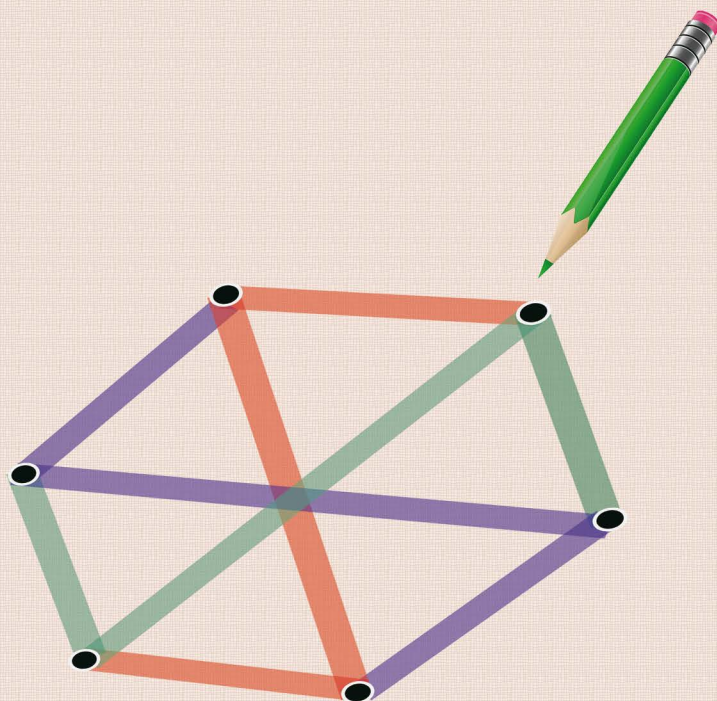


Properly colored subgraphs in edge-colored graphs

Tingting Han



**PROPERLY COLORED SUBGRAPHS
IN EDGE-COLORED GRAPHS**

Tingting Han

PROPERLY COLORED SUBGRAPHS IN EDGE-COLORED GRAPHS

DISSERTATION

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Preface

This thesis contains research results on edge-colored graphs, which were obtained by the author and different collaborators since March 2018. Apart from an introductory chapter (Chapter 1), the reader will find five closely related technical chapters (Chapters 2–6). The first two chapters (Chapters 2 and 3) are mainly based on the research results that the author obtained while she was working as a PhD student in Northwestern Polytechnical University in Xi'an, China. The final three chapters (Chapters 4–6) are mainly based on the research results that the author obtained during her stay at the University of Twente. This thesis is devoted to developing sufficient conditions for the existence of specific properly colored cycles in edge-colored graphs and related cyclic properties, like the existence of properly colored cycle-factors and properly colored cycles of different or all lengths.

Papers underlying this thesis

- [1] Edge-Colored complete graphs containing no properly colored odd cycles, *Graphs and Combinatorics* **37** (2021), 1129–1138 (with S. Zhang, Y. Bai and R. Li). (Chapter 2)
- [2] Sufficient conditions for properly colored C_3 's and C_4 's in edge-colored complete graphs, *Discrete Applied Mathematics* **327** (2023), 101–109 (with H.J. Broersma, Y. Bai and S. Zhang). (Chapter 3)
- [3] Properly colored cycles of different lengths in edge-colored complete graphs, *Discrete Mathematics* **346** (2023), 113653 (with S. Zhang and Y. Bai). (Chapter 4)

- [4] Properly colored cycles in edge-colored balanced bipartite graphs, submitted (with H.J. Broersma, Y. Bai and S. Zhang). (Chapters 3, 5 and 6)

Another recent joint paper by the author

- [1] Color neighborhood union conditions for proper edge-pancyclicity of edge-colored complete graphs, *Discrete Applied Mathematics* **307** (2022), 145–152 (with F. Wu, S. Zhang and B. Li).

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Chapter 1

Introduction

We start this chapter with a short intuitive introduction to the topic of this thesis, avoiding the formal definitions for the moment.

A graph in its most basic form is a collection of vertices together with a set of edges that join certain pairs of these vertices. Therefore graphs can model any practical problem in which one considers certain objects, and the essence of the problem is whether there exists a relationship between pairs of these objects or not. Typical examples are all kinds of networks, varying from road networks to communication networks, and from social networks to neural networks, to name just a few. Graphs provide an approach to study various phenomena that can be represented by such networks. Since the early applications of graphs, graph theory has developed into an independent and mature mathematical branch. It plays an important role in other research and application areas like combinatorics, computer science, chemistry, physics, biology and so on.

For a graph, the edges can be undirected or directed, and may have other attributes associated with them. Many real world problems have been modeled by different types of graphs. If there are two possible options for a specific relationship between every pair of objects, then this can be modeled as a graph. Edges then indicate which pairs are related. In case the relationship is not symmetric, this does not work, and we have to use directed edges (arcs) to indicate which ordered pairs are related. However, in more complex situations,

either of these models may fail to capture the essence of the relationship. For example, consider a graph that models a network of relay stations. Let us assume that, in order to avoid interference between the received and transmitted radio signals, a relay station should receive and transmit radio signals on different frequencies. In such cases, we can use vertices to represent relay stations, edges to represent the transmission lines between pairs of relay station, and colors to represent the frequencies that are used to receive or transmit a radio signal through the corresponding transmission lines. In this setting, it is natural that the concept of an edge-colored graph is introduced, that is, a graph together with an assignment of colors to its edges, one color per edge. We will define these concepts formally in the next section.

Edge-colored graphs often have dual properties of graphs and directed graphs (digraphs). On one hand, edge-colored graphs generalize undirected graphs by assigning a color to each edge. On the other hand, edge-colored graphs generalize digraphs. To see this, we consider a simple transformation from digraphs to edge-colored graphs, which can be found in [63]. Let D be a digraph with vertex set $\{v_1, v_2, \dots, v_n\}$ and let $\{c_1, c_2, \dots, c_n\}$ be a color set. By replacing each arc $v_i v_j$ with an edge and coloring it with color c_j , we obtain an edge-colored graph G^c ; see Figure 1.1 for an example, in which c_3 is yellow, c_6 is blue, etc.

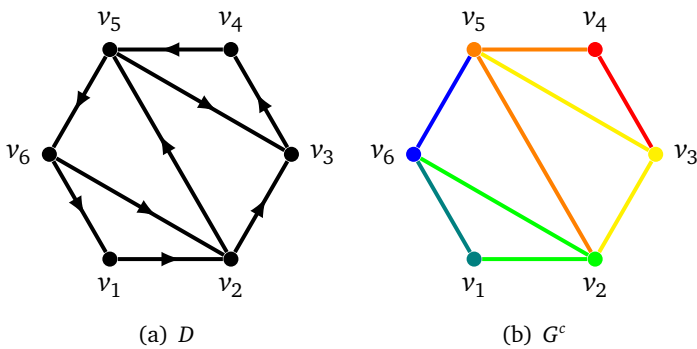


Figure 1.1: An example

This is why approaches for solving problems on undirected and directed

graphs can often be generalized to solve problems on edge-colored graphs. For example, any graph with n vertices and n edges contains a cycle, and similarly, any edge-colored graph with n vertices and n colors contains a rainbow cycle; if a directed cycle has a chord, then one can find a shorter directed cycle, and similarly, if a rainbow cycle has a chord, then one can find a shorter rainbow cycle. These statements are easy to check if one applies the definitions of cycle, directed cycle and rainbow cycle, which we postpone to the next section.

There are two typical classes of problems in the research on edge-colored graphs. One class is determining the minimum number of colors such that each pair of (adjacent) edges are assigned distinct colors in edge-colored graphs. A typical example is *anti-Ramsey theory*, determining the minimum number of colors such that each pair of edges are assigned distinct colors in an edge-colored complete graph. The other class is to study the existence of subgraphs with specific colorings in an edge-colored graph. Typical examples are *Ramsey theory*, dealing with the existence of monochromatic subgraphs in edge-colored graphs, and the *rainbow Turán problem*, dealing with the existence of rainbow subgraphs in properly edge-colored graphs. In this thesis, we concentrate on the second class and we would like to deal with the existence of properly colored cycles (i.e. with all adjacent edges colored differently) in edge-colored graphs.

Cycles play an important role in graph theory. Properly colored cycles have been applied in many fields, such as communication science [45], social science [30] and molecular biology [34]. For more results and applications on properly colored cycles, we refer the reader to two surveys [12, 50]. Here we focus on theoretical aspects.

The study of properly colored cycles (PC cycles for short) is closely related to directed cycles in directed graphs. On one hand, finding directed cycles can be seen as a special case of finding PC cycles. To see this, we reconsider the transformation introduced by Li [63] that we illustrated before with an example. It is not hard to see that there is a natural one-to-one correspondence between PC cycles in G^c and directed cycles in D . On the other hand, directed graphs also can be used as auxiliary tools to find PC cycles, as we will see

in later chapters. However, determining the existence of PC cycles in edge-colored graphs is more difficult than the existence of cycles in directed graphs. For example, every graph with minimum degree at least two contains a cycle and every digraph with minimum out-degree at least one contains a directed cycle. However, for any positive integer k , Wang and Li [88] constructed an edge-colored graph G^c with color degree more than k that contains no PC cycles. It is known that if a tournament contains a directed Hamilton cycle, then it contains directed cycles of all possible lengths. However, there exist edge-colored graphs containing a properly colored Hamilton cycle but no properly colored cycles of all possible lengths. To see this, consider a 2-edge-colored graph with a properly colored Hamilton cycle. Clearly, it contains no PC cycles of odd length.

Despite the tremendous progress that has been made on PC cycles in edge-colored graphs, there are still many problems and conjectures that remain open. In this thesis, we mainly focus on extending the existing theory related to this topic. After introducing some basic terminology and notations in Section 1.1, we will give exact statements of the research background and our contributions in Section 1.2.

1.1 Terminology and notations

All graphs considered in this thesis are simple and finite undirected graphs unless specified explicitly. For terminology and notations not defined here, we refer the reader to [21].

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and edges of G , respectively. For a vertex $v \in V(G)$, the *neighborhood* of v , denote by $N_G(v)$, is the set of all vertices adjacent to v ; the *degree* of v , denoted by $d_G(v)$, is the number of edges of G incident with v . Note that if G is a simple graph, then $d_G(v) = |N_G(v)|$. When there is no ambiguity, we often write $N(v)$ for $N_G(v)$ and $d(v)$ for $d_G(v)$. For nonempty subsets $X, Y \subseteq V(G)$, let $E(X)$ denote the set of edges with both end-vertices in X , and $E(X, Y)$ the set of edges with one end-vertex in X and the other end-vertex in Y , respectively; let $G[X]$ and $G[X, Y]$ denote the subgraph of G induced by vertex set X and

edge set $E(X, Y)$, respectively. Let C be a cycle, i.e., a (sub)graph consisting of distinct vertices v_1, v_2, \dots, v_k and edges $v_i v_{i+1}$ for $i = 1, 2, \dots, k-1$ and $v_k v_1$. We can consider a fixed direction on C , either oriented from v_1 to v_2 , etc., or oriented in the opposite direction. If we have fixed one of these directions, then for two vertices $u, v \in V(C)$, we use uCv to denote the segment between u and v along the direction of C , and use $u\bar{C}v$ to denote the segment between u and v along the opposite direction of C . In particular, if $u = v$, then $uCv = u$ and $u\bar{C}v = u$. A *chord* of a cycle C is an edge between two distinct vertices of C that are nonadjacent (not consecutive) on C . A cycle of length ℓ is called an ℓ -cycle.

An *edge-coloring* of G is defined as a mapping $C : E(G) \rightarrow \mathbb{N}$, where \mathbb{N} is the set of natural numbers. If G has such an edge-coloring, then G is an *edge-colored graph*. We say that a cycle C is *properly colored* (or simply *PC*) if all of its adjacent edges have distinct colors, and we say that a cycle C is *rainbow* if all of its edges have distinct colors. A *properly colored cycle-factor* of an edge-colored graph G is a spanning subgraph of G such that each component is a properly colored cycle, i.e., a set of vertex-disjoint properly colored cycles covering the vertex set. An edge-colored graph G is *properly colored (even) vertex-pancyclic* if every vertex of G is contained in properly colored cycles of all possible (even) lengths.

Let G be an edge-colored graph. We use $C(G)$ and $c(G)$ to denote the set and the number of colors appearing on the edges of G , respectively. If $c(G) = k$, then G is called *k-edge-colored*. For nonempty sets $X, Y \subseteq V(G)$, let $C(X)$ denote the set of colors appearing on the edges with both end-vertices in X , and $C(X, Y)$ denote the set of colors appearing on the edges with one end-vertex in X and the other vertex in Y . For simplicity, we use $C(uv)$ for $C(\{u\}, \{v\})$ and $C(u, Y)$ for $C(\{u\}, Y)$. For a vertex $v \in V(G)$, a color $i \in C(G)$ and a subgraph H of G , the *color degree of v to H* , denoted by $d_H^c(v)$, is defined as the number of colors appearing on the edges incident with v and with other end-vertices in $V(H)$; the *i-colored degree of v to H* , denoted by $d_H^i(v)$, is the number of edges of color i incident with v and with other end-vertices in $V(H)$; and the *maximum monochromatic degree of v to H* , denoted by $\Delta_H^{mon}(v)$, is the maximum number of edges of the same color incident with v and with other end-vertices in H . Note that it is possible that $v \in V(H)$.

Define $\delta^c(G) = \min\{d_G^c(v) : v \in V(G)\}$, $\Delta^c(G) = \max\{d_G^c(v) : v \in V(G)\}$ and $\Delta^{mon}(G) = \max\{d_G^i(v) : i \in C(G), v \in V(G)\}$. For a color $i \in C(G)$, denote by G^i the subgraph of G induced by all edges with color i . We say that color i is a *spanning and connected color* of G if $V(G^i) = V(G)$ and G^i is connected. Denote by $SC(G)$ the set of all spanning and connected colors of G .

We use K_n^c to denote an edge-colored complete graph with n vertices, G_n^c to denote an edge-colored graph with n vertices and $G_{n,n}^c$ to denote a balanced edge-colored bipartite graph with $2n$ vertices.

A *Gallai coloring* is an edge-coloring of a complete graph containing no PC triangles. Let K_n^c be an edge-colored complete graph and let V_1, V_2, \dots, V_q be q nonempty subsets of $V(K_n^c)$, where $q \geq 2$. We say that $\{V_1, V_2, \dots, V_q\}$ is a *partition* of $V(K_n^c)$ if $V(K_n^c) = \cup_{1 \leq i \leq q} V_i$ and $V_i \cap V_j = \emptyset$ for every $1 \leq i < j \leq q$. A partition $\{V_1, V_2, \dots, V_q\}$ of $V(K_n^c)$ is a *Gallai partition* if

$$|C(V_i, V_j)| = 1 \text{ for } 1 \leq i < j \leq q \text{ and } |\cup_{1 \leq i < j \leq q} C(V_i, V_j)| \leq 2.$$

We need to introduce some terminology and notation of directed graphs since they are often used as auxiliary tools in proofs. Let D be a directed graph. We use $V(D)$ and $A(D)$ to denote the sets of vertices and arcs of D , respectively. For a vertex $v \in V(D)$, denote by $N_D^+(v)$ the *out-neighborhood* of v , i.e., $N_D^+(v) = \{u \in V(D) : vu \in A(D)\}$; denote by $N_D^-(v)$ the *in-neighborhood* of v , i.e., $N_D^-(v) = \{u \in V(D) : uv \in A(D)\}$; denote by $d_D^+(v)$ the *out-degree* of v , i.e., $d_D^+(v) = |\{u \in V(D) : vu \in A(D)\}|$; and denote by $d_D^-(v)$ the *in-degree* of v , i.e., $d_D^-(v) = |\{u \in V(D) : uv \in A(D)\}|$. For nonempty sets $X, Y \subseteq V(D)$, we use $A(X, Y)$ to denote the set of arcs from X to Y ; if X and Y are vertex-disjoint and there are no arcs from Y to X , and $xy \in A(D)$ for all $x \in X$ and $y \in Y$, then we say X completely dominates Y and denote this by $X \Rightarrow Y$. For simplicity, we use $x \Rightarrow Y$ for $\{x\} \Rightarrow Y$.

1.2 PC cycles

We start with the following problem.

Problem 1.1. Given an edge-colored graph G^c , check whether G^c contains a PC cycle.

Yeo [90] in 1997 characterized edge-colored graphs containing no PC cycles.

Theorem 1.1 (Yeo [90]). *If G^c contains no PC cycles, then G^c contains a vertex v such that no components of $G^c - v$ are joined to v with edges of more than one color.*

Note that one can delete such vertices one by one without destroying any PC cycle. It is easy to see that Problem 1.1 is polynomially solvable.

Fujita et al. [41] in 2018 obtained a sharp color degree condition for the existence of PC cycles.

Theorem 1.2 (Fujita et al. [41]). *Let k, n be two positive integers. If $n < k! \sum_{i=0}^k 1/i!$ and $\delta^c(G_n^c) \geq k$, then G_n^c contains a PC cycle.*

The problem of giving sufficient conditions for the existence of specific PC cycles seems more difficult. It was studied extensively by various researchers in the last decades.

1.2.1 Short PC cycles

Firstly we focus on short PC cycles in general edge-colored graphs. We start by showing the Caccetta-Häggkvist Conjecture [24], which is one of the best known in graph theory.

Conjecture 1.1 (Caccetta and Häggkvist [24]). *For every positive integer r , every digraph on n vertices with minimum outdegree at least n/r has a directed cycle of length at most r .*

This conjecture is trivial for $r \leq 2$ but for $r \geq 3$ it remains open. Seymour and Spirk [82] in 2020 considered a bipartite version of the Caccetta-Häggkvist Conjecture. See [85] for more partial results. Aharoni et al. [3] in 2019 considered a generalized version of the Caccetta-Häggkvist Conjecture, and obtained some results for the existence of rainbow triangles.

Conjecture 1.2 (Aharoni et al. [3]). Let r be a positive integer. If G_n^c has n color classes, each of which has size at least n/r , then G_n^c contains a rainbow cycle of length at most r .

H. Li [63] in 2013 showed that every edge-colored graph of order n with minimum color degree at least $(n+1)/2$ contains a rainbow triangle. B. Li et al. [62] in 2014 independently obtained this result by proving a stronger form, and characterized the extremal graphs.

Theorem 1.3 (B. Li et al. [62]). Every G_n^c with $\sum_{v \in V(G_n^c)} d^c(v) \geq n(n+1)/2$ contains a rainbow triangle.

Theorem 1.4 (B. Li et al. [62]). If $\delta^c(G_n^c) \geq n/2$ and G_n^c contains no rainbow triangles, then n is even and G_n^c is a PC complete bipartite graph $K_{n/2, n/2}$, unless G_n^c is a PC $K_4 - e$ when $n = 4$.

R. Li et al. [70] in 2016 gave a color degree sum condition for adjacent vertices that guarantees the existence of rainbow triangles, and characterized the extremal graphs. H. Li [63] in 2013 gave a minimum color degree condition for the existence of rainbow 4-cycles in edge-colored balanced bipartite graphs. Ning and Ge [79] in 2016 generalized Li's result about rainbow 4-cycles to edge-colored unbalanced bipartite graphs. Čada et al. [25] in 2016 gave a minimum color degree condition for the existence of rainbow 4-cycles in triangle-free edge-colored graphs. Ding et al. [33] in 2022 gave an asymptotically sharp color degree condition for the existence of rainbow 4-cycles in edge-colored graphs.

Next we focus on short PC cycles in edge-colored (bipartite) complete graphs. Gallai [46] in 1967 characterized edge-colored complete graphs containing no PC triangles.

Theorem 1.5 (Gallai [46]). In every Gallai coloring of a complete graph, there exists a Gallai partition.

Gyárfás and Simonyi [53] in 2004 gave a maximum monochromatic degree condition ($\Delta^{mon}(K_n^c) < 2n/5$) for rainbow triangles in edge-colored complete graphs. Fujita et al. [41] in 2018 gave a minimum color degree condition ($\delta^c(K_n^c) > \log_2 n$) for rainbow triangles in edge-colored complete

graphs. Axenovich et al. [6] in 2003 gave a minimum color degree condition ($\delta^c(K_n^c) \geq 3$) for PC 4-cycles in edge-colored complete graphs. In Chapter 3, we give a maximum monochromatic degree condition ($\Delta^{mon}(K_n^c) < 2n/3 - 1$) for PC 4-cycles in edge-colored complete graphs and characterize the extremal graphs. We also characterize all edge-colored complete graphs K_n^c 's containing no rainbow triangles but satisfying $\delta^c(K_n^c) = \log_2 n$ and $\Delta^{mon}(K_n^c) = 2n/5$, respectively. In addition, we give maximum monochromatic degree conditions for an edge-colored complete graph guaranteeing that every vertex is contained in a rainbow triangle and a PC 4-cycle, respectively. Furthermore, we consider total monochromatic degree conditions for an edge-colored complete graph guaranteeing that every vertex is contained in a rainbow triangle and a PC 4-cycle, respectively.

Fujita et al. [41] in 2018 and Čada et al. [26] in 2020 independently characterized edge-colored bipartite complete graphs containing no PC 4-cycles. Based on this characterization, they gave a minimum color degree condition and a maximum monochromatic degree condition for PC 4-cycles in edge-colored complete bipartite graphs. They also gave a minimum color degree condition and a maximum monochromatic degree condition for an edge-colored complete graph guaranteeing that every vertex is contained in a PC 4-cycle, respectively.

Finally, we focus on PC even cycles and PC odd cycles in edge-colored complete graphs. R. Li et al. [67] characterized edge-colored complete graphs containing no PC even cycles, which is equivalent to characterizing edge-colored bipartite complete graphs containing no PC 4-cycles and PC 6-cycles. In Chapter 2, we characterize edge-colored complete graphs containing no PC odd cycles, which is equivalent to characterizing edge-colored complete graphs containing no PC triangles and PC 5-cycles. This implies that an edge-colored complete graph contains a PC cycle of length $\ell \equiv k \pmod{2}$, where $k \in \{1, 2\}$, if and only if it contains a PC cycle of length $k + 2$ or $k + 4$. We also propose the following problem, which is a generalization of this result and give a solution for the case that $k - 1$ and m are relatively prime, where $m \geq k \geq 3$.

Problem 1.2. Let k, m be two positive integers with $m \geq k \geq 3$. Does there

exist an integer $c = c(m, k)$ such that each edge-colored complete graph containing a PC cycle of length $\ell \equiv k \pmod{m}$ has a PC cycle of length $\ell' \in \{k, k + m, \dots, k + cm\}$?

1.2.2 Long PC cycles

For general edge-colored graphs, Lo [73] in 2014 conjectured that every edge-colored graph G_n^c with $\delta^c(G_n^c) \geq 2n/3$ contains a PC Hamilton cycle. In [74], Lo solved this conjecture asymptotically.

Theorem 1.6 (Lo [74]). *For any $\epsilon > 0$, there exists an integer $n_0 = n_0(\epsilon)$ such that every G_n^c with $n \geq n_0$ and $\delta^c(G_n^c) \geq (2/3 + \epsilon)n$ is pancyclic.*

For edge-colored complete graphs, Bollobás and Erdős [20] in 1976 proposed the following well-known conjecture.

Conjecture 1.3 (Bollobás and Erdős [20]). *If $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$, then K_n^c contains a PC Hamilton cycle.*

In the same paper, they proved a weaker version, that is, if $\Delta^{mon}(K_n^c) < \alpha n$, then K_n^c contains a PC Hamilton cycle, where $\alpha = 1/69$. Chen and Daykin [27] in 1976 improved this bound to $\alpha = 1/17$. Shearer [83] in 1979 improved this bound to $\alpha = 1/7$. Alon and Gutin [5] in 1997 improved the bound to $\alpha = 1 - 1/\sqrt{2} - \epsilon$ for sufficient large n , where ϵ is an arbitrary real number. Lo [75] in 2016 solved this conjecture asymptotically.

Theorem 1.7 (Lo [75]). *For any $\epsilon > 0$, there exists an integer $n_0 = n_0(\epsilon)$ such that every K_n^c with $\Delta^{mon}(K_n^c) \leq (1/2 - \epsilon)n$ and $n \geq n_0$ contains a PC Hamilton cycle.*

Abouelaoualim et al. [1] in 2010 proposed the following conjecture, which is a weaker version of Bollobás and Erdős's conjecture for regular edge-colored complete graphs.

Conjecture 1.4 (Abouelaoualim et al. [1]). *Every k -edge-colored complete regular graph with $k \geq 3$, has a PC Hamilton cycle.*

Abouelaoualim et al. [1] also obtained a colored degree condition for PC pancyclicity in edge-colored multigraphs.

Theorem 1.8 (Abouelaoualim et al. [1]). *Let G_n^c be a k -edge-colored multi-graph, $k \geq 2$. Assume that for each $x \in V(G_n^c)$, $d^i(x) \geq \lceil (n+1)/2 \rceil$ for each color $i \in \{1, 2, \dots, k\}$.*

(i) *If $k = 2$, then G_n^c is PC even-pancyclic.*

(ii) *If $k \geq 3$, then G_n^c is PC pancyclic.*

Fujita and Magnant [42] in 2011 proposed the following conjecture.

Conjecture 1.5 (Fujita and Magnant [42]). *If $\delta^c(K_n^c) \geq (n+1)/2$, then K_n^c is PC vertex-pancyclic.*

R. Li et al. [66] in 2017 showed that each vertex of K_n^c is contained in PC cycles of length at least $\delta^c(K_n^c)$. Chen et al. [28] in 2019 and L. Li et al. [64] in 2022 confirmed this conjecture when no monochromatic triangles and monochromatic paths of length three exist, respectively. R. Li [65] in 2021 showed that an edge-colored complete graph containing no monochromatic triangles is PC vertex-pancyclic, unless it belongs to two special classes of edge-colored graphs.

In Chapter 6, we obtain a necessary and sufficient condition for edge-colored complete bipartite graphs containing no monochromatic paths of length three to be PC even vertex-pancyclic. In particular, an edge-colored complete bipartite graph containing no monochromatic paths of length three is PC even vertex-pancyclic unless it belongs to two special classes of edge-colored graphs. This result can be seen as a generalization of the following result of Häggkvist and Manoussakis on bipartite tournaments. We postpone the explanation of the generalization in Chapter 6.

Theorem 1.9 (Häggkvist and Manoussakis [54]). *If a bipartite tournament has a Hamilton cycle, then either it is even vertex-pancyclic or it belongs to one special class of digraphs.*

1.2.3 Vertex-disjoint PC cycles

Let k be a positive integer. Denote by $f(k)$ the minimum integer such that every digraph with minimum outdegree at least $f(k)$ contains k vertex-disjoint cycles. In view of the complete symmetric digraph on $2k - 1$ vertices, we

have $f(k) \geq 2k - 1$. In 1981, Bermond and Thomassen [18] proposed the following well known conjecture.

Conjecture 1.6 (Bermond and Thomassen [18]). $f(k) = 2k - 1$.

It is trivial for $k = 1$. Thomassen [86] in 1983 showed that $f(2) = 3$. Lichiardopol, Pór and Sereni [71] showed that $f(3) = 5$. Alon [4] in 1996 showed that $f(k) \leq 64k$. Bucić [23] in 2018 improved on this bound to show $f(k) \leq 18k$. For more related work, see [7, 11, 19] for examples.

For general edge-colored graphs, Hu et al. [57] in 2020 proposed the following conjecture and almost solved the case $k = 2$.

Conjecture 1.7 (Hu et al. [57]). For all positive integers n and k with $n \geq 3k$, if $\delta^c(G_n^c) \geq (n + k)/2$, then G_n^c contains k vertex-disjoint rainbow triangles.

For edge-colored complete graphs, Li et al. [68] in 2020 proposed the following conjecture and proved it for the case $k = 2$.

Conjecture 1.8 (Li et al. [68]). If $\Delta^{mon}(K_n^c) \leq n - 3k + 1$, then K_n^c contains k vertex-disjoint PC cycles.

Motivated by these results and conjectures, in Chapter 4, we consider maximum monochromatic degree conditions for (vertex-disjoint) PC cycles of different lengths in edge-colored complete graphs. We obtain a sharp maximum monochromatic degree condition for k PC cycles of different lengths. We also obtain a maximum monochromatic degree condition for k vertex-disjoint PC cycles of different lengths and give a optimal solution for the case $k = 2$.

1.2.4 PC cycle-factors

For ease of explanation, we give the following two definitions.

Definition 1.1. A pair of vertices x, y of an edge-colored multigraph G^c is called color-connected if there exist PC paths $P = xx' \cdots y'y$ and $P' = xu' \cdots v'y$ such that $C(xx') \neq C(xu')$ and $C(y'y) \neq C(v'y)$. We say that G^c is color-connected if every pair of vertices of G^c is color-connected.

Definition 1.2. Let G^c be a 2-edge-colored multigraph and let $P = x_1x_2x_3x_4$ be a PC path of length three. We say that P is closed-alternating if there exist $y, w \in V(G^c)$ such that $x_1yx_4x_1$ is a PC cycle. We say that G^c is closed-alternating if each PC path of length three is closed-alternating.

We firstly present some results on edge-colored multigraphs. Bang-Jensen and Gutin [12] in 1997 proved that one can construct a maximum cycle subgraph and a maximum 1-path-cycle subgraph in an edge-colored multigraph with n vertices in time $O(n^3)$ respectively. Saad [81] in 1996 proved that a 2-edge-colored complete multigraph contains a PC Hamilton cycle if and only if it is color-connected and has a PC cycle-factor. Bang-Jensen and Gutin [13] in 1998 proved that a 2-edge-colored complete bipartite multigraph contains a PC Hamilton cycle if and only if it is color-connected and has a PC cycle-factor. Contreras-Balbuena et al. [31] in 2017 proved that for a 2-edge-colored multigraph G^c , if G^c is closed-alternating and it has a PC cycle-factor, then G^c has a PC Hamilton cycle.

Next we present some results on simple edge-colored graphs. Bang-Jensen et al. [15] in 1998 proved that if an edge-colored complete graph contains a PC cycle-factor then it contains a PC Hamilton path. Feng et al. [38] in 2006 proved that an edge-colored complete graph contains a PC Hamilton path if and only if it contains a spanning PC 1-path-cycle factor, i.e., a spanning subgraph which is the disjoint union of one PC path and a collection of PC cycles. Lo [76] in 2014 gave the following color degree condition for PC cycle-factors in edge-colored graphs.

Theorem 1.10 (Lo [76]). *If $\delta^c(G_n^c) \geq 2n/3$, then G_n^c contains a PC cycle-factor.*

Guo et al. [49] in 2022 gave a minimum color degree condition for PC cycle-factors in edge-colored complete bipartite graphs.

Theorem 1.11 (Guo et al. [49]). *If $\delta^c(K_{n,n}^c) > 3n/4$, then $K_{n,n}^c$ contains a PC cycle-factor.*

In Chapter 5, we give a minimum color degree condition for PC cycle-factors in general edge-colored bipartite graphs, which is essentially sharp.

The remainder of this thesis consists of the five technical chapters, followed by a summary, a Dutch summary, the bibliography, acknowledgements, and

some information about the author. In the chapters that follow, we state our contributions to this field explicitly and prove them.

Chapter 2

Edge-colored complete graphs containing no PC odd cycles

In this chapter, we characterize edge-colored complete graphs containing no PC odd cycles and give an efficient algorithm with complexity $O(n^3)$ for deciding the existence of PC odd cycles in an edge-colored complete graph of order n . Moreover, we show that for two integers k, m with $m \geq k \geq 3$, where $k - 1$ and m are relatively prime, an edge-colored complete graph contains a PC cycle of length $\ell \equiv k \pmod{m}$ if and only if it contains a PC cycle of length $\ell' \equiv k \pmod{m}$, where $\ell' < 2m^2(k - 1) + 3m$.

2.1 Introduction

In 1967, Gallai proved the following classical theorem.

Theorem 2.1 (Gallai [46]). *In every Gallai coloring of a complete graph, there exists a Gallai partition.*

This theorem has naturally led to a research on edge-colored complete graphs free of fixed subgraphs other than rainbow triangles(see [39,43]), and has also been generalized to noncomplete graphs [52] and hypergraphs [77]. Yeo [90] in 1997 proved that every edge-colored graph G^c containing no

PC cycles contains a vertex z such that each component of $G^c - z$ is joined to z with edges of more than one color. This implies that the vertices of an edge-colored complete graph containing no PC cycles can be ordered in such a way that the color of every edge is determined by its end-vertex of smaller index. Li et al. [63] in 2020 characterized edge-colored complete graphs containing no PC even cycles. For an edge-colored complete bipartite graph G^c containing no PC 4-cycles, a local coloring of G^c is given by Fujita et al. [41] in 2018, and recently a full characterization of G^c is obtained by Čada et al. [26] in 2019. Hoffman et al. [56] in 2019 characterized edge-colored graphs containing no rainbow cycles in which the maximum number of colors appears.

It is well known that a graph is bipartite if and only if it contains no odd cycles. One may naturally ask whether an analogous result holds in edge-colored complete graphs. In this Chapter, we shall characterize edge-colored complete graphs containing no PC odd cycles.

For convenience, we introduce a definition of degenerate sets given by Li et al. [69]. Let G be an edge-colored complete graph, if there exists a nonempty set $S \subset V(G)$ such that $C(S) \subseteq C(S, V(G) \setminus S)$ and $|C(S, V(G) \setminus S)| = 1$, then we say that S is a 1-degenerate set of G ; if there exists a nonempty set $S \subset V(G)$ with $S = X \cup Y$ such that $C(X) \subseteq C(X, V(G) \setminus S) = \{c_1\}$, $C(Y) \subseteq C(Y, V(G) \setminus S) = \{c_2\}$ and $C(X, Y) = \{c_1, c_2\}$, then we say that S is a 2-degenerate set of G . For an edge-colored complete graph which has a 1-degenerate or 2-degenerate set, we have the following observation.

Observation 2.1. *Let G be an edge-colored complete graph. If G has a 1-degenerate or 2-degenerate set S , then each PC odd cycle of G is contained in $G - S$.*

Proof. If S is a 1-degenerate set, then the conclusion trivially holds. If S is a 2-degenerate set with $S = X \cup Y$ such that $C(X) \subseteq C(X, V(G) \setminus S) = \{c_1\}$, $C(Y) \subseteq C(Y, V(G) \setminus S) = \{c_2\}$ and $C(X \cup Y) = \{c_1, c_2\}$. Let $U = V(G) \setminus S$. Note that $G[S]$ contains no PC odd cycles. So it suffices to show that there exist no PC cycles C such that $V(C) \cap (X \cup Y) \neq \emptyset$ and $V(C) \cap U \neq \emptyset$. Suppose to the contrary that there exists a PC cycle C such that $V(C) \cap (X \cup Y) \neq \emptyset$ and $V(C) \cap U \neq \emptyset$. Let $P = uv_1v_2 \cdots v_mw$ be a segment of C with $u, w \in U$ and

$v_i \in X \cup Y$ for each $i \in \{1, 2, \dots, m\}$ (possibly $u = w$). By symmetry between X and Y , assume that $v_1 \in X$. If $m = 1$, then we have $C(uv_1) = C(wv_1) = \{c_1\}$. If $m \geq 2$, then $C(uv_1) = \{c_1\}$, $C(v_i v_{i+1}) = \{c_2\}$ for odd i and $C(v_i v_{i+1}) = \{c_1\}$ for even i . Recall that $C(X) = \{c_1\}$, $C(Y) = \{c_2\}$ and $C(X, Y) \subseteq \{c_1, c_2\}$. So we have $v_i \in X$ for odd i and $v_i \in Y$ for even i . If m is odd, then $C(wv_m) = C(v_{m-1}v_m) = \{c_1\}$. If m is even, then $C(wv_m) = C(v_{m-1}v_m) = \{c_2\}$. In both cases, the vertex v_m is incident to two edges of C with a same color, a contradiction. \square

Now we define three typical classes of edge-colored complete graphs. Let \mathcal{G}_1 be the set of all edge-colored complete graphs satisfying that $G \in \mathcal{G}_1$ if one color forms a spanning subgraph and other colors span pairwise vertex-disjoint subgraphs (possibly empty). Let \mathcal{G}_2 be the set of all edge-colored complete graphs satisfying that $G \in \mathcal{G}_2$ if there exists a 1-degenerate set S and $G - S$ contains no PC odd cycles (as shown in Figure 2.1 (a)). Let \mathcal{G}_3 be the set of all edge-colored complete graphs satisfying that $G \in \mathcal{G}_3$ if there exists a 2-degenerate set S and $G - S$ contains no PC odd cycles (as shown in Figure 2.1 (b)).

