ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

Guillaume Cébron and Moritz Weber

Quantum groups based on spatial partitions

Tome XXXII, nº 4 (2023), p. 727–768.

https://doi.org/10.5802/afst.1750

Publication membre du centre
Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse Volume XXXII, n° 4, 2023
pp. 727-768

Quantum groups based on spatial partitions [(*)]

Guillaume Cébron [(1)] and Moritz Weber [(2)]

Abstract. — We define new compact matrix quantum groups whose intertwiner spaces are dual to tensor categories of three-dimensional set partitions (which we call *spatial partitions*). This extends substantially Banica and Speicher's approach of the so called easy quantum groups: It enables us to find new examples of quantum subgroups of Wang's free orthogonal quantum group O_n^+ which do *not* contain the symmetric group S_n; we may define new kinds of products of quantum groups coming from new products of categories of partitions; and we give a quantum group interpretation of certain categories of partitions which do neither contain the pair partition nor the identity partition.

Résumé. — Nous définissons de nouveaux groupes quantiques compacts de matrices dont les espaces d'entrelaceurs sont en dualité avec des catégories tensorielles de partitions d'ensembles tri-dimensionels (que nous appelons *partitions spatiales*). Cela généralise de manière conséquente l'approche de Banica et Speicher dite des groupes quantiques « easy »: cela nous permet d'exhiber de nouveaux exemples de sous-groupes quantiques du groupe quantique orthogonal libre O_n^+ de Wang qui ne contiennent *pas* le groupe symétrique S_n; nous pouvons définir de nouveaux types de produits de groupes quantiques, venant de nouveaux produits de catégories de partitions; et nous donnons une interprétation en terme de groupe quantique de certaines catégories de partitions qui ne contiennent ni la partition paire, ni la partition identité.

(*) Reçu le 26 mars 2021, accepté le 20 juillet 2021.

Keywords: set partitions, three-dimensional partitions, spatial partitions, compact matrix quantum groups, easy quantum groups, partition quantum groups, Banica–Speicher quantum groups, free orthogonal quantum groups, tensor categories, Kronecker product.

2020 *Mathematics Subject Classification:* 20G42, 05A18, 05E10, 46L54.

(1) Institut de Mathématiques de Toulouse; UMR5219; Université de Toulouse; CNRS; UPS, 31062 Toulouse, France — guillaume.cebron@math.univ-toulouse.fr

(2) Saarland University, Fachbereich Mathematik, Postfach 151150, 66041 Saarbrücken, Germany — weber@math.uni-sb.de

This research was funded by the ERC Advanced Grant NCDFP held by Roland Speicher.

Article proposé par Vincent Guedj.

Introduction

Compact matrix quantum groups have been defined by Woronowicz in the 1980's [33]. In the 1990's, Wang [29] gave a definition of a free quantum version O_n^+ of the orthogonal group $O_n \subseteq M_n(\mathbb{C})$. The idea is basically to replace the scalar entries u_{ij} of an orthogonal matrix by noncommuting variables. One can think of the u_{ij} as operators on a Hilbert space, for instance. The quantum group O_n^+ contains the group O_n, hence there are somehow more orthogonal rotations in the quantum world than in the classical world.

In order to understand quantum subgroups of O_n^+, Banica and Speicher [3] developed the theory of easy quantum groups. They are based on set partitions which are decompositions of finite ordered sets into disjoint subsets. In a Tannaka–Krein (or Schur–Weyl) sense, the intertwiner spaces of easy quantum groups are dual to categories of partitions [3, 25, 34]. More precisely, to each partition p we associate a linear map T_p. A category of partitions is a set of partitions which is closed under taking tensor products, composition and involution of partitions. These operations on partitions p correspond exactly to canonical operations on the linear maps T_p turning the linear span of these T_p into a tensor category. A quantum subgroup $G \subseteq O_n^+$ of O_n^+ is called an easy quantum group [3], if its intertwiners are given by such a linear span of maps T_p indexed by partitions p coming from a category of partitions. Hence, easy quantum groups (operator algebraic objects) are in one-to-one correspondence to categories of partitions (combinatorial objects).

The motivation for our article came from the following three questions.

Firstly, any category of partitions is required to contain two particular partitions as a base case: the pair partition $\sqcap$ and the identity partition $|$ (in order to obtain a quantum subgroup of O_n^+).

Question A. — *Can we replace these base partitions by other base partitions and still associate quantum groups to such categories?*

From a combinatorial point of view, there is no problem in studying categories of partitions with different base cases, but the interpretation of such objects on the quantum group side is a priori not clear.

Secondly, given two categories of partitions $\mathcal{C}_1$ and $\mathcal{C}_2$.

Question B. — *Can we form a new category of partitions out of two given ones by some product construction which resembles product constructions on the level of quantum groups?*

Thirdly, the approach to construct quantum subgroups G of O_n^+ via easy quantum groups comes with the restriction that G contains the symmetric group S_n.

Question C. — *How can we extend the machinery of easy quantum groups in order to cover quantum groups $S_n \not\subseteq G \subseteq O_n^+$?*

We can give answers to all three questions at the same time, with our new machinery. On the way, we define new products of general quantum subgroups of O_n^+ and we find many new examples of quantum subgroups of O_n^+.

Easy quantum groups have links to Voiculescu's free probability theory [20, 28], for instance via de Finetti theorems [2, 12]. See also [4, 6, 8, 9, 14, 17, 21, 22, 25, 32] as an incomplete list for recent work on easy quantum groups or particularly on O_n^+. Question C has also been tackled in [23]. Moreover, see [7, 10, 15] for further extensions of the setting of easy quantum groups.

1. Main ideas and main results

The key point of Banica and Speicher's approach is to consider a partition $p \in P(k,l)$ of a set with $k+l$ elements (k "upper" ones and l "lower" ones) and to associate a linear map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ to it, for a fixed $n \in \mathbb{N}$. If the number n can be written as a product $n = n_1 \cdots n_m$ for $n_i \in \mathbb{N}$, we obtain

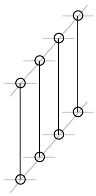
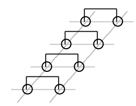
$$T_p : (\mathbb{C}^{n_1 \cdots n_m})^{\otimes k} \to (\mathbb{C}^{n_1 \cdots n_m})^{\otimes l} \quad \text{governed by } p \in P(k,l).$$

Our main tool is derived from the following simple observation. If we consider partitions in $P(km, lm)$ and apply the assignment $p \mapsto T_p$, we obtain a map

$$T_p : (\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_m})^{\otimes k} \to (\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_m})^{\otimes l} \qquad \text{governed by } p \in P(km, lm).$$

Under the isomorphism $\mathbb{C}^n = \mathbb{C}^{n_1 \cdots n_m} \cong \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_m}$, this enables us to find many more maps from $(\mathbb{C}^n)^{\otimes k}$ to $(\mathbb{C}^n)^{\otimes l}$ compared to Banica and Speicher's approach, since we may use partitions on more points (Section 3.4).

On a technical level, it is convenient to view partitions in $P(km, lm)$ as three-dimensional partitions (on $k \times m$ "upper" points and $l \times m$ "lower" points) and to deal with the set $P^{(m)}(k,l)$ of "spatial partitions" (see Section 2.2 for a definition). Then, spatial partition quantum groups are defined as quantum subgroups of $O^+_{n_1\cdots n_m}$ whose intertwiner spaces are given by maps indexed by spatial partitions (Section 3.5). They correspond to categories of spatial partitions: sets of spatial partitions which are closed under tensor product, composition and involution, and which contains the base partitions

$|^{(m)} :=$  $\in P^{(m)}(1,1)$ and $\sqcap^{(m)} :=$  $\in P^{(m)}(0,2)$

(see Section 2.3). Note that we do not require the containment of $| \in P(1,1)$ and $\sqcap \in P(0,2)$, on the contrary to the categories of partitions of Banica and Speicher. This answers Question A (see also Remark 2.9).

In order to prepare an answer to Question B observe that given two categories $\mathcal{C}_i \subseteq P$ for $i = 1,2$, we may form the category $\mathcal{C}_1 \times \mathcal{C}_2 \subseteq P^{(2)}$ by placing partitions from $\mathcal{C}_1$ on the first level and partitions from $\mathcal{C}_2$ on the second one in our three-dimensional picture. On the other hand, given two compact matrix quantum groups (G,u) and (H,v) such that the matrices u and v have the same size, we can form the glued direct product $(G,u)\tilde{\times}(H,v)$ of [25, Def. 6.4], and more generally, the glued direct product $(G,u)\tilde{\times}_p(H,v)$ with amalgamation over a partition $p \in P^{(2)}$ (see Definition 4.6; see also the work in [7]). The latter one is the compact matrix quantum group given by

$$\begin{aligned} &C^*(u_{ij}v_{kl}) \\ &\subseteq C(G) \otimes_{\max} C(H)/\langle u_{ij}v_{kl} \text{ satisfy intertwiner relations associated to } p\rangle \end{aligned}$$

and the matrix $u\tilde{\times}_p v = (u_{ij}v_{kl})$. We then have the following answer to Question B.

Theorem A (Thm. 4.4, Thm. 4.8). — *Let $(G_i,u_i) \subseteq O_n^+$ be easy quantum groups with categories $\mathcal{C}_i \subseteq P$ for $i = 1,2$. Then,*

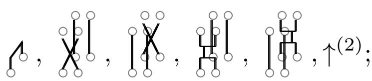
$$\begin{array}{lll} \mathcal{C}_1 \times \mathcal{C}_2 & \textit{corresponds to} & (G_1,u_1)\tilde{\times}(G_2,u_2) \subseteq O^+_{n^2}; \\ \langle \mathcal{C}_1 \times \mathcal{C}_2, p\rangle & \textit{corresponds to} & (G_1,u_1)\tilde{\times}_p(G_2,u_2) \subseteq O^+_{n^2}. \end{array}$$

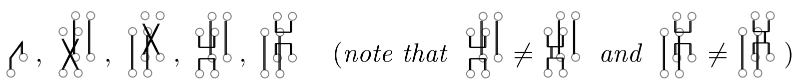
Regarding Question C, we have the following result.

Theorem B (Thm. 5.3). — *For $n_1 = \ldots = n_m = n$ the maximal category $P^{(m)}$ of all spatial partitions corresponds to $S_n \subseteq O^+_{n^m}$. We thus have $S_n \subseteq G \subseteq O^+_{n^m}$ for all spatial partition quantum groups; in particular $S_{n^m} \not\subseteq G \subseteq O^+_{n^m}$ is possible.*

Hence, while for instance easy quantum groups $G \subseteq O^+_{1024}$ come with the restriction $S_{1024} \subseteq G$, our approach only requires $\mathbb{Z}/2\mathbb{Z} \subseteq G$. The next two theorems of combinatorial type show that the step from $m = 1$ to $m = 2$ is huge.

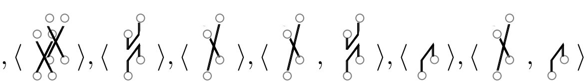
Theorem C (Thm. 2.18, Cor. 2.19, Thm. 2.20). — *The category $P^{(2)}$ (resp. $P_2^{(2)}$) of all spatial (resp. spatial pair) partitions is generated by the partitions $|^{(2)}$, $\sqcap^{(2)}$ and*

$P^{(2)}$: 

$P_2^{(2)}$: 

Recall that for $m = 1$, we have $P^{(1)} = \langle \times, \sqcup\sqcap, \uparrow \rangle$ and $P_2^{(1)} = \langle \times \rangle$ (see [32]).

Theorem D (Thm. 5.8). — *The following subcategories of $P_2^{(2)}$ are all distinct:*

$\langle \emptyset \rangle, \langle \ast^{(2)} \rangle,$  $, P_2^{(2)}, \mathcal{C}_1 \times \mathcal{C}_2$

with $\mathcal{C}_i \in \{NC_2, \langle \ast \rangle, P_2\}$ (non-exhaustive list).

Recall that in the case $m = 1$, we have exactly three subcategories of $P_2^{(1)}$, namely $NC_2 = \langle \emptyset \rangle, \langle \ast \rangle$ and $P_2^{(1)} = \langle \times \rangle$ (see [32]).

We provide also two outlooks in the unitary case. We define a free product $\mathcal{C}_1 * \mathcal{C}_2$ of two categories due to a certain noncrossing condition[(1)] between the levels (Section 4.4). Considering particular finite quantum spaces (B, ψ) in the sense of Wang [31], namely $B = \bigoplus_{l=1}^n M_N(\mathbb{C})$ and $\psi(x_1 \oplus \cdots \oplus x_n) = \frac{1}{nN}\sum_{l=1}^n \mathrm{Tr}_N(x_l)$, where Tr_N is the unnormalized trace on $M_N(\mathbb{C})$, we observe that the relations of the quantum automorphism group of (B, ψ) may be expressed in terms of spatial partitions (see Section 5.5).

[(1)] Note that it is not so clear a priori how to define noncrossing three-dimensional partitions.

2. The combinatorics: spatial partitions and categories

Let us first introduce the combinatorics of our objects.

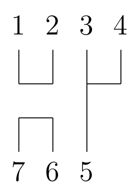
2.1. Partitions

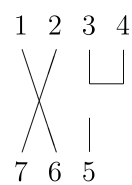
Let $k, l \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and consider the ordered set $\{1, \ldots, k, k+1, \ldots, k+l\}$. A *(set) partition* is a decomposition of this set into disjoint subsets, the *blocks*. We usually speak of the points $1, \ldots, k$ as "upper points" while $k+1, \ldots, k+l$ are "lower points". We identify a partition with the picture placing k points on an upper line, l points on a lower line and connecting these points by strings according to the block pattern (where the upper points are numbered from left to right whereas the lower points are numbered from right to left). The set of all partitions with k upper and l lower points is denoted by $P(k, l)$ and we put $P := \bigcup_{k,l \in \mathbb{N}_0} P(k, l)$.

Example 2.1. — Let $k = 4$ and $l = 3$. The partitions

$$p = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7\}\} \qquad \text{and} \qquad q = \{\{1, 6\}, \{2, 7\}, \{3, 4\}, \{5\}\}$$

in $P(4, 3)$ are represented by the following pictures.

$p =$ 

$q =$ 

We usually omit to write the numbers in the picture. If the strings of a partition may be drawn in such a way that they do not cross, we call it a *noncrossing partition*, denoting by $NC \subseteq P$ the subset of all noncrossing partitions. Note that in Example 2.1, the partition p is in NC while q is not.

Example 2.2. — Here are some examples of partitions in P.

(a) The *identity partition* $| \in P(1, 1)$.
(b) The *pair partitions* $\sqcap \in P(0, 2)$ and $\sqcup \in P(2, 0)$.
(c) The *singleton partitions* $\uparrow \in P(0, 1)$ and $\downarrow \in P(1, 0)$.

Partitions are well-known objects in mathematics, see for instance [3, 20, 24, 26].

