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Abstract. There are many strategic situations at which a game theoretical framework should be used to analyze the equilibrium decisions at which the incomplete information annoy the process of deriving the certain rules for making decisions. In these cases, players use signals of each other's to get proper decisions. For example, in economic environment, some macro-economic latent variables induce incomplete information. Morris and Shin (2000) referred this type of game as global game and studied one-shot type of it. However, in practical situations, it is a type of repeated game. In the current paper, following notations of Morris and Shin (2000), repeated games in the presence of incomplete information are studied. Using stochastic approximation technique for M-estimation of latent variable, the learning phenomenon is observed. Financial applications of proposed problem is presented. Finally conclusions are given.
Keywords: M-estimation; Nash equilibriums; Repeated game; Stochastic approximation

1 Introduction. Naturally, game of incomplete information is useful tool for modeling some problems in economics. There, payoff of each player depends on his and his opponent player actions and as well as some unknown economic fundamentals which are considered as a latent parameter where utilities of players relate to this parameter. This phenomenon happens in some economical game such as many accounts of currency attacks, bank runs, and liquidity crises. This type of game is referred as global game. These games, first studied by Carlsson and van Damme (1993), at which uncertain economic fundamentals are summarized by a state θ and each player observes a different signal of the state with a small amount of disturbance. They assumed that the disturbance is common knowledge among the players, each player's signal generates beliefs about fundamentals, beliefs about other players' beliefs about fundamentals, and so on. The game is played one-shot time. Here, this game is studied under repeated game framework.

The repeated games that are played out over and over for a period of time. They are represented using the extensive form. In spite of one-shot games, repeated games makes a new series of decisions: that is the possibility of cooperating means that we may decide to compromise in order to carry on receiving a payoff over time, knowing that if we do not uphold our end of the deal, our opponent may decide not to either.

1.1 Preliminaries. In the current paper, repeated global games are studied. Using the stochastic approximation technique, recursive estimation are constructed and repeated games are defined in the context of global game framework. To this end, consider a two players game where there are two strategies $s_i, i = 1,2$ at which player 1 has utility function u_θ and the second has utility function v_θ. Denote

$$A_{ij} = A_\theta(s_i, s_j), i, j = 1,2, A = u, v.$$

As described before, utilities of both players depend on latent (for example, macro-economic) variable θ. At i-th stage of game, player 1,2 receive signal $x_i, y_i, i \geq 1$ where they are modeled as a error measurement model

$$x_i = \theta + \varepsilon_i,\ y_i = \theta + \zeta_i, i \geq 1$$

where ε_i's and ζ_i's are independent random variables which are mutually independent of each other and they have continuous distribution with density functions f, g. Indeed, signals x_i's and y_i's have location family of distributions. It is assumed that, at each stage each player uses information set $\{A_j, j = 1,2, \dots, i\}$ to make inference about θ. This game may be considered as a repeated global game (see Fudenberg and Tirole, 1991) and would be applied to well-known games such as prisoner's dilemma, battle of the sexes, matching the pennies, chicken game and *etc.*, see Osborne (2003). Consider the mixed strategies $(p, 1-p), (q, 1-q)$ for strategy profile (s_1, s_2) for player 1,2, respectively. It is easy to see that

$$\begin{cases} p = \frac{v_{22}-v_{21}}{v_{11}-v_{21}-v_{21}+v_{22}} = h_1(\theta), \\ q = \frac{u_{22}-u_{12}}{u_{11}-u_{21}-u_{12}+u_{22}} = h_2(\theta). \end{cases} \quad (1)$$