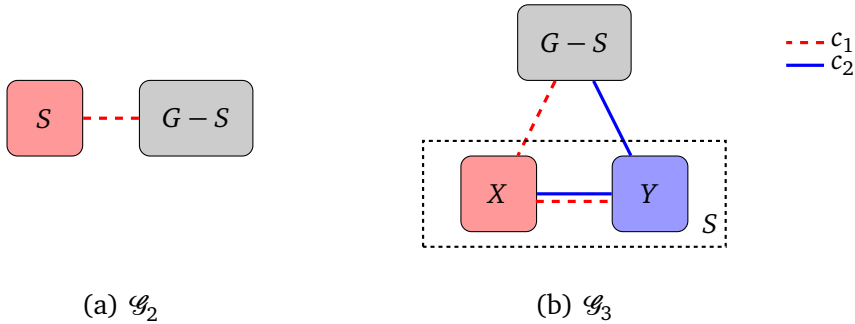


Figure 2.1: Two typical classes of edge-colored complete graphs containing no PC odd cycles

It is no hard to check that every edge-colored complete graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ contains no PC odd cycles. Indeed, every edge-colored complete graph containing no PC odd cycles is contained in $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Theorem 2.2. *Let G be an edge-colored complete graph. Then the following three statements are equivalent.*

- (i) G contains no PC odd cycles.
- (ii) G contains no PC triangles or PC 5-cycles.
- (iii) $G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

We postpone the proof of this theorem to Section 2.3. According to this characterization, we obtain the following immediate corollary.

Corollary 2.1. *Let G be an edge-colored complete graph. If $\delta^c(G) \geq 3$, then G contains a PC odd cycle.*

This constant 3 is best possible in view of every 2-edge-colored complete graph G with $\delta^c(G) = 2$ containing no PC odd cycles.

Gutin et al. [51] in 2017 studied the problem of deciding the existence of a PC odd cycle in an edge-colored graph. Until now, the existence of a deterministic polynomial time algorithm for a PC odd cycle in an edge-colored graph is still an open question. But for edge-colored complete graphs, we can find a polynomial time algorithm.

Theorem 2.3. *For each edge-colored complete graph on n vertices, there exists an algorithm to decide the existence of PC odd cycles in time $O(n^3)$.*

We postpone the proof of this theorem to Section 2.3. Moreover, Li et al. [67] proved that an edge-colored complete graph contains a PC even cycle if and only if it contains a PC cycle of length 4 or 6. Combining this with Theorem 2.2, we have that an edge-colored complete graph contains a PC cycle of length $\ell \equiv k \pmod{2}$, where $k \in \{1, 2\}$, if and only if it contains a PC cycle of length $k+2$ or $k+4$. It would be interesting to consider the following problem which is a generalization of this result.

Problem 2.1. Let k, m be two positive integers with $m \geq k \geq 3$. Does there exist an integer $c(m, k)$ such that each edge-colored complete graph G containing a PC cycle of length $\ell \equiv k \pmod{m}$ has a PC cycle of length $\ell' \in \{k, k+m, \dots, k+cm\}$?

Here we give a solution to the case that $k-1$ and m are relatively prime, where $m \geq k \geq 3$.

Theorem 2.4. *Let k, m be two integers with $m \geq k \geq 3$, where $k - 1$ and m are relatively prime. An edge-colored complete graph contains a PC cycle of length $\ell \equiv k \pmod{m}$ if and only if it contains a PC cycle of length $\ell' \equiv k \pmod{m}$, where $\ell' < 2m^2(k - 1) + 3m$.*

We postpone the proof of this theorem to Section 2.4. In next section, we present some preliminaries.

2.2 Three lemmas

In the following, we present three lemmas which will be frequently used in the proofs of Theorems 2.2 and 2.3.

Lemma 2.1. *Let G be an edge-colored complete graph with $\delta^c(G) = \Delta^c(G) = 2$. If G contains no PC triangles or PC 5-cycles, then $G \in \mathcal{G}_1$.*

Proof. By Theorem 2.1, G has a Gallai partition U_1, U_2, \dots, U_p , where $p \geq 2$. Assume that $\cup_{1 \leq i < j \leq p} C(U_i, U_j) \subseteq \{c_1, c_2\}$. If G contains no monochromatic edge-cuts, then $C(U_i, V(G) \setminus U_i) = \{c_1, c_2\}$ for each $i \in \{1, 2, \dots, p\}$. This implies that $\{c_1, c_2\} \subseteq C(v, V(G))$ for each $v \in V(G)$. It follows from $\Delta^c(G) = 2$ that G is 2-colored, which implies that $G \in \mathcal{G}_1$. If G contains a monochromatic edge-cut, then G has a partition $\{U_1, U_2\}$ with $|C(U_1, U_2)| = 1$. Assume without loss of generality that $C(U_1, U_2) = \{c_1\}$. Note that the color c_1 appears at every vertex in G . Since $d_G^c(v) = 2$ for each vertex $v \in V(G)$, all colors different from c_1 span pairwise vertex-disjoint subgraphs. This implies $G \in \mathcal{G}_1$. \square

Lemma 2.2. *Let G be an edge-colored complete graph with $\delta^c(G) \geq 2$ and $\Delta^c(G) \geq 3$. Let $U = \{u \in V(G) : d_G^c(u) \geq 3\}$ and $W = V(G) \setminus U$. If G contains no PC triangles or PC 5-cycles, then the following statements hold.*

- (i) $\delta^c(G) = 2$;
- (ii) $|C(w, U)| = 1$ for each $w \in W$;
- (iii) $|C(W, U)| = 2$.

Proof. (i) Suppose to the contrary that $\delta^c(G) \geq 3$. If G contains a monochromatic edge-cut, then G has a partition $\{U_1, U_2\}$ with $|C(U_1, U_2)| = 1$. Let

$C(U_1, U_2) = \{c_1\}$. Since $d_G^c(v) \geq 3$ for each $v \in V_1$, there exist two adjacent edges, say u_1v_1 and v_1w_1 , in $E(U_1)$ such that $C(u_1v_1) \neq C(v_1w_1)$, $C(u_1v_1) \neq \{c_1\}$ and $C(v_1w_1) \neq \{c_1\}$. Since $d_G^c(v) \geq 3$ for each $v \in V_2$, there exists an edge $u_2v_2 \in E(U_2)$ such that $C(u_2v_2) \neq \{c_1\}$. Now $u_1v_1w_1u_2v_2u_1$ is a PC 5-cycle, a contradiction. If G contains no monochromatic edge-cuts, then since G contains no PC triangles, G has a Gallai partition U_1, U_2, \dots, U_p such that $C(U_i, V(G) \setminus U_i) = \{c_1, c_2\}$ for each $i \in \{1, 2, \dots, p\}$. It is clear that $p \geq 3$. Assume without loss of generality that $C(U_1, U_2) = \{c_1\}$ and $C(U_1, U_3) = \{c_2\}$. By symmetry between c_1 and c_2 , we can assume that $C(U_2, U_3) = \{c_2\}$. Since $d_G^c(v) \geq 3$ for each $v \in V(G)$, there exist edges $u_2u'_2 \in E(U_2)$ and $u_3u'_3 \in E(U_3)$ such that $C(u_2u'_2) \notin \{c_1, c_2\}$ and $C(u_3u'_3) \notin \{c_1, c_2\}$. Let $u_1 \in U_1$. Then $u_1u_2u'_2u_3u'_3u_1$ is a PC 5-cycle, a contradiction.

(ii) Suppose to the contrary that there exists a vertex w in W such that $|C(w, U)| \neq 1$. Assume $C(w, U) = \{c_1, c_2\}$. Let $U_1 = \{u \in U : C(uw) = c_1\}$ and $U_2 = \{u \in U : C(uw) = c_2\}$. If there exists an edge $u_1u_2 \in E(U_1, U_2)$ such that $C(u_1u_2) \notin \{c_1, c_2\}$, then wu_1u_2w is a PC triangle, a contradiction. So $C(U_1, U_2) \subseteq \{c_1, c_2\}$. Since $d_G^c(u_1) \geq 3$ and $d_G^c(u_2) \geq 3$, there exist vertices $u'_1 \in U_1, u'_2 \in U_2$ such that $C(u_1u'_1) \notin \{c_1, c_2\}$ and $C(u_2u'_2) \notin \{c_1, c_2\}$. This implies that $wu_1u'_1u'_2u_2w$ is a PC 5-cycle, a contradiction.

(iii) Suppose to the contrary that $|C(W, U)| \neq 2$. If $|C(W, U)| = 1$, then we can assume $C(W, U) = \{c_1\}$. Since $d_G^c(u) \geq 3$ for each $u \in U$, there exist two adjacent edges, say u_1u_2 and u_2u_3 , in $E(U)$ such that $C(u_1u_2) \neq C(u_2u_3)$, $C(u_1u_2) \neq c_1$ and $C(u_2u_3) \neq c_1$. Since $\delta^c(G) \geq 2$, there exists an edge $w_1w_2 \in E(W)$ such that $C(w_1w_2) \neq c_1$. Now $u_1u_2u_3w_1w_2u_1$ is a PC 5-cycle, a contradiction. If $|C(W, U)| \geq 3$, then there exist three distinct vertices w_1, w_2, w_3 such that $C(w_i, U) = \{c_i\}$ for each $i \in \{1, 2, 3\}$. Let u be a vertex in U . Note that $C(w_iw_j) \in \{c_i, c_j\}$ for every pair i, j with $1 \leq i < j \leq 3$ since otherwise uw_iw_ju is a PC triangle. Without loss of generality, assume that $C(w_1w_2) = c_1$. Since $d_G^c(w_2) = 2$, we have $C(w_2w_3) = c_2$. Since $d_G^c(w_3) = 2$, we have $C(w_1w_3) = c_3$. This implies that $w_1w_2w_3w_1$ is a PC triangle, a contradiction. \square

Lemma 2.3. *Let G be an edge-colored complete graph with $\delta^c(G) = 2$ and $\Delta^c(G) \geq 3$. Suppose that G has a partition $\{X, Y, U\}$ such that $U = \{u \in V(G) :$*

$d_G^c(u) \geq 3$, $C(X, U) = \{c_1\}$ and $C(Y, U) = \{c_2\}$. Let $X_1 = \{x \in X : C(x, X) \not\subseteq \{c_1\}\}$ and $Y_1 = \{y \in Y : C(y, Y) \not\subseteq \{c_2\}\}$. If G contains no PC triangles or PC 5-cycles, then the following statements hold.

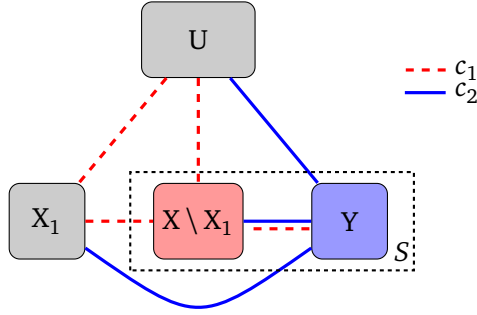
- (i) $C(X \cup Y) \subseteq \{c_1, c_2\}$;
- (ii) either $X_1 = \emptyset$ or $Y_1 = \emptyset$;
- (iii) $C(X_1, Y) \subseteq \{c_2\}$ and $C(Y_1, X) \subseteq \{c_1\}$;
- (iv) $(X \setminus X_1) \cup (Y \setminus Y_1)$ is a 2-degenerate set.

Proof. (i) Suppose to the contrary that $C(X, Y) \not\subseteq \{c_1, c_2\}$, then there exists an edge $v_1 v_2 \in E(X \cup Y)$ such that $C(v_1 v_2) \not\subseteq \{c_1, c_2\}$. Let $u \in U$. If $v_1 v_2 \in E(X, Y)$, then $uv_1 v_2 u$ is a PC triangle, a contradiction. By symmetry between X and Y , we can assume $v_1 v_2 \in E(X)$. Let $u_1 u_2 \in E(U)$ such that $C(u_1 u_2) \not\subseteq \{c_1, c_2\}$ and $y \in Y$. Since $v_1 v_2 u_1 u_2 y v_1$ is not PC 5-cycles, we have $C(v_1 y) = c_2$, which implies that $d_G^c(v_1) = 3$. By the definition of X , we have that $d_G^c(v_1) = 2$, a contradiction.

(ii) Suppose to the contrary that $X_1 \neq \emptyset$ and $Y_1 \neq \emptyset$, then there exists an edge $x_1 x_2 \in E(X_1)$ and $y_1 y_2 \in E(Y_1)$ such that $C(x_1 x_2) \neq c_1$ and $C(y_1 y_2) \neq c_2$. By the definition of U , there exists an edge $u_1 u_2 \in E(U)$ such that $C(u_1 u_2) \not\subseteq \{c_1, c_2\}$. Recall that $C(x_1 y_1) \in \{c_1, c_2\}$. So either $u_1 u_2 x_2 x_1 y_1 u_1$ or $u_1 u_2 x_1 y_1 y_2 u_1$ is a PC 5-cycles, a contradiction.

(iii) By symmetry between X and Y , we only need to show that $C(X_1, Y) \subseteq \{c_2\}$ when $X_1 \neq \emptyset$. Recall that $C(X, Y) \subseteq \{c_1, c_2\}$. If $C(X_1, Y) \not\subseteq \{c_2\}$, then there exists an edge xy with $x \in X_1$ and $y \in Y$ such that $C(xy) = c_1$. By the definition of X_1 and U , there exists a vertex $x_1 \in X_1$ such that $C(xx_1) \neq c_1$ and an edge $u_1 u_2 \in E(U)$ such that $C(u_1 u_2) \not\subseteq \{c_1, c_2\}$. This implies that $x_1 x y u_1 u_2 x_1$ is a PC 5-cycles, a contradiction.

(iv) According to (ii), assume without loss of generality that $Y_1 = \emptyset$. By the definition of X_1 , it is clear that $C(X_1, X \setminus X_1) = \{c_1\}$. It follows from (i), (ii) and (iii) that the coloring of G must be as shown in Figure 2.2. Clearly, $(X \setminus X_1) \cup (Y \setminus Y_1)$ is a 2-degenerate set. \square


 Figure 2.2: The coloring of G in Lemma 2.3.

2.3 PC odd cycles

Proof of Theorem 2.2. We show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: It is trivial.

$(ii) \Rightarrow (iii)$: If $\delta^c(G) \geq 2$, then it follows from Lemma 2.2 that $\delta^c(G) = 2$. If $\delta^c(G) = \Delta^c(G) = 2$, then it follows from Lemma 2.1 that $G \in \mathcal{G}_1$, we are done. If $\delta^c(G) = 2$ and $\Delta^c(G) \geq 3$, then it follows from Lemmas 2.2 and 2.3 that G has a 2-degenerate set S and $G - S$ contains no PC triangles or PC 5-cycles. If $\delta^c(G) = 1$, then G has a 1-degenerate set S and $G - S$ contains no PC triangles or PC 5-cycles. In both cases, let $G' = G - S$. By Observation 2.1 and definitions of \mathcal{G}_1 , \mathcal{G}_2 and \mathcal{G}_3 , it suffices to show that G' contains no PC odd cycles. Now repeat the above arguments for G' similar to G and continue recursively. Note that the procedure will stop within a finite number of steps and the remaining graph G^* satisfying that $|V(G^*)| \leq 2$ or $G^* \in \mathcal{G}_1$, and since G^* contains no PC odd cycles, neither does G' . Now S is a 1-degenerate or 2-degenerate set and G' contains no PC odd cycles, which implies that $G \in \mathcal{G}_2 \cup \mathcal{G}_3$.

$(iii) \Rightarrow (i)$: If $G \in \mathcal{G}_1$, then it is not hard to check that G contains no PC odd cycles. If $G \in \mathcal{G}_2 \cup \mathcal{G}_3$, then G has a 1-degenerate or 2-degenerate set S . By Observation 2.1 and the definitions of \mathcal{G}_2 and \mathcal{G}_3 , G contains no PC odd cycles. \square

Proof of Theorem 2.3. Firstly we design Algorithm 1 for deciding the existence of PC odd cycles in edge-colored complete graphs. The basic idea is as follows. Let G be a given edge-colored complete graph. If $\delta^c(G) = 1$, then it is clear that G has a 1-degenerate set. We can find out the 1-degenerate set S of G and let $G' = G - S$. If $\delta^c(G) \geq 3$, then it follows from Corollary 2.1 that G contains a PC odd cycle. If $\delta^c(G) = \Delta^c(G) = 2$, then we can check that whether $G \in \mathcal{G}_1$. If $G \in \mathcal{G}_1$, then it is clear that G contains no PC odd cycles. If $G \notin \mathcal{G}_1$, then it follows from Lemma 2.1 that G contains a PC odd cycle. If $\delta^c(G) = 2$ and $\Delta^c(G) \geq 3$, then we can check that whether G contains a 2-degenerate set. It follows from Lemmas 2.2 and 2.3 that if G contains no PC odd cycles, then G contains a 2-degenerate set; otherwise, G contains a PC odd cycles. If we find out a 2-degenerate set S of G , then let $G' = G - S$. By Observation 2.1, we only need check whether G' contains a PC odd cycle. Then repeat the above procedure for G' similar to G and continue recursively. Since $|V(G)|$ is finite, the procedure will stop within a finite number of steps.

It is not hard to check the correctness of this algorithm. Now we calculate the overall running time. Note that the total number of iterations is no more than n . In each iteration, the most complicated procedures are Step 1-4 and Step 3. Both of them can be done in time $O(n^2)$. Thus the PC odd cycle problem in edge-colored complete graphs can be solved in time $O(n^3)$. \square

Algorithm 1 PC Odd Cycle Problem in Edge-colored Complete Graphs**Input:** An edge-colored complete graph G .**Output:** If G contains a PC odd cycle, then output YES. Otherwise, output NO.

Step 0. Set $V = V(G), X = \emptyset, Y = \emptyset, U = \emptyset, U_0 = \emptyset$ and v_1, \dots, v_k be an arbitrary ordering of $V(G)$ such that $\delta^c(G) = d_G^c(v_1) \leq \dots \leq d_G^c(v_k) = \Delta^c(G)$. If $|V| \leq 2$, then output NO. Otherwise turn to Step 1.

Step 1. Consider the values of $\delta^c(G)$ and $\Delta^c(G)$.

1-1. If $\delta^c(G) = 1$, then set $G = G - v_1$ and turn to Step 0 (by Observation 2.1).

1-2. If $\delta^c(G) \geq 3$, then output YES (by Corollary 2.1).

1-3. If $\delta^c(G) = \Delta^c(G) = 2$, then calculate $\cap_{v \in V(G)} C(v, V)$. If $\cap_{v \in V(G)} C(v, V) \neq \emptyset$, then output NO. Otherwise, output YES (by Lemma 2.1).

1-4. If $\delta^c(G) = 2$ and $\Delta^c(G) \geq 3$, then set $U = \{v \in V(G) : d_G^c(v) \geq 3\}$ and $W = V \setminus U$. If $|C(w, U)| = 1$ for each $w \in W$, then turn to Step 2. Otherwise, output YES (by (ii) of Lemma 2.2).

Step 2. If $|C(W, U)| = 2$, then turn to Step 3. Otherwise, output YES (by (iii) of Lemma 2.2).

Step 3. For each vertex $v_i \in V \setminus U$, if $C(v_i, U) = C(v_1, U)$, put v_i into X ; otherwise, put v_i into Y . If $C(X \cup Y) \not\subseteq C(W, U)$, then output YES (by (i) of Lemma 2.3). Otherwise, do the following:

3-1. If $C(X) \neq C(X, U)$ and $C(Y) \neq C(Y, U)$, then output YES (by (ii) of Lemma 2.3).

3-2. If $C(X) = C(X, U)$ and $C(Y) = C(Y, U)$, then set $G = G[U]$ and turn to Step 0 (by Observation 2.1).

3-3. If $C(X) \neq C(X, U)$ and $C(Y) = C(Y, U)$, then set $X_1 = \{v \in X : d_{G[X]}^c(v) = 2\}$. If $C(X_1, Y) = C(Y)$, then set $G = G[U \cup X_1]$ and turn to Step 0 (by Observation 2.1 and (iv) of Lemma 2.3). Otherwise, output YES (by (iii) of Lemma 2.3).

3-4. If $C(X) = C(X, U)$ and $C(Y) \neq C(Y, U)$, then set $Y_1 = \{v \in Y : d_{G[Y]}^c(v) = 2\}$. If $C(Y_1, X) = C(X)$, then set $G = G[U \cup Y_1]$ and turn to Step 0 (by Observation 2.1 and (iv) of Lemma 2.3). Otherwise, output YES (by (iii) of Lemma 2.3).

2.4 PC cycles of length k modulo m

Firstly, we need the following auxiliary terminology to deliver the proof of Theorem 2.4.

Let P be a path. For two vertices $u, v \in V(P)$, we use uPv to denote the segment on P between u and v . Let C be a cycle. Give C a direction. For two vertices $u, v \in V(C)$, we use uCv to denote the segment between u and v along the direction of C and use $u\bar{C}v$ to denote the segment between u and v in the opposite direction of C . In particular, if $u = v$, then $uPv = u$, $uCv = u$ and $u\bar{C}v = u$.

Let C be a PC cycle in an edge-colored graph G . Give C a direction. Let u_1, u_2, \dots, u_s ($s \geq 2$) be some vertices on C ordered along the direction of C such that $u_j u_{j+1}$ is a chord of C for all $j \in \{1, 2, \dots, s-1\}$. If $C' = u_s C u_1 u_2 \dots u_s$ is a PC cycle in G and $|V(C')| < |V(C)|$, then we say $u_1 u_2 \dots u_s$ is a *skipper* of C . In particular, if $|V(C)| - |V(C')| = r > 0$, then we also say $u_1 u_2 \dots u_s$ is an r -*skipper* of C .

Proof of Theorem 2.4. Let $C = v_0 v_1 \dots v_{\ell-1} v_0$ be a shortest PC cycle in G satisfying that $\ell \equiv k \pmod{m}$. Suppose to the contrary that $\ell \geq 2m^2(k-1) + 3m$. Let $p = 2m(k-1) + 3$. Then $\ell \geq mp$ and segments $v_{ip-p} C v_{ip-1}$ ($i \in \{1, 2, 3, \dots, m\}$) are vertex-disjoint.

Claim 1. There exists a skipper P_i of C such that $V(P_i)$ is contained in the segment $v_{ip-p} C v_{ip-1}$ for all $i \in \{1, 2, 3, \dots, m\}$.

Proof. It suffices to prove for the segment $v_0 C v_{p-1}$. Note that $p-1$ is even. Let $q = (p-1)/2$. Then $q = m(k-1) + 1$. Suppose to the contrary that there exist no skippers on the segment $v_0 C v_{p-1}$. Since $v_q v_{q+1} v_{q+2} \dots v_{q+k-1} v_q$ is not a PC cycle of length k , either $C(v_q v_{q+k-1}) = C(v_q v_{q+1})$ or $C(v_q v_{q+k-1}) = C(v_q v_{q+k-1}) = C(v_{q+k-2} v_{q+k-1})$. If $C(v_q v_{q+k-1}) = C(v_q v_{q+1})$, then let $w_{i-q} = v_i$ for all $i \in \{q, q+1, \dots, p-1\}$. Otherwise, let $w_i = v_{q+k-i-1}$ for all $i \in \{0, 1, \dots, q+k-1\}$. In this way, we will either obtain a segment $w_0 C w_h$ or a segment $w_0 \bar{C} w_h$, which are contained in the segment $v_0 C v_{p-1}$ with $h \geq m(k-1) + 1$ and $C(w_0 w_1) = C(w_0 w_{k-1})$.

Without loss of generality, assume that the obtained segment is $w_0 C w_h$ and $C(w_0 w_1) = c_1$. Since $w_0 w_{k-1}$ is not a skipper of C , we have $C(w_{k-1} w_k) = c_1$. If $C(w_{k-1} w_{2k-2}) \neq c_1$, then either $w_0 w_{k-1} w_{2k-2}$ is a skipper of C (when $C(w_{k-1} w_{2k-2}) \neq C(w_{2k-2} w_{2k-1})$) or $w_{k-1} w_k w_{k+1} \cdots w_{2k-2} w_{k-1}$ is a PC cycle of length k (when $C(w_{k-1} w_{2k-2}) = C(w_{2k-2} w_{2k-1})$), a contradiction. So $C(w_{k-1} w_{2k-2}) = c_1$. Again, since $w_{k-1} w_{2k-2}$ is not a skipper of C , we have $C(w_{2k-2} w_{2k-1}) = c_1$. By repeating this argument, we have

$$C(w_{i(k-1)} w_{i(k-1)+1}) = c_1$$

for all $i \in \{1, 2, \dots, m\}$. Note that $w_0 w_{i(k-1)+1}$ is not a skipper of C . We have $C(w_0 w_{i(k-1)+1}) \neq c_1$, and thus $w_0 w_1 \cdots w_{i(k-1)+1} w_0$ is a PC cycle of length $i(k-1) + 2$ for all $i \in \{1, 2, \dots, m\}$. Recall that $k-1$ and m are relatively prime. There must exist an integer $i_0 \in \{1, 2, \dots, m\}$ such that $i_0(k-1) + 2 \equiv k \pmod{m}$. Now $w_0 w_1 \cdots w_{i_0(k-1)+1} w_0$ is a PC cycle of length $\ell' \equiv k \pmod{m}$ with $\ell' < \ell$, which contradicts the choice of C . \square

According to Claim 1, for each $i \in \{1, 2, 3, \dots, m\}$, assume that P_i is an r_i -skipper of C with $V(P_i)$ contained in the segment $v_{ip-p} C v_{ip-1}$, and x_i, y_i are initial vertex and terminal vertex of P_i , respectively. An application of the pigeonhole principle (see [22]) states that for m integers r_1, r_2, \dots, r_m , there exist indices s and t with $0 \leq s < t \leq m$ such that $\sum_{i=s}^t r_i$ is divisible by m . Using skippers P_s, P_{s+1}, \dots, P_t , each segment $v_{jp-p} C v_{jp-1}$ with $s \leq j \leq t$ can be replaced by a PC path $v_{jp-p} C x_j P_j y_j C v_{jp-1}$. Thus we obtain a shorter PC cycle of length $\ell' \equiv k \pmod{m}$, where $\ell' = \ell - \sum_{i=s}^t r_i$, a contradiction. \square

Chapter 3

Short PC cycles in edge-colored graphs

In this chapter, we firstly investigate sufficient conditions in terms of the minimum color degree and maximum monochromatic degree for the existence of short PC cycles in edge-colored complete graphs. In particular, we obtain sharp results for the existence of PC 4-cycles, and we characterize the extremal graphs for several known results on the existence of PC triangles. Then we obtain sharp sufficient conditions guaranteeing that every vertex is contained in a PC triangle or PC 4-cycle, respectively. In addition, for edge-colored bipartite graphs, we give a sufficient condition for PC cycles of length at most 6.

3.1 Introduction

The work reported here is motivated by recent results from several different groups of researchers on the existence of short PC cycles in edge-colored graphs.

The problem of obtaining sufficient conditions for the existence of short PC cycles in edge-colored graphs has been studied extensively, especially during the last decades. H. Li [63] in 2013 and B. Li et al. [62] in 2014

independently obtained a minimum color degree condition for the existence of PC triangles in edge-colored graphs. H. Li [63] in 2013 obtained a minimum color degree condition for the existence of rainbow 4-cycle (i.e. all edges colored differently) in edge-colored balanced bipartite graphs. Ning and Ge [79] in 2016 generalized H. Li's result about rainbow 4-cycles into edge-colored unbalanced bipartite graphs. Fujita et al. [41] in 2018 characterized edge-colored complete bipartite graphs containing no PC 4-cycles, and obtained a sharp minimum color degree condition and a sharp maximum monochromatic degree condition for the existence of PC 4-cycles in edge-colored complete bipartite graphs. Independently, Čada et al. [26] in 2020 characterized edge-colored complete bipartite graphs containing no PC 4-cycles. Ding et al. [33] in 2022 obtained an asymptotically sharp color degree condition for the existence of PC 4-cycles and rainbow 4-cycles in edge-colored graphs. For more related work, see [2, 35, 36, 57] for the existence of PC triangles and [25, 44, 89] for PC (or rainbow, i.e., with all edges colored differently) 4-cycles, respectively.

When restricting the host graph to an edge-colored complete graph K_n^c , already back in 1967 Gallai [46] characterized edge-colorings of a K_n^c containing no PC triangles; Erdős and Tuza [37] in 1993 proved that every K_n^c with $\delta^c(K_n^c) > \log_2 n$ contains a PC triangle; Gyárfás and Simonyi [53] in 2004 showed that every K_n^c with $\Delta^{mon}(K_n^c) < 2n/5$ contains a PC triangle; Axenovich et al. [6] in 2003 proved that every K_n^c with $\delta^c(K_n^c) \geq 3$ contains a PC 4-cycle; Fujita and Magnant [42] in 2011 conjectured that if $\delta^c(K_n^c) \geq (n+1)/2$, then every vertex of K_n^c is contained in PC cycles of all possible lengths, and showed that the conjecture holds for PC triangles and PC 4-cycles.

In this chapter, we firstly continue the investigation of sufficient conditions for the existence of short PC cycles in a K_n^c . We start by characterizing the extremal graphs for several known results on the existence of PC triangles. Then we obtain sharp results for the existence of PC 4-cycles. Next we obtain sufficient conditions guaranteeing that every vertex of a K_n^c is contained in short PC cycles. In addition, for edge-colored bipartite graphs, we give a sufficient condition for the existence of PC cycle of length at most 6.

For illustrating the sharpness of our results and for describing the extremal graphs, we define the following three classes of graphs.

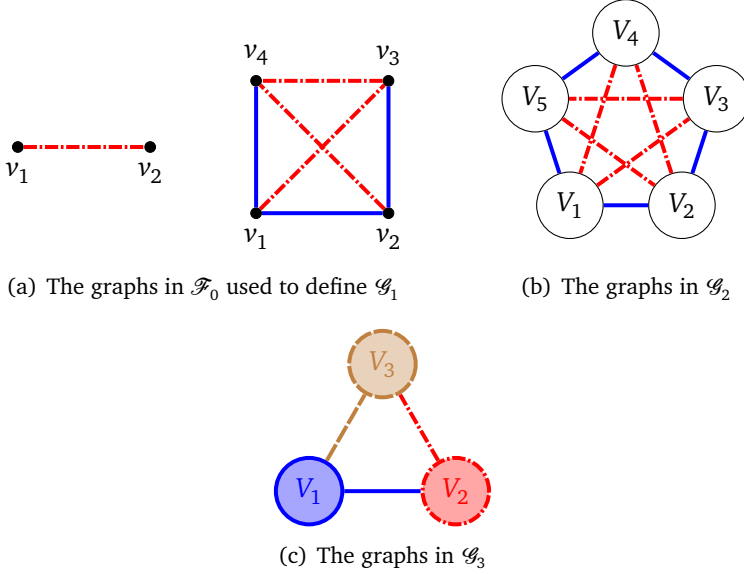


Figure 3.1: Three classes of edge-colored complete graphs

Construction 3.1. Let \mathcal{F}_0 consist of all K_2^c 's, and all K_4^c 's in which the edges of each of the two colors form a path of length 3, as shown in Figure 3.1(a). For any positive integer $i \geq 1$, let \mathcal{F}_i be the set of all edge-colored complete graphs that can be constructed recursively in the following way: blow up all vertices of a graph $F_{i-1} \in \mathcal{F}_{i-1}$ into classes of the same cardinality two or four; the edges joining any two classes inherit the color from the edge of F_{i-1} joining the corresponding vertices; inside of each class we assign colors according to an edge-colored complete graph in \mathcal{F}_0 avoiding the colors on the edges of F_{i-1} . Let $\mathcal{G}_1 = \cup_{i=0}^{\infty} \mathcal{F}_i$.

Construction 3.2. Let F be a red-blue coloring of a K_5 in which both color classes form 5-cycles, as illustrated in Figure 3.1(b). Blow up the five vertices of F into five classes of the same cardinality. The edges joining any two classes inherit the color of the corresponding edge of F . Inside of each class

we assign colors according to an edge-colored complete graph containing no PC triangles, avoiding the colors red and blue. Let \mathcal{G}_2 be the set of all edge-colored complete graphs constructed this way.

Construction 3.3. Let V_1, V_2, V_3 be three disjoint nonempty vertex sets of the same cardinality. Let G be an edge-colored complete graph with vertex set $V_1 \cup V_2 \cup V_3$ such that all edges with one end in V_i and the other end in $V_i \cup V_{i+1}$ are colored with c_i for each $1 \leq i \leq 3$, where subscripts are taken modulo 3, as illustrated in Figure 3.1(c). Let \mathcal{G}_3 be the set of all edge-colored complete graphs constructed this way.

We start with the following two known sufficient conditions in terms of the minimum color degree and the maximum monochromatic degree for the existence of PC triangles in an edge-colored complete graph K_n^c .

Theorem 3.1 (Erdős and Tuza [37]). *If $\delta^c(K_n^c) > \log_2 n$, then K_n^c contains a PC triangle.*

Theorem 3.2 (Grossman and Häggkvist [48]). *If $\Delta^{mon}(K_n^c) < 2n/5$, then K_n^c contains a PC triangle.*

It is a natural problem to characterize all edge-colored complete graphs K_n^c containing no PC triangles but satisfying $\delta^c(K_n^c) = \log_2 n$ and $\Delta^{mon}(K_n^c) = 2n/5$, respectively. We firstly give a complete solution to this problem.

Theorem 3.3.

- (i) *If $\delta^c(K_n^c) = \log_2 n$, then K_n^c contains a PC triangle unless $K_n^c \in \mathcal{G}_1$;*
- (ii) *If $\Delta^{mon}(K_n^c) = 2n/5$, then K_n^c contains a PC triangle unless $K_n^c \in \mathcal{G}_2$.*

We postpone all proofs of our new contributions to later sections, not to interrupt the narrative. Axenovich et al. in 2003 gave the following minimum color degree condition for the existence of a PC 4-cycle in an edge-colored complete graph.

