $X = V$ otherwise (i.e., if $\lambda_{G'}(w, v) > \ell$). Then for any vertex $u \in \Gamma_{G'}(v) \cap X$, shifting $\{(w, u)\}/\{e\}$ preserves the ℓ-edge-connectivity of G'. Furthermore, such u can be chosen so that $\{v, u\}$ is not a disconnecting pair in G'.*

Proof: If $\lambda_{G'}(w, v) > \ell$, then $G' - (w, v) + (w, u)$ clearly remains ℓ-edge-connected for any vertex $u \in E_{G'}(v) - w$, where $E_{G'}(v) - w \neq \emptyset$ by $\kappa(G') \geq 2$. Then assume $\lambda_{G'}(w, v) = \ell$, and choose a cut X stated in the claim. Note that $|X| \geq 2$ by $c_{G'}(v) > \ell$. Since $G'[X]$ is connected by Lemma 2, we have $X \cap \Gamma_{G'}(v) \neq \emptyset$. We claim that $G' - (w, v) + (w, u)$ remains ℓ-edge-connected for any $u \in X \cap \Gamma_{G'}(v)$. If not, then G' has a cut Y such that $v \in Y$, $\{w, u\} \subseteq V - Y$ and $c_{G'}(Y) = \ell$. Observe that $Y - X \neq \emptyset$ (by the minimality of X), $X - Y \neq \emptyset$ (by $u \in X - Y$), $X \cap Y \neq \emptyset$ (by $v \in X \cap Y$) and $V - (X \cup Y) \neq \emptyset$ (by $w \in V - (X \cup Y)$) hold. That is, X and Y cross each other, and by (4), we have $c_{G'}(X \cap Y) = \ell$ and $v \in X \cap Y$. This however contradicts the minimality of X.

Next we show that $X \cap \Gamma_{G'}(v)$ contains at least one vertex u such that $\{v, u\}$ is not a disconnecting pair in G' (if such u exists then $u \in X \cap \Gamma_{G'}(v)$ implies that shifting $\{(w, u)\}/\{e\}$ preserves the ℓ-edge-connectivity of G'). This is clear if X contains no vertex z that gives rise to a disconnecting pair $\{v, z\}$ in G'. Thus, assume that $\{v, z\}$ is a disconnecting pair in G' for all vertices $z \in X \cap \Gamma_{G'}(v)$. Clearly, the disconnecting pair $\{v, z\}$ has at least two $\{v, z\}$-components, one of which does not contain w and is denoted by T_z. Note that both $G'[X]$ and $G'[V - X]$ must be connected by Lemma 2. From $\{v, z\} \subseteq X$ and the connectedness of $G'[V - X]$, we see that $T_z \subset X$ must hold. Clearly, T_z contains at least one minimal tight set T in the 2-vertex-connected graph G'. By Lemma 4, T contains no vertex z' such that $\{v, z'\}$ is a disconnecting pair in G'. ■

Proof of Property 5: Assume that $c_{G_3}(v^*) > \ell$ and $E_{G_3}(v^*) \cap F_3 \neq \emptyset$. Choose a disconnecting pair $S_1 = \{v^*, v_1\}$ in G_3. By (10) and Property 4, we see that, for each S_1-component T, $G_3[T + v^*]$ contains at least one edge from F_3. Now we choose an edge $e_1 = (w_1, v^*) \in F_3$ and a vertex $u_1 \in E_{G_3}(v^*) - w_1$, which satisfies Claim 4. The proof is done in two stages (1) and (2).

(1) Assume that there is no S_1-component T such that $G_3[T]$ contains an edge from F_3. Then we can choose two S_1-components T_1 and T_2 so that e_1 (resp., (u_1, v^*)) belongs to $G_3[T_1+v^*]$ (resp., $G_3[T_2+v^*]$). This is because, if $\lambda_{G_3}(v^*, w) = \ell$ holds for some $(v^*, w) \in F_3$ with $w \in T$ for an S_1-component T (otherwise it is trivial), then, for other S_1-component T', the cut X of Claim 4 contains a vertex $u \in \Gamma_{G_3}(v^*) \cap T'$ or separate the end vertices of an edge $(v^*, w') \in F_3$ with $w' \in T'$; in the latter case, we choose (v^*, w') as e_1 again.

We now show that $G_3' = G_3 - (v^*, w_1) + (u_1, w_1)$ satisfies $\mathcal{P}_3(G_3') \supseteq \mathcal{P}_3(G_3)$. Obviously $\mathcal{P}_3(G_3 + (u_1, w_1)) \supseteq \mathcal{P}_3(G_3)$ holds. Hence it is sufficient to show $\kappa_{G_3'}(v^*, w_1) \geq 3$, because this implies that no two vertices $x, y \in V$ with $\kappa_{G_3}(x, y) \geq 3$ are disconnected by any disconnecting pair newly created by the deletion of edge (v^*, w_1) from $G_3 + (u_1, w_1)$. Since $\kappa(G_3 - e_1) \geq 2$, $G_3'[T_1 \cup S_1]$ has two pairwise internally disjoint paths P_1 and P_2 such that P_1 connects v^* and w_1 and P_2 connects v_1 and w_1. We can show that $G_3 - T_1$ has pairwise disjoint path P_3 between v^* and u_1 and path P_4 between v^* and v_1, such that P_3 and P_4 are also disjoint with both P_1 and P_2 (except at vertices v^* and v_1), since otherwise $\{v^*, u_1\}$ or $\{u_1, v_1\}$ is a disconnecting pair in G_3, a contradiction to the choice of u_1 or the assumption on G_3 (recall that from the choice of u_1 in Claim 4, $\{v^*, u_1\}$ is not a disconnecting pair in G_3). The details of this can be found in [16]. Note that the shifting $(u_1, w_1)/(v^*, w_1)$ decreases the number of S_1-components since $(u_1, w_1) \in E_{G_3'}(T_1, T_2)$ holds and T_1 remains to be an S_1-component in $G_3 - e_1$.

(2) Assume that there is an S_1-component T such that $G_3[T]$ contains an edge from F_3 (an example is given in Figure A.3 in the appendix). Then we can choose two edges $e_1 = (v^*, w_1)$ and $e_2 = (u_2, w_2) \in F_3$, and two S_1-components T_1 and T_2 so that e_1 (resp., e_2) belongs to $G_3[T_1 + v^*]$ (resp., $G_3[T_2]$). For such $e_1 = (v^*, w_1)$, we choose the vertex $u_1 \in \Gamma_{G_3}(v^*) - w_1$ stated in Claim 4. If u_1 belongs to an S_1-component T with $T \neq T_1$, then we can apply the above argument in (1) and see that shifting $(u_1, w_1)/(v^*, w_1)$ is improving and decreases the number of S_1-components. Thus, $u_1 \in T_1$ is assumed.

First we consider the ℓ-edge-connected graph $G_3' = G_3 - e_1 + e_1'$ obtained by shifting e_1'/e_1, where $e_1 = (v^*, w_1)$ and $e_1' = (u_1, w_1)$. Now G_3' may have a vertex pair $x, y \in V$ with $\kappa_{G_3'}(x, y) = 2 < \kappa_{G_3}(x, y) \geq 3$; i.e., the shifting $(u_1, w_1)/(v^*, w_1)$ may not be improving in G_3. Note that e_1' is s-admissible with respect to S_1 in G_3 since $G_3[T_1] - e_1$ is connected. Since neither e_1' nor e_2 is incident to S_1, one of the switchings in Property 2 (a) preserves ℓ-edge-connectivity of G_3'. Without loss of generality assume that the graph $G_3'' = G_3 \cup \{(u_1, u_2), (w_1, w_2)\} - \{(v^*, w_1), (u_2, w_2)\}$ obtained by switching $\{(u_1, u_2), (w_1, w_2)\}/\{e_1', e_1\}$ is ℓ-edge-connected.

