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Research paper

Mechanism balancing taxonomy

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ABSTRACT

The balancing of mechanisms consists in distributing their moving masses, inertias, and elastic components in order to achieve key mechanical properties, such as the elimination of the shaking forces and moments exported onto their supporting structure or the insensitivity of the mechanism to gravity and to the motions of its chassis. This article introduces a new refined systematic taxonomy for the classification of multi-degree-of-freedom (DoF) 3-dimensional passively balanced mechanisms. The taxonomy is composed of 15 distinct types – compared to only 4 types described in the literature. Each type is provided with its definition, the necessary conditions to be satisfied, the list of resulting mechanical properties, and a 1-DoF mechanism example. This taxonomy applies to mechanisms subject to gravity and mounted onto fixed or mobile chassis undergoing linear and angular accelerations, and, newly, also angular velocities. It is represented as a 4-set Venn diagram built on 4 *primary* balancing types: *static, force, moment,* and a newly introduced *inertial invariance*. This theoretical work allows for a refined categorization of the broad spectrum of balanced mechanisms while alleviating some inconsistencies observed in the existing literature.

1. Introduction

1.1. Literature review of mechanism balancing classifications

Balancing is used to improve the static or dynamic performance of mechanisms by, e.g., counteracting gravitational forces to which they may be subjected or canceling vibrations and shaking forces exported by the mechanisms onto their support structure. This article[1] treats the main types of *passive* balancing and does not cover *partial, approximate* [1], or *harmonic* [2] balancing. *Passive balancing* is considered to be achieved either (1) when the springs, magnets, masses, inertia, etc. of a mechanism are added or redistributed [3–5] or (2) when the initial mechanism kinematics is complexified (e.g., by adding new links to make it symmetric) as long as the initial number of DoF of the mechanism remains unchanged [6–8]. On the other hand, *active balancing,* although not the focus of this article, is considered to be achieved (1) when separate active units are used to actively cancel shaking forces and moments exported to the frame of the mechanism [9,10], or (2) when the trajectory of the mechanism is planned to follow a reactionless path [11,12].

Four different balancing types – *static balancing, shaking force balancing, shaking moment balancing,* and *dynamic balancing* – are

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presented in the literature known to the authors, and various classifications have been proposed [13–20] covering the most well-known balancing types. The taxonomy introduced in the present article allows for a more refined sorting of balanced mechanisms into 15 rigorously defined types with unequivocal mechanical properties, including 11 balancing types that have not been mentioned in the known balancing classifications. It constitutes a useful tool both for the analysis and design of mechanisms and also brings important conceptual clarifications, resolving some inconsistencies found in the literature.

An example of such inconsistencies are the non-zero-force force balanced mechanisms, such as the mechanical oscillators described in [7,21], which cannot be classified within the *static balancing* and *force balancing* categories as described in [13,22–24]. Indeed, as will be shown below, *force balancing* is not a subset of *static balancing* but an overlapping separate set.

Other inconsistencies exist between [19] and [24, p. 3] versus [4] and [25, Ch. 2]: the former state that dynamically balanced mechanisms are dynamically decoupled from the motions of their base, thus behaving as a single rigid body, while the latter introduce a supplementary required condition – a constant inertia tensor – for achieving complete decoupling.

1.2. Statement of results

The central contribution of this article is a univocal categorization of 3-dimensional balanced mechanisms with their associated definitions and properties. This classification is meant to assist designers in the selection of the most suitable balancing type for their specific applications, as well as to analyze the mechanical properties of existing mechanisms. Each balancing definition addresses the sensitivity of the balanced mechanism to the linear and angular accelerations, as well as the angular velocities, arising from motion of its chassis. Additionally, a new primary balancing type termed *Inertial invariance* is introduced, which, when combined with dynamic balancing, extends the reactionless properties of dynamic balance to rotating bases. The taxonomy is organized around 4 *primary balancing types* (static, force, moment and inertial invariance), whose 15 combinations represent the entire set of balancing types, which includes 4 *pure balancing types* plus 11 *blended balancing types.* To the knowledge of the authors, the proposed taxonomy, along with the introduction of *Inertial invariance* and an exhaustive study of the influence of the base motion on the pose of a mechanism, has not been done in the existing literature. This limited-length conceptual article focuses on definitions and respective classifications while excluding the historical aspects of mechanism balancing [18,26] as well as the analytical proofs and experimental methods [7,21,27].

1.3. Outline of the paper

In Section 2.3, we present and define the most important balancing types, which are: *Static balancing* ($\mathbf{S}^*$), *Force balancing* ($\mathbf{F}^*$), *Moment balancing* ($\mathbf{M}^*$), *Dynamic balancing* ($\mathbf{FM}^*$), *Inertial invariance* ($\mathbf{I}^*$) and *Inertial balancing* ($\mathbf{FMI}^*$). In Section 2.4, we graphically illustrate the new balancing taxonomy and summarize all the introduced physical properties. Section 2.5 summarizes balancing concepts from the designer's point of view, treating the decoupling of a mechanism and its base from two different perspectives: how to prevent a mechanism's movements from disturbing its base, the latter being fixed or mobile, and how to prevent a mechanism from being disturbed by the motions of its base. In Section 2.6, we illustrate how elementary 1-DoF rotating mechanisms can be balanced using the different concepts and definitions given in the previous section. In Section 3, the contributions of the proposed classification to the field of mechanism balancing are then discussed.

2. Balancing taxonomy

2.1. Notation used in the definitions

$\mathscr{R}$	Inertial frame of reference.
$\mathscr{F}$	Frame of reference attached to the chassis of the mechanism.
V	Total potential energy of a mechanism.
q_i	Generalized coordinate for the i^{th} joint of mechanism.
τ_i	Generalized force (or torque) for the i^{th} joint of a mechanism.
m_i	Total mass of the i^{th} rigid link.
$\boldsymbol{c}_{\mathrm{A},i}$	Center of mass vector of the i^{th} link of a mechanism relative to point A.
$\dot{\boldsymbol{c}}_{i/\mathscr{F}}$	Linear velocity vector of the center of mass of the i^{th} link of the mechanism relative to $\mathscr{F}$.
$\boldsymbol{p}_{i/\mathscr{F}}$	Linear momentum vector of the i^{th} link of a mechanism relative to $\mathscr{F}$.
$\boldsymbol{p}_{\mathrm{tot}/\mathscr{F}}$	Total linear momentum vector of a mechanism relative to $\mathscr{F}$.
$\boldsymbol{J}_i$	Inertia tensor of the i^{th} link of a mechanism expressed at its center of mass relative to its attached frame of reference.
$\boldsymbol{J}_{\mathrm{A,tot}/\mathscr{F}}$	Total inertia tensor of a mechanism expressed at point A relative to $\mathscr{F}$.
$\boldsymbol{\omega}_{i/\mathscr{F}}$	Angular velocity vector of the i^{th} link of a mechanism relative to $\mathscr{F}$.
$\boldsymbol{\sigma}_{\mathrm{A},i/\mathscr{F}}$	Angular momentum vector of the i^{th} link of a mechanism expressed at point A relative to $\mathscr{F}$.
$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{F}}$	Total angular momentum vector of a mechanism expressed at point A relative to $\mathscr{F}$.
$\boldsymbol{\delta}_{\mathrm{A},i/\mathscr{F}}$	Shaking moment of the i^{th} link of a mechanism expressed at point A relative to $\mathscr{F}$.
$\boldsymbol{\delta}_{\mathrm{A,tot}/\mathscr{F}}$	Total shaking moment of a mechanism expressed at point A relative to $\mathscr{F}$.
	Bold characters stand for vectors and **BOLD** capitals for tensors.

2.2. Mechanism definition

In this article, mechanisms are considered to be articulated structures composed of rigid links connected together by ideal joints and attached to a single chassis with a given orientation with respect to the field of gravity. Each rigid link is characterized by its mass, center of mass (CoM), and mass moments of inertia.

As an example, the generic mechanism illustrated in Fig. 1 is composed of two massless rigid links denoted 1 and 2. Link 1 is articulated relative to the chassis along the z-axis by a pivot joint located in O_1, and θ_1 represents its angular displacement. Link 2 is articulated relative to link 1 along the z-axis by a pivot joint located in O_2, and θ_2 represents its angular displacement. As a single point mass m_{11} is attached to link 1 in c_{11}, its CoM is located at c_{11}. Likewise, as a single point mass m_{21} is attached to link 2 in c_{21}, its CoM is located in c_{21}. In order to simplify the readability of some equations, we use the notation c_{θ_i} and s_{θ_i} instead of $\cos(\theta_i)$ and $\sin(\theta_i)$. For the graphical legend, refer to Fig. 12.

2.3. Balancing types

In this section, we identify and define the main balancing types that may be sources of confusion in the literature. For each balancing type that is developed, we give the definition, the conditions to be satisfied, the achieved properties, a discussion on the chassis motion invariance, remarks, and applications. This format should help the reader better understand the intrinsic properties of each balancing type as well as their limits. For conciseness reason, we state the mechanical properties without formally demonstrating them. Note that the balancing types noted with a star (*) cover all the combinations including these types. The other balancing types (i. e., those without a star) represent particular *pure* or *blended balancing types*.

2.3.1. Type $\mathbf{S}^*$: static balancing

Definition. *Static balancing* is achieved when a mechanism has a total potential energy which is constant over its workspace. $\mathscr{F}$ is considered fixed relative to $\mathscr{R}$.

Condition.

$$V = \text{const.} \tag{2.1}$$

Property. A type $\mathbf{S}^*$ mechanism is said to be a *zero-force* mechanism. Indeed, the static force τ_i at joint i required to actuate or hold a type $\mathbf{S}^*$ mechanism in position can be expressed as:

$$\tau_i = -\frac{\partial V}{\partial q_i} = 0. \tag{2.2}$$

The potential energy of a mechanism can be due to gravitational, elastic, magnetic, electrical, or other effects. Either the entire mechanism can be subjected to the potential (such as gravity) or only part of it (e.g., springs between two rigid segments of the mechanism).