Theorem 3.4 (Axenovich et al. [6]). *If $\delta^c(K_n^c) \geq 3$, then K_n^c contains a PC 4-cycle.*

Note that the condition in the above theorem is sharp in view of all graphs $G \in \mathcal{G}_3$. It is natural to ask for a sharp maximum monochromatic degree condition for a K_n^c to contain a PC 4-cycle. We obtain the following result.

Theorem 3.5.

- (i) If $\Delta^{mon}(K_n^c) < 2n/3 - 1$, then K_n^c contains a PC 4-cycle;
- (ii) If $\Delta^{mon}(K_n^c) = 2n/3 - 1$, then K_n^c contains a PC 4-cycle unless $K_n^c \in \mathcal{G}_3$.

Fujita and Magnant [42] in 2011 conjectured that if $\delta^c(K_n^c) \geq (n+1)/2$, then every vertex of K_n^c is contained in a PC cycle of all possible lengths, and showed that the conjecture holds for PC triangles and PC 4-cycles. Motivated by these results, we next consider sharp maximum monochromatic degree conditions for a K_n^c guaranteeing that every vertex is contained in a PC triangle and PC 4-cycle, respectively. We obtain the following results.

Theorem 3.6.

- (i) If $\Delta^{mon}(K_n^c) < (n+1)/3$, then every vertex of K_n^c is contained in a PC triangle;
- (ii) If $\Delta^{mon}(K_n^c) < n/2$, then every vertex of K_n^c is contained in a PC 4-cycle.

The conditions in Theorem 3.6 are sharp. For (i) of Theorem 3.6, consider a K_n^c on vertex set $\{v_0\} \cup V_1 \cup V_2 \cup V_3$, for four disjoint nonempty vertex sets with $|\{v_0\}| = 1$, $|V_1| = (n+1)/3$ and $|V_2| = |V_3| = (n-2)/3$, for suitable values of n . Color all edges between V_i and $\{v_0\} \cup V_{i+1}$ with color c_i for each $i \in \{1, 2, 3\}$, where $V_4 = V_1$, and color all remaining edges with $3\binom{(n-2)/3}{2} + (n-2)/3$ new colors. One can check that $\Delta^{mon}(K_n^c) = (n+1)/3$, but v_0 is not contained in any PC triangle. For (ii) of Theorem 3.6, consider a K_n^c on vertex set $\{v_0, u_1, u_2, \dots, u_{n-1}\}$, where n is even. Color edges $u_i v_0, u_i u_{i+1}, \dots, u_i u_{i+(n-2)/2}$ with color c_i for each $i \in \{1, 2, \dots, n-1\}$, where subscripts are taken modulo $n-1$. One can check that $\Delta^{mon}(K_n^c) = n/2$, but v_0 is not contained in any PC 4-cycle.

Furthermore, we give a sharp total monochromatic degree condition for a K_n^c guaranteeing that every vertex is contained in a PC triangle and PC 4-cycle.

Theorem 3.7. If $n \geq 4$ and $\sum_{v \in V(K_n^c)} \Delta^{mon}(v) \leq 2n - 3$, then every vertex of K_n^c is contained in a PC triangle and PC 4-cycle.

Since $\Delta^{mon}(v) + d^c(v) \leq n$ for each $v \in V(K_n^c)$, we can obtain the following corollary.

Corollary 3.1. *If $n \geq 4$ and $\sum_{v \in V(K_n^c)} d^c(v) \geq n^2 - 2n + 3$, then every vertex of K_n^c is contained in a PC triangle and PC 4-cycle.*

The conditions in Theorem 3.7 and Corollary 3.1 are sharp. Consider a K_n^c on vertex set V . Let v_0 be a vertex in V . Color all edges with both end-vertices in $V \setminus \{v_0\}$ with $\binom{n-1}{2}$ colors and all remaining edges with a new color. One can check that $\sum_{v \in V(K_n^c)} \Delta^{mon}(v) = 2n - 2$ and $\sum_{v \in V(K_n^c)} d^c(v) = (n-1)^2 + 1$, but v_0 is not contained in any PC cycle.

In addition, for edge-colored bipartite graphs, we give a minimum color degree condition for the existence of PC cycles of length at most 6.

Theorem 3.8. *If $\delta^c(G_{n,n}^c) > \frac{n}{3} + 1$, then $G_{n,n}^c$ contains a PC cycle of length at most 6.*

The remainder of this chapter is organized as follows. In Section 3.2, we give further known results and lemmas that will be used later. In Section 3.3, we present the proofs of Theorems 3.3, 3.5, 3.6 and 3.7, respectively. In section 3.4, we present the proof of Theorem 3.8.

3.2 Preliminaries

The following two known results and four lemmas will be used in our later proofs.

Theorem 3.9 (Gallai [46]). *For every Gallai coloring of a complete graph, there exists a Gallai partition.*

Theorem 3.10 (Yeo [90]). *Let G be an edge-colored graph containing no PC cycles. Then G contains a vertex v such that no component of $G - v$ is joined to v with edges of more than one color.*

Lemma 3.1. *Let G be an edge-colored complete graph. Suppose that V_1, V_2 are two disjoint vertex sets of G and $C(V_1, V_2) = \{c_1, c_2\}$. If one of the following two statements holds, then G contains a PC 4-cycle.*

- (i) $d_{G[V_2]}^{c_1}(v_1) \geq 1$ for each vertex $v_1 \in V_1$ and $d_{G[V_1]}^{c_2}(v_2) \geq 1$ for each vertex $v_2 \in V_2$.
(ii) $d_{G[V_2]}^{c_2}(v_1) \geq d$ for each vertex $v_1 \in V_1$ and there exists no subset $V_2' \subseteq V_2$ with $|V_2'| \geq d$ such that $C(V_1, V_2') = \{c_2\}$, where d is a positive integer.

Proof. (i) Construct a bipartite tournament D with bipartition $\{V_1, V_2\}$. For each edge $uv \in E(V_1, V_2)$, if $C(uv) = c_1$, then let $uv \in A(D)$; otherwise, let $vu \in A(D)$. It follows that D is a bipartite tournament with $\delta^+(D) \geq 1$. So D contains a directed cycle of length 4, which corresponds to a PC 4-cycle in G , a contradiction.

(ii) Let $V_2' \subseteq V_2$ be the maximum subset such that $C(V_1, V_2') = \{c_2\}$. Set $V_2'' = V_2 \setminus V_2'$. If $|V_2'| \leq d - 1$, then note that $d_{G[V_2'']}^{c_2}(v_1) \geq 1$ for each vertex $v_1 \in V_1$ and $d_{G[V_1]}^{c_1}(v_2) \geq 1$ for each vertex $v_2 \in V_2''$. Construct a bipartite tournament D with bipartition $\{V_1, V_2''\}$. By similar arguments as in (i), G contains a PC 4-cycle. \square

Lemma 3.2. *Let G be an edge-colored complete graph with $|C(G)| \leq 2$. If $\delta^c(G) \geq 2$, then G contains a PC 4-cycle.*

Proof. By Theorem 4.5, $\delta^c(G) \geq 2$ implies G contains a PC cycle. Let $C = v_1 v_2 \cdots v_p v_1$ be a shortest PC cycle. Since $|C(G)| \leq 2$, it is clear that G contains no PC triangles, so $p \neq 3$. Clearly, $|C(G)| \leq 2$ also implies that $C(v_1 v_2) = C(v_3 v_4)$. If $p \geq 5$, then either $v_1 v_2 v_3 v_4 v_1$ is a PC 4-cycle or $v_1 v_4 v_5 \cdots v_p v_1$ is a shorter PC cycle than C , a contradiction. \square

Lemma 3.3. *Let q, r, s, t be positive integers with $s > q$, and $s = tq + r$, where $0 < r \leq q$. If a_1, a_2, \dots, a_s are non-negative integers satisfying that $\sum_{1 \leq i \leq s} a_i = s$ and $a_i \leq q$ for each $i \in \{1, 2, \dots, s\}$, then $\sum_{1 \leq i \leq s} a_i^2 \leq tq^2 + r^2$.*

Proof. Assume without loss of generality (abbreviated to w.l.o.g. in the sequel) that $a_1 \geq a_2 \geq \cdots \geq a_s$. It suffices to show that $\sum_{1 \leq i \leq s} a_i^2$ attains its maximum value when $a_1 = a_2 = \cdots = a_t = q$, $a_{t+1} = r$ and $a_{t+2} = \cdots = a_s = 0$. Suppose to the contrary that a_1, a_2, \dots, a_s is a series of numbers such that $\sum_{1 \leq i \leq s} a_i^2$ attains its maximum value such that there exists some $i_0 \in \{1, 2, \dots, t\}$ with $a_{i_0} < q$ or $i_0 = t + 1$ with $a_{i_0} < r$. Let p be the maximum subscript such that $a_p \neq 0$. It is clear that $p > i_0$. Let b_1, b_2, \dots, b_s be a series of numbers such

that $b_{i_0} = a_{i_0} + 1$, $b_p = a_p - 1$ and $b_i = a_i$ for each $i \in \{1, 2, \dots, s\} \setminus \{i_0, p\}$. It is clear that $\sum_{1 \leq i \leq s} a_i^2 < \sum_{1 \leq i \leq s} b_i^2$, which contradicts the assumption that $\sum_{1 \leq i \leq s} a_i^2$ attains its maximum value. \square

Lemma 3.4. *Any bipartite graph $G_{n,n}$ with $\sum_{v \in V(G_{n,n})} \binom{d(v)}{2} > 2 \binom{n}{2}$ contains a 4-cycle.*

Proof. Denote by p_2 the number of paths of length two in $G_{n,n}$. Firstly we count the number of paths of length two according to their central vertex. It is clear that $p_2 = \sum_{v \in V(G_{n,n})} \binom{d(v)}{2}$. Note that each path of length two has a unique pair of ends. According to their ends, the set of all paths of length two can be partitioned into $2 \binom{n}{2}$ subsets. By the pigeonhole principle, the condition $\sum_{v \in V(G_{n,n})} \binom{d(v)}{2} > 2 \binom{n}{2}$ implies that there exist two paths of length two with the same pair of ends. The union of the two paths forms a 4-cycle. \square

It is worth remarking that Lemma 3.4 is a bipartite version of a result of Reiman in [80], which states that any graph G with $\sum_{v \in V(G)} \binom{d(v)}{2} > \binom{n}{2}$ contains a 4-cycle.

3.3 PC triangles and PC 4-cycles in edge-colored complete graphs

Proof of Theorem 3.3. We prove the two statements in the same order.

(i) Suppose that the statement is false, and that G is a counterexample on n vertices such that n is as small as possible. By Theorem 4.4, G has a Gallai partition. Let $\{U_1, U_2, \dots, U_p\}$ be a Gallai partition such that p is as small as possible. We distinguish the cases that G does or does not contain a monochromatic edge-cut.

Suppose first that G contains a monochromatic edge-cut. Then $p = 2$. If $|U_i| < n/2$ for some $i \in \{1, 2\}$, then $\delta^c(G[U_i]) \geq \log_2 n - 1 = \log_2 n/2 > \log_2 |U_i|$. It follows from Theorem 3.1 that $G[U_i]$ contains a PC triangle, a contradiction. Hence $|U_1| = |U_2| = n/2$. Now we have $\delta^c(G[U_1]) =$

$\delta^c(G[U_2]) = \log_2 n/2$. By the choice of n , we have $G[U_1], G[U_2] \in \mathcal{G}_1$, which implies that $G \in \mathcal{G}_1$.

Next suppose that G contains no monochromatic edge-cuts. Then it is clear that $p \geq 4$. If $|U_i| < n/4$ for some $i \in \{1, 2, \dots, p\}$, then $\delta^c(G[U_i]) \geq \log_2 n - 2 = \log_2 n/4 > \log_2 |U_i|$. It follows from Theorem 3.1 that $G[U_i]$ contains a PC triangle, a contradiction. Hence $|U_1| = |U_2| = |U_3| = |U_4| = n/4$. Since G contains no monochromatic edge-cuts, we can assume w.l.o.g. that $C(U_1, U_2) = C(U_2, U_4) = C(U_3, U_4) = \{c_1\}$ and $C(U_1, U_3) = C(U_1, U_4) = C(U_2, U_3) = \{c_2\}$. Now we have $\delta^c(G[U_1]) = \delta^c(G[U_2]) = \delta^c(G[U_3]) = \delta^c(G[U_4]) = \log_2 n/4$. By the choice of n , we have $G[U_i] \in \mathcal{G}_1$ for each $i \in \{1, 2, 3, 4\}$, which implies that $G \in \mathcal{G}_1$.

(ii) Let $G = K_n^c$. Since G contains no PC triangles, it follows from Theorem 3.2 that $\Delta^{mon}(G) = 2n/5$. By Theorem 4.4, G has a Gallai partition $\{U_1, U_2, \dots, U_p\}$, where $p \geq 2$. Since $\Delta^{mon}(G) = 2n/5$, G contains no monochromatic edge-cuts. So we can assume that $C(U_i, V(G) \setminus U_i) = \{c_1, c_2\}$ for each $i \in \{1, 2, \dots, p\}$. Note that $p \geq 4$. If $|U_i| < n/5$ for some $i \in \{1, 2, \dots, p\}$, then either $d_G^{c_1}(v) > 2n/5$ or $d_G^{c_2}(v) > 2n/5$ for $v \in U_i$, a contradiction. So $|U_i| \geq n/5$ for each $i \in \{1, 2, \dots, p\}$. This implies that $p \leq 5$.

If $p = 4$, then we can assume w.l.o.g. that $C(U_1, U_2) = C(U_2, U_4) = C(U_3, U_4) = \{c_1\}$ and $C(U_1, U_3) = C(U_1, U_4) = C(U_2, U_3) = \{c_2\}$. Since $d_G^{c_1}(v) \leq 2n/5$ for an arbitrary vertex $v \in U_4$, we have $|U_2| + |U_3| \leq 2n/5$, which implies that $|U_1| + |U_4| \geq 3n/5$. But now $d_G^{c_1}(v) \geq 3n/5 > 2n/5$ for each vertex $v \in U_2$, a contradiction. So we have $p = 5$.

Since $p = 5$, it follows that $|U_i| = n/5$ for each $i \in \{1, 2, \dots, 5\}$ and $\Delta^{mon}(G) = 2n/5$. Assume w.l.o.g. that $C(U_1, U_2) = C(U_1, U_5) = \{c_1\}$ and $C(U_1, U_3) = C(U_1, U_4) = \{c_2\}$. By symmetry between c_1 and c_2 , we can assume that $C(U_2, U_3) = \{c_1\}$. This implies that $C(U_2, U_4) = C(U_2, U_5) = \{c_2\}$. Hence $C(U_3, U_4) = C(U_4, U_5) = \{c_1\}$, which yields that $C(U_3, U_5) = \{c_2\}$. Now $C(U_i, U_{i+1}) = \{c_1\}$ and $C(U_i, U_{i+2}) = \{c_2\}$ for each $i \in \{1, 2, \dots, 5\}$, where subscripts are taken modulo 5. Note that $c_1, c_2 \notin C(U_i)$ for each $i \in \{1, 2, \dots, 5\}$, since otherwise $\Delta^{mon}(G) > 2n/5$. So we have $G \in \mathcal{G}_2$. \square

Proof of Theorem 3.5. It suffices to show that if $\Delta^{mon}(K_n^c) \leq 2n/3 - 1$, then K_n^c contains a PC 4-cycle unless $K_n^c \in \mathcal{G}_3$. Suppose that the statement is false,

and G is a counterexample on n vertices. By Theorem 3.4, there exists a vertex $v_0 \in V(G)$ such that $d_G^c(v_0) = 2$. Let $C(v_0, V(G)) = \{c_1, c_2\}$ and $U_i = \{u \in V(G) : C(uv_0) = c_i\}$ for each $i \in \{1, 2\}$. Since $1 \leq \Delta^{\text{mon}}(G) \leq 2n/3 - 1$, we have $n \geq 3$. If $n = 3$, then G is a PC triangle, which implies that $G \in \mathcal{G}_3$, a contradiction. So we can assume that $n \geq 4$. It follows that $|U_i| \geq n/3 > 1$ for each $i \in \{1, 2\}$. Let $X_i = \{x \in U_i : C(x, U_i) \setminus \{c_i\} \neq \emptyset\}$ and $Y_i = U_i \setminus X_i$ for each $i \in \{1, 2\}$.

Now we proceed by showing the following three claims.

Claim 1. Either $X_1 \neq \emptyset$ or $X_2 \neq \emptyset$, but not both.

Proof. (1) Firstly we assert that at least one of X_1 and X_2 is nonempty. Suppose that $X_1 = \emptyset$ and $X_2 = \emptyset$. Since $\Delta^{\text{mon}}(G) \leq 2n/3 - 1 < n - 1$, we have $\delta^c(G) \geq 2$. This implies that $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$. By Lemma 3.2, we have $|C(G)| \geq 3$. Let H be the subgraph induced by all edges between Y_1 and Y_2 with colors which are different from c_1 and c_2 . If H contains two nonadjacent edges y_1y_2 and $y'_1y'_2$ with $y_1, y'_1 \in Y_1$ and $y_2, y'_2 \in Y_2$, then $y_1y_2y'_2y'_1y_1$ is a PC 4-cycle, a contradiction. Hence H is either a star or a K_2 . Let y be the center of the star H or just one vertex of the K_2 . By symmetry, assume that $y \in Y_1$. Let $Y'_1 = Y_1 \setminus \{y\}$. Consider the graph $G[Y'_1, Y_2]$. Since $\Delta^{\text{mon}}(G) \leq 2n/3 - 1$, we have $d_G^{c_2}(y'_1, Y_2) \geq n/3 > 1$ for each vertex $y'_1 \in Y'_1$ and $d_G^{c_1}(y_2, Y'_1) \geq n/3 - 1 > 0$ for each vertex $y_2 \in Y_2$. It follows from Lemma 3.1 that $G[Y'_1, Y_2]$ contains a PC 4-cycle, which is also a PC 4-cycle in G , a contradiction.

(2) Next we assert that at most one of X_1 and X_2 is nonempty. Suppose that $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$.

Firstly we show that there exists a new color, say c_3 , such that $C(X_1, X_2) = \{c_3\}$, $C(X_1) \subseteq \{c_1, c_3\}$ and $C(X_2) \subseteq \{c_2, c_3\}$. To see this, choose arbitrary vertices $x_1 \in X_1$ and $x_2 \in X_2$. By the definition of X_1 and X_2 , there exist vertices $x'_1 \in X_1$ and $x'_2 \in X_2$ such that $C(x_1x'_1) \neq c_1$ and $C(x_2x'_2) \neq c_2$. Consider cycles $v_0x'_1x_1x_2v_0$ and $v_0x_1x_2x'_2v_0$ for every pair of vertices $x_1 \in X_1$ and $x_2 \in X_2$. We have $c_1, c_2 \notin C(X_1, X_2)$. If $|C(X_1, X_2)| \neq 1$, then there exist two adjacent edges between X_1 and X_2 with different colors, say c_3 and c_4 . Assume w.l.o.g. that $C(x_1x_2) = c_3$ and $C(x_1x'_2) = c_4$, where $x_1 \in X_1$ and $x_2, x'_2 \in X_2$. It follows that either $v_0x'_1x_1x_2v_0$ or $v_0x'_1x_1x'_2v_0$ is a PC 4-cycle, a

contradiction. So we have $C(X_1, X_2) = \{c_3\}$. Moreover, by considering cycles $v_0x'_1x_1x_2v_0$ for every pair of vertices $x_1, x'_1 \in X_1$ and $v_0x_1x_2x'_2v_0$ for every pair of vertices $x_2, x'_2 \in X_2$, we have $C(X_1) \subseteq \{c_1, c_3\}$ and $C(X_2) \subseteq \{c_2, c_3\}$.

Next we show that $Y_1 \cup Y_2 \neq \emptyset$. Let $X'_i = \{x_i \in X_i : C(x_i, X_i) = \{c_3\}\}$ and $X''_i = X_i \setminus X'_i$ for each $i \in \{1, 2\}$. By Lemma 3.2, we have $\delta^c(G[X_1]) = \delta^c(G[X_2]) = 1$. It follows from the definition of X_1 and X_2 that there exists at least one vertex $x_i \in X_i$ such that $C(x_i, X_i) = \{c_3\}$ for each $i \in \{1, 2\}$. So $X'_i \neq \emptyset$ for each $i \in \{1, 2\}$. Since $d_G^{c_3}(x'_i) \leq 2n/3 - 1 < n - 2$ for an arbitrary vertex $x'_i \in X'_i$, we have $Y_1 \cup Y_2 \neq \emptyset$.

By symmetry between Y_1 and Y_2 , we can assume that $Y_1 \neq \emptyset$. Note that $C(Y_1, X''_2) \subseteq \{c_1\}$. Indeed, if $X''_2 \neq \emptyset$, then choose an arbitrary vertex $x''_2 \in X''_2$. It follows from the definition of X''_2 that there exists a vertex $x \in X''_2$ such that $C(xx''_2) = c_2$. Choose arbitrary vertices $x_1 \in X_1$ and $y'_2 \in Y'_2$. Recall that $C(X_1, X_2) = \{c_3\}$. So we have $C(x_1x) = c_3$. By the definition of X'_2 , we have $C(x'_2x''_2) = c_3$. Consider cycles $y_1v_0x'_2x''_2y_1$ and $y_1x_1xx''_2y_1$ for every pair of vertices $y_1 \in Y_1$ and $x''_2 \in X''_2$. We have $C(Y_1, X''_2) \subseteq \{c_1\}$.

To derive a contradiction, we distinguish two cases depending on the emptiness or non-emptiness of Y_2 .

a) If $Y_2 = \emptyset$, then it follows from $X'_2 \neq \emptyset$ that for each vertex $x'_2 \in X'_2$, there exists a vertex $x_2 \in X_2$ such that $C(x_2x'_2) = \{c_3\}$. Consider cycles $y_1v_0x_2x'_2y_1$ for every pair of vertices $y_1 \in Y_1$ and $x'_2 \in X'_2$. We have $C(Y_1, X'_2) \subseteq \{c_1, c_3\}$. Consider the graph $G[Y_1, X'_2]$. Recall that $C(Y_1, X''_2) \subseteq \{c_1\}$. Note that $d_{G[X'_2]}^{c_3}(y_1) \geq n/3 > 1$ for each vertex $y_1 \in Y_1$ and $d_{G[Y_1]}^{c_1}(x'_2) \geq n/3 - 1 > 0$ for each vertex $x'_2 \in X'_2$. It follows from Lemma 3.1 that $G[Y_1, X'_2]$ contains a PC 4-cycle, a contradiction.

b) If $Y_2 \neq \emptyset$, then choose an arbitrary vertex $y_2 \in Y_2$. For arbitrary vertices $x_1 \in X_1$ and $x_2 \in X_2$, there exist vertices $x'_1 \in X_1$ and $x'_2 \in X_2$ such that $C(x_1x'_1) = C(x_2x'_2) = c_3$. Consider cycles $x_1x'_1v_0y_2x_1$ for every pair of vertices $x_1 \in X_1, y_2 \in Y_2$ and $x_2x'_2v_0y_1x_2$ for every pair of vertices $x_2 \in X_2, y_1 \in Y_1$, respectively. We have $C(X_1, Y_2) \subseteq \{c_2, c_3\}$ and $C(X_2, Y_1) \subseteq \{c_1, c_3\}$. Consider cycles $x_1y_1x_2y_2x_1$ for every quadruple of vertices $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1$ and $y_2 \in Y_2$. We have that either $C(X_1, Y_2) = \{c_2\}$ or $C(X_2, Y_1) = \{c_1\}$. Assume w.l.o.g. that $C(X_1, Y_2) = \{c_2\}$.

Consider cycles $y_1x_1x_2y_2y_1$ for every pair of vertices $y_1 \in Y_1$ and $y_2 \in Y_2$. We have $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$. Note that $d_{G[Y_1]}^{c_1}(y_2) \geq n/3$ for each vertex $y_2 \in Y_2$. Since $G[Y_1, Y_2]$ contains no PC 4-cycles, it follows from Lemma 3.1 that there exists a subset $Y'_1 \subseteq Y_1$ with $|Y'_1| \geq n/3$ such that $C(Y'_1, Y_2) = \{c_1\}$. Recall that $C(Y_1, X_2'') \subseteq \{c_1\}$. We have $|X'_2| \geq n/3$. It follows that

$$\begin{aligned} d_{G[Y'_1]}^{c_3}(x'_2) &\leq d_G^{c_3}(x'_2) - d_{G[X_1 \cup X'_2]}^{c_3}(x'_2) \\ &\leq \Delta^{mon}(G) - n/3 \\ &\leq n/3 - 1 \end{aligned}$$

for each vertex $x'_2 \in X'_2$. So we have $d_{G[Y'_1]}^{c_1}(x'_2) = |Y'_1| - d_{G[Y'_1]}^{c_3}(x'_2) \geq 1$ for each vertex $x'_2 \in X'_2$. Consider the graph $G[Y'_1, X'_2]$. Note that $d_{G[X'_2]}^{c_3}(y'_1) \geq n/3 > 0$ for each vertex $y'_1 \in Y'_1$. It follows from Lemma 3.1 that $G[Y'_1, X'_2]$ contains a PC 4-cycle, a contradiction.

From (1) and (2) we conclude that exactly one of X_1 and X_2 is a nonempty set. \square

Assume w.l.o.g. that $X_1 \neq \emptyset$ and $X_2 = \emptyset$. Recall that $|U_2| = |X_2| + |Y_2| \geq n/3$. So we have $|Y_2| \geq n/3$.

Claim 2. There exists a new color, say c_3 , such that $\{c_3\} \subseteq C(X_1, Y_2) \subseteq \{c_2, c_3\}$ and the graph induced by all edges between X_1 and Y_2 with color c_3 is not a star. In particular, if $Y_1 \neq \emptyset$, then $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$.

Proof. (1) Firstly we assert that $c_1 \notin C(X_1, Y_2)$ and there exists a new color, say c_3 , such that $c_3 \in C(X_1, Y_2)$.

Let x_1 be an arbitrary vertex in X_1 . By the definition of X_1 , there exists a vertex $x'_1 \in X_1$ such that $C(x_1x'_1) \neq c_1$. Consider cycles $v_0x'_1x_1y_2v_0$ for every pair of vertices $x_1 \in X_1$ and $y_2 \in Y_2$. We have $c_1 \notin C(X_1, Y_2)$.

If there exists no new color $c_3 \in C(X_1, Y_2)$, then it follows from $c_1 \notin C(X_1, Y_2)$ that $C(X_1, Y_2) = \{c_2\}$. Since $d_G^{c_2}(y_2) \leq 2n/3 - 1$ for an arbitrary vertex $y_2 \in Y_2$, we have $Y_1 \neq \emptyset$. If $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$, then consider the graph $G[Y_1, Y_2]$. Note that $d_{G[Y_2]}^{c_2}(y_1) \geq n/3 > 0$ for each vertex $y_1 \in Y_1$ and $d_{G[Y_1]}^{c_1}(y_2) \geq n/3 > 0$ for each vertex $y_2 \in Y_2$. By Lemma 3.1, $G[Y_1, Y_2]$

contains a PC 4-cycle, a contradiction. So $C(Y_1, Y_2) \not\subseteq \{c_1, c_2\}$. Let H be the subgraph induced by all edges between Y_1 and Y_2 with colors which are different from c_1 and c_2 . If H contains two nonadjacent edges y_1y_2 and $y'_1y'_2$ with $y_1, y'_1 \in Y_1$ and $y_2, y'_2 \in Y_2$, then $y_1y_2y'_2y'_1$ is a PC 4-cycle, a contradiction. Hence H is either a star or a K_2 . Let y be the center of the star H or just one vertex of the K_2 . By symmetry, assume that $y \in Y_1$. Let $Y'_1 = Y_1 \setminus \{y\}$. Consider the graph $G[Y'_1, Y_2]$. Note that $d_{G[Y_2]}^{c_2}(y'_1) \geq n/3 > 1$ for each vertex $y'_1 \in Y'_1$ and $d_{G[Y'_1]}^{c_1}(y_2) \geq n/3 - 1 > 0$ for each vertex $y_2 \in Y_2$. It follows from Lemma 3.1 that $G[Y'_1, Y_2]$ contains a PC 4-cycle, which is also a PC 4-cycle in G , a contradiction.

(2) Next we assert that $C(X_1, Y_2) \subseteq \{c_2, c_3\}$ and the graph induced by all edges between X_1 and Y_2 with color c_3 is not a star. Let H be the subgraph induced by all edges between X_1 and Y_2 with colors different from c_2 . Recall that $c_1 \notin C(X_1, Y_2)$. It suffices to show that H is not a star and $|C(H)| = 1$.

Firstly we show that H is not a star. To this end, we will show the following two properties.

a) H is not a star with center in X_1 . If we assume H is a star with center in X_1 , then it follows from $c_3 \in C(X_1, Y_2)$ that there exists an edge x_1y_2 with $x_1 \in X_1$ and $y_2 \in Y_2$ such that $C(x_1y_2) = c_3$. Recall that $n \geq 4$. So we have $\Delta_G^{mon}(y_2) \leq 2n/3 - 1 < n - 2$ for an arbitrary vertex $y_2 \in Y_2$. It follows that $Y_1 \neq \emptyset$. Let $Y'_2 = Y_2 \setminus \{y_2\}$. Consider the graph $G[Y_1, Y'_2]$. By considering cycles $y_1x_1y_2y'_2y_1$ for every pair of vertices $y_1 \in Y_1$ and $y'_2 \in Y'_2$, we have $C(Y_1, Y'_2) \subseteq \{c_1, c_2\}$. Note that $d_{G[Y'_2]}^{c_2}(y_1) \geq n/3 - 1 > 0$ for each vertex $y_1 \in Y_1$ and $d_{G[Y_1]}^{c_1}(y'_2) \geq n/3 - 1 > 0$ for each vertex $y'_2 \in Y'_2$. By Lemma 3.1, we have that $G[Y_1, Y'_2]$ contains a PC 4-cycle, a contradiction.

b) H is not a star with center in Y_2 . If we assume H is a star with center in Y_2 , then let $y_2 \in Y_2$ be the center of H and $Y'_2 = Y_2 \setminus \{y_2\}$. It follows from $\Delta_G^{mon}(y) \leq 2n/3 - 1$ for an arbitrary vertex $y \in Y_2 \setminus \{y_2\}$ that $Y_1 \neq \emptyset$. Consider the graph $G[Y_1, Y'_2]$. By considering cycles $y_1x_1y_2y'_2y_1$ for every pair of vertices $y_1 \in Y_1$ and $y'_2 \in Y'_2$, we have $C(Y_1, Y'_2) \subseteq \{c_1, c_2\}$. Note that $d_{G[Y'_2]}^{c_2}(y_1) \geq n/3 - 1 > 0$ for each vertex $y_1 \in Y_1$ and $d_{G[Y_1]}^{c_1}(y'_2) \geq n/3 > 1$ for each vertex $y'_2 \in Y'_2$. By Lemma 3.1, we have that $G[Y_1, Y'_2]$ contains a PC 4-cycle, a contradiction.

Next we show that $|C(H)| = 1$. To this end, we will show the following three properties.

a) Any two nonadjacent edges x_1y_2 and $x'_1y'_2$ with $x_1, x'_1 \in X_1$ and $y_2, y'_2 \in Y_2$ have the same color. Suppose to the contrary that $C(x_1y_2) = \{c_3\}$ and $C(x'_1y'_2) = \{c_4\}$. Since $x_1y_2y'_2x'_1x_1$ is not a PC 4-cycle, we have $C(x_1x'_1) \subseteq \{c_3, c_4\}$. Assume w.l.o.g. that $C(x_1x'_1) = c_3$. This implies that $v_0x_1x'_1y_2v_0$ is a PC 4-cycle, a contradiction.

b) Any two adjacent edges x_1y_2 and $x_1y'_2$ with $x_1 \in X_1$ and $y_2, y'_2 \in Y_2$ have the same color. Suppose to the contrary that $C(x_1y_2) = c_3$ and $C(x_1y'_2) = c_4$. By the definition of X_1 , there exists an edge $x_1x'_1 \in E(X_1)$ with $C(x_1x'_1) \neq c_1$. Then either $v_0x'_1x_1y_2v_0$ or $v_0x'_1x_1y'_2v_0$ is a PC 4-cycle, a contradiction.

c) Any two adjacent edges x_1y_2 and x'_1y_2 with $x_1, x'_1 \in X_1$ and $y_2 \in Y_2$ have the same color. Let x_1y_2 and x'_1y_2 be two adjacent edges with $x_1, x'_1 \in X_1$ and $y_2 \in Y_2$ such that $C(x_1y_2) = \{c_3\}$ and $C(x'_1y_2) = \{c_4\}$. Since H is not a star, there exists an edge $xy \in E(H)$ with $x \in X_1$ and $y \in Y_2 \setminus \{y_2\}$. Note that if $x \notin \{x_1, x'_1\}$, then either x_1y_2, xy or x'_1y_2, xy are two nonadjacent edges in H with different colors, which contradicts property a). So we have $x \in \{x_1, x'_1\}$. Assume w.l.o.g. that $x = x_1$. It follows from property b) that $C(xy) = c_3$. But now xy and x'_1y_2 are two nonadjacent edges in H with different colors, a contradiction.

(3) Finally we assert that if $Y_1 \neq \emptyset$, then $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$. Choose arbitrary vertices $y_1 \in Y_1$ and $y_2 \in Y_2$. Let H be the graph induced by all edges between X_1 and Y_2 with color c_3 . Since H is not a star, there exists an edge $x_1y'_2$ with $x_1 \in X_1$ and $y'_2 \in Y_2 \setminus \{y_2\}$ such that $C(x_1y'_2) = c_3$. Consider cycles $y_1x_1y'_2y_2y_1$ for every pair of vertices $y_1 \in Y_1$ and $y_2 \in Y_2$. We have $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$. \square

Let $Y'_2 \subseteq Y_2$ be a maximum subset such that $C(Y'_2, V(G) \setminus X_1) = \{c_2\}$, and let $X'_1 = \{x_1 \in X_1 : c_3 \in C(x_1, Y'_2)\}$. Choose an arbitrary pair of vertices $x'_1, x''_1 \in X'_1$. By the definition of X'_1 , for each $x'_1 \in X'_1$, there exists a vertex $y'_2 \in Y'_2$ such that $C(x'_1y'_2) = c_3$. Consider cycles $v_0x'_1x''_1y'_2v_0$ for every pair of vertices $x'_1, x''_1 \in X'_1$. We have $C(X'_1) \subseteq \{c_1, c_3\}$.

Claim 3. $|X_1| = |X'_1| = |Y'_2| = |Y_2| = n/3$ and $C(X_1) = \{c_3\}$.