We now consider whether $\mathcal{P}_3(G_3'') \supseteq \mathcal{P}_3(G_3)$ holds or not. For this, we first prove that (a) $\kappa_{G_3'' \cup (u_2, w_2)}(v^*, w_1) \geq 3$ holds (which implies $\mathcal{P}_3(G_3'' + e_2) \supseteq \mathcal{P}_3(G_3)$), and then consider whether (b) $\kappa_{G_3''}(u_2, w_2) \geq 3$ (hence, $\mathcal{P}_3(G_3'') \supseteq \mathcal{P}_3(G_3)$) holds or not; in either case, one of (ii) and (iii) of Property 5 results. These properties can be proved from $\kappa(G_3 - \{(v^*, w_1), (u_2, w_2)\}) \geq 2$, the choice of u_1 and the assumption on G_3, in a similar way to (1). Here the details are omitted (see [16]).

■

Proof of Property 7: Let T_1 be an S-component in G_4 containing an edge $e_1 = (u_1, w_1) \in F_4$ s-admissible with respect to S, such that no S'-component $T' \subset T_1$ with $S' \neq S$ contains an edge in F_4 s-admissible with respect to S', where S and S' are disconnecting pairs in G_4. Let $S = \{v^*, v_1\}$. Assume that there is an edge $e_2 = (v^*, w_2) \in F_4$ which connects v^* and a vertex w_2 in another S-component T_2 ($\neq T_1$). By Claim 2, we can assume that the switching $\{(u_1, v^*), (w_1, w_2)\}/\{e_1, e_2\}$ preserves the ℓ-edge-connectivity of G_4 without loss of generality; the resulting graph is denoted $G'_4 := (G_4 - \{e_1, e_2\}) + \{(u_1, v^*), (w_1, w_2)\}$. There are the following three possible cases (1)-(3).

(1) $\mathcal{P}_3(G'_4) \supseteq \mathcal{P}_3(G_4)$. In this case, we are done. Hence consider the case $\mathcal{P}_3(G'_4) \not\supseteq \mathcal{P}_3(G_4)$. Since $\mathcal{P}_3(G'_4 + \{e_1, e_2\}) \supseteq \mathcal{P}_3(G_4)$ holds, it must hold either (2) $\mathcal{P}_3(G'_4 + e_1) \not\supseteq \mathcal{P}_3(G_4)$ or (3) $\mathcal{P}_3(G'_4 + e_1) \supseteq \mathcal{P}_3(G_4)$, $\mathcal{P}_3(G'_4) \not\supseteq \mathcal{P}_3(G_4)$. We consider these cases separately.

(2) $\mathcal{P}_3(G'_4 + e_1) \not\supseteq \mathcal{P}_3(G_4)$ implies that $\kappa_{G'_4+e_1}(v^*, w_2) = 2$. Then we claim that for any cut X with $\{u_1, w_2\} \subseteq X$ and $\{v^*, w_1\} \subseteq V - X$, at least one of $G_4[X]$ and $G_4[V - X]$ is not connected; hence, by Claim 1, switching $\{(u_1, w_2), (v^*, w_1)\}/\{e_1, e_2\}$ is improving in G_4 and decreases the number of S-components by one, proving the property. If there is a cut X with $\{u_1, w_2\} \subseteq X$ and $\{v^*, w_1\} \subseteq V - X$ such that both of $G_4[X]$ and $G_4[V - X]$ are connected, then $v_1 \in X$ holds and there is a path between v^* and w_1 in $G'_4[(V - X) \cap (T_1 \cup \{v^*\})]$, since u_1 and w_2 are disconnected by $S = \{v^*, v_1\}$. Hence we have two pairwise disjoint paths P_1 between u_1 and v_1 in $G'_4[X \cap (T_1 \cup \{v_1\})]$ and P_2 between v^* and w_1 in $G'_4[(V - X) \cap (T_1 \cup \{v^*\})]$. Now by $\kappa(G_4 - \{e_1, e_2\}) \geq 2$, $G'_4[T_2 \cup S]$ contains two pairwise disjoint paths P_3 between w_2 and v^* and P_4 between w_2 and v_1. Now $G'_4 + e_1$ has three pairwise disjoint paths $\{(v^*, u_1)\} \cup P_1 \cup P_4$, $P_2 \cup \{(w_1, w_2)\}$ and P_3 between v^* and w_2, contradicting $\kappa_{G'_4+e_1}(v^*, w_2) = 2$.

(3) $\mathcal{P}_3(G'_4 + e_1) \supseteq \mathcal{P}_3(G_4)$, $\mathcal{P}_3(G'_4) \not\supseteq \mathcal{P}_3(G_4)$. Thus G'_4 has a disconnecting pair $S' = \{v', v''\}$ which disconnects u_1 and w_1 (otherwise, $\kappa_{G'_4}(u_1, w_1) \geq 3$ would imply $\mathcal{P}_3(G'_4) \supseteq \mathcal{P}_3(G_4)$). We first show that

$$v' \in T_1 \text{ and } v'' = v^* \text{ hold for every disconnecting pair } S' = \{v', v''\} \text{ which disconnects } u_1 \text{ and } w_1 \text{ in } G'_4. \tag{14}$$

Now $\{v', v''\} \cap T_1 \neq \emptyset$ holds since the s-admissibility of e_1 with respect to S in G_4 implies that both u_1 and w_1 are contained in the same S-component in $G_4 - e_1$. Assume $v' \in T_1$ without loss of generality. It is easy to see that $v'' \in \{v_1, v^*\} \cup T_1 \cup T_2$ holds. If $v'' \in T_1 \cup \{v_1\}$, then S' cannot disconnect v^* and w_2 in G'_4, since $G'_4[T_2] = G_4[T_2] - e_2$ contains a path between v^* and w_2. Hence, $v'' \in T_2 \cup \{v^*\}$ holds.

If $v'' \in T_2$ holds, then we see that any two vertices in $\{v^*, v_1, v', v''\}$ is a disconnecting pair in $G_4 - \{e_1, e_2\}$. Hence, for any cut X with $\{u_1, w_2\} \subseteq X$ and

$\{v^*, w_1\} \subseteq V - X$, at least one of $G_4[X]$ and $G_4[V - X]$ is not connected. By Claim 1, switching $\{(u_1, w_2), (v^*, w_1)\}/\{e_1, e_2\}$ is improving in G_4 and decreases the number of S-components by one, proving the property.

Thus assume that $v'' = v^*$. We denote all disconnecting pairs in G'_4 that disconnect u_1 and w_1 by $S'_i = \{v^*, v'_i\}$, $i = 1, \cdots, p$, where $v'_i \in T_1$ for all i. Now we can observe the followings:

$$\begin{array}{l} p(G'_4 - S'_i) = p(G_4 - S'_i) + 1 \text{ for all } S'_i \\ p(G'_4 - S'') \leq p(G_4 - S'') \text{ for the rest of disconnecting pairs } S'' \text{ of } G_4. \end{array} \quad (15)$$

Note that v'_i is a disconnecting vertex in $G_4[T_1] - e_1$, since each S'_i is also a disconnecting pair in $G_4 - e_1$ and consists of v^* and a vertex in T_1. Let T_{11} (resp., T_{12}) be a subset of T_1 with $u_1 \in T_{11}$ (resp., $w_1 \in T_{12}$) such that T_{11} (resp., T_{12}) is a component of $(G_4[T_1] - e_1) - \{v'_1, \cdots, v'_p\}$. Clearly, $T_{11} \neq T_{12}$ holds, since each S'_i disconnects u_1 and w_1 in $G_4 - e_1$. Note that $c_{G_4}(v^*, T_{11}) > 0$ and $c_{G_4}(\{v_1, v^*\}, T_{12}) > 0$ hold, because otherwise G_4 would have a disconnecting pair S'' with $v^* \notin S''$. Now $|\Gamma_{G_4 - e_1}(T_{11}) \cap T_1| = 1$ holds, since otherwise $|\Gamma_{G_4 - e_1}(T_{11}) \cap T_1| \geq 2$ and $\Gamma_{G_4 - e_1}(T_{11}) \cap T_1 \subseteq \{v'_1, \cdots, v'_p\}$ contradict that each of v'_i disconnects u_1 and w_1 in $G_4[T_1] - e_1$. Let $\Gamma_{G_4 - e_1}(T_{11}) \cap T_1 = \{v'_1\}$ without loss of generality (see Figure 7). We shall show in the following Claims 5 and 6 that if $|E(G_4[T_1]) \cap F_4| \geq \min\{\ell - 1, n - 2\}$ holds, then there is an improving switching of three edges in F_4 that decreases the number of S-components by one in G_4. In Claim 5, we show that if $G_4[T_{11} \cup \{v'_1\}]$ has an edge $e'_1 = (u'_1, w'_1) \in F_4$, then we can obtain G''_4 with $\mathcal{P}_3(G''_4) \supseteq \mathcal{P}_3(G_4)$ and $p(G''_4 - S) < p(G_4 - S)$, where G''_4 denotes the multigraph resulting from the switching of two edges $e'_1, (w_1, w_2)$ or of three edges $e'_1, (u_1, v^*), (w_1, w_2)$ in G'_4. In Claim 6, we show that if $|E(G_4[T_1]) \cap F_4| \geq \min\{\ell - 1, n - 2\}$ holds, then there are at least two edges $e^*_1, e^{**}_1 \in E(G_4[T_1]) \cap F_4$ such that there is an improving switching of e^*_1, e^{**}_1 and e_2 in F_4, according to Claim 5. This will then complete the proof of Property 7. ■