It is assumed that parameters $A_{ij}, i, j = 1,2, A = u, v$ is well-selected to make sure that $p, q \in (0,1)$. Here, some results about the maximum likelihood of location parameter in a location family of distribution are proposed. Suppose that random variable z has location family

$$k_\theta(z) = k(z - \theta), \theta \in (-\infty, \infty).$$

The logarithm of likelihood function of θ based on n observation $z_i, i = 1, \dots, n$ is given by

$$\sum_{i=1}^{n} \rho(z_i - \theta)$$

where $\rho(z) = \log(k(z))$. Suppose that

$$z_i = \theta + \pi_i,$$

for some decentralized random variable π_i. This is a type of M-estimation of θ (see Wilcox, 2012). The M-estimation is useful when π_i's are heavy tailed distributed random variables. Here, it is assumed that ρ is a convex function and twice differentiable. For example, when $\rho(z) = z^2$, then the maximum likelihood of θ is

$$\bar{z}_n = (1 - \lambda_n)\bar{z}_{n-1} + \lambda_n z_n,$$

where $\lambda_n = \frac{1}{n}$. This equation is a type of stochastic approximation recursive (SAR) (see Borkar, 2088) which is useful to study the repeated game and learning process of it. To derive the SAR of M-estimate of θ, i.e., $\hat{\theta}_n$ consider the maximized log-likelihood function

$$l = \sum_{i=1}^{n} \rho(z_i - \hat{\theta}_n)$$

and notice that $e = \sqrt{n}(\hat{\theta}_n - \theta) = O_p(1)$. Then,

$$l = \sum_{i=1}^{n} \rho\left(z_i - \theta - \frac{e}{\sqrt{n}}\right) = \sum_{i=1}^{n} \rho\left(\pi_i - \frac{e}{\sqrt{n}}\right).$$

The last equation is well approximated by second order Taylor expansion as follows

$$l \approx \sum_{i=1}^{n} \rho(\pi_i) - \frac{e}{\sqrt{n}} \sum_{i=1}^{n} \rho'(\pi_i) + \frac{e^2}{2n} \sum_{i=1}^{n} \rho''(\pi_i).$$

Maximizing l with respect to e, it is seen that

$$e = \sqrt{n} \frac{\sum_{i=1}^{n} \rho'(\pi_i)}{\sum_{i=1}^{n} \rho''(\pi_i)} = \sqrt{n}(\hat{\theta}_n - \theta).$$

It is seen that

$$\hat{\theta}_n - \theta = \frac{\sum_{i=1}^{n} \rho'(\pi_i)}{\sum_{i=1}^{n} \rho''(\pi_i)} = (1 - \lambda_n)(\hat{\theta}_{n-1} - \theta) + \lambda_n \frac{\rho'(\pi_n)}{\rho''(\pi_n)},$$

where $\lambda_n = \frac{\rho''(\pi_n)}{\sum_{i=1}^{n} \rho''(\pi_i)}$. Since, ρ is a convex function, $\lambda_n \in [0,1]$ at the above equation is a type of SAR. This equation is used to define a repeated game.

Before going ahead, first, the work of Morris and Shin (2000) is reviewed as follows, in general notations, to set a well-defined framework for defining the repeated game. Following Carlsson and Damme (1993) and Morris and Shin (2001), up to i-th stage of game, suppose that two players are received random signals $x_j, y_j, j = 1,2, \dots, i$, respectively. Let $\boldsymbol{A_i} = (A_1, \dots, A_i), A = x, y$ and let $f_\theta(\boldsymbol{x_i}), g_\theta(\boldsymbol{y_i})$ denote the joint probability densities of $\boldsymbol{x_i}, \boldsymbol{y_i}$, respectively. These densities relate to unknown common parameter θ. There are some priors (possibly improper) $\pi(\theta)$ exist for θ. Then, both players combine their private know with prior to derive Bayesian posteriors $\pi(\theta|\boldsymbol{x_i}), \pi(\theta|\boldsymbol{y_i})$. The main mechanism for combining distributions is Bayesian theorem. Again, players 1,2 combine these knowledge with partner knowledge (for example, player 1 uses $f_\theta(\boldsymbol{x_i})$ and $\pi(\theta|\boldsymbol{y_i})$ to make conditional distribution $f(\boldsymbol{x_i}|\boldsymbol{y_i})$). The following schematic figure shows these steps, visually. For simplicity arguments, the suffix i is dropped.