Proof. (1) Firstly we assert that $|X'_1| = |Y'_2| = n/3$. To this end, it suffices to show the following two properties.

a) $|Y'_2| \geq n/3$ and $|X'_1| \geq n/3$. Note that if $Y_1 = \emptyset$, then it is clear that $Y'_2 = Y_2$, which implies that $|Y'_2| \geq n/3$. If $Y_1 \neq \emptyset$, then it follows from Claim 2 that $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$. Since $G[Y_1, Y_2]$ contains no PC 4-cycles, it follows from Lemma 3.1 that there exists a subset $Y''_2 \subseteq Y_2$ with $|Y''_2| \geq n/3 - 1 > 0$ such that $C(Y''_2, Y_1) = \{c_2\}$. One can check that $Y''_2 \subseteq Y'_2$, which implies that $|Y'_2| \geq n/3$. By the definition of Y'_2 , we have $C(Y'_2, V(G) \setminus X_1) = \{c_2\}$. By the definition of X'_1 , we have $C(X_1 \setminus X'_1, Y'_2) \subseteq \{c_2\}$. Since $d_G^{c_2}(y'_2) \leq 2n/3 - 1$ for an arbitrary vertex $y'_2 \in Y'_2$, we have $|X'_1| \geq n/3$.

b) $|X'_1| + |Y'_2| \leq 2n/3$. Consider the graph $G[X'_1, Y'_2]$. By Claim 2, $C(X'_1, Y'_2) \subseteq \{c_2, c_3\}$. Let $S = \{x'_1 \in X'_1 : C(x'_1, X'_1) = \{c_3\}\}$ and $T = X'_1 \setminus S$. By Lemma 3.2, we have $\delta^c(G[X'_1]) = 1$. It follows from the definition of X'_1 that there exists at least one vertex $x'_1 \in X'_1$ such that $C(x'_1, X'_1) = \{c_3\}$. So we have $S \neq \emptyset$. If $|X'_1| + |Y'_2| \geq 2n/3 + 1$, then it follows from $d_G^{c_3}(s) \leq 2n/3 - 1$ for each vertex $s \in S$ that $d_{G[Y'_2]}^{c_2}(s) \geq 1$ for each vertex $s \in S$. If $T \neq \emptyset$, then choose an arbitrary vertex $t \in T$. By the definition of T , there exists a vertex $t' \in T$ such that $C(tt') = c_1$. By the definition of X'_1 , there exists a vertex $y'_2 \in Y'_2$ such that $C(t'y'_2) = c_3$. Recall that $|Y_2| \geq n/3 > 1$. Consider cycles $tt'y'_2y'_2t$ for every vertex $y_2 \in Y'_2 \setminus \{y'_2\}$. We have $C(T, Y'_2 \setminus y'_2) = \{c_2\}$, which implies that $d_{G[Y'_2]}^{c_2}(t) \geq |Y'_2| - 1 \geq n/3 - 1 > 0$ for each vertex $t \in T$. Recall that $X'_1 = S \cup T$. So $d_G^{c_2}(x'_1, Y'_2) > 0$ for each vertex $x'_1 \in X'_1$. Note that $d_{G[X'_1]}^{c_3}(y_2) \geq n/3 > 0$ for each vertex $y_2 \in Y_2$. By Lemma 3.1, $G[X'_1, Y'_2]$ contains a PC 4-cycle, a contradiction. So we have $|X'_1| + |Y'_2| \leq 2n/3$.

(2) Next we assert that $X_1 = X'_1$ and $C(X_1) = \{c_3\}$. By Claim 2, we have $C(X_1, Y_2) \subseteq \{c_2, c_3\}$. Since $d_G^{c_2}(y_2) \leq 2n/3 - 1$ for every vertex $y_2 \in Y_2$, we have $C(X'_1, Y_2) = \{c_3\}$. Choose arbitrary vertices $y_2, y'_2 \in Y_2$. Consider cycles $x_1x'_1y'_2y_2x_1$ for an arbitrary pair of vertices $x_1, x'_1 \in X'_1$. We have $C(X'_1) = \{c_3\}$. Consider cycles $v_0x_1x'_1y_2v_0$ for every pair of vertices $x_1 \in X_1 \setminus X'_1$ and $x'_1 \in X'_1$. We have $C(X'_1, X_1 \setminus X'_1) \subseteq \{c_1, c_3\}$. Since $d_G^{c_3}(x'_1) \leq 2n/3$ for every vertex $x'_1 \in X'_1$, we have $C(X_1, X_1 \setminus X'_1) = \{c_1\}$. If $|X_1 \setminus X'_1| > 0$, then it follows from the definition of X'_1 that there exists an edge uv in $E(X_1 \setminus X'_1)$ with

$C(uv) \neq \{c_1\}$. This implies that $uvx_1x'_1u$ is a PC 4-cycle for an arbitrary pair of vertices $x_1, x'_1 \in X'_1$, a contradiction. So we have $X'_1 = X_1$. It follows from the definition of X'_1 that $C(X_1) = \{c_3\}$.

(3) Moreover, we assert that $Y_2 = Y'_2$. By Claim 2, we have $C(X_1, Y_2) \subseteq \{c_2, c_3\}$. If $Y_2 \setminus Y'_2 \neq \emptyset$, then it follows from $d_G^{c_3}(x_1) \leq 2n/3 - 1$ for every $x_1 \in X_1$ that $C(X_1, Y_2 \setminus Y'_2) = \{c_2\}$. Now $d_G^{c_2}(y_2) \geq 2n/3 + 1$ for each vertex $y_2 \in Y_2 \setminus Y'_2$, a contradiction. So we have $Y_2 = Y'_2$. \square

By Claim 3, we have $|Y_1| = n/3 - 1 > 0$. It follows from Claim 2 that $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$. Since $\Delta_G^{mon}(y_1) \leq 2n/3 - 1$ for every vertex $y_1 \in Y_1$, we have $C(Y_1, Y_2) = \{c_2\}$. Recall that $C(X_1, Y_2) \subseteq \{c_2, c_3\}$. Since $d_G^{c_2}(y_2) \leq 2n/3 - 1$ for every vertex $y_2 \in Y_2$, we have $C(X_1, Y_2) = \{c_3\}$. Now it is clear that $G \in \mathcal{G}_3$, a contradiction. This completes the proof of Theorem 3.5. \square

Proof of Theorem 3.6. We prove the statements in the same order.

(i) Suppose that the statement is false, and that G is a counterexample on n vertices and v_0 is a vertex of G which is not contained in any PC triangle. Let $C(v_0, V(G)) = \{c_1, c_2, \dots, c_k\}$ and $U_i = \{u \in V(G) \setminus \{v_0\} : C(uv_0) = c_i\}$ for each $i \in \{1, 2, \dots, k\}$. Note that $k \geq 2$. Denote $|U_i| = n_i$ for each $i \in \{1, 2, \dots, k\}$. Assume w.l.o.g. that $n_1 \geq n_2 \geq \dots \geq n_k$. Denote $\Delta^{mon}(G) = \Delta$. Let $t = \lceil (n-1)/\Delta \rceil - 1$ and $r = n-1 - t\Delta$. Note that $0 < r \leq \Delta$.

Now we define a digraph D on vertex set $V(G) \setminus \{v_0\}$. For every edge $v_i v_j$ with $v_i \in U_i$ and $v_j \in U_j$, we add the arc $v_i v_j \in A(D)$ if and only if $C(vv_j) = C(v_i v_j)$. Since v_0 is not contained in any PC triangle, we have that $C(U_i, U_j) \subseteq \{c_i, c_j\}$ for every $1 \leq i < j \leq k$. Note that D is a multipartite

tournament. It follows from Lemma 3.3 that

$$\begin{aligned}
 (\Delta - 1)(n - 1) &\geq \sum_{v \in V(G) \setminus v_0} d^-(v) = |A(D)| \\
 &= \binom{n-1}{2} - \sum_{1 \leq i \leq k} \binom{n_i}{2} \\
 &\geq [(n-1)^2 - \sum_{1 \leq i \leq k} n_i^2]/2 \\
 &\geq [(n-1)^2 - t\Delta^2 - r^2]/2 \\
 &\geq [(n-1)^2 - (n-1-r)\Delta - r^2]/2 \\
 &\geq [(n-1)^2 - (n-1)\Delta]/2.
 \end{aligned}$$

This implies that $\Delta \geq (n+1)/3$, a contradiction.

(ii) Suppose that the statement is false, and that G is a counterexample on n vertices and v_0 is a vertex of G which is not contained in any PC 4-cycle. Let $C(v_0) = \{c_1, c_2, \dots, c_k\}$ and $U_i = \{u \in V(G) \setminus \{v_0\} : C(uv_0) = \{c_i\}\}$ for each $i \in \{1, 2, \dots, k\}$. Since $\Delta^{mon}(v_0) \leq (n-2)/2$, we have $k \geq 3$. Denote $|U_i| = n_i$ for each $i \in \{1, 2, \dots, k\}$.

Now we define a digraph D on vertex set $V(G) \setminus \{v_0\}$. For every edge $v_i v_j$ with $v_i \in V_i, v_j \in V_j$ and $i \neq j$, we add the arc $v_i v_j \in A(D)$ if and only if $C(v_0 v_j) = C(v_i v_j)$. We assert that $d_D^-(v_i) \leq \Delta^{mon}(G) - n_i$ for each vertex $v_i \in V_i$, where $i \in \{1, 2, \dots, k\}$. Indeed, if $d_D^-(v_i) = 0$ for some $v_i \in V_i$, where $i \in \{1, 2, \dots, k\}$, then it is clear that $d_D^-(v_i) \leq \Delta^{mon}(G) - n_i$. If $d_D^-(v_i) \neq 0$ for some $v_i \in V_i$, where $i \in \{1, 2, \dots, k\}$, then there exists a vertex $v_j \in V_j$ with $i \neq j$ such that $C(v_i v_j) = c_i$. Since $v_0 w v_i v_j v_0$ is not a PC 4-cycle for each vertex $w \in V_i \setminus \{v_i\}$, we have $C(v_i, V_i) = \{c_i\}$ for each vertex $v_i \in V_i$. This implies that $d_D^-(v_i) \leq \Delta^{mon}(G) - n_i$ for each vertex $v_i \in V_i$. Since $\sum_{v \in V(G) \setminus \{v_0\}} d_D^-(v) = \sum_{v \in V(G) \setminus \{v_0\}} d_D^+(v)$, there exists a vertex $v_{i_0} \in V_{i_0}$ for some $i_0 \in \{1, 2, \dots, k\}$ such that $d_D^+(v_{i_0}) \leq d_D^-(v_{i_0}) \leq \Delta^{mon}(G) - n_{i_0}$.

Moreover, we define a graph H on vertex set $V(G) \setminus \{v_0\}$. For every edge $v_i v_j$ with $v_i \in V_i, v_j \in V_j$ and $i \neq j$, we add the edge $v_i v_j \in E(H)$ if $C(v_i v_j) \notin \{c_i, c_j\}$. It follows that

$$d_H(v_{i_0}) + d_D^-(v_{i_0}) = (n-1-n_{i_0}) - d_D^+(v_{i_0}) \geq n-1 - \Delta^{mon}(G) \geq \Delta^{mon}(G) + 1.$$

If $d_D^-(v_{i_0}) = 0$, then there exist two adjacent edges $v_{i_0}v_s \in E(H)$ and $v_{i_0}v_t \in E(H)$ with $s \neq t$ such that $C(v_{i_0}v_s) \neq C(v_{i_0}v_t)$. If $d_D^-(v_{i_0}) \neq 0$, then there exists an edge $v_{i_0}v_s \in E(H)$ and an arc $v_tv_{i_0} \in A(D)$ such that $s \neq t$. It follows from the definitions of D and H that $C(v_{i_0}v_s) \neq C(v_{i_0}v_t)$. In both cases, v_0 is contained in the PC cycle $v_0v_sv_{i_0}v_tv_0$, a contradiction. \square

Proof of Theorem 3.7. We prove the statements in the same order.

(i) Suppose that the statement is false, and that G is a counterexample on n vertices and v_0 is a vertex of G not contained in any PC triangle. Let $C(v_0, V(G)) = \{c_1, c_2, \dots, c_k\}$ and $U_i = \{u \in V(G) \setminus \{v_0\} : C(uv_0) = c_i\}$ for each $i \in \{1, 2, \dots, k\}$. Note that $k \geq 2$. Indeed, if not, then $d_G^c(v_0) = 1$, which implies that

$$(n-1)^2 + 2 \leq \sum_{v \in V(G)} d_G^c(v) \leq (n-1) \cdot \Delta^{mon}(G) + d_G^c(v_0) \leq (n-1)^2 + 1,$$

a contradiction.

Now we define a digraph D on vertex set $V(G) \setminus \{v_0\}$. For every edge v_iv_j with $v_i \in V_i$ and $v_j \in V_j$, we add the arc $v_iv_j \in A(D)$ if and only if $C(vv_j) = C(v_iv_j)$. Since v_0 is not contained in any PC triangle, we have that $C(U_i, U_j) \subseteq \{c_i, c_j\}$ for every $1 \leq i < j \leq k$. So D is a k -partite tournament with k -partition $\{U_1, U_2, \dots, U_k\}$. This implies that

$$\sum_{v \in V(G) \setminus \{v_0\}} d_D^-(v) = |A(D)| \geq k(k-1)/2 \geq (k-1).$$

It follows that

$$\begin{aligned} (n-1)^2 + 2 &\leq \sum_{v \in V(G)} d_G^c(v) \\ &\leq d_G^c(v_0) + \sum_{v \in V(G) \setminus \{v_0\}} (n - \Delta_G^{mon}(v)) \\ &\leq d_G^c(v_0) + \sum_{v \in V(G) \setminus \{v_0\}} (n-1 - d_D^-(v)) \\ &\leq k + (n-1)^2 - (k-1) \\ &= (n-1)^2 + 1, \end{aligned}$$

a contradiction.

(ii) Suppose that the statement is false, and that G is a counterexample on n vertices and v_0 is a vertex of G not contained in any PC 4-cycle. Let $\Delta_G^{mon}(v_0) = r$ and $U = \{v \in V(G) \setminus \{v_0\} : \Delta_G^{mon}(v) = 1\}$. Note that

$$\sum_{v \in V(G) \setminus \{v_0\}} \Delta_G^{mon}(v) \leq 2n - 3 - r = 2(n - 1) - (r + 1).$$

By the pigeonhole principle, we have $|U| \geq r + 1 \geq 2$.

If $|U| = 2$, then $\Delta_G^{mon}(v_0) = r = 1$. Let $U = \{u_1, u_2\}$ and $W = V(G) \setminus U$. Since $v_0 u_1 u_2 w v_0$ and $v_0 u_2 u_1 w v_0$ are not PC 4-cycles for each vertex $w \in W$, we have $C(v_0 w) = C(w u_1) = C(w u_2)$. It follows that $\Delta_G^{mon}(w) \geq 3$ for each vertex $w \in W$. So

$$3 + 3(n - 3) \leq \sum_{v \in V(G)} \Delta_G^{mon}(v) \leq 2n - 3,$$

which implies that $n \leq 3$, a contradiction.

If $|U| \geq 3$, then it follows from $\Delta_G^{mon}(v_0) = r \leq |U| - 1$ that $|C(v_0, U)| \geq 2$. This implies that there exist two distinct vertices $u, v \in U$ such that $C(v_0 u) \neq C(v_0 v)$. Let $w \in U \setminus \{u, v\}$. By the definition of U , we have $\Delta_G^{mon}(u) = \Delta_G^{mon}(v) = \Delta_G^{mon}(w) = 1$. It is clear that $v_0 u w v v_0$ is a PC 4-cycle, a contradiction. \square

3.4 PC cycles of length at most 6 in edge-colored bipartite graphs

Proof of Theorem 3.8. Suppose to the contrary that the statement is false and G is a counterexample chosen such that $|E(G)|$ is as small as possible. Let $\{X, Y\}$ be the bipartition of G . Set $\delta = \delta^c(G)$. By the choice of G , we know that G contains no monochromatic paths of length three. Indeed, if not, then let $v_0 v_1 v_2 v_3$ be such a path and $G' = G - v_1 v_2$. Note that $\delta^c(G') = \delta^c(G) > \frac{n}{3} + 1$ but $|E(G')| < |E(G)|$, contradicting the choice of G .

Let v be a vertex and let c_0 be a color of edges incident with v in G

such that $\Delta^{mon}(G) = |\{u \in N(v) : C(uv) = c_0\}|$. Assume without loss of generality (abbreviated to w.l.o.g. in the sequel) that $v \in X$. Let $Y_0 = \{u \in N(v) : C(uv) = c_0\}$. Choose Y_1 to be a subset of $N(v) \setminus Y_0$ such that $C(vy_1) \neq C(vy_2)$ for all distinct $y_1, y_2 \in N(v) \setminus Y_0$ and $|Y_1| = \delta - 1$. For each vertex $y \in Y_1$, choose X_y to be a maximal subset of neighbors of y such that $C(x_1y) \neq C(x_2y)$ for all distinct $x_1, x_2 \in N(y) \setminus \{v\}$. Set $X_1 = \cup_{y \in Y_1} X_y$. For each vertex $x \in X_1$, let $Y_x = \{y \in Y \setminus Y_1 : C(xy) \notin C(x, Y_1)\}$. Set $Y_2 = \cup_{x \in X_1} Y_x$.

If $\Delta^{mon}(G) = 1$, then G is properly colored. It follows from Lemma 3.4 that G contains a PC 4-cycle. So we can assume that $\Delta^{mon}(G) \geq 2$. Since G contains no monochromatic paths of length three, it follows from the definitions of Y_0 and Y_2 that $Y_0 \cap Y_2 = \emptyset$. We define a digraph D on vertex set $X_1 \cup Y_1 \cup Y_2$. For each edge xy with $x \in X_1$ and $y \in Y_1$, we add the arc $xy \in A(D)$ if $C(vy) = C(xy)$ and the arc $yx \in A(D)$ if $C(vy) \neq C(xy)$; for each edge xy with $x \in X_1$ and $y \in Y_2$, we add the arc $xy \in A(D)$ if $C(xy) \notin C(x, Y_1)$.

Claim 4. (i) For each vertex $x \in X_1$, $|C(x, N_D^-(x))| \leq 1$.

(ii) For each vertex $y \in Y_2$, $|C(y, N_D^-(y))| \leq 1$.

Proof. (i) If there exists $x \in X_1$ such that $|C(x, N_D^-(x))| \geq 2$, then assume that $y_1, y'_1 \in Y_1$ are two distinct vertices such that $C(xy_1) \neq C(xy'_1)$. It follows that $vy_1xy'_1v$ is a PC 4-cycle, a contradiction. So we have $|C(x, N_D^-(x))| \leq 1$ for each vertex $x \in X_1$.

(ii) If there exists $y \in Y_2$ such that $|C(y, N_D^-(y))| \geq 2$, then assume that $x_1, x'_1 \in X_1$ are two distinct vertices such that $C(x_1y) \neq C(x'_1y)$. If $x_1, x_2 \in N_{y_1}$ for some $y_1 \in Y_1$, then $yx_1y_1x_2y$ is a PC 4-cycle, a contradiction. So we can assume that there exist two distinct vertices $y_1, y'_1 \in Y_1$ such that $x_1 \in N_{y_1}$ and $x'_1 \in N_{y'_1}$. It follows that $vy_1x_1yx'_1y'_1v$ is a PC 6-cycle, a contradiction. So we have $|C(y, N_D^-(y))| \leq 1$ for each vertex $y \in Y_2$. \square

Note that

$$|A(Y_1, X_1)| \geq \sum_{y \in Y_1} d_D^+(y) \geq (\delta - 1)|Y_1| = (\delta - 1)^2.$$

By (i) of Claim 4, we have $|C(x, Y_1)| \leq |N_D^+(x) \cap Y_1| + 1$ for each $x \in X_1$. Recall that $|Y_1| = \delta - 1$. It follows that

$$\begin{aligned}
 |A(X_1, Y_2)| &= \sum_{x \in X_1} |N_D^+(x) \cap Y_2| \geq \sum_{x \in X_1} (\delta - |C(x, Y_1)|) \\
 &\geq \sum_{x \in X_1} (\delta - (|N_D^+(x) \cap Y_1| + 1)) \\
 &= (\delta - 1)|X_1| - |A(X_1, Y_1)| \\
 &\geq (\delta - 1)|X_1| - (|X_1| \cdot |Y_1| - |A(Y_1, X_1)|) \\
 &= |A(Y_1, X_1)| \geq (\delta - 1)^2.
 \end{aligned}$$

By (ii) of Claim 4, we have $|Y_2| \geq |A(X_1, Y_2)|/\Delta \geq (\delta - 1)^2/\Delta$. But now

$$n < 3(\delta - 1) \leq \Delta + (\delta - 1) + (\delta - 1)^2/\Delta \leq |Y_0| + |Y_1| + |Y_2| \leq n,$$

a contradiction. □

Chapter 4

PC cycles of different lengths in edge-colored complete graphs

In this chapter, we show that (i) if $\Delta^{mon}(K_n^c) \leq n - 2k$, then K_n^c contains k PC cycles of different lengths and the bound is sharp; (ii) if $\Delta^{mon}(K_n^c) \leq n - 2^{k+1} - 2k + 4$, then K_n^c contains k vertex-disjoint PC cycles of different lengths; in particular, $\Delta^{mon}(K_n^c) \leq n - 6$ suffices for the existence of two vertex-disjoint PC cycles of different lengths.

4.1 Introduction

The existence of cycles with different constraints in graphs and digraphs have been extensively studied, see [18, 87] for cycles of prescribed lengths, [18, 29] for vertex-disjoint cycles and [47, 61] for edge-disjoint cycles. For more recent work, we refer the reader to [8, 9, 16, 58, 60, 72].

Edge-colored graphs have received considerable attention, see Chapter 16 in [14] for a survey and [10, 41, 59, 68, 76, 84] for more recent work. It is worth noting that edge-colored graphs can be viewed as a generalization of digraphs, and PC cycles in edge-colored graphs are closely related to directed cycles in digraphs. To see this, we consider a digraph D with vertex set $\{v_1, v_2, \dots, v_n\}$. By replacing each arc $v_i v_j$ with an edge and coloring it with

color c_j , we obtain an edge-colored graph G^c . It is not hard to see that there is a natural one-to-one correspondence between PC cycles in G^c and directed cycles in D . So many concepts and results on cycles in graphs and digraphs can often be generalized to PC cycles in edge-colored graphs.

Let k be a positive integer. It is widely known that every graph with minimum degree at least $k + 1$ contains k cycles of different lengths. Corrái and Hajnal [32] in 1963 proved that every graph of order $3k$ and minimum degree at least $2k$ contains k vertex-disjoint cycles. Bensmail et al. [17] in 2017 showed that every graph of order $7 \cdot \lfloor k^2/4 \rfloor$ and minimum degree at least $(k^2 + 3k)/2$ contains k vertex-disjoint cycles of different lengths. Analogously, every digraph with minimum out-degree at least k contains k directed cycles of different lengths. In 1981, Bermond and Thomassen [18] conjectured that every digraph with minimum out-degree at least $2k - 1$ contains k vertex-disjoint cycles. Lichiardopol [71] in 2014 conjectured that for every integer k , there exists an integer $f(k)$ such that every digraph with minimum out-degree at least $f(k)$ contains k vertex-disjoint cycles of different lengths. In 2020, Li et al. [68] conjectured that every edge-colored complete graph with n vertices and maximum monochromatic degree at most $n - 3k + 1$ contains k vertex-disjoint PC cycles and proved it for $k = 2$.

Motivated by above results and conjectures, we consider (vertex-disjoint) PC cycles of different lengths in edge-colored graphs.

Problem 4.1. For every positive integer k , does there exist an integer $g(k)$ such that every edge-colored graph G with $\delta^c(G) \geq g(k)$ contains k PC cycles of different lengths?

Problem 4.2. For every positive integer k , does there exist an integer $h(k)$ such that every edge-colored graph G with $\delta^c(G) \geq h(k)$ contains k vertex-disjoint PC cycles of different lengths?

For any positive integer d , Wang and Li [88] constructed an edge-colored graph G with $\delta^c(G) \geq d$ that contains no PC cycles. This implies that neither Problem 4.1 nor Problem 4.2 has a positive answer. One may naturally ask what happens if the host graphs are edge-colored complete graphs. Denote by K_n^c an edge-colored complete graph with n vertices. In this Chapter, we obtain

the following sufficient conditions in terms of maximum monochromatic degree for the existence of (vertex-disjoint) PC cycles of different lengths. Sufficient conditions in terms of minimum color degree for the existence of (vertex-disjoint) PC cycles of different lengths will be obtained as a corollary.

Theorem 4.1. *If $\Delta^{mon}(K_n^c) \leq n - 2k$, then K_n^c contains k PC cycles of different lengths.*

Theorem 4.2. *If $\Delta^{mon}(K_n^c) \leq n - 2^{k+1} - 2k + 4$, then K_n^c contains k vertex-disjoint PC cycles of different lengths.*

In particular, for the case $k = 2$ of Theorem 4.2, we obtain the following stronger result by weakening the maximum monochromatic degree condition.

Theorem 4.3. *If $\Delta^{mon}(K_n^c) \leq n - 6$, then K_n^c contains two vertex-disjoint PC cycles of different lengths.*

We postpone the proofs of Theorems 4.1, 4.2 and 4.3 in Sections 4.2, 4.3 and 4.4, respectively.

Remark 4.1. Let V_1 and V_2 be two disjoint nonempty vertex sets. Consider a K_n^c with vertex set $V_1 \cup V_2$ such that all edges between V_1 and V_2 are colored with c_1 and the remaining edges are colored with c_2 . If $|V_1| = 2k - 1$ and $|V_2| \geq 2k - 1$, then $\Delta^{mon}(K_n^c) = n - 2k + 1$. Let C be an arbitrary PC cycle in K_n^c . Note that every edge of C between V_1 and V_2 must be immediately followed by one edge with both end-vertices in V_i with $i \in \{1, 2\}$ and every edge of C with both end-vertices in V_i with $i \in \{1, 2\}$ must be immediately followed by one edge between V_1 and V_2 . It follows that $4 \leq |V(C)| \leq 4k - 4$ and $|V(C)| \equiv 0 \pmod{4}$. This implies that K_n^c contains exactly $k - 1$ PC cycles of different lengths. So the bound of Theorem 4.1 is sharp. If $|V_1| = 5$ and $|V_2| \geq 5$, then $\Delta^{mon}(K_n^c) = n - 5$ and K_n^c contains no two vertex-disjoint PC cycles of different lengths. So the bound of Theorem 4.3 is sharp.

Since $\Delta^{mon}(K_n^c) + \delta^c(K_n^c) \leq n$, we can obtain the following sufficient conditions in terms of minimum color degree for the existence of (vertex-disjoint) PC cycles of different lengths.

Corollary 4.1. (i) If $\delta^c(K_n^c) \geq 2k$, then K_n^c contains k PC cycles of different lengths.

(ii) If $\delta^c(K_n^c) \geq 2^{k+1} + 2k - 4$, then K_n^c contains k vertex-disjoint PC cycles of different lengths.

(iii) If $\delta^c(K_n^c) \geq 6$, then K_n^c contains two vertex-disjoint PC cycles of different lengths.

The remainder of this paper is organized as follows. In Sections 4.2, 4.3 and 4.4, we present the proofs of Theorems 4.1, 4.2 and 4.3, respectively.

The following two known results will be used in our later proofs.

Theorem 4.4 (Gallai [46]). *For every Gallai coloring of a complete graph, there exists a Gallai partition.*

Theorem 4.5 (Yeo [90]). *Let G be an edge-colored graph containing no PC cycles. Then G contains a vertex v such that no component of $G - v$ is joined to v with edges of more than one color.*

4.2 PC cycles of different lengths

Proof of Theorem 4.1. Suppose to the contrary that the statement is false and let G be a counterexample. By Theorem 4.5, if $\Delta^{mon}(G) \leq n - 2$, then G contains a PC cycle. So we have $k \geq 2$. Let $P = v_1 v_2 \cdots v_p$ be a longest PC path in G . Denote $C(v_1 v_2) = c_1$, $C(v_{p-1} v_p) = c_2$ and $C(v_1 v_p) = c_3$. Observe that for each vertex $v \in V(G) \setminus V(P)$, $C(v v_1) = c_1$ and $C(v v_p) = c_2$, otherwise a longer PC path would result from it. Since $d_G^{c_1}(v_1) \leq n - 2k$ and $d_G^{c_2}(v_p) \leq n - 2k$, we have that

$$|\{v_i \in V(P) \setminus v_1 : C(v_1 v_i) \neq c_1\}| = n - 1 - d_G^{c_1}(v_1) \geq 2k - 1,$$

$$|\{v_i \in V(P) \setminus v_p : C(v_p v_i) \neq c_2\}| = n - 1 - d_G^{c_2}(v_p) \geq 2k - 1.$$

Note that

$$p \geq |\{v_1, v_2\}| + |\{v_i \in V(P) \setminus v_1 : C(v_1 v_i) \neq c_1\}| \geq 2 + (2k - 1) \geq 5.$$

Let

$$\begin{aligned} S_1 &= \{v_i \in V(P) \setminus v_1 : C(v_1 v_i) \neq c_1 \text{ and } C(v_1 v_i) \neq C(v_i v_{i-1})\}, \\ T_1 &= \{v_i \in V(P) \setminus v_1 : C(v_1 v_i) \neq c_1 \text{ and } C(v_1 v_i) = C(v_i v_{i-1})\}, \\ S_2 &= \{v_i \in V(P) \setminus v_p : C(v_p v_i) \neq c_2 \text{ and } C(v_p v_i) \neq C(v_i v_{i+1})\}, \text{ and} \\ T_2 &= \{v_i \in V(P) \setminus v_p : C(v_p v_i) \neq c_2 \text{ and } C(v_p v_i) = C(v_i v_{i+1})\}. \end{aligned}$$

Then $|S_i| + |T_i| \geq 2k - 1$ for each $i \in \{1, 2\}$. Note that $v_2 \notin T_1$ and $v_{p-1} \notin T_2$. Since the set of cycles of the type $v_1 v_i v_{i-1} \cdots v_2 v_1$, where $v_i \in S_1$, contains no k PC cycles of different lengths, we have $|S_1| \leq k - 1$. Similarly, since the set of cycles of the type $v_p v_i v_{i+1} \cdots v_{p-1} v_p$, where $v_i \in S_2$, contains no k PC cycles of different lengths, we have $|S_2| \leq k - 1$. It follows that $|T_i| \geq k$ for each $i \in \{1, 2\}$.

Claim 1. The following two statements hold.

- (i) $C(v_1 v_2)$, $C(v_{p-1} v_p)$ and $C(v_1 v_p)$ are pairwise different, i.e., c_1, c_2 and c_3 are pairwise distinct;
- (ii) $C(v_1 v_2) = C(v_1 v_{p-1}) = C(v_{p-1} v_{p-2}) = c_1$ and $C(v_p v_{p-1}) = C(v_p v_2) = C(v_2 v_3) = c_2$.

Proof. (i) Firstly we assert that $C(v_1 v_2) \neq C(v_{p-1} v_p)$ (i.e., $c_1 \neq c_2$). If $C(v_1 v_2) = C(v_{p-1} v_p)$, then $v_p \notin T_1$ and $v_1 \notin T_2$. Recall that $v_2 \notin T_1$ and $v_{p-1} \notin T_2$. If $C(v_2 v_p) = c_1$, then $v_2 \notin T_2$. Since $|T_2| \geq k$, the set of cycles of the type $v_p v_i v_{i-1} \cdots v_2 v_p$, where $v_i \in T_2$, contains at least k PC cycles of different lengths, a contradiction. If $C(v_2 v_p) \neq c_1$, then it follows from $|T_1| \geq k$ that the set of cycles of the type $v_1 v_i v_{i+1} \cdots v_p v_2 v_1$, where $v_i \in T_1$, contains at least k PC cycles of different lengths, a contradiction.

Next we assert that $C(v_1 v_2) \neq C(v_1 v_p)$ and $C(v_p v_{p-1}) \neq C(v_1 v_p)$. By symmetry, we only need to show that $C(v_1 v_2) \neq C(v_1 v_p)$ (i.e., $c_1 \neq c_3$). Indeed, if $C(v_1 v_p) = C(v_1 v_2)$, then we have $C(v_1 v_p) \neq C(v_{p-1} v_p)$, which implies that $v_p \notin T_1$. Recall that $v_2 \notin T_1$. Since $|T_1| \geq k$, the set of cycles of the type $v_1 v_i v_{i+1} \cdots v_p v_1$, where $v_i \in T_1$, contains at least k PC cycles of different lengths, a contradiction.

(ii) By symmetry, we only need to show that $C(v_2v_p) = C(v_2v_3) = c_2$. Let

$$S_3 = \{v_i \in V(P) : C(v_1v_i) \neq c_3 \text{ and } C(v_1v_i) \neq C(v_1v_{i+1})\}, \text{ and}$$

$$T_3 = \{v_i \in V(P) : C(v_1v_i) \neq c_3 \text{ and } C(v_1v_i) = C(v_1v_{i+1})\}.$$

It follows that $|S_3| + |T_3| \geq 2k - 1$. Note that $v_2, v_p \notin T_3$. Since the set of cycles of the type $v_1v_iv_{i+1} \cdots v_pv_1$, where $v_i \in S_3$, contains no k PC cycles of different lengths, we have $|S_3| \leq k - 1$. It follows that $|T_3| \geq k$.

Firstly we assert that $C(v_2v_p) = c_2$. Suppose to the contrary that $C(v_2v_p) \neq c_2$. Recall that $v_2, v_p \notin T_1$. If $C(v_2v_p) \neq c_1$, then it follows from $|T_1| \geq k$ that the set of cycles of the type $v_1v_iv_{i+1} \cdots v_pv_2v_1$, where $v_i \in T_1$, contains at least k PC cycles of different lengths, a contradiction. If $C(v_2v_p) = c_1$, then it follows from $|T_3| \geq k$ that the set of cycles of the type $v_1v_iv_{i-1} \cdots v_2v_pv_1$, where $v_i \in T_3$, contains k PC cycles of different lengths, a contradiction.

Next we assert that $C(v_2v_3) = c_2$. Suppose to the contrary that $C(v_2v_3) \neq c_2$. Recall that $v_2, v_p \notin T_3$. It follows from $|T_3| \geq k$ that the set of cycles of the type $v_1v_iv_{i-1} \cdots v_2v_pv_1$, where $v_i \in T_3$, contains at least k PC cycles of different lengths, a contradiction. \square

By (i) of Claim 1, we have $v_1 \notin T_2$ and $v_p \notin T_1$. By (ii) of Claim 1, we have $v_2, v_{p-1} \notin T_1$ and $v_2, v_{p-1} \notin T_2$. It follows that if $v_i \in T_1$ or $v_i \in T_2$ for some i , then $3 \leq i \leq p - 2$. Note that all paths of the type $v_2v_1v_iv_{i+1} \cdots v_{p-1}$ for $v_i \in T_1$ are PC paths of different lengths with $C(v_1v_2) = C(v_{p-2}v_{p-1}) = c_1$. Note also that all paths of the type $v_2v_3 \cdots v_iv_pv_{p-1}$ for $v_i \in T_2$ are PC paths of different lengths with $C(v_2v_3) = C(v_{p-1}v_p) = c_2$. Recall that $|T_i| \geq k$ for each $i \in \{1, 2\}$. If $C(v_2v_{p-1}) \neq c_1$, then $|T_1| \geq k$ implies that the set of cycles of the type $v_1v_iv_{i+1} \cdots v_{p-1}v_2v_1$ for $v_i \in T_1$, contains at least k PC cycles of different lengths, a contradiction. If $C(v_2v_{p-1}) = c_1$, then $|T_2| \geq k$ implies that the set of cycles of the type $v_pv_iv_{i-1} \cdots v_2v_{p-1}v_p$ for $v_i \in T_2$, contains at least k PC cycles of different lengths, a contradiction. \square

4.3 Vertex-disjoint PC cycles of different lengths

Proof of Theorem 4.2. Suppose to the contrary that the statement is false and let G be a counterexample chosen such that k is as small as possible. By Theorem 4.5, if $\Delta^{mon}(G) \leq n - 2$, then G contains a PC cycle, which implies that $k \geq 2$. By the choice of G , we can assume that G contains at least $k - 1$ vertex-disjoint PC cycles of different lengths. Let C_1, C_2, \dots, C_{k-1} be such PC cycles satisfying that $|V(C_1)| < |V(C_2)| < \dots < |V(C_{k-1})|$ and $\sum_{i=1}^{k-1} |V(C_i)|$ is as small as possible. We will get a contradiction by showing that G contains k vertex-disjoint PC cycles of different lengths.

