Weighted Multivariate Mean Reversion for Online Portfolio Selection

Boqian Wu[1], Benmeng Lyu[2,3], and Jiawen Gu[2](✉)

[1] University of Twente, Enschede, The Netherlands
b.wu@utwente.nl
[2] Southern University of Science and Technology, Shenzhen, China
benmeng@connect.hku.hk,gujw@sustech.edu.cn
[3] The University of Hong Kong, Hong Kong, Hong Kong

Abstract. Portfolio selection is a fundamental task in finance and it is to seek the best allocation of wealth among a basket of assets. Nowadays, Online portfolio selection has received increasing attention from both AI and machine learning communities. Mean reversion is an essential property of stock performance. Hence, most state-of-the-art online portfolio strategies have been built based on this. Though they succeed in specific datasets, most of the existing mean reversion strategies applied the same weights on samples in multiple periods and considered each of the assets separately, ignoring the data noise from short-lived events, trend changing in the time series data, and the dependence of multi-assets. To overcome these limitations, in this paper, we exploit the reversion phenomenon with multivariate robust estimates and propose a novel online portfolio selection strategy named "Weighted Multivariate Mean Reversion" (WMMR) (Code is available at: https://github.com/boqian333/WMMR).. Empirical studies on various datasets show that WMMR has the ability to overcome the limitations of existing mean reversion algorithms and achieve superior results.

Keywords: portfolio selection · online learning · multivariate robust estimates

1 Introduction

Portfolio selection, which has been explored in both finance and quantitative fields, is concerned with determining a portfolio for allocating the wealth among a set of assets to achieve some financial objectives such as maximizing cumulative wealth or risk-adjusted return, in the long run. There are two main mathematical theories for this problem: the mean-variance theory [22] and the Kelly investment [17]. Mean-variance theory proposed by Markowitz trades off between the expected return (mean) and risk (variance) of a portfolio in a single-period framework. Contrarily, the Kelly investment aims to maximize the expected log return in a multi-period setting. Online portfolio selection (PS), which follows

B. Wu and B. Lyu—Equal contribution.

D. Koutra et al. (Eds.): ECML PKDD 2023, LNAI 14173, pp. 255–270, 2023.
https://doi.org/10.1007/978-3-031-43424-2_16

the Kelly investment and investigates the sequential portfolio selection strategies, is attracting increasing interest from AI and machine learning communities. Based on the Kelly investment model, some state-of-the-art online PS strategies [10] assume that current best-performing stocks would also perform well in the next trading period. However, empirical evidence indicates that such assumptions may often be violated especially in the short term. This observation of an asset's price tends to converge to the average price over time, leading to strategies of buying poor-performing stocks and selling those with good performance. This trading principle is known as the "mean reversion" principle.

In recent years, by exploiting the multi-period mean reversion principle, several online PS strategies [5,13,18] have been proposed and achieved encouraging results when applied to many datasets. However, the existing studies ignored the data noise from short-lived events, trend changes in the time series data, and the dependence of multi-assets [18,21], while these are important properties of stock movements. To overcome these drawbacks, different methods have been proposed [26]. For instance, a new PS strategy has been proposed, which more accurately estimates parameters via subset resampling. This approach is particularly useful when the number of assets is large. An ensemble learning method has also been proposed for Kelly's growth optimal portfolio to mitigate estimation errors [24]. Additionally, [28] introduced a novel Relation-aware Transformer (RAT) method to simultaneously model complex sequential patterns and varying asset correlations for PS.

In this paper, we propose a multi-period online PS strategy named "Weighted Multivariate Mean Reversion" (WMMR) without requiring subset resampling demanding thousands of loops or model training requiring sufficient data. The basic idea of WMMR is to update the next price prediction via robust multivariate estimates with exponential decay. By capturing the correlation between multiple assets, robust multivariate estimates could reduce or remove the effect of outlying data points, which are produced by the short-lived events in the financial market and may lead to incorrect forecasts or predictions. We determine the portfolio selection strategies via online learning techniques. The experimental results show that WMMR can achieve greater profits than several existing algorithms. Moreover, it is robust to different parameter values and its performance is consistently well when considering reasonable transaction costs.

2 Problem Setting

Let us consider a financial market with m assets for n periods. On the t^{th} period, the assets' prices are represented by a close price vector $\mathbf{p}_t \in \mathbb{R}^m_+$ and each element $p_{t,i}$ represents the close price of asset i. The changes of asset prices for n trading periods are represented by a sequence of non-negative, non-zero price relative vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m_+$. Let us use $\mathbf{x}^n = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ to denote such a sequence of price relative vectors for n periods and $x_{t,i} = \frac{p_{t,i}}{p_{t-1,i}}$. Thus, an investment in asset i on the t^{th} period increases by a factor of $x_{t,i}$.

At the beginning of the t^{th} period, we diversify our capital among the m assets specified by a portfolio vector $\mathbf{b}_t = (b_{t,1}, \ldots, b_{t,m})$, where $b_{t,i}$ represents the proportion of wealth invested in asset i. Typically, we assume the portfolio is self-financed and no short selling is allowed, which means each entry of a portfolio is non-negative and adds up to one, that is, $\mathbf{b}_t \in \Delta_m$, where $\Delta_m = \{\mathbf{b}_t : \mathbf{b}_t \in \mathbb{R}_+^m, \sum_{i=1}^m b_{t,i} = 1\}$. The investment procedure is represented by a portfolio strategy, that is, $\mathbf{b}_1 = \frac{1}{m}\mathbf{1}$ and following sequence of mappings $f : \mathrm{R}_+^{m(t-1)} \rightarrow \Delta_m, t = 2, 3, \ldots,$ where $\mathbf{b}_t = f(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1})$ is the t^{th} portfolio given past market sequence $\mathbf{x}^{t-1} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}\}$. Let us denote $\mathbf{b}^n = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ as the portfolio strategy for n trading period.

On the t^{th} trading period, an investment according to portfolio $\mathbf{b}_t$ results in a portfolio daily return s_t, that is, the wealth increases by a factor of $s_t = \mathbf{b}_t^T \mathbf{x}_t = \sum_{i=1}^m b_{ti} x_{ti}$. Since we reinvest and adopt price relative, the portfolio wealth would grow multiplicatively. Thus, after n trading periods, the investment according to a portfolio strategy $\mathbf{b}_n$ results in portfolio cumulative wealth S_n, which increases the initial wealth by a factor of $\prod_{t=1}^n \mathbf{b}_t^T \mathbf{x}_t$, that is,

$$S_n = S_0 \prod_{t=1}^{n} \mathbf{b}_t^T \mathbf{x}_t, \tag{1}$$

where S_0 denotes the initial wealth and is set to \$1 for convenience.