CLAIM 5 *Assume that $G_4[T_{11} \cup \{v'_1\}]$ has an edge $e'_1 = (u'_1, w'_1) \in F_4$. Then in G'_4 $(= G_4 - \{e_1, e_2\} \cup \{(u_1, v^*), (w_1, w_2)\})$, there is an improving switching of two edges $e'_1, (w_1, w_2)$ or of three edges $e'_1, (u_1, v^*), (w_1, w_2)$ in G'_4. Moreover, this resulting graph G''_4 satisfies $\mathcal{P}_3(G''_4) \supseteq \mathcal{P}_3(G_4)$ and $p(G''_4 - S) < p(G_4 - S)$.*

Proof: We first show that edge (w_1, w_2) is s-admissible with respect to all S'_i in G'_4. Note that $\{w_1, w_2\} \cap S'_i = \emptyset$ holds for all S'_i, and $G'_4[T_1 + v_1]$ contains a path P_1 between w_1 and v_1 which passes none of v'_i, since otherwise $\Gamma_{G_4 - e_1}(T_{12}) = \{v^*, v'_p\}$ implies that $\{v^*, v'_p\}$ is a disconnecting pair in G_4 and that the $\{v^*, v'_p\}$-component containing e_1 is contained in T_1, contradicting the minimality of T_1. Also, $G'_4[T_2 + v_1]$ contains a path between w_2 and v_1. Then for each S'_i, G'_4 has an S'_i-components T'_i which contains both w_1 and w_2. Further $G'_4[T'_i] - (w_1, w_2)$ is also connected by path $P_1 \cup P_2$. Therefore edge (w_1, w_2) is s-admissible with respect to each S'_i in G'_4.

If $e'_1 = (u'_1, w'_1)$ is not incident to any v'_i, then by Property 2 (a), at least one of the switchings $\{(w_1, u'_1), (w_2, w'_1)\}/\{(w_1, w_2), e'_1\}$ and $\{(w_2, u'_1), (w_1, w'_1)\}/$

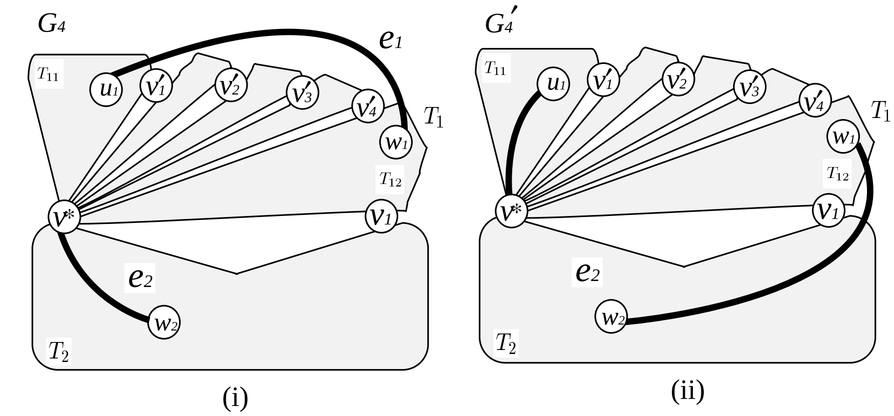

Figure 7. Illustrations of the graph G_4 and G_4' such that $\mathcal{P}_3(G_4) \not\subseteq \mathcal{P}_3(G_4')$ holds, where G_4' denotes the graph obtained from G_4 (which satisfies the conditions in Property 7) by switching $\{(u_1, v^*), (w_1, w_2)\}/\{e_1, e_2\}$. (i) $e_1 = (u_1, w_1)$ is s-admissible with respect to $S = \{v^*, v_1\}$ and e_2 is incident to v^*. (ii) $S_i' = \{v^*, v_i'\}$ denotes all disconnecting pair in G_4', $i = 1, 2, 3, 4$, for which (15) holds.

$\{(w_1, w_2), e_1'\}$ is improving in G_4', and $p(G_4'' - S_i') = p(G_4' - S_i') - 1$ holds for all S_i', since e_1' is not incident to S_i', and two edges e_1' and (w_1, w_2) are contained in different S_i'-components for each S_i'. Hence by (15), $p(G_4'' - S_i') = p(G_4 - S_i')$ holds.

We then consider the case in which e_1' is incident to some v_i'. Then $i = 1$ holds since $\Gamma_{G_4 - e_1}(T_{11}) \cap T_1 = \{v_1'\}$. Assume $e_1' = (v_1', w_1')$ without loss of generality. Then by Property 3, at least one of the switchings $\{(v^*, v_1'), (u_1, w_2), (u_1', w_1)\}/\{(w_1, w_2), e_1', (u_1, v^*)\}$ and $\{(v^*, v_1'), (u_1', w_2), (u_1, w_1)\}/ \{(w_1, w_2), e_1', (u_1, v^*)\}$ is improving in G_4', and $p(G_4'' - S_1') = p(G_4' - S_1') - 1$ holds. For each $S_i' = \{v^*, v_i'\}$, observe that G_4'' can be regarded as a graph obtained from $(G_4' - \{e_1', (u_1, v^*)\}) + \{(u_1, w_1'), (v^*, v_1')\}$ by switching edges (u_1, w_1') and (w_1, w_2). Since (w_1, w_2) is s-admissible with respect to S_i' in G_4', so is it in $G_4' - \{e_1', (u_1, v^*)\} \cup \{(u_1, w_1'), (v^*, v_1')\}$. Since switching two edges which are contained in different S-components and one of which is s-admissible with respect to S decreases the number of S-components by one by Property 2, $p(G_4'' - S_i') = p((G_4' - \{e_1', (u_1, v^*)\} \cup \{(u_1, w_1'), (v^*, v_1')\}) - S_i') - 1$ $= p(G_4' - S_i') - 1$ holds for each S_i'. Hence $p(G_4'' - S_i') = p(G_4 - S_i')$ follows from (15).

Now $p(G_4'' - S_i') = p(G_4 - S_i')$ holds for all i. Moreover, vertices u_1 and w_1 are contained in the same S_1'-component in G_4''. Therefore $\mathcal{P}_3(G_4'') \supseteq \mathcal{P}_3(G_4)$ holds.

Finally, we observe that $T_1 \cup T_2$ is an S-component in G_4'', since $E_{G_4''}(T_1, T_2) \neq \emptyset$ and both of $G_4[T_1 - e_1]$ and $G_4[T_2 - e_2]$ are connected. Therefore $p(G_4'' - S) < p(G_4 - S)$ holds. ■

Claim 6 *If* $G_4[T_1]$ *contains at least* $\min\{\ell-1, n-2\}$ *edges in* F_4*, then we can execute an improving switching of edges in* F_4 *that decreases the number of* S*-components by one.*

Proof: Let $E(G_4[T_1]) \cap F_4 = \{e'_1 = (u'_1, w'_1), \cdots, e'_r = (u'_r, w'_r)\}$ $(r \geq \ell-1)$. From the above arguments, for each e'_i, $i = 1, \ldots, r$, the switching $\{(u'_i, v^*), (w'_i, w_2)\}/\{e'_i, e_2 = (v^*, w_2)\}$ with $w_2 \in T_2$ preserves the ℓ-edge-connectivity without loss of generality, but generates disconnecting pairs $S_{ij} = \{v^*, v_{ij}\}$ with $v_{ij} \in T_1$, $j = 1, \ldots, p_i$, which disconnect u'_i and w'_i in $G_4 - e'_i$. Then for each i, let $T_{i1} \subset T_1$ denote the S_{i1}-component in $G_4 - e'_i$ such that $T_{i1} \cup \{v^*\}$ contains no disconnecting pair S_{ij} (i.e., $T_{i1} \cap \{v_{i1}, \ldots, v_{ip_i}\} = \emptyset$). As observed in the proof of Claim 5, we see that $u'_i \in T_{i1}$ and $c_{G_4}(v^*, T_{i1}) > 0$ hold.