$$\begin{cases} f_\theta(\boldsymbol{x}) + \pi(\theta) \to \pi(\theta|\boldsymbol{x}) \\ g_\theta(\boldsymbol{y}) + \pi(\theta) \to \pi(\theta|\boldsymbol{y}) \end{cases} \to \begin{cases} f_\theta(\boldsymbol{x}) + \pi(\theta|\boldsymbol{y}) \to \pi(\boldsymbol{x}|\boldsymbol{y}) \\ g_\theta(\boldsymbol{y}) + \pi(\theta|\boldsymbol{x}) \to \pi(\boldsymbol{y}|\boldsymbol{x}) \end{cases}$$

Figure 1. Steps of combining distributions

This paper is organized as follows. In the next section, the repeated game is applied to famous two player's game such as prisoner's dilemma to see the results of repeated games simulated situations. An application is proposed in section 3. Finally, section 4 concludes

2 Repeated game. Here, using the SAR technique defined in above section, the repeated game is defined. Following Morris and Shin (2000) and Carlsson and van Damme (1993), at i −th stage of game, then consider the threshold strategies

$$T_{x_i} = \begin{cases} s_1 & w_i > L_i \\ s_2 & w_i \le L_i \end{cases},$$

where $w_i = w(\boldsymbol{x_i})$ is a suitable function for player 1 to make decision at i-th stage of game. Then, at stage i-th,

$$h_1(\theta) = P(w_i > L_i | \boldsymbol{y_i}), \qquad (2)$$

where the joint density $f(\boldsymbol{x_i}|\boldsymbol{y_i})$ is used to compute the right hand side of equation. In the above equation, to find threshold, L_i, unknown parameter θ is replaced by its M-estimate as described in preliminary section, by replacing $k(z)$ with $f_\theta(\boldsymbol{y_i})$. Similarly, notations and computations for player 2 may be defined. Next, to define the repeated game, it is enough to find $h_j(\theta), j = 1,2$, at stage i-th, using formula (1), then use data $\boldsymbol{x_i}, \boldsymbol{y_i}$ to find these values using formula (2), then equalize these values and replace θ by its M-estimates. In this way, since there is recursive relation for M-estimates, then a learning structure is defined for each player at every stage.

Here, the repeated game is applied to famous games such as prisoner's dilemma, battle of the sexes, matching the pennies. Indeed, the above mathematical results are applied to some simulated examples, to see the results, empirically.

Example 1 (*Prisoner's dilemma*). Consider the following prisoner's dilemma game as follows:

$2 \;\; 2$	$0 \;\; \theta+1$
$\theta+1 \;\; 0$	$\theta \;\; \theta$

The first row and column relate to state "confess" denoted by C and the second row and column relate to "not-confess" denoted by N. Let p, q be the mixed probability of mixture equilibrium strategies of player 1, 2, respectively. It is easy to check that $p = q = \theta$. Suppose that game is repeated until stage n and observations $A_i, i = 1,2, \ldots, n, A = x, y$ are derived. Notice that, under the repeated measurement error model, it is seen that $\bar{x}_n$ has $N(\theta, \frac{\sigma^2}{n})$ and assuming prior $\theta \sim U(-\infty, \infty)$, then θ has posterior distribution $N(\bar{x}_n, \frac{\sigma^2}{n})$. Also, notice that $\bar{y}_n = \theta + \zeta_n$. Thus, $\bar{y}_n$ given $\bar{x}_n$ has normal distribution with mean $\bar{x}_n$ and variance $\frac{\sigma^2+v^2}{n}$. Consider the threshold strategy

$$s_n^x = \begin{cases} C & \bar{x}_n > L_n^x \\ N & \bar{x}_n \le L_n^x \end{cases}$$

It is seen that

$$p = \theta = \Phi\left(\frac{\sqrt{n}(\bar{y}_n - L_n^x)}{\sqrt{\sigma^2+v^2}}\right).$$

By replacing θ with $\bar{y}_n$, one can see that

$$L_n^x = \bar{y}_n - \sqrt{\frac{\sigma^2+v^2}{n}}\,\Phi^{-1}(\bar{y}_n).$$

Similar arguments show that

$$L_n^y = \bar{x}_n - \sqrt{\frac{\sigma^2+v^2}{n}}\,\Phi^{-1}(\bar{x}_n).$$

Since, there is a SA recursive as $\bar{z}_n = (1-\lambda_n)\bar{z}_{n-1} + \lambda_n z_n, z = x, y$, learning process happen in the simultaneous equations.