Finally, let us formulate the online portfolio selection problem as a sequential decision problem. In this task, the portfolio manager is a decision maker whose goal is to make a portfolio strategy $\mathbf{b}^n$ on financial markets to maximize the portfolio cumulative wealth S_n. He computes the portfolios sequentially. On each trading period t, the portfolio manager has access to the sequences of previous price relative vectors $\mathbf{x}^{t-1} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}\}$, and previous sequences of portfolio vectors $\mathbf{b}^{t-1} = \{\mathbf{b}_1, \ldots, \mathbf{b}_{t-1}\}$. Based on historical information, the portfolio manager computes a new portfolio vector $\mathbf{b}_t$ for the next price relative vector $\mathbf{x}_t$, where the decision criterion varies among different managers. The resulting portfolio $\mathbf{b}_t$ is scored based on the portfolio period return of S_t. The procedure repeats until the end of trading periods and the portfolio strategy is finally scored by the cumulative wealth S_n.

3 Related Work and Motivation

3.1 Related Work

Following the principle of the Kelly investment [17], many kinds of portfolio selection methods have been proposed. Online learning portfolio selection maximizes the expected return with sequential decision-making. The most common and well-known benchmark is the *Buy-And-Hold* (BAH) strategy, that is, one invests his/her wealth in the market with an initial portfolio and holds it within his/her investment periods. The BAH strategy with a uniform initial portfolio $\mathbf{b}_1 = (1/m, 1/m, \ldots, 1/m)^T$ is called uniform BAH strategy, which is adopted as market strategy producing the market index in our study. Contrary to the static nature of the BAH strategy, active trading strategies usually change portfolios regularly during trading periods. A classical active strategy is *Constant Rebalanced Portfolios* (CRP) [6], which rebalances a fixed portfolio every trading period. The Best CRP (BCRP) is the best CRP strategy over the entire trading period, which is only a high-sight strategy.

Several portfolio strategies assume that past well-performing securities would still perform well in the future. These strategies are called momentum strategies, which approximate the expected logarithmic cumulative return of BCRP. The portfolio in *Universal portfolios* (UP) [5] is the historical performance weighted average of all possible CRP experts. The Semi-Universal Portfolio(SUP) strategies with transaction cost [14] consider Cover's moving target portfolio with occasional rebalancing. *Exponential Gradient* (EG) [12] is based on multiplicative updates.

Empirical evidence indicates that opposite trends may often happen in the financial market, which is a common and famous principle called mean reversion. Based on the idea of mean reversion, [3] proposed the *Anticorrelation* (Anticor) strategy. It calculates a cross-correlation matrix between two specific market windows and transfers the wealth from winning assets to losing assets, and adjusts the corresponding amounts based on the cross-correlation matrix. [21] proposed the *Passive Aggressive Mean Reversion* (PAMR) strategy, which only considers the single periodical mean reversion property. [9] proposed the *Passive Aggressive Combined Strategy* (PACS), which combines price reversion and momentum via a multipiece-wise loss function. [18] proposed the *Online Moving Average Reversion* (OLMAR) strategy, which exploits mean reversion's multi-period nature via moving average prediction. [13] proposed the *Robust Median Reversion* (RMR) strategy which exploits the reversion phenomenon by robust L_1-median estimator. All in all, mean reversion is crucial for designing online portfolio selection strategies.

3.2 Motivation

The existing moving average reversion strategies, i.e. OLMAR [18] and RMR [13], exploits the mean reversion in the following ways. OLMAR assumes that the stock price of $(t+1)^{th}$ period will revert to the moving average (mean) of the prices in the previous periods with a w-window, that is, the update for prediction becomes $\hat{\mathbf{p}}_{t+1} = \frac{1}{w}\sum_{i=t-w+1}^{i=t} \mathbf{p}_i$. Considering the noises and outliers in real market data, RMR exploits the multi-period reversion property via the robust median reversion, that is,

$$\hat{\mathbf{p}}_{t+1} = \arg\min_{\boldsymbol{\mu}} \sum_{i=t}^{t-w+1} \left\|\mathbf{p}_t - \boldsymbol{\mu}\right\|, \tag{2}$$

where $\|\cdot\|$ denotes the Euclidean norm. The robust median is a L_1-median in statistics [27], which is of less sensitivity to the outliers and noisy data compared to the mean. Empirical results of RMR on various datasets are significantly better than OLMAR, which inspires us to explore the robust estimates [1,15,16] in online portfolio selection.

We assume that the stock prices $\mathbf{p}_t$ satisfy $\mathbf{p}_t = \boldsymbol{\mu}_t + \boldsymbol{\sigma}_t \odot \boldsymbol{u}_t$, where $\boldsymbol{\mu}_t = (\mu_{t,1}, ..., \mu_{t,m}) \in \mathbb{R}^m$ and $\boldsymbol{u}_t = (u_{t,1}, ..., u_{t,m}) \in \mathbb{R}^m$ represent the real price behind and the noise contaminating the real price respectively. It is noticed that $\odot$ represents the element-wise multiplication. Let $u_{t,1}, ..., u_{t,m}$ for $t = 1, ..., n$ are i.i.d with the density f. $\boldsymbol{\sigma}_t = (\sigma_{t,1}, ..., \sigma_{t,m}) \in \mathbb{R}_+^m$ is the unknown parameter to measure the contamination scale on the corresponding asset. Thus, the density of $\mathbf{p}_t$ can be defined as $\frac{1}{\boldsymbol{\sigma}_t} f\left(\frac{\mathbf{p}_t - \boldsymbol{\mu}_t}{\boldsymbol{\sigma}_t}\right)$. Note that $\mathbf{p}_t$, $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are all vectors and the above operations are element-wise. The maximum likelihood estimation (MLE) of $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ is:

$$\begin{aligned}(\hat{\boldsymbol{\mu}}_t, \hat{\boldsymbol{\sigma}}_t) &= \arg\max_{\boldsymbol{\mu}_t, \boldsymbol{\sigma}_t} \frac{1}{\boldsymbol{\sigma}^n} \prod_{i=t-w+1}^{t} f\left(\frac{\mathbf{p}_i - \boldsymbol{\mu}_t}{\boldsymbol{\sigma}_t}\right) \\ &= \arg\min_{\boldsymbol{\mu}_t, \boldsymbol{\sigma}_t} \left\{\frac{1}{n} \sum_{i=t-w+1}^{t} \rho\left(\frac{\mathbf{p}_i - \boldsymbol{\mu}_t}{\boldsymbol{\sigma}_t}\right) + \log \boldsymbol{\sigma}_t\right\},\end{aligned} \tag{3}$$

where $\rho(.) = -\log f(.)$, since $f(.)$ is everywhere positive and the logarithm is an increasing function. If $\rho(.)$ is differentiable and $\rho''(0)$ exists, first order optimization for (3) yields:

$$\begin{cases} \hat{\boldsymbol{\mu}}_t = (\sum_{i=t-w+1}^{t} \mathbf{p}_i W_1(\frac{\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t}{\hat{\boldsymbol{\sigma}}_t}))/(\sum_{i=t-w+1}^{t} W_1(\frac{\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t}{\hat{\boldsymbol{\sigma}}_t})), \\ \hat{\boldsymbol{\sigma}}_t^2 = \frac{1}{w} \sum_{i=t-w+1}^{t} W_2\left(\frac{\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t}{\hat{\boldsymbol{\sigma}}_t}\right)(\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)^2, \end{cases} \tag{4}$$

where $\begin{cases} W_1(x) = \begin{cases} -\rho'(x)/x & \text{if} \quad x \neq 0 \\ -\rho''(0) & \text{if} \quad x = 0 \end{cases}, \\ W_2(x) = \begin{cases} -\rho'(x)/x & \text{if} \quad x \neq 0 \\ -\rho''(0)/2 & \text{if} \quad x = 0 \end{cases}. \end{cases}$