Assume that none of the edges in $F_4 \cap E(G_4[T_1])$ satisfies condition of Claim 5 in G_4 (otherwise, we can switch some edges in F_4, as stated in the claim). This implies that $G_4[T_{i1} \cup \{v_{i1}\}]$ contains no edge in F_4 for each i. Furthermore, we see that an edge in $F_4 \cap E(G_4[T_1])$, at least one of whose end vertices is contained in T_{i1}, is only the edge e'_i, since otherwise $\{v_{i1}, v^*\}$ would not be a disconnecting pair in $G_4 - e'_i$. This implies that any two sets T_{i1} and T_{j1} $(1 \leq i < j \leq r)$ are disjoint, and hence $r \leq \min\{c_G(v^*, T_1), |\Gamma_{G_4}(v^*) \cap T_1|\}$ holds.

Since at least one edge in F_4 is incident to v^*, we have $c_{G_4}(v^*) = \ell$ from Property 5. Note that $c_G(v^*, T_1) \leq \ell - 2$ holds by $(w_2, v^*) \in F_4 \cap E_{G_4}(v^*, T_2)$ and $c_G(v^*, T_2) \geq 1$. Also, $|\Gamma_{G_4}(v^*) \cap T_1| \leq n-3$ holds. Thus, $r \leq \min\{\ell-2, n-3\}$. However, this contradicts the assumption that $E(G_4[T_1]) \cap F_4$ contains at least $\min\{\ell-1, n-2\}$ edges s-admissible with respect to S. Therefore, we can conclude that there is an edge e'_i for which Claim 5 holds, a contradiction. ■

We now show that an improving switching in Property 7 can be found in $O(nm + n^2 \log n)$ time. We first choose a set $F \subset F_4$ of $\min\{\ell-1, n-2\}$ edges s-admissible with respect to a disconnecting pair $S = \{v^*, v_1\}$ in G_4. Then choose an edge $e_2 = (v^*, w_2) \in F_4$ such that $V[F] \subseteq T_1$ and $w_2 \in T_2$ hold for two S-components T_1 and T_2. For each edge $e = (u, w) \in F$, we test whether $G_4 - e + (w, w_2)$ satisfies (14) or not. This can be done in $O(|F|m) = O(nm)$ time by using a linear time triconnected component algorithm [10]. If (14) does not hold for some edge $e \in F$, then one of the cases (1)-(2) of the proof of Property 7 holds, and a desired switching can be found in $O(nm + n^2 \log n)$ time. If (14) holds for all edges in F, then Claim 5 is applicable to at least one of edges in F as observed in the proof of Claim 6.

Since the number of S-components decrease after executing an improving switching in Property 7, the total time complexity for Step IV is $O(n^2 m + n^3 \log n)$. Also, it is easy to see that the number of pairs of vertices which newly became adjacent after Step IV is $O(n)$.

Proof of Property 8: In G_5, the number of edges in F_5 incident to v^* is $\max\{0, \ell - c_G(v^*)\}$ by Property 5. By Property 7, the number of s-admissible edges with respect to a disconnecting pair S is at most $\ell - 2$. These and Property 4 imply Property 8. ■

8. Justification of Step V

Let $G_5 = (V, E \cup F_5)$ be obtained from G_4 by Step IV. Now G_5 is ℓ-edge-connected and has disconnecting pairs $S_i = \{v^*, v_i\}$ $(i = 1, \cdots, q)$, where the number of edges in F_5 incident to v^* is $\max\{0, \ell - c_G(v^*)\}$. Moreover, G_5 satisfies one of (i) and (ii) in Property 8.

We will prove Properties 9 and 10, from which the correctness of Step V follows.

Proof of Property 9: Let $F_5' \subseteq F_5$ be the set of edges incident to S_1, and let $F_5'' = F_5 - F_5'$. By Property 8, the number of edges in F_5 incident to v^* is $\max\{\ell - c_G(v^*), 0\}$. By Property 8 and choice of v^* in the case of $q = 1$, any edge in F_5 incident to v_1 is also incident to v^*. Hence the number of edges in F_5 incident to S_1 is $\max\{\ell - c_G(v^*), \ell - c_G(v_1), 0\}$. Hence it is sufficient to show $p(G - S_1) = p(G_5 - S_1) + |F_5''|$. If $p(G - S_1) < p(G_5 - S_1) + |F_5''|$ holds, then there is at least one edge $e \in F_5''$ such that $p((G_5 - e) - S_1) = p(G_5 - S_1)$. Then this e is s-admissible with respect to S_1 since e is not incident to S. This contradicts Property 8(i) of G_5. ■

Proof of Property 10: Assume that G_5 satisfies (ii) of Property 8. We first show that

$$|\mathcal{T}(G_5)| \leq (\ell + 1)/2. \tag{16}$$

By $\kappa(G) \geq 2$, each minimal tight set $T \in \mathcal{T}(G_5)$ contains a vertex u which is adjacent to v^* by some edge in $E(G)$, implying that $|\mathcal{T}(G_5)| \leq |E_G(v^*)|$. By Property 8, there is at most one minimal tight set $T' \in \mathcal{T}(G_5)$ which contains an edge in F_5. For other minimal tight set $T \in \mathcal{T}(G_5)$, a vertex $v \in T$ is adjacent to v^* by some edge in F_5 (by (10)). Thus, $|\mathcal{T}(G_5)| - 1 \leq \ell - |E_G(v^*)|$. From these inequalities, we have (16).

We now prove the following inequalities.

$$|F_5| + |\mathcal{T}(G_5)| - 1 \leq 2\ell - 3, \tag{17}$$

$$|F_5| \leq 2\ell - 4. \tag{18}$$

Let T_1 be an S-component in G_5 containing an edge in F_5 s-admissible with respect to S, such that no S'-component $T' \subset T_1$ with $S' \neq S$ contains an edge in F_5 s-admissible with respect to S', where S and S' are disconnecting pairs in G_5. Let F_5^* be the set of edges in F_5 which are incident to v^*, and F_5^a and F_5^n be the sets of edges in $F_5 - F_5^*$ which are s-admissible with respect to S, and not s-admissible with respect to S, respectively.

By the proof of Property 8 (Claim 6), $|F_5^a| \leq c_G(v^*, T_1)$ holds. Now $|F_5^n| \leq c_G(v^*, T_1) - 1$ holds, since the removal of an edge in F_5^n increases the number of S-components and each S-component T in $G_5 - F_5^n$ with $T \subset T_1$ satisfies $c_G(v^*, T) \geq 1$ by $\kappa(G) \geq 2$. Since all minimal tight sets $T \in \mathcal{T}(G_5)$ in G_5 except T_1 contain no edge in F_5 and satisfy $|E_{G_5}(v^*, T) \cap F_5| \geq 1$ by (10), we have $|\mathcal{T}(G_5)| \leq |F_5^*| + 1$. Note that $c_G(v^*, T_1) + c_G(v^*, V - T_1) + |F_5^*| = \ell$ holds.