Chassis motion invariance. The static balancing condition of a mechanism that relies on the compensation of its gravitational potential energy by any different potential energy depends on the orientation of its chassis with respect to the direction of gravity. Other internal sources of potential energy, such as springs, are invariant to the orientation of the chassis.

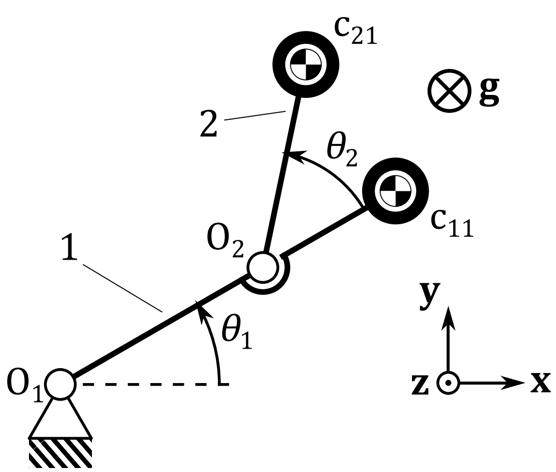


Fig. 1. Example of a generic mechanism as considered in this article.

Remark 1. A *zero-force* mechanism is also a *zero-stiffness* mechanism. The opposite is not true: a *zero-stiffness* mechanism can be a non-zero force mechanism. In that case, it is a *constant force* mechanism (which can be used, for example, to counteract gravitational pull).

Remark 2. In the absence of friction, actuators of a *zero-force* mechanism require energy only to accelerate the inertia of the various bodies composing the mechanism. As no energy is needed to hold the mechanism in any pose, all poses are stable: the mechanism is at *neutral* (or *indifferent*) *equilibrium*.

Remark 3. A mechanism having only gravitational potential energy is statically balanced if its CoM is fixed relative to its chassis, or mobile but remaining at a constant altitude. This is the case for a planar mechanism that solely use masses subject to a gravitational field normal to its plane of motion.

Remark 4. A mechanism without any potential energy is statically balanced. This is the case for a mechanism that solely use masses evolving in a micro-gravity environment.

Applications. From the aforementioned properties, it is obvious that *static balancing* can be used to reduce the size of actuators by designing them only according to the desired dynamics. This is a useful property in household products, orthopedics, or vacuum environments. *Static balancing* can also be used to simplify the control of a mechanism where the static forces are intrinsically compensated. Additionally, *static balancing* is used in vibration isolation systems, where low stiffness leads to high attenuation of chassis vibrations [28].

2.3.2. Type $\mathbf{F}^*$: force balancing

Definition. *Force balancing* is achieved when a mechanism has a constant linear momentum over its workspace. $\mathscr{F}$ is considered fixed relative to $\mathscr{R}$.

Condition.

$$\boldsymbol{p}_{\text{tot}/\mathscr{F}} = \sum_i \boldsymbol{p}_{i/\mathscr{F}} = \sum_i m_i \dot{\boldsymbol{c}}_{i/\mathscr{F}} = \mathbf{const}. \tag{2.3}$$

Under the condition that the masses of the mechanism links remain the same (which is often the case), conservation of the linear momentum of the mechanism implies that its total CoM has a constant speed. However, it is often more convenient to consider the total CoM to be fixed in the workspace of the mechanism so that:

$$\boldsymbol{p}_{\text{tot}/\mathscr{F}} = \mathbf{0}. \tag{2.4}$$

Property 1. According to Newton's second law, the time derivative of the linear momentum of a mechanism is the shaking force it exports to its frame. As for a force balanced mechanism, its linear momentum is constant (or zero), and there are no exported shaking forces:

$$\boldsymbol{F}_{\text{ext}} = \frac{d\boldsymbol{p}_{\text{tot}/\mathscr{F}}}{dt} = \mathbf{0}. \tag{2.5}$$

Property 2. The CoM of a type $\mathbf{F}^*$ mechanism being fixed within its workspace, its gravitational potential energy is constant. A type $\mathbf{F}^*$ mechanism is then independent of gravity as well as any constant linear accelerations.

Chassis motion invariance. The motion of a type $\mathbf{F}^*$ mechanism is insensitive to all linear accelerations coming from a rectilinear displacement of its chassis. By insensitive, we mean that the mentioned disturbances, in this case external linear accelerations, have no effect on the pose and motion of the mechanism. The motion of a type $\mathbf{F}^*$ mechanism may, however, be impacted by angular velocities and accelerations coming from a rotational displacement of its chassis.

Remark 1. One may notice that we used the formulation *Force balancing* instead of *Shaking force balancing* for the sake of compactness.

Remark 2. Note that in Property 2, we assume the gravitational field to be constant over the workspace of the mechanism, which is often a valid approximation. Therefore, the center of gravity of a mechanism coincides with its center of mass.

Remark 3. A type $\mathbf{F}^*$ mechanism is often considered to be statically balanced. However, that does not hold when other forms of potential energy might be present, such as elastic springs or magnets. Therefore, force balancing is not a subset of static balancing. For example, a typical wristwatch oscillator composed of a balanced wheel and a balance spring has a fixed CoM but a non-constant potential energy due to the winding and unwinding of its spring [29, Ch. 5], [30, Ch. 7.4 & 7.10].

Applications. Force balancing is used to reduce the vibrations that a mechanism exports to its chassis. By doing so, it enables a reduction of its noise, wear, and fatigue problems, and it may increase its dynamic performances (maximum velocities and accelerations) [5]. For robotic applications, the reduction of exported forces also enables better repeatability as well as a simplified control command [31]. For sensing applications, such as in gyroscopic MEMS, force balanced designs are used to restrict the sensitivity of the device to angular velocities or accelerations [32–34]. Furthermore, wristwatch mechanical oscillators are typically force balanced in order to be insensitive to gravity effects as well as linear accelerations caused by chassis motion [7,21,35].

2.3.3. Type $\mathbf{M}^*$: moment balancing

Definition. *Moment balancing* around point A is achieved when a mechanism has a total angular momentum constant over its workspace when evaluated at point A; where A is fixed with respect to the chassis of the mechanism. $\mathscr{T}$ is considered fixed relative to $\mathscr{R}$.

Condition.

$$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{T}} = \sum_i \boldsymbol{\sigma}_{\mathrm{A},i/\mathscr{T}} = \sum_i \left(m_i \boldsymbol{c}_{\mathrm{A},i} \times \dot{\boldsymbol{c}}_{i/\mathscr{T}} + \boldsymbol{J}_i \boldsymbol{\omega}_{i/\mathscr{T}} \right) = \mathbf{const}. \tag{2.6}$$

Moment balancing imposes a constant angular momentum, see Eq. (2.6). However, it is often more convenient to consider it equal to zero, so it can be expressed as:

$$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{T}} = \mathbf{0}. \tag{2.7}$$

Property. A type $\mathbf{M}^*$ mechanism around point A does not export any shaking moments to its frame at point A. The shaking moments around point A of a moment balanced mechanism around point A can be expressed as:

$$\boldsymbol{\delta}_{\mathrm{A,tot}/\mathscr{T}} = \sum_i \boldsymbol{\delta}_{\mathrm{A},i/\mathscr{T}} = \sum_i \frac{d^{\mathscr{T}}}{dt} \boldsymbol{\sigma}_{\mathrm{A},i/\mathscr{T}} = \mathbf{0}. \tag{2.8}$$

Chassis motion invariance. The motion of a type $\mathbf{M}^*$ mechanism around point A is sensitive to all linear and angular accelerations, except angular accelerations around point A, as well as all angular velocities coming from rectilinear and rotational displacements of its chassis. Note that the angular acceleration of the chassis always comes along with the angular velocity. However, one may understand that a type $\mathbf{M}^*$ mechanism around point A is insensitive at point A to angular acceleration effects but sensitive at point A as well as everywhere else to Coriolis and centrifugal forces.

Remark 1. One may notice that we used the formulation *Moment balancing* instead of *Shaking moment balancing* for the sake of compactness.

Remark 2. A type $\mathbf{M}^*$ mechanism is not necessarily a type $\mathbf{F}^*$ mechanism, so it may export forces and be excited by external linear and angular accelerations.

Applications. To the knowledge of the authors, type $\mathbf{M}^*$ mechanisms do not have any dedicated applications in literature. Hypothetical applications might be the elimination of shaking moments in a rotationally compliant chassis, such as a cable-suspended robot.

2.3.4. Type $\mathbf{FM}^*$: dynamic balancing

Definition. *Dynamic balancing* is achieved when a mechanism's linear and angular momenta are constant over its workspace. $\mathscr{T}$ is considered fixed relative to $\mathscr{R}$.

Conditions.

$$\boldsymbol{p}_{\mathrm{tot}/\mathscr{T}} = \sum_i \boldsymbol{p}_{i/\mathscr{T}} = \sum_i m_i \dot{\boldsymbol{c}}_{i/\mathscr{T}} = \mathbf{const}.$$

and

$$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{T}} = \sum_i \boldsymbol{\sigma}_{\mathrm{A},i/\mathscr{T}} = \sum_i \left(m_i \boldsymbol{c}_{\mathrm{A},i} \times \dot{\boldsymbol{c}}_{i/\mathscr{T}} + \boldsymbol{J}_i \boldsymbol{\omega}_{i/\mathscr{T}} \right) = \mathbf{const}.$$

Property 1. A type $\mathbf{FM}^*$ mechanism does not export shaking forces to its chassis.

Property 2. A type $\mathbf{FM}^*$ mechanism has a fixed center of mass relative to its chassis.

Property 3. A type $\mathbf{FM}^*$ mechanism does not export shaking moments to its chassis.