We use $\hat{\boldsymbol{\mu}}_t$ as the updated prediction for $\mathbf{p}_{t+1}$. It's noted that $\frac{\mathbf{p}_i - \hat{\boldsymbol{\mu}_t}}{\hat{\boldsymbol{\sigma}}_t}$ is the outlyingness measure adjusting the weights on sample $\mathbf{p}_{t+1}$ in i-th period and the next estimated stock price as a weighted mean. In general $W(x)$ is a non-increasing function of $|x|$, so outlying observations will receive smaller weights. It is worth noting that $W_1(x)$ and $W_2(x)$ are equal except when $x = 0$.

Table 1. Examples of $W(d_i)$ functions

	$W(d_i)$
HUBER	$\begin{cases} k/\sqrt{d_i}, & \sqrt{d_i} \leq k \\ 1, & \sqrt{d_i} > k \end{cases}$
BISQUAR	$\begin{cases} (1 - \frac{d_i}{k^2})^2, & \sqrt{d_i} \leq k \\ 0, & \sqrt{d_i} > k \end{cases}$
SHR	$\begin{cases} 1, & d_i \leq 4 \\ q(d_i), & 4 < d_i \leq 9 \\ 0, & d_i > 9 \end{cases}$
	$q(d) = -1.944 + 1.728d - 0.312d^2 + 0.016d^3$

In most cases of interest, it is known or assumed that some form of dependence between stocks exists, and hence that considering each of them separately would entail a loss of information. In the univariate case, $\frac{\mathbf{p}_i - \hat{\boldsymbol{\mu}_t}}{\hat{\boldsymbol{\sigma}_t}}$ measures the univariate outlyingness. In the multivariate case,

the squared Mahalanobis Dis-
tance [7] between the vectors $\mathbf{p}_i$ and $\boldsymbol{\mu}_t$ with respect to the covariance matrix $\boldsymbol{\Sigma}_t$ is used to measure the multivariate outlyingness, which is defined as $d_i(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t) = (\mathbf{p}_i - \boldsymbol{\mu}_t)^T \boldsymbol{\Sigma}_t^{-1}(\mathbf{p}_i - \boldsymbol{\mu}_t)$, that is, the normalized squared distance between $\mathbf{p}_i$ and $\boldsymbol{\mu}_t$. In general, the dependence of multiple assets is taken into consideration and we derive the updated prediction for the mean and covariance matrix of return by MLE.

Assumption 1 *Suppose that:* ***i.*** *The observations* $\mathbf{p}_i$ *are the i.i.d samples from multivariate probability density* $f(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$*.* ***ii.*** *The probability density* $f(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$ *has the form of* $f(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t) = \frac{1}{\sqrt{|\boldsymbol{\Sigma}_t|}} h(d_i(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t))$*, where* $|\boldsymbol{\Sigma}_t|$ *is the determinant of* $\boldsymbol{\Sigma}_t$*.* ***iii.*** $\ln f$ *is differentiable.*

Theorem 1. *Under Assumption 1, the updated prediction is given by:*

$$\begin{cases} \hat{\boldsymbol{\mu}}_t = \sum_{i=t-w+1}^{t} W(d_i)\mathbf{p}_i / \sum_{i=t-w+1}^{t} W(d_i), \\ \hat{\boldsymbol{\Sigma}}_t = \frac{1}{w} \sum_{i=t-w+1}^{t} W(d_i)(\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)(\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)^T, \end{cases} \quad (5)$$

where $W(d_i) = (-2\ln h(d_i))'$ *and* $d_i(\mathbf{p}_i, \hat{\boldsymbol{\mu}}_t, \hat{\boldsymbol{\Sigma}}_t) = (\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)^T \hat{\boldsymbol{\Sigma}}_t^{-1}(\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)$*, which are different from the univariate case.*

Proof. Let $\mathbf{p}_i$ are the i.i.d sample from $f(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t) = \frac{1}{\sqrt{|\boldsymbol{\Sigma}_t|}} h(d_i(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t))$, for $i = t-w+1, ..., t$. The MLE of $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ is

$$\hat{\boldsymbol{\mu}}_t, \hat{\boldsymbol{\Sigma}}_t = \underset{\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t}{\operatorname{argmax}} \frac{1}{|\boldsymbol{\Sigma}_t|^{w/2}} \prod_{i=t-w+1}^{t} h(d_i(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)). \quad (6)$$

It is noted that Since h is everywhere positive and the logarithm is an increasing function, thus, Eq. 6 can be written as

$$\hat{\boldsymbol{\mu}}_t, \hat{\boldsymbol{\Sigma}}_t = \underset{\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t}{\operatorname{argmin}}\, w \ln |\hat{\boldsymbol{\Sigma}}_t| + \sum_{i=t-w+1}^{t} \rho(d_i), \quad (7)$$

where $\rho(d_i) = -2\ln h(d_i)$ and $d_i = d\left(\mathbf{p}_i, \hat{\boldsymbol{\mu}}_t, \hat{\boldsymbol{\Sigma}}_t\right) = (\mathbf{p}_i - \boldsymbol{\mu}_t)^T \boldsymbol{\Sigma}_t^{-1}(\mathbf{p}_i - \boldsymbol{\mu}_t)$. Differentiating with respect to $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ yields

$$\sum_{i=t-w+1}^{t} W(d_i)(\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t) = \mathbf{0}, \frac{1}{w} \sum_{i=t-w+1}^{t} W(d_i)(\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)(\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)^T = \hat{\boldsymbol{\Sigma}}_t$$

with $W(d_i) = \rho'(d_i)$. If we knew $f(.)$ exactly, the $W(d_i)$ would be "optimal", but since we only know $f(.)$ approximately, our goal is to find estimators that are "nearly optimal". For simplicity, we will consider two cases:

- Multivariate Normal: $f(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t) = \frac{exp(-\frac{1}{2}d_i)}{\sqrt{|\Sigma|}}$, then $W(di)$ is a constant.

– m (the number of stocks) multivariate Student distribution with v degrees: $f(\mathbf{p}_i, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t) = \frac{exp(-\frac{1}{2}(m+v))}{(d_i+v)\sqrt{|\Sigma|}}$, then $W(d_i) = (m+v)/(d_i+v)$. If the value of v is large, then $W(d_i)$ is a constant; v is 0, then $W(d_i) = m/d_i$.