From the above relations, we have $|F_5| = |F_5^a| + |F_5^n| + |F_5^*| \leq 2c_G(v^*, T_1) - 1 + |F_5^*| \leq \ell - 1 + (c_G(v^*, T_1) - c_G(v^*, V - T_1))$. Now from $c_G(v^*) < \ell$, $c_G(v^*, T_1) > 0$ and $c_G(v^*, V - T_1) > 0$, it follows $c_G(v^*, T_1) - c_G(v^*, V - T_1) \leq \ell - 3$. So $|F_5| \leq 2\ell - 4$ holds, proving (18). Similarly, $|F_5| + |\mathcal{T}(G_5)| - 1 = |F_5^a| + |F_5^n| + |F_5^*| + |\mathcal{T}(G_5)| - 1 \leq 2c_G(v^*, T_1) - 1 + 2|F_5^*| \leq 2\ell - 1 - 2c_G(v^*, V - T_1)$ holds. Together with $c_G(v^*, V - T_1) > 0$, this implies $|F_5| + |\mathcal{T}(G_5)| - 1 \leq 2\ell - 3$, proving (17). ■

8.1. Enumerative approach

In this section, we show that if G_4 satisfies (ii) of Property 8, then there is a subset $Z \subseteq V$ with size $|Z| \leq (2\ell + 1)(4\ell - 8)$ such that there is an optimum solution F of EVAP$(\ell, 3)$ satisfying $V(F) \subseteq Z$. We also show that such a Z can be computed in $O(n^3)$ time. Now $opt(G) \leq 2\ell - 3$ holds by Property 10. Therefore, in Step V, we can find an optimum solution F by inspecting all possible choices of end vertices of F from Z. The number of such choices is $\sum_{1 \leq i \leq 2\ell - 3} \{ \binom{|Z|}{2} \}^i = O((32\ell^4)^{(2\ell - 3)})$.

For two vertices $x, y \in V$, testing whether $\kappa_G(x, y) \geq h$ can be done in $O(h^2 n)$ time using a maximum flow technique [4] on a sparsified spanning subgraph of G' with $O(hn)$ edges, where such sparsification takes $O(m + n \log n)$ time [25, 26]. Therefore we can test the $(\ell, 3)$-connectivity in $O(nm + n^2 \log n + n^3) = O(n^3)$ time. Thus, an optimal solution F with $V(F) \subseteq Z$ can be obtained in $O(32^{2\ell - 3} \ell^{4(2\ell - 3) + 1} n^3)$ time, polynomial if ℓ is considered to be a constant.

We first describe how to specify the above set in Z in $O(n^3)$ time. The Z consists of the following three vertex sets $Z_1, Z_2, Z_3 \subseteq V$.

(i) For a given multigraph $G = (V, E)$, we add a new vertex s and compute a subpartition $\mathcal{Z}$ and a subset Z_1 of V in Lemma 9. For an edge set F_1 computed in Step I for the G, $|Z_1| \leq |F_1| \leq 2|F_5| \leq 2(2\ell - 4)$ holds by Lemma 9(i) and Property 10.

(ii) We set $Z_2 = \cup_{X \in \mathcal{Z}} V[E_G(X)]$. Since $c_G(X) < \ell$ for all $X \in \mathcal{Z}$ by Lemma 9(i), it holds $|\Gamma_G(X)| + |\Gamma_G(V - X)| \leq 2\ell - 2$. Thus, $|Z_2| \leq (2\ell - 2)|Z_1| \leq (2\ell - 2)(4\ell - 8)$.

(iii) For an ordered pair (s', t') of vertices, we call a subset $X \subset V$ an (s', t')-*cut* if $s' \in X$ and $t' \notin X$. For each minimal tight set $T_i \in \mathcal{T}(G)$, denote $\Gamma_G(T_i)$ by $\{s_i, t_i\}$, and consider the graph $G[T_i \cup \{s_i, t_i\}]$ induced from G by $T_i \cup \{s_i, t_i\}$. Let S_{s_i} (resp., S_{t_i}) be a minimum (s_i, t_i)-cut (resp., a minimum (t_i, s_i)-cut) in $G[T_i \cup \{s_i, t_i\}]$, where we choose S_{s_i} (resp., S_{t_i}) so that no (s_i, t_i)-cut $X \subset S_{s_i}$ is a minimum (s_i, t_i)-cut (resp., (t_i, s_i)-cut $X \subset S_{t_i}$ is a minimum (t_i, s_i)-cut). For each T_i, we define vertices $z_{s,i}$ and $z_{t,i}$ as follows. Let $z_{s,i}$ (resp., $z_{t,i}$) be an arbitrary vertex in $S_{s_i} - s_i$ (resp., in $S_{t_i} - t_i$) if $|S_{s_i}| \geq 2$ (resp., $|S_{t_i}| \geq 2$), or an arbitrary vertex in T_i otherwise, where possibly $z_{s,i} = z_{t,i}$. Let $Z_3 = \cup_{T_i \in \mathcal{T}(G)} \{z_{s,i}, z_{t,i}\}$. Note that $|Z_3| \leq 2|\mathcal{T}(G)| = 2|F_1| \leq 8\ell - 16$ holds.

By Lemma 9, those $\mathcal{Z}$, Z_1 and Z_2 can be obtained in $O(nm + n^2 \log n) = O(n^3)$ time. To compute Z_3, we need to solve a maximum flow problem for every minimal tight set $T \in \mathcal{T}(G)$. This can be done in $O(n^3)$ time [22] in total. Then we define $Z = Z_1 \cup Z_2 \cup Z_3$. Clearly $|Z| \leq (2\ell+1)(4\ell-8)$ follows from the above arguments. Now we prove the next property, based on Claims 7 – 10, whose proofs are described in the appendix.

PROPERTY 15 *There is an optimum solution F of EVAP$(\ell,3)$ such that $V(F) \subseteq Z$.*

Proof: Let $G^* = (V, E \cup F^*)$, where an edge set F^* is an optimum solution of EVAP$(\ell,3)$. We show that there is an optimal solution F^* such that $V(F^*) \subseteq Z$. For this, we choose an optimal solution F^* that maximizes

$$\Phi(F^*) \triangleq |V(F^*) \cap Z|.$$

Let us assume that F^* contains an edge $e^* = (u,w)$ with $w \notin Z$ (otherwise we are done), and derive a contradiction.

We first show that

$$\lambda(G^* - e^*) < \ell \text{ and } \kappa(G^* - e^*) = 2.$$

If $\lambda(G^* - e^*) \geq \ell$, then the optimality of F^* implies $\kappa(G^* - e^*) = 2$. Thus, there is a disconnecting pair which disconnects u and w in $G^* - e^*$, and there is a minimal tight set T_w of $G^* - e^*$ with $w \in T_w$ and $u \in V - T_w - \Gamma_{G^*-e^*}(T_w)$, where $\Gamma_G(T_w) = \Gamma_{G^*-e^*}(T_w)$ holds by $\kappa(G) = 2$. From the minimality of T_w, no $T' \subset T_w$ is a tight set in $G^* - e^*$. Note that T_w contains a minimal tight set $T_i \in \mathcal{T}(G)$. From construction of Z_3, it holds $T_w \cap Z_3 \neq \emptyset$. Thus, for every vertex $z \in T_w \cap Z_3$, we see that the shifting $\{(u,z)\}/\{e^*\}$ preserves the $(\ell,3)$-connectivity of G^*. The resulting optimal solution $F = F^* \cup \{(u,z)\} - \{e^*\}$ satisfies $\Phi(F) > \Phi(F^*)$, contradicting the maximality of $\Phi(F^*)$. Thus, $\lambda(G^* - e^*) < \ell$ holds.

Now we show that $\kappa(G^* - e^*) = 2$ holds. Assume $\kappa(G^* - e^*) \geq 3$. Now since $\lambda(G^* - e^*) < \ell$ holds, $G^* - e^*$ has a cut Y_w such that $w \in Y_w$, $u \notin Y_w$ and $c_{G^*}(Y_w) = \ell$ (i.e., $c_{G^*-e^*}(Y_w) < \ell$). Choose the Y_w such that $c_{G^*}(Y') > \ell$ holds for all cuts $Y' \subset Y_w$ with $w \in Y'$ and $u \notin Y'$. Note that $c_{G^*-e^*}(Y_w) = \ell - 1$ implies $c_G(Y_w) < \ell$, and hence by Lemma 9(ii) we have

$$Y_w \cap Z_1 \neq \emptyset. \tag{19}$$

Here we observe that the following claim holds.

CLAIM 7 *For any vertex $z \in Y_w \cap Z_1$, the shifting $\{(u,z)\}/\{e^*\}$ preserves the ℓ-edge-connectivity of G^*.* □

Hence, for any vertex $z \in Y_w \cap Z_1$, the set $F = F^* \cup \{(u,z)\} - \{e^*\}$ is also an optimal solution, contradicting the maximality of $\Phi(F^*)$. Therefore, $\lambda(G^* - e^*) < \ell$ and $\kappa(G^* - e^*) = 2$ hold for the edge $e^* = (u,w) \in F^*$.

