# Edge colorings of planar graphs 



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# EDGE COLORINGS OF PLANAR GRAPHS 

## DISSERTATION

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by

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## Preface

This thesis contains new graph theoretical research results on three different types of edge colorings of planar graphs, which were obtained by the author with collaborators between March 2019 and March 2023.

Apart from an introductory chapter (Chapter 1), the readers will find four closely related chapters (Chapters 2-5). In the introductory chapter, we give a short introduction to the topic, with some background and the necessary terminology and notation to understand the rest of the thesis.

In the four technical chapters we focus on three different types of edge colorings, and we restrict ourselves throughout the thesis to planar graphs. Chapter 2 focuses on list edge coloring of planar graphs. Chapters 3 and 4 focus on signed edge coloring of planar graphs. Chapter 5 focuses on edge DP-coloring of planar graphs.

The results in Chapter 5, the first main lemma in Chapter 3, and the first theorem in Chapter 4 were obtained while the author of this thesis was working as a PhD student in Northwestern Polytechnical University in Xi'an, China. The results in Chapter 2, the second main lemma in Chapter 3, and the second theorem in Chapter 4 were obtained while she was working as a visiting joint PhD student at the University of Twente.

The papers associated with these four research chapters have been listed below and have been published in (or submitted to) scientific journals.

## Papers underlying this thesis

[1] List edge colorings of planar graphs without non-induced 7-cycles, submitted (with H.J. Broersma, Y. Lu and S. Zhang).
(Chapter 2)
[2] Signed planar graphs with $\Delta \geq 8$ are $\Delta$-edge-colorable, Discrete Mathematics 344 (2023), 112567 (with Y. Lu and S. Zhang).
(Chapters 3 and 4)
[3] Edge coloring of signed planar graphs in which each 6-cycle has at most one chord, submitted (with H.J. Broersma, Y. Lu and S. Zhang). (Chapters 3 and 4)
[4] Edge DP-coloring in planar graphs, Discrete Mathematics 344 (2021), 112314 (with Y. Lu and S. Zhang).
(Chapter 5)

## Another recent joint paper by the author

[1] Edge coloring of signed graphs, Discrete Applied Mathematics 182 (2020), 234-242 (with Y. Lu, R. Luo, D. Ye and S. Zhang).

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## Chapter 1

## Introduction

In this chapter we will introduce the central concepts of this thesis, but we start with some background and intuition on the historical motivation behind the topic of graph coloring. The formal definitions will follow later.

## Graph Theory

Graph Theory is a relatively young branch of mathematics, but has been seriously studied for several hundreds of years. It has developed into a mature research field, and it has many applications in other scientific disciplines and different application areas.

Graph Theory is concerned with the study of (binary) relationships between objects. These objects are usually represented by vertices, with one vertex for each object; the binary relationships between pairs of objects are represented by edges, with one edge for each pair of related objects. The mathematical notion of a graph then consists of these two sets: a set of vertices and a set of edges, where each edge consists of a pair of vertices representing a related pair of objects. We assume here that the relationship is symmetric; otherwise, we have to take care of the direction of the relationship. As an example of a situation in which graphs turn up naturally, we next consider an application which has been the main driving force behind graph coloring. In fact, it looks more like a toy problem.

## Map coloring

Suppose we have a map with many regions, and we want to distinguish the regions by assigning a color to each region in such a way that neighboring regions (which share a stretch of border, not just one common point) receive different colors. How many colors suffice to obtain a valid coloring of the regions?

If we represent the regions by vertices of a graph and the pairs of neighboring regions by edges of the graph, we can translate the above problem into a graph coloring problem in a straightforward way. We have to assign a color to each vertex in such a way that there are no edges for which both vertices receive the same color. We come back to this example later.

So far the translation into a graph problem seems like a cheap trick or a trivial exercise, but the nice thing is that the same model can be used for many different settings, not only for any kind of map. We give another example.

Suppose we consider a number of base stations for mobile communication, and we want to avoid interference by assigning different operating frequencies to base stations that are so close to one another that they are likely to interfere. To use the frequencies in an economic way, we want to use as few different frequencies as possible. If we represent the base stations by vertices, the interfering pairs by edges, and the frequencies by colors, we are in the same situation as in the above map coloring problem. It is not hard to imagine other settings in which conflicting pairs of objects lead to similar graph coloring problems.

## Planar graphs

Let us turn back to the map coloring problem in order to introduce the notion of a planar graph in an intuitive way. In this map coloring problem, we are basically considering a 2D map (apart from the thickness of the piece of paper on which the map has been printed), showing the regions and their borders. Suppose we draw one point in the interior of every region and we connect two points by one line segment or curve if the associated regions share a stretch of border. Then under mild assumptions we can do this in such a way that
different line segments or curves do not intersect, except at their endpoints. The drawing we obtain is then referred to as a plane embedding of the graph that represents the map. The vertices of this graph are drawn as distinct points in the plane, and the edges of this graph are drawn as line segments or curves in the plane, in such a way that different line segments or curves do not intersect, except at their endpoints. If a graph admits such a plane embedding, then it is called a planar graph. Planar graphs have nice structural properties which are known since Leonhard Euler in 1758 established what is now commonly known as Euler's Formula. We will come back to this later and use this formula frequently throughout this thesis.

## The Four Color Theorem

The study of graph coloring has historically been closely linked to the study of planar graphs and the Four Color Theorem. The Four Color Theorem states that any map (planar graph) can be colored with at most four colors in such a way that no two adjacent regions (vertices) receive the same color. This theorem was first conjectured in 1852 by Francis Guthrie. In 1879 Alfred Kempe [39] claimed to have proven this theorem with a proof that relied on a technique called "Kempe chains", but his proof was later found to contain an error by Percy Heawood in 1890. Despite this, the method of Kempe chains remains a key ingredient in the theory of graph coloring. In fact, by using this technique one can relatively easily confirm that the vertices of a planar graph can be colored with five colors in such a way that no two adjacent vertices receive the same color. Perhaps surprisingly, it took another almost hundred years before the Four Color Theorem was confirmed. The Four Color Theorem was lacking a formal proof until 1976, when a computer-assisted proof was published by Kenneth Appel and Wolfgang Haken [2]. In fact, all the currently known proofs rely partly on computer-assisted checks of many configurations, and it is still an open problem to find a pure combinatorial proof.

The Four Color Theorem (or even more its conjectured validity) has inspired a lot of research in the field of graph theory. Many variants and generalizations of the graph coloring problem have been proposed, involving concepts like edge coloring, total coloring, list (edge) coloring, signed (edge)
coloring, DP (edge) coloring, among others. These variants and generalizations have their own unique properties and applications. We should emphasize here that the results of this thesis are mainly of theoretical relevance. Therefore we refrain from giving details about applications. It might be clear from the two examples we gave before that graph coloring problems turn up in many different application areas.

Apart from the above example of frequency assignment, graph coloring problems have applications in wavelength assignment, network flow optimization, routing and traffic scheduling, social network analysis, and resource allocation in parallel computing, among others. The study of graph coloring in all its forms continues to be an active area of research in graph theory and related fields.

In the sequel, we will encounter several different variants of edge coloring. Before we are going to explore the relevant concepts and background in different sections, we first need to introduce some essential terminology and notation.

### 1.1 Terminology and notation

All the ordinary graphs (or simply, graphs) and signed graphs we consider in this thesis are finite and contain no loops or parallel edges. For terminology and notations not defined here we follow the modern textbook of Bondy and Murty [6].

A graph is a pair $G=(V, E)$, where $V$ is a nonempty set whose elements are called vertices, and $E$ is a set of (unordered) pairs of vertices whose elements are called edges. We take the liberty to use $u v$ to denote the edge consisting of the pair $\{u, v\}$ if no confusion can arise. The vertices $u$ and $v$ of an edge $u v$ are called the endpoints of the edge. The endpoints of an edge are said to be incident with the edge, and an edge is also said to be incident with its endpoints. Two vertices which are incident with a common edge are called adjacent, as are two edges which are incident with a common vertex. Two distinct adjacent vertices are also called neighbors.

Let $G=(V, E)$ be a graph. Then the degree $d_{G}(x)$ of $x \in V$ is the number of edges of $G$ incident with $x$. We use $\Delta(G)=\max \left\{d_{G}(v) \mid v \in V\right\}$ to denote the maximum degree of $G$, and $\delta(G)=\min \left\{d_{G}(v) \mid v \in V\right\}$ to denote the minimum degree of $G$. A graph $G$ is regular if all the vertices of $G$ are of equal degree. In particular, if every vertex of $G$ has degree $r$, then $G$ is called $r$-regular.

For a vertex $v \in V(G)$, we use $N_{G}(v)$ and $E_{G}(v)$ (or simply $N(v)$ and $E(v)$ if no confusion can occur) to denote the set of neighbors of $v$, and the set of edges incident with $v$, respectively. For two vertices $u, v$, let $N_{G}(u, v)=$ $N_{G}(u) \cup N_{G}(v)$. For $S \subseteq V(G)$, let $N_{G}(S)=\bigcup_{v \in S} N_{G}(v)$. We use the shorthand $d$-vertex ( $d^{+}$-vertex, $d^{-}$-vertex, respectively) to denote a vertex with degree $d$ (at least $d$, at most $d$, respectively), and we let $V_{d}(G)\left(V_{d^{+}}(G), V_{d^{-}}(G)\right.$, respectively) be the set of $d$-vertices ( $d^{+}$-vertices, $d^{-}$-vertices, respectively) in $G$. If an edge of $G$ has exactly one endpoint with degree 1 in $G$, we call it a pendant edge.

A walk in $G$ is a sequence $v_{0} e_{1} v_{1} \ldots v_{k-1} e_{k} v_{k}$ of vertices and edges such that the edge $e_{i}$ is incident with the vertices $v_{i-1}$ and $v_{i}$ for $i=1, \ldots, k$. The walk is closed if $v_{0}=v_{k}$ and is open otherwise. A trail is a walk in which all edges are distinct. A path is a trail with no repeated vertex. A cycle is a closed walk of length at least three in which the vertices are distinct except for the first and the last vertex. The length of a cycle is the number of its edges. A cycle of length $k$ is called a $k$-cycle. A chord of a cycle $C$ is an edge in $E(G) \backslash E(C)$ both of whose ends lie on $C$.

A graph is called complete if all its vertices are pairwise adjacent, and it is called non-complete otherwise. We use $K_{n}$ to denote a complete graph on $n$ vertices.

A matching in a graph is a set of edges no two of which share an endpoint. This is sometimes called a set of independent edges.

A graph is said to be connected if there exists a path between every pair of vertices in the graph. Throughout the thesis we assume that the graphs we consider are connected.