2.2. Spatial partitions

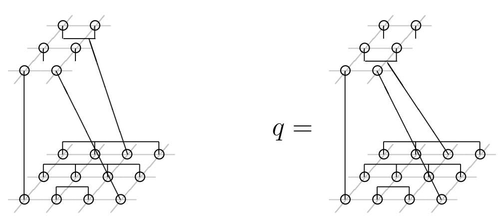
Let us now introduce the new notion of spatial partitions. Let $m \in \mathbb{N}$ and $k, l \in \mathbb{N}_0$. Consider the set

$$\{1, \dots, k, k+1, \dots, k+l\} \times \{1, \dots, m\}.$$

A *spatial partition (on m levels)* is a decomposition of this set into disjoint subsets (*blocks*). We sometimes also simply write *partition*, when it is clear that we speak of spatial partitions. The set of all such spatial partitions is denoted by $P^{(m)}(k,l)$ and we put $P^{(m)} := \bigcup_{k,l\in\mathbb{N}_0} P^{(m)}(k,l)$. Again, the points $(1,y), \dots, (k,y)$ are seen as upper points and the points $(k+1,y), \dots, (k+l,y)$ as lower ones, for $y \in \{1, \dots, m\}$. Furthermore, if $(x,y) \in \{1, \dots, k, k+1, \dots, k+l\} \times \{1, \dots, m\}$ is a point of a partition $p \in P^{(m)}(k,l)$, we call its second component y the *level* of the point. We say that a partition $p \in P^{(m)}$ *respects the levels*, if whenever two points (x_1, y_1) and (x_2, y_2) are in the same block of p, then $y_1 = y_2$.

We view a spatial partition as a three-dimensional partition having an upper plane consisting of $k \times m$ points and a lower plane of $l \times m$ points. Thus, the m levels are nothing but a new dimension in our pictorial representation.

Example 2.3. — Let $m = 3$, $k = 2$ and $l = 4$. The following partitions are in $P^{(3)}(2,4)$ and p respects the levels while q does not.

$p =$  $q =$

We observe that there is no canonical definition of noncrossing partitions in three dimensions.

Remark 2.4. — For any $m \in \mathbb{N}$, $k, l \in \mathbb{N}_0$, the sets

$$A := \{1, \dots, km, km+1, \dots, km+lm\}$$

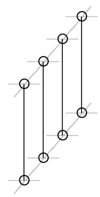
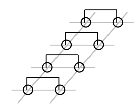
and

$$B := \{1, \dots, k, k+1, \dots, k+l\} \times \{1, \dots, m\}$$

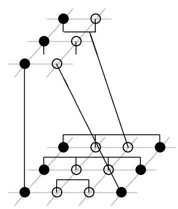
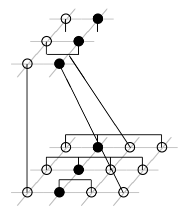
are in bijective correspondence by identifying a point $(x-1)m + y \in A$, $1 \leqslant x \leqslant k+l$, $1 \leqslant y \leqslant m$ with the point $(x,y) \in B$. Thus, the sets $P^{(m)}(k,l)$ and $P(km, lm)$ are isomorphic. In particular, for $m = 1$, spatial partitions (on one level) are simply the well-known partitions in the sense of Section 2.1.

Definition 2.5. — *If* $p \in P(k,l)$ *is a partition, then* $p^{(m)} \in P^{(m)}(k,l)$ *given by repeating* p *on each level* $1 \leqslant s \leqslant m$ *is the* amplified version of p (on m levels)*. It respects the levels.*

Example 2.6. — The amplified partitions $|^{(4)}$ and $\sqcap^{(4)}$ are the following partitions.

$|^{(4)} =$  $\sqcap^{(4)} =$ 

Remark 2.7. — In the spirit of [7, 25, 26], we may also add colors to the points of our partitions. However, for later purposes, we will require the following rule: firstly, we color each point of the *first* level of a spatial partition $p \in P^{(m)}(k,l)$ either in white ($\circ$) or in black ($\bullet$); secondly, we then copy this color pattern to all other levels. In other words, we do not color all points arbitrarily, the colors of all points $(x,y) \in \{1,\dots,k+l\} \times \{1,\dots,m\}$ for a fixed x coincide.

$p =$  $q =$ 

However, we will mostly work with uncolored partitions in this article.

2.3. Categories of spatial partitions

For a fixed $m \in \mathbb{N}$, we have the following operations on the set $P^{(m)}$, the so called *category operations.*

- The *tensor product* of two spatial partitions $p \in P^{(m)}(k,l)$ and $q \in P^{(m)}(k',l')$ is the spatial partition $p \otimes q \in P^{(m)}(k+k',l+l')$ obtained by writing p and q side by side.
- The *composition* of two spatial partitions $q \in P^{(m)}(k,r)$ and $p \in P^{(m)}(r,l)$ is the spatial partition $pq \in P^{(m)}(k,l)$ obtained by writing p below q, joining their strings by identifying the lower resp. upper $r \times m$-planes of points, and erasing the strings which are disconnected from the upper $k \times m$-plane and the lower $l \times m$-plane.

- The *involution* of a spatial partition $p \in P^{(m)}(k,l)$ is given by the spatial partition $p^* \in P^{(m)}(l,k)$ obtained when swapping the upper with the lower plane. In particular, the involution respects the levels.

Definition 2.8. — *A subset $\mathcal{C} \subseteq P^{(m)}$ is a* category of spatial partitions*, if $\mathcal{C}$ is closed under tensor product, composition and involution, and if it contains the amplified identity partition $|^{(m)} \in P^{(m)}(1,1)$ and the amplified pair partition $\sqcap^{(m)} \in P^{(m)}(0,2)$.*

We write $\mathcal{C} = \langle p_1, \ldots, p_n \rangle$, if $\mathcal{C}$ is the smallest category containing $p_1, \ldots, p_n \in P^{(m)}$. We then speak of the category *generated by* $p_1, \ldots, p_n$. We omit to write $\sqcap^{(m)}$ and $|^{(m)}$ as generators since they are always contained in a category.

Remark 2.9. — In the case $m = 1$, the above category operations as well as categories of partitions were first introduced by Banica and Speicher [3]; see also [26, 28] for concrete examples of these operations in that case. We now extend their definition to the three-dimensional setting in a canonical way, but let us note another aspect of the passage from $m = 1$ to arbitrary $m \in \mathbb{N}$. Observe that the isomorphism $P^{(m)}(k,l) \cong P(km, lm)$ of Remark 2.4 respects the category operations. Hence, if we view $P^{(m)}$ as a subset of P, a category $\mathcal{C} \subseteq P^{(m)}$ of spatial partitions corresponds to a set $\mathcal{C}' \subseteq P$ which is closed under the category operations (as operations in P). However, $\mathcal{C}'$ is *not* a category of partitions in Banica–Speicher's sense, since it does not contain the base partitions $\sqcap$ nor $|$. From this point of view, we somehow modified Banica and Speicher's definition of categories of partitions $\mathcal{C} \subseteq P$ by simply replacing the base partitions $\sqcap \in P$ and $| \in P$ by different ones, namely by

$$\sqcap^{(m)} \in P^{(m)}(0,2) \quad \longleftrightarrow \quad \{\{1, m+1\}, \{2, m+2\}, \ldots, \{m, 2m\}\} \in P(0, 2m)$$

and

$$|^{(m)} \in P^{(m)}(1,1) \quad \longleftrightarrow \quad |^{\otimes m} \in P(m,m)$$

using the isomorphism $P^{(m)}(k,l) \cong P(km, lm)$ of Remark 2.4. From the combinatorial point of view, there is no difficulty in choosing different base partitions for Banica and Speicher's categories of partitions, but so far a quantum group interpretation of such categories was missing. In this article, we provide one for the case of $\sqcap^{(m)}$ and $|^{(m)}$.

Remark 2.10. — In view of Remark 2.7, we define a category of colored spatial partitions as in Definition 2.8 replacing $\sqcap^{(m)}$ and $|^{(m)}$ by $\sqcap_{\circ\bullet}^{(m)}$, $\sqcap_{\bullet\circ}^{(m)}$, $\overset{\circ}{\underset{\circ}{|}}{}^{(m)}$ and $\overset{\bullet}{\underset{\bullet}{|}}{}^{(m)}$. We will leave the colored case as a side remark, for the moment.

Definition 2.11. — *Let $\mathcal{C} \subseteq P$ be a set of partitions. Using the notation of Definition 2.5, we denote by*

$$[\mathcal{C}]^{(m)} := \{p^{(m)} \mid p \in \mathcal{C}\} \subseteq P^{(m)}$$

the amplification *of $\mathcal{C}$.*

Lemma 2.12. — *If $\mathcal{C} \subseteq P$ is a category of partitions, then the amplification $[\mathcal{C}]^{(m)} \subseteq P^{(m)}$ is a category of spatial partitions.*

Proof. — A direct proof is straightforward. Alternatively, one can use the fact that the isomorphism of Remark 2.4 respects the category operations. □

Example 2.13. — Here are examples of categories of spatial partitions. We will see more exotic ones in Section 5.

(a) The set $P^{(m)}$ of all spatial partitions is a category of spatial partitions. It is maximal in the sense that it contains all other categories of spatial partitions. We have $[P]^{(m)} \neq P^{(m)}$ for $m \neq 1$, since a spatial partition in the amplification $[P]^{(m)}$ of P consists of m copies of a partition from P to all levels; the set $P^{(m)}$ in turn is much larger containing *any* spatial partition.

(b) The set $P_2^{(m)}$ of all spatial pair partitions (i.e. all blocks consist of exactly two points) is a category of spatial partitions. Again, we have $[P_2]^{(m)} \neq P_2^{(m)}$ for $m \neq 1$.

(c) The amplification $[NC_2]^{(m)}$ of NC_2 is the minimal category of spatial partitions. It is generated by $|^{(m)}$ and $\sqcap^{(m)}$. Note that $[NC_2]^{(m)}$ is *not* the set of all noncrossing pair partitions in $P^{(m)}$. In fact, it is not clear in the three-dimensional picture what a noncrossing partition is supposed to be, only the identification $P^{(m)}(k,l) \cong P(km,lm)$ allows for a notion of noncrossing partitions. However, the set $\bigcup_{k,l\in\mathbb{N}_0} NC(km,lm)$ seen as a subset of $P^{(m)}$ is not a category of spatial partitions. It is closed under the category operations, but it does not contain $\sqcap^{(m)}$ (see Remark 2.9).

Let us mention another useful operation on the set $P^{(m)}$. We define the m-*rotation* by the following. Let $p \in P^{(m)}(k,l)$ and consider the upper plane of points of p consisting of k rows, each row consisting of m points. Let $q \in P^{(m)}(k-1,l+1)$ be the spatial partition obtained from p by shifting the leftmost upper row of p to the left of the lower plane without changing the order of the points nor the strings attached to these points. We say that q is a rotated version of p. Likewise we may rotate on the right hand side and we may rotate lower points to the upper plane. As an example, observe that $\sqcap^{(m)}$ is obtained from $|^{(m)}$ by m-rotation. We refer to [26, §1.2] for examples of 1-rotation. Note that the m-rotation does not affect the level of a point when being rotated.

LEMMA 2.14. — *Every category of spatial partitions is closed under m-rotation.*

Proof. — Let $p \in P^{(m)}(k,l)$. The spatial partition $q := (|^{(m)} \otimes p)(\sqcap^{(m)} \otimes |^{((k-1)m)})$ arises from p by m-rotating the leftmost upper row of p to the lower plane. Similarly for the rotations from the lower plane to the upper plane, and for rotations on the right hand side. See also [26, Lem. 1.1] □

2.4. π-graded spatial partitions and categories

Later, we will need spatial partitions that are allowed to mix certain levels (but not all levels). This is captured by the following definition. The idea is to cluster into a block each set of levels that are allowed to be interchanged and to decompose the set $\{1,\dots,m\}$ accordingly. This is encoded in a partition $\pi \in P(m)$, the grading partition.

DEFINITION 2.15. — *Let $m \in \mathbb{N}$ and let $\pi \in P(m)$ be a partition of m points. A spatial partition $p \in P^{(m)}$ is* π-graded, *if whenever two points (x_1, y_1) and (x_2, y_2) are in the same block of p, then y_1 and y_2 are in the same block of π. The partition π is called the* grading partition. *We denote by $P_\pi^{(m)}$ the set of all π-graded (spatial) partitions in $P^{(m)}$.*

If π consists only of singletons, the π-graded partitions are exactly those that respect the levels. If π is the one block partition, then every partition in $P^{(m)}$ is π-graded. As a nontrivial example, let $m = 4$ and $\pi = \{\{1,3\},\{2,4\}\}$. Then a partition $p \in P^{(m)}$ is π-graded if and only if no block of p contains points from an odd level *and* from an even level.

DEFINITION 2.16. — *A category of spatial partitions $\mathcal{C}$ is* π-graded, *if all partitions in $\mathcal{C}$ are π-graded.*

LEMMA 2.17. — *Let $m \in \mathbb{N}$.*

(a) *Let $\pi \in P(m)$ be a grading partition. If p and q in $P^{(m)}$ are π-graded, then so are $p \otimes q, pq, p^*$ or any m-rotation of p or q.*
(b) *The set $P_\pi^{(m)}$ of all π-graded partitions in $P^{(m)}$ is a category of spatial partitions.*
(c) *If $p_1,\dots,p_k$ are π-graded, so is the category $\langle p_1,\dots,p_k\rangle$ generated by them.*

Proof. — The proof of (a) is straightforward, and (b) and (c) follow immediately. □

2.5. Generators of $P_\pi^{(m)}$, $P^{(2)}$ and $P_2^{(2)}$

In the case $m = 1$, it is not difficult to see that P is generated by , $\uparrow$ and . This allows us to define natural further categories like $\langle \rangle$ or $\langle , \uparrow\rangle$, see for instance [32]. We are thus interested in finding canonical generators of the category $P^{(m)}$, the maximal category of spatial partitions. We refine the statement by considering π-graded partitions, including the case $P^{(m)}$ when π is the one block partition on m points.

Theorem 2.18. — *Let $\pi \in P(m)$ be a grading partition. The category $P_\pi^{(m)}$ of all π-graded partitions is generated by the following partitions besides the base partitions $|^{(m)}$ and $\sqcap^{(m)}$:*

(i) *The singleton partition $\uparrow^{(m)}$.*

(ii) *For $i = 1, \dots, m$, the partition given by on level i and $| \otimes |$ on all other levels. For $m = 2$ this amounts to $\in P^{(2)}(2,2)$ and $\in P^{(2)}(2,2)$.*

(iii) *For $i = 1, \dots, m$, the partition given by on level i and $| \otimes |$ on all other levels. For $m = 2$ this amounts to $\in P^{(2)}(2,2)$ and $\in P^{(2)}(2,2)$.*

(iv) *For $1 \leqslant i < j \leqslant m$ two points in the same block of π, the partition given by on the levels i and j and $|$ on the others. For $m = 2$ and $\pi = \sqcap$ this amounts to $\in P^{(2)}(1,1)$.*

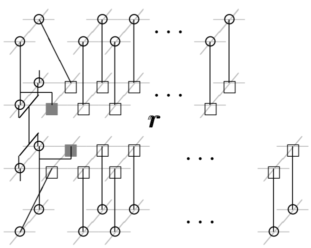
Proof. — We give a proof for $m = 2$ and $\pi = \sqcap$, the general case being a straightforward adaption.

Let $\mathcal{C} \subseteq P^{(2)}$ be the category generated by (i) to (iv). Let p_1 and q_2 be partitions in $P(k,l)$. Using , , , the base partitions, and the category operations, we may construct a partition $p \in \mathcal{C}$ respecting the levels, such that on level one, we have p_1 (since $P = \langle , \uparrow, \rangle$). Likewise, we produce a partition $q \in \mathcal{C}$ respecting the levels, such that on level two, we have q_2. Using (iii), we may permute the points of $p \otimes q \in \mathcal{C}$ in order to obtain a partition $r \otimes s \in \mathcal{C}$ respecting the levels with $r, s \in P^{(2)}(k,l)$ such that r restricts to p_1 on level one and to q_2 on level two. Composing this partition with $\uparrow^{(2)}$ and its adjoint, we infer $r \in \mathcal{C}$.