In our paper, we use classical functions (Huber [2], Bisquare [11], and the weighting function (we shall use SHR here) employed for time series estimation [23]) in robust regression to approximate the true $W(d_i)$ under unknown $f(.)$, as in Table 1. These functions assign smaller weights to outlying observations, and some may even be removed (except for the Huber function). In a time series of financial data, there will be trend changes that cannot be ignored even in a short period. Thus, the exponential decay is adopted in Eq. 5, that is,

$$\begin{cases} \hat{\boldsymbol{\mu}}_t = (\sum_{i=t-w+1}^{t} (1-\alpha)^{t-i} \mathbf{p}_i W(d_i))/(\sum_{i=t-w+1}^{t} (1-\alpha)^{t-i} W(d_i)), \\ \hat{\boldsymbol{\Sigma}}_t = \frac{1}{w} \sum_{i=t-w+1}^{t} W(d_i) (\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t) (\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)^T, \end{cases} \tag{8}$$

where α is the decaying factor. $\hat{\boldsymbol{\mu}}_t$ is the predicted price vector for the $(t+1)^{th}$ period.

4 Multi-variate Robust Mean Reversion

4.1 Formulation

The proposed formulation, WMMR, is to find the optimal portfolio by weighted multivariate mean reversion and passive-aggressive online learning. The basic idea is to obtain the estimate of the next price relative $\mathbf{x}_{t+1}$ via robust multivariate estimates, and then maximize the expected return $\mathbf{b}^T\mathbf{x}_{t+1}$ with the hope that the new portfolio is not far away from the previous one.

Most of $W(d_i)$ in Table 1 depend on the constant $k \in R$. Here a rescaled d_i, i.e., d_i/S is applied to the $W(d_i)$, that is,

$$\begin{cases} \hat{\boldsymbol{\mu}}_t = (\sum_{i=t-w+1}^{t} (1-\alpha)^{t-i} \mathbf{p}_i W(d_i/S))/(\sum_{i=t-w+1}^{t} (1-\alpha)^{t-i} W(d_i/S)), \\ \hat{\boldsymbol{\Sigma}}_t = 1/w \sum_{i=t-w+1}^{t} (1-\alpha)^{t-i} W(d_i/S) (\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t) (\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)^T, \end{cases} \tag{9}$$

where $S = \mathbf{MED}([d_{t-w+1}, ..., d_t])$ and $d_i(\mathbf{p}_i, \hat{\boldsymbol{\mu}}_t, \hat{\boldsymbol{\Sigma}}_t) = (\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)^T \hat{\boldsymbol{\Sigma}}_t^{-1} (\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)$. In this formulation of WMMR, different $W(d_i)$ and $\hat{\boldsymbol{\Sigma}}_t$ are discussed as follows:

– **Case 1**: $W(d_i) \equiv 1$, $\hat{\boldsymbol{\Sigma}}_t$ is not considered and $\alpha \equiv 0$.
– **Case 2**: $W(d_i) = \frac{1}{\sqrt{d_i}}$, $\hat{\boldsymbol{\Sigma}}_t \equiv \boldsymbol{I}$ and $\alpha \equiv 0$.
– **Case 3**: $W(d_i)$ is the HUBER weighting function, $\hat{\boldsymbol{\Sigma}}_t$ is computed via Eq. 5 and α is a parameter.
– **Case 4**: $W(d_i)$ is the BISQUA weighting function, $\hat{\boldsymbol{\Sigma}}_t$ is computed via Eq. 5 and α is a parameter.
– **Case 5**: $W(d_i)$ is the SHR weighting function, $\hat{\boldsymbol{\Sigma}}_t$ is computed via Eq. 5 and α is a parameter.

Algorithm 1 . WMMR($\mathbf{p}_t$, $\mathbf{p}_{t-1}$, . . . , $\mathbf{p}_{t-w+1}$,τ, k, α)

Input: Current stock price sequence $\mathbf{p}_t$, $\mathbf{p}_{t-1}$, . . . , $\mathbf{p}_{t-w+1}$; Toleration level τ; Iteration maximum K; Decaying factor α.
Output: estimated $\hat{\mathbf{x}}_{t+1}$
Procedure:
Initialize $j \leftarrow 0$, $\hat{\boldsymbol{\mu}}_t \leftarrow \frac{1}{m}\mathbf{1}$ and $\hat{\boldsymbol{\Sigma}}_t = \mathbf{1}$
The estimation of next period price: $\hat{\mathbf{p}}_{t+1} \leftarrow \hat{\boldsymbol{\mu}}_t$
while $j < K$ **do**
  Calculate the following variables:
  The multivariate outlyingness: $d_i \leftarrow (\mathbf{p_i} - \hat{\boldsymbol{\mu}}_t)^T \hat{\boldsymbol{\Sigma}}_t^{-1} (\mathbf{p_i} - \hat{\boldsymbol{\mu}}_t)$ (i = t-w+1,...,t)
  The error scale: $S \leftarrow \mathbf{MED}([d_{t-w+1}, ...d_t])$
  The weight: $W_i \leftarrow W(d_i/S)$ (i = t-w+1,...,t)
  The estimation of $\hat{\boldsymbol{\mu}}_t$ in j^{th} iteration :

$$\hat{\boldsymbol{\mu}}_t \leftarrow \sum_{i=t-w+1}^{t} (1-\alpha)^{t-i} W_i \mathbf{p}_i / \sum_{i=t-w+1}^{t} (1-\alpha)^{t-i} W_i$$

  The estimation of $\hat{\boldsymbol{\Sigma}}_t$ in j^{th} iteration :

$$\hat{\boldsymbol{\Sigma}}_t \leftarrow \frac{1}{w} \sum_{i=t-w+1}^{t} W_i (\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)(\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t)^T$$

  if $|\hat{\boldsymbol{\mu}}_t - \hat{\mathbf{p}}_{t+1}| < \tau |\hat{\boldsymbol{\mu}}_t|$ **then** break
  end if
  $\hat{\mathbf{p}}_{t+1} \leftarrow \hat{\boldsymbol{\mu}}_t$
end while
The price relative vectors in $(t+1)^{th}$period: $\hat{\mathbf{x}}_{t+1} \leftarrow \hat{\mathbf{p}}_{t+1}/\mathbf{p}_t$

Note that in Case 1, $\hat{\boldsymbol{\mu}}_t = \frac{1}{w}\sum_{i=t-w+1}^{t} \mathbf{p}_i$, which is the moving average mean used in OLMAR; In Case 2, $\hat{\boldsymbol{\mu}}_t$=$(\sum_{i=t-w+1}^{t} \frac{\mathbf{p}_i}{\sqrt{\|\mathbf{p}_i - \hat{\boldsymbol{\mu}}_t\|_2}})/(\sum_{i=t-w+1}^{t} \frac{1}{\sqrt{\|\mathbf{p}_{t-i} - \hat{\boldsymbol{\mu}}_t\|_2}})$, which is the robust median used in RMR. OLMAR and RMR strategies are subsamples of WMMR. In this paper, the effectiveness of Case 3, Case 4, and Case 5 are mainly explored, which are denoted by WMMR-HUBER, WMMR-BIS, and WMMR-SHR respectively.