Suppose that as $n \to \infty$, then $\sqrt{\frac{\sigma^2+v^2}{n}} \to c$, thus, $L_n^y \to \mu_x - c\Phi^{-1}(\mu_x)$ and $L_n^x \to L_x = \mu_y - c\Phi^{-1}(\mu_y)$. When $c = 0$, then $P(\bar{x}_n > L_x) = \Phi(\sqrt{n}\frac{(\mu_y-\mu_x)}{\sigma})$. If $\mu_y - \mu_x > 0$ (< 0), then probability of C (N) state is greater than N (C). When, $\mu_y - \mu_x = 0$, the both players are indifferent with respect to C and N.

Example 2 (*Matching pennies*). Consider the matching pennies game with payoff matrix which depends to unknown parameter θ, as follows

$\theta \;\; -1$	$-1 \;\; \theta$
$-1 \;\; 1$	$1 \;\; -1$

The first row and column relate to state "head" denoted by H and the second row and column relate to "tail" denoted by T. Again, it is seen that $p = q = \frac{2}{\theta+3}$.

Suppose that $\frac{x_i}{\theta}$ has uniform distribution $u(1-\delta, 1+\delta)$ for some known fixed pre-determined $\delta \in (0,1)$. Also, assume that θ has improper uniform distribution on $(0,\infty)$. Then, the posterior distribution of θ given $x_j, j = 1,2,\dots,n$ is $u(a,b)$ where $a = \frac{x_{(n)}}{1+\delta}$ and $b = \frac{x_{(1)}}{1-\delta}$. Here, $x_{(1)}, x_{(n)}$ are the minimum and maximum of $x_j, j = 1,2,\dots,n$, respectively. It is easy to see that

$$x_{(n)} = (1-\delta)\theta + (1+\delta)\theta v_{n,}$$

where $v_{n,}$is the maximum of n independent and identically uniform distributed on (0,1) random variables. Notice that $E\left(x_{(n)}\right) = (1-\delta)\theta + (1+\delta)\theta\frac{n}{n+1} = A\theta$, at which $A = (1-\delta) + \frac{n}{n+1}(1+\delta)$. So, $\frac{x_{(n)}}{A}$ is an unbiased estimate of θ. Also, suppose that $\frac{y_i}{\theta}$ has uniform distribution $u(1-\gamma, 1+\gamma)$ for some known γ. One can see that

$$P\left(y_{(n)} \le z\right) = (\frac{z}{2\gamma\theta} - \frac{1-\gamma}{2\gamma})^n.$$

Next, consider the threshold strategy

$$s_n^y = \begin{cases} H & y_{(n)} > L \\ T & y_{(n)} \le L \end{cases}$$

Indeed, $P_\theta\left(y_{(n)} > L\right) = 1 - (\frac{L}{2\gamma\theta} - \frac{1-\gamma}{2\gamma})^n$. Assuming $P_\theta\left(y_{(n)} > L\right) = 0.95$, then

$$L = \theta\left(1 - \gamma + 2\gamma(0.05)^{\frac{1}{n}}\right).$$

Then,

$$P\left(y_{(n)} > L | x_j, j = 1,2,\dots,n\right) = \frac{\int_a^b P_\theta\left(y_{(n)} > L\right)d\theta}{b-a} = E_\theta\left\{P_\theta\left(y_{(n)} > L\right)\right\}.$$

Here, E_θ is the expectation under posterior distribution θ which is $u(a,b)$ where $a = \frac{x_{(n)}}{1+\delta}$ and $b = \frac{x_{(1)}}{1-\delta}$. The above mentioned expectation can be estimated using the Monte Carlo simulation. Hence, to find L, it is enough to let

$$\frac{\int_a^b P_\theta\left(y_{(n)} > L\right)d\theta}{b-a} = \frac{2}{\hat{\theta}+3}, \hat{\theta} = \frac{x_{(n)}}{A}.$$

In above formula, we have $= \frac{x_{(n)}}{1+\delta}$, $b = \frac{x_{(1)}}{1-\delta}$ and $A = (1-\delta) + \frac{n}{n+1}(1+\delta)$.