For the edge $e^* = (u,w) \in F^*$, let T_w and Y_w be the minimal tight set and cut defined in the above argument. Let $\Gamma_{G^*-e^*}(T_w) = \{v_1, v_2\}$. Without loss of generality, we assume that the edge $e^* = (u,w) \in F^*$ is chosen so that $G^*[T_w \cup \{v_1,v_2\}]$ contains no other edge $e \in F^*$ with $V(e) - \{v_1,v_2\} \subseteq V - Z$ (otherwise we can rechoose e^* as such an edge). From this assumption, we see that for any edge $e' = (v,v') \in F^*$ in $G^*[T_w \cup \{v_1,v_2\}]$ with $\{v,v'\} \cap T_w \neq \emptyset$, at least one vertex in $\{v,v'\} - \{v_1,v_2\}$ is in Z (otherwise, if none of vertices in $\{v,v'\} - \{v_1,v_2\}$ is contained in Z, then there would be a vertex w' such that $|T_{w'}| < |T_w|$).

It is easy to see that, if $(Y_w \cap T_w) \cap Z \neq \emptyset$, then for any vertex $z \in (Y_w \cap T_w) \cap Z$, $F = F^* \cup \{(u,z)\} - \{e^*\}$ would be an optimal solution with $\Phi(F) > \Phi(F^*)$. Thus, it holds

$$(Y_w \cap T_w) \cap Z = \emptyset. \tag{20}$$

By Lemma 2, both subgraphs $G^*[Y_w]$ and $G^*[V - Y_w]$ are connected, and $Y_w \cap \{v_1,v_2\} \neq \emptyset \neq (V - Y_w) \cap \{v_1,v_2\}$ holds; $v_1 \in Y_w$ and $v_2 \in V - Y_w$ are assumed without loss of generality.

Now we show that $Z_1 \cap (Y_w - T_w - v_1) \neq \emptyset$ holds. If $Z_1 \cap (Y_w - T_w - v_1) = \emptyset$, then v_1 is the unique vertex in $Z_1 \cap Y_w$ from (19) and (20). Let X_{v_1} be the cut in $\mathcal{Z}$ containing v_1. Since Y_w contains no vertex in Z_1 other than v_1, we see that $X_{v_1} \subseteq Y_w$ holds by Lemma 9(ii),(iii). We present the next Claim 8, which proves $Z_1 \cap (Y_w - T_w - v_1) \neq \emptyset$ since $(T_w \cap Y_w) \cap Z_2 \neq \emptyset$ will contradict (20).

Claim 8 *If there is a cut $X \in \mathcal{Z}$ with $v_1 \in X \subseteq Y_w$, then $(T_w \cap Y_w) \cap Z_2 \neq \emptyset$.* □

Moreover, G^* has an edge $e' = (u_1,w_1) \in F^* - e^*$ with $u_1 \in T_w$, since otherwise the following Claim 9 implies $(T_w \cap Y_w) \cap Z_3 \neq \emptyset$, which will contradict (20). Therefore, we have $w_1 \in T_w \cup \{v_1,v_2\}$.

Claim 9 *If $G^*[T_w \cup \{v_1,v_2\}]$ contains no edge e' in $F^* - e^*$ such that $V[e'] \cap T_w \neq \emptyset$, then $(T_w \cap Y_w) \cap Z_3 \neq \emptyset$.* □

Let $z_1 \in Z_1 \cap (Y_w - T_w - v_1)$. In the following claim, we show that we can obtain another optimal solution $F \neq F^*$ by shifting some edges in F^*.

Claim 10 *One of the shiftings $\{(u,u_1),(z_1,w_1)\}/\{e^*,e'\}$, $\{(u,w_1),(z_1,u_1)\}/\{e^*, e'\}$ and $\{(z', u_1)\}/\{e'\}$ for some $z' \in (Y_w - T_w - \{v_1,v_2\}) \cap Z_1$ preserves the $(\ell,3)$-connectivity of G^*.* □

If a shifting $\{(u,u_1),(z_1,w_1)\}/\{e^*,e'\}$ (resp., $\{(u,w_1),(z_1,u_1)\}/\{e^*,e'\}$) preserves the $(\ell,3)$-connectivity of G^*, then an optimal solution F^* which contains e^* with $V(e^*) - Z \neq \emptyset$ can be modified into an optimal solution $F = F^* \cup \{(u,u_1), (z_1, w_1)\} - \{e^*,e'\}$ (resp., $F = F^* \cup \{(u,w_1),(z_1,u_1)\} - \{e^*,e'\}$) with $\Phi(F) > \Phi(F^*)$, contradicting the maximality of $\Phi(F^*)$.

On the other hand, if a shifting $\{(z', u_1)\}/\{e'\}$ for some $z' \in (Y_w - T_w - \{v_1,v_2\}) \cap Z_1$ preserves the $(\ell,3)$-connectivity of G^*, the optimal solution $F^{**} := F^* \cup \{(z',u_1)\} - \{e'\}$ obtained by Claim 10 satisfies $\Phi(F^{**}) = \Phi(F^*)$. Let $G^{**} = G^* \cup \{(z',u_1)\} - \{e'\}$. In G^{**}, $w \notin Z$ still holds. Let T_w^{**} denote a tight set in $G^{**} - e^*$ with $w \in T_w^{**}$ and $u \in V - T_w^{**} - \Gamma_{G^{**}-e^*}(T_w^{**})$ such that there is no tight set $T' \subset T_w^{**}$ in $G^* - e^*$. Since $\{v_1, v_2\}$ is not a disconnecting pair in $G^{**} - e^*$, we have $T_w \subset T_w^{**}$. Therefore by repeatedly applying Claim 10 to the edge $e^* = (u, w)$, we can keep enlarging the size of the corresponding T_w until one of the first two shiftings in Claim 10 occurs. Therefore, there exits an optimal solution F' with $\Phi(F') > \Phi(F^*)$, contradicting the maximality of $\Phi(F^*)$.

This completes the proof of Property 15. ■

Acknowledgments

This research was partially supported by the Scientific Grant-in-Aid from Ministry of Education, Science, Sports and Culture of Japan, and the subsidy from the Inamori Foundation.

Notes

1. An extended abstract of this paper was presented at 9th Annual ACM-SIAM Symposium on Discrete Algorithms, San Francisco January 1998.

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Appendix

Proof of Claim 7: Assume that the shifting $\{(u,z)\}/\{e^*\}$ violates the ℓ-edge-connectivity of G^* for a vertex $z \in Y_w \cap Z_1$. Then there would be a cut X with $w \in X$, $u, z \notin X$ and $c_{G^*}(X) = \ell$. By the minimality of $|Y_w|$, we have $X \not\subseteq Y_w$. Thus, by $z \in Y_w - X$, X and Y_w cross each other. By (4), $2\ell = c_{G^*}(Y_w) + c_{G^*}(X) \geq c_{G^*}(Y_w - X) + c_{G^*}(X - Y_w) + 2c_{G^*}(u,w) \geq 2\ell + 2$, a contradiction. ■

Proof of Claim 8: There are two possible cases (1) $w \in X$, and (2) $w \notin X$.

(1) $w \in X$. By $\kappa(G) \geq 2$, G has a $\{v_1, v_2\}$-component T' containing w, and $G[T' \cup \{v_2\}]$ contains a path P from w to v_2. In $G^*[T_w \cup \{v_2\}]$, the path P must go through an edge (u_a, u_b) in $E_G(X)$ visiting u_a before u_b. By definition of Z_2, the first end vertex $u_a \in X \cap T_w \subseteq Y_w \cap T_w$ belongs to Z_2.