4.2 Online Portfolio Selection

$$\mathbf{b}_{t+1} = \arg\min_{\mathbf{b}} \frac{1}{2}\|\mathbf{b} - \mathbf{b}_t\|^2 + \frac{\theta}{2}\|\mathbf{b}\|^2 \ \text{s.t.} \begin{cases} \mathbf{b}^T \hat{\mathbf{x}}_{t+1} \geq \epsilon, \\ \mathbf{b}^T \mathbf{1} = \mathbf{1} \end{cases} \tag{10}$$

where $\hat{\mathbf{x}}_{t+1}$ is the price relative estimated via weighted multivariate mean reversion and $\theta > 0$ is the regularization parameter and is manually tuned. The above formulation attempts to find a portfolio satisfying the condition of $\mathbf{b}^T \hat{\mathbf{x}}_{t+1} \geq \epsilon$ while not far away from the last portfolio. On one side, when the expected return is larger than a threshold ϵ, the investment strategy will passively keep the last portfolio. On another side, when the constraint $\mathbf{b}^T \hat{\mathbf{x}}_{t+1} \geq \epsilon$ is not satisfied, the portfolio will be aggressively updated by forcing expected return is larger than the threshold ϵ. By adding the regularization $\|\mathbf{b}\|^2$ under the constrain $\mathbf{b}^T\mathbf{1} = \mathbf{1}$, we push the new portfolio move forward to $\frac{\mathbf{1}}{m}$ and prevent the solution from over-fitted.

Algorithm 2 . Online Portfolio Selection(ϵ, w, $\hat{\mathbf{x}}_{t+1}$, $\mathbf{b}_t$)

Input: Reversion threshold: $\epsilon > 1$; Window size: w; Predicted price relatives :$\hat{\mathbf{x}}_{t+1}$; Current portfolio: $\mathbf{b}_t$.
Output: Next portfolio $\mathbf{b}_{t+1}$.
Procedure: Calculate the following variables:
$\eta_{t+1} = \mathbf{max}(0, \frac{(1+\theta)\epsilon - \hat{\mathbf{x}}_{t+1}^T(\mathbf{b}_t + \theta\mathbf{1})}{\|\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}\|^2}))$
Update the portfolio: $\mathbf{b} = \frac{1}{1+\theta}[\mathbf{b}_t + \eta_{t+1}(\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1})] + \frac{\theta}{(1+\theta)}\frac{\mathbf{1}}{m}$
Normalize $\mathbf{b}_t$: $\mathbf{b}_{t+1} = \arg\min_{\mathbf{b}\in\Delta_m} \|\mathbf{b} - \mathbf{b}_{t+1}\|^2$

Algorithm 3 . Portfolio Selection with WMMR

Input: Reversion threshold: $\epsilon > 1$; Window size: w; Iteration maximum k; Toleration level τ; Decaying factor α; Market Sequence $\mathbf{P}^n$.
Output: Cumulative wealth after n^{th} periods
Procedure:
Initialization: Initial portfolio: $b_1 = \frac{1}{m}\mathbf{1}$; Initial wealth: $S_0 = 1$.
for $t = w$ **to** n **do**
    Predict next price relative vector according **Algorithm** 1:
    $\hat{\mathbf{x}}_{t+1} \leftarrow$ WMMR($\mathbf{p}_t$, $\mathbf{p}_{t-1}$, . . . , $\mathbf{p}_{t-w+1}$,τ, k, α).
    Update the portfolio according **Algorithm** 2:
    $\mathbf{b}_{t+1} \leftarrow$ Online Portfolio Selection(ϵ, w, $\hat{\mathbf{x}}_{t+1}$, $\mathbf{b}_t$).
    Receive stock price: $\mathbf{P}_{t+1}$.
    Update cumulative return: $S_{t+1} \leftarrow S_t \times \left(\mathbf{b}_{t+1}{}^T \frac{\mathbf{p}_{t+1}}{\mathbf{p}_t}\right)$.
end for

4.3 Algorithms

From the formulation of WMMR(Eq. 9), the weights $W(d_i/S)$ depend also on $\hat{\boldsymbol{\mu}}_t$ and $\hat{\boldsymbol{\Sigma}}_t$, hence Eq. 9 is not an explicit expression for $\hat{\boldsymbol{\mu}}_t$ and $\hat{\boldsymbol{\Sigma}}_t$. The solution of weighted multivariate estimation could be calculated through iteration, and the iteration process is described in **Algorithm** 1. Once the constraint $\|\boldsymbol{\mu}_{t+1} - \boldsymbol{\mu}_t\|_1 \le \tau \|\boldsymbol{\mu}_t\|_1$ is satisfied, or the number of iteration is larger than the threshold k, the iteration is terminated, where τ is a toleration level and k is the maximum iteration number.

The constrained optimization problem (10) can be solved by the technique of convex optimization [4]. The solution of (10) without considering the non-negativity constraint is

$$\mathbf{b}_{t+1} = \frac{1}{1+\theta}[\mathbf{b}_t + \eta(\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1})] + \frac{\theta}{(1+\theta)}\frac{\mathbf{1}}{m}, \tag{11}$$

where $\eta = \mathbf{max}(0, \frac{(1+\theta)\epsilon - \hat{\mathbf{x}}_{t+1}^T(\mathbf{b}_t + \theta\mathbf{1})}{\|\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}\|^2})$.

Proof. Define the Lagrangian of the problem (10) to be:

$$\mathcal{L}(\mathbf{b}, \eta, \lambda) = \frac{1}{2}\left\|\mathbf{b}-\mathbf{b}_t\right\|^2 + \frac{1}{2}\theta\left\|\mathbf{b}\right\|^2 - \eta\left(\hat{\mathbf{x}}_{t+1}^T\mathbf{b}-\epsilon\right) + \lambda(\mathbf{1}^T\mathbf{b}-1). \quad (12)$$

Setting the partial derivatives of $\mathcal{L}$ with respect to the elements of $\mathbf{b}$ to zero, yields:

$$0 = \frac{\partial \mathcal{L}}{\partial \mathbf{b}} = ((\theta+1)\mathbf{b}-\mathbf{b}_t) - \eta\hat{\mathbf{x}}_{t+1} + \lambda\mathbf{1}. \quad (13)$$

Multiplying both sides of Eq. 13 with $\mathbf{1}^T$, and $\mathbf{1}^T\mathbf{b}=1$, $\mathbf{1}^T\mathbf{1}=m$, we can get $\lambda = -\frac{\theta}{m} + \frac{\eta}{m}\mathbf{1}^T\hat{\mathbf{x}}_{t+1}$. Define $\bar{x}_{t+1} = \frac{\mathbf{1}^T\hat{\mathbf{x}}_{t+1}}{m}$ as the mean of the price relatives in the period $(t+1)^{th}$. Then, λ can be rewritten as $\lambda = -\frac{\theta}{m} + \eta\bar{x}_{t+1}$, and the solution for $\mathcal{L}$ is

$$\mathbf{b} = \frac{\mathbf{b}_t}{1+\theta} + \frac{\theta\mathbf{1}}{(1+\theta)m} + \frac{\eta}{1+\theta}(\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}). \quad (14)$$