Chassis motion invariance. A type $\mathbf{FM}^*$ mechanism is insensitive to linear and angular accelerations of its chassis relative to $\mathscr{R}$. A type $\mathbf{FM}^*$ mechanism may, however, be excited by the angular velocities of its chassis. One may understand that a type $\mathbf{FM}^*$ mechanism is insensitive to angular acceleration effects but is sensitive to Coriolis and centrifugal forces.

Remark 1. The angular momentum of a type $\mathbf{F}^*$ mechanism no longer depends on the point at which it is evaluated. Therefore, *force balancing* is often achieved before *moment balancing.*

Remark 2. From the combination of *force* and *moment balancing, dynamic balancing* is a subset of *force* and *moment balancing.*

Remark 3. *Dynamic balancing* may often result in an increase of mass, inertia, size or complexity of the mechanism which requires a trade-off.

Applications. As with *force balancing*, the reduction of exported shaking forces and moments to the chassis of the mechanism reduces vibrations and their inherent problems. *Dynamic balancing* may also aim to enhance the repeatability of a robot or manipulator [5]. In watchmaking, there is an interest in finding dynamically balanced oscillators because they are less sensitive to angular accelerations and theoretically have a higher quality factor than force balanced oscillators [21,36–38].

2.3.5. Type $\mathbf{I}^*$: inertial invariance

Definition. *Inertial invariance* is achieved around point A when a mechanism's angular momentum and total inertia tensor are configuration-invariant over its workspace when evaluated at point A; A being fixed relative to $\mathscr{F}$.

Conditions.

$$\frac{\partial \boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{F}}}{\partial q_i} = \mathbf{0}, \tag{2.9}$$

$$\frac{\partial \boldsymbol{J}_{\mathrm{A,tot}/\mathscr{F}}}{\partial q_i} = 0. \tag{2.10}$$

Chassis motion invariance. The motion of a type $\mathbf{I}^*$ mechanism around point A is sensitive to all linear and angular accelerations as well as all angular velocities of its chassis, except angular velocities around point A.

Remark 1. A type $\mathbf{FI}^*$ or $\mathbf{SFI}^*$ mechanism will have a total inertia tensor configuration-invariant no matter the stationary point in which it is expressed.

Remark 2. When a type $\mathbf{M}^*$ mechanism is made *inertially invariant* (type $\mathbf{I}^*$) it will become insensitive to angular velocities and accelerations around A.

Remark 3. For types **I**, **SI**, **MI,** or **SMI** mechanisms the balancing properties are satisfied only at a specific set of points (the same applies for types **M** and **SM** mechanisms). When type **I** is combined with type **F**, the point-dependence disappears.

Applications. Serial robots and manipulators whose inertia tensors are decoupled and configuration invariant are useful when their chassis are fixed relative to $\mathscr{R}$. These specific mechanisms have an actuation torque or force τ_i of their i^{th} link that is decoupled from the motion of the links both upstream and downstream. Such mechanisms are studied because they are easier to control [31,39, Ch. 5.2], [40]. An application of inertial invariance concerns the well-known balance spring oscillators that are type **FI** 1-DoF rotating mechanisms. Firstly, their CoM is fixed along their axis of revolution, which renders them *force balanced* (type **F**). Practically, they will be insensitive to any linear accelerations coming from different gravity orientations or motions from the wristwatch holder. Secondly, one of their principal axes of inertia is aligned with their axis of rotation, causing their angular momentum to be configuration-invariant so they do not to export any centrifugal torques along axes that are orthogonal to their axis of rotation (see *rigid rotor balancing* references [41, Ch. 1], [42] to understand the exportation of centrifugal torques). Thirdly, their homogeneous mass distribution (close to a cylinder) makes their total inertia tensor configuration-invariant in addition to being force balanced. As the balance-spring oscillator has a configuration-invariant angular momentum as well as inertia tensor, it is of type **FI**, and is insensitive to all angular velocities. Note that the balance spring oscillators are still sensitive to angular accelerations, see the **FI** mechanism studied in Section 2.6.6.

2.3.6. Type *FMI**: inertial balancing

Definition. *Inertial balancing* is achieved when a mechanism has constant linear and angular momenta and an invariant total inertia tensor configuration over its workspace. The chassis of the mechanism is considered fixed relative to an inertial frame $\mathscr{R}$.

Conditions.

$$\boldsymbol{p}_{\mathrm{tot}/\mathscr{F}} = \sum_i \boldsymbol{p}_{i/\mathscr{F}} = \sum_i m_i \dot{\boldsymbol{c}}_{i/\mathscr{F}} = \mathbf{const}.$$

$$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{F}} = \sum_i \boldsymbol{\sigma}_{\mathrm{A},i/\mathscr{F}} = \sum_i \left(m_i \boldsymbol{c}_{\mathrm{A},i} \times \dot{\boldsymbol{c}}_{i/\mathscr{F}} + \boldsymbol{J}_i \boldsymbol{\omega}_{i/\mathscr{F}}\right) = \mathbf{const}.$$

and

$$\frac{\partial \boldsymbol{J}_{\mathrm{A,tot}/\mathscr{T}}}{\partial q_i} = 0.$$

Property 1. A type $\mathbf{FMI}^*$ mechanism has a fixed center of mass relative to its chassis.

Property 2. A type $\mathbf{FMI}^*$ mechanism does not export shaking forces to its chassis.

Property 3. A type $\mathbf{FMI}^*$ mechanism does not export shaking moments to its chassis.

Property 4. A type $\mathbf{FMI}^*$ mechanism has a total inertia tensor that is configuration-invariant.

Chassis motion invariance. The motion of a type $\mathbf{FMI}^*$ mechanism is insensitive to all linear and angular accelerations and velocities of its chassis.

Remark 1. *Inertial balancing* is a subset of *dynamic balancing* and *inertial invariance*.

Remark 2. *Inertial balancing*, as well as *dynamic balancing*, may often result in an increase of mass, inertia, size or complexity of the final mechanism which requires a trade-off.

Applications. For space applications, reactionless mechanisms that are insensitive to all linear and angular motions coming from their chassis are of interest, as they will not affect or be affected by the motion of their satellite frame, for example, by creating a ripple in its rotation speed. In a horological context, *inertially balanced* mechanical oscillators are of interest as they will not be affected by any linear or angular perturbations that could degrade their chronometric performances [37,38,43]. The inertia tensors of $\mathbf{FM}^*$ parallel mechanisms or sub-mechanisms must also be made constant if they undergo spatial motions. Hence, they will be decoupled from parts that support them by not exporting any inertial forces and moments and being motion-insensitive to any linear and rotational motions [4,44, Ch. 5]. These types of mechanisms are also used in synthesis because they are dynamically equal to a rigid body and can, therefore, be stacked into dynamically balanced mechanisms of higher DoF [3,45].

2.4. Proposed balancing taxonomy

In Section 2.3, we presented the different types of balancing. We introduced the properties of each of them and emphasized how they can be combined. In Fig. 2, we now illustrate graphically the nesting of these balancing types. One may see that from the 4 *primary balancing types* ($\mathbf{S}^*$, $\mathbf{F}^*$, $\mathbf{M}^*$ and $\mathbf{I}^*$), there are a total of 15 different balancing types (unbalanced mechanisms being excluded), with 4 *pure balancing types* plus 11 *blended balancing types*. Among the *blended types*, we highlighted two important subsets: the $\mathbf{FM}^*$ subset, called *dynamic balancing*, and the $\mathbf{FMI}^*$ subset, called *inertial balancing*.

Notation. Balancing types (*primary* or not) denoted with a star (*) cover all the subsets that can be generated by a combination of one

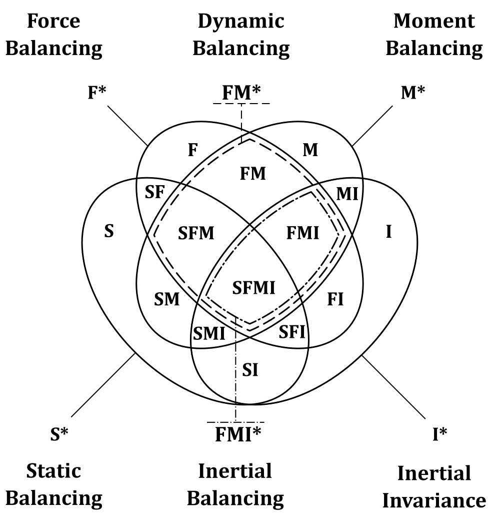


Fig. 2. Mechanism balancing taxonomy represented as a 4-set Venn diagram composed of 4 *primary balancing types*: $\mathbf{S}^*$, $\mathbf{F}^*$, $\mathbf{M}^*$, $\mathbf{I}^*$; 4 *pure balancing types*: **S**, **F**, **M**, **I**; and 11 *blended balancing types*: **SF**, **SFM**, **SFMI**, **SFI**, **SM**, **SMI**, **SI**, **FM**, **FMI**, **FI**, **MI**. The $\mathbf{FM}^*$subset constitutes the well-known *dynamic balancing* type. The $\mathbf{FMI}^*$ subset is an important new balancing type called *inertial balancing*, which is introduced in this article.

or more balancing types. A balancing type not denoted with a star (*) defines a specific *pure* or *blended balancing type*. As examples:

- $\mathbf{FM} = (\mathbf{F}^* \cap \ \mathbf{M}^*) \setminus (\mathbf{S}^* \cup \ \mathbf{I}^*)$ and $\mathbf{FM}^* \ = \mathbf{FM} \cup \mathbf{FMI} \cup \mathbf{SFM} \cup \mathbf{SFMI}$
- $\mathbf{FMI} = (\mathbf{F}^* \cap \ \mathbf{M}^* \cap \ \mathbf{I}^*) \setminus \mathbf{S}^*$ and $\mathbf{FMI}^* \ = \mathbf{FMI} \cup \mathbf{SFMI}$

Remark 1. *Dynamic balancing* (type $\mathbf{FM}^*$) results from the intersection of *force* (type $\mathbf{F}^*$) and *moment* (type $\mathbf{M}^*$) *balancing*. Likewise, *inertial balancing* (type $\mathbf{FMI}^*$) is a subset of *dynamic balancing* (type $\mathbf{FM}^*$) and *inertial invariance* (type $\mathbf{I}^*$).