Claim 1. $|V(C_1)| \leq 4$.

Proof. Suppose to the contrary that $|V(C_1)| \geq 5$. It suffices to show that there exists a shorter PC cycle C^* such that $V(C^*) \subset V(C_1)$. Denote $C_1 = v_1 v_2 \dots v_t v_1$. If $C(v_1 v_2) \neq C(v_1 v_3)$ and $C(v_2 v_3) \neq C(v_1 v_3)$, then $v_1 v_2 v_3 v_1$ is such a shorter PC cycle C^* , a contradiction. So we can assume w.l.o.g. that $C(v_1 v_2) = C(v_1 v_3)$. Since $v_1 v_3 v_4 \dots v_t v_1$ is not such a shorter PC cycle C^* , we have $C(v_1 v_3) = C(v_3 v_4)$. It follows that either $v_1 v_2 v_3 v_4 v_1$ or $v_1 v_4 v_5 v_6 \dots v_t v_1$ is such a shorter PC cycle C^* , a contradiction. \square

Claim 2. $|V(C_i)| \leq 2|V(C_{i-1})|$ for each $i \in \{2, 3, \dots, k-1\}$.

Proof. Suppose to the contrary that there exists some $i_0 \in \{2, 3, \dots, k-1\}$ such that $|V(C_{i_0})| \geq 2|V(C_{i_0-1})| + 1$. It suffices to show that there exists a PC cycle C^* such that $V(C^*) \subset V(C_{i_0})$ and $|V(C_{i_0-1})| < |V(C^*)| < |V(C_{i_0})|$. Denote $|V(C_{i_0-1})| = s$ and $|V(C_{i_0})| = t$. Then it is clear that $t \geq 2s + 1$. Let $C_{i_0} = v_1 v_2 \dots v_t v_1$. If $C(v_1 v_2) \neq C(v_1 v_{s+1})$ and $C(v_s v_{s+1}) \neq C(v_1 v_{s+1})$, then $v_1 v_2 \dots v_{s+1} v_1$ is such a PC cycle C^* , a contradiction. So we can assume w.l.o.g. that $C(v_1 v_2) = C(v_1 v_{s+1})$. Since $v_1 v_{s+1} v_{s+2} \dots v_t v_1$ is not such a PC cycle C^* , we have $C(v_1 v_{s+1}) = C(v_{s+1} v_{s+2})$. It follows that either $v_1 v_2 \dots v_{s+1} v_{s+2} v_1$ or $v_1 v_{s+2} v_{s+3} \dots v_t v_1$ is such a PC cycle C^* , a contradiction. \square

Now $|V(C_i)| \leq 2^{i+1}$ for each $i \in \{1, 2, \dots, k-1\}$, which implies that $\sum_{i=1}^{k-1} |V(C_i)| \leq 2^{k+1} - 4$. Let $G' = G - V(C_1) \cup V(C_2) \cup \dots \cup V(C_{k-1})$. Then

$$\begin{aligned} \Delta^{mon}(G') &\leq n - 2^{k+1} - 2k + 4 \\ &= (n - \sum_{i=1}^{k-1} |V(C_i)|) + (\sum_{i=1}^{k-1} |V(C_i)| - 2^{k+1} - 2k + 4) \\ &\leq |V(G')| - 2k. \end{aligned}$$

By Theorem 4.1, we have that G' contains at least one PC cycle C^* of length different from C_1, C_2, \dots, C_{k-1} . This implies that $C^*, C_1, C_2, \dots, C_{k-1}$ are k vertex-disjoint PC cycles of different lengths in G , a contradiction. \square

4.4 Two vertex-disjoint PC cycles of different lengths

In this section, we present the proof of Theorem 4.3. Firstly we give five lemmas which will be used in the proof.

Lemma 4.1. *Let K_n^c be an edge-colored complete graph containing no PC odd cycles. If $\delta^c(K_n^c) \geq 2$ and $\Delta^c(K_n^c) \geq 3$, then K_n^c has a partition $\{U, X, Y\}$ such that $C(X) \subseteq C(X, U) = \{c_1\}$, $C(Y) \subseteq C(Y, U) = \{c_2\}$ and $C(X, Y) \subseteq \{c_1, c_2\}$, where c_1 and c_2 are two distinct colors.*

Proof. Note that Theorem 2.2 states that if K_n^c contains no PC odd cycles, then K_n^c is in three special classes of graphs. If $\Delta^c(K_n^c) \geq 3$, then K_n^c cannot be in the first class and if $\delta^c(K_n^c) \geq 2$, then K_n^c cannot be in the second class. Therefore K_n^c is in the third class, as stated in Lemma 4.1. \square

Lemma 4.2. *If $\Delta^{mon}(K_n^c) \leq n - 3$, then K_n^c contains a PC cycle of length at least 4.*

Proof. Suppose to the contrary that the statement is false and G is a counterexample. A result of Axenovich et al. (see [6], Theorem 4.10) states that every edge-colored complete graph with minimum color degree at least 3 contains a PC 4-cycle. So there exists a vertex $v \in V(G)$ such that $d_G^c(v) = 2$. Let $C(v, V(G)) = \{c_1, c_2\}$ and $U_i = \{u \in V(G) : C(uv) = c_i\}$ for each $i \in \{1, 2\}$.

Since $d_G^{c_1}(v) \leq n-3$ and $d_G^{c_2}(v) \leq n-3$, we have $|U_1| \geq 2$ and $|U_2| \geq 2$. Let $X_i = \{x \in U_i : C(x, U_i) \neq \{c_i\}\}$ and $Y_i = U_i \setminus X_i$ for each $i \in \{1, 2\}$. Note that for each $i \in \{1, 2\}$, if $X_i \neq \emptyset$ and $Y_i \neq \emptyset$, then it follows from the definition of X_i that $C(X_i, Y_i) = \{c_i\}$.

Claim 1. For any two disjoint subsets $V_1, V_2 \subseteq V(G)$ with $C(V_1, V_2) = \{c_1, c_2\}$, if $d_{G[V_1]}^{c_1}(v_1) \geq 1$ for each $v_1 \in V_1$ and $d_{G[V_2]}^{c_2}(v_2) \geq 1$ for each $v_2 \in V_2$, then G contains a PC 4-cycle.

Proof. Construct a bipartite tournament (i.e. an oriented complete bipartite graph) D with a bipartite sets V_1, V_2 and arc set $A(D)$ as follows. For each edge $v_1 v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$, let $v_1 v_2 \in A(D)$ if $C(v_1 v_2) = c_1$ in G ; otherwise, let $v_2 v_1 \in A(D)$. Note that D is a bipartite tournament with $\delta^+(D) \geq 1$. So D contains a directed cycle. Let $C = x_1 x_2 \cdots x_t x_1$ be a shortest directed cycle in D . If $t > 4$, then C have a chord, say $x_i x_j$. Note that $x_j x_{j+1} \cdots x_i x_j$ is a shorter directed cycle, where subscripts are taken modulo t , a contradiction. So C is a directed 4-cycle, which corresponds to a PC 4-cycle in G . \square

Now we distinguish three cases.

Case 1. $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$.

Choose arbitrary vertices $x_1 \in X_1$ and $x_2 \in X_2$. By the definition of X_1 and X_2 , there exist edges $x_1 x'_1$ in $E(X_1)$ and $x_2 x'_2$ in $E(X_2)$ such that $C(x_1 x'_1) \neq c_1$ and $C(x_2 x'_2) \neq c_2$. Since $v x'_1 x_1 x_2 v$ and $v x_1 x_2 x'_2 v$ are not PC 4-cycles, we have $c_1 \notin C(X_1, X_2)$ and $c_2 \notin C(X_1, X_2)$.

Firstly we assert that there exists a new color c_3 such that $C(X_1, X_2) = \{c_3\}$. Indeed, if not, then there exist two adjacent edges in $E(X_1, X_2)$ with different colors, say c_3 and c_4 . By symmetry, assume that $C(x x') = c_3$ and $C(x x'') = c_4$, where $x \in X_1$ and $x', x'' \in X_2$. By the definition of X_1 , there exists a vertex $x''_1 \in X_1 \setminus x$ such that $C(x x''_1) \neq c_1$. It follows that either $v x''_1 x x' v$ or $v x''_1 x x'' v$ is a PC 4-cycle, a contradiction.

Next we assert that $C(X_1) = C(X_2) = \{c_3\}$. By symmetry, we only need to show that $C(X_1) = \{c_3\}$. Since $v x'_1 x_1 x_2 v$ is not PC 4-cycle, we have $C(X_1) \subseteq \{c_1, c_3\}$. By the definition of X_1 , we have that $c_3 \in C(X_1)$. If $C(X_1) \neq \{c_3\}$, then there exist two adjacent edges $x_1 x'_1, x'_1 x''_1$ in $E(X_1)$ such

that $C(x_1x'_1) = c_3$ and $C(x'_1x''_1) = c_1$. It follows that $vx_1x'_1x''_1x_2v$ is a PC 5-cycle, a contradiction.

If $Y_1 \cup Y_2 = \emptyset$, then $d_G^{c_3}(x) = n - 2$ for each vertex $x \in X_1 \cup X_2$, a contradiction. Assume w.l.o.g that $Y_1 \neq \emptyset$. Choose an arbitrary vertex $y_1 \in Y_1$. Since neither $vx_1x'_1y_1x_2v$ nor $vx_1x'_1y_1x_2x'_2v$ is a PC cycle of length at least 4, we have $C(X_2, Y_1) = \{c_1\}$. If $Y_2 = \emptyset$, then $d_G^{c_1}(y_1) = n - 1$, a contradiction. So we can assume that $Y_2 \neq \emptyset$. Since neither $vx_1y_2x_2x'_2v$ nor $vx'_1x_1y_2v$ is a PC cycle of length at least 4, we have $C(X_1, Y_2) = \{c_2\}$. Moreover, since $x_1y_1y_2x_2x_1$ is not a PC 4-cycle, we have $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$. By the definition of Y_1 and Y_2 , we have $C(Y_1) \subseteq \{c_1\}$ and $C(Y_2) \subseteq \{c_2\}$. It follows that $C(Y_1, U_1 \cup X_2 \cup \{v\}) = \{c_1\}$ and $C(Y_2, X_1 \cup U_2 \cup \{v\}) = \{c_2\}$. Since $\Delta^{mon}(G) \leq n - 3$, we have $d_{G[Y_2]}^{c_2}(y_1) \geq 1$ for each $y_1 \in Y_1$ and $d_{G[Y_1]}^{c_1}(y_2) \geq 1$ for each $y_2 \in Y_2$. By Claim 1, $G[Y_1, Y_2]$ contains a PC 4-cycle, a contradiction.

Case 2. $X_1 \neq \emptyset$ or $X_2 \neq \emptyset$, but not both.

By symmetry between X_1 and X_2 , assume that $X_1 \neq \emptyset$ and $X_2 = \emptyset$. Note that $Y_2 = U_2 \setminus X_2 \neq \emptyset$. Choose arbitrary vertices $x_1 \in X_1$ and $y_2 \in Y_2$. By the definition of X_1 , there exists an edge $x_1x'_1$ in $E(X_1)$ such that $C(x_1x'_1) \neq c_1$.

Firstly we assert that $Y_1 = \emptyset$. Suppose to the contrary that $Y_1 \neq \emptyset$. Choose an arbitrary vertex $y_1 \in Y_1$. Since $vx_1x'_1y_1y_2v$ is not a PC 5-cycle, we have $C(Y_1, Y_2) \subseteq \{c_1, c_2\}$. The condition $d_G^{c_1}(y_1) \leq n - 3$ implies that $d_{G[Y_2]}^{c_2}(y_1) \geq 2$ for each $y_1 \in Y_1$. Let Y'_2 be a maximal subset of Y_2 such that $C(Y_1, Y'_2) \subseteq \{c_2\}$. Note that if $|Y'_2| \leq 1$, then it follows from Claim 1 that $G[Y_1, Y_2 \setminus Y'_2]$ contains a PC 4-cycle, a contradiction. So $|Y'_2| \geq 2$, which implies that there exist at least two distinct vertices $y, y' \in Y_2$ such that $C(\{y, y'\}, Y_1) = \{c_2\}$. Since $d_G^{c_2}(y) \leq n - 3$ and $d_G^{c_2}(y') \leq n - 3$, there exist two nonadjacent edges $xy, x'y'$ in $E(X_1, Y_2)$ such that $C(xy) \neq c_2$ and $C(x'y') \neq c_2$. By the definition of X_1 , there exists a vertex $x_0 \in X_1 \setminus x$ such that $C(xx_0) \neq c_1$ and a vertex $x'_0 \in X_1 \setminus x'$ such that $C(x'_0x') \neq c_1$. Since vx_0xyv and $vx'_0x'y'v$ are not PC 4-cycles, we have $C(xy) \neq c_1$ and $C(x'y') \neq c_1$. Recall that $C(X_1, Y_1) = \{c_1\}$. Let y_1 be a vertex in Y_1 . Then $xyy_1x'y'v$ is a PC 6-cycle, a contradiction.

Next we assert that there exists a vertex $x \in X_1$ and a new color, say c_3 , such that $C(x, X_1) = \{c_3\}$. Since $vx'_1x_1y_2v$ is not a PC 4-cycle, we have $c_1 \notin C(X_1, Y_2)$. It follows from $d_G^{c_2}(y_2) \leq n - 3$ for each $y_2 \in Y_2$ that there

exists an edge xy with $x \in X_1$ and $y \in Y_2$ such that $C(xy) \notin \{c_1, c_2\}$. Let $C(xy) = c_3$. We will show that x is a vertex such that $C(x, X_1) = \{c_3\}$. Let x_1 be an arbitrary vertex in $X_1 \setminus x$. Since, for each $x_1 \in X_1 \setminus x$, vx_1xyv is not a PC 4-cycle, we have $C(x, X_1 \setminus x) \subseteq \{c_1, c_3\}$. If there exists a vertex $x' \in X_1 \setminus x$ such that $C(xx') = c_1$, then it follows from the definition of X_1 that there exists a vertex $x'' \in X_1 \setminus x$ such that $C(x'x'') \neq c_1$. This implies that $vx''x'xyv$ is a PC 5-cycle, a contradiction. So $C(x, X_1) = \{c_3\}$.

Moreover, we assert that there exists a vertex $x' \in X_1 \setminus x$ such that $C(x', Y_2) = \{c_3\}$. Recall that $C(x, X_1) = \{c_3\}$. So, for each $x_1 \in X_1$, there exists an edge $x_1x'_1$ in $E(X_1)$ such that $C(x_1x'_1) = c_3$. Let y_2 be an arbitrary vertex in Y_2 . Since $vx'_1x_1y_2v$ is not a PC 4-cycle, we have $C(X_1, Y_2) \subseteq \{c_2, c_3\}$. Recall that $Y_1 = \emptyset$. Consider the complete bipartite graph $G[X_1 \setminus x, Y_2]$. For each vertex $y_2 \in Y_2$, since $d_G^{c_2}(y_2) \leq n-3$, we have $d_{G[X_1 \setminus x]}^{c_3}(y_2) \geq 1$. It follows from Claim 1 that there exists a vertex $x' \in X_1 \setminus x$ such that $C(x', Y_2) = \{c_3\}$.

Let x be a vertex in X_1 such that $C(x, X_1) = \{c_3\}$ and x' a vertex in $X_1 \setminus x$ such that $C(x', Y_2) = \{c_3\}$. It follows from $d_G^{c_3}(x') \leq n-3$ that $C(x', X_1) \neq \{c_3\}$. This implies that there exists an edge $x'x''$ with $x'' \in X_1 \setminus x$ such that $C(x'x'') \neq c_3$. Since $C(x, X_1) = \{c_3\}$, we have $C(xx'') = c_3$. It follows that $vx'x''x'y_2v$ is a PC 5-cycle, a contradiction.

Case 3. $X_1 = \emptyset$ and $X_2 = \emptyset$.

Note that $Y_1 = U_1 \neq \emptyset$ and $Y_2 = U_2 \neq \emptyset$. If $|C(G)| \leq 2$, then G contains no PC odd cycles. Since $\Delta^{mon}(G) \leq n-3$, it follows from Theorem 4.5 that G contains a PC cycle of length at least 4, a contradiction. So $|C(G)| \geq 3$. Let H be the subgraph induced by all edges with colors different from c_1 and c_2 . If H contains two nonadjacent edges y_1y_2 and $y'_1y'_2$ with $y_1, y'_1 \in Y_1$ and $y_2, y'_2 \in Y_2$, then $y_1y_2y'_2y'_1$ is a PC 4-cycle, a contradiction. So H is a star. Let u be the center of star H and let $G' = G - u$. Then $\Delta^{mon}(G') \leq n-3 = |V(G')| - 2$, which implies that $|C(G')| = 2$. Note that G' contains no PC odd cycles. By Theorem 4.5, G' contains a PC cycle of length at least 4, which is also a PC cycle of G , a contradiction. \square

Lemma 4.3. Let $C = xyzwx$ be a PC 4-cycle in K_n^c , where $C(xy) = C(zw) = c_1$ and $C(yz) = C(xw) = c_2$. If uv is an edge vertex-disjoint with C such that

$C(uv) = c_1$ and $d_C^{c_2}(u) + d_C^{c_2}(v) \geq 5$, then $K_n^c[V(C) \cup \{u, v\}]$ contains a PC 6-cycle.

Proof. Suppose to the contrary that $K_n^c[V(C) \cup \{u, v\}]$ contains no PC 6-cycles. Assume w.l.o.g. that $d_C^{c_2}(u) \geq d_C^{c_2}(v)$. Note that $3 \leq d_C^{c_2}(u) \leq 4$ since $d_C^{c_2}(u) + d_C^{c_2}(v) \geq 5$. If $d_C^{c_2}(u) = 3$, then assume that $C(u, \{x, y, z\}) = \{c_2\}$. Since $uxyzwvu$, $uyxwzvu$ and $uzwxyvu$ are not PC 6-cycles, we have $C(v, \{y, z, w\}) = \{c_1\}$, which implies $d_C^{c_2}(v) \leq 1$. If $d_C^{c_2}(u) = 4$, then $C(u, V(C)) = \{c_2\}$. Since $uxyzwvu$, $uyxwzvu$, $uzwxyvu$ and $uwzyxvu$ are not PC 6-cycles, we have $C(v, V(C)) = \{c_1\}$, which implies $d_C^{c_2}(v) = 0$. In both cases, $d_C^{c_2}(u) + d_C^{c_2}(v) \leq 4$, a contradiction. \square

Lemma 4.4. Suppose that K_n^c has a bipartition $\{V_1, V_2\}$ such that $C(V_1, V_2) = \{c_1\}$, where $|V_1| = n_1$ and $|V_2| = n_2$. If $\Delta^{mon}(K_n^c) \leq n - 2$ and K_n^c contains only PC 4-cycles, then for each $i \in \{1, 2\}$, there exists a new color c'_i such that $\{c'_i\} \subseteq C(V_i) \subseteq \{c_1, c'_i\}$. Moreover, if $C(V_i) = \{c_1, c'_i\}$, then $K_n^{c'_i}[V_i] \cong K_{1, n_i-1}$.

Proof. By symmetry between V_1 and V_2 , we only need to show that the statement holds for $i = 1$. Let H_1 be an edge-colored graph induced by all edges in $E(V_1)$ with colors different from c_1 . It suffices to show that either H_1 is a complete graph or H_1 is a star, and $|C(H_1)| = 1$ in both cases.

If $c_1 \notin C(V_1)$, then it is clear that H_1 is a complete graph. If $c_1 \in C(V_1)$, then there exists an edge xy in $E(V_1)$ with $C(xy) = c_1$. Since $d^{c_1}(y) \leq n - 2$, there exists a vertex $z \in V_1 \setminus \{x, y\}$ such that $C(yz) \neq c_1$. Let $W = V_1 \setminus \{x, y, z\}$. Since $\Delta^{mon}(K_n^c) \leq n - 2$, there exists an edge $x_2y_2 \in E(V_2)$ such that $C(x_2y_2) \neq c_1$. If there exists a vertex $w \in W$ such that $C(xw) \neq c_1$, then $x_2y_2zyxwx_2$ is a PC 6-cycle, a contradiction. So we have $C(x, W) \subseteq \{c_1\}$. It follows from $d^{c_1}(x) \leq n - 2$ that $C(xz) \neq c_1$. Since $x_2y_2wyxzx_2$ is not a PC 6-cycle for each vertex $w \in W$, we have $C(y, W) \subseteq \{c_1\}$. If there exists an edge ww' in $E(W)$ such that $C(ww') \neq c_1$, then $x_2y_2ww'xzx_2$ is a PC 6-cycle. So we have $C(W) \subseteq \{c_1\}$. Since $d^{c_1}(w) \leq n - 2$ for each $w \in W$, we have $c_1 \notin C(z, W)$. Hence H_1 is a star. In both cases, H_1 is a connected graph. If $|C(H_1)| \neq 1$, then there exist two adjacent edges x_1y_1 and y_1z_1 in H_1 such that $C(x_1y_1) \neq C(y_1z_1)$, then $x_1y_1z_1x_2y_2x_1$ is a PC 5-cycle, a contradiction. So we have $|C(H_1)| = 1$. \square

Lemma 4.5. *Suppose that K_n^c has a partition $\{U, X, Y\}$ with $U \neq \emptyset$ and $X \cup Y \neq \emptyset$ such that $C(X) \subseteq C(X, U) = \{c_1\}$, $C(Y) \subseteq C(Y, U) = \{c_2\}$ and $C(X, Y) \subseteq \{c_1, c_2\}$, where c_1 and c_2 are two distinct colors. If $K_n^c[X \cup Y]$ contains no PC cycles, then there exists either a vertex $x \in X$ such that $C(x, V(K_n^c)) = \{c_1\}$ or a vertex $y \in Y$ such that $C(y, V(K_n^c)) = \{c_2\}$.*

Proof. Denote $G = K_n^c$. If $X = \emptyset$, then there exists a vertex $y \in Y$ such that $C(y, V(G)) = \{c_2\}$. If $Y = \emptyset$, then there exists a vertex $x \in X$ such that $C(x, V(G)) = \{c_1\}$. So we can assume that $X \neq \emptyset$ and $Y \neq \emptyset$. Define a digraph D on vertex set $X \cup Y$ and arc set $A(D)$ as follows. If $C(xy) = c_1$ for $x \in X$ and $y \in Y$, then let $yx \in A(D)$; if $C(xy) = c_2$ for $x \in X$ and $y \in Y$, then let $xy \in A(D)$. Note that D contains no directed cycles if and only if $G[X, Y]$ contains no PC cycles. So D contains no directed cycles. This implies that there exists a vertex $x \in X$ such that $d_D^+(x) = 0$ or a vertex $y \in Y$ such that $d_D^+(y) = 0$. By the definition of D , there exists a vertex $x \in X$ such that $C(x, Y) = \{c_1\}$ or a vertex $y \in Y$ such that $C(y, X) = \{c_2\}$. Since $C(X) \subseteq C(X, U) = \{c_1\}$ and $C(Y) \subseteq C(Y, U) = \{c_2\}$, there exists a vertex $x \in X$ such that $C(x, V(G)) = \{c_1\}$ or a vertex $y \in Y$ such that $C(y, V(G)) = \{c_2\}$. \square

For convenience, we need the following auxiliary terminology. Let

$$C_1, C_2, \dots, C_m$$

be m vertex-disjoint PC 4-cycles, where $m \geq 2$ and $C_i = x_i y_i z_i w_i x_i$ for each $i \in \{1, 2, \dots, m\}$. For a pair of vertex-disjoint cycles $C_i = x_i y_i z_i w_i x_i$ and $C_j = x_j y_j z_j w_j x_j$ with $1 \leq i \neq j \leq m$, let $F^*(C_i, C_j)$ be an edge-colored graph obtained in the following way (as shown in Figure 4.1): $V(F^*) = V(C_i) \cup V(C_j)$; add edges $x_i z_i, y_i w_i$ and all possible edges between $V(C_i)$ and $V(C_j)$; let

$$\begin{aligned} C(x_i z_i) &= C(x_i y_i) = C(z_i w_i) = C(x_j y_j) = C(z_j w_j) = c_1, \\ C(y_i w_i) &= C(x_i w_i) = C(z_i y_i) = C(x_j w_j) = C(z_j y_j) = c_2, \\ C(\{x_i, z_i\}, V(C_j)) &= \{c_1\}, C(\{y_i, w_i\}, V(C_j)) = \{c_2\}. \end{aligned}$$

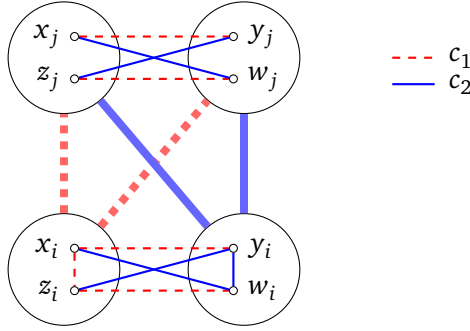


Figure 4.1: The edge-colored graph $F^*(C_i, C_j)$.

Lemma 4.6. Suppose that $C(K_n^c) = \{c_1, c_2\}$ and C_1, C_2, \dots, C_m are vertex-disjoint PC 4-cycles in K_n^c , where $m \geq 3$ and $C_i = x_i y_i z_i w_i x_i$ for each $i \in \{1, 2, \dots, m\}$. If K_n^c contains no two vertex-disjoint PC cycles of different lengths, then up to renaming the vertices, for every pair i, j of integers with $1 \leq i < j \leq m$,

$$\begin{aligned} C(x_i z_i) = C(x_i y_i) = C(z_i w_i) = c_1, C(\{x_i, z_i\}, V(C_j)) = \{c_1\}, \text{ and} \\ C(y_i w_i) = C(x_i w_i) = C(z_i y_i) = c_2, C(\{y_i, w_i\}, V(C_j)) = \{c_2\}. \end{aligned} \quad (4.1)$$

Proof. Suppose to the contrary that the statement is false and G is a counterexample. Since $C(K_n^c) = \{c_1, c_2\}$, we have that C_i is a PC cycle with exactly two edges of color c_1 and exactly two edges of color c_2 for each $i \in \{1, 2, \dots, m\}$. By renaming the vertices, we can assume that $C(x_i y_i) = C(z_i w_i) = c_1$ and $C(x_i w_i) = C(z_i y_i) = c_2$ for each $i \in \{1, 2, \dots, m\}$. And if $\{C(x_i z_i), C(y_i w_i)\} = \{c_1, c_2\}$, then we assume that $C(x_i z_i) = c_1$ and $C(y_i w_i) = c_2$.

Claim 1. For every pair i, j of integers with $1 \leq i \neq j \leq m$, $F^*(C_i, C_j)$ or $F^*(C_j, C_i)$ is a spanning subgraph of $G[V(C_i) \cup V(C_j)]$.

Proof. If $G[V(C_i) \cup V(C_j)]$ contains a PC cycle C^* of length greater than 4, then since $m \geq 3$, we have that G contains two vertex-disjoint PC cycles C^* and C_k of different lengths, where $k \neq i$ and $k \neq j$, a contradiction. So $G[V(C_i) \cup V(C_j)]$ contains only PC 4-cycles.

By symmetry, assume that $C(x_i x_j) = c_1$. Firstly we assert that

$$C(\{x_i, z_i\}, \{x_j, z_j\}) = \{c_1\} \text{ and } C(\{y_i, w_i\}, \{y_j, w_j\}) = \{c_2\}.$$

Since $y_i z_i w_i x_i x_j w_j y_i$, $w_i x_i x_j w_j z_j y_j w_i$ and $z_i w_i x_i x_j w_j z_i$ are not PC 6-cycles, we have $C(y_i w_j) = C(w_i y_j) = c_2$ and $C(z_i z_j) = c_1$. If $C(x_i z_j) = c_2$, then since $y_i x_i z_j w_j x_j y_j y_i$, $w_i z_i y_i x_i z_j w_j w_i$ and $z_i y_i x_i z_j w_j x_j z_i$ are not PC 6-cycles, we have $C(y_i y_j) = C(w_i w_j) = c_1$ and $C(z_i x_j) = c_2$. This implies that $x_i y_i z_i w_i y_j x_j w_j z_j x_i$ is a PC 8-cycle, a contradiction. So we have $C(x_i z_j) = c_1$. Since $y_i z_i w_i x_i z_j y_j y_i$, $w_i x_i z_j y_j x_j w_j w_i$ and $z_i w_i x_i z_j y_j x_j z_i$ are not PC 6-cycles, we have $C(y_i y_j) = C(w_i w_j) = c_2$ and $C(z_i x_j) = c_1$.

If $C(\{x_i, z_i\}, \{y_j, w_j\}) = \{c_2\}$ and $C(\{x_j, z_j\}, \{y_i, w_i\}) = \{c_2\}$, then

$$x_i y_i z_j w_j z_i w_i x_j y_j x_i$$

is a PC 8-cycle, a contradiction. So either there exists an edge with color c_1 between $\{x_i, z_i\}$ and $\{y_j, w_j\}$ or there exists an edge with color c_1 between $\{x_j, z_j\}$ and $\{y_i, w_i\}$. By symmetry between C_i and C_j , we can assume that there exists one edge with color c_1 between $\{x_i, z_i\}$ and $\{y_j, w_j\}$. By symmetry between x_i and z_i and symmetry between y_j and w_j , we can assume that $C(x_i y_j) = c_1$. We assert that $C(\{x_i, z_i\}, \{y_j, w_j\}) = \{c_1\}$ and $C(\{y_i, w_i\}, \{x_j, z_j\}) = \{c_2\}$. Since $w_i x_i y_j z_j w_j x_j w_i$, $y_i z_i w_i x_i y_j z_j y_i$ and $z_i w_i x_i y_j z_j w_j z_i$ are not PC 6-cycles, we have $C(w_i x_j) = C(y_i z_j) = c_2$ and $C(z_i w_j) = c_1$. If $C(x_i w_j) = c_2$, then since $y_i x_i w_j z_j y_j x_j y_i$, $w_i z_i y_i x_i w_j z_j w_i$ and $z_i y_i x_i w_j z_j y_j z_i$ are not PC 6-cycles, we have $C(y_i x_j) = C(w_i z_j) = c_1$ and $C(z_i y_j) = c_2$. This implies that $x_i y_j z_i w_j x_j y_i z_j w_i x_i$ is a PC 8-cycle, a contradiction. So we can assume that $C(x_i w_j) = c_1$. Since $w_i x_i w_j x_j y_j z_j w_i$, $y_i z_i w_i x_i w_j x_j y_j y_i$ and $z_i w_i x_i w_j x_j y_j z_i$ are not PC 6-cycles, we have $C(w_i z_j) = C(y_i x_j) = c_2$ and $C(z_i y_j) = c_1$.

Now we have $C(\{x_i, z_i\}, V(C_j)) = \{c_1\}$ and $C(\{y_i, w_i\}, V(C_j)) = \{c_2\}$. Recall that $C(x_i z_i) = c_1$ and $C(y_i w_i) = c_2$. So $F^*(C_i, C_j)$ is a spanning subgraph of $G[V(C_i) \cup V(C_j)]$. \square

For convenience, we define a digraph D with vertex set

$$V(D) = \{v_1, v_2, \dots, v_m\}$$

and arc set $A(D)$ as follows. For every pair i, j of integers with $1 \leq i < j \leq m$, if $F^*(C_i, C_j)$ is a spanning subgraph of $G[V(C_i) \cup V(C_j)]$, then let $v_i v_j \in A(D)$; if $F^*(C_j, C_i)$ is a spanning subgraph of $G[V(C_i) \cup V(C_j)]$, then let $v_j v_i \in A(D)$. By Claim 1, it is clear that D is a tournament. We assert that D is a tournament containing no directed cycles. Suppose to the contrary that D contains a directed cycle. Let $C = v_1 v_2 \cdots v_t v_1$ be a shortest directed cycle. If $t > 3$, then C has a chord, say $v_i v_j$. Note that $v_j v_{j+1} \cdots v_i v_j$ is a shorter directed cycle, where subscripts are taken modulo t , a contradiction. So C is a directed triangle. Assume w.l.o.g. that $C = v_1 v_2 v_3 v_1$. It follows from the definition of D that

$$C(\{x_1, z_1\}, V(C_2)) = C(\{x_2, z_2\}, V(C_3)) = C(\{x_3, z_3\}, V(C_1)) = \{c_1\},$$

$$C(\{y_1, w_1\}, V(C_2)) = C(\{y_2, w_2\}, V(C_3)) = C(\{y_3, w_3\}, V(C_1)) = \{c_2\}.$$

Then we can check that $x_1 y_2 z_2 w_3 x_1$ and $z_1 y_1 z_3 y_3 x_3 w_2 z_1$ are two vertex-disjoint PC cycles of different lengths, a contradiction. Thus D contains no directed cycles.

By renaming the cycles C_1, C_2, \dots, C_m , we can assume w.l.o.g. that

$$C_1, C_2, \dots, C_m$$

is an order corresponding to v_1, v_2, \dots, v_m such that $v_i v_j \in A(D)$ for every pair i, j of integers with $1 \leq i < j \leq m$. It follows from the definition of D and C_1, C_2, \dots, C_m that (4.1) holds. The proof is complete. \square

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Suppose to the contrary that the statement is false and G is a counterexample.

Claim 1. The graph G contains no PC triangles or monochromatic edge-cuts.

Proof. If G contains a PC triangle, then we can obtain a subgraph G' from G by removing a PC triangle C^* . Note that $\Delta^{mon}(G') \leq n - 6 = |V(G')| - 3$. It follows from Lemma 4.2 that G' contains a PC cycle of length greater than 3, which is also a PC cycle in G vertex-disjoint with C^* , a contradiction.

If G contains a monochromatic edge-cut, then G has a bipartition $\{V_1, V_2\}$ such that $|C(V_1, V_2)| = 1$. Assume w.l.o.g. that $C(V_1, V_2) = \{c_1\}$. It follows from $d_G^{c_1}(v) \leq n - 6$ for each $v \in V(G)$ that there exist three nonadjacent edges u_1u_2, u_3u_4, u_5u_6 in $E(V_1)$ and three nonadjacent edges v_1v_2, v_3v_4, v_5v_6 in $E(V_2)$ with colors different from c_1 . This implies that $u_1u_2v_1v_2u_1$ and $u_3u_4v_3v_4u_5u_6v_5v_6u_3$ are two vertex-disjoint PC cycles of different lengths, a contradiction. \square

It follows from Theorem 4.4 that G has a Gallai partition. Let

$$\{U_1, U_2, \dots, U_p\}$$

be a Gallai partition of G such that p is as large as possible. Since G contains no monochromatic edge-cuts, we have that $p \geq 4$. In particular, we can assume that $\cup_{1 \leq i < j \leq p} C(U_i, U_j) = \{c_1, c_2\}$.