Use  and (iii) to connect arbitrary upper points of p_1 with arbitrary upper points of q_2, and likewise for connecting lower points with lower points. As for building a string between an upper point of p_1 and a lower point of q_2, assume that both are leftmost within their level (possibly using (iii)). Let $v \in P^{(2)}(1,2)$ be the partition consisting of a three block on level one and $\uparrow \otimes |$ on level two. Let $w \in P^{(2)}(2,1)$ be the partition consisting of $\downarrow \otimes |$ on level one and a three block on level two. By the preceding considerations, v and w are in $\mathcal{C}$. We conclude that the partition

$$r' := (w \otimes (|^{(2)})^{\otimes l-1})($$  $$\otimes r)(v \otimes (|^{(2)})^{\otimes k-1})$$

is in $\mathcal{C}$.

$r' =$ 

$(v \otimes (|^{(2)})^{\otimes k-1})$

( $\otimes r)$

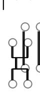
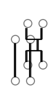
$(w \otimes (|^{(2)})^{\otimes l-1})$

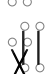
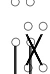
r acts on all square points

It coincides with the partition obtained when connecting the leftmost upper point of r on level one with the leftmost lower point of r on level two (gray points in the picture). We infer that we may connect arbitrary blocks of p_1 with arbitrary blocks of q_2, such that we may construct any partition $p \in P^{(2)}$ in $\mathcal{C}$. $\square$

Corollary 2.19. — *For $m = 2$ and $\pi = \sqcap$, the category $P^{(2)}$ is generated by the following partitions besides the base partitions $|^{(2)}$ and $\sqcap^{(2)}$:*

(i) $\uparrow^{(2)} \in P^{(2)}(0,1)$,

(ii)  $\in P^{(2)}(2,2)$ *and*  $\in P^{(2)}(2,2)$,

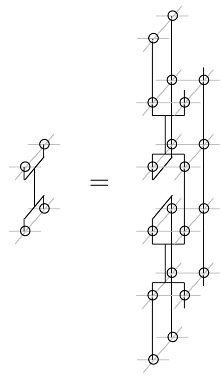
(iii)  $\in P^{(2)}(2,2)$ *and*  $\in P^{(2)}(2,2)$,

(iv)  $\in P^{(2)}(1,1)$.

Moreover, we can replace the partition of item (iv) by

(iv′)  $\in P^{(2)}(0,1)$.

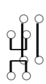
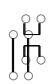
Proof. — By Theorem 2.18, the category $P^{(2)}$ is generated by (i)–(iv). Thus, all we have to prove is that  is in the category generated by (i)–(iii) and (iv′), which is the case as can be seen by the following picture.

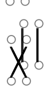
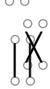


□

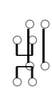
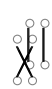
For $m = 1$, we have $P_2 = \langle$  $\rangle$, see [28]. For $m = 2$, the situation is more complicated.

Theorem 2.20. — *For $m = 2$ and $\pi = \sqcap$, the category $P_2^{(2)}$ consisting of all spatial pair partitions on two levels (see also Example 2.13 (b)) is generated by the following partitions besides the base partitions $|^{(2)}$ and $\sqcap^{(2)}$:*

(i)  $\in P^{(2)}(2,2)$ *and*  $\in P^{(2)}(2,2)$,

(ii)  $\in P^{(2)}(2,2)$ *and*  $\in P^{(2)}(2,2)$,

(iii)  $\in P^{(2)}(0,1)$.

Proof. — Similar to the proof of Theorem 2.18, we use ,  and the base partitions in order to construct arbitrary pair partitions $pp^* \in P_2(k,k)$ (where $p \in P_2(0,k)$) on level one. Tensoring several such partitions and several copies of $|^{(2)}$, using (ii) to permute the points and finally composing them with suitable tensor powers of $\sqcap^{(2)}$ and its adjoint, we obtain any arbitrary partition in $P_2^{(2)}$ respecting the levels. We may mix the levels using the partition which may be constructed from (ii) and (iii). $\square$

3. Spatial partition quantum groups

We will now associate quantum groups to categories of spatial partitions. We first recall some basics about compact matrix quantum groups and Woronowicz's Tannaka–Krein result.

3.1. Compact matrix quantum groups

The following definition of a compact matrix quantum group is due to Woronowicz [33, 35]. It is a special case of his theory of compact quantum groups. See also [19, 27] for more details.

Definition 3.1. — *Let $n \in \mathbb{N}$. A* compact matrix quantum group *is a tupel (A, u) such that*

- *A is a unital C^*-algebra generated by n^2 elements u_{ij}, $1 \leqslant i, j \leqslant n$,*
- *the matrices $u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ are invertible in $M_n(A)$,*
- *and the map $\Delta : A \to A \otimes_{\min} A$ given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ is a *-homomorphism.*

If $G \subseteq M_n(\mathbb{C})$ is a compact matrix group, then $C(G)$ gives rise to a compact matrix quantum group in the above sense. We therefore write $A = C(G)$ even if the C^*-algebra A from Definition 3.1 is noncommutative, and we speak of G as the compact matrix quantum group (which is only defined via $C(G)$), sometimes specifying (G, u) in order to keep track of the generating matrix u.

Definition 3.2. — *Let (G, u) with $u = (u_{ij})_{i,j=1,\ldots,n}$ and (H, v) with $v = (v_{ij})_{i,j=1,\ldots,m}$ be two compact matrix quantum groups.*

(a) *We say that G is a* quantum subgroup of H as a compact matrix quantum group, *if there is a surjective *-homomorphism $\varphi : C(H) \to C(G)$ mapping $\varphi(v_{ij}) = u_{ij}$. In particular, we require $n = m$.*

(b) *We say that G is a* quantum subgroup of H as a compact quantum group*, if there is a surjective *-homomorphism $\varphi : C(H) \to C(G)$ such that $\Delta_G(\varphi(v_{ij})) = \sum_{k=1}^m \varphi(v_{ik}) \otimes \varphi(v_{kj})$. In general, we may have $n \neq m$.*

We simply speak of a quantum subgroup $G \subseteq H$ *if there is no confusion with the above cases (a) and (b); in particular, since we will always apply throughout the article (a) in the case $n = m$ and (b) in the case $n \neq m$.*

Example 3.3. —

(a) Wang [29] defined the *free orthogonal quantum group O_n^+* as the compact matrix quantum group given by the universal C^*-algebra

$$C(O_n^+) := C^*\left(u_{ij}, i,j = 1,\dots,n \;\middle|\; u_{ij} = u_{ij}^*, \sum_k u_{ik}u_{jk} = \sum_k u_{ki}u_{kj} = \delta_{ij}\right).$$

If we take the quotient of $C(O_n^+)$ by the relations that all u_{ij} commute, we obtain the algebra of functions $C(O_n)$ over the orthogonal group $O_n \subseteq M_n(\mathbb{C})$. Thus, we have $O_n \subseteq O_n^+$ as quantum subgroups in the sense of Definition 3.2 (a).

Wang also defined the *free unitary quantum group U_n^+* via

$$C(U_n^+) := C^*\left(u_{ij} \;\middle|\; \sum_k u_{ik}u_{jk}^* = \sum_k u_{ki}^* u_{kj} = \sum_k u_{ik}^* u_{jk} = \sum_k u_{ki}u_{kj}^* = \delta_{ij}\right).$$

(b) Wang [31] also defined the *free symmetric quantum group S_n^+* via

$$C(S_n^+) := C^*\left(u_{ij}, i,j = 1,\dots,n \;\middle|\; u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1\right).$$

The quotient by the commutator ideal yields $C(S_n)$, where $S_n \subseteq M_n(\mathbb{C})$ is the symmetric group, thus $S_n \subseteq S_n^+$. It is not difficult to check, that we have $u_{ik}u_{jk} = 0$ and $u_{ki}u_{kj} = 0$ for $i \neq j$. Hence S_n^+ is a quantum subgroup of O_n^+ in the sense of Definition 3.2 (a).

(c) We may view the symmetric group S_n as a quantum subgroup of $O_{n^2}^+$ in the sense of Definition 3.2 (b). Indeed, let u_{ij}, $i,j = 1,\dots,n$ be the generators of $C(S_n)$. For $i_1, i_2, j_1, j_2 \in \{1,\dots,n\}$, we put

$$v'_{(i_1,i_2)(j_1,j_2)} := u_{i_1j_1}u_{i_2j_2} \in C(S_n).$$

Labeling the generators of $C(O_{n^2}^+)$ by $v_{(i_1,i_2)(j_1,j_2)}$, it is easy to verify that we have a surjection from $C(O_{n^2}^+)$ to $C(S_n)$ mapping $v_{(i_1,i_2)(j_1,j_2)}$ to $v'_{(i_1,i_2)(j_1,j_2)}$. It respects the comultiplication map Δ of S_n.

3.2. Tannaka–Krein duality

Similar to Schur–Weyl duality for groups, we may reconstruct a compact matrix quantum group from its intertwiner spaces. This is due to Woronowicz's Tannaka–Krein result [34]. Let us briefly sketch it, referring to [16, 19, 25] for more details. We restrict to the case $u = \bar{u}$.

For $k \in \mathbb{N}_0$, the matrix

$$u^{\otimes k} \in M_{n^k}(C(G)) \cong M_n(\mathbb{C}) \otimes \cdots \otimes M_n(\mathbb{C}) \otimes C(G)$$

is a *(tensor) representation* of a compact matrix quantum group (G, u). The set of *intertwiners* between $u^{\otimes k}$ and $u^{\otimes l}$ is the set of linear map $T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ such that $Tu^{\otimes k} = u^{\otimes l}T$. It is denoted by $\mathrm{Hom}_G(k, l)$. The collection of spaces $(\mathrm{Hom}_G(k, l))_{k,l\in\mathbb{N}_0}$ forms a *C^*-tensor category* or rather a *concrete monoidal W^*-category* in the sense of Woronowicz. A simplifed version of his Tannaka–Krein Theorem is the following.

Theorem 3.4 (Tannaka–Krein Theorem [34]). — *Let* $(\mathrm{Hom}(k, l))_{k,l\in\mathbb{N}_0}$ *be an (abstract) C^*-tensor category which is generated by an element $f = \bar{f}$. Then, there exists a compact matrix quantum group (G, u) with $u = \bar{u}$ such that* $\mathrm{Hom}_G(k, l) = \mathrm{Hom}(k, l)$ *for all $k, l \in \mathbb{N}_0$. It is universal in the sense that whenever (H, v) is a compact matrix quantum group such that $Tv^{\otimes k} = v^{\otimes l}T$ for all* $T \in \mathrm{Hom}(k, l)$ *and all k, l, then H is a quantum subgroup of G.*

We conclude that compact matrix quantum groups are determined by their intertwiner spaces, hence all we need to know about a compact matrix quantum group (G, u) is $(\mathrm{Hom}_G(k, l))_{k,l\in\mathbb{N}_0}$.

3.3. Linear maps associated to partitions

Banica and Speicher [3] associated linear maps to partitions $p \in P$ in order to obtain quantum groups whose intertwiner spaces are of a combinatorial form. Let $n \in \mathbb{N}$ and $p \in P(k, l)$. Let $i = (i_1, \ldots, i_k)$ and $j = (j_1, \ldots, j_l)$ be multi indices whose components range in $\{1, \ldots, n\}$. We decorate the k upper points of p from left to right with the entries of i, and likewise for the l lower points (from left to right) using j. If the strings of p connect only equal indices, then $\delta_p(i, j) := 1$ and $\delta_p(i, j) := 0$ otherwise. See [25, Ex. 4.2] or [28] for examples.

Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{C}^n$. For $p \in P(k, l)$, we define the linear map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ by setting

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) := \sum_{j_1,\ldots,j_l=1}^{n} \delta_p(i, j) e_{j_1} \otimes \cdots \otimes e_{j_l}.$$

The convention is to put $(\mathbb{C}^n)^{\otimes 0} = \mathbb{C}$. The maps T_p behave nicely with respect to the category operations.

Proposition 3.5 ([3]). — *We have:*

(a) $T_p \otimes T_q = T_{p\otimes q}$,
(b) $(T_p)^* = T_{p^*}$,
(c) $T_q T_p = n^{b(q,p)} T_{qp}$, *where* $b(q,p)$ *is the number of disconnected strings arising in the composition of* p *and* q,
(d) $T_| = \mathrm{id} : \mathbb{C}^n \to \mathbb{C}^n$,
(e) $T_{\sqcap} = \sum_i e_i \otimes e_i \in (\mathbb{C}^n)^{\otimes 2}$.

3.4. Linear maps associated to spatial partitions

We now extend Section 3.3 to spatial partitions. Let $m \in \mathbb{N}$ and let $n_1, \dots, n_m \in \mathbb{N}$. By $\ker(n_1, \dots, n_m)$ we denote the unique partition π in $P(m)$ with the property that s and t are in the same block of π if and only if $n_s = n_t$. Let $p \in P^{(m)}(k,l)$ be $\ker(n_1, \dots, n_m)$-graded, i.e. the strings of p connect different levels only if the "dimensions" n_i of these levels coincide. We put

$$[n_1 \times \ldots \times n_m] := \{1, \dots, n_1\} \times \{1, \dots, n_2\} \times \ldots \times \{1, \dots, n_m\}.$$

Let I be a multi index in $[n_1 \times \ldots \times n_m]^k$ and J be a multi index in $[n_1 \times \ldots \times n_m]^l$. Hence I is of the form

$$I = (I_1, \dots, I_k) = \left((i_1^1, \dots, i_m^1), \dots, (i_1^k, \dots, i_m^k)\right).$$

We define $\delta_p(I,J)$ as before, decorating the upper plane of p by I and the lower plane by J, i.e. under the identification $P^{(m)}(k,l) \cong P(km, lm)$, we simply apply the former definition of δ_p. We may find a natural orthonormal basis of $\mathbb{C}^{n_1 n_2 \dots n_m}$ using the following isomorphism:

$$\begin{aligned} \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_m} \quad &\cong \quad \mathbb{C}^{n_1 n_2 \dots n_m} \\ e_{i_1} \otimes \ldots \otimes e_{i_m} &\longleftrightarrow e_{(i_1, \dots, i_m)}. \end{aligned}$$

We assign the following linear map S_p to p:

$$S_p : (\mathbb{C}^{n_1 n_2 \dots n_m})^{\otimes k} \to (\mathbb{C}^{n_1 n_2 \dots n_m})^{\otimes l}$$

$$e_{(i_1^1, \dots, i_m^1)} \otimes \ldots \otimes e_{(i_1^k, \dots, i_m^k)} \mapsto \sum_{\substack{j_1^1, \dots, j_m^1, \dots, \\ j_1^l, \dots, j_m^l}} \delta_p(I,J) e_{(j_1^1, \dots, j_m^1)} \otimes \ldots \otimes e_{(j_1^l, \dots, j_m^l)}.$$

For $m = 1$ and $p \in P^{(1)}(k,l) = P(k,l)$, the constructions of T_p and S_p coincide.