Remark 2. A mechanism may not belong to any of the introduced balancing types. In this case, the mechanism does not have any balancing properties and is said to be unbalanced. The unbalanced type is represented by a slashed O (Ø), see Table 5.

To summarize the properties of each balancing type, we gathered them all in Table 1 and attributed a specific symbol to each of them. Then, using the symbolic representation from this latter table, we exhaustively assigned each balancing type shown in Fig. 2 with its respective intrinsic balancing properties in Table 2. Note that each *pure balancing type* (**S**, **F**, **M**, **I**) has a specific property or group of properties that blend when multiple balancing types are performed at once. This is actually the case for *pure static* and *force balancing*, whose properties combine without any change. However, *pure moment balancing* and *inertial invariance*, have properties that become point-invariant when combined with *force balancing* (types **FI**, **SFI**, **FM**, **SFM**, **FMI**, and **SFMI**).

2.5. Balancing from the designer's point of view

From a designer's point of view, two main cases arise:

1 The requirement is to design a mechanism which is insensitive to the motions of its chassis. For this case, Table 3 lists the required balancing properties as a function of the type of motions of the chassis. Application example: wrist-watch oscillators, gyroscopic MEMS, robotic grippers.
2 The requirement is to design a mechanism whose motions do not export any forces and torques to its chassis. For this case Table 4 lists the required balancing type as a function of the type of motions of the chassis. Application examples: high speed rotating rotors like reaction wheels used for satellite attitude control or turbine shafts, car engines, pick-and-place robotic manipulators, aerospace mechanisms.

These conditions depend on the type of motion which the chassis is undergoing:

- Translation at constant speed (e.g., inertial frame, train in rectilinear motion at constant speed).
- Translation at accelerated speeds (e.g., rectilinear accelerating train).
- Rotation at constant angular velocity around a fixed axis (e.g., carousel at constant speed).
- Rotation at constant angular velocity around any axis (e.g., spinning of a satellite).
- Accelerated angular velocity at very low angular speed (negligible centrifugal forces and non-negligible Euler forces) around a given axis (e.g., sudden start of a carousel).

Table 1
List of balancing property symbols.

Symbol	Property	Symbol	Property
V	Constant potential energy	σ_A	Constant angular momentum at a specific set of points
Z	Zero-force mechanism	M	No shaking moments exported to the chassis
Eq	Neutral equilibrium	M_A	No shaking moments exported to the chassis at a specific set of points
p	Constant linear momentum	α	Insensitivity to angular accelerations
G	Fixed CoM relative to the chassis	α_A	Insensitivity to angular accelerations around a specific set of points
φ_g	Insensitivity to gravity orientations	J	Configuration-invariant angular momentum and total inertia tensor
F	No shaking forces exported to the chassis	J_A	Configuration-invariant angular momentum and total inertia tensor at a specific set of points
V_g	Constant gravitational potential energy	Ω	Insensitivity to angular velocities
γ	Insensitivity to linear accelerations	Ω_A	Insensitivity to angular velocities around a specific set of points
σ	Constant angular momentum		

Table 2
Balancing properties of all balancing types.

BALANCING PROPERTIES		I	S	SI	F	FI	SF	SFI	M	MI	SM	SMI	FM	FMI	SFM	SFMI
σ, M, α	**M***								σ_A M_A α_A	σ_A M_A α_A	σ_A M_A α_A	σ_A M_A α_A	σ M α	σ M α	σ M α	σ M α
φ_g, p, G, F, V_g, γ	**F***				φ_g p G F V_g γ	φ_g p G F V_g γ	φ_g p G F V_g γ	φ_g p G F V_g γ					φ_g p G F V_g γ	φ_g p G F V_g γ	φ_g p G F V_g γ	φ_g p G F V_g γ
V, z, Eq	**S***		V z Eq	V z Eq			V z Eq	V z Eq			V z Eq	V z Eq			V z Eq	V z Eq
J, Ω	**I***	J_A Ω_A		J_A Ω_A		J Ω		J Ω		J_A Ω_A		J_A Ω_A		J Ω		J Ω

Table 3
Balancing conditions required for any mechanism to be insensitive to the motions of its chassis.

Type of motion of the chassis	Translation	Rotation around a fixed axis	Rotation around any axis
Constant speed	None	**I**	***FI***
Acceleration at low speed	**F**	**M**	**FM**
Acceleration at high speed		**MI**	**FMI**

Table 4
Balancing conditions required for any mechanism in order for it not to export any shaking forces and moments to its chassis moving in translation or rotation.

Type of motion of the chassis	Translation	Rotation
Constant speed	**FM**	**FMI**
Acceleration		

- Accelerated angular velocity at very low angular speed (negligible centrifugal forces and non-negligible Euler forces) around any axis (e.g., tangential shock on a wrist-watch).
- Accelerated angular velocity at high angular speed around a given axis.
- Accelerated angular velocity at high angular speed around any axis.

These two tables show the important role played by the inertial balancing (type **I***) type newly introduced in this article for the design of mechanism which are embarked on a base rotating at non-negligible angular velocities, i.e., leading to significant centrifugal forces. More specifically, these tables illustrate that dynamically balanced mechanisms designed for applications where their chassis is rotating may not be reactionless as they do not necessarily maintain their inertia tensor constant. Additionally, Table 3 shows that it is possible to selectively cancel angular couplings between a mechanism and its frame around specific axes by satisfying appropriate balancing conditions. Identifying the minimum balancing conditions required to achieve precise, rather than global, decoupling would make it possible to design new mechanisms with reduced mass and volume, dedicated to very specific applications.

2.6. Balancing examples

In order to illustrate the different balancing types, we propose in this section to follow the balancing of a 1-DoF baseline mechanism from its initial unbalanced configuration to its *inertial balancing* (type **FMI***). This sequence eases the understanding of the mechanical properties as well as the limitations of the different balancing types introduced.

2.6.1. Example of unbalanced mechanism

The total potential energy of M0, shown in Fig. 3, can be expressed as:

$$V = m_{11} g l_{c11} s_{\theta_1}, \tag{2.11}$$

where θ_1 is the angular displacement of the massless link 1 relative to its chassis; m_{11} is the added point mass attached to link 1 at c_{11}; l_{c11} is the distance between the CoM c_{11} and joint center O_1; g is the gravitational acceleration. One can see from Eq. (2.11) that the total potential energy of M0 is purely gravitational. Also, the gravitational potential energy of M0 is not constant, so the mechanism is not type $\mathbf{S}^*$. The linear momentum of M0, whose chassis is fixed relative to an inertial frame $\mathscr{R}$, can be expressed as:

$$\boldsymbol{p}_{\mathrm{tot}/\mathscr{R}} = m_{11} l_{c11} \dot{\theta}_1 (-s_{\theta_1} \boldsymbol{x} + c_{\theta_1} \boldsymbol{y}). \tag{2.12}$$

According to Eq. (2.12), M0's linear momentum varies with $\dot{\theta}_1$, indicating that M0 is not type $\mathbf{F}^*$. The angular momentum around an arbitrary point A with coordinates (X, Y, Z) that belongs to the chassis of M0 can be expressed as:

$$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{R}} = m_{11} l_{c11} \dot{\theta}_1 (Z(c_{\theta_1} \boldsymbol{x} + s_{\theta_1} \boldsymbol{y}) + (l_{c11} - X c_{\theta_1} - Y s_{\theta_1}) \boldsymbol{z}). \tag{2.13}$$

One can see from Eq. (2.13) that the angular momentum at point A varies with $\dot{\theta}_1$ no matter X, Y or Z, so M0 is not type $\mathbf{M}^*$. Finally, it is obvious that the inertia tensor of M0 is configuration-variant no matter the point A it is expressed, so M0 is not type $\mathbf{I}^*$. One can observe that the mechanism M0 is inertially invariant in the *xy*-plane at joint center O_1. At this latter point, the inertia along the *z*-axis of M0 reduces to $\boldsymbol{J}_{zz} = m_{11} l_{c11}^2$ with a configuration-invariant angular momentum $\boldsymbol{\sigma} = m_1 l_{c11}^2 \dot{\theta}_1 \boldsymbol{z}$. Therefore, at point O_1, it may be shown that M0 is insensitive to angular velocities around the *z*-axis.

2.6.2. Example of type S balanced mechanism

The potential energy of M1, shown in Fig. 4, can be expressed as:

$$V = \frac{1}{2} k (l_{12}^2 + l_{13}^2) + (m_{11} g l_{c11} - k l_{12} l_{13}) s_{\theta_1}, \tag{2.14}$$

where θ_1 is the angular displacement of the massless link 1 relative to its chassis; m_{11} is the added point mass attached to link 1 at c_{11}; k is the stiffness of the zero-length spring that connects link 1 to its chassis between point B and point A; l_{c11} is the distance between the CoM c_{11} and joint center O_1; l_{12} is the distance between joint centers O_1 and O_2; l_{13} is the distance between joint centers O_1 and O_3; g is the amplitude of the gravitational acceleration. According to Eq. (2.1) and Eq. (2.14), the condition to statically balance M1 is:

$$m_{11} g l_{c11} = k l_{12} l_{13}. \tag{2.15}$$

The velocity of the CoM of M1 is the same as that of M0, hence M1 is not type $\mathbf{F}^*$. Similarly, the angular momentum and total inertia tensor of M1 are equal to those of M0, hence M1 is neither type $\mathbf{M}^*$ nor type $\mathbf{I}^*$. Finally, Eq. (2.15) shows that M1 is only type **S**. Note that the static balancing condition of M1 on Earth will not be the same on Mars. Indeed, just as the amplitudes of the gravitational accelerations will differ, so will the static balancing conditions.