If $e$ is an edge of a given graph $G=(V, E)$, we use the notation $G-e$ to indicate the graph obtained from $G$ by removing the edge $e$. If $G$ is connected and $G-e$ is disconnected, then we call $e$ a cut-edge of $G$. Similarly, if $v$ is a
vertex of a graph $G$ on at least two vertices, then we use the notation $G-v$ to indicate the graph obtained from $G$ by removing the vertex $v$ together with all the edges incident to $v$. If $G$ is connected and $G-v$ is disconnected, then we call $v$ a cut-vertex of $G$.

By contracting an edge $e$ of a graph $G$, we mean deleting $e$ from $G$ and identifying its endpoints, replacing any resulting multiple edges by single edges. A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from $G$ by repeatedly deleting edges, deleting vertices and contracting edges. A graph $G$ is called $H$-minor free if $G$ has no minor which is isomorphic to $H$.

A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane (using points to represent its vertices and line segments or curves to represent its edges) in such a way that (the line segments or curves representing) its edges intersect only at their endpoints. Given such a plane embedding of a planar graph $G$, the plane is divided into connected regions which are separated from each other by the (line segments or curves that represent the) edges of $G$. These regions are called the faces of $G$. Note that the faces depend on the embedding, so a planar graph can have different sets of faces. One of these faces is always unbounded, and called the outer face.

Let $F(G)$ be the set of faces of (a fixed plane embedding of) a planar graph $G$. The boundary of a face $f \in F(G)$ is a shortest closed walk along the vertices and edges of $f$. The length of this shortest closed walk is called the degree of $f$, denoted by $d_{G}(f)$ or simply $d(f)$. Note that this implies that all edges on the boundary of $f$ contribute 1 to its degree, except for cut-edges; the latter contribute 2 to $d(f)$. As we did for vertices, we use the shorthand $d$-face ( $d^{+}$-face, $d^{-}$-face, respectively) to denote a face with degree $d$ (at least $d$, at most $d$, respectively).

If the boundary of $f$ is a cycle $x_{1} x_{2} \ldots x_{k} x_{1}$, then we call this boundary a facial cycle. If $d(f) \leq 4$ and $G$ contains no 1 -vertex, then the boundary of $f$ is a cycle. In that case $f$ is denoted by $f=\left[x_{1} x_{2} \ldots x_{k}\right]$, and is also referred to as a $\left(d\left(x_{1}\right), d\left(x_{2}\right), \ldots, d\left(x_{k}\right)\right)$-face, according to the degrees $d\left(x_{i}\right)$ of its vertices.

For a vertex $v \in V(G)$, let $N(v)=\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$. We assume that all $v_{i}$ are arranged in a clockwise order around $v$ in the plane embedding, i.e., $v_{i+1}$ is the
immediate successor of $v_{i}$ for $1 \leq i \leq k-1$, and $v_{1}$ is the immediate successor of $v_{k}$ in this order. If $v$ is not a cut-vertex, then every two consecutive incident edges with $v$ in the clockwise ordering are on a common face. In this case, we use $f_{i}$ to denote the face which is incident with the edges $v v_{i}$ and $v v_{i+1}$ for $i \leq k-1$, and with $v v_{k}$ and $v v_{1}$ for $i=k$. Under these assumptions, we let $F_{v}=\left\{f_{1}, f_{2}, \ldots, f_{k}\right\}$. We use $\lambda_{i}(v)\left(\lambda_{i^{+}}(v), \lambda_{i^{-}}(v)\right)$ to denote the number of $i$-faces ( $i^{+}$-faces, $i^{-}$-faces) of $G$ incident with $v$ and $n_{i}(v)\left(n_{i^{+}}(v), n_{i^{-}}(v)\right)$ to denote the number of $i$-vertex ( $i^{+}$-vertex, $i^{-}$-vertex) of $G$ adjacent to $v$.

For two integers $a, b$ with $a \leq b$, we let $[a, b]=\{a, a+1, \ldots, b\}$.
In the next four sections, we will give an overview of four different types of edge colorings, including the main open conjectures involving these concepts, the progress that has been established over the years, and our contributions to the field. We start with a natural counterpart of the graph coloring problem we introduced at the beginning of this chapter. Instead of coloring the vertices of a graph, we aim to color the edges of the graph in such a way that no vertex is incident with two edges of the same color.

### 1.2 Edge coloring

Let us start this section with another small example of an application in which the concept of edge coloring appears naturally. Suppose we consider a number of teachers and a number of classes, and assume we know which teachers have to take care of which classes. We want to find out how may time slots we need to schedule these classes without conflicts. The restrictions are that each teacher can take care of at most one class in each time slot, and that each class requires one teacher and one time slot. We can model this as a graph problem by representing each teacher and each class by one vertex, and using edges to indicate which teachers have to take care of which classes. To obtain a feasible schedule, we have to assign the time slots to the edges in such a way that no time slot appears twice at a vertex, i.e., at two (or more) edges incident with the same vertex. If we interpret the time slots as colors, this problem is a special case of edge coloring.

Definition 1.1. A $k$-edge-coloring of a graph $G$ is a coloring of the edges of
$G$ with $k$ colors such that two edges that incident receive distinct colors. A graph is $k$-edge-colorable if it has a $k$-edge-coloring.

Definition 1.2. The edge chromatic number, denoted by $\chi^{\prime}(G)$ is the minimum $k$ such that $G$ is $k$-edge-colorable.

The first paper dealing with the edge coloring problem was written by Tait [60] in 1880, so about the same time Kempe thought he had a valid proof of the Four Color Theorem. In fact, the results in [60] were inspired by the map coloring problem. In [60], Tait presents a rather surprising relationship between face colorings and edge colorings of 3-connected cubic (3-regular) plane graphs. Here a face coloring is an assignment of colors to the faces of a plane graph with the property that neighboring faces receive different colors. Hence, a face coloring of a plane graph is equivalent to a vertex coloring of its dual graph. This dual graph is obtained by defining one vertex for each face and adding an edge between two vertices whenever the corresponding faces share an edge on their boundary. We omit further details, since they are not relevant for the results of this thesis.

In 1916, König [40] published his celebrated result on bipartite graphs. Recall that a graph is called bipartite if its vertex set can be partitioned into two disjoint sets in such a way that every edge has one endpoint in both of these sets. König's Theorem states that every bipartite graph can be edge colored using exactly $\Delta$ colors. In 1949, Shannon [57] proved that every graph can be edge colored with at most $\left\lfloor\frac{3 \Delta}{2}\right\rfloor$ colors. An important theorem due to Vizing [61], and independently Gupta [23], asserts that for any (simple) graph $G$, either $\chi^{\prime}(G)=\Delta$ or $\chi^{\prime}(G)=\Delta+1$, and became known as Vizing's Theorem.

Theorem 1.1 (Vizing [61]). For any (simple) graph $G, \chi^{\prime}(G) \leq \Delta+1$.
According to the Vizing-Gupta bound, a (simple) graph can be classified into one of two classes: Class 1 graphs for which $\Delta$ colors suffice, and Class 2 graphs for which $\Delta+1$ colors are necessary (See Figure 1.1 for an example). This means a gap of at most one between an edge coloring obtained by using a constructive proof of Vizing's Theorem and an optimal edge coloring.

However, by a result of Holyer [29], the determination of the chromatic index is an NP-hard optimization problem.


A Class 1 graph


Figure 1.1: An example of Class 1 and Class 2 graphs.

In 1977, Erdős and Wilson [18] showed that almost all graphs are Class 1. Specifically, they considered the Erdős-Rényi model of random graphs. This result has important implications for the study of random graphs, as it suggests that most random graphs possess certain desirable properties such as planarity, colorability, and so on. We omit further details on random graphs and their properties.

In 1965, Vizing [62] showed that any planar graph with maximum degree at least eight is Class 1. This is a stronger result than the general result that almost all graphs are Class 1, and it is specific to planar graphs. On the other hand, Vizing observed that for any maximum degree in the range from two to five, there exist planar graphs which are Class 2. He also conjectured that all (simple) planar graphs with maximum degree six or seven are Class 1, which is known as Vizing's Planar Graph Conjecture.

Conjecture 1.1 (Vizing [62]). Every (simple) planar graph of maximum degree 6 or 7 is Class 1.

Independently, in 2000, Zhang [80], and in 2001, Sanders and Zhao [54] confirmed that Conjecture 1.1 is true for planar graphs of maximum degree 7.

It is well-known that every planar graph contains neither a $K_{5}$-minor nor a $K_{3,3}$-minor. Therefore, the family of $K_{5}$-minor free graphs is a generalization of planar graphs. Recently, Feng et al. [19] extended the above results by showing that every $K_{5}$-minor free graph with maximum degree $\Delta \geq 7$ is $\Delta$-edge-colorable.

However, Conjecture 1.1 remains open for planar graphs with maximum degree 6 . Nevertheless, many interesting results have been reached in recent years, confirming the conjecture for planar graphs with maximum degree 6 subject to various conditions and constraints. We refer the interested reader to the following sources for more details [ $9,20,27,43,44,49,65,72,75,83,84]$.

In the next section, we turn to list edge coloring, a generalization of the above concept of edge coloring.

### 1.3 List edge coloring

The concept of list edge coloring was introduced by Vizing in 1976, as a generalization of edge coloring. We can use the scheduling example with the teachers and classes of the previous section to introduce this concept.

In many practical settings such scheduling problems involve additional restrictions. In our example, suppose there are restrictions on the availability of the teachers in certain time slots, or likewise on the choice of the time slots for the classes. This can be modeled by assigning a list of colors to each edge of the graph, indicating which time slots are in principle available to choose from for the associated teacher and class. An edge coloring would then only correspond to an eligible solution if each of the colors assigned to the edges is chosen from the list of colors assigned to that edge. This leads naturally to the concept of list edge coloring.

Definition 1.3. An edge list assignment for a graph $G$ is a function $L$ that assigns to each edge $e \in E(G)$ a list of colors (integers) $L(e)$. If $G$ has a proper edge coloring $\phi$ such that $\phi(e) \in L(e)$ for each $e$ of $G$, then we say that $G$ is edge-L-colorable, and that $\phi$ is an edge-L-coloring of $G$. The graph $G$ is said to be edge- $k$-choosable if $G$ is edge- $L$-colorable for any list assignment $L$ with $|L(e)| \geq k$ for any edge $e \in E(G)$.

Definition 1.4. The list edge chromatic number, denote by $\chi_{\ell}^{\prime}(G)$ is the minimum $k$ such that $G$ is edge- $k$-choosable.

From the definitions it is immediately clear that edge coloring is a special case of list edge coloring, where all the lists are equal (and sufficiently large). Thus $\chi^{\prime}(G) \leq \chi_{\ell}^{\prime}(G)$.

The most famous open problem about list edge coloring is probably the following list coloring conjecture, which was proposed independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris (See [35] for more details on the history of the conjecture).

Conjecture 1.2. If $G$ is a graph, then $\chi_{\ell}^{\prime}(G)=\chi^{\prime}(G)$.
This conjecture has been confirmed for several classes of graphs, including $d$-regular $d$-edge-colorable planar graphs [17], graphs with $\Delta \geq 12$ which can be embedded in a surface of non-negative characteristic [8], outerplanar graphs [68], bipartite multigraphs [8,21], complete graphs of odd order [24], and complete graphs of prime degree [55]. We omit the details, but the results show that list edge coloring has received considerable attention over the years.