Claim 2. $n \geq 11$.

Proof. Suppose to the contrary that $n < 11$. Since $\Delta^{mon}(G) \leq n - 6$, we have $n \geq 7$. If $n = 7$, then $\Delta^{mon}(G) \leq 1$ implies $\delta^c(G) \geq 6$. If $n = 8$, then $\Delta^{mon}(G) \leq 2$ implies that $\delta^c(G) \geq 4$. In both cases, we have $|U_i| \geq 3$ for each $i \in \{1, 2, \dots, p\}$. Since $p \geq 4$, we have $n = \sum_{i=1}^p |U_i| \geq 12$, a contradiction. If $9 \leq n \leq 10$, then $\Delta^{mon}(G) \leq n - 6$ implies that $\delta^c(G) \geq 3$. It follows that there exists an edge x_iy_i in $E(U_i)$ with $C(x_iy_i) \notin \{c_1, c_2\}$ for each $i \in \{1, 2, \dots, p\}$. If $p \geq 5$, then $x_1y_1x_2y_2x_1$ and $x_3y_3x_4y_4x_5y_5x_3$ are two vertex-disjoint PC cycles of different lengths, a contradiction. So we have $p = 4$ (recall that $p \geq 4$). Since $9 \leq n \leq 10$, there exists a vertex in $U_i \setminus \{x_i, y_i\}$ for some $i \in \{1, 2, 3, 4\}$, say $z_1 \in U_1 \setminus \{x_1, y_1\}$. Recall that G contains no monochromatic edge-cuts. So we can assume w.l.o.g. that $C(U_1, U_2) = \{c_1\}$ and $C(U_1, U_3) = \{c_2\}$. Then $z_1x_2y_2x_3y_3z_1$ and $x_1y_1x_4y_4x_1$ are two vertex-disjoint PC cycles of different lengths, a contradiction. \square

Claim 3. The graph G contains a PC 4-cycle C_0 with vertices from four distinct sets of U_1, U_2, \dots, U_p .

Proof. Construct an auxiliary graph H with $V(H) = \{u_1, u_2, \dots, u_p\}$ and $C(u_i u_j) = C(U_i, U_j)$ for $1 \leq i < j \leq p$. It suffices to show that H contains a PC 4-cycle. Since G contains no monochromatic edge-cuts, the auxiliary graph H contains no monochromatic edge-cuts. In particular, $\Delta^{\text{mon}}(H) \leq p - 2$ and $|C(H)| = 2$. By Theorem 4.5, H contains at least one PC cycle. Let $C^* = v_1 v_2 \cdots v_t v_1$ be a shortest PC cycle in H . Assume w.l.o.g. that $C(v_1 v_2) = C(v_3 v_4) = c_1$. If $t > 4$, then either $v_1 v_2 v_3 v_4 v_1$ or $v_1 v_4 v_5 v_6 \cdots v_t v_1$ is a shorter PC cycle, which contradicts to the choice of C^* . So we have $t = 4$, which implies that H contains a PC 4-cycle. \square

Denote $C_0 = x_0 y_0 z_0 w_0 x_0$, where $C(x_0 y_0) = C(z_0 w_0) = c_1$ and $C(x_0 w_0) = C(y_0 z_0) = c_2$. Note that $C(V(C_0)) \subseteq \{c_1, c_2\}$. Let $G' = G - V(C_0)$. Then it is clear that G' contains only PC 4-cycles.

Claim 4. The following holds.

- (i) $SC(G') \subseteq \{c_1, c_2\}$.
- (ii) If G' contains a PC 4-cycle, say C^* , then $C(V(C_0), V(G') \setminus V(C^*)) \subseteq \{c_1, c_2\}$.

Proof. (i) Suppose to the contrary that $SC(G') \not\subseteq \{c_1, c_2\}$. Then we have $V(G') \subseteq U_{i_0}$ for some $i_0 \in \{1, 2, \dots, p\}$. By the definition of C_0 , there exists a vertex $v \in \{x_0, y_0, z_0, w_0\}$ such that $v \notin U_{i_0}$. This implies that $C(v, U_{i_0}) = \{c_1\}$ or $C(v, U_{i_0}) = \{c_2\}$. But now $d_G^{c_1}(v) \geq p - 3$ or $d_G^{c_2}(v) \geq p - 3$, a contradiction.

(ii) Suppose to the contrary that $C(V(C_0), V(G') \setminus V(C^*)) \not\subseteq \{c_1, c_2\}$. Assume w.l.o.g. that there exists a vertex $v \in V(G') \setminus V(C^*)$ such that $C(v x_0) \notin \{c_1, c_2\}$. It follows that $\{v, x_0\} \subseteq U_i$ for some $i \in \{1, 2, \dots, p\}$. Since x_0, y_0, w_0 are from three distinct sets of U_1, U_2, \dots, U_p , we have $C(y_0 v) = c_1$ and $C(w_0 v) = c_2$. So $v x_0 y_0 z_0 w_0 v$ and C^* are two vertex-disjoint PC cycles of different lengths, a contradiction. \square

In the following, we distinguish two cases depending on the existence of monochromatic edge-cuts of G' .

Case 1. G' contains a monochromatic edge-cut.

Let $\{V_1, V_2\}$ be a bipartition of G' such that $|C(V_1, V_2)| = 1$, where $|V_1| \leq |V_2|$. Denote $|V_1| = n_1$ and $|V_2| = n_2$. Recall that $SC(G') \subseteq \{c_1, c_2\}$. So we can assume w.l.o.g. that $C(V_1, V_2) = \{c_1\}$. By Lemma 4.4, we can assume $\{c'_i\} \subseteq C(V_i) \subseteq \{c_1, c'_i\}$ for each $i \in \{1, 2\}$. Firstly, we show that G' has the following property which will be frequently used.

- For each $i \in \{1, 2\}$, if $n_i \geq 4$, then $C(V_i) = \{c'_i\}$.

By symmetry between V_1 and V_2 , we only need to show that the statement holds for $i = 1$. Suppose to the contrary that $C(V_1) \neq \{c'_1\}$. Then it follows from Lemma 4.4 that $C(V_1) = \{c_1, c'_1\}$ and $G^{c'_1}[V_1] \cong K_{1, n_1-1}$. Let x_1 be the center and y_1, z_1, w_1 be three distinct leaves of $G^{c'_1}[V_1]$. It is clear that $C(y_1 z_1) = c_1$. Since $d_G^{c_1}(y_1) \leq n - 6$ and $d_G^{c_1}(z_1) \leq n - 6$, we have $c_1 \notin C(\{y_1, z_1\}, V(C_0))$. Since $\Delta^{mon}(G') \leq |V(G')| - 2$, there exists an edge $x_2 y_2$ in $E(V_2)$ such that $C(x_2 y_2) = c'_2$. Then $y_1 z_1 x_0 y_0 z_0 w_0 y_1$ and $x_1 w_1 x_2 y_2 x_1$ are two vertex-disjoint PC cycles of different lengths, a contradiction.

Now we distinguish three cases depending on the value of n_1 . Since $\Delta^{mon}(G') \leq n - 2 = |V(G')| - 2$, we have $n_1 \geq 2$.

Subcase 1.1. $n_1 = 2$.

By Claim 2, we have that $n_2 \geq 5$. It follows that $C(V_2) = \{c'_2\}$. Let $V_1 = \{x_1, y_1\}$. Since $\Delta^{mon}(G') \leq |V(G')| - 2$, we have $C(x_1 y_1) = c'_1$. Note that, for each $v \in V_2$, there exist two distinct vertices x_2, y_2 in $V_2 \setminus v$ such that $x_1 y_1 x_2 y_2 x_1$ is a PC 4-cycle. It follows from (ii) of Claim 4 that $C(V(C_0), V_2) \subseteq \{c_1, c_2\}$.

If $C(V_2) \neq \{c_2\}$, then it follows from $\cup_{1 \leq i < j \leq q} C(U_i, U_j) = \{c_1, c_2\}$ that $V_2 \subseteq U_{i_0}$ for some $i_0 \in \{1, 2, \dots, q\}$. By the definition of C_0 , there exists at least one vertex $v \in \{x_0, y_0, z_0, w_0\}$ such that $v \notin U_{i_0}$. So we have $C(v, V_2) = \{c_1\}$ or $C(v, V_2) = \{c_2\}$. But now $d_G^{c_1}(v) \geq n - 5$ or $d_G^{c_2}(v) \geq n - 5$, a contradiction.

If $C(V_2) = \{c_2\}$, then let x_2 and y_2 be two distinct vertices in V_2 . Note that $d_G^{c_2}(x_2) = n - 7$ and $d_G^{c_2}(y_2) = n - 7$. Since $d_G^{c_2}(x_2) \leq n - 6$ and $d_G^{c_2}(y_2) \leq n - 6$, we have $d_{C_0}^{c_1}(x_2) \geq 3$ and $d_{C_0}^{c_1}(y_2) \geq 3$. By Lemma 4.3, we have that $G[V(C_0) \cup \{x_2, y_2\}]$ contains a PC 6-cycle, which is also a PC cycle in G , say C^* . Let z_2, w_2 be two distinct vertices in $V_2 \setminus \{x_2, y_2\}$. So C^* and $x_1 y_1 z_2 w_2 x_1$ are two vertex-disjoint PC cycles of different lengths, a contradiction.

Subcase 1.2. $n_1 = 3$.

By Claim 2, we have $n_2 \geq 4$. It follows that $C(V_2) = \{c'_2\}$. Let $V_1 = \{x_1, y_1, z_1\}$. Since $\Delta^{mon}(G') \leq |V(G')| - 2$, we can assume that $C(x_1y_1) = C(x_1z_1) = c'_1$. It follows from $d_G^{c_1}(y_1) \leq n - 6$ that $d_{C_0}^{c_1}(y_1) \leq 1$. Assume w.l.o.g. that $C(x_0y_1) \neq c_1$.

If $C(w_0, V_2) \neq \{c_1\}$, then there exists a vertex $x_2 \in V_2$ such that $C(w_0x_2) \neq c_1$. Let y_2 and z_2 be two distinct vertices in $V_2 \setminus x_2$. Then $x_2y_1x_0y_0z_0w_0x_2$ and $x_1z_1y_2z_2x_1$ are two vertex-disjoint PC cycles of different lengths, a contradiction.

If $C(w_0, V_2) = \{c_1\}$, then since $d_G^{c_1}(w_0) \leq n - 6$, we have $C(y_0w_0) \neq c_1$. Let x_2y_2 and z_2w_2 be two nonadjacent edges in $E(V_2)$. Then $w_0x_2y_2y_1x_0y_0w_0$ and $x_1z_1z_2w_2x_1$ are two vertex-disjoint PC cycles of different lengths, a contradiction.

Subcase 1.3. $n_1 \geq 4$.

Note that $n_2 \geq n_1 \geq 4$. It follows that $C(V_1) = \{c'_1\}$ and $C(V_2) = \{c'_2\}$. So for each $i \in \{1, 2\}$, there exist two nonadjacent edges x_iy_i and z_iw_i in $E(V_i)$ with colors different from c_1 . Now $x_1y_1x_2y_2z_1w_1z_2w_2x_1$ and C_0 are two vertex-disjoint PC cycles of different lengths, a contradiction.

Case 2. G' contains no monochromatic edge-cuts.

Recall that $SC(G') \subseteq \{c_1, c_2\}$. It follows from G' containing no monochromatic edge-cuts that $SC(G') = \{c_1, c_2\}$. In the following, we distinguish two cases.

Subcase 2.1. $\Delta^c(G') = 2$.

By the definition of G' , we have $\Delta^{mon}(G') \leq n - 6 = |V(G')| - 2$. It follows from Theorem 4.5 that G' contains at least one PC cycle. Recall that G' contains only PC 4-cycles. Let C_1, C_2, \dots, C_m be vertex-disjoint PC 4-cycles in G' such that m is as large as possible. It is clear that $m \geq 1$. Since $SC(G') = \{c_1, c_2\}$, it follows from $\Delta^c(G') = 2$ that $C(G') = \{c_1, c_2\}$. Denote $C_i = x_iy_iz_iw_ix_i$ for each $i \in \{1, 2, \dots, m\}$, where $C(x_iy_i) = C(z_iw_i) = c_1$ and $C(x_iw_i) = C(y_iz_i) = c_2$. Let $W = V(C_0) \cup V(C_1) \cup \dots \cup V(C_m)$ and $G'' = G - W$. Note that G'' contains no PC cycles.

In the following, we distinguish two cases depending on the value of m .

(1) If $m = 1$, then it follows from Claim 2 that $|V(G'')| = n - 8 \geq 3$. Since G'' contains no PC cycles, it follows from Theorem 4.5 that there exists a vertex $v \in V(G'')$ such that $d_{G''}^c(v) = 1$. Note that $C(G'') \subseteq C(G') = \{c_1, c_2\}$. Assume w.l.o.g. that $C(v, V(G'')) = \{c_1\}$. Since C_1 is a PC 4-cycle, it follows from (ii) of Claim 4 that $C(v, V(C_0)) \subseteq \{c_1, c_2\}$ for each $v \in V(G')$. Note that $C(v, V(C_1)) \subseteq C(G') = \{c_1, c_2\}$. So $C(v, V(C_0) \cup V(C_1)) \subseteq \{c_1, c_2\}$. Since $d_G^{c_1}(v) \leq n - 6$, we have $d_{G[W]}^{c_2}(v) \geq 5$. It follows that either $d_{C_0}^{c_2}(v) \geq 3$ or $d_{C_1}^{c_2}(v) \geq 3$.

Assume w.l.o.g. that $d_{C_0}^{c_2}(v) \geq 3$ and $C(v, \{x_0, y_0, z_0\}) = \{c_2\}$. We assert that $C(V(G'') \setminus v, \{y_0, z_0, w_0\}) = \{c_1\}$ and $C(V(G'')) = \{c_1\}$. Since, for each $u \in V(G'') \setminus v$, $uvx_0y_0z_0w_0u$, $uvy_0x_0w_0z_0u$ and $uvz_0w_0x_0y_0u$ are not PC 6-cycles which is vertex-disjoint with C_1 , we have $C(V(G'') \setminus v, \{y_0, z_0, w_0\}) = \{c_1\}$. This implies that $d_{C_0}^{c_1}(u) \geq 3$ for each vertex $u \in V(G'') \setminus v$. By Claim 2, we have $|V(G'')| = n - 8 \geq 3$. If $C(V(G'')) \neq \{c_1\}$, then there exists an edge v_1v_2 in $E(V(G'') \setminus v)$ such that $C(v_1v_2) = c_2$. Note that $d_{C_0}^{c_1}(v_1) \geq 3$ and $d_{C_0}^{c_1}(v_2) \geq 3$. It follows from Lemma 4.3 that $G[V(C_0) \cup \{v_1, v_2\}]$ contains a PC 6-cycle, say C^* . This implies that C^* and C_1 are two vertex-disjoint PC cycles of different lengths, a contradiction. So we have $C(V(G'')) = \{c_1\}$.

Let v_1 and v_2 be two distinct vertices in $V(G'') \setminus v$. Recall that $C(V(G'') \setminus v, \{y_0, z_0, w_0\}) = \{c_1\}$. So $d_{C_0}^{c_1}(v_1) \geq 3$ and $d_{C_0}^{c_1}(v_2) \geq 3$. Since $\Delta^{mon}(G) \leq n - 6$, we have $d_{G[W]}^{c_2}(v_1) \geq 5$ and $d_{G[W]}^{c_2}(v_2) \geq 5$. It follows that $C(\{v_1, v_2\}, V(C_1)) = \{c_2\}$. This implies that $v_1v_2x_1y_1z_1w_1v_1$ and C_0 are two vertex-disjoint PC cycles of different lengths, a contradiction.

(2) If $m \geq 2$, then it follows from (ii) of Claim 4 that $C(V(C_0), V(G') \setminus V(C_1)) \subseteq \{c_1, c_2\}$ and $C(V(C_0), V(G') \setminus V(C_2)) \subseteq \{c_1, c_2\}$. This implies that $C(V(C_0), V(C_i)) \subseteq \{c_1, c_2\}$ for every $i \in \{1, 2, \dots, m\}$. Hence $C(W) = \{c_1, c_2\}$. Since $G[W]$ contains no two vertex-disjoint PC cycles of different lengths, it follows from Lemma 4.6 that $\Delta^{mon}(G[W]) = |W| - 2$. If $|V(G'')| \leq 1$, then $\Delta^{mon}(G) \geq |W| - 2 = n - |V(G'')| - 2 \geq n - 3$, a contradiction. So we have $|V(G'')| \geq 2$. Recall that G'' contains no PC cycles. It follows from Theorem 4.5 that there exists a vertex $v \in V(G'')$ such that $d_{G''}^c(v) = 1$. Assume w.l.o.g. that $C(v, V(G'')) = \{c_1\}$. Since $G[W]$ contains no two vertex-disjoint PC

cycles of different lengths, it follows from Lemma 4.6 that there exists some $i_0 \in \{0, 1, \dots, m\}$ such that $d_{G[W]}^{c_2}(x_{i_0}) = d_{G[W]}^{c_2}(z_{i_0}) = 1$. Since $d_G^{c_1}(z_{i_0}) \leq n - 6$, there exists a vertex $u \in V(G'')$ such that $C(z_{i_0}u) \neq c_1$. Since $d_G^{c_1}(v) \leq n - 6$, there exists a vertex $w \in V(C_{j_0})$ with $j_0 \neq i_0$ such that $C(wv) \neq c_1$. Now $uvw x_{i_0} w_{i_0} z_{i_0} u$ is a PC 6-cycle, which is vertex-disjoint with C_i , where $i \neq i_0$ and $i \neq j_0$, a contradiction.

Subcase 2.2. $\Delta^c(G') \geq 3$.

Recall that G' contains only PC 4-cycles and $SC(G') \subseteq \{c_1, c_2\}$. By Lemma 4.1, we have that G' has a partition $\{U, X, Y\}$ such that $C(X) \subseteq C(X, U) = \{c_1\}$, $C(Y) \subseteq C(Y, U) = \{c_2\}$ and $C(X, Y) \subseteq \{c_1, c_2\}$. Since $\Delta^c(G') \geq 3$, there exists an edge $u_1 u_2$ in $E(U)$ such that $C(u_1 u_2) \notin \{c_1, c_2\}$. It follows from $\Delta^{mon}(G') \leq |V(G')| - 2$ that $d_{G'}^c(v) = 2$ for each vertex $v \in X \cup Y$. Note that $d_{G[X \cup Y]}^c(v) = 2$ for each $v \in X \cup Y$. It follows from Theorem 4.5 that $G[X \cup Y]$ contains at least one PC cycle. Note that $G[X \cup Y]$ contains only PC 4-cycles. Let C_1, C_2, \dots, C_m be vertex-disjoint PC 4-cycles in $G[X \cup Y]$ such that m is as large as possible. It is clear that $m \geq 1$. Denote $C_i = x_i y_i z_i w_i x_i$ for each $i \in \{1, \dots, m\}$, where $C(x_i y_i) = C(z_i w_i) = c_1$ and $C(x_i w_i) = C(y_i z_i) = c_2$. Assume w.l.o.g. that $x_i \in X$ for each $i \in \{1, \dots, m\}$. Since $C(X) \subseteq \{c_1\}$ and $C(x_i w_i) = c_2$, we have $w_i \in Y$. Since $C(Y) \subseteq \{c_2\}$ and $C(z_i w_i) = c_1$, we have $z_i \in X$. Since $C(X) \subseteq \{c_1\}$ and $C(y_i z_i) = c_2$, we have $y_i \in Y$. Let $W = V(C_0) \cup V(C_1) \cup \dots \cup V(C_m)$ and $G'' = G - W$.

In the following, we distinguish two cases depending on the value of m .

(1) If $m = 1$, then C_1 is a PC 4-cycle. Firstly we assert that $V(C_1) \neq X \cup Y$. If $V(C_1) = X \cup Y$, then $X = \{x_1, z_1\}$ and $Y = \{y_1, w_1\}$. Recall that $C(X) \subseteq C(X, U) = \{c_1\}$, $C(Y) \subseteq C(Y, U) = \{c_2\}$, $C(x_i y_i) = C(z_i w_i) = c_1$ and $C(x_i w_i) = C(y_i z_i) = c_2$. By the definition of G' , we have $|V(G')| = n - 4$. This implies that $d_{G'}^{c_1}(x_1) = d_{G'}^{c_1}(z_1) = d_{G'}^{c_2}(y_1) = d_{G'}^{c_2}(w_1) = n - 6$. It follows from $\Delta^{mon}(G) \leq n - 6$ that $c_1 \notin C(\{x_1, z_1\}, V(C_0))$ and $c_2 \notin C(\{y_1, w_1\}, V(C_0))$. Let $u_1 u_2$ be an edge in $E(U)$ such that $C(u_1 u_2) \notin \{c_1, c_2\}$. Recall that C_1 is a PC 4-cycle. It follows from (ii) of Claim 4 that $C(v, V(C_0)) \subseteq \{c_1, c_2\}$ for each $v \in V(G'')$. This implies $C(x_0 u_1) \subseteq \{c_1, c_2\}$. If $C(x_0 u_1) = c_1$, then $x_0 u_1 u_2 x_1 x_0$ and $y_0 z_0 w_0 z_1 w_1 y_1 y_0$ are two PC cycles of different lengths, a contradiction.

If $C(x_0u_1) = c_2$, then $x_0u_1u_2y_1x_0$ and $y_0z_0w_0z_1x_1w_1y_0$ are two PC cycles of different lengths, a contradiction. So we have $V(C_1) \neq X \cup Y$.

Now consider the graph G'' with partition $\{U, X \setminus V(C_1), Y \setminus V(C_1)\}$. By Lemma 4.5, we can assume w.l.o.g. that there exists a vertex $x \in X \setminus V(C_1)$ such that $C(x, V(G'')) = \{c_1\}$. Recall that $C(v, V(C_0)) \subseteq \{c_1, c_2\}$ for each vertex $v \in V(G'')$. So we have $C(x, V(C_0)) \subseteq \{c_1, c_2\}$. By the definition of C_1 , we have $C(x, V(C_1)) \subseteq C(X \cup Y) \subseteq \{c_1, c_2\}$. So $C(x, W) \subseteq \{c_1, c_2\}$. Since $d_G^{c_1}(x) \leq n - 6$, we have $d_{G[W]}^{c_2}(x) \geq 5$. It follows that either $d_{C_0}^{c_2}(x) \geq 3$ or $d_{C_1}^{c_2}(x) \geq 3$.

Assume w.l.o.g. that $d_{C_0}^{c_2}(x) \geq 3$ and $C(x, \{x_0, y_0, z_0\}) = \{c_2\}$. Since, for each $u \in V(G'') \setminus x$, $uxx_0y_0z_0w_0u$, $uxy_0x_0w_0z_0u$ and $uxz_0w_0x_0y_0u$ are not PC 6-cycles which are vertex-disjoint with C_1 , we have $C(V(G'') \setminus x, \{y_0, z_0, w_0\}) = \{c_1\}$. Let u_1u_2 be an edge in $E(U)$ such that $C(u_1u_2) \notin \{c_1, c_2\}$. Note that $d_{C_0}^{c_1}(u_1) \geq 3$ and $d_{C_0}^{c_1}(u_2) \geq 3$. It follows from Lemma 4.3 that $G[V(C_0) \cup \{u_1, u_2\}]$ contains a PC 6-cycle, which is vertex-disjoint with C_1 , a contradiction.

(2) If $m \geq 2$, then firstly we assert that $C(W) = \{c_1, c_2\}$. By the definition of C_0 , we have $C(V(C_0)) = \{c_1, c_2\}$. By the definition of C_i for every $1 \leq i \leq m$, we have $C(W \setminus V(C_0)) \subseteq C(X \cup Y) = \{c_1, c_2\}$. Since C_1 and C_2 are PC 4-cycles, it follows from (ii) of Claim 4 that $C(V(C_0), V(G') \setminus V(C_1)) \subseteq \{c_1, c_2\}$ and $C(V(C_0), V(G') \setminus V(C_2)) \subseteq \{c_1, c_2\}$. This implies that $C(V(C_0), V(C_i)) \subseteq \{c_1, c_2\}$ for every $i \in \{1, 2, \dots, m\}$. Hence $C(W) = \{c_1, c_2\}$.

Next we assert that $W \neq V(C_0) \cup X \cup Y$. Since $G[W]$ contains no two vertex-disjoint PC cycles of different lengths, it follows from Lemma 4.6 that there exists a PC 4-cycle C_{i_0} such that $d_{G[W]}^{c_2}(x_{i_0}) = 1$ and $d_{G[W]}^{c_1}(w_{i_0}) = 1$. Recall that $x_{i_0} \in X$. If $i_0 \in \{1, 2, \dots, m\}$, then $C(x_{i_0}, U) \subseteq C(X, U) = \{c_1\}$, which implies that $d_G^{c_1}(x_{i_0}) = n - 2$, a contradiction. If $i_0 = 0$, then it follows from $d_G^{c_1}(x_0) \leq n - 6$ that there exists a vertex $u \in U$ such that $C(x_0u) \neq c_1$. Recall that $C(x_1u) \in C(X, U) = \{c_1\}$. Now $ux_1w_0z_0y_0x_0u$ and C_2 are two vertex-disjoint PC cycles of different lengths, a contradiction. So we have $W \neq V(C_0) \cup X \cup Y$.

Now consider the graph G'' with partition $\{U, X \setminus W, Y \setminus W\}$. By Lemma 4.5, we can assume w.l.o.g. that x is a vertex in $x \in X \setminus W$ such that $C(x, V(G'')) =$

$\{c_1\}$. Consider the graph $G[W]$. By Lemma 4.6, there exists a PC 4-cycle C_{i_0} such that $d_{G[W]}^{c_2}(x_{i_0}) = 1$ and $d_{G[W]}^{c_2}(z_{i_0}) = 1$. Since $d_G^{c_1}(x) \leq n - 6$, there exists a vertex $u \in V(C_{j_0})$, where $j_0 \neq i_0$, such that $C(ux) \neq c_1$. Since $d_G^{c_1}(x_{i_0}) \leq n - 6$, there exists a vertex $w \in V(G'')$ such that $C(x_{i_0}w) \neq c_1$. Now $uxwx_{i_0}y_{i_0}z_{i_0}u$ is a PC 6-cycle, which is vertex-disjoint with C_i , where $i \neq i_0$ and $i \neq j_0$, a contradiction.

The proof of Theorem 4.3 is complete. □

Chapter 5

PC cycle-factors in edge-colored bipartite graphs

In this chapter, we consider PC cycle-factors in balanced edge-colored bipartite graphs. We show that if $\delta^c(G_{n,n}^c) \geq 2n/3 + 3$, then $G_{n,n}^c$ contains a PC cycle-factor, i.e., a set of vertex-disjoint properly colored cycles covering the vertex set, and this bound is essentially sharp.

5.1 Introduction

A properly colored cycle-factor of an edge-colored graph is a spanning subgraph such that each component is a properly colored cycle, i.e., a set of vertex-disjoint properly colored cycles covering the vertex set. The study of PC cycle-factors also received a lot of attention. Bang-Jensen et al. [15] in 1998 proved that if an edge-colored complete graph contains a PC cycle-factor, then it contains a PC Hamilton path. Feng et al. [38] in 2006 proved that an edge-colored complete graph contains a PC Hamilton path if and only if it contains a spanning PC 1-path-cycle factor, i.e., a spanning subgraph which is the disjoint union of one PC path and a collection of PC cycles. Lo [76] in 2014 gave the following color degree condition for the existence of a PC cycle-factor in edge-colored graphs.

Theorem 5.1 (Lo [76]). *If $\delta^c(G_n^c) \geq 2n/3$, then G_n^c contains a PC cycle-factor.*

Recently, Guo et al. [49] gave the following color degree condition for the existence of a PC cycle-factor in edge-colored complete balanced bipartite graphs.

Theorem 5.2 (Guo [49]). *If $\delta^c(K_{n,n}^c) > 3n/4$, then $K_{n,n}^c$ contains a PC cycle-factor.*

It is a natural problem to determine a minimum color degree condition for the existence of a PC cycle-factor in general edge-colored balanced bipartite graphs. It follows from Theorem 5.1 that $\delta^c(G_{n,n}^c) \geq 4n/3$ suffices. But it is far from being best possible for $G_{n,n}^c$. In our next main result, we give the following essentially sharp color degree condition.

Theorem 5.3. *If $\delta^c(G_{n,n}^c) \geq 2n/3 + 3$, then $G_{n,n}^c$ contains a PC cycle-factor.*

We postpone the proof of this theorem to Section 5.3. The color degree condition in the above theorem is essentially sharp (possibly apart from the additive constant 3), in view of the following construction.

Construction 5.1. Let k and n be two integers with $k < 2n/3$. Let X_1, Y_1, X_2, Y_2 be four disjoint vertex set, where $X_1 = \{x_1, x_2, \dots, x_k\}, Y_1 = \{y_1, y_2, \dots, y_k\}$ and $|X_2| = |Y_2| = n - k$. Add all edges between X_1 and Y_1 and color these edges rainbow. For each $1 \leq i \leq k$, add all the edges with a new color c_i between x_i and Y_2 , and all the edges with a new color c'_i between y_i and X_2 . Note that the resulting edge-colored graph G is balanced bipartite with $\delta^c(G) < 2n/3$. Let C be a PC cycle in G and give C a direction. Note that every vertex $v \in V(C) \cap (X_2 \cup Y_2)$ must be followed on C by at least two consecutive vertices in $X_1 \cup Y_1$. Let F be a subgraph induced by vertex-disjoint PC cycles in G . It is clear that $|V(F)| \leq |X_1 \cup Y_1| + |X_1 \cup Y_1|/2 < 2n$, which implies that G contains no PC cycle-factors. So $\delta^c(G) \geq 2n/3$ is necessary.

In next section, we present some preliminaries.

5.2 Preliminaries

Let G^c be an edge-colored graph and C a cycle. Give C a direction. For two vertices $u, v \in V(C)$, we use uCv to denote the segment between u and v along the direction of C , and we use $u\bar{C}v$ to denote the segment between u and v in the opposite direction. In particular, if $u = v$, then $uCv = u$ and $u\bar{C}v = u$. For a vertex $x \in V(C)$, we use x^+ and x^- to denote the immediate successor and ancestor of x on C , respectively. We write x^{--} for $(x^-)^-$, x^{++} for $(x^+)^+$, x^{3-} for $(x^{--})^-$ and x^{3+} for $(x^{++})^+$. We use $P(u, v)$ to denote a path in G^c with initial vertex u and terminal vertex v . Similarly, for a vertex $x \in V(P(u, v))$, we use x^+ and x^- to denote the immediate successor (if $x \neq v$) and ancestor of x (if $x \neq u$) on $P(u, v)$.

5.3 Proof of the main result

Proof of Theorem 5.3. Suppose to the contrary that the statement is false and G is a counterexample with $|E(G)|$ is as small as possible, with bipartition $\{X, Y\}$, and we let $\delta = \delta^c(G)$. By the choice of G , we again know that G contains no monochromatic paths of length at least 3.

Let C_1, C_2, \dots, C_k be vertex-disjoint PC cycles of G such that (i) $|V(\cup_{i=1}^k C_i)|$ is as large as possible; (ii) subject to (i) $|E(G')|$ is as large as possible, where $F = G[\cup_{i=1}^k V(C_i)]$ and $G' = G - F$. Note that $|V(F)| < 2n$. Give a direction to each cycle C_1, C_2, \dots, C_k in F .

For any fixed vertex $u \in V(G')$, choose $N^c(u)$ to be a subset of neighbors of u such that $C(uv_1) \neq C(uv_2)$ for all distinct $v_1, v_2 \in N^c(u)$. Based on $N^c(u)$ and the direction of each cycle C_i , we define the following sets.

$$W(u) = \{v : v \in V(G') \cap N^c(u)\},$$

$$R^{++}(u) = \{v^{++} : v \in N^c(u) \text{ and } C(uv) \neq C(vv^+)\},$$

$$R^{--}(u) = \{v^{--} : v \in N^c(u) \text{ and } C(uv) \neq C(vv^-)\},$$

$$R^+(u) = \{v^+ : v \in N^c(u) \text{ and } C(uv) \neq C(vv^-)\},$$

$$R^-(u) = \{v^- : v \in N^c(u) \text{ and } C(uv) \neq C(vv^+)\},$$

$$R^{**}(u) = R^{++}(u) \cap R^{--}(u),$$

$$S^{**}(u) = R^{++}(u) \setminus R^{--}(u), \quad T^{**}(u) = R^{--}(u) \setminus R^{++}(u).$$

$$R^*(u) = R^+(u) \cap R^-(u), \quad S^*(u) = R^+(u) \setminus R^-(u), \quad T^*(u) = R^-(u) \setminus R^+(u),$$

Note that

$$2|R^*(u)| + |S^*(u)| + |T^*(u)| + |W(u)| \geq \delta$$

and

$$2|R^{**}(u)| + |S^{**}(u)| + |T^{**}(u)| + |W(u)| \geq \delta.$$

For each $v \in S^{**}(x_0) \cup T^{**}(x_0) \cup S^{**}(y_0) \cup T^{**}(y_0)$, let

$$f(v) = \begin{cases} v^{--} & \text{if } v \in S^{**}(x_0) \cup S^{**}(y_0), \\ v^{++} & \text{if } v \in T^{**}(x_0) \cup T^{**}(y_0), \end{cases}$$

and

$$f'(v) = \begin{cases} v^- & \text{if } v \in S^{**}(x_0) \cup S^{**}(y_0), \\ v^+ & \text{if } v \in T^{**}(x_0) \cup T^{**}(y_0). \end{cases}$$

For each $v \in S^*(x_0) \cup T^*(x_0) \cup S^*(y_0) \cup T^*(y_0)$, let

$$g(v) = \begin{cases} v^- & \text{if } v \in S^*(x_0) \cup S^*(y_0), \\ v^+ & \text{if } v \in T^*(x_0) \cup T^*(y_0), \end{cases}$$

and

$$g'(v) = \begin{cases} v^{--} & \text{if } v \in S^*(x_0) \cup S^*(y_0), \\ v^{++} & \text{if } v \in T^*(x_0) \cup T^*(y_0). \end{cases}$$

Now we proceed by distinguishing the following four cases, based on the (non)existence of PC paths of length three or less in G' . In all cases we will

derive a contradiction.

Case 1. There exists a PC path of length three, say $x_0y_1x_1y_0$, in G' , where $x_0, x_1 \in X$ and $y_0, y_1 \in Y$.

Let H be the edge-colored subgraph of G induced by

$$R^{**}(x_0) \cup S^{**}(x_0) \cup T^{**}(x_0) \cup W(x_0) \cup R^{**}(y_0) \cup S^{**}(y_0) \cup T^{**}(y_0) \cup W(y_0).$$

Note that $|V(H) \cap X| \geq \delta/2$ and $|V(H) \cap Y| \geq \delta/2$. We define a vertex-coloring of $V(H)$ as follows: for each $v \in V(H)$,

$$C(v) = \begin{cases} \gamma & \text{if } v \in R^{**}(x_0) \cup R^{**}(y_0), \\ C(vv^+) & \text{if } v \in S^{**}(x_0) \cup S^{**}(y_0), \\ C(vv^-) & \text{if } v \in T^{**}(x_0) \cup T^{**}(y_0), \\ C(x_0v) & \text{if } v \in W(x_0) \setminus \{y_0, y_1\}, \\ C(y_0v) & \text{if } v \in W(y_0) \setminus \{x_0, x_1\}, \end{cases}$$

where γ is a new color that does not appear in G .