Remark 3.6. — The definition of S_p constitutes the technical key observation of this article. It looks quite simple, but let us discuss it from a different perspective. Observe that the map $T_p : (\mathbb{C}^{n_1\cdots n_m})^{\otimes k} \to (\mathbb{C}^{n_1\cdots n_m})^{\otimes l}$ for $p \in P(k,l)$ coincides with $S_{p^{(m)}} : (\mathbb{C}^{n_1\cdots n_m})^{\otimes k} \to (\mathbb{C}^{n_1\cdots n_m})^{\otimes l}$ for $p^{(m)} \in P^{(m)}(k,l)$. Hence, with the maps S_p for general spatial partitions $p \in P^{(m)}$, we can go "finer" than T_p, making use of the decomposition of $n_1 \cdots n_m$ into factors. Since not all spatial partitions in $P^{(m)}$ come from amplifications, the assignment $p \mapsto S_p$ is richer than the assignment $p \mapsto T_p$.

Proposition 3.5 translates to the following.

Proposition 3.7. — *We have:*

(a) $S_p \otimes S_q = S_{p\otimes q}$,
(b) $(S_p)^* = S_{p^*}$,
(c) $S_q S_p = (n_1 \cdots n_m)^{b(q,p)} S_{qp}$,
(d) $S_{|^{(m)}} = \mathrm{id} : \mathbb{C}^{n_1 n_2 \ldots n_m} \to \mathbb{C}^{n_1 n_2 \ldots n_m}$,
(e) $S_{\sqcap^{(m)}} = \sum_{(i_1,\ldots,i_m)} e_{(i_1,\ldots,i_m)} \otimes e_{(i_1,\ldots,i_m)} \in (\mathbb{C}^{n_1 n_2 \ldots n_m})^{\otimes 2}$.

Proof. — Let $p \in P^{(m)}(k,l)$. Viewing it as a partition in $P(km, lm)$, we may assign the following map to it:

$$T_p : (\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_m})^{\otimes k} \to (\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_m})^{\otimes l}$$
$$e_{i_1} \otimes \ldots \otimes e_{i_{km}} \mapsto \sum_{j_1,\ldots,j_{lm}} \delta_p(I,J) e_{j_1} \otimes \ldots \otimes e_{j_{lm}}.$$

Under the isomorphism

$$\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_m} \cong \mathbb{C}^{n_1 n_2 \ldots n_m}$$
$$e_{i_1} \otimes \ldots \otimes e_{i_m} \longleftrightarrow e_{(i_1,\ldots,i_m)},$$

it coincides with the map

$$S_p : (\mathbb{C}^{n_1 n_2 \ldots n_m})^{\otimes k} \to (\mathbb{C}^{n_1 n_2 \ldots n_m})^{\otimes l}$$
$$e_{(i_1^1,\ldots,i_m^1)} \otimes \ldots \otimes e_{(i_1^k,\ldots,i_m^k)} \mapsto \sum_{\substack{j_1^1,\ldots,j_m^1,\ldots,\\ j_1^l,\ldots,j_m^l}} \delta_p(I,J) e_{(j_1^1,\ldots,j_m^1)} \otimes \ldots \otimes e_{(j_1^l,\ldots,j_m^l)}.$$

Thus, the assertions (a), (b) and (c) follow directly from Proposition 3.5. The assertions (d) and (e) follow from Remark 3.6. □

3.5. Definition of spatial partition quantum groups

The properties of Proposition 3.7 ensure that $\mathrm{span}\{S_p \mid p \in \mathcal{C}(k,l)\}$ is an abstract C^*-tensor category in Woronowicz's sense. Hence we may apply the Tannaka–Krein Theorem to it in order to obtain a quantum group. This motivates the following definition generalizing the one by Banica and Speicher [3]. See also [22, 28, 32] for more on easy quantum groups and [25] for an explicit transition from categories of partitions to quantum groups via Tannaka–Krein.

Definition 3.8. — *Let $m \in \mathbb{N}$ and let $n_1, \dots, n_m \in \mathbb{N}$. A compact matrix quantum group (G, u) with $G \subseteq O^+_{n_1 \cdots n_m}$ and $u \in M_{n_1 \cdots n_m}(C(G))$ is a* spatial partition quantum group*, if there is a category of* $\ker(n_1, \dots, n_m)$*-graded partitions $\mathcal{C} \subseteq P^{(m)}$ such that for all $k, l \in \mathbb{N}_0$, the intertwiner spaces of G are of the form*

$$\mathrm{Hom}_G(k, l) = \mathrm{span}\{S_p \mid p \in \mathcal{C}(k, l)\}.$$

It is convenient to use multi indices from $[n_1 \times \ldots \times n_m]$ for the matrix $u \in M_{n_1 \cdots n_m}(C(G))$, i.e. $u = (u_{IJ})_{I,J \in [n_1 \times \ldots \times n_m]}$. Banica and Speicher [3] defined easy quantum groups using the maps T_p for $p \in P$. For $m = 1$, their quantum groups and our spatial partition quantum groups coincide.

Remark 3.9. — Again, let us briefly comment on an extension in the line of Remarks 2.7 and 2.10 and [7, 25, 26].

Given a category of colored spatial partitions as in Remark 2.10, we associate linear maps S_p to such a colored partition p exactly as in Section 3.4. The colorization of p does not play any role for this definition. However, for the interpretation of S_p as an intertwiner we *do* need the colors of the points: if the color pattern of the upper first level of p is the word $w = (w_1, \dots, w_k) \in \{\circ, \bullet\}^k$ whereas $s = (s_1, \dots, s_l) \in \{\circ, \bullet\}^l$ colors the lower first level, the map S_p is supposed to be an intertwiner of the representations $u^w = u^{w_1} \otimes \cdots \otimes u^{w_k}$ and $u^s = u^{s_1} \otimes \cdots \otimes u^{s_l}$, where $u^\circ = u$ and $u^\bullet = \bar{u}$.

By Tannaka–Krein duality, we then obtain a unitary spatial quantum group $G \subseteq U^+_{n_1 \cdots n_m}$ (we do not assume $u = \bar{u}$ anymore).

3.6. C^*-algebraic relations associated to spatial partitions

The equations $S_p u^{\otimes k} = u^{\otimes l} S_p$, for $p \in P^{(m)}(k, l)$ give rise to relations on the u_{IJ}. They are the following.

Definition 3.10. — *Let $m \in \mathbb{N}$, $n_1, \dots, n_m \in \mathbb{N}$ and let $p \in P^{(m)}(k, l)$. We say that elements u_{IJ}, $I, J \in [n_1 \times \ldots \times n_m]$* satisfy the relations $R(p)$*, if, for all choices of multi indices $I = (I_1, \dots, I_k) \in [n_1 \times \ldots \times n_m]^k$ and*

$J = (J_1, \dots, J_l) \in [n_1 \times \dots \times n_m]^l$, *we have*

$$\sum_{A_1,\dots,A_k \in [n_1 \times \dots \times n_m]} \delta_p(A, J) u_{A_1 I_1} \dots u_{A_k I_k} = \sum_{B_1,\dots,B_l \in [n_1 \times \dots \times n_m]} \delta_p(I, B) u_{J_1 B_1} \dots u_{J_l B_l}.$$

LEMMA 3.11. — *We have $S_p u^{\otimes k} = u^{\otimes l} S_p$ if and only if the relations $R(p)$ are satisfied.*

Proof. — Using the matrix units $e_{JI} \in M_{n_1 \cdots n_m}(\mathbb{C})$ for $J, I \in [n_1 \times \dots \times n_m]$, we write

$$u^{\otimes k} = \sum_{\substack{I_1,\dots,I_k \\ J_1,\dots,J_k}} e_{J_1 I_1} \otimes \dots \otimes e_{J_k I_k} \otimes u_{J_1 I_1} \cdots u_{J_k I_k} \in M_{n_1 \cdots n_m}(\mathbb{C}) \otimes \dots \otimes M_{n_1 \cdots n_m}(\mathbb{C}) \otimes C(G).$$

Applying it to a vector $e_{I_1} \otimes \dots \otimes e_{I_k} \otimes 1$, we obtain

$$u^{\otimes k}(e_{I_1} \otimes \dots \otimes e_{I_k} \otimes 1) = \sum_{A_1,\dots,A_k} e_{A_1} \otimes \dots \otimes e_{A_k} \otimes u_{A_1 I_1} \cdots u_{A_k I_k}.$$

Thus

$$S_p u^{\otimes k}(e_{I_1} \otimes \dots \otimes e_{I_k} \otimes 1) = \sum_{J_1,\dots,J_k} e_{J_1} \otimes \dots \otimes e_{J_l} \otimes \left(\sum_{A_1,\dots,A_k} \delta_p(A, J) u_{A_1 I_1} \cdots u_{A_k I_k} \right),$$

$$u^{\otimes k} S_p(e_{I_1} \otimes \dots \otimes e_{I_k} \otimes 1) = \sum_{J_1,\dots,J_k} e_{J_1} \otimes \dots \otimes e_{J_l} \otimes \left(\sum_{B_1,\dots,B_l} \delta_p(I, B) u_{J_1 B_1} \cdots u_{J_l B_l} \right).$$

Hence, $S_p u^{\otimes k} = u^{\otimes l} S_p$ if and only if the relations $R(p)$ hold. □

PROPOSITION 3.12. — *Let $G \subseteq O^+_{n_1 \dots n_m}$ be a compact matrix quantum group. If $C(G)$ is the universal unital C^*-algebra generated by self-adjoint elements u_{IJ} such that u is orthogonal and the relations $R(p)$ are satisfied for all $p \in \mathcal{C}$ for some* $\ker(n_1, \dots, n_m)$*-graded category $\mathcal{C} \subseteq P^{(m)}$, then G is a spatial partition quantum group.*

In particular, if $\mathcal{C} = \langle p_1, \dots, p_k \rangle$, then the relations $R(p)$ are satisfied for all $p \in \mathcal{C}$ if and only if the relations $R(p_1), \dots, R(p_k)$ and $R(\sqcap^{(m)})$ are satisfied.

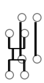
Proof. — The proof is similar to [25, Prop. 5.7] (see also the appendix of the arXiv version of [25]): The space $W := \mathrm{span}\{S_p \mid p \in \mathcal{C}\}$ is a W^*-tensor category in the sense of Woronowicz; hence we may associate a quantum group H to it and the generators of $C(H)$ satisfy all relations $R(p)$, by Lemma 3.11. We thus have a surjection from $C(G)$ to $C(H)$. Conversely, $C(G)$ is a model of W which yields a map from $C(H)$ to $C(G)$ by universality of H. Hence $G = H$.

It is a direct algebraic computation to check that the relations $R(p \otimes q)$, $R(pq)$ and $R(p^*)$ hold, whenever $R(p)$ and $R(q)$ hold. Thus, the relations $R(p_1), \ldots, R(p_k)$ and $R(\sqcap^{(m)})$ imply the relations $R(p)$ for all $p \in \mathcal{C}$. $\square$

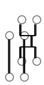
Here is a list of the relations associated to the generators of $P^{(2)}$ and $P_2^{(2)}$.

$$|^{(2)} : u_{(i_1 i_2)(j_1 j_2)} = u_{(i_1 i_2)(j_1 j_2)};$$

$$\sqcap^{(2)}, \sqcap^{(2)^*} : \sum_{g_1,g_2} u_{(i_1,i_2)(g_1,g_2)} u_{(j_1,j_2)(g_1,g_2)} = \sum_{g_1,g_2} u_{(g_1,g_2)(i_1,i_2)} u_{(g_1,g_2)(j_1,j_2)} = \delta_{i_1 j_1} \delta_{i_2 j_2};$$



$$: u_{(k_1,k_2)(i_1,i_2)} u_{(k_3,k_4)(i_1,i_3)} = 0 \text{ if } k_1 \neq k_3, \quad u_{(i_1,i_2)(k_1,k_2)} u_{(i_1,i_3)(k_3,k_4)} = 0 \text{ if } k_1 \neq k_3;$$



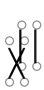
$$: u_{(k_1,k_2)(i_1,i_2)} u_{(k_3,k_4)(i_3,i_2)} = 0 \text{ if } k_2 \neq k_4, \quad u_{(i_1,i_2)(k_1,k_2)} u_{(i_3,i_2)(k_3,k_4)} = 0 \text{ if } k_2 \neq k_4;$$

$$: \sum_g u_{(g,b_2)(i_1,i_2)} = \sum_h u_{(b_1,b_2)(h,i_2)} \text{ (in particular independent of } i_1, b_1);$$

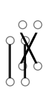
$$: \sum_g u_{(b_1,g)(i_1,i_2)} = \sum_h u_{(b_1,b_2)(i_1,h)} \text{ (in particular independent of } i_2, b_2);$$

,  *

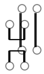
$$: \sum_{g_1,g_2} u_{(i_1,i_2)(g_1,g_2)} = \sum_{g_1,g_2} u_{(g_1,g_2)(j_1,j_2)} = 1;$$

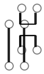


$$: u_{(b_1,b_2)(i_1,i_2)} u_{(b_3,b_4)(i_3,i_4)} = u_{(b_3,b_2)(i_3,i_2)} u_{(b_1,b_4)(i_1,i_4)};$$



$$: u_{(b_1,b_2)(i_1,i_2)} u_{(b_3,b_4)(i_3,i_4)} = u_{(b_1,b_4)(i_1,i_4)} u_{(b_3,b_2)(i_3,i_2)};$$

 $: \delta_{j_1 j_3} \sum_g u_{(g,j_2)(i_1,i_2)} u_{(g,j_4)(i_3,i_4)} = \delta_{i_1 i_3} \sum_h u_{(j_1,j_2)(h,i_2)} u_{(j_3,j_4)(h,i_4)};$

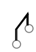
 $: \delta_{j_2 j_4} \sum_g u_{(j_1,g)(i_1,i_2)} u_{(j_3,g)(i_3,i_4)} = \delta_{i_2 i_4} \sum_h u_{(j_1,j_2)(i_1,h)} u_{(j_3,j_4)(i_3,h)};$

 $: \delta_{b_1 b_2} \sum_g u_{(g,g)(i_1,i_2)} = \delta_{i_1 i_2} \sum_h u_{(b_1,b_2)(h,h)};$

 $: \sum_g u_{(b_1,b_2)(g,g)} = \delta_{b_1 b_2};$

 $: \delta_{i_1 i_2} \sum_g u_{(g,g)(j_1,j_2)} = \delta_{j_1 j_2} \sum_g u_{(i_1,i_2)(g,g)};$

 $: u_{(i_1,i_2)(j_1,j_2)} = u_{(i_2,i_1)(j_2,j_1)};$

Remark 3.13. — Inspired from the above relations for , we view them more generally for $(u_{ij})_{i,j=1,\ldots,n}$ as

$$\sum_{k \in I} u_{ik} = \delta_{i \in I}$$

for some subset $I \subseteq \{1,\ldots,n\}$. For instance:

$$\sum_{k \text{ even}} u_{ik} = \delta_{i \text{ even}}.$$

It is easy to check that these relations pass through the comultiplication. Hence, one can define some partial versions of quantum permutation groups.