2.6.3. Example of type I mechanism balanced around a single point

To make M0 or M1 inertially invariant, one can add a second point mass m_{12} located in c_{12} so that O_1c_{12} is perpendicular to O_1c_{11},

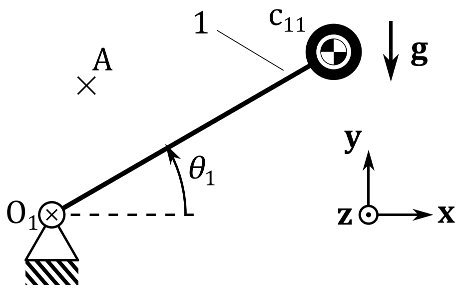


Fig. 3. Mechanism M0 – unbalanced baseline mechanism with no particular properties.

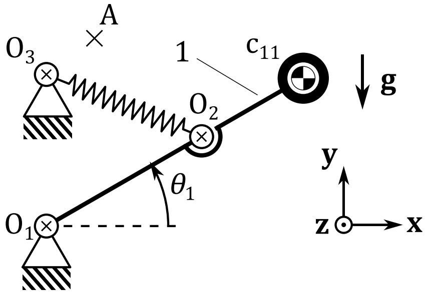


Fig. 4. Mechanism M1 - Type **S**; the pull of gravity is counteracted by a tensioned spring, resulting in a statically balanced mechanism.

which leads to mechanism M2 (Fig. 5). Note that the zero-length spring of M1 has been replaced by a spiral spring of angular stiffness k_θ so that the total fluctuating potential energy of M2 can be expressed as:

$$V = \frac{1}{2} k_\theta \theta_1^2. \tag{2.16}$$

We can also write the linear momentum of M2, which is of the form:

$$\boldsymbol{p}_{\mathrm{tot}/\mathscr{R}} = [-(m_{11} l_{\mathrm{c}11} \mathrm{s}_{\theta_1} + m_{12} l_{\mathrm{c}12} \mathrm{c}_{\theta_1})\boldsymbol{x} + (m_{11} l_{\mathrm{c}11} \mathrm{c}_{\theta_1} - m_{12} l_{\mathrm{c}12} \mathrm{s}_{\theta_1})\boldsymbol{y}]\dot{\theta}_1. \tag{2.17}$$

According to Eqs. (2.16) and (2.17), the total potential energy as well as the linear momentum of M2 are not constant. Hence, M2 is neither type $\mathbf{S}^*$ nor type $\mathbf{F}^*$. The angular momentum of M2 at joint center O_1, belonging to the chassis of the mechanism and fixed relative to the inertial frame $\mathscr{R}$, is:

$$\boldsymbol{\sigma}_{\mathrm{O}_1,\mathrm{tot}/\mathscr{R}} = (m_{11} l_{\mathrm{c}11}^2 + m_{12} l_{\mathrm{c}12}^2)\dot{\theta}_1 \boldsymbol{z}. \tag{2.18}$$

Eq. (2.18) shows that the angular momentum of M2 at point O_1 is configuration-invariant but still varies with $\dot{\theta}_1$ so M2 is not type $\mathbf{M}^*$. Finally, the total inertia tensor of M2 at point O_1 written in the base of the inertial frame $\mathscr{R}$ is:

$$\boldsymbol{J}_{\mathrm{O}_1,\mathrm{tot}/\mathscr{R}} = \begin{bmatrix} m_{11} l_{\mathrm{c}11}^2 + (m_{12} l_{\mathrm{c}12}^2 - m_{11} l_{\mathrm{c}11}^2)\mathrm{c}_{\theta_1}^2 & (m_{12} l_{\mathrm{c}12}^2 - m_{11} l_{\mathrm{c}11}^2)\mathrm{s}_{\theta_1}\mathrm{c}_{\theta_1} & 0 \\ (m_{12} l_{\mathrm{c}12}^2 - m_{11} l_{\mathrm{c}11}^2)\mathrm{s}_{\theta_1}\mathrm{c}_{\theta_1} & m_{12} l_{\mathrm{c}12}^2 - (m_{12} l_{\mathrm{c}12}^2 - m_{11} l_{\mathrm{c}11}^2)\mathrm{c}_{\theta_1}^2 & 0 \\ 0 & 0 & m_{11} l_{\mathrm{c}11}^2 + m_{12} l_{\mathrm{c}12}^2 \end{bmatrix}_{\mathscr{R}} \tag{2.19}$$

From Eq. (2.19), one can see that the condition for M2 to be type $\mathbf{I}^*$ is:

$$m_{12} = m_{11} \frac{l_{\mathrm{c}11}^2}{l_{\mathrm{c}12}^2}, \tag{2.20}$$

Injecting Eq. (2.20) into Eq. (2.19), the expression of the total inertia of M2 simplifies and becomes:

$$\boldsymbol{J}_{\mathrm{O}_1,\mathrm{tot}/\mathscr{R}} = \begin{bmatrix} m_{11} l_{\mathrm{c}11}^2 & 0 & 0 \\ 0 & m_{12} l_{\mathrm{c}12}^2 & 0 \\ 0 & 0 & m_{11} l_{\mathrm{c}11}^2 + m_{12} l_{\mathrm{c}12}^2 \end{bmatrix}_{\mathscr{R}},$$

which is then independent of θ_1 parameter; hence, M2 is type $\mathbf{I}^*$. Note that as M2 is type $\mathbf{I}^*$ in O_1 and its angular momentum is configuration-invariant, one may show that the former is insensitive to all angular velocities from its chassis around any axis passing through point O_1.

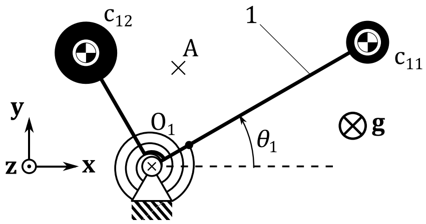


Fig. 5. Mechanism M2 - Type **I** mechanism balanced around joint center: When evaluated at an arbitrary point A, the total inertia tensor of the mechanism varies, i.e., the mechanism is not inertially invariant. When evaluated at joint center O_1, the total inertia tensor is constant, i.e., the mechanism is inertially invariant.

2.6.4. Example of type **F** *balanced mechanism*

To force balance M1, one can add a second point mass m_{12} located in c_{12} so that it belongs to the O_1c_{11} line, see mechanism M3 in Fig. 6. Note that the zero-length spring of M1 has been replaced by a spiral spring of angular stiffness k_θ. The linear momentum of M3 can then be expressed as:

$$\boldsymbol{p}_{\mathrm{tot}/\mathscr{R}} = (m_{11}l_{c11} - m_{12}l_{c12})(-s_{\theta_1}\boldsymbol{x} + c_{\theta_1}\boldsymbol{y})\dot{\theta}_1. \tag{2.21}$$

One can see from Eq. (2.21) that the linear momentum of M3 is equal to zero if:

$$m_{11}l_{c11} = m_{12}l_{c12}. \tag{2.22}$$

As the CoM of M3 is fixed relative to its chassis, its fluctuating total potential energy can be expressed as:

$$V = \frac{1}{2}k_\theta\theta_1^2. \tag{2.23}$$

According to Eq. (2.23), the total potential energy of M3 varies with θ_1, hence M3 is not type $\mathbf{S}^*$. From Eq. (2.22), M3 is at least type $\mathbf{F}^*$. As M3 is of type $\mathbf{F}^*$, its angular momentum evaluated at any point A belonging to its chassis can be expressed as:

$$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{R}} = \left(m_{11}l_{c11}^2 + m_{12}l_{c12}^2\right)\dot{\theta}_1\boldsymbol{z}. \tag{2.24}$$

Eq. (2.24) shows that the angular momentum of M2 varies with $\dot{\theta}_1$. M2 is not type $\mathbf{M}^*$, hence not type $\mathbf{FM}^*$. Finally, as for M0 and M1, M3 has an inertia tensor that is configuration-variant no matter the point A it is expressed, so M3 is not type $\mathbf{I}^*$.

2.6.5. Example of type **M** *balanced mechanism*

To moment balance M1, one can couple it with another mechanism that performs an opposite rotational motion, see Fig. 7. M4's total angular momentum at point A with coordinates (X, Y, Z) can be expressed as:

$$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{R}} = \boldsymbol{\sigma}_{\mathrm{A},1/\mathscr{R}} + \boldsymbol{\sigma}_{\mathrm{A},2/\mathscr{R}} = \mathbf{0}. \tag{2.25}$$

Assuming that:

$$m_{11} = m_{21},$$

$$l_{c11} = l_{c21},$$

the linear momentum of M4 can be expressed as:

$$\boldsymbol{p}_{\mathrm{tot}/\mathscr{R}} = -2m_{11}l_{c11}\dot{\theta}_1 s_{\theta_1}\boldsymbol{y}. \tag{2.26}$$

Remark that M4 is not force balanced (type $\mathbf{F}^*$) as the expression of its linear momentum, see Eq. (2.26), varies with $\dot{\theta}_1$. The angular momentum at point A can be expressed as:

$$\boldsymbol{\sigma}_{\mathrm{A,tot}/\mathscr{R}} = 2m_{11}l_{c11}\dot{\theta}_1 s_{\theta_1}(Z\boldsymbol{x} - X\boldsymbol{z}). \tag{2.27}$$

Eq. (2.27) shows that the angular momentum of M4 is zero for any point A belonging to line $\Delta(\mathrm{O}, \boldsymbol{y})$, i.e., where $X = Z = 0$. M4 is thus moment balanced (type **M**) at any point A belonging to $\Delta(\mathrm{O}, \boldsymbol{y})$ but not at any other point. Finally, as for M0, M2 and M3, M4 has an inertia tensor that is configuration-variant no matter the point A it is expressed, so M4 is not type $\mathbf{I}^*$. One might imagine that M4 could be suspended in a gravitational field in the direction of Δ by a single thread colinear with Δ. In that case, the chassis would not move under the motion of the mechanism as long as $\ddot{\theta}_1$ remains limited.