Claim 1. Let xy be an arbitrary edge in H with $x \in X, y \in Y$ and $x \notin \{x_0, x_1\}, y \notin \{y_0, y_1\}$.

(i) If $x \in S^{**}(y_0) \cup T^{**}(y_0) \cup W(y_0)$ and $y \in S^{**}(x_0) \cup T^{**}(x_0) \cup W(x_0)$, then $C(xy) = C(x)$ or $C(xy) = C(y)$.

(ii) If $x \in R^{**}(y_0)$ or $y \in R^{**}(x_0)$, say $x \in R^{**}(y_0)$, then $y \notin R^{**}(x_0)$ and $C(xy) = C(y)$, which implies that $R^{**}(x_0) \cup R^{**}(y_0)$ is an independent set in G .

Proof. (i) Suppose to the contrary that $C(xy) \neq C(x)$ and $C(xy) \neq C(y)$. In each possible case, we will indicate a subgraph F' of G consisting of vertex-disjoint PC cycles with $|V(F')| > |V(F)|$, which contradicts the maximality of $|V(F)|$. We distinguish the following two cases and their subcases.

a) $x \in W(y_0) \setminus \{x_0, x_1\}$ or $y \in W(x_0) \setminus \{y_0, y_1\}$. Assume w.l.o.g. that $x \in W(y_0) \setminus \{x_0, x_1\}$. If $y \in W(x_0) \setminus \{y_0, y_1\}$, then $C = xyx_0y_1x_1y_0x$ is a PC

cycle. Then set $F' = F + C$. If $y \in S^{**}(x_0) \cup T^{**}(x_0)$, then set

$$F' = F - \{f'(y)\} + f(y)x_0y_1x_1y_0xy.$$

b) $x \in S^{**}(y_0) \cup T^{**}(y_0)$ and $y \in S^{**}(x_0) \cup T^{**}(x_0)$. Let

$$P = f(y)x_0y_1x_1y_0f(x).$$

If $x \notin \{y^+, y^-, y^{3+}, y^{3-}\}$, then set $F' = F - \{f'(x), f'(y)\} + \{xy\} + P$.

If $x \in \{y^-, y^+\}$, then assume w.l.o.g. that $x = y^-$. If $x \in S^{**}(y_0)$ or $y \in T^{**}(x_0)$, then $C(xy) = C(x)$ or $C(xy) = C(y)$; if $x \in T^{**}(y_0)$ and $y \in S^{**}(x_0)$, set $F' = F - \{f'(x), f'(y)\} + P$.

In the following we consider the case that $x \notin \{y^-, y^+\}$ and $x \in \{y^{3-}, y^{3+}\}$. We may assume w.l.o.g. that $x = y^{3-}$.

For the case $x = y^{3-}$ and $x \neq y^{3+}$, if $x \in S^{**}(y_0)$ or $y \in T^{**}(x_0)$, then set $F' = F - \{f'(x), f'(y)\} + \{xy\} + P$; if $x \in T^{**}(y_0)$ and $y \in S^{**}(x_0)$, then set $F' = F - \{f(x)f(y)\} + P$.

For the case $x = y^{3-}$ and $x = y^{3+}$, if $x \in S^{**}(y_0)$ and $y \in T^{**}(x_0)$, then set $F' = F - \{f(x)f(y)\} + P$; if $x \in T^{**}(y_0)$ and $y \in S^{**}(x_0)$, then set $F' = F - \{f(x)f(y)\} + P$; otherwise, set $F' = F - \{f'(x), f'(y)\} + \{xy\} + P$.

(ii) If $x \in R^{**}(y_0)$, then we can treat $x \in S^{**}(y_0)$ and $x \in T^{**}(y_0)$, respectively.

a) Firstly we assert that if $y \in S^{**}(x_0) \cup T^{**}(x_0) \cup W(x_0)$, then $C(xy) = C(y)$. If we treat $x \in S^{**}(y_0)$, then it follows from (i) and the definition of $C(x)$ that $C(xy) = C(xx^+)$ or $C(xy) = C(y)$. If we treat $x \in T^{**}(y_0)$, then it follows from (i) and the definition of $C(x)$ that $C(xy) = C(xx^-)$ or $C(xy) = C(y)$. So we have $C(xy) = C(y)$.

b) Next we assert that $y \notin R^{**}(x_0)$. If $y \in R^{**}(x_0)$, then we can treat $y \in S^{**}(x_0)$ and $y \in T^{**}(x_0)$, respectively. If we treat $y \in S^{**}(x_0)$, then it follows from a) and the definition of $C(y)$ that $C(xy) = C(yy^+)$. If we treat $y \in T^{**}(x_0)$, then it follows from a) and the definition of $C(y)$ that $C(xy) = C(yy^-)$. Now $C(xy) = C(yy^+) \neq C(yy^-) = C(xy)$, a contradiction. So we have $y \notin R^{**}(x_0)$. \square

Let $X_1 = (V(H) \cap X) \setminus \{x_0, x_1\}$ and $Y_1 = (V(H) \cap Y) \setminus \{y_0, y_1\}$. Define D to be the directed graph on vertex set $X_1 \cup Y_1$ such that there is a directed edge from u to v if $uv \notin E(H)$ or $C(uv) = C(v)$. If $X_1 \subseteq R^{**}(y_0)$ and $Y_1 \subseteq R^{**}(x_0)$, then it follows from (ii) of Claim 1 that there exists a vertex in $R^{**}(x_0)$ or $R^{**}(y_0)$, say $y' \in R^{**}(x_0)$, such that $d_H^c(y') \leq |\{x_0, x_1\}| \leq 2$. Recall that $|V(H) \cap X| \geq \delta/2$. It follows that

$$d_G^c(y') \leq d_H^c(y') + n - |V(H) \cap X| \leq 2 + n - \delta/2 \leq 2n/3 + 1/2 < \delta,$$

a contradiction. So we have $(S^{**}(x_0) \cup T^{**}(x_0)) \cap X_1 \neq \emptyset$ or $(S^{**}(y_0) \cup T^{**}(y_0)) \cap Y_1 \neq \emptyset$.

Let $(S^{**}(y_0) \cup T^{**}(y_0)) \cap X_1 = X'_1$ and $(S^{**}(x_0) \cup T^{**}(x_0)) \cap Y_1 = Y'_1$. Define D' to be the subgraph of D induced by vertex set $X'_1 \cup Y'_1$. Note that $|A(D')| \geq |X'_1| \cdot |Y'_1|$. By an averaging argument, we may assume w.l.o.g. that there exists a vertex $x' \in X'_1$ such that $d_{D'}^-(x') \geq |Y'_1|/2$. It follows from (i) and (ii) of Claim 1 that

$$\begin{aligned} d_D^-(x') &\geq |Y'_1|/2 + |R^{**}(x_0) \cap Y_1| \\ &= |(S^{**}(x_0) \cup T^{**}(x_0)) \cap Y_1|/2 + |R^{**}(x_0) \cap Y_1| \\ &\geq \delta/2 - 2. \end{aligned}$$

Then $d_G^c(x') \leq n - d_D^-(x') + 1 \leq n - \delta/2 + 3 \leq 2n/3 + 3/2 < \delta$, a contradiction.

Case 2. There exists a PC path of length two, say $x_0 y_0 x_1$, in G' , where $x_0, x_1 \in X$ and $y_0 \in Y$.

We may assume that G' contains no PC paths of length three, since otherwise we are back in Case 1. Let H be an edge-colored subgraph of G induced by

$$R^{**}(x_0) \cup S^{**}(x_0) \cup T^{**}(x_0) \cup R^*(x_1) \cup S^*(x_1) \cup T^*(x_1).$$

Note that $2|R^{**}(x_0)| + |S^{**}(x_0)| + |T^{**}(x_0)| \geq \delta - 1$ and $2|R^*(x_1)| + |S^*(x_1)| + |T^*(x_1)| \geq \delta - 1$, since $W(x_0) = W(x_1) \subseteq \{y_0\}$. This implies that $|V(H) \cap X| \geq (\delta - 1)/2$ and $|V(H) \cap Y| \geq (\delta - 1)/2$. We define a vertex-coloring of $V(H)$ as follows: for each $v \in V(H)$,

$$C(v) = \begin{cases} \gamma & \text{if } v \in R^{**}(x_0) \text{ or } v \in R^*(x_1), \\ C(vv^+) & \text{if } v \in S^{**}(x_0) \text{ or } v \in S^*(x_1), \\ C(vv^-) & \text{if } v \in T^{**}(x_0) \text{ or } v \in T^*(x_1), \end{cases}$$

where γ is a new color that does not appear in G .

Claim 2. Let xy be an arbitrary edge in H with $x \in X, y \in Y$ and $x \notin \{y^{3-}, y^{3+}\}$.

(i) If $x \in S^*(x_1) \cup T^*(x_1)$ and $y \in S^{**}(x_0) \cup T^{**}(x_0)$, then $C(xy) = C(x)$ or $C(xy) = C(y)$.

(ii) If $x \in R^*(x_1)$ or $y \in R^{**}(x_0)$, say $x \in R^*(x_1)$, then $y \notin R^{**}(x_0)$ and $C(xy) = C(y)$, which implies that $R^*(x_0) \cup R^{**}(x_1)$ is an independent set in G .

Proof. (i) Suppose to the contrary that $C(xy) \neq C(x)$ and $C(xy) \neq C(y)$. We will indicate a subgraph F' in G consisting of vertex-disjoint PC cycles with $|V(F')| > |V(F)|$, which contradicts the maximality of $|V(F)|$.

If $x \notin \{y^+, y^-, y^{3+}, y^{3-}\}$, then $x, g(x), y, f(y)$ are pairwise different. Let $P = f(y)x_0y_0x_1g(x)$. Set

$$F' = F - \{f'(y)\} - \{xg(x)\} + \{xy\} + P.$$

If $x = y^+$ or $x = y^-$, then assume w.l.o.g. that $x = y^-$. If $x \in S^{**}(x_1)$ or $y \in T^*(x_0)$, then $C(xy) = C(x)$ or $C(xy) = C(y)$; if $x \in T^{**}(x_1)$ and $y \in S^*(x_0)$, then set $F' = F - \{x\} + P$.

(ii) If $x \in R^*(x_1)$, then we can treat $x \in S^*(x_1)$ and $x \in T^*(x_1)$, respectively. The proof is similar to the proof of (ii) of Claim 1. We omit the details. \square

Let $X_1 = V(H) \cap X$ and $Y_1 = V(H) \cap Y$. Define D to be the directed graph on vertex set $X_1 \cup Y_1$ such that there is a directed edge from u to v if $uv \notin E(H)$ or $C(uv) = C(v)$. If $X_1 \subseteq R^*(x_1)$ and $Y_1 \subseteq R^{**}(x_0)$, then it follows from (ii) of Claim 1 that there exists a vertex $x' \in R^*(x_1)$ such that $d_H^c(x') = 0$ or $y' \in R^{**}(x_0)$ such that $d_H^c(y') = 0$. Recall that $|V(H) \cap X| \geq (\delta - 1)/2$

and $|V(H) \cap Y| \geq (\delta - 1)/2$. If there exists a vertex $x' \in R^*(x_1)$ such that $d_H^c(x') = 0$, then it follows that

$$d_G^c(x') \leq d_H^c(x') + n - |V(H) \cap Y| \leq d_H^c(x') + n - (\delta - 1)/2 \leq 2n/3 - 1 < \delta,$$

a contradiction. If there exists a vertex $y' \in R^{**}(x_0)$ such that $d_H^c(y') = 0$, then it follows that

$$d_G^c(y') \leq d_H^c(y') + n - |V(H) \cap X| \leq d_H^c(y') + n - (\delta - 1)/2 \leq 2n/3 - 1 < \delta,$$

a contradiction. So we have $(S^*(x_1) \cup T^*(x_1)) \cap X_1 \neq \emptyset$ or $(S^{**}(x_0) \cup T^{**}(x_0)) \cap Y_1 \neq \emptyset$.

Let $(S^*(x_1) \cup T^*(x_1)) \cap X_1 = X'_1$ and $(S^{**}(x_0) \cup T^{**}(x_0)) \cap Y_1 = Y'_1$. Define D' to be the subgraph of D induced by vertex set $X'_1 \cup Y'_1$. Note that $|A(D')| \geq |X'_1| \cdot |Y'_1| - |X'_1| - |Y'_1|$. By an averaging argument, there exists a vertex $x' \in X'_1$ such that $d_{D'}^-(x') \geq |Y'_1|/2 - 1$ or a vertex $y' \in Y'_1$ such that $d_{D'}^-(y') \geq |X'_1|/2 - 1$. If there exists a vertex $x' \in X'_1$ such that $d_{D'}^-(x') \geq |Y'_1|/2 - 1$, then it follows from (i) and (ii) of Claim 2 that

$$\begin{aligned} d_D^-(x') &\geq d_{D'}^-(x') + |R^{**}(x_0) \cap Y_1| \\ &\geq |Y'_1|/2 - 1 + |R^{**}(x_0) \cap Y_1| \\ &= |(S^{**}(x_0) \cup T^{**}(x_0)) \cap Y_1|/2 + |R^{**}(x_0) \cap Y_1| - 1 \\ &\geq (\delta - 3)/2. \end{aligned}$$

Then $d_G^c(x') \leq n - d_D^-(x') + 1 \leq n - (\delta - 3)/2 + 1 \leq 2n/3 + 1 < \delta$, a contradiction. If there exists a vertex $y' \in Y'_1$ such that $d_{D'}^-(y') \geq |X'_1|/2 - 1$, then it follows from (i) and (ii) of Claim 2 that

$$\begin{aligned} d_D^-(y') &\geq d_{D'}^-(y') + |R^*(x_1) \cap X_1| \\ &\geq |X'_1|/2 - 1 + |R^*(x_1) \cap X_1| \\ &= |(S^*(x_1) \cup T^*(x_1)) \cap X_1|/2 + |R^*(x_1) \cap X_1| - 1 \\ &\geq (\delta - 3)/2. \end{aligned}$$

Then $d_G^c(y') \leq n - d_D^-(y') + 1 \leq n - (\delta - 3)/2 + 1 \leq 2n/3 + 1 < \delta$, a contradiction.

Case 3. There exists an edge, say x_0y_0 , in G' , where $x_0 \in X$ and $y_0 \in Y$.

We may assume that G' contains no PC paths of length two, since otherwise we are back in Case 1 or Case 2. Let H be an edge-colored subgraph of G induced by

$$R^*(x_0) \cup S^*(x_0) \cup T^*(x_0) \cup R^*(y_0) \cup S^*(y_0) \cup T^*(y_0).$$

Note that $2|R^*(x_0)| + |S^*(x_0)| + |T^*(x_0)| \geq \delta - 1$ since $W(x_0) \subseteq \{y_0\}$ and $W(y_0) \subseteq \{x_0\}$. This implies that $|V(H) \cap X| \geq (\delta - 1)/2$ and $|V(H) \cap Y| \geq (\delta - 1)/2$. We define a vertex-coloring of $V(H)$ as follows: for each $v \in V(H)$,

$$C(v) = \begin{cases} \gamma & \text{if } v \in R^*(x_0) \text{ or } v \in R^*(y_0), \\ C(vv^+) & \text{if } v \in S^*(x_0) \text{ or } v \in S^*(y_0), \\ C(vv^-) & \text{if } v \in T^*(x_0) \text{ or } v \in T^*(y_0), \end{cases}$$

where γ is a new color that does not appear in G .

Claim 3. Let xy be an arbitrary edge in H with $x \in X, y \in Y$ and $x \notin \{y^{3-}, y^{3+}\}$.

(i) If $x \in S^*(x_0) \cup T^*(x_0)$ and $y \in S^*(y_0) \cup T^*(y_0)$, then $C(xy) = C(x)$ or $C(xy) = C(y)$.

(ii) If $x \in R^*(x_0)$ or $y \in R^*(y_0)$, say $x \in R^*(x_0)$, then $y \notin R^*(x_0)$ and $C(xy) = C(y)$, which implies that $R^*(x_0) \cup R^*(y_0)$ is an independent set in G .

Proof. (i) Suppose to the contrary that $C(xy) \neq C(x)$ and $C(xy) \neq C(y)$. We will indicate a subgraph F' in G consisting of vertex-disjoint PC cycles with $|V(F')| > |V(F)|$, which contradicts the maximality of $|V(F)|$. Let $P = g(x)x_0y_0g(y)$.

If $x \notin \{y^+, y^-, y^{3+}, y^{3-}\}$, then $x, g(x), y, g(y)$ are pairwise different. Set

$$F' = F - \{xg(x), yg(y)\} + \{xy\} + P.$$

If $x = y^+$ or $x = y^-$, then assume w.l.o.g. that $x = y^-$. If $x \in S^*(x_0)$ or $y \in T^*(y_0)$, then $C(xy) = C(x)$ or $C(xy) = C(y)$; if $x \in T^*(x_0)$ and $y \in S^*(y_0)$, then set $F' = F - \{xy\} + P$.

(ii) If $x \in R^*(x_0)$, then we can treat $x \in S^*(x_0)$ and $x \in T^*(x_0)$, respectively. The proof is similar to the proof of (ii) of Claim 1. We omit the details. \square

Let $X_1 = V(H) \cap X$ and $Y_1 = V(H) \cap Y$. Define D to be the directed graph on vertex set $X_1 \cup Y_1$ such that there is a directed edge from u to v if $uv \notin E(H)$ or $C(uv) = C(v)$. If $X_1 \subseteq R^*(x_0)$ or $Y_1 \subseteq R^*(y_0)$, then it follows from (ii) of Claim 3 that there exists a vertex in $R^*(x_0)$ or $R^*(y_0)$, say $x' \in R^*(x_0)$, such that $d_H^c(x') = 0$. Recall that $|V(H) \cap Y| \geq (\delta - 1)/2$. It follows that

$$d_G^c(x') \leq d_H^c(x') + n - |V(H) \cap Y| \leq d_H^c(x') + n - (\delta - 1)/2 \leq 2n/3 - 1 < \delta,$$

a contradiction. So we have $(S^*(x_0) \cup T^*(x_0)) \cap X_1 \neq \emptyset$ or $(S^*(y_0) \cup T^*(y_0)) \cap Y_1 \neq \emptyset$.

Let $S^*(x_1) \cup T^*(x_1) \cap X_1 = X'_1$ and $(S^{**}(x_0) \cup T^{**}(x_0)) \cap Y_1 = Y'_1$. Define D' to be the subgraph of D induced by vertex set $X'_1 \cup Y'_1$. Note that $|A(D')| \geq |X'_1| \cdot |Y'_1| - |X'_1| - |Y'_1|$. By an averaging argument, we may assume w.l.o.g. that there exists a vertex $x' \in X'_1$ such that $d_{D'}^-(x') \geq |Y'_1|/2 - 1$. It follows from (i) and (ii) of Claim 3 that

$$\begin{aligned} d_D^-(x') &\geq d_{D'}^-(x') + |R^*(y_0) \cap Y_1| \\ &\geq |Y'_1|/2 - 1 + |R^*(y_0) \cap Y_1| \\ &= |(S^*(y_0) \cup T^*(y_0)) \cap Y_1|/2 + |R^*(y_0) \cap Y_1| - 1 \\ &\geq (\delta - 3)/2. \end{aligned}$$

Then $d_G^c(x') \leq n - d_D^-(x') + 1 \leq n - (\delta - 3)/2 + 1 \leq 2n/3 + 1 < \delta$, a contradiction.

Case 4. There exist no edges in G' .

Let x_0 and y_0 be a pair of vertices in $V(G')$ with $x_0 \in X$ and $y_0 \in Y$. Define H to be an edge-colored subgraph of G induced by $R^*(x_0) \cup S^*(x_0) \cup$

$T^*(x_0) \cup R^*(y_0) \cup S^*(y_0) \cup T^*(y_0)$. Note that $2|R^*(x_0)| + |S^*(x_0)| + |T^*(x_0)| \geq \delta$, since $W(x_0) = \emptyset$ and $W(y_0) = \emptyset$. This implies that $|V(H) \cap X| \geq \delta/2$ and $|V(H) \cap Y| \geq \delta/2$. We define a vertex-coloring of $V(H)$ as follows: for each $v \in V(H)$,

$$C(v) = \begin{cases} \gamma & \text{if } v \in R^*(x_0) \text{ or } v \in R^*(y_0), \\ C(vv^-) & \text{if } v \in S^*(x_0) \text{ or } v \in S^*(y_0), \\ C(vv^+) & \text{if } v \in T^*(x_0) \text{ or } v \in T^*(y_0), \end{cases}$$

where γ is a new color that does not appear in G .

Firstly we assert that there exists an edge, say x_1y_1 , such that $x_1y_1 \in E(H) \setminus E(\cup_{i=1}^k C_i)$. Indeed, if not, then we can consider an arbitrary vertex $v \in V(G') \cap X$. Since $d_{G'}^c(v) \geq 2n/3 + 3$ and G' contains no edges, we have $|V(F) \cap Y| \geq 2n/3 + 3$, which implies that $|V(G') \cap Y| \leq n/3 - 3$. Now for an arbitrary vertex $x \in V(F) \cap X$, we have $d_G^c(x) \leq |\{xx^-, xx^+\}| + |V(G') \cap Y| \leq n/3 - 1$, a contradiction.

Let x_1y_1 be such an edge. We use $g'(x_1, S^*(x_0))$ to denote the corresponding vertices $g'(x_1)$ when $x_1 \in R^*(x_0)$ and we treat $x_1 \in S^*(x_0)$, and use $g'(y_1, S^*(y_0))$ to denote the corresponding vertices $g'(y_1)$ when $y_1 \in R^*(y_0)$ and we treat $y_1 \in S^*(y_0)$. Define $X_1 = \{x_1, \hat{g}(x_1), y_1^-, y_1^+\}$, $Y_1 = \{y_1, \hat{g}(y_1), x_1^-, x_1^+\}$, where

$$\hat{g}(x_1) = \begin{cases} g'(x_1) & \text{if } x_1 \in S^*(x_0) \cup T^*(x_0), \\ g'(x_1, S^*(x_0)) & \text{if } x_1 \in R^*(x_0), \end{cases}$$

and

$$\hat{g}(y_1) = \begin{cases} g'(y_1) & \text{if } y_1 \in S^*(y_0) \cup T^*(y_0), \\ g'(y_1, S^*(y_0)) & \text{if } y_1 \in R^*(y_0). \end{cases}$$

Let $X_2 = (V(H) \cap X) \setminus X_1$ and $Y_2 = (V(H) \cap Y) \setminus Y_1$.

Claim 4. Let x_2y_2 be an arbitrary edge in H such that $x_2 \in X_2$ and $y_2 \in Y_2$.

(i) If $x_2 \in S^*(x_0) \cup T^*(x_0)$ and $y_2 \in S^*(y_0) \cup T^*(y_0)$, then $C(x_2y_2) = C(x_2)$

or $C(x_2y_2) = C(y_2)$.

(ii) If $x_2 \in R^*(x_0)$ or $y_2 \in R^*(y_0)$, say $x_2 \in R^*(x_0)$, then $y_2 \notin R^*(x_0)$ and $C(x_2y_2) = C(y_2)$, which implies that $R^*(x_0) \cup R^*(y_0)$ is an independent set in G .

(iii) $x_2 \notin \{y_2^-, y_2^+\}$.

Proof. (i) Suppose to the contrary that $C(x_2y_2) \neq C(x_2)$ and $C(x_2y_2) \neq C(y_2)$. We will indicate a subgraph F' in G consisting of vertex-disjoint PC cycles with $|V(F')| > |V(F)|$, which contradicts the maximality of $|V(F)|$.

If $x_2 \notin \{y_2^-, y_2^+\}$, then it is clear that

$$x_1, \hat{g}(x_1), y_1, \hat{g}(y_1), x_2, g(x_2), y_2, g(y_2)$$

are pairwise different. Let $P_1 = \hat{g}(x_1)x_0g(x_2)$ and $P_2 = \hat{g}(y_1)y_0g(y_2)$. Set

$$F' = F - \{x_1\hat{g}(x_1), x_2g(x_2), y_1\hat{g}(y_1), y_2g(y_2)\} + \{x_1y_1, x_2y_2\} + P_1 + P_2.$$

For the case $x_2 = y_2^-$, if $x_2 \in S^*(x_0)$ or $y_2 \in T^*(y_0)$, then $C(x_2y_2) = C(x_2)$ or $C(x_2y_2) = C(y_2)$; if $x_2 \in T^*(x_0)$ and $y_2 \in S^*(y_0)$, then set

$$F' = F - \{x_1\hat{g}(x_1), y_1\hat{g}(y_1), x_2y_2\} + \{x_1y_1\} + P_1 + P_2.$$

For the case $x_2 = y_2^+$, the proof is similar. We omit the details.

(ii) If $x_2 \in R^*(x_0)$, then we can treat $x_2 \in S^*(x_0)$ and $x_2 \in T^*(x_0)$, respectively. The proof is similar to the proof of (ii) of Claim 1. We omit the details.

(iii) Firstly we consider the case $x_2 = y_2^-$. Let $P_1 = \hat{g}(x_1)x_0g(x_2)$ and $P_2 = \hat{g}(y_1)y_0g(y_2)$. If $x_2 \in S^*(x_0)$ and $y_2 \in T^*(y_0)$, then set

$$F' = F - \{x_2, y_2\} - \{x_1\hat{g}(x_1), y_1\hat{g}(y_1)\} + \{x_1y_1\} + P_1 + P_2.$$

Note that F' consists of vertex-disjoint PC cycles with $|V(F')| = |V(F)|$ and x_2y_2 is an edge in $G - F'$, which contradicts the choice of F . If $x_2 \in T^*(x_0) \cup$

$R^*(x_0)$ and $y_2 \in S^*(y_0) \cup R^*(y_0)$, then set

$$F' = F - \{x_1\hat{g}(x_1), y_1\hat{g}(y_1), x_2y_2\} + \{x_1y_1\} + P_1 + P_2.$$

Note that F' consists of vertex-disjoint PC cycles with $|V(F')| > |V(F)|$, a contradiction.

For the case $x_2 = y_2^+$, the proof is similar. We omit the details. \square

Define D to be the directed graph on vertex set $X_2 \cup Y_2$ such that there is a directed edge from u to v if $uv \notin E(H)$ or $C(uv) = C(v)$. If $X_2 \subseteq R^*(x_0)$ and $Y_2 \subseteq R^*(y_0)$, then it follows from (ii) of Claim 4 that there exists a vertex in $R^*(x_0)$ or $R^*(y_0)$, say $x' \in R^*(x_0)$, such that $d_H^c(x') \leq |Y_1| = 4$. Recall that $|V(H) \cap Y| \geq \delta/2$. It follows that

$$d_G^c(x') \leq d_H^c(x') + n - |V(H) \cap Y| \leq d_H^c(x') + n - \delta/2 \leq 2n/3 + 5/2 < \delta,$$

a contradiction. So we have $(S^*(x_0) \cup T^*(x_0)) \cap X_2 \neq \emptyset$ or $(S^*(y_0) \cup T^*(y_0)) \cap Y_2 \neq \emptyset$.

Let $(S^*(x_0) \cup T^*(x_0)) \cap X_2 = X'_2$ and $(S^*(y_0) \cup T^*(y_0)) \cap Y_2 = Y'_2$. Define D' to be the subgraph of D induced by vertex set $X'_2 \cup Y'_2$. Note that $|A(D')| \geq |X'_2| \cdot |Y'_2|$. By an averaging argument, we may assume w.l.o.g. that there exists a vertex $x' \in X'_2$ such that $d_{D'}^-(x') \geq |Y'_2|/2$. It follows from (i) and (ii) of Claim 4 that

$$\begin{aligned} d_D^-(x') &\geq |Y'_2|/2 + |R^*(y_0) \cap Y_2| \\ &= |(S^*(y_0) \cup T^*(y_0)) \cap Y_2|/2 + |R^*(y_0) \cap Y_2| \\ &\geq \delta/2 - 4. \end{aligned}$$

By (iii) of Claim 4, we have

$$|\{z \in V(G) : C(x'z) = C(x')\}| \geq d_D^-(x') + 1 \geq \delta/2 - 3.$$

So

$$d_G^c(x') \leq n - |\{z \in V(G) : C(x'z) = C(x')\}| + 1 \leq 2n/3 + 5/2 < \delta,$$

a contradiction.

This completes the proof of Theorem 5.3.

□

Chapter 6

PC even vertex-pancyclicity of edge-colored complete bipartite graphs

In this chapter, we obtain a necessary and sufficient condition for edge-colored complete balanced bipartite graphs containing no monochromatic paths of length three to be PC even vertex-pancyclic. Our result is the following generalization of a result of Häggkvist and Manoussakis on bipartite tournaments in [Combinatorica 9 (1989) 33–38]. If $K_{n,n}^c$ contains no monochromatic paths of length three and $K_{n,n}^c$ has a PC Hamilton cycle, then $K_{n,n}^c$ is PC even vertex-pancyclic, unless $K_{n,n}^c$ belongs to two special classes of edge-colored graphs.

6.1 Introduction

Recall the definition that an edge-colored graph G is properly colored (even) vertex-pancyclic if every vertex of G is contained in properly colored cycles of all possible (even) lengths. Our main result deals with the PC even vertex-pancyclicity of edge-colored balanced complete bipartite graphs. In recent years, the PC vertex-pancyclicity of edge-colored complete graphs containing

no specific monochromatic subgraphs has been studied extensively (See [28, 64, 65] for examples). In order to motivate the condition of containing no monochromatic paths of length three in our next result, we note the following. If there exists a monochromatic path of length three in an edge-colored bipartite graph, then we can delete its middle edge without changing the color degrees of the vertices of the graph. From this point of view, it is of independent interest to study the properties of edge-colored graphs containing no monochromatic paths of length three. In our main result, we show that edge-colored balanced complete bipartite graphs containing no monochromatic paths of length three are PC even vertex-pancyclic if and only if they have a PC Hamilton cycle, unless they belong to one of the following two special families of edge-colored graphs.

Before presenting our result, we need to define the classes \mathcal{G}_1^* and \mathcal{G}_2^* of exceptional graphs, as subclasses of \mathcal{G}_1 and \mathcal{G}_2 below, respectively. One can observe that every graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ contains only PC cycles of lengths which are multiples of 4, and G contains a PC Hamilton cycle if and only if $G \in \mathcal{G}_1^* \cup \mathcal{G}_2^*$. For both classes, consider the set of all edge-colored complete bipartite graphs with vertex set $V = V_1 \cup V_2 \cup V_3 \cup V_4$, for disjoint sets $V_i = \{v_i^1, v_i^2, \dots, v_i^{|V_i|}\}$.

Construction 6.1. For graphs in \mathcal{G}_1 , let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ be a color set, where $\Gamma_i = \{c_i^1, \dots, c_i^{|V_i|}\}$ and $\Gamma_i \cap \Gamma_{i+1} = \emptyset$ for each $i \in \{1, 2, 3, 4\}$; for every pair i, j with $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \dots, |V_i|\}$, add all possible edges between V_{i+1} and v_i^j and color them with color c_i^j , where subscripts are taken modulo 4. Let \mathcal{G}_1^* be the set of all graphs in \mathcal{G}_1 satisfying $|V_1| = |V_2| = |V_3| = |V_4|$.

Construction 6.2. For graphs in \mathcal{G}_2 , let $\Gamma = \Gamma_1 \cup \Gamma'_1 \cup \Gamma_3 \cup \Gamma'_3$ be a color set, where $\Gamma_i = \{c_i^1, \dots, c_i^{|V_i|}\}$, $\Gamma'_i = \{\bar{c}_i^1, \dots, \bar{c}_i^{|V_i|}\}$ for each $i \in \{1, 3\}$ and $\Gamma_1 \cap \Gamma'_3 = \emptyset$, $\Gamma'_1 \cap \Gamma_3 = \emptyset$; for every pair i, j with $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \dots, |V_i|\}$, add all possible edges between V_{i+1} and v_i^j and color them with color c_i^j , and add all possible edges between V_{i-1} and v_i^j and color them with color \bar{c}_i^j , where subscripts are taken modulo 4. Let \mathcal{G}_2^* be the set of all graphs in \mathcal{G}_2 satisfying that $|V_1| = |V_3|$ and $|V_1| + |V_3| = |V_2| + |V_4|$.

In this chapter, we obtain the following necessary and sufficient condition for an edge-colored complete balanced bipartite graph $K_{n,n}^c$ containing no monochromatic paths of length three to be PC even pancyclic and PC even vertex-pancyclic, respectively.

Theorem 6.1. *Suppose that $K_{n,n}^c$ contains no monochromatic paths of length three. Then $K_{n,n}^c$ is PC even pancyclic if and only if it contains a PC Hamilton cycle, unless $K_{n,n}^c \in \mathcal{G}_1^* \cup \mathcal{G}_2^*$.*

Theorem 6.2. *Suppose that $K_{n,n}^c$ contains no monochromatic paths of length three. Then $K_{n,n}^c$ is PC even vertex-pancyclic if and only if it has a PC Hamilton cycle, unless $K_{n,n}^c \in \mathcal{G}_1^* \cup \mathcal{G}_2^*$.*

In Section 6.2, we present the proof of Theorem 6.1. Based on Theorem 6.1, in Section 6.3, we present the proof of Theorem 6.2. It is interesting to note that Theorem 6.2 can be seen as a generalization of the following theorem. Here we refrain from defining the class of exceptional graphs, since they are not relevant in this context.

Theorem 6.3 (Häggkvist and Manoussakis [54]). *If a bipartite tournament has a Hamilton cycle, then either it is even vertex-pancyclic or it belongs to a well-defined class of digraphs.*

Here a bipartite tournament is defined as an orientation of a complete bipartite graph. To explain the generalization, we recall the aforementioned simple transformation introduced by Li [63]. Suppose G^c is obtained from a bipartite tournament T with a Hamilton cycle by this transformation. It is not hard to see that (i) for each vertex v in G^c , at most one color appearing on the edges incident with v appears more than once; (ii) G^c contains no monochromatic paths of length three; (iii) there is a natural one-to-one correspondence between PC cycles in G^c and directed cycles in T . So, in this sense Theorem 6.2 can be seen as a generalization of Theorem 6.3 by removing condition (i).

6.2 PC even pancyclicity

Proof of Theorem 6.1. Let $G = K_{n,n}^c$ contain no monochromatic paths of length three, and suppose $G \notin \mathcal{G}_1^* \cup \mathcal{G}_2^*$. One direction is trivial. It suffices to show that if G contains a PC Hamilton cycle, then it contains PC cycles of all possible even lengths. Let $C = v_0 v_1 \cdots v_{2k-1} v_0$ be a PC cycle of length $2k$ in G for an arbitrary integer $k \geq 3$. We will show that G contains a PC cycle of length $2k - 2$.