Remark 3.14. — In the setting of colored partitions, the relations associated to partitions read as follows. Let $m \in \mathbb{N}$, $n_1,\ldots,n_m \in \mathbb{N}$ and let p be a colored partition with upper color pattern $w = (w_1,\ldots,w_k)$ and lower color pattern $s = (s_1,\ldots,s_l)$. We say that elements u_{IJ}, $I, J \in [n_1 \times \ldots \times n_m]$ satisfy the (colored) relations $R(p)$, if, for all choices of multi indices $I = (I_1,\ldots,I_k) \in [n_1 \times \ldots \times n_m]^k$ and $J = (J_1,\ldots,J_l) \in [n_1 \times \ldots \times n_m]^l$, we have

$$\sum_{A_1,\dots,A_k\in[n_1\times\dots\times n_m]} \delta_p(A,J)u^{w_1}_{A_1 I_1}\dots u^{w_k}_{A_k I_k}$$
$$= \sum_{B_1,\dots,B_l\in[n_1\times\dots\times n_m]} \delta_p(I,B)u^{s_1}_{J_1 B_1}\dots u^{s_l}_{J_l B_l}.$$

Here, we understand $u^x_{A_1 I_1} = u_{A_1 I_1}$ if $x = \circ$ and $u^x_{A_1 I_1} = u^*_{A_1 I_1}$ if $x = \bullet$.

As an example:

 $: u^*_{(i_1,i_2)(j_1,j_2)} = u_{(i_2,i_1)(j_2,j_1)}$

4. Products of categories

Given two categories of partitions $\mathcal{C}_1$ and $\mathcal{C}_2$, how can we form a new one from this data? Several possibilities will be developped in the sequel.

4.1. Kronecker product of categories

In the setting of spatial partition quantum groups, we have an obvious possibility to form a new category out of two given categories $\mathcal{C}_1 \subseteq P$ and $\mathcal{C}_2 \subseteq P$: We simply put $\mathcal{C}_1$ on level one and $\mathcal{C}_2$ on level two. More generally, we have the following setup.

Definition 4.1. — *Let $s \in \mathbb{N}$. Let $m_i \in \mathbb{N}$ and let $\pi_i \in P(m_i)$ for $i = 1,\dots,s$.*

(a) *Let $k,l \in \mathbb{N}_0$. Let $p_i \in P^{(m_i)}_{\pi_i}(k,l)$ for $i = 1,\dots,s$. We denote by*
$$\begin{pmatrix} p_s \\ \vdots \\ p_1 \end{pmatrix} \in P^{(m_1+\dots+m_s)}_{\pi_1\otimes\dots\otimes\pi_s}(k,l)$$
the partition given by placing p_1 on the levels 1 to m_1, placing p_2 on the levels m_1+1 to m_1+m_2, and so on.

(b) *Let $\mathcal{C}_i \subseteq P^{(m_i)}_{\pi_i}$ be sets of π_i-graded partitions, for $i = 1,\dots,s$. We denote by*
$$\mathcal{C}_1\times\dots\times\mathcal{C}_s := \left\{ \begin{pmatrix} p_s \\ \vdots \\ p_1 \end{pmatrix} \in P^{(m_1+\dots+m_s)}_{\pi_1\otimes\dots\otimes\pi_s} \;\middle|\; p_i \in \mathcal{C}_i \text{ for all } i = 1,\dots,s \right\}$$
$$\subseteq P^{(m_1+\dots+m_s)}_{\pi_1\otimes\dots\otimes\pi_s}$$
the Kronecker product *of the sets $\mathcal{C}_i$, $i = 1,\dots,s$.*

Lemma 4.2. — *Let $\mathcal{C}_i \subseteq P_{\pi_i}^{(m_i)}$ be categories of π_i-graded spatial partitions, for $i = 1, \ldots, s$. Then $\mathcal{C}_1 \times \ldots \times \mathcal{C}_s \subseteq P_{\pi_1 \otimes \ldots \otimes \pi_s}^{(m_1 + \ldots + m_s)}$ is category of $\pi_1 \otimes \ldots \otimes \pi_s$-graded spatial partitions.*

Proof. — The proof is straightforward. □

4.2. Glued tensor products of spatial partition quantum groups

It is natural to ask for the quantum group picture of the above Kronecker product of categories. Recall the following product of quantum groups from [25, Def. 6.4].

Definition 4.3. — *Let (G, u) and (H, v) be two compact matrix quantum groups with $n \times n$ matrix $u = (u_{ij})_{i,j=1,\ldots,n}$ and $m \times m$ matrix $v = (v_{kl})_{k,l=1,\ldots,m}$. The* glued direct product *$(G, u) \tilde{\times} (H, v)$ of (G, u) and (H, v) is the compact matrix quantum group given by the C^*-subalgebra*

$$C(G \tilde{\times} H) := C^*(u_{ij} v_{kl} \mid i, j = 1, \ldots, n \text{ and } k, l = 1, \ldots m) \subseteq C(G) \otimes_{\max} C(H)$$

and the $nm \times nm$ matrix $u \tilde{\times} v := (u_{ij} v_{kl})$. Here, we identify $C(G) \otimes_{\max} C(H)$ with the universal C^*-algebra generated by elements $u_{ij} \in C(G)$ and $v_{kl} \in C(H)$ such that all u_{ij} commute with all v_{kl}. We also write $G \tilde{\times} H$ short for $(G, u) \tilde{\times} (H, v)$.

Theorem 4.4. — *Let $(G_i, u_i) \subseteq O_{N_i}^+$ be spatial partition quantum groups with categories $\mathcal{C}_i \subseteq P^{(m_i)}$ for $i = 1, 2$ and $N_i = n_1^i \cdots n_{m_i}^i$. The spatial partition quantum group associated to the category $\mathcal{C}_1 \times \mathcal{C}_2$ is $G_1 \tilde{\times} G_2 \subseteq O_{N_1 N_2}^+$.*

Proof. — It is straightforward to check that $G_1 \tilde{\times} G_2$ is indeed a quantum subgroup of $O_{N_1 N_2}^+$ in the sense of Definition 3.2 (a) mapping the generators $w_{(i_1,i_2)(j_1,j_2)}$ of $O_{N_1 N_2}^+$ to $u_{i_1 j_1} v_{i_2 j_2} \in C(G_1 \tilde{\times} G_2)$. Moreover, let G be the compact matrix quantum group associated to the category $\mathcal{C}_1 \times \mathcal{C}_2$. Thus its intertwiner space is

$$\mathrm{Hom}_G(k, l) = \mathrm{span}\{S_p \mid p \in \mathcal{C}_1 \times \mathcal{C}_2\}.$$

By definition, the intertwiner space of $G_1 \tilde{\times} G_2$ is

$$\begin{aligned} &\mathrm{Hom}_{G_1 \tilde{\times} G_2}(k, l) \\ &= \left\{T : (\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2})^{\otimes k} \to (\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2})^{\otimes l} \;\middle|\; T(u_1 \otimes u_2)^{\otimes k} = (u_1 \otimes u_2)^{\otimes l} T\right\}. \end{aligned}$$

To prove the statement of the theorem, it suffices to prove that

$$\mathrm{Hom}_G(k, l) = \mathrm{Hom}_{G_1 \tilde{\times} G_2}(k, l) \qquad \text{for all } k, l \in \mathbb{N}_0.$$

For doing so, let us first consider $S_p \in \mathrm{Hom}_G(k,l)$. Thus $p = \left(\begin{smallmatrix} p_2 \\ p_1 \end{smallmatrix}\right)$ with $p_i \in \mathcal{C}_i$. Reordering the elements of the tensor product $(\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2})^{\otimes k} \cong (\mathbb{C}^{N_1})^{\otimes k} \otimes (\mathbb{C}^{N_2})^{\otimes k}$, we observe that $S_p \cong S_{p_1} \otimes S_{p_2}$. Thus

$$\begin{aligned} S_p(u_1 \otimes u_2)^{\otimes k} \cong (S_{p_1} \otimes S_{p_2})(u_1^{\otimes k} \otimes u_2^{\otimes k}) = (u_1^{\otimes k} \otimes u_2^{\otimes k})(S_{p_1} \otimes S_{p_2}) \\ \cong (u_1 \otimes u_2)^{\otimes k} S_p. \end{aligned}$$

Consequently, $S_p \in \mathrm{Hom}_{G_1 \tilde{\times} G_2}(k,l)$ and by linearity, $\mathrm{Hom}_G(k,l) \subseteq \mathrm{Hom}_{G_1 \tilde{\times} G_2}(k,l)$.

We conclude by a dimension argument. Recall that the dimension of $\mathrm{Hom}_H(k,l)$ of a compact matrix quantum group (H,w) is given by $h_H(\chi_w^{k+l})$ where h_H is the Haar measure on H and $\chi_w = \sum_i w_{ii}$. The Haar measure of $G_1 \widetilde{\times} G_2$ is given by $h_{G_1} \otimes h_{G_2}$ by [30]. We have

$$\begin{aligned} \dim \mathrm{Hom}_{G_1 \tilde{\times} G_2}(k,l) &= h_{G_1} \otimes h_{G_2}(\chi_{u_1 \otimes u_2}^{k+l}) \\ &= h_{G_1} \otimes h_{G_2}(\chi_{u_1}^{k+l} \chi_{u_2}^{k+l}) \\ &= h_{G_1}(\chi_{u_1}^{k+l}) h_{G_2}(\chi_{u_2}^{k+l}) \\ &= \dim \mathrm{Hom}_{G_1}(k,l) \cdot \dim \mathrm{Hom}_{G_2}(k,l) \\ &= \dim \mathrm{span}\{S_{p_1} \mid p_1 \in \mathcal{C}_1\} \cdot \dim \mathrm{span}\{S_{p_2} \mid p_2 \in \mathcal{C}_2\} \\ &= \dim \mathrm{span}\{S_{p_1} \mid p_1 \in \mathcal{C}_1\} \otimes \mathrm{span}\{S_{p_2} \mid p_2 \in \mathcal{C}_2\} \\ &= \dim \mathrm{span}\{S_{p_1} \otimes S_{p_2} \mid p_1 \in \mathcal{C}_1, p_2 \in \mathcal{C}_2\}. \end{aligned}$$

Once again, reordering the elements of the tensor product $(\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2})^{\otimes k} \cong (\mathbb{C}^{N_1})^{\otimes k} \otimes (\mathbb{C}^{N_2})^{\otimes k}$, we observe that $S_{p_1} \otimes S_{p_2} \cong S_p$ whenever $p = \left(\begin{smallmatrix} p_2 \\ p_1 \end{smallmatrix}\right)$. It allows us to conclude:

$$\begin{aligned} \dim \mathrm{Hom}_{G_1 \tilde{\times} G_2}(k,l) &= \dim \mathrm{span}\{S_{p_1} \otimes S_{p_2} \mid p_1 \in \mathcal{C}_1, p_2 \in \mathcal{C}_2\} \\ &= \dim \mathrm{span}\{S_p \mid p = \left(\begin{smallmatrix} p_2 \\ p_1 \end{smallmatrix}\right), p_1 \in \mathcal{C}_1, p_2 \in \mathcal{C}_2\} \\ &= \dim \mathrm{span}\{S_p \mid p \in \mathcal{C}_1 \times \mathcal{C}_2\} \\ &= \dim \mathrm{Hom}_G(k,l). \end{aligned}$$

□

Corollary 4.5. — *In particular, the quantum groups $S_n \widetilde{\times} S_n$, $S_n \widetilde{\times} S_n^+$, $S_n^+ \widetilde{\times} S_n^+$ or $O_n \widetilde{\times} S_n^+$ etc. are spatial partition quantum groups.*

We observe that the class of spatial partition quantum groups is closed under taking the glued direct product (increasing the number m). This is not the case for easy quantum groups: taking the glued direct product, we leave the class of easy quantum groups entering the class of spatial partition quantum groups.

4.3. Glued tensor products with amalgamation over partitions

Forming the Kronecker product $\mathcal{C}_1 \times \mathcal{C}_2$ of two categories $\mathcal{C}_i \subseteq P$ (with easy quantum groups (G,u) and (H,v)), we obtain a category of spatial partitions respecting the (two) levels. We will obtain further relations for the generators $u_{ij}v_{kl}$ of $C(G\tilde{\times}H)$, if we throw in partitions mixing the levels. See also [7] for further ways of amalgamation.

Definition 4.6. — *Let (G,u) and (H,v) be compact matrix quantum groups such that the matrices u and v have the same size. Put $w_{(i,k)(j,l)} := u_{ij}v_{kl} \in C(G) \otimes_{\max} C(H)$. Let $p \in P^{(2)}$. The* glued direct product over p *(or the p*-amalgamated glued direct product*) $(G,u)\tilde{\times}_p(H,v)$ (or short $G\tilde{\times}_pH$) is given by the C^*-subalgebra*

$$\begin{aligned}&C^*(u_{ij}v_{kl} \mid i,j,k,l = 1,\dots,n)\\ &\subseteq C(G) \otimes_{\max} C(H)/\langle \textit{the elements } w_{(i,k)(j,l)} \textit{ satisfy the relations } R(p)\rangle.\end{aligned}$$

Lemma 4.7. — *The C^*-algebra in Definition 4.6 admits a comultiplication turning $(G,u)\tilde{\times}_p(H,v)$ into a compact matrix quantum group with fundamental representation $u\tilde{\times}_pv := (u_{ij}v_{kl})_{i,j,k,l=1,\dots,n}$.*

Proof. — The C^*-algebra $C(G\tilde{\times}H)$ of Definition 4.3 admits a comultiplication due to [25]. We can view $C(G\tilde{\times}_pH)$ as a quotient of $C(G\tilde{\times}H)$, and consider the following diagram.

$$\begin{tikzcd} C(G\tilde{\times}H) \arrow[r, "\Delta"] \arrow[d, "\alpha"'] & C(G\tilde{\times}H) \otimes C(G\tilde{\times}H) \arrow[d, "\alpha\otimes\alpha"] \\ C(G\tilde{\times}_pH) & C(G\tilde{\times}_pH) \otimes C(G\tilde{\times}_pH) \end{tikzcd}$$

Hence, all we have to check is that the map $(\alpha \otimes \alpha) \circ \Delta$ factorizes through α. For doing so, we only need to check that the elements

$$\sum_{s,t} w_{(i,k)(s,t)} \otimes w_{(s,t)(j,l)} \in C(G\tilde{\times}_pH) \otimes C(G\tilde{\times}_pH)$$

satisfy the relations $R(p)$, which is the case. $\square$

Theorem 4.8. — *Let $(G_i,u_i) \subseteq O_n^+$ be easy quantum groups with categories $\mathcal{C}_i \subseteq P$ for $i = 1,2$. Let $p \in P^{(2)}$. The spatial partition quantum group associated to the category $\langle \mathcal{C}_1 \times \mathcal{C}_2, p\rangle$ is $G_1\tilde{\times}_pG_2 \subseteq O_{n^2}^+$.*

Proof. — Let (G,v) be the spatial partition quantum group associated to the category $\langle \mathcal{C}_1 \times \mathcal{C}_2, p\rangle$. We denote by u the generating matrix of $G_1\tilde{\times}_pG_2$

and by w the generating matrix of $G_1 \widetilde{\times} G_2$. Because the relations $R(p)$ only involve elements of $C(G_1 \widetilde{\times} G_2)$, we have

$$C(G_1 \widetilde{\times}_p G_2) \cong C(G_1 \widetilde{\times} G_2)/\langle \text{the elements } w_{(i,k)(j,l)} \text{ satisfy the relations } R(p)\rangle$$

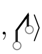
via the canonical isomorphism $w_{IJ} \mapsto u_{IJ}$.