Vizing [63] proposed the following conjecture, which is weaker than Conjecture 1.2.

Conjecture 1.3 (Vizing [63]). If $G$ is a graph, then $\chi_{\ell}^{\prime}(G) \leq \Delta+1$.
Harris [26] showed that $\chi_{\ell}^{\prime}(G) \leq 2 \Delta-2$ if $G$ is a graph with $\Delta \geq 3$. This implies Conjecture 1.3 for the case $\Delta=3$. Juvan et al. [37] settled the case for $\Delta=4$ in 1999. Several other special cases of Conjecture 1.3 have been confirmed. It is known to hold for complete graphs [24], graphs with girth at least $8 \Delta(\ln \Delta+1.1)$ [41], planar graphs with $\Delta \geq 9$ [7,14], and planar graphs with $\Delta \geq 8$ [5].

In Chapter 2, we focus on planar graphs and list a number of references to sources of recent work related to Conjecture 1.3. In all this work, the authors confirm special cases of Conjecture 1.3 for planar graphs, involving restrictions on the cycle structure. With our main result of Chapter 2, we confirm Conjecture 1.3 for planar graphs with $\Delta \geq 6$ in which every 7-cycle (if any) induces a $C_{7}$. This means that any existing cycles on exactly 7 vertices in the graph have the property that there are no additional edges in the graph between pairs of vertices on the cycle, apart from the cycle edges.

In the next section, we focus on another generalization of edge coloring, with similar features, conjectures and results. Instead of assigning lists to the edges of a graph, we next consider the situation in which signs are assigned to its edges, in order to indicate a positive or negative relationship between their endpoints.

### 1.4 Signed edge coloring

Signed graphs were initially introduced by Harary [25] in 1953 to study social psychology, and since then in certain application areas they have become a natural generalization of ordinary graphs. The presence of edge signs in signed graphs makes them more complicated than their unsigned counterparts. However, as one can imagine this added complexity allows for various phenomena that are unseen in the world of ordinary graphs. This may lead to interesting insights concerning both ordinary graphs and signed graphs. Before we give more background and list several results and conjectures, we start with some definitions.

Definition 1.5. A signed graph $(G, \sigma)$ is a graph $G$ with a signature $\sigma: E(G) \rightarrow$ $\{+,-\}$, in which case $G$ is called the underlying graph of $(G, \sigma)$. An edge $e \in E(G)$ is called positive if $\sigma(e)=+$ and negative otherwise. A signed graph is called all-negative if all edges are negative.

In a signed graph, switching at a vertex means reversing all signs of the edges incident with it (See Figure 1.2 for an example, where the negative edges are dashed, and we first switch at $v_{2}$ and then at $v_{4}$, or vice versa). Two signed graphs are said to be switching equivalent if one can be obtained from the other via a sequence of switchings.

The first very fundamental study on the structure of signed graphs and related matroids and polynomials was done by Zaslavsky in [77]. There he also introduced a notion of vertex coloring for signed graphs, which he studied in more depth in two later papers [78, 79] of the early 1980s. In order to define a chromatic number for signed graphs that is more in line with the chromatic number of (unsigned) graphs, Máčajová et al. [52] diverged from Zaslavsky's definition of vertex coloring, and investigated relationships


Figure 1.2: An example of the switching operation.
between the chromatic number of a signed graph and various graph invariants. They also obtained a Brooks-type theorem for signed graphs. We omit the details. Interested readers can refer to the following literatures for the latest research on signed graph coloring [ $32,36,38,42,53,56$ ].

Motivated by the works of Zaslavsky [77] and Máčajová et al. [52], Behr [3] recently introduced the following concept of edge coloring for signed graphs, as a natural extension of edge coloring for graphs.

In order to introduce this concept, it is convenient to treat an edge $e=$ $u v \in E(G)$ as two half edges $h_{e}^{u}$ and $h_{e}^{v}$, where $h_{e}^{u}$ is incident (only) with $u$ and $h_{e}^{v}$ is incident (only) with $v$. Let $H(G)$ be the set of all half edges of $G$.

Definition 1.6. Let $k$ be a positive integer. An $M_{k}$-edge-coloring of a signed graph ( $G, \sigma$ ) is a mapping

$$
\varphi: H(G) \rightarrow M_{k}= \begin{cases}\left\{ \pm 1, \ldots, \pm \frac{k}{2}\right\} & \text { if } k \text { is even } ; \\ \left\{ \pm 1, \ldots, \pm \frac{k-1}{2}, 0\right\} & \text { if } k \text { is odd. }\end{cases}
$$

satisfying $\varphi\left(h_{e}^{u}\right)=-\sigma(u v) \varphi\left(h_{e}^{v}\right)$, in which half edges incident with $u$ receive distinct colors for all $u v \in E(G)$ and $u \in V(G)$.

The colors $+a$ and $-a$ are said to have the same magnitude denoted by $|a|$, and are sometimes called opposite.

In order to enable appealing formulations of analogues of well-known conjectures and results on ordinary edge coloring for unsigned graphs, we prefer to follow the earlier terminology introduced by Behr [3]. For this reason, we say that a signed graph is $k$-edge-colorable if it has an $M_{k}$-edge-coloring.

With the above in mind, we also take the liberty to use $k$-edge-coloring instead of $M_{k}$-edge-coloring throughout the rest of the thesis.

Definition 1.7. The signed edge chromatic number, denoted by $\chi_{ \pm}^{\prime}(G, \sigma)$ is the minimum $k$ such that $(G, \sigma)$ is $k$-edge-colorable.

For convenience, we view the colors of two half edges of $e$ as the color of $e$ when $e$ is a negative edge. From this it is clear that ordinary edge coloring is a special case of signed edge coloring. It is of (mainly theoretical) interest to explore whether results and conjectures on edge coloring have natural counterparts on signed edge coloring.

We note here that Behr gave an equivalent definition of edge coloring of signed graphs by coloring edges rather than by coloring half edges (See [3] or [82]). By applying this equivalent definition, Zhang et al. [82] noted that for any signed graph $G$, if $\sigma^{\prime}$ is obtained from $\sigma$ by one switching operation at a vertex $v \in V(G)$, then for any integer $k$, " $(G, \sigma)$ admits a $k$-edge-coloring" if and only if " $\left(G, \sigma^{\prime}\right)$ admits a $k$-edge-coloring". This is clear, since one of these $k$-edge-colorings can be obtained from the other one by reversing the colors on the half edges incident to $v$. This fact implies that the edge chromatic number and criticality of signed graphs are preserved under switchings.

In [82], they also showed that every signed planar graph is ( $\Delta+1$ )-edgecolorable. And they proposed the following signed version of Conjecture 1.3.

Conjecture 1.4 (Zhang et al. [82]). Every signed planar graph with maximum degree $\Delta$ is $\Delta$-edge-colorable for all $\Delta \geq 6$.

Zhang et al. [82] used the concept of linear coloring to partially confirm Conjecture 1.4, by establishing the following result. We omit the details.

Theorem 1.2 (Zhang et al. [82]). Every signed planar graph $G$ with maximum degree $\Delta$ is $\Delta$-edge-colorable if either $\Delta \geq 10$ or $\Delta \in\{8,9\}$ and $G$ does not contain any adjacent triangles.

Behr [3] showed that the minimum number of colors required for an edge coloring of a signed graph is bounded from above by $\Delta+1$, thus obtaining the following analogue of Vizing's Theorem.

Theorem 1.3 (Behr [3]). Every signed graph with maximum degree $\Delta$ is $\Delta$ - or ( $\Delta+1$ )-edge-colorable.

In Chapters 3 and 4, we study the structure of critical signed graphs, and we confirm the signed planar graph conjecture for signed planar graphs with $\Delta \geq 8$, and for signed planar graphs with $\Delta \geq 6$ in which every 6-cycle has at most one chord.

The final variant of edge coloring we will encounter in this thesis is the rather technical concept of edge DP-coloring, which is the edge analogue of what is known as correspondence coloring.

### 1.5 Edge DP-coloring

Before we recall the definition of edge DP-coloring given by Bernshteyn and Kostochka in [4], let us start with some remarks adopted from a paper on correspondence coloring due to Dvořák and Postle [16].

Many proofs of colorability results are based on a method involving socalled reducible configurations, including the known existing proofs of the Four Color Theorem. The essence of this approach is to reduce any given graph step by step to one of a bounded size for which its colorability can be checked (in principle by hand, or by computer). In each step, colorability should be preserved by using a suitable operation on the (intermediate) graph. One of the commonly used operations within this framework is the identification of two vertices with similar coloring properties. We omit the details, since this operation depends on the type of coloring one considers. However, we note here that this approach is in general not applicable in the context of list coloring, since the vertices we want to identify might have different lists. Hence, this type of argument for ordinary coloring does not translate directly to the list coloring setting.

Motivated by this, to enable reductions in the list coloring domain, Dvořák and Postle [16] introduced a generalization of list coloring which they called correspondence coloring. They showed the relevance of this new concept by resolving a conjecture on list coloring due to Borodin. We omit the details, since we are focused on edge colorings instead of vertex colorings.

In a recent paper [4], Bernshteyn and Kostochka examined the difference between list coloring and correspondence coloring, and introduced the edge coloring analogue of correspondence coloring, which they named edge DPcoloring in recognition of Dvořák and Postle. Their definition is based on the correspondence coloring of line graphs, and leads to the following rather technical definition.

Let $L$ be an edge list assignment of a graph $G$. Define a graph $\tilde{G}$ as follows:
(i) $V(\tilde{G})=\cup_{e \in E(G)}(\{e\} \times L(e))$, where $\times$ denotes the Cartesian product;
(ii) For $e \in E(G)$, the subgraph of $\tilde{G}$ induced by $\{e\} \times L(e)$ is a complete graph;
(iii) For $e \sim_{G} e^{\prime}$, the edges of $\tilde{G}$ between $\{e\} \times L(e)$ and $\left\{e^{\prime}\right\} \times L\left(e^{\prime}\right)$ consist of a matching, denoted by $M_{L, e e^{\prime}}$.

Here the notation $e \sim_{G} e^{\prime}$ (or simply, $e \sim e^{\prime}$ ) means that edge $e$ is adjacent to edge $e^{\prime}$ (they share an endpoint) in $G$. We call

$$
\mathscr{M}_{L}=\left\{M_{L, e e^{\prime}}: e \sim_{G} e^{\prime}\right\}
$$

a matching assignment over $L$, and the graph $\tilde{G}$ an $\mathscr{M}_{L}$-cover of $G$.
Definition 1.8. For an edge list assignment $L$ of $G$ and a matching assignment $\mathscr{M}_{L}$, if the $\mathscr{M}_{L}$-cover $\tilde{G}$ of $G$ has an independent set $I$ with $|I|=|E(G)|$, then we call $I$ an $\mathscr{M}_{L}$-coloring of $G$.