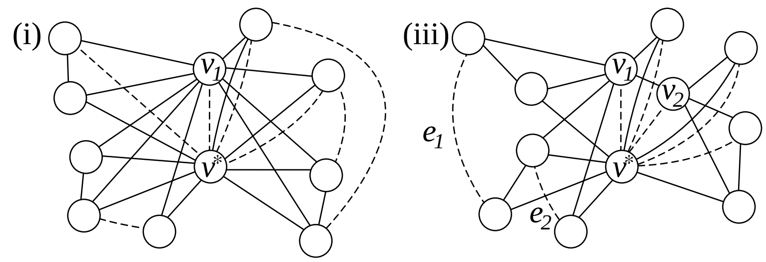

Figure A.1. Illustrations of the graph G_3 satisfying Property 4 (i) and (iii). Edges in F_3 are drawn by broken lines. For each disconnecting pair $\{v^*, v_i\}$, any edge in F_3 incident to v_i is also incident to v^*. (i) The number of disconnecting pairs in G_3 is exactly one. Every edge in F_3 is not s-admissible with respect to the disconnecting pair $\{v^*, v_1\}$ in G_3. (iii) Both of edges e_1 and e_2 in F_3 are s-admissible with respect to the disconnecting pair $S_2 = \{v^*, v_2\}$ in G_3, and are containing in the same S_2-component T. Moreover, any edge in F_3, not contained in the S_2-component T, is incident to v^*.

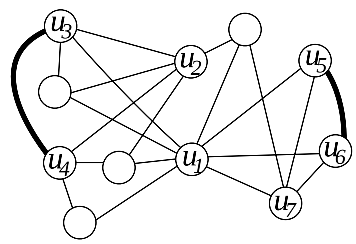

Figure A.2. Illustrations of the edge (u_3, u_4) which is not s-admissible with respect to the disconnecting pair $\{u_1, u_2\}$, and the edge (u_5, u_6) which is s-admissible with respect to $\{u_1, u_2\}$ but is not s-admissible with respect to $\{u_1, u_7\}$.

(2) $w \notin X$. We first show that w and v_1 are connected in the induced graph $G[(Y_w \cap T_w) \cup \{v_1\}]$. Otherwise then there is a partition $\{V_1, V_2\}$ of $(Y_w \cap T_w) \cup \{v_1\}$ such that $v_1 \in V_1$ and $w \in V_2$, and $E_{G^*}(V_1, V_2) = \emptyset$ implies that $c_{G^*}(Y_w - V_2) < c_{G^*}(Y_w) = \ell$ (which contradicts the ℓ-edge-connectivity of G^*). Thus $E_{G^*}(V_1, V_2)$ contains an edge $(v, v') \in F^*$. By the choice of w, at least one vertex in $\{v, v'\}$ must be contained in Z, which contradicts (20). Thus, w and v_1 are connected in $G[(Y_w \cap T_w) \cup \{v_1\}]$, which contains a path P from v_1 to w. By $X \subseteq Y_w - w$, the path P must go through an edge $(u_a, u_b) \in E_G(X)$ visiting u_a before u_b. By definition of Z_2, the second end vertex $u_b \in Y_w \cap T_w$ belongs to Z_2. ■

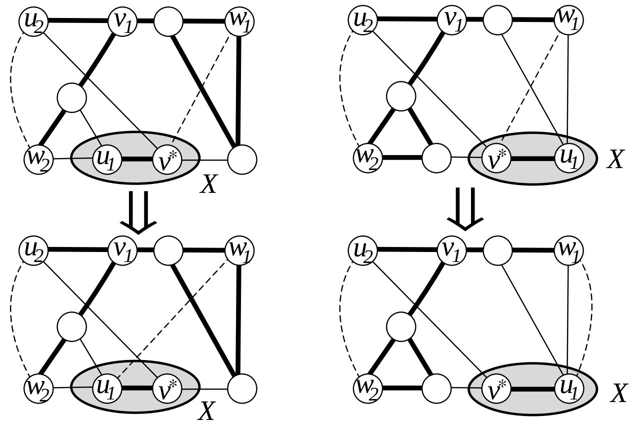

Figure A.3. Illustrations of shifting operations in the proof of Property 5, where X denotes a cut defined in Lemma 4.

Proof of Claim 9: Assume $V[F^* - e^*] \cap T_w = \emptyset$. Hence, $V[F^*] \cap T_w = \{w\}$ and $G^*[T_w \cup \{v_1, v_2\}] = G[T_w \cup \{v_1, v_2\}]$ hold. Let $T_i \in \mathcal{T}(G)$ be a minimal tight set in G with $T_i \subseteq T_w$. Thus $\kappa(G^*) \geq 3$ implies $V[F^*] \cap T_i \neq \emptyset$. From this, we have $V[F^*] \cap T_i = \{w\}$ and $T_i = T_w$ from the minimality of T_w. Let S_1 be a minimum (v_1, v_2)-cut in $G[T_w \cup \{v_1, v_2\}]$ such that $|S_1|$ is minimum among those (v_1, v_2)-cuts. Since S_1 is a minimum (v_1, v_2)-cut in $G[T_w \cup \{v_1, v_2\}]$, we have $c_G(S_1, T_w \cup \{v_1, v_2\} - S_1) \leq c_G(Y_w \cap (T_w \cup \{v_1\}), T_w \cup \{v_2\} - Y_w)$. If $w \in Y_w - S_1$, then we would have $c_{G^*}(S_1 \cup (Y_w - T_w - v_1)) + 1 \leq c_{G^*}(Y_w) = \ell$, contradicting the ℓ-edge-connectivity of G^*. Hence $w \in Y_w \cap S_1$. Thus, $c_{G^*}(S_1 \cup (Y_w - T_w - v_1)) = c_{G^*}(Y_w) = \ell$ by the ℓ-edge-connectivity of G^*. From the minimality of $|S_1|$, we see that $S_1 \subseteq Y_w$. By $w \in S_1$, we have $|S_1| \geq 2$. This, however, implies that Y_w contains a vertex in Z_3 which has been chosen from S_1, a contradiction to (20).

■

Proof of Claim 10: Shifting $\{(u, z_1)\}/\{e^*\}$ preserves the ℓ-edge-connectivity of G^*, since $z_1 \in Y_w$. Let us denote the resulting edge and graph by $e_1 = (u, z_1)$ and $G' = G^* - e^* + e_1$. Since $\{u, z_1\} \subseteq V - T_w - \{v_1, v_2\}$ holds, $\kappa(G^*) \geq 3$ implies that $V - T_w - \{v_1, v_2\}$ induces a connected component in $G^* - e^*$. Thus the edge e_1 is s-admissible with respect to $\{v_1, v_2\}$ in G'. There are three possible cases (1) $w_1 \in T_w$, (2) $w_1 = v_2$ and (3) $w_1 = v_1$.

(1) $w_1 \in T_w$. In this case, switching $\{(u, u_1), (z_1, w_1)\}/\{e_1, e'\}$ or $\{(u, w_1), (z_1, u_1)\}/\{e_1, e'\}$ is improving in G' by Property 2(a). Moreover, no disconnecting pair exists in the graph obtained from this switching, since every disconnecting pair in

$G^* - e^*$ disconnects $\{u\}$ and $\{u_1, w_1\}$. Therefore shifting $\{(u, u_1), (z_1, w_1)\}/\{e^*, e'\}$ or $\{(u, w_1), (z_1, u_1)\}/\{e, e'\}$ preserves the $(\ell, 3)$-connectivity of G^*.

(2) $w_1 = v_2$. We show that switching $\{(u, u_1), (z_1, v_2)\}/\{e_1, e'\}$ is improving in G'. We first claim that $c_{G'}(X) \geq \ell+2$ holds for all cuts X that separate $\{z_1, v_2\}$ and $\{u_1, u\}$. Assume that G' has a cut X' such that $\{z_1, v_2\} \subseteq X'$, $\{u_1, u\} \subseteq V - X'$ and $c_{G'}(X') \leq \ell+1$. By Lemma 2, both subgraphs $G'[X']$ and $G'[V - X']$ are connected, and hence $v_1 \notin X'$ holds (otherwise if $\{u_1, u\} \subseteq V - X'$, then $G'[V - X']$ would not be connected). Now Y_w and X' cross each other by $z_1 \in X' \cap Y_w$, $v_2 \in X' - Y_w$, $v_1 \in Y_w - X'$ and $u \in V - (X' \cup Y_w)$. By (4), $(\ell + 1) + \ell \geq c_{G'}(X') + c_{G'}(Y_w) = c_{G'}(X' - Y_w) + c_{G'}(Y_w - X') + 2c_{G'}(X' \cap Y_w, V - (X' \cup Y_w))$ holds. Now $c_{G'}(X' - Y_w) \geq \ell$ and $c_{G'}(Y_w - X') \geq \ell$ hold by $\lambda(G') \geq \ell$, and $c_{G'}(X' \cap Y_w, V - (X' \cup Y_w)) \geq 1$ holds by $(u, z_1) \in E_{G'}(X' \cap Y_w, V - (X' \cup Y_w))$. This implies $2\ell + 1 \geq \ell + \ell + 2$, a contradiction. Hence $c_{G'}(X) \geq \ell + 2$ holds for all cuts X that separate $\{z_1, v_2\}$ and $\{u_1, u\}$. Therefore a switching $\{(u, u_1), (z_1, v_2)\}/\{e_1, e'\}$ preserves the ℓ-edge-connectivity of G'.