By definition, the space of intertwiners of G contains the one of $G_1 \widetilde{\times} G_2$. By universality of Tannaka–Krein theorem, it means that G is a quantum subgroup of $G_1 \widetilde{\times} G_2$, or more precisely, that there exists a surjective $*$-homomorphism $\varphi : C(G_1 \widetilde{\times} G_2) \to C(G)$ mapping w_{IJ} to v_{IJ}. Because v_{IJ} satisfy the relation $R(p)$, this homomorphism can be quotiented into a surjective $*$-homomorphism $\varphi : C(G_1 \widetilde{\times}_p G_2) \to C(G)$ mapping u_{IJ} to v_{IJ}, meaning that G is a quantum subgroup of $G_1 \widetilde{\times}_p G_2$.

Conversely, the space of intertwiners of $G_1 \widetilde{\times}_p G_2$ is bigger than the one of G, because it contains S_p and S_q for $q \in \mathcal{C}_1 \times \mathcal{C}_2$, which means that it contains $\operatorname{span}\{S_q \mid q \in \langle \mathcal{C}_1 \times \mathcal{C}_2, p\rangle\}$. By universality of Tannaka–Krein theorem, it implies that $G_1 \widetilde{\times}_p G_2$ is a quantum subgroup of G. □

It is straightforward to generalize Definition 4.6 to products of an arbitrary finite number $m \in \mathbb{N}$ of quantum groups, allowing for an amalgamation with arbitrary partitions $P^{(m)}$. One can also choose G_1 and G_2 of the above proposition to be spatial partition quantum groups rather than easy quantum groups, after extending Definition 4.6 to an amalgamation of $G \subseteq O^+_{n_1^1 \cdots n_{m_1}^1}$ and $H \subseteq O^+_{n_1^2 \cdots n_{m_2}^2}$ with respect to a partition $p \in P_\pi^{(m_1+m_2)}$, where $\pi = \ker(n_1^1, \ldots, n_{m_1}^1, n_1^2, \ldots, n_{m_2}^2)$. Observe that an amalgamation with a partition respecting the levels boils down to the glued direct product without amalgamation.

Example 4.9. — Let $\mathcal{C}_1 = \mathcal{C}_2 = NC_2$.

(a) For $p =$ , the category $\langle NC_2 \times NC_2,$ $\rangle$ corresponds to $O_n^+ \widetilde{\times}_p O_n^+$ with

$$C(O_n^+ \widetilde{\times}_p O_n^+) = C^*\left(u_{ij}v_{kl} \;\middle|\; (u_{ij}), (v_{kl}) \text{ are orth.}, u_{ij}v_{kl} = v_{kl}u_{ij}, \sum_k u_{i_1k}v_{i_2k} = \delta_{i_1i_2} \right).$$

(b) For $p =$ , the category $\langle NC_2 \times NC_2,$ $\rangle$ corresponds to $O_n^+ \widetilde{\times}_p O_n^+$ with

$$C(O_n^+ \widetilde{\times}_p O_n^+) = C^*\left(u_{ij}v_{kl} \;\middle|\; \begin{matrix} (u_{ij}), (v_{kl}) \text{ are orth.}, \\ u_{ij}v_{kl} = v_{kl}u_{ij},\ u_{ij}v_{kl} = u_{kl}v_{ij} \end{matrix} \right).$$

4.4. Noncrossing product of categories

Building on an extension of the setting of spatial partition quantum groups to the colored situation (Remark 3.9), we may give another construction for defining new categories from old ones, in the sense of Thm, 4.4. We first need a notion of non-crossing colored spatial partitions.

For this purpose, we extend the isomorphism of Remark 2.4 from white points to white and black points: for any $m \in \mathbb{N}$, $k, l \in \mathbb{N}_0$ and a fixed color pattern on $\{1, \ldots, k, k+1, \ldots, k+l\} \times \{1, \ldots, m\}$, the sets

$$A := \{1, \ldots, km, km+1, \ldots, km+lm\}$$

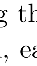
and

$$B := \{1, \ldots, k, k+1, \ldots, k+l\} \times \{1, \ldots, m\}$$

are in bijective correspondence by identifying a point $(x-1)m+y \in A$, $1 \leqslant x \leqslant k+l$, $1 \leqslant y \leqslant m$ with the point $(x, y) \in B$ if (x, y) is white and by identifying a point $xm+1-y \in A$, $1 \leqslant x \leqslant k+l$, $1 \leqslant y \leqslant m$ with the point $(x, y) \in B$ if (x, y) is black. This reverse order on black points reflects the identity $\overline{u \otimes v} = \overline{v} \otimes \overline{u}$.

We can define two different products using this isomorphism. The definition of the noncrossing product $\mathcal{C}_1 \times_{\mathrm{nc}} \mathcal{C}_2$ and of the free product $\mathcal{C}_1 * \mathcal{C}_2$ of categories $\mathcal{C}_1 \subseteq P$ and $\mathcal{C}_2 \subseteq P$ of colored partitions follow Definition 4.1 with additional conditions of being noncrossing:

- We place a partition p from $\mathcal{C}_1$ on level one, and a partition q from $\mathcal{C}_2$ on level two and consider the resulting partition $\binom{q}{p}$.
- For the definition of $\mathcal{C}_1 \times_{\mathrm{nc}} \mathcal{C}_2$, we require that the partitions p and q do not cross each other under the above isomorphism. More precisely, $\binom{q}{p}$ is in $\mathcal{C}_1 \times_{\mathrm{nc}} \mathcal{C}_2$ whenever there exists a partition $r \leqslant \binom{q}{p}$ which respects the levels and which is noncrossing under the above isomorphism.
- For the definition of $\mathcal{C}_1 * \mathcal{C}_2$, we require in addition that for each block of r on the first level, the restriction of p to this block is in $\mathcal{C}_1$, and for each block of r on the second level, the restriction of q to this block is in $\mathcal{C}_2$.

As a consequence, $\mathcal{C}_1 * \mathcal{C}_2 \subseteq \mathcal{C}_1 \times_{\mathrm{nc}} \mathcal{C}_2$. For example, putting  on both level gives us the noncrossing partition $\{\{1,4\},\{2,3\}\}$, meaning that  $^{(2)}$ is an element of the noncrossing product $\mathcal{C}_1 \times_{\mathrm{nc}} \mathcal{C}_2$. In addition, each restriction is , which is in every category of colored partitions, meaning that  $^{(2)} \in$ $\mathcal{C}_1 * \mathcal{C}_2$. Similarly, we can verify that  $^{(2)}$, $^{(2)}$ and $^{(2)}$ are in the free product $\mathcal{C}_1 * \mathcal{C}_2$. Since the conditions are maintained under the category operations, we deduce that $\mathcal{C}_1 \times_{\mathrm{nc}} \mathcal{C}_2 \subseteq P^{(2)}$ and $\mathcal{C}_1 * \mathcal{C}_2 \subseteq P^{(2)}$ are categories of colored spatial partitions on two levels. Remark that for some categories (including P_2 and NC_2), we have $\mathcal{C}_1 \times_{\mathrm{nc}} \mathcal{C}_2 = \mathcal{C}_1 * \mathcal{C}_2$.

Theorem 4.10 is the unitary version of Theorem 4.4, where we used the glued free product of [25, Def. 6.4]: if (G,u) and (H,v) are two compact matrix quantum groups with $u = (u_{ij})_{i,j=1,\ldots,n}$ and $v = (v_{kl})_{k,l=1,\ldots,m}$, the *glued free product* $G \tilde{*} H$ of G and H is given by the C^*-subalgebra

$$C^*(u_{ij}v_{kl} \mid i,j = 1,\ldots,n \text{ and } k,l = 1,\ldots m) \subseteq C(G) * C(H).$$

Theorem 4.10. — *Let $(G_i,u_i) \subseteq O_n^+$ be easy quantum groups with categories $\mathcal{C}_i \subseteq P$ for $i = 1,2$. The spatial partition quantum group associated to the category $\mathcal{C}_1 * \mathcal{C}_2$ is $G_1 \tilde{*} G_2 \subseteq U_{n^2}^+$.*

Proof. — The intertwiners between tensor products of the representations $u_1, \bar{u}_1, u_2$ and $\bar{u}_2$ of $G_1 * G_2$ are explicitely given by [13, Prop. 2.15]: they are linear combinations of compositions of morphisms of the type $\mathrm{id} \otimes R \otimes \mathrm{id}$ where R is either an intertwiner between tensor products of the representations $u_1, \bar{u}_1$ of G_1 or an intertwiner between tensor products of the representations $u_2, \bar{u}_2$ of G_2.

Let us describe this set of intertwiners in a different way. Two tensor products of the representations $u_1, \bar{u}_1, u_2$ and $\bar{u}_2$ of length k and l can be seen as a decoration of k upper points and l lower points by $u_1, \bar{u}_1, u_2$ and $\bar{u}_2$. Given such a decoration, we can consider a partition p of $k + l$ points which does not connect the points decorated by $u_1, \bar{u}_1$ with the points decorated by $u_2, \bar{u}_2$ such that:

- there exists a noncrossing partition r which is coarser than p and which does not connect the points decorated by $u_1, \bar{u}_1$ with the points decorated by $u_2, \bar{u}_2$,
- the restriction of p to each block of r decorated by $u_1, \bar{u}_1$ is in $\mathcal{C}_1$, and the restriction of p to each block of r decorated by $u_2, \bar{u}_2$ is in $\mathcal{C}_2$.

Let us call such a partition an admissible partition. They form a category of colored partitions. We claim that the intertwiners between tensor products of the representations $u_1, \bar{u}_1, u_2$ and $\bar{u}_2$ of $G_1 * G_2$ are exactly given by the linear combinations of the morphisms T_p for p an admissible partition in the sense above.

Let us briefly sketch the proof. On one hand, if p is in $\mathcal{C}_1$ or $\mathcal{C}_2$, the morphism $\mathrm{id} \otimes T_p \otimes \mathrm{id}$ can be written as $T_{|^{\otimes a} \otimes p \otimes |^{\otimes b}}$ with the admissible partition $|^{\otimes a} \otimes p \otimes |^{\otimes b}$. Taking the closure by linear combination and composition gives us that every intertwiner between tensor products of the representations $u_1, \bar{u}_1, u_2$ and $\bar{u}_2$ is given by the linear combinations of the morphisms T_p for p an admissible partition. Conversely, if p is an admissible partition, there exists a noncrossing partition r which is as described above. There exists at least one interval block in r. Doing some rotation if necessary, we can assume that this block is supported on consecutive points on the upper left corner. Because p is admissible, the restriction of p to this block of r is a partition p_1 of $\mathcal{C}_1$ or $\mathcal{C}_2$, and we can decompose $p = p_1 \otimes p_2$ with p_2 admissible. But T_{p_1} is an intertwiner between tensor products of the representations $u_1, \bar{u}_1, u_2$ and $\bar{u}_2$ of $G_1 \tilde{*} G_2$. We conclude by induction on the number of blocks of the admissible partitions.

Thus we can describe the set of intertwiners between tensor products of the representations $u_1 \otimes u_2$ and $\overline{u_1 \otimes u_2} = \bar{u}_2 \otimes \bar{u}_1$ of $G_1 * G_2$, or equivalently between tensor products of the fundamental representation of $G_1 \tilde{*} G_2$ and its adjoint, as linear combination of T_p with p an admissible partition. Using the isomorphism described at the beginning of the section, we see that it coincides exactly with

$$\operatorname{span}\{S_p \mid p \in \mathcal{C}_1 * \mathcal{C}_2\}. \qquad \square$$

5. Examples of spatial partition quantum groups

In this section, several examples of spatial partition quantum groups are given. Of course, easy quantum groups are such examples. Interestingly, spatial partition quantum groups are also related to other known quantum groups. For instance, the new machine of spatial partition quantum groups covers some quantum symmetry groups of finite quantum spaces.

We first take a look at the natural cornerstones of the theory.

5.1. Amplifications of easy quantum groups

A trivial class of spatial partition quantum groups is obtained by the amplification of easy quantum groups.

Proposition 5.1. — *Let $n_1, \ldots, n_m \in \mathbb{N}$. Let $G_n \subseteq O_n^+$ be an easy quantum group with category $\mathcal{C} \subseteq P$. Then, the spatial partition quantum group associated to $[\mathcal{C}]^{(m)}$ is the easy quantum group $G_{n_1 \cdots n_m} \subseteq O^+_{n_1 \cdots n_m}$ with category $\mathcal{C}$.*

Proof. — By Lemma 2.12, $[\mathcal{C}]^{(m)} \subseteq P^{(m)}$ is a category of spatial partitions; it thus gives rise to a spatial partition quantum group. For $p \in \mathcal{C}(k,l)$, the maps

$$\begin{aligned} T_p &: (\mathbb{C}^{n_1 n_2 \dots n_m})^{\otimes k} \to (\mathbb{C}^{n_1 n_2 \dots n_m})^{\otimes l} \\ \text{and} \quad S_{p^{(m)}} &: (\mathbb{C}^{n_1 n_2 \dots n_m})^{\otimes k} \to (\mathbb{C}^{n_1 n_2 \dots n_m})^{\otimes l} \end{aligned}$$

coincide by Remark 3.6. As a consequence, the intertwiner spaces of the spatial partition quantum group associated to $[\mathcal{C}]^{(m)}$ and of the easy quantum group with category $\mathcal{C}$ are the same, which allows us to conclude by the Tannaka–Krein theorem. ☐

Corollary 5.2. — *We have the following correspondences of spatial partition quantum groups:*

$$\begin{array}{lcllcl} S^+_{n_1\cdots n_m} & \longleftrightarrow & [NC]^{(m)}, & \qquad O^+_{n_1\cdots n_m} & \longleftrightarrow & [NC_2]^{(m)}, \\ S_{n_1\cdots n_m} & \longleftrightarrow & [P]^{(m)}, & \qquad O_{n_1\cdots n_m} & \longleftrightarrow & [P_2]^{(m)}. \end{array}$$

5.2. Minimal and maximal spatial partition quantum groups

In the case $m = 1$, we have $S_n \longleftrightarrow P$ and $O_n^+ \longleftrightarrow NC_2$. Since every category of partitions satisfies $P \supseteq \mathcal{C} \supseteq NC_2$, we have

$$S_n \subseteq G \subseteq O_n^+$$

for easy quantum groups G. The case $m \geqslant 2$ is different, since we may have

$$S_{n_1\cdots n_m} \nsubseteq G \subseteq O^+_{n_1\cdots n_m}$$

for spatial partition quantum groups G, whenever $[P]^{(m)} \nsupseteq \mathcal{C} \supseteq [NC_2]^{(m)}$ (see Example 2.13). We are thus interested in finding the minimal spatial partition quantum group corresponding to the maximal category of spatial partitions $P^{(m)}$.

Theorem 5.3. — *Let $n_1, \dots, n_m \in \mathbb{N}$ and let $\pi = \ker(n_1, \dots, n_m)$. The category $P_\pi^{(m)}$ of all π-graded partitions corresponds to the spatial partition quantum group*

$$S_{n_{i_1}} \tilde{\times} \dots \tilde{\times} S_{n_{i_r}} \subseteq O^+_{n_1\cdots n_m},$$

where $\{n_{i_1}, \dots, n_{i_r}\}$ is the set $\{n_1, \dots, n_m\}$ without repetitions.