Definition 1.9. The edge DP-chromatic number of $G$, denoted by $\chi_{D P}^{\prime}(G)$, is the minimum integer $k$ such that $G$ has an $\mathscr{M}_{L}$-coloring for any edge list assignment $L$ with $|L(e)| \geq k$ for each $e \in E(G)$ and any matching assignment $\mathscr{M}_{L}$. If $\chi_{D P}^{\prime}(G) \leq k$, then we say that $G$ is edge DP-k-colorable.

Here is a small example to illustrate the above concepts, in which all edges have assigned lists of two colors.

Example 1.1. Figure 1.3 shows two distinct $\mathscr{M}_{L}$-covers of the 4-cycle $C_{4}$. Note that according to the above definitions $C_{4}$ admits an $\mathscr{M}_{L_{1}}$-coloring but not an $\mathscr{M}_{L_{2}}$-coloring.



Figure 1.3: Two distinct $\mathscr{M}_{L}$-covers of a 4-cycle
In fact, by Theorem 1.5 below, we know that $\chi_{D P}^{\prime}\left(C_{4}\right) \geq 3$, whereas the next result due to Galvin [21] implies that $\chi_{\ell}^{\prime}\left(C_{4}\right)=2$.
Theorem 1.4 (Galvin [21]). For every bipartite graph $G, \chi_{\ell}^{\prime}(G)=\chi^{\prime}(G)=$ $\Delta(G)$.

By the definitions of edge $L$-coloring and $\mathscr{M}_{L}$-coloring, when

$$
M_{L, e e^{\prime}}=\left\{(e, c)\left(e^{\prime}, c\right): c \in L(e) \cap L\left(e^{\prime}\right)\right\}, \text { for all } e \sim e^{\prime},
$$

then $G$ has an edge $L$-coloring if and only if $G$ has an $\mathscr{M}_{L}$-coloring. Thus $\chi_{\ell}^{\prime}(G) \leq \chi_{D P}^{\prime}(G)$.

It is interesting to analyze the counterparts of results and conjectures on list edge coloring for edge DP-coloring. Bernshteyn and Kostochka [4] proved the following theorem, showing that the counterpart of Theorem 1.4 does not hold for edge DP-coloring. More in particular, their result implies that it is impossible for a $d$-regular graph $G$ with $d \geq 2$ to have edge $D P$-chromatic number $d$.

Theorem 1.5 (Bernshteyn and Kostochka [4]). For all integers $d \geq 2$, every $d$-regular graph $G$ satisfies $\chi_{D P}^{\prime}(G) \geq d+1$.

In [4], they also formulated the following open problem.
Problem 1.1 (Bernshteyn and Kostochka [4]). Does there exist a graph $G$ with $\chi_{D P}^{\prime}(G) \geq \Delta(G)+2$ ?

This question is closely related to the earlier Conjecture 1.3 due to Vizing [63] stating that $\chi_{\ell}^{\prime}(G) \leq \Delta(G)+1$ for any graph $G$. The results we
present in Chapter 5 are partial answers to the above question and conjecture. As in the other chapters, we restrict ourselves in Chapter 5 to planar graphs. We show that $\chi_{D P}^{\prime}(G) \leq \Delta(G)+1$ for any planar graph $G$ with $\Delta \geq 9$. Moreover, we prove that $\chi_{D P}^{\prime}(G)=\Delta(G)$ for any planar graph $G$ with $\Delta \geq 8$ which contains no 3 -cycles, as well as for any planar graph $G$ with $\Delta \geq 7$ which contains no 4-cycles.

In the final section of this introductory chapter, we give a short explanation of one of the key ingredients in most of our proofs.

### 1.6 The discharging method

The discharging technique we are going to explain here is a powerful and widely used method for proving theorems about planar graphs by contradiction.

It is based on the idea of assigning initial charges to the vertices and faces of a plane embedding of a planar graph, and then redistributing the charges in a way that preserves the total charge while satisfying certain rules. The aim is to reach a contradiction with Euler's Formula: any plane embedding of a connected planar graph with vertex set $V$, edge set $E$ and face set $F$ satisfies: $|V|-|E|+|F|=2$. Using the facts that $\sum_{v \in V} d(v)=2 m$ and $\sum_{f \in F} d(f)=2 m$, by straightforward calculations one can obtain

$$
\sum_{v \in V}(3 d(v)-10)+\sum_{f \in F}(2 d(f)-10)=-20
$$

or

$$
\sum_{v \in V}(2 d(v)-6)+\sum_{f \in F}(d(f)-6)=-12
$$

to give two examples we will use in the thesis.
Based on these equalities, one can assign the initial charge $w: V \cup F \rightarrow \mathbb{Z}$ defined by

$$
\left\{\begin{array}{lll}
w(v)=3 d(v)-10 & (\text { or } 2 d(v)-6) & \text { for } v \in V \\
w(f)=2 d(f)-10 & (\text { or } d(f)-6) & \text { for } f \in F
\end{array}\right.
$$

to the vertices and faces of $G$. In order to reach a contradiction, the idea is to redistribute the charges among the vertices and the faces in $G$ by a number of discharging rules. The aim is to show that the new charges of all the vertices and faces are nonnegative, contradicting Euler's Formula.

Hence, the method is typically used to prove theorems by contradiction. To do this, we start with a minimum counterexample and assume that it satisfies certain conditions. We then study some reducible configurations that cannot occur in a minimum counterexample, and assign initial charges to the vertices and faces of the embedding of the graph.

The key property of the charges is that the sum of all charges in the graph must be negative. This property allows us to apply the rules for shifting the charges, which involve moving charges from high-charge vertices or faces to low-charge ones while maintaining the total charge.

The final step of the method involves verifying that the final charges of every face and every vertex are nonnegative. If we can do this, then we have shown that the minimum counterexample is not a planar graph, which contradicts our assumption.

One of the strengths of the discharging technique is that it can be used to prove theorems about planar graphs with local constraints, such as graphs that satisfy certain edge or vertex conditions. By identifying and analyzing the set of reducible configurations, we can often prove theorems that would be difficult or impossible to prove by other methods. Overall, the discharging technique is a powerful tool for proving theorems about planar graphs, and it has been used to make significant contributions to the field of graph theory.

## Chapter 2

## List edge coloring of planar graphs

In this chapter, we confirm the Conjecture 1.3 holds for planar graphs with $\Delta \geq 6$ in which every 7-cycle (if any) induces a $C_{7}$ (so, without chords).

### 2.1 Introduction

As we mentioned in the previous chapter, Conjecture 1.3 has been proved for planar graph with maximum degree condition $\Delta \geq 8$. There are also lots of related results on Conjecture 1.3 by adding restrictions which can be found in $[10,11,13,22,30,31,33,45-48,51,58,59,64,66,67,69-71,73,81]$. Here, we list one result obtained by Dong, Liu and Li [15].

Theorem 2.1 (Dong, Liu and Li [15]). Let $G$ be a planar graph where all 7 -cycles are induced. If $\Delta \geq 7$, then $\chi_{\ell}^{\prime}(G) \leq \Delta+1$.

The additional condition implies that any existing 7-cycles in $G$ are induced, i.e., contain no chords. In this chapter, we strengthen Theorem 2.1 and obtain the following result, showing that this additional condition allows a further relaxation of the maximum degree condition.

Theorem 2.2. Let $G$ be a planar graph in which any existing 7-cycles contain no chords. If $\Delta \geq 6$, then $\chi_{\ell}^{\prime}(G) \leq \Delta+1$.

In fact, in Section 2.3 we prove the following result.
Theorem 2.3. Let $G$ be a planar graph in which any existing 7-cycles contain no chords. If $\Delta \leq 6$, then $\chi_{\ell}^{\prime}(G) \leq 7$.

It is clear that Theorem 2.2 is a direct consequence of Theorem 2.1 and Theorem 2.3. We postpone our proof of Theorem 2.3 to Section 2.3. Just like in most of the proofs in this area, the main ingredient of the proof is discharging. In the next section we introduce and apply a Combinatorial Nullstellensatz due to Alon [1]. We use it there in order to determine several configurations which cannot appear in an assumed minimal counterexample to Theorem 2.3. Then, we use discharging rules to complete our proof in Section 2.3.

### 2.2 A useful polynomial for list edge coloring

In this section, we will use the following theorem, known as the Combinatorial Nullstellensatz, and apply it to a polynomial associated with the edges of a graph.

Theorem 2.4 (Alon [1]). Let $\mathbb{F}$ be a field, and let $P=P\left(x_{1}, \ldots, x_{m}\right)$ be a polynomial in $\mathbb{F}\left[x_{1}, \ldots, x_{m}\right]$ with degree $\operatorname{deg}(P)=\sum_{j=1}^{m} i_{j}$, where each $i_{j}$ is a nonnegative integer. If the coefficient of the monomial $x_{1}^{i_{1}} \cdots x_{m}^{i_{m}}$ in $P$ is nonzero, and if $S_{1}, \ldots, S_{m}$ are subsets of $\mathbb{F}$ with $\left|S_{j}\right|>i_{j}$, then there are $s_{1} \in S_{1}, \ldots, s_{m} \in S_{m}$ such that $P\left(s_{1}, \ldots, s_{m}\right) \neq 0$.

In fact, the stated version is the second one of the two versions that appeared in Alon's paper [1]. Theorem 2.4 and its variants have been applied to obtain new results in many different areas, including graph coloring. In particular, they have been used to resolve conjectures on list (edge) coloring for special classes of graphs, by applying them to certain polynomials associated with the vertices and edges of the graph. For our purposes, we consider the
following polynomial which is based on variables associated with the edges of the graph.

Let $H$ be a subgraph of a graph $G$, and let $E(H)=\left\{e_{1}, e_{2}, \ldots, e_{m}\right\}$. Assign a variable $x_{i}$ to the edge $e_{i}$ for each $i \in[1, m]$, and define the polynomial $P_{H}=P_{H}\left(x_{1}, x_{2}, \ldots, x_{m}\right)$ by

$$
\begin{gather*}
P_{H}=\prod_{\substack{w \in V(H) \\
d_{H}(w)>1}} P_{H, w} \text {, with } \\
P_{H, w}=\prod_{\substack{e_{i}, e_{j} \in E_{H}(w) \\
i<j}}\left(x_{i}-x_{j}\right) \text { for } w \in V(H) \text { with } d_{H}(w)>1 \tag{2.1}
\end{gather*}
$$

We next demonstrate how this polynomial can be used in our setting if a certain condition on a monomial is met. In particular, we show that in such cases Theorem 2.4 can be applied to guarantee that a list edge coloring of a subgraph can be extended to a list edge coloring of the whole graph. This will help us to identify a large number of configurations that cannot appear in an assumed minimal counterexample to Theorem 2.3.