2.6.6. Example of type **FI** *balanced mechanism*

To force balance M1 while keeping it inertially invariant, one solution is to introduce a third mass and redistribute all of them every 120°, having $m_{11} = m_{12} = m_{13}$ and $l_{c11} = l_{c12} = l_{c13}$, see Fig. 8. The CoM of M5 is fixed relative to its chassis due to axial symmetry reasons, and its configuration-invariant inertia tensor in O_1 expressed in the inertial reference frame $\mathscr{R}$ is equal to:

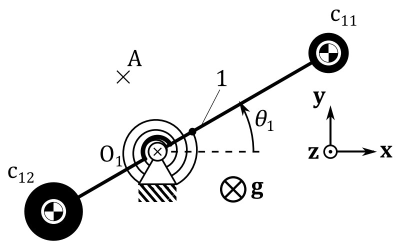


Fig. 6. Mechanism M3 – Type **F**; the CoM displacement is canceled by the introduction of a counterweight, resulting in a force balanced mechanism.

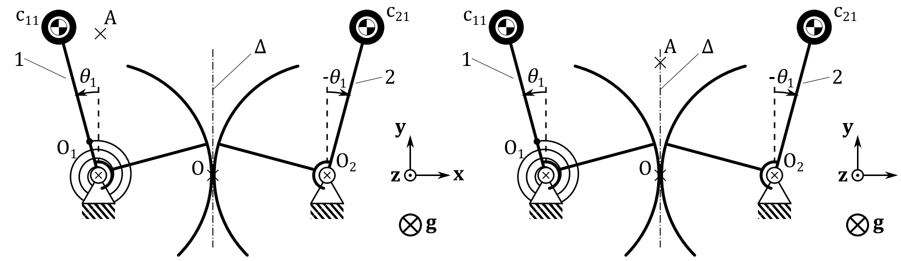


Fig. 7. Mechanism M4 – Type **M**; the introduction of a counter-rotating mechanism cancels the angular momentum at a specific set of points, forming line Δ, resulting in a moment balanced mechanism at points belonging to line Δ. **Left**: the angular momentum of the mechanism varies when evaluated at point A. **Right**: the angular momentum of the mechanism is constant when evaluated at point A, the latter belonging to line Δ.

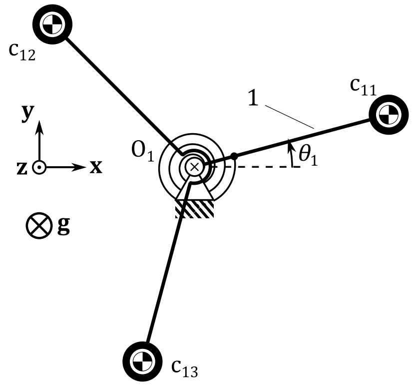


Fig. 8. Mechanism M5 – Type **FI**: the CoM displacement is canceled by disposing three identical masses distributed at 120° at equal distance from point O_1, resulting in a force balanced and inertially invariant mechanism.

$$\boldsymbol{J}_{O_1,\mathrm{tot}/\mathscr{R}} = \begin{bmatrix} \frac{3}{2}m_{11}l_{c11}^2 & 0 & 0 \\ 0 & \frac{3}{2}m_{11}l_{c11}^2 & 0 \\ 0 & 0 & 3m_{11}l_{c11}^2 \end{bmatrix}_{\mathscr{R}} . \tag{2.28}$$

The angular momentum of M5 at any point belonging to its chassis can be expressed as:

$$\boldsymbol{\sigma}_{O_1,\mathrm{tot}/\mathscr{R}} = 3m_{11}l_{c11}^2\dot{\theta}_1 z. \tag{2.29}$$

From symmetry reasons and results from Eqs. (2.28) and (2.29), M5 is at least type **FI***. As the angular momentum of M5 is configuration-invariant but still varies with $\dot{\theta}_1$, it is not type **M**. Note that because its CoM is fixed and its inertia tensor and angular momentum are configuration-invariant, M5 is insensitive to all linear accelerations as well as all angular velocities. Observe that the M5 mechanism is typically a balance-spring oscillator used in most mechanical watches today.

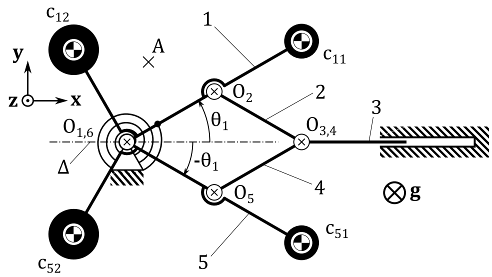


Fig. 9. Mechanism M6 – Type **MI**: the introduction of a counter-rotating mechanism cancels the angular momentum at a specific set of points, forming line Δ, resulting in a moment balanced mechanism at points belonging to line Δ. The inertial invariance of M6 is only satisfied at joint centers O_1 and O_6 (noted $O_{1,6}$).

2.6.7. Example of type **MI** mechanism balanced around a single point

Mechanism M6 presents a way to moment balance the inertially invariant mechanism M2 seen in Section 2.6.3 by coupling two of them performing opposite rotations, see Fig. 9. Section 2.6.3 already explained that M6 is inertially invariant at joint centers $O_{1,6}$, if:

$$m_{12} = m_{11}\left(\frac{l_{c11}}{l_{c12}}\right)^2 \tag{2.30}$$

$$m_{52} = m_{51}\left(\frac{l_{c51}}{l_{c52}}\right)^2 \tag{2.31}$$

Using mass conditions from Eqs. (2.30) and (2.31), the angular momentum of M6 at the same joint center is equal to:

$$\boldsymbol{\sigma}_{O_1,\mathrm{tot}/\mathscr{R}} = \boldsymbol{\sigma}_{O_1,1/\mathscr{R}} + \boldsymbol{\sigma}_{O_1,2/\mathscr{R}} = 2\left(m_{11}l_{c11}^2 - m_{51}l_{c51}^2\right)\dot{\theta}_1\boldsymbol{z}. \tag{2.32}$$

One can see that Eq. (2.32) is equal to zero if:

$$m_{11}l_{c11}^2 = m_{51}l_{c51}^2. \tag{2.33}$$

If the conditions given by Eqs. (2.30),(2.31) and (2.33) are satisfied, M6 is then of type **MI*** and is insensitive to angular velocities and accelerations around any axis through point O_1. One may observe that M6 is still not force balanced as its linear momentum varies with $\dot{\theta}_1$:

$$\boldsymbol{p}_{\mathrm{tot}/\mathscr{R}} = \boldsymbol{p}_{1/\mathscr{R}} + \boldsymbol{p}_{5/\mathscr{R}} = m_{11}l_{c11}\left[\left(-s_{\theta_1}\left(1+\frac{l_{c51}}{l_{c11}}\right) - c_{\theta_1}\left(\frac{l_{c11}}{l_{c12}} + \frac{l_{c51}^2}{l_{c11}l_{c52}}\right)\right)\boldsymbol{x} + \left(c_{\theta_1}\left(1-\frac{l_{c51}}{l_{c11}}\right) + s_{\theta_1}\left(-\frac{l_{c11}}{l_{c12}} + \frac{l_{c51}^2}{l_{c11}l_{c52}}\right)\right)\boldsymbol{y}\right]\dot{\theta}_1. \tag{2.34}$$

2.6.8. Example of type **FM** balanced mechanism

A solution to dynamically balance M4 is to force balance it as presented in M3 and moment balance it as in M4, see mechanism M7 in Fig. 10.

M7 force balancing

M7 force balancing starts by writing the force balancing condition:

$$\boldsymbol{p}_{\mathrm{tot}/\mathscr{R}} = -(m_{11}l_{c11} - m_{12}l_{c12})\dot{\theta}_1(c_{\theta_1}\boldsymbol{x} + s_{\theta_1}\boldsymbol{y}) + (m_{21}l_{c21} - m_{22}l_{c22})\dot{\theta}_2(c_{\theta_2}\boldsymbol{x} + s_{\theta_2}\boldsymbol{y}) = \boldsymbol{0}, \tag{2.35}$$

where $\dot{\theta}_2 = r_1\dot{\theta}_1/r_2$ comes from the rolling without slipping condition in O between links 1 and 2; r_1 and r_2 are the distances between point O and joint centers O_1 and O_2. Hence, using Eq. (2.35) the force balance conditions for M7 become:

$$m_{11}l_{c11} = m_{12}l_{c12}, \tag{2.36}$$

$$m_{21}l_{c21} = m_{22}l_{c22}. \tag{2.37}$$

M7 moment balancing

M7 moment balancing at O starts by writing the moment balancing condition:

$$\boldsymbol{\sigma}_{O,\mathrm{tot}/\mathscr{R}} = \left(\left(m_{11}l_{c11}^2 + m_{12}l_{c12}^2\right)\dot{\theta}_1 - \left(m_{21}l_{c21}^2 + m_{22}l_{c22}^2\right)\dot{\theta}_1\frac{r_1}{r_2}\right)\boldsymbol{z} = \boldsymbol{0}. \tag{2.38}$$

Hence, using Eq. (2.38) to moment balanced M7, one can find that:

$$m_{11}l_{c11}^2 + m_{12}l_{c12}^2 = \left(m_{21}l_{c21}^2 + m_{22}l_{c22}^2\right)\frac{r_1}{r_2}. \tag{2.39}$$

M7 dynamic balancing

M7 dynamic balancing is achieved when Eqs. (2.36), (2.37) and (2.39) are satisfied. One can note that we have to solve a system