Plugging Eq. 14 to $\frac{\partial\frac{1}{2}\|\mathbf{b}-\mathbf{b}_t\|^2}{\partial\eta} + \frac{\partial\frac{\theta}{2}\|\mathbf{b}\|^2}{\partial\eta}$, noting that, $\frac{1}{m}\mathbf{1}^T(\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}) = 0$, yields: Thus,

$$\frac{\partial\frac{1}{2}\left\|\mathbf{b}-\mathbf{b}_t\right\|^2}{\partial\eta} + \frac{\partial\frac{\theta}{2}\left\|\mathbf{b}\right\|^2}{\partial\eta} = \frac{1}{(1+\theta)}\eta\|\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}\|^2 \quad (15)$$

Plugging Eq. 14 to $\frac{\partial\eta(\hat{\mathbf{x}}_{t+1}^T\mathbf{b}-\epsilon)}{\partial\eta}$, yields

$$\frac{\partial\eta(\hat{\mathbf{x}}_{t+1}^T\mathbf{b}-\epsilon)}{\partial\eta} = \hat{\mathbf{x}}_{t+1}^T(\frac{\mathbf{b}_t}{1+\theta} + \frac{\theta\mathbf{1}}{1+\theta}) - \epsilon + \frac{2}{1+\theta}\eta\|\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}\|^2 \quad (16)$$

Plugging the expression of λ and Eq. 14 to $\frac{\partial\lambda(\mathbf{1}^T\mathbf{b}-1)}{\partial\eta}$, we get,

$$\frac{\partial\lambda(\mathbf{1}^T\mathbf{b}-1)}{\partial\eta} = 0, \quad (17)$$

From Eq. 15, Eq. 16 and Eq. 17, we get,

$$0 = \frac{\partial\mathcal{L}}{\partial\eta} = \epsilon - \hat{\mathbf{x}}_{t+1}^T(\frac{\mathbf{b}_t}{1+\theta} + \frac{\theta\mathbf{1}}{1+\theta}) - \frac{\eta}{(1+\theta)}\|\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}\|^2, \quad (18)$$

then,

$$\eta = \frac{(1+\theta)\epsilon - \hat{\mathbf{x}}_{t+1}^T(\mathbf{b}_t + \theta\mathbf{1})}{\|\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}\|^2}. \quad (19)$$

It is noted that $\eta > 0$, so

$$\eta = \mathbf{max}(0, \frac{(1+\theta)\epsilon - \hat{\mathbf{x}}_{t+1}^T(\mathbf{b}_t + \theta\mathbf{1})}{\|\hat{\mathbf{x}}_{t+1} - \bar{x}_{t+1}\mathbf{1}\|^2}) \quad (20)$$

For simplicity, the non-negativity constraint of portfolio $\mathbf{b}$ is not considered in the above formulation. It is possible that the resulting portfolio calculated from Eq. 11 is not non-negative. Thus, the projection of the solution to the simplex domain [8] is necessary, as shown in **Algorithm** 2. Finally, the online portfolio selection algorithm based on the Weighted Multivariate Mean Reversion is described in **Algorithm** 3. Unlike the regret minimization approaches, the WMMR strategy takes advantage of the statistical properties (mean reversion) of the financial market, which is difficult to provide a traditional regret bound. [3] failed to provide a regret bound for the Anticor strategy, which passively exploits the mean reversion idea. Although we cannot prove the traditional regret bound, the proposed algorithms do provide strong empirical evidence, which sequentially advances the state of the art.

5 Experiments

The effectiveness of the proposed portfolio strategies is tested on four public datasets from real markets, whose information is summarized in Table 2. NYSE(O), which is a benchmark dataset pioneered by [5]. Considering amalgamation and bankruptcy, the second dataset NYSE(N) consists of 23 stocks from dataset NYSE(O) including 36 stocks and was collected by Li et al. [19]. The third dataset is DJIA collected by Borodin et al. [3]. MSCI is a dataset that is collected from global equity indices that constitute the MSCI World Index. Several research studies and the state-of-art model RMR also utilize these four datasets in their experiments.

Table 2. Summary of the four real datasets in our numerical experiments.

dataset	Market	Region	Time frame	Trading days	Assets
NYSE(o)	Stock	US	Jul.3rd 1962-Dec.31st 1984	5651	36
NYSE(N)	Stock	US	Jan.1st 1985-Jun.30th 2010	6431	23
DJIA	Stock	US	Jan.14th 2001-Jan.14th 2003	507	30
MSCI	Index	Global	Apr.1st 2006-Mar.31st 2010	1043	24

Cumulative wealth is the most common and significant metric and is used to measure investment performance in this paper. To be consistent comparison with other different methods, we implement the proposed WMMR-HUBER (with $k = 0.95$), WMMR-BIS (with $k = 3.85$), WMMR-SHR and set the parameters empirically without tuning for each dataset separately as follows: $w = 5$, $\epsilon = 100$, $\alpha = 0.85$ and $\theta = 0.1$. It is worth noting that choices of parameters are not always optimal for WMMR, though these parameters can be tuned to obtain optimal results. The sensitivities of these parameters will be evaluated in the next section. It is necessary to note that the parameters in Algorithm 1, iteration maximum K, are fixed to 50.

5.1 Cumulative Wealth

The cumulative wealth achieved by various methods is summarized in Table 3. On dataset NYSE(O), NYSE(N) and DJIA, WMMR (WMMR-HUBER, WMMR-BISQUARE, and WMMR-SHR) outperform the state-of-the-art. On dataset MSCI, WMMR beats the existing algorithm RMR. By tuning different values of parameter w, ϵ, α, and θ for the corresponding dataset, we also refer to the best performance (in hindsight) shown as WMMR(max) in Table 3. Besides, WMMR(max) is showing the potential of the proposed method by tuning the optimal parameter.

Finally, Table 4 shows some statistics of WMMR. We only present the results achieved by WMMR-HUBER since the effect of WMMR-BISQUARE and WMMR-SHR, are quite similar to that of WMMR. From the results, a small p-value reveals that WMMR's excellent performance is owed to the strategy principle but not due to luck.

Table 3. Cumulative wealth achieved by various strategies on the four datasets.