Example 3 (Battle of the sexes). An example of this game in a parametric form is given by

$\theta+1 \quad \theta$	$1 \quad 1$
$0 \quad 0$	$\theta \quad \theta+1$

where $\theta > 1$. The mixed strategies of player 1, 2 are $p = 0.5\left(1+\frac{1}{\theta}\right), q = 0.5\left(1-\frac{1}{\theta}\right)$. Suppose that x_i, y_i have exponential distributions with parameters $a\theta, b\theta$, respectively. Also, assume that θ has gamma distribution with parameters α, β. Then, the posterior distribution of θ given $x_j, j = 1,2,\dots,n$ is gamma with parameters $n+\alpha$ and $\beta + na\bar{x}$. Also, $\bar{y}$ given θ has gamma distribution with parameters $n+1, nb\theta$. Next, consider the threshold strategy

$$s_n^y = \begin{cases} B & \bar{y} > L \\ S & \bar{y} \le L \end{cases}$$

Notice that $P_\theta(\bar{y} > L)$ is attainable from gamma distribution and $P\left(\bar{y} > L | x_j, j = 1,2,\dots,n\right) = E_\theta\{P_\theta(\bar{y} > L)\}$ is estimable by the Monte Carlo method.

The following proposition summarizes the above discussion.

Proposition. The equilibrium threshold strategies are given in the mentioned games.

Game	*Prisoner's dilemma*	*Matching pennies*	*Battle of the sexes*
Threshold strategy	$s_n^y = \begin{cases} C & \bar{y}_n > L_n^y \\ N & \bar{y}_n \le L_n^y \end{cases}$	$s_n^y = \begin{cases} H & y_{(n)} > L \\ T & y_{(n)} \le L \end{cases}$	$s_n^y = \begin{cases} B & \bar{y} > L \\ S & \bar{y} \le L \end{cases}$

3 An application. Suppose that two firms produce a same product and it is expected they contribute to determine the price of product and their market share. However, their decision depends on a macro-economic variable θ. Indeed, based on various values of θ and its distribution, their decisions differ and may play prisoner's dilemma, matching the pennies or battle of sexes games. Indeed, when there is no information about the sign of θ, they play the prisoner's dilemma, if both firms believes that θ is positive but non-informative on $(0, \infty)$, then they play the matching the pennies and if they agree that as θ gets large, the probability of its occurrence becomes small, they play battle of sexes game.
The following table gives the threshold determined by three types of games based on selected values for existing hyper-parameters.

Table 1: Thresholds for different games

Game	Hyper-parameter	Threshold
Prisoner's dilemma	$\begin{cases} \sigma^2 = \nu^2 = 0.01n \\ c = 0.1 \\ \mu_x = \mu_y = 1 \end{cases}$	$L_x = L_y = 0.916$
Matching pennies	$\begin{cases} n = 50 \\ \hat{\theta} = 0.2 \\ \gamma = 0.01 \end{cases}$	$L = 0.2017$
Battle of the sexes	$\begin{cases} n = 5 \\ \hat{\theta} = 0.2 \\ b = 0.1 \end{cases}$	$L = 1.0513$

The following plots gives the minus of logarithm of probability of first firm accepting to contribute in pricing the product based on various games, the same plots may be given for the second firm.