Lemma 2.5. Let $G$ be a graph with an edge list assignment L. Let $H$ be a subgraph of $G$, and let $\left.L\right|_{E(\bar{H})}$ denote the restriction of $L$ to $E(\bar{H})$, where $\bar{H}=G-E(H)$. Let $P_{H}$ be defined as in Eq. (2.1) with $E(H)=\left\{e_{1}, \ldots, e_{m}\right\}$. Then $G$ is edge-L-colorable if the following two conditions are satisfied:
(1) $\bar{H}$ is edge- $\left.L\right|_{E(\bar{H})}$-colorable;
(2) there is a monomial $\prod_{e_{i} \in E(H)} x_{i}^{t_{i}}$ in $P_{H}$ with a nonzero coefficient and such that $0 \leq t_{i}<\left|L\left(e_{i}\right)\right|-d_{\bar{H}}(u)-d_{\bar{H}}(w)$ for each $e_{i}=u w \in E(H)$.

Proof. Assume that (1) and (2) hold. Let $\phi$ be an edge- $\left.L\right|_{E(\bar{H})}$-coloring of $\bar{H}$. Denote by $S_{\phi}$ the edge list assignment of $H$ satisfying that, for every $e_{i}=u w \in E(H)$,

$$
S_{\phi}\left(e_{i}\right)=L\left(e_{i}\right) \backslash\left\{\phi(h): h \in E_{\bar{H}}(u) \cup E_{\bar{H}}(w)\right\} .
$$

Then $\left|S_{\phi}\left(e_{i}\right)\right| \geq\left|L\left(e_{i}\right)\right|-d_{\bar{H}}(u)-d_{\bar{H}}(w)$. By Theorem 2.4, there exist $c_{i} \in$ $S_{\phi}\left(e_{i}\right)$ for all $e_{i} \in E(H)$ such that $P_{H}\left(c_{1}, \ldots, c_{m}\right) \neq 0$. This implies $G$ has an edge- $L$-coloring obtained from $\phi$ by coloring each $e_{i} \in E(H)$ with color $c_{i}$.

It will be shown in the next lemma that the graphs illustrated in Figure 2.1 cannot appear as a subgraph in a minimal counterexample $G$ to Theorem 2.3. It should be noted here that the integers in Figure 2.1 indicate the largest degree the associated vertices can attain in $G$. Hence, each of the graphs of Figure 2.1 represents a larger number of forbidden configurations.

In the sequel, with an edge $k$-list assignment we mean a list assignment in which each edge has an assigned list with exactly $k$ colors. The next lemma is a key ingredient for showing that the discharging rules we define in the final part of our proof of Theorem 2.3 lead to a contradiction, if we assume there exists a counterexample to Theorem 2.3. For the statement of the lemma, let us assume that $G$ is a graph with maximum degree $\Delta \leq 6$ and an edge 7-list assignment $L$, and that $H$ is a subgraph of $G$. We write $H \in\left\{F_{1}, \ldots, F_{14}\right\}$ if $H$ is one of the graphs of Figure 2.1 such that the degree of each vertex of $H$ in $G$ does not exceed the value indicated in the associated circle in Figure 2.1. Lemma 2.5 has the following consequence.

Lemma 2.6. If $H \in\left\{F_{1}, \ldots, F_{14}\right\}$ and $G-E(H)$ is edge- $\left.L\right|_{E(G) \backslash E(H)}$-colorable, then $G$ is edge-L-colorable.

Proof. We only illustrate the proof for $H=F_{1}$; the other cases can be treated in a similar way. For refereeing purposes we added an appendix, but we suggest to omit it from the final version. Suppose $H=F_{1}$ and let the vertices be labeled as in Figure 2.1. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,3]$, and $x_{4}$ to $v_{1} v_{2}$. Using Eq. (2.1),

$$
P_{H}=P_{H, v} \cdot P_{H, v_{1}} \cdot P_{H, v_{2}}=\left(x_{1}-x_{2}\right)\left(x_{2}-x_{3}\right)\left(x_{1}-x_{3}\right) \cdot\left(x_{1}-x_{4}\right) \cdot\left(x_{2}-x_{4}\right)
$$

Straightforward calculations show that the coefficient of the monomial $x_{1}^{2} x_{2}$ $x_{3} x_{4}$ in $P_{H}$ is 1 , hence nonzero. In order to verify that condition (2) of Lemma 2.5 holds, we refer to Figure 2.1 to check whether $0 \leq t_{i}<\left|L\left(e_{i}\right)\right|-$ $d_{\bar{H}}(u)-d_{\bar{H}}(w)$ for each $e_{i}=u w \in E(H)$, where $t_{i}$ is the exponent of $x_{i}$

$F_{1}$

$F_{4}$





Figure 2.1: Fourteen forbidden configurations.
associated with the edge $e_{i}$ in the above monomial. Below are the details for the edges $v v_{1}, v v_{2}, v v_{3}$ and $v_{1} v_{2}$, respectively, showing that (2) indeed holds.

$$
\begin{aligned}
& 7-d_{\bar{H}}(v)-d_{\bar{H}}\left(v_{1}\right)=7-\left(d_{G}(v)-d_{H}(v)\right)-\left(d_{G}\left(v_{1}\right)-d_{H}\left(v_{1}\right)\right) \geq 3>2 \\
& 7-d_{\bar{H}}(v)-d_{\bar{H}}\left(v_{2}\right)=7-\left(d_{G}(v)-d_{H}(v)\right)-\left(d_{G}\left(v_{2}\right)-d_{H}\left(v_{2}\right)\right) \geq 2>1 \\
& 7-d_{\bar{H}}(v)-d_{\bar{H}}\left(v_{3}\right)=7-\left(d_{G}(v)-d_{H}(v)\right)-\left(d_{G}\left(v_{3}\right)-d_{H}\left(v_{3}\right)\right) \geq 2>1 \\
& 7-d_{\bar{H}}\left(v_{1}\right)-d_{\bar{H}}\left(v_{2}\right)=7-\left(d_{G}\left(v_{1}\right)-d_{H}\left(v_{1}\right)\right)-\left(d_{G}\left(v_{2}\right)-d_{H}\left(v_{2}\right)\right) \geq 2>1
\end{aligned}
$$

Since $\bar{H}$ is edge- $\left.L\right|_{E(\bar{H})}$-colorable, $G$ is edge- $L$-colorable by Lemma 2.5.
Proofs of other cases are given in Appendix A.

### 2.3 Proof of Theorem 2.3

In this section, we will prove Theorem 2.3 by contradiction. Suppose that $G$ is a planar graph with maximum degree $\Delta \leq 6$, and let $L$ be an edge 7-list assignment of $G$ such that
(a) every 7-cycle of $G$ (if any) contains no chords;
(b) $G$ is not edge- $L$-colorable;
(c) every proper subgraph $G^{\prime}$ of $G$ is edge- $\left.L\right|_{E\left(G^{\prime}\right)}$-colorable.

In fact, we may assume $\Delta=6$, since it was recently shown in [28] that $\chi_{\ell}^{\prime}(G) \leq \Delta+2$ for every finite simple graph $G$. Clearly, we may also assume that $G$ is connected. We next prove the following two claims.

Claim 1. For any $u v \in E(G), d_{G}(u)+d_{G}(v) \geq \Delta+3$.
Proof. Suppose not, let $e=u v \in E(G)$ such that $d_{G}(u)+d_{G}(v) \leq \Delta+2$. Using (c), assume $\phi$ is an edge- $\left.L\right|_{E(G-e)}$-coloring of $G-e$. Then $S_{\phi}(e)=$ $L(e) \backslash\left\{\phi(h): h \in E_{G-e}(u) \cup E_{G-e}(v)\right\} \neq \emptyset$. Hence, $G$ has an edge- $L$-coloring obtained from $\phi$ by coloring $e$ with a color $c \in S_{\phi}(e)$, a contradiction to (b).

Claim 2. There is no even cycle $H$ in $G$ such that $d_{G}(u)+d_{G}(v)=\Delta+3$ for every $u v \in E(H)$.

Proof. Suppose to be contrary that $H$ is an even cycle of $G$ such that $d_{G}(u)+$ $d_{G}(v)=\Delta+3$ for every $u v \in E(H)$. Using (c), assume $\phi$ is an edge- $\left.L\right|_{E(\bar{H})^{-}}$ coloring of $\bar{H}=G-E(H)$. For $u v \in E(H)$, let

$$
S_{\phi}(u v)=L(u v) \backslash\left\{\phi(h): h \in E_{\bar{H}}(u) \cup E_{\bar{H}}(v)\right\} .
$$

Then $\left|S_{\phi}(u v)\right| \geq(\Delta+1)-\left(d_{G}(u)+d_{G}(v)\right)+\left(d_{H}(u)+d_{H}(v)\right)=2$. Since $H$ is an even cycle, it has a proper edge coloring $\varphi$ with $\varphi(u v) \in S_{\phi}(u v)$ for $u v \in E(H)$. Hence $G$ has an edge-L-coloring $\varphi$ defined by $\varphi(e)=\phi(e)$ if $e \in E(\bar{H})$ and $\phi(e)=\varphi(e)$ if $e \in E(H)$, a contradiction to (b).

Note that $\delta=\delta(G) \geq 3$ by Claim 1. Recall that $V_{i}$ is the set of $i$-vertices of $G$ for $i \in[3, \Delta]$. Let $G_{3}$ be the subgraph of $G$ induced by all edges incident with $V_{3}$, i.e., all edges that have at least one end vertex in $V_{3}$ (if we assume $V_{3} \neq \emptyset$ ). Noting that $\Delta=6$ and using Claim 1, all edges of $G_{3}$ in fact have exactly one end vertex in $V_{3}$. Moreover, by Claim 1 and $2, G_{3}$ is a forest. Based on this, we next define the concept of a 3-master, which will be relevant in the discharging. For any component $T$ of $G_{3}$, pick a vertex $r \notin V_{3}$ as the root of $T$. Then every 3-vertex $x$ in $T$ has exactly two children (i.e., neighbors of $x$ in $T$ that are not on the path from $r$ to $x$ in $T$ ). Such a child of a 3-vertex $x$ in $T$ is called a 3-master of $x$. Note that every edge $u v \in E\left(G_{3}\right)$ with $u \in V_{3}$ is part of such a rooted tree. Hence, each 3-vertex of $G$ has exactly two 3-masters, and each 3 -master (which is in fact a $\Delta$-vertex) is the 3 -master of exactly one 3-vertex.

Recall that, for a vertex $v \in V(G), N(v)=\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}, f_{i}$ is the face which is incident with the edges $v v_{i}$ and $v v_{i+1}$ for $i \leq k-1$, and with $v v_{k}$ and $v v_{1}$ for $i=k, \lambda_{i}(v)\left(\lambda_{i^{+}}(v), \lambda_{i^{-}}(v)\right)$ denotes the number of $i$-faces $\left(i^{+}\right.$-faces $i^{-}$-faces ) of $G$ incident with $v$. Before we define the discharging rules, we first prove four additional claims. The first claim follows directly from (a).