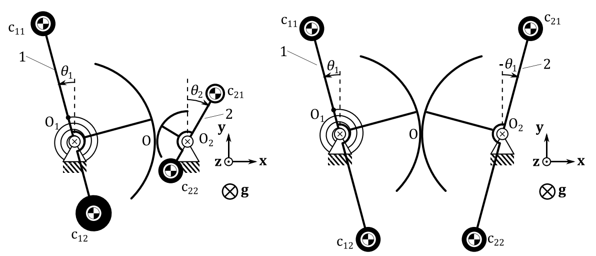


Fig. 10. Mechanism M7 – Type **FM**; linear and angular momenta are canceled by redistributing mass and inertia as well as introducing a counter-rotating mechanism, resulting in a dynamically balanced mechanism. **Left**: with an arbitrary transmission ratio. **Right**: with a unitary transmission ratio.

with the three aforementioned equations and ten unknowns (m_{11}, m_{12}, m_{21}, m_{22}, l_{c11}, l_{c12}, l_{c21}, l_{c22}, r_1, r_2), so there is an infinite number of possible configurations to dynamically balance M7. To see if M7 is type **FMI,** one can write its total inertial tensor at point O expressed in $\mathscr{F}$:

$$\boldsymbol{J}_{\mathrm{O,tot}/\mathscr{F}} = \begin{bmatrix} \boldsymbol{J}_{\mathrm{O},xx} & \boldsymbol{J}_{\mathrm{O},xy} & 0 \\ \boldsymbol{J}_{\mathrm{O},xy} & \boldsymbol{J}_{\mathrm{O},yy} & 0 \\ 0 & 0 & \boldsymbol{J}_{\mathrm{O},zz} \end{bmatrix}_{\mathscr{F}}, \tag{2.40}$$

where:

$$\boldsymbol{J}_{\mathrm{O},xx} = \left(m_{11}l_{c11}^2 + m_{12}l_{c12}^2\right)\mathrm{c}_{\theta_1}^2 + \left(m_{21}l_{c21}^2 + m_{22}l_{c22}^2\right)\mathrm{c}_{\theta_2}^2, \tag{2.41}$$

$$\boldsymbol{J}_{\mathrm{O},yy} = \left(m_{11}l_{c11}^2 + m_{12}l_{c12}^2\right)\mathrm{s}_{\theta_1}^2 + \left(m_{21}l_{c21}^2 + m_{22}l_{c22}^2\right)\mathrm{s}_{\theta_2}^2 + (m_{11}+m_{12})r_1^2 + (m_{21}+m_{22})r_2^2, \tag{2.42}$$

$$\boldsymbol{J}_{\mathrm{O},zz} = m_{11}l_{c11}^2 + m_{12}l_{c12}^2 + m_{21}l_{c21}^2 + m_{22}l_{c22}^2 + (m_{11}+m_{12})r_1^2 + (m_{21}+m_{22})r_2^2, \tag{2.43}$$

$$\boldsymbol{J}_{\mathrm{O},xy} = \left(m_{11}l_{c11}^2 + m_{12}l_{c12}^2\right)\mathrm{s}_{\theta_1}\mathrm{c}_{\theta_1} + \left(m_{21}l_{c21}^2 + m_{22}l_{c22}^2\right)\mathrm{s}_{\theta_2}\mathrm{c}_{\theta_2}. \tag{2.44}$$

The components of the total inertia tensor of M7 vary with θ_1 and θ_2 except $\boldsymbol{J}_{\mathrm{O},zz}$, hence M7 is not inertially balanced (not type **FMI**). Also, remark that if we consider $r_1 = r_2$ (see Fig. 10 right), the total tensor of inertia simplifies and becomes diagonal. However, M7's moments of inertia will continue to vary with θ_1. This means that rotations of the chassis of M7 around the x and y axes may influence its pose and dynamics, while rotations around axes normal to its plane (z-axis) may not.

2.6.9. Example of type **FMI** *balanced mechanism*

A solution to inertially balance M7 is to add a third mass to links 1 and 2, as explained in Section 2.6.6 and illustrated by mechanism M8 in Fig. 11. This will make the moment of inertia of links 1 and 2 invariant to rotations about any axis. In order to reach inertial balancing, the following conditions must be respected:

$$\left.\begin{aligned} m_{11} &= m_{12} = m_{13} \\ m_{21} &= m_{22} = m_{23} \\ l_{c11} &= l_{c12} = l_{c13} \\ l_{c21} &= l_{c22} = l_{c23} \end{aligned}\right\} \begin{array}{c}\text{Force balancing} \\ \& \\ \text{Inertial invariance}\end{array} \tag{2.45}$$

$$\left. m_{11}l_{c11}^2 = \frac{r_1}{r_2}m_{21}l_{c21}^2 \right\} \text{Moment balancing} \tag{2.46}$$

The total inertia tensor in O of the mechanism M8 is then of the form:

$$\boldsymbol{J}_{\mathrm{O,tot}/\mathscr{R}} = \begin{bmatrix} \boldsymbol{J}_{\mathrm{O},xx} & 0 & 0 \\ 0 & \boldsymbol{J}_{\mathrm{O},yy} & 0 \\ 0 & 0 & \boldsymbol{J}_{\mathrm{O},zz} \end{bmatrix}_{\mathscr{R}}, \tag{2.47}$$

where:

$$\boldsymbol{J}_{\mathrm{O},xx} = \frac{3}{2}\left(m_{11}l_{c11}^2 + m_{21}l_{c21}^2\right), \tag{2.48}$$

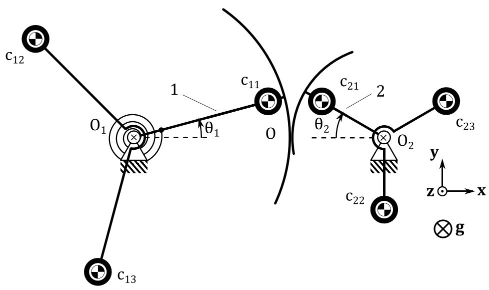


Fig. 11. Mechanism M8 – Type **FMI**; linear and angular momenta are canceled by redistributing mass and inertia and introducing a counter-rotating mechanism, the mechanism's total tensor of inertia is also made configuration invariant resulting in an inertially balanced mechanism.

$$J_{O,yy} = \frac{3}{2}\left(m_{11}l_{c11}^2 + m_{21}l_{c21}^2\right) + 3\left(m_{11}r_1^2 + m_{21}r_2^2\right), \tag{2.49}$$

$$J_{O,zz} = 3\left(m_{11}l_{c11}^2 + m_{21}l_{c21}^2 + m_{11}r_1^2 + m_{21}r_2^2\right). \tag{2.50}$$

From Eqs. (2.45) and (2.46), M8 is dynamically balanced (type **FM***) and from Eqs. (2.47) to (2.50), one can see that the total inertia tensor of M8 is configuration invariant, so M8 is inertially invariant (type **I***). Hence, M8 is a type **FMI** mechanism. As a result, the motion of the chassis has no effect on this mechanism.

2.6.10. *Synthetic balancing taxonomy illustrated with examples*

Fig. 12 shows where the mechanism examples described in Section 2.5 lie within the taxonomy. Each mechanism in Fig. 12 is represented with respect to the field of gravity.

From Table 5, one can read that the represented type **FI** mechanism, which is force balanced and inertially invariant, (1) is insensitive to gravity change of orientation, (2) has a constant linear momentum, (3) has a constant gravitational potential, (4) does not transmit shaking forces to its chassis, (5) has a fixed CoM relative to its chassis, (6) is insensitive to linear accelerations, (7) has a constant total inertia tensor, and (8) is insensitive to angular velocities.

3. Discussion

3.1. *Situation of the state-of-the-art within the proposed taxonomy*

As early as the 1960s [46], the terms *shaking force balance* and *shaking moment balance* were established. At the same time, literature on the balancing of rotary axes uses the term *static balance* to denote a CoM of an axle that is aligned with the rotation axis, which in that case coincides with the meaning of force balance. Later, the term static balance was used in mechanisms to refer to mechanisms with a fixed CoM [3]. In these cases, it clearly referred to *static balancing* without using other forms of potential energy. However, this distinction is sometimes omitted [13], leading to undesirable confusion. There, it is claimed that *static balancing* is a superset of *force balancing*, implying that *static balancing* is a necessary condition for *force balancing*. Here we reaffirm the definition of *shaking force balancing*, which considers solely the linear momentum conservation of a mechanism. *Static balance* and *force balance* are disjointed concepts that can be brought about by the same procedure, provided that there are no other potential energy sources than gravity.