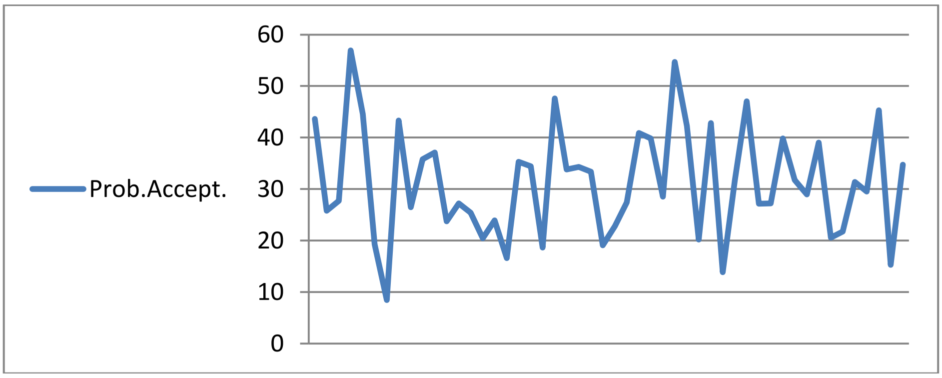

Fig.1: Minus of logarithm of accepting, Prisoner's dilemma

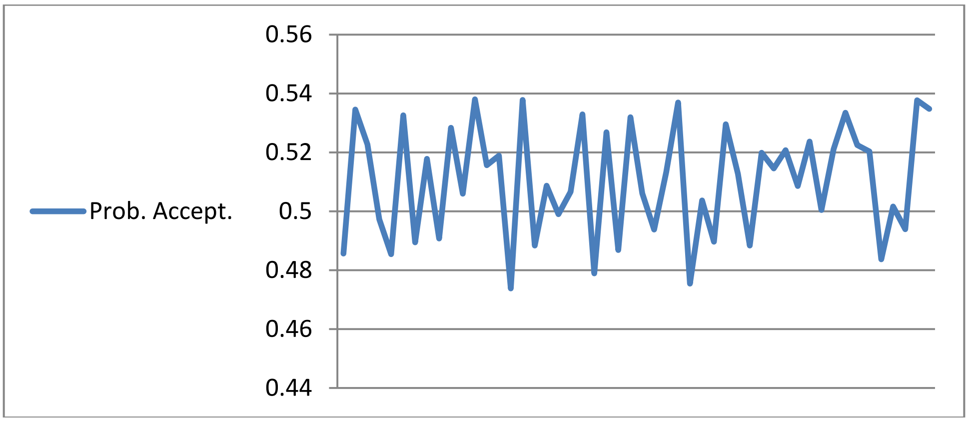

Fig.2: Minus of logarithm of accepting, Matching the pennies

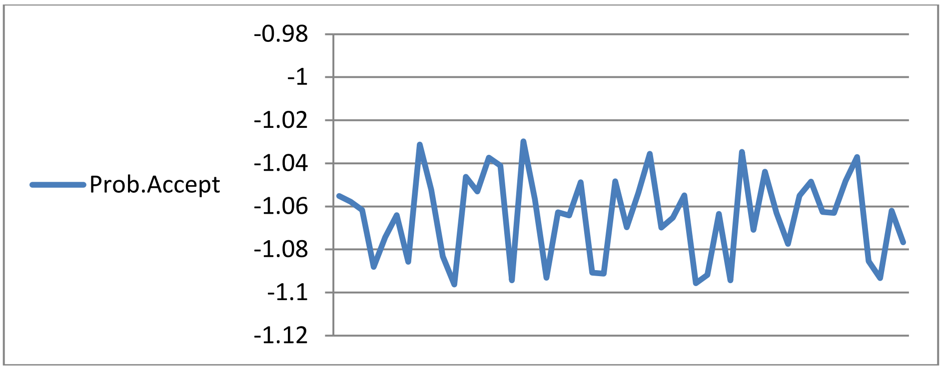

Fig.2: Minus of logarithm of accepting, Battle of sexes

The best strategy that two firm may choose which leads to contribute is the matching the pennies, the second best are the battle of sexes. If they select the prisoner's dilemma, with a high probability, they will never contribute. This action which leads to the deletion of one of them, ultimately.

4 Conclusions. Using a recursive relation for M-estimate of an unknown common parameter exist in utilities of both players, a learning structure is defined in a repeated game in global framework of game theory. This unknown parameter may be a macro-economic fundamental parameter. For updating distributions, Bayesian mechanism is applied. It is seen that the game is well defined and repeated decisions converges to ultimate Nash equilibrium of specified game. This type of equilibrium helps firms in determining the price of same byproduct.

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