Claim 3. Let $v$ be a 6 -vertex of $G$. Then $\lambda_{3}(v) \leq 4$.
Proof. Recall that by (a), any 7-cycle of $G$ contains no chords in $G$. Hence, the induced subgraph $G[N(v)]$ contains no paths with length 5 , and so $\lambda_{3}(v) \leq 4$.

The remaining claims reveal more details on the local structure around a vertex of $G$.

Claim 4. Let $v$ be a $k$-vertex of $G$ with $5 \leq k \leq 6$. Then the following statements hold.
(1) Let $\lambda_{3}(v)=k-2$ and $\lambda_{4}(v)=1$. Then either $\lambda_{7^{+}}(v)=1$ or $\lambda_{6}(v)=1$, and the local structure around $v$ is as illustrated in one of the configurations in Fig. 2.2.
(2) Let $k=5, \lambda_{3}(v)=4$, and $\lambda_{4}(v)=1$. Then $\lambda_{4^{-}}(x) \leq 2$ for any 4neighbor $x$ of $v$.
(3) Let $k=5$ and $\lambda_{3}(v) \geq 4$. Then $\lambda_{3}(x) \leq 3$ for any 5-neighbor $x$ of $v$.
(4) Let $k=6$ and $\lambda_{3}(v)=\lambda_{4}(v)=3$. Then the degrees of any two consecutive faces around $v$ are different (so they alternate between 3 and 4).

(a)

(d)

(b)

(e)

(c)

(f)

Figure 2.2: Possible local configurations around a $k$-vertex $v$ with $\lambda_{3}(v)=k-2$ and $\lambda_{4}(v)=1$.

Proof. We assume that $N_{G}(v)=\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$ and $F_{v}=\left\{f_{1}, f_{2}, \ldots, f_{k}\right\}$ in cases when $v$ is not a cut vertex. By Claim 1 , we have $d\left(v_{i}\right) \geq \Delta+3-k=9-k$.
(1) Since $\lambda_{3}(v)=k-2$ and $\lambda_{4}(v)=1$, $v$ is not a cut vertex. Without loss of generality, assume that the unique 4-face in $F_{v}$ is $f_{k}$, and the unique $5^{+}$-face in $F_{v}$ is $f_{\ell}$ with $\ell \in[3, k-1]$. Let $C=v v_{1} u v_{k} v$ be the facial cycle of $f_{k}$. Since any 7-cycle of $G$ contains no chords and $d\left(v_{1}\right) \geq 3, u \in\left\{v_{3}, \ldots, v_{k-1}\right\}$. Furthermore, we can deduce the following facts. If $k=5$ and $\ell=3$, then $u \neq v_{4}$; otherwise $d\left(v_{5}\right)=2$. If $k=6$ and $\ell=3$, then $u \neq v_{5}$; otherwise $d\left(v_{6}\right)=2$. Moreover then $u \neq v_{4}$; otherwise $v v_{6} v_{5} v_{4} v_{1} v_{2} v_{3} v$ is a 7 -cycle with chords. Furthermore $u \neq v_{3}$; otherwise $v v_{1} v_{2} v_{3} v_{6} v_{5} v_{4} v$ is a 7-cycle with chords. If $k=6$ and $\ell=4$, then $u \neq v_{5}$; otherwise $d\left(v_{6}\right)=2$. Moreover then $u \neq v_{4}$; otherwise $v v_{5} v_{6} v_{4} v_{1} v_{2} v_{3} v$ is a 7-cycle with chords. If $k=6$ and $\ell=5$, then $u \neq v_{5}$; otherwise $v v_{1} v_{2} v_{3} v_{4} v_{5} v_{6} v$ is a 7 -cycle with chords. These facts together imply that the local structure around $v$ is restricted to the six possible cases (a) $\sim(\mathrm{f})$ that are illustrated in Fig. 2.2.

Suppose first that the local structure around $v$ is as in (a). Then $k=5$ and $\ell=4$. To show that (1) holds in this case, we make the following additional observations. If the boundary of $f_{4}$ does not contain the edge $v_{4} v_{5}$, then $d\left(f_{4}\right) \geq 6$; otherwise the union of the boundaries of $f_{1}, f_{2}, f_{3}, f_{4}$ contains a 7 -cycle with chords. If the boundary of $f_{4}$ contains the edge $v_{4} v_{5}$, then $v_{5}$ is a cut vertex of $G$. This implies that either $d\left(f_{4}\right) \geq 7$ or $d\left(f_{4}\right)=6$, and the boundary of $f_{4}$ consists of two 3 -cycles with a common vertex $v_{5}$. So, in all these situations referring to the local structure of (a) the statement in (1) holds.

By analogous arguments, it can be shown that $d\left(f_{\ell}\right) \geq 7$ for the local structures that are illustrated in (b) $\sim(\mathrm{f})$ of Fig. 2.2. We omit the details. This completes the proof of (1).

For (2) and (3), we note that the assumptions $k=5$ and $\lambda_{3}(v) \geq 4$ directly imply the existence of a 6 -cycle with chords. By a careful analysis of the possible local structures, (2) and (3) can be obtained in a rather straightforward way by using the assumption that $G$ contains no 7-cycle with chords. We omit the details.

For (4), note that the assumptions imply that $v$ is not a cut vertex. Suppose
that the conclusion does not hold. Without loss of generality, assume that $f_{1}, f_{2}, f_{\ell}$ are three 3 -faces with $\ell \in[3,4]$. Suppose first that $\ell=3$. Then both $f_{4}$ and $f_{6}$ are 4-faces. Let the facial cycles of $f_{4}$ and $f_{6}$ be $v v_{4} u_{1} v_{5}$ and $v v_{1} u_{2} v_{6}$, respectively. Recall that $d\left(v_{4}\right) \geq \delta(G) \geq 3$ by Claim 1. Now the assumption that $G$ contains no 7 -cycle with chords implies that $u_{1} \in\left\{v_{1}, v_{2}\right\}$. By symmetry, we also get that $u_{2} \in\left\{v_{3}, v_{4}\right\}$. Now the edges $v_{5} u_{1}$ and $v_{6} u_{2}$ contradict that $G$ is a plane embedding. The case when $\ell=4$ can be treated is a similar way to obtain a contradiction with planarity. We omit the details. So (4) holds. This completes the proof of Claim 4.

The next two claims deal with the local structure around a 6 -vertex with two 3-neighbors.

Claim 5. Let $v$ be a 6 -vertex of $G$. If $v$ is incident with two edge-disjoint triangles $v x y v$ and $v u w v$ with $d(x)=d(u)=3$, then the other neighbors of $v$ are 6-vertices.

Proof. Suppose to be contrary that $z$ is a $5^{-}$-neighbor of $v$. Let $x^{\prime}$ be the vertex in $N(x) \backslash\{v, y\}$, and let $u^{\prime}$ be the vertex in $N(u) \backslash\{v, w\}$. Then $d\left(x^{\prime}\right)=d(y)=$ $d(w)=d\left(u^{\prime}\right)=6$ by Claim 1, and $u^{\prime} \neq x^{\prime}$ by Claim 2. Let $G^{\prime}=G-\{x, u\}$.
 any $e \in E(G)$, let

$$
\begin{equation*}
S_{\phi}(e)=L(e) \backslash\left\{\phi(h): h \in E\left(G^{\prime}\right) \text { is adjacent to } e \text { in } G\right\} \tag{2.2}
\end{equation*}
$$

Then $\left|S_{\phi}(e)\right| \geq 2$ for $e \in\left\{x x^{\prime}, x y, u u^{\prime}, u w\right\}$ and $\left|S_{\phi}(e)\right| \geq 3$ for $e \in\{x v, u v\}$. If $\left|S_{\phi}(x y)\right| \geq 3$, then we can obtain an edge- $L$-coloring of $G$ from $\phi$ by choosing a color from $S_{\phi}\left(u u^{\prime}\right), S_{\phi}(u w), S_{\phi}(u v), S_{\phi}(x v), S_{\phi}\left(x x^{\prime}\right)$ and $S_{\phi}(x y)$ to color $u u^{\prime}, u w, u v, x v, x x^{\prime}$ and $x y$ respectively. This contradicts (b), and thus $\left|S_{\phi}(x y)\right|=2$. Similar arguments are used to establish that $\left|S_{\phi}\left(x x^{\prime}\right)\right|=$ $\left|S_{\phi}\left(u u^{\prime}\right)\right|=\left|S_{\phi}(u w)\right|=2$ and $\left|S_{\phi}(x v)\right|=\left|S_{\phi}(u v)\right|=3$.

We claim that $S_{\phi}(x y)=S_{\phi}\left(x x^{\prime}\right)$ and $S_{\phi}\left(u u^{\prime}\right)=S_{\phi}(u w)$. Suppose to be contrary that $S_{\phi}(x y) \backslash S_{\phi}\left(x x^{\prime}\right) \neq \emptyset$. Then we can obtain an edge- $L$-coloring of $G$ from $\phi$ by choosing a color from $S_{\phi}(x y) \backslash S_{\phi}\left(x x^{\prime}\right), S\left(u u^{\prime}\right), S_{\phi}(u w), S_{\phi}(u v)$, $S_{\phi}(x v)$ and $S_{\phi}\left(x x^{\prime}\right)$ to color $x y, u u^{\prime}, u w, u v, x v$ and $x x^{\prime}$, a contradiction to (b). By symmetry, $S_{\phi}\left(u u^{\prime}\right)=S_{\phi}(u w)$.

Since $\phi$ was chosen arbitrarily, using the above claim it is sufficient to prove that $G^{\prime}$ has an edge- $\left.L\right|_{E\left(G^{\prime}\right)}$-coloring, denoted by $\psi$, satisfying either $S_{\psi}(x y) \neq S_{\psi}\left(x x^{\prime}\right)$ or $S_{\psi}\left(u u^{\prime}\right) \neq S_{\psi}(u w)$. For this, we make use of an auxiliary directed graph.