Methods	NYSE(O)	NYSE(N)	DJIA	MSCI
Market	14.50	18.06	0.76	0.91
Best-stock	54.14	83.51	1.19	1.50
BCRP	250.60	120.32	1.24	1.51
UP	26.68	31.49	0.81	0.92
EG	27.09	31.00	0.81	0.93
ONS	109.91	21.59	1.53	0.86
B^k	1.08E+09	4.64E+03	0.68	2.64
BNN	3.35E+11	6.80E+04	0.88	13.47
CORN	1.48E+13	5.37E+05	0.84	**26.19**
Anticor	2.41E+08	6.21E+06	2.29	3.22
PAMR	5.14E+15	1.25E+06	0.68	15.23
CWMR	6.49E+15	1.41E+06	0.68	17.28
OLMAR	4.04E+16	2.24E+08	2.05	16.33
RMR	1.64E+17	3.25E+08	2.67	16.76
TCO	1.35E+14	9.15E+06	2.01	9.68
WMMR-HUBER	4.14E+17	**4.11E+08**	**3.14**	17.65
WMMR-BIS	**4.53E+17**	3.75E+08	2.91	17.02
WMMR-SHR	3.0E+17	3.43E+08	3.10	17.42
WMMR(max)	5.83E+17	3.02E+09	3.15	25.82

5.2 Computational Time

It is widely known that computational time is important to certain trading environments, we evaluate the computational time on one core of an Intel Core i5 2.3 GHz processor with 16GB, using Python on MacBook Pro. Experiments show that it takes 57.38s, 101.65s, 526.2s, and 443.3s for DJIA, MSCI, NYSE(O), and NYSE(N) respectively, which means that the computational time for each of trading periods is less than 0.1s. The computational time is acceptable even in the scenario of high-frequency trading, which occurs in fractions of a second. Such time efficiency supports WMMR's large-scale real applications.

5.3 Parameter Sensitivity

Table 4. Statistical test of our algorithms.

Stat. Attr	NYSE(O)	NYSE(N)	DJIA	MSCI
Size	5651	6431	507	1043
MER(WMMR)	0.0078	0.0037	0.0028	0.0030
MER(Market)	0.0005	0.0005	-0.0004	0.0000
t-statistics	15.2249	7.1985	2.2059	3.9214
p-value	0.0000	0.0000	0.0278	0.0000

Firstly, the effect of sensitivity parameter w on cumulative wealth is evaluated, in Fig. 1. It is obvious that in most cases, except NYSE(N), the cumulative wealth decreases with increasing w. Secondly, the effect of sensitivity parameter ϵ on cumulative wealth is evaluated. From Fig. 2, The growth of cumulative wealth is sharp as ϵ increases and turns flat when ϵ exceeds a threshold. Finally, the effect of sensitivity parameter θ and α on cumulative wealth are evaluated in Fig. 3 and Fig. 4. From the above observation, it is clear that WMMR is robust for different parameters and it is convenient to choose satisfying parameters.

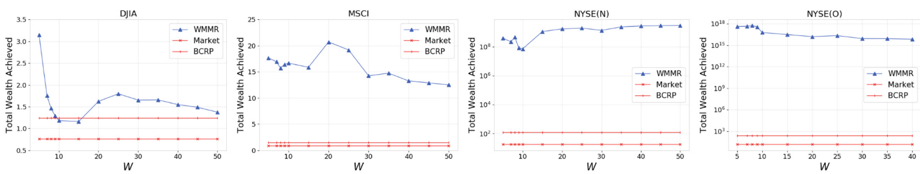

Fig. 1. Parameter sensitivity of WMMR w.r.t. w with fixed $\epsilon = 100, \alpha = 0.85, \theta = 0.1$

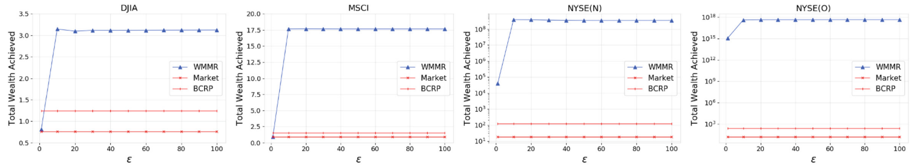

Fig. 2. Parameter sensitivity of WMMR w.r.t. ϵ with fixed $w = 5, \alpha = 0.85, \theta = 0.1$

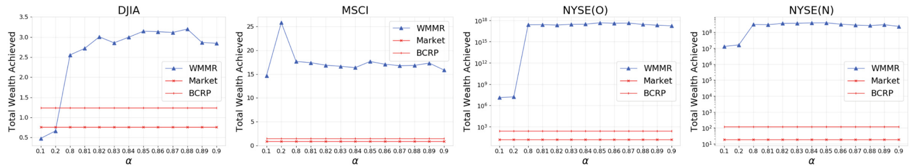

Fig. 3. Parameter sensitivity of WMMR w.r.t. α with fixed $w = 5, \epsilon = 100, \theta = 0.1$

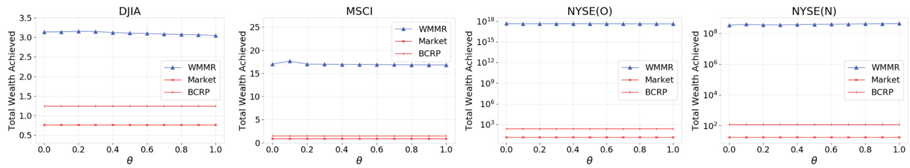

Fig. 4. 4Parameter sensitivity of WMMR w.r.t. θ with fixed $w = 5, \epsilon = 100, \alpha = 0.85$

5.4 Risk-Adjusted Returns

The risk in terms of volatility risk and drawdown risk and the risk-adjusted return in terms of annualized Sharpe ratio are evaluated in the experiment, taking two benchmarks (Market and BCRP) and two state-of-the-art algorithms (OLMAR and RMR) for comparison. The result of Risk-Adjusted Returns is shown in Fig. 5. Though the high return is associated with high risk, WMMR achieves the best performance in terms of the Sharpe ratio.

5.5 Transaction Cost Scalability

For a real-world application, the transaction cost is an important practical issue for portfolio selection. Ignoring this cost may lead to aggressive trading and bring biases into the estimation of returns. [25] proposed an approximate dynamic programming (ADP) method to tackle the multi-asset portfolio optimization problems with proportional transaction costs. [20] proposed a novel online portfolio selection framework, named Transaction Cost Optimization(TCO) to trade-off between maximizing expected log return and minimizing transaction costs. Here, the proportional transaction cost model proposed in [3] is adopted to compute the cumulative wealth:

$$\mathbf{S}_0 \prod_{t=1}^{n} \left[(\mathbf{b}_t \cdot \mathbf{x}_t) \times \left(1 - \frac{\gamma}{2} \times \sum_i \left| b_{t,i} - \hat{b}_{t-1,i} \right| \right) \right],$$

where, γ is transaction cost rate $\gamma \in (0, 0.1)$ in the experiments, $\hat{b}_{(t-1,i)} = \frac{b_{t-1,i} x_{t-1,i}}{\mathbf{b}_{t-1}^T \cdot \mathbf{x}_{t-1}}$. The cumulative wealth with transaction cost is plotted in Fig. 6. From Fig. 6, we can observe that WMMR can withstand reasonable transaction cost rates, and can beat the two benchmarks in most cases.