In the special case $n_1 = \dots = n_m = n$, we have

$$P^{(m)} \quad \longleftrightarrow \quad S_n \subseteq O^+_{n^m}.$$

Proof. — We only prove the special case $m = 2$ and $n_1 = n_2 = n$, the general case following from a straightforward adaption and an application of Theorem 4.4. Recall from Definition 3.2 and Example 3.3 that S_n can be viewed as a quantum subgroup of $O^+_{n^2}$ by mapping the generators $v_{(i_1,i_2)(j_1,j_2)}$ of $C(O^+_{n^2})$ to the product $v'_{(i_1,i_2)(j_1,j_2)} := u_{i_1j_1}u_{i_2j_2}$ in $C(S_n)$.

Let A be the C^*-algebra generated by elements $v_{(i_1,i_2)(j_1,j_2)}$ satisfying all relations (R_p) for all $p \in P^{(m)}$. By Proposition 3.12 and Corollary 2.19, this is equivalent to satisfying all relations (R_p) for all generators p of $P^{(2)}$ as listed in Corollary 2.19. It is easy to check that $v'_{(i_1,i_2)(j_1,j_2)} \in C(S_n)$ satisfies all these relations, hence a map $\varphi : A \to C(S_n)$ mapping $v_{(i_1,i_2)(j_1,j_2)} \to v'_{(i_1,i_2)(j_1,j_2)}$ exists by the universal property. Conversely, the elements $u'_{ij} := \sum_k v_{(ik)(j1)} \in A$ satisfy the relations of $C(S_n)$ as can be verified directly. This yields a map $\psi : C(S_n) \to A$ mapping u_{ij} to u'_{ij} by the universal property and we have that φ and ψ are inverse to each other. Thus, A and $C(S_n)$ are isomorphic; the isomorphism respects Δ.

An alternative proof using intertwiners is based on the observation that the map from $C(O^+_{n^2})$ to $C(S_n)$ maps the matrix v to $u^{\otimes 2}$. Thus, intertwiners between $v^{\otimes k}$ and $v^{\otimes l}$ give rise to intertwiners between $(u^{\otimes 2})^{\otimes k} = u^{\otimes 2k}$ and $(u^{\otimes 2})^{\otimes l} = u^{\otimes 2l}$. Since the linear span of $\{T_p \mid p \in P(2k, 2l)\}$ coincides with the linear span of $\{S_p \mid p \in P^{(2)}(k, l)\}$, we deduce that the intertwiners of S_n viewed as a subgroup of $O^+_{n^2}$ and the intertwiners of the spatial partition quantum group which corresponds to the category of all spatial partitions on two levels are the same, which allows us to conclude by the Tannaka–Krein theorem. ☐

We conclude that in the case $m = 2$ and $n_1 = n_2 = n$, we have, for any spatial partition quantum group G,

$$S_n \subseteq G \subseteq O^+_{n^2}.$$

Recall that the class of easy quantum groups only covers the case

$$S_{n^2} \subseteq G \subseteq O^+_{n^2}.$$

More striking, while the obstruction for easy quantum groups $G \subseteq O^+_{1024}$ is $S_{1024} \subseteq G$, we only have $\mathbb{Z}/\mathbb{Z}_2 \subseteq G$ for spatial quantum groups $G \subseteq O^+_{1024}$.

Remark 5.4. — Although our approach yields a larger class of quantum subgroups of $O^+_{n^2}$, we may not construct a quantum group G with $S_{n^2} \subseteq G \subseteq O^+_{n^2}$ which is not an easy quantum group. Indeed, if G is spatial partition with category $\mathcal{C} \subseteq P^{(2)}$ and if $S_{n^2} \subseteq G \subseteq O^+_{n^2}$, then $\mathcal{C} \subseteq [P]^{(2)}$ since S_{n^2} corresponds to $[P]^{(2)}$. But this means that any partition $p \in \mathcal{C}$ is a 2-amplification of a partition $p' \in P$. Restriction of $\mathcal{C}$ to partitions on its first level yields a category of partitions $\mathcal{C}' \subseteq P$ such that $\mathcal{C} = [\mathcal{C}']^{(2)}$, hence G is an easy quantum group by Proposition 5.1.

5.3. Examples in the case $m = 2$

In this subsection, we restrict to the case $m = 2$ and $n_1 = n_2 = n$ and we provide an incomplete list of categories $\mathcal{C} \subseteq P_2^{(2)}$ of spatial pair partitions (all blocks are of size 2). In order to distinguish them, we introduce the following five sets.

Definition 5.5. — *We define the following subsets of $P_2^{(2)}$.*

(a) *We let*

$$\mathcal{C}_{\text{resplevels}} := \{\left(\begin{smallmatrix} p_2 \\ p_1 \end{smallmatrix}\right) \mid p_1, p_2 \in P_2\} \subseteq P_2^{(2)}$$

be the set of all spatial partitions respecting the levels.

(b) *A spatial partition $p \in P_2^{(2)}$ is called* level symmetric*, if it is symmetric when swapping the levels one and two. In other words, if two points (x_1, y_1) and (x_2, y_2) form a block of p, then also $(x_1, \bar{y}_1)$ and $(x_2, \bar{y}_2)$ form a block, where $\bar{y} := \left\{\begin{smallmatrix} 1 & \text{if } y=2 \\ 2 & \text{if } y=1 \end{smallmatrix}\right.$. We put*

$$\mathcal{C}_{\text{symm}} := \{p \in P_2^{(2)} \mid p \textit{ is level symmetric}\} \subseteq P_2^{(2)}.$$

(c) *We let*

$$\mathcal{C}_{\text{nodiagonal}} := \left\{ p \in P_2^{(2)} \;\middle|\; \begin{array}{l} \textit{no two points } (x,1) \textit{ and } (y,2) \\ \textit{with } x \neq y \textit{ form a block} \end{array} \right\} \subseteq P_2^{(2)}$$

be the set of all spatial partitions having no diagonal strings between the levels. We put

$$\mathcal{C}^{\text{symm}}_{\text{nodiagonal}} := \mathcal{C}_{\text{nodiagonal}} \cap \mathcal{C}_{\text{symm}}.$$

(d) *We let*

$$\mathcal{C}_{\text{noviceversa}} := \{p \in P_2^{(2)} \mid \textit{no two points } (x,1) \textit{ and } (x,2) \textit{ form a block}\} \subseteq P_2^{(2)}$$

be the set of all spatial partitions having no geodesic strings between the levels. We put

$$\mathcal{C}^{\text{symm}}_{\text{noviceversa}} := \mathcal{C}_{\text{noviceversa}} \cap \mathcal{C}_{\text{symm}}.$$

(e) *We let*

$$\mathcal{C}_{\text{even}} := \bigcup_{k+l=2n,\ n \in \mathbb{N}} P_2^{(2)}(k,l) \subseteq P_2^{(2)}$$

be the set of all spatial partitions whose number of blocks is even.

Remark 5.6. — We have $\mathcal{C}_{\text{resplevels}} = \mathcal{C}_{\text{nodiagonal}} \cap \mathcal{C}_{\text{noviceversa}}$ and $\mathcal{C}_{\text{resplevels}} \subseteq \mathcal{C}_{\text{even}}$. Moreover, $\mathcal{C}^{\text{symm}}_{\text{noviceversa}} \subseteq \mathcal{C}_{\text{even}}$.

Lemma 5.7. — *The sets $\mathcal{C}_{\text{resplevels}}$, $\mathcal{C}_{\text{symm}}$, $\mathcal{C}^{\text{symm}}_{\text{nodiagonal}}$, $\mathcal{C}^{\text{symm}}_{\text{noviceversa}}$ and $\mathcal{C}_{\text{even}}$ are categories of spatial partitions.*

Proof. — We may use Lemma 2.17(b) with $\pi = \uparrow \otimes \uparrow$ for the set $\mathcal{C}_{\text{resplevels}}$. As for the others, one can directly verify stability under the category operations. □

Recall that there are only three subcategories of P_2 in the case $m = 1$, namely NC_2, $\langle \times \rangle$ and P_2 (see [32]). For $m = 2$ we have many more.

Theorem 5.8. — *All of the following categories are subcategories of $P_2^{(2)}$. They are all distinct.*

(a) *The amplifications $[NC_2]^{(2)} = \langle \emptyset \rangle$, $[\langle \times \rangle]^{(2)} = \langle \times^{(2)} \rangle$ and $[P_2]^{(2)} =$ $\langle$  $\rangle$ (see Proposition 5.1).*

(b) *The categories $\mathcal{C}_1 \times \mathcal{C}_2$ with $\mathcal{C}_i \in \{NC_2, \langle \times \rangle, P_2\}$ as in Section 4.*

(c) *The category $\langle$  $\rangle$.*

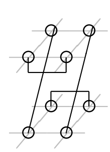
(d) *The category $\langle$  $\rangle$.*

(e) *The category $\langle$  ,  $\rangle$.*

(f) *The category $\langle$  $\rangle$.*

(g) *The category $\langle$  ,  $\rangle$.*

(h) *The category generated by the following spatial partition.*



(i) *The category $P_2^{(2)}$ itself.*

Proof. — We may distinguish the above categories using those of Lemma 5.7: We have the following containments of categories. Observe that $p \in \mathcal{C}$

if and only if $\langle p\rangle \subseteq \mathcal{C}$, since $\langle p\rangle$ is the smallest category containing p.

	$\mathcal{C}_{\text{resplevels}}$	$\mathcal{C}_{\text{symm}}$	$\mathcal{C}^{\text{symm}}_{\text{nodiagonal}}$	$\mathcal{C}^{\text{symm}}_{\text{noviceversa}}$	$\mathcal{C}_{\text{even}}$
(a)	$\subseteq$	$\subseteq$			
(b)	$\subseteq$	$\not\subseteq$			
(c)	$\not\subseteq$	$\subseteq$	$\subseteq$	$\not\subseteq$	$\subseteq$
(d)	$\not\subseteq$	$\subseteq$	$\not\subseteq$	$\subseteq$	$\subseteq$
(e)	$\not\subseteq$	$\subseteq$	$\not\subseteq$	$\not\subseteq$	$\subseteq$
(f)	$\not\subseteq$	$\subseteq$	$\subseteq$	$\not\subseteq$	$\not\subseteq$
(g)	$\not\subseteq$	$\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$
(h)	$\not\subseteq$	$\not\subseteq$			$\subseteq$
(i)	$\not\subseteq$	$\not\subseteq$			$\not\subseteq$

Hence, all of the categories (a) to (i) are distinct. □

It is very likely that the above list is not complete. However, we believe that (a) and (b) list all categories respecting the levels. The above categories are of interest since they correspond to quantizations of the orthogonal group O_n in a way. By Proposition 5.1, the amplifications $[NC_2]^{(2)}$, $[\langle \times\!\!\!\!\mid \rangle]^{(2)}$ and $[P_2]^{(2)}$ correspond to $O^+_{n^2}$, $O^*_{n^2}$ and O_{n^2} respectively. By Theorem 4.4, the categories $\mathcal{C}_1 \times \mathcal{C}_2$ with $\mathcal{C}_i \in \{NC_2, \langle \times\!\!\!\!\mid \rangle, P_2\}$ correspond to glued tensor products of O^+_n, O^*_n and O_n. As for determining the quantum groups corresponding to the categories (c-h) of Theorem 5.8, use the C^*-algebraic relations of Section 3.6. Note that the quantum groups of Example 4.9 do not come into play here, since $\langle NC_2 \times NC_2, p\rangle \neq \langle p\rangle$ in both cases due to $\mathcal{C}_{\text{symm}}$.

Concerning the quantum group G corresponding to the category $P_2^{(2)}$, it is easy to check, like in Theorem 5.3, that the elements

$$v'_{(i_1,i_2)(j_1,j_2)} := u_{i_1j_1}u_{i_2j_2} \in C(O_n)$$

satisfy all relations $R(p)$ for $p \in P_2^{(2)}$ (using Theorem 2.20). However, this map is not surjective; in particular, $u'_{ij} := \sum_k v_{(i,k)(j,1)} \in C(G)$ does *not* give rise to an orthogonal matrix (or equivalenty: $\sum_k u_{ij}u_{k1} \neq u_{ij}$ in $C(O_n)$). Thus, we have to leave the question open to which quantum group $P_2^{(2)}$ corresponds.

5.4. From quantum subgroups of $O^+_{n^2}$ to quantum subgroups of O^+_n

Starting with a quantum subgroup G of $O^+_{n^2}$ we may associate a quantum subgroup $\mathring{G}$ of O^+_n to it, under certain conditions.

Definition 5.9. — *Let $G \subseteq O_{n^2}^+$ be a compact matrix quantum group such that $C(G)$ is generated by $u_{(i,k)(j,l)}$, for $i,j,k,l = 1,\dots,n$. We put*

$$\mathring{u}_{ij} := \sum_k u_{(i,k)(j,1)}.$$

Let $C(\mathring{G}) \subseteq C(G)$ be the C^*-subalgebra of $C(G)$ generated by the elements $\mathring{u}_{ij}$ for $i,j \in \{1,\dots,n\}$.

The elements $\mathring{u}_{ij}$ are self-adjoint. We now investigate, when $C(\mathring{G})$ gives rise to a compact matrix quantum group $\mathring{G} \subseteq O_n^+$. We express the necessary condition in terms of C^*-algebraic relations $R(p)$ associated to partitions $p \in P^{(2)}$ as in Section 3.6. However, our next proposition does not only work for spatial partition quantum groups, it holds for general compact matrix quantum groups.

Proposition 5.10. — *Suppose the relations $R(p)$ for $p =$  are satisfied for the elements $u_{(i,k)(j,l)} \in C(G)$ and suppose $S_n \subseteq G \subseteq O_{n^2}^+$ (where $S_n \subseteq G$ is in the sense of Example 3.3). Then:*

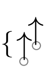
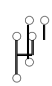
(a) *We have, independently of the choice of x and y,*

$$\mathring{u}_{ij} = \sum_k u_{(i,k)(j,x)} = \sum_k u_{(i,y)(j,k)}.$$

(b) *The map $\Delta : C(G) \to C(G) \otimes C(G)$ restricts to $\Delta : C(\mathring{G}) \to C(\mathring{G}) \otimes C(\mathring{G})$ with $\Delta(\mathring{u}_{ij}) = \sum_k \mathring{u}_{ik} \otimes \mathring{u}_{kj}$.*

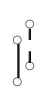
(c) *The C^*-algebra $C(\mathring{G})$ gives rise to a compact matrix quantum group $\mathring{G}$ with*

$$S_n \subseteq \mathring{G} \subseteq O_n^+.$$

(d) *If in addition the relations $R(p)$ for $p \in \{$, $\}$ are satisfied for the elements $u_{(i,k)(j,l)} \in C(G)$, then*

$$S_n \subseteq \mathring{G} \subseteq S_n^+.$$

Proof. —

(a). — This is exactly what the relations $R(p)$ for $p =$  are.