Suppose to the contrary that G contains no PC cycles of length $2k - 2$. In the remainder of the proof all subscripts are taken modulo $2k$. If there exists some $i \in \{0, 1, \dots, 2k-1\}$ such that $C(v_{i-1}v_i) \neq C(v_i v_{i+3}) \neq C(v_{i+3}v_{i+4})$, then $v_i v_{i+3} C v_i$ is a PC cycle of length $2k - 2$. So we may assume that $C(v_i v_{i+3}) = C(v_{i-1}v_i)$ or $C(v_i v_{i+3}) = C(v_{i+3}v_{i+4})$ for each $i \in \{0, 1, \dots, 2k-1\}$. By symmetry, assume that $C(v_0 v_3) = C(v_3 v_4)$. If $C(v_2 v_5) = C(v_1 v_2)$, then $v_0 v_3 v_2 v_5 C v_0$ is a PC cycle of length $2k - 2$. So we may assume that $C(v_2 v_5) = C(v_5 v_6)$. Considering the cycle $v_i v_{i+3} v_{i+2} v_{i+5} C v_i$ one by one in the order $i = 0, 2, \dots, 2k - 2$, we have that $C(v_i v_{i+3}) = C(v_{i+3} v_{i+4})$ for every $i \in \{0, 2, \dots, 2k - 2\}$. By similar arguments, we have that $C(v_i v_{i+3}) = C(v_{i+3} v_{i+4})$ for every $i \in \{1, 3, \dots, 2k - 1\}$ or $C(v_i v_{i+3}) = C(v_{i-1} v_i)$ for every $i \in \{1, 3, \dots, 2k - 1\}$.

We distinguish the following two cases.

Case 1. $C(v_i v_{i+3}) = C(v_{i+3} v_{i+4})$ for every $i \in \{1, 3, \dots, 2k - 1\}$.

Define a vertex-coloring of $V(C)$ such that $C(v_i) = C(v_i v_{i+1})$ for each $i \in \{0, 1, \dots, 2k - 1\}$. In the following, it suffices to show that $C(v_i v_{i+4m+1}) = C(v_i)$ and $C(v_i v_{i+4m+3}) = C(v_{i+4m+3})$ for every pair of i, m with $i \in \{0, 1, \dots, 2k - 1\}$ and $m \in \{0, 1, \dots, (k - 2)/2\}$, where k is a multiple of 4.

Suppose to the contrary that r_1 is the minimum value such that

$$C(v_i v_{i+4r_1+1}) \neq C(v_i)$$

and r_2 is the minimum value such that $C(v_i v_{i+4r_2+3}) \neq C(v_{i+4r_2+3})$. For each $i \in \{1, 2\}$, set $r_i = \infty$ if there exists no such r_i . Let $r = \min\{r_1, r_2\}$. Note that $C(v_i v_{i+1}) = C(v_i)$ and $C(v_i v_{i+3}) = C(v_{i+3})$ for each $i \in \{0, 1, \dots, 2k - 1\}$. Hence $r \geq 1$. If $r = r_1$, then assume w.l.o.g. that $C(v_0 v_{4r+1}) \neq C(v_0)$. By

the minimality of r , we have $C(v_{-2}v_{4r-3}) = C(v_{4r-3})$ and $C(v_{-1}v_{4r-2}) = C(v_{4r-2})$. Recall that $C(v_i v_{i+3}) = C(v_{i+3})$ for each $i \in \{0, 1, \dots, 2k-1\}$. If $C(v_0 v_{4r+1}) \neq C(v_{4r} v_{4r+1})$, then

$$v_0 v_1 v_2 C v_{4r-3} v_{-2} v_{-1} v_{-4} v_{-3} v_{-6} v_{-5} \cdots v_{4r+2} v_{4r+3} v_{4r} v_{4r+1} v_0$$

is a PC cycle of length $2k-2$, a contradiction. If $C(v_0 v_{4r+1}) = C(v_{4r} v_{4r+1})$, then

$$v_0 v_{4r+1} C v_{-1} v_{4r-2} v_{4r-3} \bar{C} v_0$$

is a PC cycle of length $2k-2$, a contradiction. If $r = r_2$, then assume w.l.o.g. that $C(v_0 v_{4r+3}) \neq C(v_{4r+3})$. By the minimality of r , we have $C(v_{-1} v_{4r}) = C(v_{-1})$. Recall that $C(v_i v_{i+3}) = C(v_{i+3})$ for each $i \in \{0, 2, \dots, 2k-2\}$. Hence

$$v_0 v_{4r+3} C v_{-1} v_{4r} v_{4r+1} v_{4r-2} v_{4r-1} v_{4r-4} v_{4r-3} \cdots v_4 v_5 v_2 v_3 v_0$$

is a PC cycle of length $2k-2$, a contradiction. So we have $G \in \mathcal{G}_1$. Moreover, since G contains a PC Hamilton cycle, we have $G \in \mathcal{G}_1^*$. This contradiction implies that k is a multiple of 4, which completes this case.

Case 2. $C(v_i v_{i+3}) = C(v_{i-1} v_i)$ for every $i \in \{1, 3, \dots, 2k-1\}$.

Define two vertex-colorings of $V(C)$ such that $C(v_i) = C(v_i v_{i+1})$ and $\bar{C}(v_i) = C(v_{i-1} v_i)$ for each $i \in \{1, 3, \dots, 2k-1\}$. Note that $C(v_i v_{i+3}) = \bar{C}(v_i)$ and $C(v_{i-3} v_i) = C(v_i)$ for every $i \in \{1, 3, \dots, 2k-1\}$. In the following, it suffices to show that $C(v_i v_{i+4m+1}) = \bar{C}(v_{i+4m+1})$ and $C(v_i v_{i+4m+3}) = C(v_{i+4m+3})$ for every $i \in \{0, 2, \dots, 2k-2\}$ and $m \in \{0, 1, \dots, (k-2)/2\}$, where k is a multiple of 4.

Suppose to the contrary that r_1 is the minimum value such that

$$C(v_i v_{i+4r_1+1}) \neq \bar{C}(v_{i+4r_1+1})$$

and r_2 is the minimum value such that $C(v_i v_{i+4r_2+3}) \neq C(v_{i+4r_2+3})$. For each $i \in \{1, 2\}$, set $r_i = \infty$ if there exists no such r_i . Let $r = \min\{r_1, r_2\}$. Note that $C(v_i v_{i+1}) = \bar{C}(v_{i+1})$ and $C(v_i v_{i+3}) = C(v_{i+3})$ for each $i \in \{1, 3, \dots, 2k-1\}$. Hence $r \geq 1$. If $r = r_1$, then assume w.l.o.g. that $C(v_0 v_{4r+1}) \neq \bar{C}(v_{4r+1})$. Since

G contains no monochromatic paths of length three, we have $C(v_0 v_{4r+1}) \neq C(v_0)$ and $C(v_0 v_{4r+1}) \neq \bar{C}(v_0)$. By the minimality of r , we have $C(v_{-2} v_{4r-3}) = C(v_{4r-3})$. Recall that $C(v_{i-3} v_i) = C(v_i)$ for each $i \in \{1, 3, \dots, 2k-1\}$. It follows that

$$v_0 v_{4r+1} v_{4r} v_{4r+3} v_{4r+2} \cdots v_{-1} v_{-2} v_{4r-3} \bar{C} v_0$$

is a PC cycle of length $2k-2$, a contradiction. If $r = r_2$, then assume w.l.o.g. that $C(v_0 v_{4r+3}) \neq C(v_{4r+3})$. Since G contains no monochromatic paths of length three, we have $C(v_0 v_{4r-3}) \neq \bar{C}(v_0)$. By the minimality of r , we have $C(v_{-2} v_{4r-1}) = \bar{C}(v_{4r-1})$. Recall that $C(v_i v_{i+3}) = C(v_{i+3})$ for each $i \in \{0, 1, \dots, 2k-1\}$. It follows that

$$v_0 v_{4r+3} C v_{-2} v_{4r-1} v_{4r} v_{4r-3} v_{4r-2} v_{4r-5} v_{4r-4} \cdots v_4 v_1 v_2 v_{-1} v_0$$

is a PC cycle of length $2k-2$, a contradiction. So we have $G \in \mathcal{G}_2$. Moreover, since G contains a PC Hamilton cycle, we have $G \in \mathcal{G}_2^*$. This contradiction implies that k is a multiple of 4, which completes the proof of this case and the lemma. \square

6.3 PC even vertex-pancyclicity

In this section, we give the proof of Theorem 6.2. Firstly, we present a lemma which will be used in the proof.

Lemma 6.1. *Suppose that $K_{n,n}^c$ contains no monochromatic paths of length three, and that C is a PC cycle of length $2n-2$ in $K_{n,n}^c$ containing a vertex x of $K_{n,n}^c$. If $K_{n,n}^c[V(C)] \in \mathcal{G}_1^* \cup \mathcal{G}_2^*$ and $K_{n,n}^c$ contains a PC Hamilton cycle, then $K_{n,n}^c$ contains PC cycles of all possible even lengths through x .*

Proof. Suppose to the contrary that the statement is false and G is a counterexample. Since $G[V(C)] \in \mathcal{G}_1^* \cup \mathcal{G}_2^*$, one can check that there exists a PC cycle of length $4m$ through x for every integer $m \in \{1, 2, \dots, (n-1)/2\}$. If there exists a PC cycle of length $4m+2$ through x for every integer $m \in \{1, 2, \dots, (n-1)/2\}$, then G is PC even vertex-pancyclic, a contradiction. So we assume that there exist no PC cycles of length $4m_0+2$ through x for some $m_0 \in \{1, 2, \dots, (n-1)/2\}$.

We distinguish two cases depending on whether

$$G[V(C)] \in \mathcal{G}_1^* \text{ or } G[V(C)] \in \mathcal{G}_2^*.$$

Case 1. $G[V(C)] \in \mathcal{G}_1^*$.

Let $\{V_1, V_2, V_3, V_4\}$ be a partition of $V(C)$ such that $|C(v_i, V_{i+1})| = 1$ for each vertex $v_i \in V_i$ with $i \in \{1, 2, 3, 4\}$, where the subscripts are taken modulo 4. Define a vertex-coloring of $V(C)$ such that $C(v_i) \in C(v_i, V_{i+1})$ for each vertex $v_i \in V_i$ with $i \in \{1, 2, 3, 4\}$.

Note that, for each pair of vertices $v_i, v'_i \in V_i$, there exists a PC path, say $P(v_i, v'_i)$, of length $4m_0$ through x such that $C(v_i v_i^+) = C(v_i^+)$ and $C(v'_i v_i'^-) = C(v'_i)$; for each pair of vertices $v_i \in V_i, v_{i+1} \in V_{i+1}$, there exists a PC path, say $P(v_i, v_{i+1})$, of length $4m_0 - 1$ through x such that $C(v_i v_i^+) = C(v_i^+)$ and $C(v_{i+1} v_{i+1}^-) = C(v_{i+1})$.

Let $\{X, Y\}$ be the bipartition of G with $V_1, V_3 \in X$ and $V_2, V_4 \in Y$. Let x' and y' be the vertices of $G - C$ with $x' \in X$ and $y' \in Y$. Define a digraph D on vertex set $V(G)$. For every edge uv with $u \in \{x', y'\}$ and $v \in V(C)$, add the arc $uv \in A(D)$ if $C(uv) \neq C(u)$ and the arc $vu \in A(D)$ if $C(uv) = C(u)$; for every edge uv with $u, v \in V(C)$, add the arc $uv \in A(D)$ if $C(uv) = C(v)$.

Firstly we assert that $u \Rightarrow V_i$ or $V_i \Rightarrow u$. Indeed, if not, then we can choose a pair of vertices $v_i, v'_i \in V_i$ such that $uv_i \in A(D)$ and $v'_i u \in A(D)$. In particular, if $x \in V_i$, then we prefer to choose $x = v_i$ or $x = v'_i$. By the definition of D , we have $C(uv_i) = C(v_i)$ and $C(v'_i u) \neq C(v'_i)$. Recall that $C(v_i v_i^+) = C(v_i^+)$ and $C(v'_i v_i'^-) = C(v'_i)$. This implies that $P(v_i, v'_i)uv_i$ is a PC cycle of length $4m_0 + 2$ through x , a contradiction. So we have $u \Rightarrow V_i$ or $V_i \Rightarrow u$.

Next we assert that if $V_i \Rightarrow u$ for some $i \in \{1, 2, 3, 4\}$, then $|C(u, V_i)| = 1$. Indeed, if $|C(u, V_i)| \geq 2$, then we can choose a pair of vertices $v_i, v'_i \in V_i$ such that $v_i u, v'_i u \in A(D)$ and $C(uv_i) \neq C(uv'_i)$. In particular, if $x \in V_i$, then we prefer to choose $x = v_i$ or $x = v'_i$. By the definition of D , we have $C(uv'_i) \neq C(v'_i v_i'^-)$. Note that $C(uv_i) \neq C(v_i v_i^+)$, since otherwise there exists a vertex $v_{i-1} \in V_{i-1}$ such that $uv_i v_{i-1} v'_i$ is a monochromatic path of length three, a contradiction. It follows that $P(v_i, v'_i)uv_i$ is a PC cycle of length $4m_0 + 2$ through x , a contradiction. So we have $|C(u, V_i)| = 1$.

Moreover, we assert that there exists some $i \in \{1, 2, 3, 4\}$ such that $V_i \Rightarrow u$ and $C(x'y') \notin C(u, V_i)$. If $x' \Rightarrow V_2 \cup V_4$ and $y' \Rightarrow V_1 \cup V_3$, then note that there exist no directed paths of length greater than one between x' and y' in D . So there exist no PC paths of length greater than one between x' and y' in G . This contradicts that G contains a PC Hamilton cycle. So there exists some $i \in \{1, 2, 3, 4\}$ such that $V_i \Rightarrow u$. If $C(x'y') \in C(u, V_i)$, then assume w.l.o.g. that $V_2 \Rightarrow x'$ and $C(x'y') \in C(x', V_2)$. Since G contains no monochromatic paths of length three, $C(x'y') \notin C(y', V_1 \cup V_3)$. So we may assume that $y' \Rightarrow V_1 \cup V_3$. Let $D' = D + y'x'$. Note that D' contains a directed Hamilton cycle if and only if G contains a PC Hamilton cycle. Since y' is not contained in any directed cycle in D' , we have that y' is not contained in any PC cycle in G . This contradicts that G contains a PC Hamilton cycle.

Finally we assert that $G \in \mathcal{G}_1$. Assume w.l.o.g. that $V_2 \Rightarrow x'$ and $C(x'y') \notin C(x', V_2)$. If $y' \Rightarrow V_1$, then $P(v_1, v_2)x'y'v_1$ is a PC cycle of length $4m_0 + 2$ through x , a contradiction. If $V_1 \Rightarrow y'$ and $C(x'y') \notin C(y', V_1)$, then $P(v_1, v_2)x'y'v_1$ is a PC cycle of length $4m_0 + 2$ through x , a contradiction. So we have $V_1 \Rightarrow y'$ and $C(x'y') \in C(y', V_1)$. Since G contains a PC Hamilton cycle, we have $d_G^c(y') \geq 2$. It follows that $C(x'y') \notin C(y', V_3)$. If $V_3 \Rightarrow y'$, then it follows that $P(v_2, v_3)y'x'v_2$ is a PC cycle of length $4m_0 + 2$ through x , a contradiction. So we have $y' \Rightarrow V_3$. Note that $C(x'y') \notin C(x', V_4)$, since otherwise $v_4x'y'v_1$ is a monochromatic path of length three for a pair of vertices $v_1 \in V_1$ and $v_4 \in V_4$. If $V_4 \Rightarrow x'$, then $P(v_3, v_4)x'y'v_3$ is a PC cycle of length $4m_0 + 2$ through x , a contradiction. So we have $x' \Rightarrow V_4$. Now one can check that $G \in \mathcal{G}_1$ with partition $\{V_1 \cup \{x'\}, V_2, V_3, V_4 \cup \{y'\}\}$.

Now $G \in \mathcal{G}_1$ and $G \notin \mathcal{G}_1^*$, which implies that G contains no PC Hamilton cycle, a contradiction.

Case 2. $G[V(C)] \in \mathcal{G}_2^*$.

Let $\{V_1, V_2, V_3, V_4\}$ be a partition of $V(C)$ such that $|C(v_i, V_{i+1})| = 1$ and $|C(v_i, V_{i-1})| = 1$ for each vertex $v_i \in V_i$ with $i \in \{1, 2, 3, 4\}$, where the subscripts are taken modulo 4. Define two vertex-colorings of $V(C)$ such that $C(v_i) \in C(v_i, V_{i+1})$ and $\bar{C}(v_i) \in C(v_i, V_{i-1})$ for each vertex $v_i \in V_i$ with $i \in \{1, 3\}$. Let $\{X, Y\}$ be the bipartition of G with $V_1, V_3 \in X$ and $V_2, V_4 \in Y$. Let x' and y' be the vertices of $G - C$ with $x' \in X$ and $y' \in Y$.

Note that, for each pair of vertices $v_i, v'_i \in V_i$, where $i \in \{1, 3\}$, there exists a PC path, say $P(v_i, v'_i)$, of length $4m_0$ through x such that $C(v_i v_i^+) = \bar{C}(v_i)$ and $C(v'_i v_i^-) = C(v'_i)$; for each pair of vertices $v_i \in V_i, v_{i+1} \in V_{i+1}$ with $i \in \{1, 3\}$, there exists a PC path, say $P(v_i, v_{i+1})$, of length $4m_0 - 1$ through x such that $C(v_i v_i^+) = \bar{C}(v_i)$; for each pair of vertices $v_i \in V_i, v_{i-1} \in V_{i-1}$ with $i \in \{1, 3\}$, there exists a PC path, say $P(v_{i-1}, v_i)$, of length $4m_0 - 1$ through x such that $C(v_i^- v_i) = C(v_i)$.

Firstly we assert that $|C(x', V_i)| = 1$ for each $i \in \{2, 4\}$. Indeed, if not, then we can choose a pair of vertices $v_i, v'_i \in V_i$ such that $C(uv_i) \neq C(uv'_i)$. In particular, if $x \in V_i$, then we prefer to choose $x = v_i$ or $x = v'_i$. Since G contains no monochromatic paths of length three, we have that $P(v_i, v'_i)x'v_i$ is a PC cycle of length $4m_0 + 2$ through x , a contradiction. So we have $|C(x', V_i)| = 1$.

Next we assert that $C(x'y') \in C(x', V_2)$ or $C(x'y') \in C(x', V_4)$. Suppose to the contrary that $C(x'y') \notin C(x', V_2 \cup V_4)$. If there exists a vertex $v_1 \in V_1$ such that $C(y'v_1) \neq C(v_1)$ and $C(y'v_1) \neq C(x'y')$, then $P(v_1, v_4)x'y'v_1$ is a PC cycle of length $4m_0 + 2$, a contradiction. So we have $C(y'v_1) = C(v_1)$ or $C(y'v_1) \neq C(x'y')$ for each vertex $v_1 \in V_1$. If there exists a vertex $v_1 \in V_1$ such that $C(y'v_1) \neq \bar{C}(v_1)$ and $C(y'v_1) \neq C(x'y')$, then $P(v_1, v_2)x'y'v_1$ is a PC cycle of length $4m_0 + 2$ through x , a contradiction. So we have $C(y'v_1) = \bar{C}(v_1)$ or $C(y'v_1) \neq C(x'y')$ for each vertex $v_1 \in V_1$. It follows that $C(y', V_1) = C(x'y')$. By symmetry between V_1 and V_3 , we have $C(y', V_3) = C(x'y')$. This implies that $d^c(y') = 1$, which contradicts that G contains a PC Hamilton cycle.

Assume w.l.o.g. that $C(x'y') \in C(x', V_2)$. Since G contains a PC Hamilton cycle, we have $d^c(x') \geq 2$. It follows from the above two assertions that $C(x'y') \notin C(x', V_4)$. Define $C(x') = C(x', V_2)$ and $\bar{C}(x') = C(x', V_4)$. One can check that $G \in \mathcal{G}_2$ with partition $\{V_1 \cup \{x'\}, V_2 \cup \{y'\}, V_3, V_4\}$. It is clear that $G \notin \mathcal{G}_2^*$, which contradicts that G contains a PC Hamilton cycle. \square

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. The sufficiency is trivial. We only need to show that if $K_{n,n}^c$ contains a PC Hamilton cycle, then it is PC even vertex-pancyclic.

Suppose to the contrary that the statement is false and G is a counterexample chosen such that n is as small as possible. It is clear that $n \geq 3$. Let x be an arbitrary vertex of G . By Theorem ??, G contains a PC cycle of length $2n - 2$. We distinguish two cases.

Case 1. There exists a PC cycle C of length $2n - 2$ through x .

The minimality of G implies that either there exist PC cycles of all possible even lengths through x in $G[V(C)]$ or $G[V(C)] \in \mathcal{G}_1^* \cup \mathcal{G}_2^*$. If there exist PC cycles of all possible even lengths through x in $G[V(C)]$, then the condition that G contains a PC Hamilton cycle implies that G is PC even vertex-pancyclic, a contradiction. So we may assume that $G[V(C)] \in \mathcal{G}_1^* \cup \mathcal{G}_2^*$. Since G contains a PC Hamilton cycle, it follows from Lemma 6.1 that G is PC even vertex-pancyclic, a contradiction.

Case 2. There exist no PC cycles of length $2n - 2$ through x .

Let $\{X, Y\}$ be the bipartition of G . Assume w.l.o.g. that $x \in X$. Let $C = v_0 v_1 \cdots v_{2n-3} v_0$ be a PC cycle of length $2n - 2$ with $v_0, v_2, \dots, v_{2n-4} \in X$ and $v_1, v_3, \dots, v_{2n-3} \in Y$, where subscripts are taken modulo $2n - 2$. Let y be the other vertex in $G - C$.

Subcase 2.1. There exists some $i \in \{1, 3, \dots, 2n - 3\}$ such that $C(xv_i) = C(v_i v_{i+1})$ or $C(xv_i) = C(v_{i-1} v_i)$.

Assume w.l.o.g. that $C(xv_1) = C(v_1 v_2)$. Firstly we assert that $C(xv_i) = C(v_i v_{i+1})$ for every $i \in \{1, 3, \dots, 2n - 3\}$ and $C(yv_i) = C(xy)$ or $C(yv_i) = C(v_i v_{i+1})$ for every $i \in \{0, 2, \dots, 2n - 4\}$. Since G contains no monochromatic paths of length three, we have $C(xv_1) \neq C(xv_3)$. If $C(xv_3) \neq C(v_3 v_4)$, then $xv_3 C v_1 x$ is a PC cycle of length $2n - 2$ through x , and we are back in Case 1. So we have $C(xv_3) = C(v_3 v_4)$. Considering the cycles $xv_{i+2} C v_i x$ one by one in the order $i = 1, 3, \dots, 2n - 3$, we have $C(xv_i) = C(v_i v_{i+1})$ for every $i \in \{1, 3, \dots, 2n - 3\}$. Since G contains no monochromatic paths of length three, we have $C(xy) \neq C(xv_i)$ for every $i \in \{1, 3, \dots, 2n - 3\}$. If $C(yv_i) \neq C(xy)$ and $C(yv_i) \neq C(v_i v_{i+1})$ for some $i \in \{0, 2, \dots, 2n - 4\}$, then $xyv_{i+3} C v_i x$ is a PC cycle of length $2n - 2$ through x , and we are back

in Case 1. So we have $C(yv_i) = C(xy)$ or $C(yv_i) = C(v_iv_{i+1})$ for every $i \in \{0, 2, \dots, 2n-4\}$.

Next we assert that for every chord v_iv_j of C , $C(v_iv_j) = C(v_iv_{i+1})$ or $C(v_iv_j) = C(v_jv_{j+1})$. If there exists a chord v_iv_j of C such that $C(v_iv_j) \neq C(v_iv_{i+1})$ and $C(v_iv_j) \neq C(v_jv_{j+1})$, then assume w.l.o.g. that $i \in \{1, 3, \dots, 2n-3\}$ and $j \in \{0, 2, \dots, 2n-4\}$. It follows that $xv_{j-1}\bar{C}v_iv_jCv_{i-2}x$ is a PC cycle of length $2n-2$ through x , and we are back in Case 1. So we have $C(v_iv_j) = C(v_iv_{i+1})$ or $C(v_iv_j) = C(v_jv_{j+1})$.

Let C' be a PC Hamilton cycle of G and let $P = xu_1u_2 \cdots u_t y$ be a segment of C' such that $C(yu_t) \neq C(xy)$. It is clear that $t \geq 1$. Define a vertex-coloring of $V(C)$ such that $C(v_i) = C(v_iv_{i+1})$ for each vertex $v \in V(C)$. It follows from the above two assertions that $C(xu_1) = C(u_1)$ and $C(u_iu_{i+1}) = C(u_{i+1})$ for each $i \in \{1, 2, \dots, t-1\}$. This implies $C(yu_t) \neq C(u_t)$. Recall that $C(yu_t) = C(xy)$ or $C(yu_t) = C(u_t)$. So we have $C(yu_t) = C(xy)$, a contradiction.

Subcase 2.2. For every $i \in \{1, 3, \dots, 2n-3\}$, $C(xv_i) \neq C(v_iv_{i+1})$ and $C(xv_i) \neq C(v_{i-1}v_i)$.

If there exists some $i \in \{1, 3, \dots, 2n-1\}$ such that $C(xv_i) \neq C(xv_{i+2})$, then it follows from Claim 2.1 that $xv_{i+2}Cv_ix$ is a PC cycle of length $2n-2$ through x , and we are back in Case 1. Hence $|C(x, V(C))| = 1$. Note that $d^c(x) \geq 2$ and $d^c(y) \geq 2$ since G contains a PC Hamilton cycle. It follows that $C(xy) \notin C(x, V(C))$ and there exists a vertex v_i for some $i \in \{0, 2, \dots, 2n-2\}$ such that $C(yv_i) \neq C(xy)$. Now either $xyv_i\bar{C}v_{i+3}x$ or $xyv_iCv_{i-3}x$ is a PC cycle of length $2n-2$ through x , and we are back in Case 1. \square

Summary

This thesis contains a number of new contributions on properly colored subgraphs (PC subgraphs for short) in edge-colored graphs. These new results involve the existence of short PC cycles, (vertex-disjoint) PC cycles of different lengths, PC cycle-factors and PC vertex-pancyclicity.

It is well-known that a graph is bipartite graph if and only if it contains no odd cycles, and checking whether a graph on n vertices and m edges is bipartite can be done in time $O(n + m)$. One natural problem in the research of edge-colored graphs is to characterize the coloring of edge-colored complete graphs containing no PC odd cycles. Based on a well-known Gallai partition theorem, in Chapter 2 we obtain a complete characterization of edge-colored complete graphs containing no PC odd cycles, and we give an effective algorithm with complexity $O(n^3)$ for deciding the existence of PC odd cycles. This is quite different from the result in uncolored graphs. It is interesting to note that analogous problems in edge-colored graphs appear to be more difficult than in uncolored graphs, even for edge-colored complete graphs.

Another natural problem in the research of edge-colored graphs is the existence of short PC cycles in edge-colored graphs. A well-known conjecture by Caccetta and Häggkvist states that every digraph on n vertices with minimum outdegree at least n/r has a directed cycle of length at most r . Motivated by minimum color degree and maximum monochromatic degree conditions for the existence of short cycles in edge-colored complete graphs, in Chapter 3 we give a maximum monochromatic degree condition for the existence of PC 4-cycles and characterize the extremal graphs for several

known results on the existence of PC triangles. Edge number conditions and degree conditions for the existence of triangles and 4-cycle in graphs are not hard to obtain. However, the theorems and proofs involving minimum color degree conditions and maximum monochromatic degree conditions for PC triangles or PC 4-cycles in edge-colored complete graphs are far from trivial.

It is widely known that every graph with minimum degree at least $k + 1$ contains k cycles of different lengths. Analogously, every digraph with minimum out-degree at least k contains k directed cycles of different lengths. For any positive integer k , Wang and Li [88] constructed an edge-colored graph G with $\delta^c(G) \geq k$ that contains no PC cycles. It is a natural approach to consider the existence of PC cycles of different lengths in edge-colored complete graphs. In Chapter 4, we give sufficient conditions for the existence of (vertex-disjoint) PC cycles of different lengths, which are far from trivial.

A classical theorem of Dirac states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle. However, as a natural analogy of Dirac's Theorem, the problem of giving a sufficient color degree condition for the existence of PC Hamilton cycles in edge-colored graphs is more difficult. Lo [74] gave an asymptotically sharp solution and gave a sharp color degree condition for the existence of a PC cycle-factor in an edge-colored graph. In Chapter 5, we give a sufficient color degree condition for the existence of PC cycle-factors in edge-colored balanced bipartite graphs.

Moon [78] established that if a tournament has a Hamilton cycle, then it is vertex-pancyclic. Häggkvist and Manoussakis [54] established that if a bipartite tournament has a Hamilton cycle, then it is even vertex-pancyclic, unless it belongs to a special class of digraphs. R. Li [65] showed that an edge-colored complete graph containing no monochromatic triangles is PC vertex-pancyclic unless it belongs to two special classes of edge-colored graphs. In Chapter 6, we show that if an edge-colored balanced complete bipartite graphs contains no monochromatic paths of length three and it has a PC Hamilton cycle, then it is PC even vertex-pancyclic, unless it belongs to two special classes of edge-colored complete graphs. Note that the analogous problems in edge-colored complete (bipartite) graphs appears to be more difficult than in (bipartite) tournaments, even after we add forbidden subgraph conditions.

Throughout this thesis, we have investigated the existence of PC subgraphs under different types of conditions, such as color degree conditions, monochromatic degree conditions and total color degree conditions. Although we have made some comparisons on the problems of different types of PC cycles between directed graphs and edge-colored graphs, there are still many interesting problems and conjectures that remain open. We hope that this will attract more attentions from many researchers.

Samenvatting

Dit proefschrift bevat een aantal nieuwe bijdragen op het gebied van de existentie van correct gekleurde deelgrafen (PC-deelgrafen in het kort) in lijngekleurde grafen. Deze nieuwe resultaten hebben betrekking op het bestaan van korte PC-cykels, (punt-disjuncte) PC-cykels van verschillende lengtes, PC-cykel-factoren en PC-punt-pancycliciteit.

Het is algemeen bekend dat een graaf bipartiet is dan en slechts dan als die graaf geen oneven cykel bevat, en dat het controleren of een graaf op n punten en m lijnen bipartiet is kan worden gedaan in tijd $O(n + m)$. Een natuurlijk probleem in het onderzoek van lijngekleurde grafen is om de kleuring te karakteriseren van lijngekleurde volledige grafen die geen oneven PC-cykels bevatten. In Hoofdstuk 2 gebruiken we een partitiestelling van Gallai om een volledige karakterisering te bepalen van lijngekleurde volledige grafen die geen oneven PC-cykels bevatten, en geven we een effectief algoritme met complexiteit $O(n^3)$ voor het detecteren van het bestaan van oneven PC-cykels. Dit wijkt nogal af van het resultaat in ongekleurde grafen. Het is interessant om op te merken dat analoge problemen in lijngekleurde grafen vaak moeilijker op te lossen lijken dan in ongekleurde grafen, zelfs voor lijngekleurde volledige grafen.

Een ander natuurlijk probleem in het onderzoek van lijngekleurde grafen is het bestaan van korte PC-cykels in lijngekleurde grafen. Een bekend vermoeden van Caccetta en Häggkvist stelt dat elke gerichte graaf op n punten met minimale uitgraad tenminste n/r een gerichte cykel van lengte ten hoogste r heeft. Gemotiveerd door kleurgraadvoorwaarden en monochromatische graadvoorwaarden voor het bestaan van korte cykels in lijngekleurde volledige

grafen, geven we in Hoofdstuk 3 een monochromatische graadvoorwaarde voor het bestaan van PC 4-cykels en karakteriseren we de extremale grafen voor verschillende bekende resultaten over het bestaan van PC driehoeken. Analoge voorwaarden voor het bestaan van driehoeken en 4-cykels in ongekleurde grafen zijn niet moeilijk af te leiden. Echter, de stellingen en bewijzen betreffende minimale kleurgraadvoorwaarden en maximale monochromatische graadvoorwaarden voor PC driehoeken of PC 4-cykels in lijngekleurde volledige grafen zijn verre van triviaal.

Het is algemeen bekend dat elke graaf met minimale graad tenminste $k + 1$ minstens k cykels van verschillende lengtes bevat. Evenzo bevat elke gerichte graaf met minimale uit-graad tenminste k minstens k gerichte cykels van verschillende lengtes. Echter, voor elk positief geheel getal k hebben Wang en Li [88] een lijngekleurde graaf G geconstrueerd met $\delta^c(G) \geq k$ die geen enkele PC-cykel bevat. Het is een natuurlijk probleem om het bestaan van PC-cykels van verschillende lengtes in lijngekleurde volledige grafen te onderzoeken. In Hoofdstuk 4 geven we voldoende voorwaarden voor het bestaan van (punt-disjuncte) PC-cykels van verschillende lengtes, wat verre van triviaal blijkt te zijn.

Een klassieke stelling van Dirac stelt dat elke graaf G op $n \geq 3$ punten met minimale graad $\delta(G) \geq n/2$ een Hamiltoncykel bevat. Echter, als een natuurlijke analogie van Dirac's stelling, is het probleem van het geven van een voldoende kleurgraadvoorwaarde voor het bestaan van een PC-Hamiltoncykel in een lijngekleurde graaf een stuk moeilijker. Lo [74] gaf een asymptotisch scherpe oplossing en gaf een scherpe kleurgraadvoorwaarde voor het bestaan van een PC-cykelfactor. In Hoofdstuk 5 geven we een voldoende kleurgraadvoorwaarde voor het bestaan van een PC-cykelfactor in een lijngekleurde gebalanceerde bipartiete graaf.

Moon [78] stelde vast dat een toernooi punt-pancyclisch is als het een Hamiltoncykel bevat. Häggkvist en Manoussakis [54] stelden vast dat een bipartiet toernooi even punt-pancyclisch is als het een Hamiltoncykel bevat, tenzij het behoort tot een speciale klasse van gerichte grafen. Li [65] toonde aan dat een lijngekleurde volledige graaf zonder monochromatische driehoeken PC-punt-pancyclisch is, tenzij deze behoort tot twee speciale

klassen van lijngekleurde grafen. In Hoofdstuk 6 laten we zien: als een lijngekleurde gebalanceerde volledige bipartiete graaf geen monochromatische paden van lengte drie bevat en een PC-Hamiltoncykel heeft, dan is die graaf even PC-punt-pancyclisch, tenzij de graaf behoort tot twee speciale klassen van grafen. We merken opnieuw op dat de analoge problemen in lijngekleurde volledige (bipartiete) grafen moeilijker lijken te zijn dan in (bipartiete) toernooien, zelfs nadat we voorwaarden toevoegen betreffende bepaalde verboden deelgrafen.

In dit proefschrift hebben we het bestaan van PC-deelgrafen onder verschillende soorten omstandigheden onderzocht, zoals in de aanwezigheid van kleurgraadvoorwaarden, monochromatische graadvoorwaarden en totale kleurgraadvoorwaarden. Hoewel we enkele vergelijkingen hebben gemaakt tussen de problemen betreffende verschillende typen PC-cykels in gerichte grafen en lijngekleurde grafen, zijn er nog veel interessante problemen en vermoedens die open blijven. We hopen dat deze onderwerpen meer aandacht zullen trekken van andere onderzoekers.

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