Let $D$ be a directed graph with vertex set $V(D)=\{v y, v z, v w\}$ and arc set $A(D)=\left\{\left(e_{1}, e_{2}\right): \phi\left(e_{1}\right) \in S_{\phi}^{\prime}\left(e_{2}\right), \forall e_{1}, e_{2} \in V(D)\right\}$, where $\left(e_{1}, e_{2}\right) \in A(D)$ means that $e_{1}$ is an in-neighbor of $e_{2}$ in $D$, and for any $e \in E(G)$,

$$
S_{\phi}^{\prime}(e)=L(e) \backslash\left\{\phi(h): h \in E\left(G^{\prime}-\{v y, v z, v w\}\right) \text { is adjacent to } e \text { in } G\right\}
$$

Note that $\left|S_{\phi}^{\prime}(e)\right| \geq 2$ for $e \in\{v y, v z, v w\}$. We construct $\psi$ as follows. If $|A(D)|=\emptyset$, then let $\psi$ be the mapping obtained from $\phi$ by recoloring $v y$ with a color in $S_{\phi}^{\prime}(v y) \backslash\{\phi(v y)\}$. If $D$ contains a directed cycle $e_{1} e_{2} \ldots e_{t} e_{t+1}$ with $t \in[2,3]$ and $e_{1}=e_{t+1}$, then let $\psi$ be the mapping obtained from $\phi$ by replacing the color of $e_{i}$ with $\phi\left(e_{i-1}\right)$ for $i \in[2, t+1]$. If $D$ contains a maximal directed path $e_{1} e_{2} \ldots e_{t}$ with $t \in[2,3]$, then let $\psi$ be the mapping obtained from $\phi$ by replacing the color of $e_{1}$ with a color in $S_{\phi}^{\prime}\left(e_{1}\right) \backslash\left\{\phi\left(e_{1}\right)\right\}$ and the color of $e_{i}$ with $\phi\left(e_{i-1}\right)$ for $i \in[2, t]$. In each case, it is not difficult to check that $\psi$ is an edge- $\left.L\right|_{E\left(G^{\prime}\right)}$-coloring of $G^{\prime}$. Moreover, either $\psi(v y) \neq \phi(v y)$, or $\psi(\nu w) \neq \phi(\nu w)$, or $\psi(v y) \neq \phi(v y)$ and $\psi(\nu w) \neq \phi(\nu w)$. Without loss of generality, assume that $\psi(v y) \neq \phi(v y)$.

Note that $S_{\psi}(x y)=\left(S_{\phi}(x y) \cup\{\phi(v y)\}\right) \backslash\{\psi(v y)\}$ and $S_{\psi}\left(x x^{\prime}\right)=$ $S_{\phi}\left(x x^{\prime}\right)$. Since $\psi(v y) \neq \phi(v y), \phi(v y) \in S_{\psi}(x y)$. Since $\left|S_{\phi}(x y)\right|=2$, $\phi(v y) \notin S_{\phi}(x y)$ by Eq. (2.2), and thus $\phi(v y) \notin S_{\phi}\left(x x^{\prime}\right)=S_{\psi}\left(x x^{\prime}\right)$. Hence $S_{\psi}(x y) \neq S_{\psi}\left(x x^{\prime}\right)$, and thus the proof of Claim 5 is complete.

The proof of the following claim is similar to the proof of Claim 5 (and to Case 3 of Theorem 6 in [34]) and therefore omitted.

Claim 6. Let $v$ be a 6 -vertex of $G$. If $v$ is incident with five neighbors $u, w, x, y, z$ such that $d(u)=d(y)=3$ and $x y, y z \in E(G)$, then $u w \notin E(G)$.

The next claim follows directly from Lemma 2.6 and assumption (b).
Claim 7. $G$ contains no $F_{1}-F_{14}$.

In the remainder of this section, we will obtain a contradiction by using the discharging method. We assign the initial charge $w(v)=3 d(v)-10$ to every vertex $v$, and $w(f)=2 d(f)-10$ to every face $f$. By straightforward calculations we obtain

$$
\sum_{v \in V(G)}(3 d(v)-10)+\sum_{f \in F(G)}(2 d(f)-10)=-20
$$

In order to reach a contradiction, we redistribute the charges among the vertices and the faces in $G$ by the following discharging rules. After that, we will show that the new charges of all the vertices and faces are nonnegative, our final contradiction.

In our discharging, we use the following five discharging rules, in which we introduce several subrules for a number of distinguished cases regarding rules 4 and 5.
(R1) Every 3-vertex receives $\frac{1}{2}$ from each of its 3-masters.
(R2) Each $6^{+}$-face gives $\frac{2 d(f)-10}{d(f)}$ to each vertex on its boundary.
(R3) Let $v$ be a 4-vertex. If $\lambda_{4^{-}}(v) \leq 2$, then $v$ gives 1 to each incident $4^{-}$-face; otherwise, $v$ gives $\frac{1}{2}$ to each incident face.
(R4) Let $v$ be a 5 -vertex.
(R4.1) $v$ gives $\frac{1}{2}$ to each incident 4-face;
(R4.2) $v$ gives the following charge to each incident 3-face $f=[v w u]$.
(R4.2.1) If $\lambda_{3}(v) \leq 2$ or $d(w)=4$ and $d(u)=5$, then $v$ gives $\frac{7}{4}$ to $f$.
(R4.2.2) If $d(w)=4$ and $d(u)=6$, then $v$ gives $a$ to $f$, where

$$
a= \begin{cases}\frac{7}{4} & \text { if } \lambda_{4^{-}}(w)>2 \text { and } u \text { has a 3-neighbor } \\ \frac{6}{4} & \text { if } \lambda_{4^{-}}(w)>2 \text { and } u \text { has no 3-neighbor; } \\ \frac{5}{4} & \text { if } \lambda_{4^{-}}(w) \leq 2\end{cases}
$$

(R4.2.3) If $d(w)=d(u)=5$, then $v$ gives $a$ to $f$, where

$$
a= \begin{cases}\frac{6}{4} & \text { if } \lambda_{3}(v)=3 \\ 1 & \text { if } \lambda_{3}(v) \geq 4\end{cases}
$$

(R4.2.4) If $d(u)=5$ and $d(w)=6$, then $v$ gives $a$ to $f$, where

$$
a= \begin{cases}\frac{6}{4} & \text { if } \lambda_{3}(v)=3 \\ \frac{3}{4} & \text { if } \lambda_{3}(v) \geq 4 \text { and } w \text { has a 3-neighbor } \\ \frac{2}{4} & \text { if } \lambda_{3}(v) \geq 4 \text { and } w \text { has no 3-neighbor }\end{cases}
$$

(R4.2.5) If $d(u)=d(w)=6$, then $v$ gives $a$ to $f$, where

$$
a= \begin{cases}\frac{2}{4} & \text { if both } u \text { and } w \text { have 3-neighbors; } \\ \frac{1}{4} & \text { if at least one of } u, w \text { has no 3-neighbor. }\end{cases}
$$

(R5) Let $v$ be a 6-vertex.
(R5.1) $v$ gives $a$ to each incident 3-face $f=[v w u]$, where $a= \begin{cases}\frac{4}{3} & \text { if } f \text { is a }(6,6,6) \text {-face; } \\ \frac{7}{4} & \text { if } f \text { contains 4- or 5-vertices and } v \text { has a 3-neighbor; } \\ 2 & \text { otherwise. }\end{cases}$
(R5.2) $v$ gives $a$ to each 4-face $f=[v x y z]$ incident with $v$, where

$$
a= \begin{cases}\frac{1}{2} & \text { if } d(x), d(y), d(z) \geq 4 \\ \frac{2}{3} & \text { if either } d(y)=3 \text { or } d(x)=3 \text { and } d(z)=6 \\ \frac{3}{4} & \text { if } d(x)=3 \text { and } d(z)<6\end{cases}
$$

We next show that the final charge, denoted by $w^{\prime}$, of every vertex and face is nonnegative.

We first consider the final charge of an arbitrary face $f \in F(G)$. When $d(f) \geq 6$, then $w^{\prime}(f) \geq w(f)-d(f) \times \frac{2 d(f)-10}{d(f)}=0$ by (R2). When $d(f)=5$, then $f$ retains its initial charge and it follows that $w^{\prime}(f)=w(f)=2 d(f)-$ $10=0$. When $d(f)=4$, then the boundary of $f$ contains at most one 3 -vertex
by Claim 2 , so $w^{\prime}(f)=w(f)+\min \left\{\frac{1}{2}+2 \times \frac{3}{4}, 3 \times \frac{2}{3}, 4 \times \frac{1}{2}\right\}=0$ by (R3), (R4.1) and (R5.2).

When $d(f)=3$, let $f=[x y z]$. If $d(x)=3$, then $d(y)=d(z)=6$ by Claim 1, so $w^{\prime}(f)=w(f)+2 \times 2=0$ by (R5.1). If $d(x)=4$ and $d(y)=d(z)=5$, then $f$ receives at least $\frac{1}{2}$ from $x$ and $\frac{7}{4}$ from each of $y$ and $z$ by (R3) and (R4.2.1), so $w^{\prime}(f) \geq w(f)+\frac{1}{2}+2 \times \frac{7}{4}=0$. If $d(x)=4, d(y)=5$ and $d(z)=6$, then $w^{\prime}(f) \geq w(f)+\min \left\{\frac{1}{2}+2 \times \frac{7}{4}, \frac{1}{2}+\frac{6}{4}+2,1+\frac{5}{4}+\frac{7}{4}\right\}=0$ by (R3), (R4.2.2) and (R5.1). Note that if $u v \in E(G)$ with $d(u)=d(v)=5$, then at most one of $u$ and $v$ is incident with at least four 3-faces by Claim 4-(4). If $d(x)=d(y)=d(z)=5$, then $w^{\prime}(f)=w(f)+\min \left\{2 \times \frac{6}{4}+1,3 \times \frac{6}{4}\right\}=0$ by (R4.2.3). If $d(x)=d(y)=5$ and $d(z)=6$, then $w^{\prime}(f)=w(f)+\min \{2 \times$ $\left.\frac{6}{4}+\frac{7}{4}, \frac{6}{4}+\frac{3}{4}+\frac{7}{4}, \frac{6}{4}+\frac{2}{4}+2\right\}=0$ by (R4.2.4) and (R5.1). If $d(x)=5$ and $d(y)=d(z)=6$, then $w^{\prime}(f)=w(f)+\min \left\{\frac{2}{4}+2 \times \frac{7}{4}, \frac{1}{4}+\frac{7}{4}+2, \frac{1}{4}+2 \times 2\right\}=0$ by (R4.2.5) and (R5.1). If $d(x)=d(y)=d(z)=6$, then $w^{\prime}(f)=w(f)+3 \times \frac{4}{3}=0$ by (R5.1). This completes the analysis for the faces and shows that indeed the final charges of all the faces are nonnegative.

Now we consider the final charge of an arbitrary $k$-vertex $v \in V(G)$. We recall the assumption on the clockwise ordering of $N_{G}(v)=\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$. Recall that $n_{i}(v)$ denotes the number of $i$-neighbors of $v$. If $k=3$, then $v$ has two 3-masters, thus $w^{\prime}(v)=w(v)+2 \times \frac{1}{2}=0$ by (R1). If $k=4$, then $w^{\prime}(v)=w(v)-\max \left\{2 \times 1,4 \times \frac{1}{2}\right\}=0$ by (R3). To complete the analysis, it is sufficient to consider the two cases $k=5$ and $k=6$.
(1) Suppose first that $k=5$.

Note that every $d$-face incident with $v$ receives 0 from $v$ if $d \geq 5, \frac{1}{2}$ from $v$ if $d=4$, and at most $\frac{7}{4}$ from $v$ if $d=3$ by (R4). It follows that $w^{\prime}(v) \geq$ $w(v)-3 \times \frac{1}{2}-2 \times \frac{7}{4}=0$ if $\lambda_{3}(v) \leq 2$. In the rest of this case, we will apply (R4) to evaluate $w^{\prime}(v)$ by distinguishing the following three subcases according to the value of $\lambda_{3}(v)$.
(1.1) $\lambda_{3}(v)=3$.