If switching $\{(u, u_1), (z_1, v_2)\}/\{e_1, e'\}$ is not improving in G', then we can see that G' has a vertex set $T' \subset V - T_w - \{v_1, v_2\}$ for which $z_1 \in T'$, $u \notin T'$, and $\Gamma_{G'-e_1}(T') = \{v_2, v_3\}$ with $v_3 \in V - T_w - \{v_1, v_2\}$ hold (see the proof of Property 7). But this contradicts $\kappa(G^*) \geq 3$, since $u \notin T'$ implies $\Gamma_{G^*}(T') = \{v_2, v_3\}$.

Consequently, switching $\{(u, u_1), (z_1, v_2)\}/\{e_1, e'\}$ is improving in G'. Moreover, there is no disconnecting pair in the graph obtained by this switching, since every disconnecting pair in $G^* - e$ disconnects vertices u and u_1. Therefore shifting $\{(u, u_1), (z_1, v_2)\}/\{e^*, e'\}$ preserves the $(\ell, 3)$-connectivity of G^*.

(3) $w_1 = v_1$. (3-a) We first assume that $G'[V - T_w]$ does not have two pairwise internally disjoint paths P_1 and P_2 such that P_1 connects v_1 and z' and P_2 connects v_1 and v_2. In this case, $G'[V - T_w]$ has a cut vertex v' that disconnects $\{v_1\}$ and $\{z', v_2\}$ or satisfies $v' \in \{z', v_2\}$. Let $W \subset V - T_w$ be a cut with $\Gamma_{G'[V-T_w]}(W) = \{v'\}$, $v_1 \in W$ and $v_2 \notin W$. Note that $z_1 \notin W$ by $z_1 \in Z_1 \cap (Y_w - T_w - v_1)$. Now $v' \in Y_w \cap (V - T_w - v_1)$ holds, since $c_{G'}(Y_w) = \ell$ and Lemma 2 imply that $G'[Y_w]$ is connected and hence there is a path from v_1 to z_1 in $G'[Y_w - T_w]$. Moreover, $W \subset Y_w$ holds, since otherwise the connectivity of $G'[V - Y_w - T_w]$ implies that there is a path P' from v_1 to v_2 with $v' \notin P'$ in $G'[V - T_w]$, contradicting the definition of v'. Hence $u \notin W$. From this, $W = \{v_1\}$ holds, since $|W| \geq 2$ implies that $\Gamma_{G^*}(W - v_1) = \Gamma_{G'}(W - v_1) = \{v_1, v'\}$, contradicting $\kappa(G^*) \geq 3$. Note that $\{v_2, v_1\}$ and $\{v_2, v'\}$ are both disconnecting pairs in G', and that neither e' nor e_1 is incident to v_2. Thus, switching $\{(u, u_1), (z_1, v_1)\}/\{e_1, e'\}$ or $\{(u, v_1), (z_1, u_1)\}/\{e_1, e'\}$ is improving in G' by Property 2(c).

Finally, we show that there is no disconnecting pair in the graph obtained by this switching. For switching $\{(u, u_1), (z_1, v_1)\}/\{e_1, e'\}$, this immediately follows since all disconnecting pairs in $G^* - e^*$ disconnect vertices u and u_1. In the case of switching $\{(u, v_1), (z_1, u_1)\}/\{e_1, e'\}$, this also follows since all disconnecting pairs in $G^* - e^*$ except $\{v_1, v_2\}$ disconnect vertices v_1 and u, and the disconnecting pair $\{v_1, v_2\}$ disconnects z_1 and u_1 in $G^* - e^*$. Therefore one of such shiftings preserves the $(\ell, 3)$-connectivity of G^*.

(3-b) $G'[V - T_w]$ has two pairwise internally disjoint paths P_1 and P_2 such that P_1 connects v_1 and z' and P_2 connects v_1 and v_2.

We first show that there is a vertex $z' \in (Y_w - T_w - \{v_1, v_2\}) \cap Z_1$ such that a shifting $\{(z', u_1)\}/\{e'\}$ preserves the ℓ-edge-connectivity of G^*. Assume that there is a cut Y' such that $c_{G^*}(Y') = \ell$, $u_1 \notin Y'$ and $v_1 \in Y'$ (otherwise we are done). Let us choose the one that minimizes $|Y'|$ (hence, all cuts $Y'' \subset Y'$ with $v_1 \in Y''$ satisfies $c_{G^*}(Y'') > \ell$) to enjoy Claim 7.

If Y' contains the vertex $z_1 \in (Y_w - T_w - v_1) \cap Z_1$, then we are done by setting $z' = z_1$. Hence, assume that $z_1 \in Y_w - Y' \neq \emptyset$. By the choice of w, $u_1 \in Z$ holds, and hence $u_1 \notin Y_w \cup Y'$ by (20).

Then we see that Y_w and Y' cannot cross each other, because otherwise $c_{G^*}(Y_w \cap Y', V - (Y_w \cup Y')) = 0$ must hold by the ℓ-edge-connectivity of G^* and (4), a contradiction to $(u_1, v_1) \in E_{G^*}(Y_w \cap Y', V - (Y_w \cup Y'))$. Hence $Y' \subset Y_w$.

By $c_G(Y') < c_{G^*}(Y') = \ell$ and (19), the Y' must contain at least one vertex $z' \in Z_1$. We show that $z' \in (Y' - T_w - v_1) \cap Z_1 \neq \emptyset$. If $(Y' - T_w - v_1) \cap Z_1 = \emptyset$, then $Y' \cap Z_1 = \{z'\} = \{v_1\}$ by (20). By Lemma 9(ii) and (iii), there is a cut $X \subseteq Y'$ with $z' = v_1 \in X \in \mathcal{Z}$. However, in this case, $X \subseteq Y' \subset Y_w$ holds, and by Claim 8, we would have $Y_w \cap Z_2 \neq \emptyset$, contradicting (20). This means that there is a vertex $z' \in Y_w - T_w - v_1 \subseteq Y_w - T_w - \{v_1, v_2\}$ by $v_2 \notin Y_w$. We show that, for such z', shifting $\{(z', u_1)\}/\{e'\}$ preserves the 3-vertex-connectivity of G^*. Let $G' = G^* + (z', u_1) - e'$ for a vertex $z' \in V - T_w - \{v_1, v_2\}$. Clearly, $\kappa(G' + e') = \kappa(G^* + (z', u_1)) \geq 3$ holds. If $\kappa(G') = 2$, then all disconnecting pairs disconnect vertices u_1 and v_1. To prove $\kappa(G') \geq 3$, it is sufficient to show $\kappa_{G'}(u_1, v_1) \geq 3$.

By assumption, $G'[V - T_w]$ has two pairwise internally disjoint paths P_1 and P_2 such that P_1 connects v_1 and z', and P_2 connects v_1 and v_2. Note that $G'[T_w \cup \{v_1, v_2\}]$ has two internally disjoint paths P_3 and P_4, such that P_3 is a path between vertices v_1 and u_1 and P_4 is a path between vertices v_2 and u_1, since T_w is a tight set in $G^* - e^*$. Now we see that G' has three internally disjoint paths $P_1 \cup (z', u_1)$, $P_2 \cup P_4$ and P_3, implying $\kappa_{G'}(u_1, v_1) \geq 3$.

Consequently, we see that shifting $\{(z', u_1)\}/\{e'\}$ preserves the $(\ell, 3)$-connectivity of G^*. ■