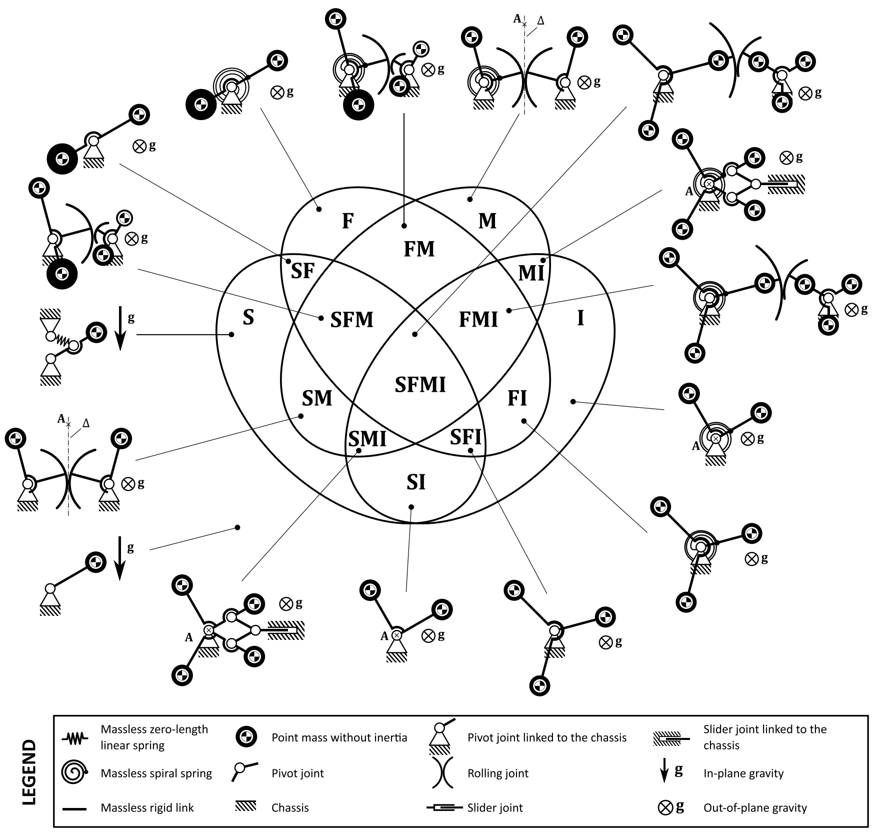
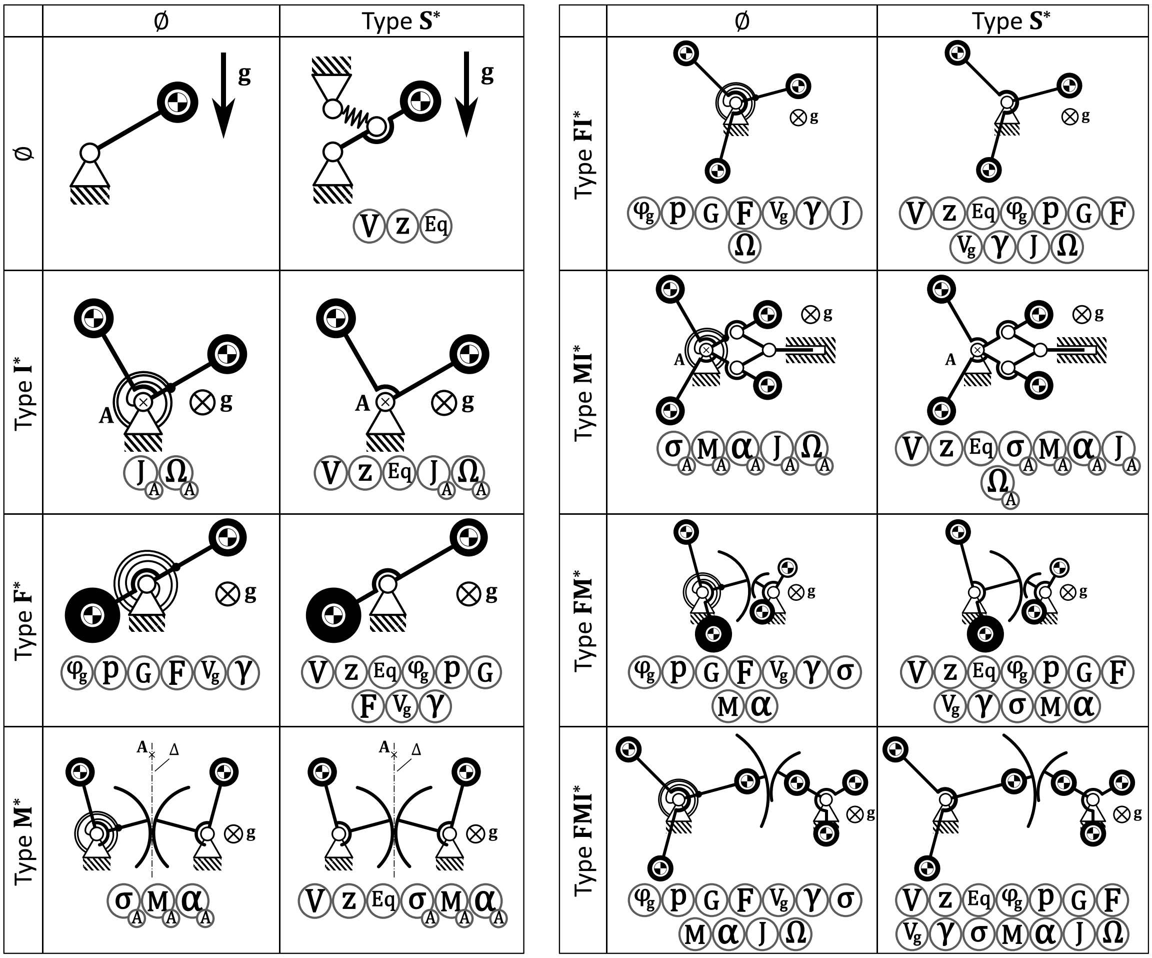


Fig. 12. Location of the mechanisms studied in Section 2.5 within the new balancing taxonomy.

Table 5
Intrinsic balancing properties of the examples of Section 2.6 classified according to the 15 balancing types.



The distinction between *force* and *static balance* is of especial relevance when studying mechanisms with multiple potential energy sources, such as force balanced oscillators [7,21].

In a landmark paper, Ricard and Gosselin [3] proposes a balancing technique that combines dynamically balanced modules with a constant total moment of inertia into larger DoF dynamically balanced mechanisms. In turn, this work was continued with spatially moving and dynamically balanced mechanisms [4]. These papers use the conditions for *inertial balance* but do not strictly define them, simply referring to them as *dynamic balance for a moving frame*, potentially leading to confusion. Some literature seems to suggest that *dynamic balance* will automatically lead to a constant total inertia in the plane of movement [4] or that a dynamically balanced mechanism will conserve its momentum in a moving frame [8]. To clarify, we propose to define *dynamic balance* with respect to the chassis of the mechanism that is fixed to an inertial frame and to add a separate condition on the total moment of inertia for inertial invariance.

One possible explanation for assigning additional attributes to *dynamic balance* is that three prominent dynamically balanced linkages - the two four-bar linkages [3] and the crank-slider mechanism [47] - also have a constant total moment of inertia in the plane of movement [20,48] and are thus *inertially balanced* in that plane. A further explanation is that linear accelerations and velocities of the base do not affect the pose of a force balanced mechanism, whereas angular velocities of the base do affect a moment balanced mechanism, unless constant inertia conditions are satisfied.

It should therefore be noted that the frame of reference with respect to which the balance conditions are formulated plays a role here. Dynamic balance conditions are typically given with respect to a fixed point or frame. This means that these conditions are no longer applicable or valid if the point or frame of reference moves.

The implications of this nuance are relevant when inertially balanced mechanisms are used in modules to form higher DoF mechanisms. Stacking dynamically balanced modules could lead to varying inertia, requiring additional balancing measures. Also, for mechanisms whose chassis undergo high angular velocities, such as watches or satellites, the difference between *dynamic* and *inertial balance* may result in significant deviations.

To the knowledge of the authors, no literature deals with *moment balance* alone, i.e., without *force balance*. The current work emphasizes that it is possible to obtain solely moment balanced **M** mechanisms but that their application is rather limited.

3.2. Scope and novelty

The proposed new taxonomy deals with the classification of exact balancing techniques but does not encompass *approximate, harmonic, active, rotary, input, optimal*, or *partial* balancing approaches. This taxonomy is useful for selecting the required balancing type required for the design of specific mechanisms according to their working environment (e.g., moving chassis potentially disturbing the mechanism motion vs highly stable chassis requiring force and moment shaking cancelling).

When combined with dynamic balancing (type **FM***), the newly introduced inertial invariance (type **I***) leads to complete insensitivity of the mechanisms to their chassis motion and to the cancelling of force and moment exportation when the chassis is in rotation, even in the presence of non-negligible centrifugal forces. We called this important new category *Inertial balancing* (type **FMI***). To the knowledge of the authors, the 8 balancing types which have inertial invariance were not addressed and classified as such in the previous literature, although they are determinant for the designer as shown in Table 3 and 4.

4. Conclusion

Although the topic of mechanism balancing has been thoroughly dealt with in the literature, this article brings additional clarity to the known categories. This is done by taking into account the effects of the angular velocities of the supporting frame of the mechanisms. These effects include the resulting fictious forces acting on the mechanisms as well as the shaking moments exported by the mechanism onto its supporting structure due to changes in its inertia during motion. This has led to the introduction of a new balancing type called *inertial invariance*.

The exploration of all the combinations of the 4 primary balancing types – *static, force, moment,* and a newly introduced *inertial invariance* – led to 15 types, i.e., 4 *pure* types and 11 *blended* types. The resulting taxonomy is represented as a 4-set Venn diagram. Each type is provided with its definition, the necessary conditions to be satisfied, the list of resulting mechanical properties, a 1-DOF mechanism example, and a label composed of four letters: S, F, M, I. This refined categorization allows for a univocal discrimination between mechanisms and the identification of their respective mechanical properties.

Within this new taxonomy, the well-known *dynamic balancing* is defined as a blended type, resulting from the intersection of *force* (**F***) and *moment* (**M***) primary types. In addition, the well-known *static balancing* (**S***) and *force balancing* (**F***) are defined as independent primary balancing types which intersect one another, constituting a new blended type labelled **SF***.

The term *inertial balancing* was introduced for the blended type labelled **FMI***. Mechanisms of this type are fully decoupled from their supporting frame, including the cases where they are subject to linear and angular accelerations as well as angular velocities. Such mechanisms are completely reactionless – they do not export any shaking forces or moments onto their chassis – and can be seen as rigid bodies regardless of the type of motion of their supporting frames.

By filling some conceptual gaps, this work also resolves some of the inconsistencies in the terms and definitions found in the literature. Thanks to elementary 1-DoF mechanisms exemplifying each balancing type and tables linking the balancing properties required for a mechanism to the corresponding balancing types, this article is meant to be a straightforward tool dedicated both to novice and expert designers.

Follow-up work will cover the publication of a complete demonstration of the properties associated with the defined balancing types, as they are beyond the scope of the present conceptual article.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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