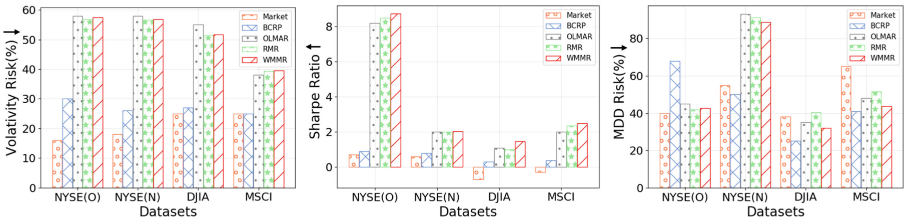

Fig. 5. Risk and risk-adjusted performance of various strategies on the four different datasets. In each diagram, the rightmost bars represent the results achieved by WMMR.

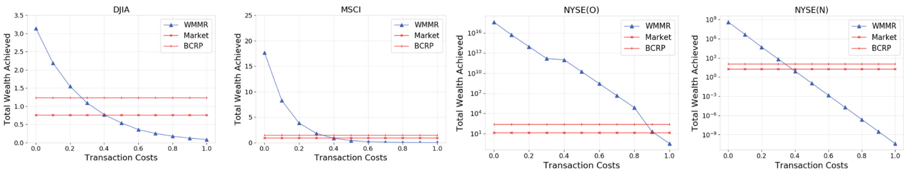

Fig. 6. Scalability of the total wealth achieved by WMMR with respect to transaction cost rate

6 Conclusion

Based on the robust multivariate estimates and PA online learning, a novel online portfolio selection strategy named "Weighted Multivariate Mean Reversion" (WMMR) is proposed in this paper. In the exploitation of "Multi-period Multivariate Average Reversion", WMMR takes data noise, trend changes, and the dependence of multi-assets into full consideration. Several cases of weighting functions with exponential decay are investigated, and the results demonstrate the effectiveness of WMMR. Moreover, extensive experiments on the real market show that the proposed WMMR can achieve satisfying performance with an acceptable run time.

Acknowledgement. J.W. Gu was supported in part by NSFC Grant 11801262, Guangdong Basic and Applied Basic Research Foundation 2023A1515030197, Shenzhen Humanities & Social Sciences Key Research Bases. We thank Qiao Xiao for helping to draw figures.

References

1. Andrews, D.F., Hampel, F.R.: Robust Estimates of Location: Survey and Advances. Princeton University Press (2015)
2. Bickel, P.J.: One-step Huber estimates in the linear model. J. Am. Stat. Assoc. **70**(350), 428–434 (1975)
3. Borodin, A., El-Yaniv, R., Gogan, V.: Can we learn to beat the best stock. J. Artif. Intell. Res. **21**, 579–594 (2004)
4. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press, Cambridge (2004)
5. Cover, T.M.: Universal portfolios. Math. Financ. **1**(1), 1–29 (1991)
6. Cover, T.M., Gluss, D.H.: Empirical Bayes stock market portfolios. Adv. Appl. Math. **7**(2), 170–181 (1986)
7. De Maesschalck, R., Jouan-Rimbaud, D., Massart, D.L.: The mahalanobis distance. Chemom. Intell. Lab. Syst. **50**(1), 1–18 (2000)
8. Duchi, J., Shalev-Shwartz, S., Singer, Y., Chandra, T.: Efficient projections onto the l 1-ball for learning in high dimensions. In: Proceedings of the 25th International Conference on Machine Learning, pp. 272–279 (2008)
9. Gao, L., Zhang, W., Tang, Q.: Passive aggressive algorithm for online portfolio selection with piecewise loss function. In: Motoda, H., Wu, Z., Cao, L., Zaiane, O., Yao, M., Wang, W. (eds.) ADMA 2013. LNCS (LNAI), vol. 8347, pp. 360–371. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-53917-6_32
10. Györfi, L., Lugosi, G., Udina, F.: Nonparametric kernel-based sequential investment strategies. Math. Financ. Int. J. Math. Statist. Financ. Econ. **16**(2), 337–357 (2006)
11. Heiberger, R.M., Becker, R.A.: Design of an s function for robust regression using iteratively reweighted least squares. J. Comput. Graph. Stat. **1**(3), 181–196 (1992)
12. Helmbold, D.P., Schapire, R.E., Singer, Y., Warmuth, M.K.: On-line portfolio selection using multiplicative updates. Math. Financ. **8**(4), 325–347 (1998)
13. Huang, D., Zhou, J., Li, B., HOI, S., Zhou, S.: Robust median reversion strategy for on-line portfolio selection (2013). In: Proceedings of the Twenty-Third International Joint Conference on Artificial Intelligence: IJCAI 2013: Beijing, 3–9 August 2013 (2012)
14. Huang, D., Zhu, Y., Li, B., Zhou, S., Hoi, S.C.: Semi-universal portfolios with transaction costs (2015)
15. Huber, P.J.: Robust estimation of a location parameter. In: Kotz, S., Johnson, N.L. (eds.) Breakthroughs in Statistics. Springer Series in Statistics. Springer, New York (1992). https://doi.org/10.1007/978-1-4612-4380-9_35
16. Huber, P.J.: Robust Statistics, vol. 523. John Wiley & Sons, New York (2004)
17. Kelly, J., Jr.: A new interpretation of information rate. Bell Syst. Tech. J. **35**(4), 917–926 (1956)
18. Li, B., Hoi, S.C.: On-line portfolio selection with moving average reversion. In: Proceedings of the Twenty-Ninth International Conference on Machine Learning: June 26–July 1, Edinburgh, Scotland. Citeseer (2012)
19. Li, B., Hoi, S.C., Zhao, P., Gopalkrishnan, V.: Confidence weighted mean reversion strategy for online portfolio selection. ACM Trans. Knowl. Disc. Data (TKDD) **7**(1), 1–38 (2013)
20. Li, B., Wang, J., Huang, D., Hoi, S.C.: Transaction cost optimization for online portfolio selection. Quant. Financ. **18**(8), 1411–1424 (2018)

21. Li, B., Zhao, P., Hoi, S.C., Gopalkrishnan, V.: PAMR: passive aggressive mean reversion strategy for portfolio selection. Mach. Learn. **87**(2), 221–258 (2012)
22. Markowitz, H.: Portfolio selection (1959)
23. Muler, N., Yohai, V.J.: Robust estimates for arch processes. J. Time Ser. Anal. **23**(3), 341–375 (2002)
24. Shen, W., Wang, B., Pu, J., Wang, J.: The Kelly growth optimal portfolio with ensemble learning. In: Proceedings of the AAAI Conference on Artificial Intelligence, vol. 33, pp. 1134–1141 (2019)
25. Shen, W., Wang, J.: Transaction costs-aware portfolio optimization via fast Lowner-John ellipsoid approximation. In: Proceedings of the AAAI Conference on Artificial Intelligence, vol. 29 (2015)
26. Shen, W., Wang, J.: Portfolio selection via subset resampling. In: Proceedings of the AAAI Conference on Artificial Intelligence, vol. 31 (2017)
27. Vardi, Y., Zhang, C.H.: The multivariate l1-median and associated data depth. Proc. Natl. Acad. Sci. **97**(4), 1423–1426 (2000)
28. Xu, K., Zhang, Y., Ye, D., Zhao, P., Tan, M.: Relation-aware transformer for portfolio policy learning. In: IJCAI (2020)