(b). — We compute, using (a):

$$\Delta(\mathring{u}_{ij}) = \sum_l \Delta(u_{(i,l)(j,1)}) = \sum_{l,k,m} u_{(i,l)(k,m)} \otimes u_{(k,m)(j,1)}$$
$$= \sum_{k,m} \mathring{u}_{ik} \otimes u_{(k,m)(j,1)} = \sum_k \mathring{u}_{ik} \otimes \mathring{u}_{kj}.$$

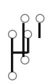
(c). — By (b), $\mathring{G}$ is a compact matrix quantum group. The matrix $\mathring{u} = (\mathring{u}_{ij})$ is orthogonal due to the following computation using (a) and $G \subseteq O^+_{n^2}$:

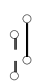
$$\sum_k \mathring{u}_{ik}\mathring{u}_{jk} = \sum_{k,m} \mathring{u}_{ik} u_{(j,1)(k,m)} = \sum_{k,l,m} u_{(i,l)(k,m)} u_{(j,1)(k,m)} = \sum_l \delta_{ij}\delta_{l1} = \delta_{ij}.$$

Similarly $\sum_k \mathring{u}_{ki}\mathring{u}_{kj} = \delta_{ij}$. Hence, $\mathring{G} \subseteq O^+_n$. As for proving $S_n \subseteq \mathring{G}$, note that by assumption we have a *-homomorphism $\varphi : C(G) \to C(S_n)$ mapping $u_{(i,k)(j,l)}$ to $v_{ij}v_{kl}$, where we denote the generators of $C(S_n)$ by v_{ij}. Thus, $\varphi(\mathring{u}_{ij}) = v_{ij}$ which proves $S_n \subseteq \mathring{G}$.

(d). — All we have to check is that the elements $\mathring{u}_{ij}$ satisfy $\mathring{u}_{ij}^2 = \mathring{u}_{ij}$ and $\sum_l \mathring{u}_{il} = \sum_l \mathring{u}_{lj} = 1$. This follows directly from (a) and the relations $R(p)$ which we list below.

$R(p)$ for $p =$  $: \sum_{g_1,g_2} u_{(b_1,b_2)(g_1,g_2)} = \sum_{g_1,g_2} u_{(g_1,g_2)(b_1,b_2)} = 1;$

$R(p)$ for $p =$  $: \sum_g u_{(b_1,b_2)(i_1,i_2)} u_{(b_1,g)(i_3,i_4)} = \delta_{i_1 i_3} u_{(b_1,b_2)(i_1,i_2)}.$ $\square$

Remark 5.11. — We may also define $\mathring{G}$ via $\mathring{u}_{ij} := \sum_k u_{(k,i)(1,j)}$ and require the relations $R(p)$ with $p =$  in Proposition 5.10; this will yield an analogue result.

We conclude that we may produce quantum groups $\mathring{G}$ which are intermediate between S_n and O^+_n, just like easy quantum groups. However, it is not clear for the moment whether or not they yield quantum groups which are not easy quantum groups. Non-easy quantum groups have been studied only very recently, see [10, 15].

5.5. Links with quantum symmetries of finite quantum spaces; case $m = 3$

In [31], Wang investigated quantum symmetry groups of finite quantum spaces, see also [1, 5, 18]. More precisely, let Tr_N denote the unnormalized trace on $M_N(\mathbb{C})$, i.e. $\mathrm{Tr}_N(x) = \sum_{i=1}^N x_{ii}$ for $x = (x_{ij}) \in M_N(\mathbb{C})$. A finite quantum space (B, ψ) consists in a finite dimensional C^*-algebra B and a state (to be more precise: a δ-form) $\psi : B \to \mathbb{C}$ given as follows:

$$B = \bigoplus_{l=1}^n M_{N_l}(\mathbb{C}), \qquad \psi(x_1 \oplus \cdots \oplus x_n) = \sum_{l=1}^n \mathrm{Tr}_{N_l}(Q_l x_l)$$

Here, the matrices $Q_l \in M_{N_l}(\mathbb{C})$ are invertible and positive satisfying $\sum_{l=1}^n \mathrm{Tr}_{N_l}(Q_l) = 1$ and $\mathrm{Tr}_{N_l}(Q_l^{-1}) = \dim(B)$.

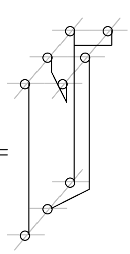
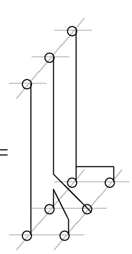
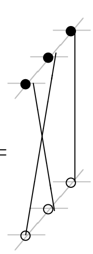
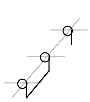
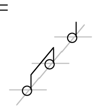
In [31], Wang computed the maximal quantum group acting on B in a ψ preserving way. Let us call it the quantum automorphism group of (B, ψ) and denote it by $G^+(B, \psi)$. If $N_1 = \cdots = N_n = 1$, $B = \mathbb{C}^n$ and $Q_l = \frac{1}{n}$, then S_n^+ is the resulting quantum automorphism group. In the general case, $G^+(B, \psi)$ can be described in terms of generators and relations, see [18]: the underlying C^*-algebra is the universal C^*-algebra generated by elements $u_{(i,j,a)(r,s,b)}$ with $a, b \in \{1, \dots, n\}$, $i, j \in \{1, \dots, N_a\}$, $r, s \in \{1, \dots, N_b\}$ and relations

(A1a) $\sum_{w=1}^{N_c} u_{(x,w,c)(k,l,a)} u_{(w,y,c)(r,s,b)} = \delta_{ab}\delta_{lr} u_{(x,y,c)(k,s,a)}$
(A1b) $\sum_{w=1}^{N_c} (Q_c^{-1})_{ww} u_{(s,r,b)(y,w,c)} u_{(l,k,a)(w,x,c)} = \delta_{ab}\delta_{lr} (Q_a^{-1})_{ll} u_{(s,k,a)(y,x,c)}$
(A2) $u^*_{(x,y,c)(k,l,a)} = u_{(y,x,c)(l,k,a)}$
(A3a) $\sum_{b=1}^n \sum_{x=1}^{N_b} (Q_b)_{xx} u_{(x,x,b)(k,l,a)} = \delta_{kl}(Q_a)_{kk}$
(A3b) $\sum_{a=1}^n \sum_{k=1}^{N_a} u_{(x,y,b)(k,k,a)} = \delta_{xy}$

Let us consider the special case $N := N_1 = \cdots = N_n$, $n \in \mathbb{N}$ and $Q_l := \frac{1}{nN} E$, where $E \in M_N(\mathbb{C})$ denotes the $N \times N$ unit matrix. So, we have:

$$B = \bigoplus_{l=1}^n M_N(\mathbb{C}), \qquad \psi(x_1 \oplus \cdots \oplus x_n) = \frac{1}{nN} \sum_{l=1}^n \mathrm{Tr}_N(x_l)$$

Let $m := 3$, $n_1 := n_2 := N$ and $n_3 := n$. Let π be the partition $\pi = \{1, 2\}\{3\} \in P(3)$. Consider the category $\mathcal{C}$ of colored spatial partitions generated by the following π-graded partitions:

$p_{1a} =$  $p_{1b} =$  $p_2 =$  $p_{3a} =$  $p_{3b} =$ 

By a direct computation using Definition 3.10 and its extension in the colored case (Remark 3.14), we observe that the spatial partition quantum group associated to $\mathcal{C}$ coincides with $G^+(B,\psi)$. Note that we have $p_{1b} = p_{1a}^*$ and $p_{3b} = p_{3a}^*$, so as generators, the partitions p_{1b} and p_{3b} are redundant.

In [1] (see also [5, §3.1]), another description by diagrams was given, namely by 2-cabled Temperley–Lieb diagrams. Denoting the matrix units in $B = \bigoplus_{l=1}^n M_{N_l}(\mathbb{C})$ by $e_{ij}^{(k)}$, the unit map $\nu : \mathbb{C} \to B$, $1 \mapsto \sum_{i,k} e_{ii}^{(k)}$ corresponds to $\delta^{-\frac{1}{2}} \sqcap$, the multiplication map $\mu : B \otimes B \to B$, $e_{ij}^{(k)} \otimes e_{rs}^{(t)} \mapsto \delta_{kt}\delta_{jr}e_{is}^{(k)}$ corresponds to $\delta^{\frac{1}{2}} | \otimes \sqcup \otimes |$ and the identity map $\mathrm{id}_B : B \to B$ corresponds to $| \otimes |$ in the Temperley–Lieb picture; the maps ν, μ and id_B are understood as intertwiners in $\mathrm{Hom}(0,1)$, $\mathrm{Hom}(2,1)$ and $\mathrm{Hom}(1,1)$ for $G^+(B,\psi)$ giving rise to the relations (A1a) and (A3b) resp. We observe that restricting the partitions p_{1a} and p_{3b} to the levels $k = 1$ and $k = 2$ and rearranging the points, our diagrams coincide with the Temperley–Lieb picture.

In any case, we conclude that spatial quantum groups provide a fine tuned way of describing the representation categories of quantum automorphism groups of finite quantum spaces, at least in the special case $B = \bigoplus_{l=1}^n M_N(\mathbb{C})$.

6. Open questions

Throughout this article, a number of questions arose. They can be the starting point for further investigations. For the convenience of the reader, we list them here.

(1) Classify all categories $\mathcal{C} \subseteq P_2^{(2)}$; see also Theorem 5.8. In particular, are there other categories apart those from Theorem 5.8 (a) and (b), who respect the levels?
(2) Are the quantum groups arising from Theorem 5.8 related to other known quantum groups (in terms of certain product constructions, for instance)? In particular, what does the quantum group corresponding to $P_2^{(2)}$ look like?
(3) Are all quantum groups $S_n \subseteq \mathring{G} \subseteq O_n^+$ arising from Proposition 5.10 easy quantum groups or can we produce non-easy ones this way?
(4) Recall that we have $S_n \subseteq O_{n^2}^+$, see Example 3.3 (c), but $S_n^+ \not\subseteq O_{n^2}^+$. However, $S_n^+ \subseteq U_{n^2}^+$ holds. Thus, it would be interesting to classify all spatial partition quantum groups $S_n^+ \subseteq G \subseteq U_{n^2}^+$ since these are the ones relevant for free probability in the sense of [2, 12, 28].

(5) Determine the category generated by the partitions p_{1a}, p_{1b}, p_2, p_{3a}, p_{3b} in Section 5.5.
(6) Can we adapt the definition of spatial quantum groups such that they also cover general quantum automorphism groups of finite quantum spaces (see Section 5.5)?
(7) Given that Section 5.5 provides quantum spaces which spatial quantum groups act on, can we find canonical quantum spaces which general spatial quantum groups act on? See also [11].

Bibliography

[1] T. Banica, "Symmetries of a generic coaction", *Math. Ann.* **314** (1999), no. 4, p. 763-780.

[2] T. Banica, S. Curran & R. Speicher, "De Finetti theorems for easy quantum groups", *Ann. Probab.* **40** (2012), no. 1, p. 401-435.

[3] T. Banica & R. Speicher, "Liberation of orthogonal Lie groups", *Adv. Math.* **222** (2009), no. 4, p. 1461-1501.

[4] M. Brannan, "Approximation properties for free orthogonal and free unitary quantum groups", *J. Reine Angew. Math.* **672** (2012), p. 223-251.

[5] ———, "Reduced operator algebras of trace-preserving quantum automorphism groups", *Doc. Math.* **18** (2013), p. 1349-1402.

[6] A. Freslon, "Fusion (semi)rings arising from quantum groups", *J. Algebra* **417** (2014), p. 161-197.

[7] ———, "On two-coloured noncrossing partition quantum groups", *Trans. Am. Math. Soc.* **320** (2019), p. 4471-4508.

[8] A. Freslon & M. Weber, "On the representation theory of partition (easy) quantum groups", *J. Reine Angew. Math.* **720** (2016), p. 155-197.

[9] D. Gromada, "Classification of globally colorized categories of partitions", *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **21** (2018), no. 4, article no. 1850029 (25 pages).

[10] D. Gromada & M. Weber, "Intertwiner spaces of quantum group subrepresentations", *Commun. Math. Phys.* **376** (2020), no. 1, p. 81-115.

[11] S. Jung & M. Weber, "Partition quantum spaces", *J. Noncommut. Geom.* **14** (2020), no. 3, p. 821-859.

[12] C. Köstler & R. Speicher, "A noncommutative de Finetti theorem: invariance under quantum permutations is equivalent to freeness with amalgamation", *Commun. Math. Phys.* **291** (2009), no. 2, p. 473-490.

[13] F. Lemeux, "The fusion rules of some free wreath product quantum groups and applications", *J. Funct. Anal.* **267** (2014), no. 7, p. 2507-2550.

[14] F. Lemeux & P. Tarrago, "Free wreath product quantum groups: the monoidal category, approximation properties and free probability", *J. Funct. Anal.* **270** (2016), no. 10, p. 3828-3883.

[15] L. Maassen, "The intertwiner spaces of non-easy group-theoretical quantum groups", *J. Noncommut. Geom.* **14** (2020), no. 3, p. 987-1017.

[16] S. Malacarne, "Woronowicz's Tannaka–Krein duality and free orthogonal quantum groups", *Math. Scand.* **122** (2018), p. 151-106.

[17] A. Mang & M. Weber, "Categories of two-colored pair partitions. Part II: Categories indexed by semigroups", *J. Comb. Theory, Ser. A* **180** (2021), article no. 105409 (44 pages).

[18] C. Mrozinski, "Quantum automorphism groups and $so(3)$-deformations", *J. Pure Appl. Algebra* **219** (2015), no. 1, p. 1-32.

[19] S. Neshveyev & L. Tuset, *Compact quantum groups and their representation categories*, Cours Spécialisés (Paris), Société Mathématique de France, 2013.

[20] A. Nica & R. Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series, vol. 335, London Mathematical Society, 2006.

[21] S. Raum & M. Weber, "Easy quantum groups and quantum subgroups of a semi-direct product quantum group", *J. Noncommut. Geom.* **9** (2015), no. 4, p. 1261-1293.

[22] ———, "The full classification of orthogonal easy quantum groups", *Commun. Math. Phys.* **341** (2016), no. 3, p. 751-779.

[23] R. Speicher & M. Weber, "Quantum groups with partial commutation relations", *Indiana Univ. Math. J.* **68** (2019), no. 6, p. 1849-1883.

[24] R. P. Stanley, *Enumerative combinatorics. Volume 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, 2012.

[25] P. Tarrago & M. Weber, "Unitary easy quantum groups: the free case and the group case", *Int. Math. Res. Not.* **18** (2017), p. 5710-5750.

[26] ———, "The classification of tensor categories of two-colored noncrossing partitions", *J. Comb. Theory, Ser. A* **154** (2018), p. 464-506.

[27] T. Timmermann, *An invitation to quantum groups and duality: From Hopf algebras to multiplicative unitaries and beyond*, EMS Textbooks in Mathematics, European Mathematical Society, 2008.

[28] D.-V. Voiculescu, N. Stammeier & M. Weber, *Free probability and operator algebras*, Münster Lectures in Mathematics, European Mathematical Society, 2016.

[29] S. Wang, "Free products of compact quantum groups", *Commun. Math. Phys.* **167** (1995), no. 3, p. 671-692.

[30] ———, "Tensor products and crossed products of compact quantum groups", *Proc. Lond. Math. Soc.* **71** (1995), no. 3, p. 695-720.

[31] ———, "Quantum symmetry groups of finite spaces", *Commun. Math. Phys.* **195** (1998), no. 1, p. 195-211.

[32] M. Weber, "On the classification of easy quantum groups", *Adv. Math.* **245** (2013), p. 500-533.

[33] S. L. Woronowicz, "Compact matrix pseudogroups", *Commun. Math. Phys.* **111** (1987), no. 4, p. 613-665.

[34] ———, "Tannaka–Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups", *Invent. Math.* **93** (1988), no. 1, p. 35-76.

[35] ———, "A remark on compact matrix quantum groups", *Lett. Math. Phys.* **21** (1991), no. 1, p. 35-39.