Then $\lambda_{4}(v) \leq 1$ since every 7 -cycle of $G$ contains no chords. No matter whether $v$ is a cut vertex or not, we may assume that $f_{1}, f_{2}, f_{\ell}$ are three 3 -faces incident with the edges $v v_{1}$ and $v v_{2}, v v_{2}$ and $v \nu_{3}$, and $v v_{\ell}$ and $v v_{\ell+1}$, respectively, with $\ell \in\{3,4\}$. Assume first that $\ell=3$. Then there are at
most two 4 -vertices in $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ by Claim 1. Note that by (R4), every 4 -face incident with $v$ receives $\frac{1}{2}$ from $v$; every $\left(d_{1}, d_{2}, d_{3}\right)$-face incident with $v$ receives at most $\frac{2}{4}$ from $v$ if $\left(d_{1}, d_{2}, d_{3}\right)=(5,6,6), \frac{6}{4}$ from $v$ if either $\left(d_{1}, d_{2}, d_{3}\right) \in\{(5,5,5),(5,5,6)\}$ or $\left(d_{1}, d_{2}, d_{3}\right)=(4,5,6)$ in which the unique 6 -vertex has no 3 -neighbor, and at most $\frac{7}{4}$ from $v$ if either $\left(d_{1}, d_{2}, d_{3}\right)=(4,5,5)$ or $\left(d_{1}, d_{2}, d_{3}\right)=(4,5,6)$ in which the unique 6 -vertex has at least one 3 neighbor. If there is no 4-vertex in $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$, then $w^{\prime}(v) \geq w(v)-\left(3 \times \frac{6}{4}+\right.$ $\left.\frac{1}{2}\right)=0$. If there are two 4 -vertices in $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$, then by the symmetry of $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{4}, v_{3}\right\}, d\left(v_{1}\right)=d\left(v_{4}\right)=4$ and $d\left(v_{2}\right)=d\left(v_{3}\right)=6$, for otherwise $G$ contains $F_{1}$ or $F_{5}$ of Fig. 2.1, and so $w^{\prime}(v) \geq w(v)-\left(2 \times \frac{7}{4}+\frac{2}{4}+\frac{1}{2}\right)>0$. Thus we may assume that there is one 4-vertex in $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ below. By the symmetry of $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{4}, v_{3}\right\}$, assume further that $d\left(v_{1}\right)=4$ or $d\left(v_{2}\right)=$ 4. Then $v$ is incident with at most two $\left(4,5,5^{+}\right)$-faces. When $\lambda_{4}(v)=0$, $w^{\prime}(v) \geq w(v)-\left(2 \times \frac{7}{4}+\frac{6}{4}\right)=0$. When $\lambda_{4}(v)=1, \lambda_{6^{+}}(v)=1$ by Claim 4(1), and thus it follows from (R2) that $v$ receives at least $\frac{1}{3}$ from the unique incident $6^{+}$-face. If $d\left(v_{1}\right)=4$, then $v$ is incident with one $\left(4,5,5^{+}\right)$-face, and thus $w^{\prime}(v) \geq w(v)+\frac{1}{3}-\left(\frac{7}{4}+2 \times \frac{6}{4}+\frac{1}{2}\right)>0$. If $d\left(v_{2}\right)=4$ and there is no 5 -vertex in $\left\{v_{1}, v_{3}, v_{4}\right\}$, then $w^{\prime}(v) \geq w(v)+\frac{1}{3}-\left(2 \times \frac{7}{4}+\frac{2}{4}+\frac{1}{2}\right)>0$. If $d\left(v_{2}\right)=4$ and there is at least one 5-vertex in $\left\{v_{1}, v_{3}, v_{4}\right\}$, then $w^{\prime}(v) \geq$ $w(v)+\frac{1}{3}-\left(\frac{7}{4}+2 \times \frac{6}{4}+\frac{1}{2}\right)>0$. This is for the following reasons. If $d\left(v_{1}\right)=5$, then $d\left(v_{3}\right)=6$ and $v_{3}$ has no 3 -neighbor, for otherwise $G$ contains $F_{3}$ or $F_{4}$; if $d\left(v_{3}\right)=5$, then $d\left(v_{1}\right)=d\left(v_{4}\right)=6$ and $v_{1}$ has no 3-neighbor, for otherwise $G$ contains one of $\left\{F_{2}, F_{3}, F_{4}\right\}$; if $d\left(v_{4}\right)=5$, then $d\left(v_{3}\right)=6$ and $v_{3}$ has no 3-neighbor, for otherwise $G$ contains $F_{2}$ or $F_{8}$.

Now assume that $\ell=4$. By Claim 4-(1), $\lambda_{7^{+}}(v)=1$ when $\lambda_{4}(v)=1$. Note that $v$ receives at least $\frac{4}{7}$ from the incident $7^{+}$-face, and sends at most $\frac{7}{4}$ to each ( $4,5,5$ )-face and each ( $4,5,6$ )-face in which the unique 6 -vertex has a 3 -neighbor, $\frac{6}{4}$ to each $\left(5,5^{+}, 5^{+}\right)$-face and each $(4,5,6)$-face in which the unique 6 -vertex has no 3 -neighbor, and $\frac{2}{4}$ to each $(5,6,6)$-face. If there is one of $f_{1}, f_{2}, f_{4}$ which is not a $\left(4,5,5^{+}\right)$-face, then $w^{\prime}(v) \geq w(v)-\left(2 \times \frac{7}{4}+\frac{6}{4}\right)=0$ when $\lambda_{4}(v)=0$, and $w^{\prime}(v) \geq w(v)+\frac{4}{7}-\left(2 \times \frac{7}{4}+\frac{6}{4}+\frac{1}{2}\right)=0$ when $\lambda_{4}(v)=1$. If each of $f_{1}, f_{2}, f_{4}$ is a $\left(4,5,5^{+}\right)$-face, since $G$ contains no $F_{1}$, all of $f_{1}, f_{2}, f_{4}$ are ( $4,5,6$ )-faces. Further, since $G$ contains no $F_{5}$, either $d\left(v_{2}\right)=d\left(v_{4}\right)=4$
or $d\left(v_{2}\right)=d\left(v_{5}\right)=4$. Moreover, $d\left(v_{1}\right)=d\left(v_{3}\right)=6$ and $v_{1}, v_{3}$ have no 3neighbor, for otherwise $G$ contains $F_{7}$. Thus $w^{\prime}(v) \geq w(v)-\left(2 \times \frac{6}{4}+\frac{7}{4}\right)>0$ when $\lambda_{4}(v)=0$, and $w^{\prime}(v) \geq w(v)+\frac{4}{7}-\left(2 \times \frac{6}{4}+\frac{7}{4}+\frac{1}{2}\right)>0$ when $\lambda_{4}(v)=1$.
(1.2) $\lambda_{3}(v)=4$.

Clearly, $v$ is not a cut vertex. Without loss of generality, assume that $f_{1}, f_{2}, f_{3}$, $f_{4}$ are four 3-faces. Since $G$ contains no $F_{6}, n_{4}(v) \leq 2$.
(1.2.1) Assume $n_{4}(v)=0$. Then $f_{i}$ for $i \in[1,4]$ is a $\left(5,5^{+}, 5^{+}\right)$-face and thus receives at most 1 from $v$. Hence $w^{\prime}(v) \geq w(v)-4 \times 1-\frac{1}{2}>0$, since $f_{5}$ is a $4^{+}$-face.
(1.2.2) Assume $n_{4}(v)=1$. By the symmetry of $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{5}, v_{4}\right\}$, assume that $d\left(v_{1}\right)=4$, or $d\left(v_{2}\right)=4$, or $d\left(v_{3}\right)=4$. Note that $n_{5}(v) \leq 2$ since $G$ contains none of $\left\{F_{2}, F_{3}, F_{10}, F_{11}, F_{12}\right\}$.

When $\lambda_{4}(v)=0, v$ gives at most $\frac{7}{4}$ to each $(4,5,5)$-face and each $(4,5,6)$ face in which the unique 6 -vertex has a 3 -neighbor, 1 to each $(5,5,5)$-face, and $\frac{3}{4}$ to each $(5,5,6)$-face. If $d\left(v_{1}\right)=4$, then $v$ is incident with at most one $\left(4,5,5^{+}\right)$-face, thus $w^{\prime}(v) \geq w(v)-\frac{7}{4}-3 \times 1>0$. If $d\left(v_{3}\right)=4$, then $v$ is incident with no $(5,5,5)$-face, for otherwise $G$ contains $F_{1}$, thus $w^{\prime}(v) \geq$ $w(v)-2 \times \frac{7}{4}-2 \times \frac{3}{4}=0$. If $d\left(v_{2}\right)=4$ and $n_{5}(v) \leq 1$, then there are two $\left(4,5,5^{+}\right)$-faces and two $\left(5,5^{+}, 6\right)$-faces in $\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}$, and thus $w^{\prime}(v) \geq$ $w(v)-2 \times \frac{7}{4}-2 \times \frac{3}{4}=0$. Next, assume that $d\left(v_{2}\right)=4$ and $n_{5}(v)=2$ below. Since $G$ contains no $F_{2}$ and $F_{3}$, either $d\left(v_{1}\right)=d\left(v_{4}\right)=5$, or $d\left(v_{1}\right)=d\left(v_{5}\right)=5$, or $d\left(v_{3}\right)=d\left(v_{5}\right)=5$, or $d\left(v_{4}\right)=d\left(v_{5}\right)=5$. For the former three cases, there are two $\left(4,5,5^{+}\right)$-faces and two $\left(5,5^{+}, 6\right)$-faces in $\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}$. For the last case, there are two $(4,5,6)$-faces, one $(5,5,6)$-face and one $(5,5,5)$-face in $\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}$, and $v_{3}$ has no 3-neighbor, for otherwise $G$ contains $F_{8}$. Hence $w^{\prime}(v) \geq w(v)-\max \left\{2 \times \frac{7}{4}+2 \times \frac{3}{4}, \frac{7}{4}+\frac{6}{4}+\frac{2}{4}+1\right\}=0$.

When $\lambda_{4}(v)=1$, the facial cycle of $f_{5}$ is the 4-cycle $v v_{1} v_{3} v_{5}$, since every 7cycle of $G$ contains no chords. So $d\left(v_{3}\right) \geq 5$, and $d\left(v_{1}\right)=4$ or $d\left(v_{2}\right)=4$. Since every 4-neighbor of $v$ is incident with at most two $4^{-}$-faces by Claim 4-(2), $v$ sends at most $\frac{7}{4}$ to each $(4,5,5)$-face, $\frac{5}{4}$ to each $(4,5,6)$-face, 1 to each $(5,5,5)$ face, and at most $\frac{3}{4}$ to each $(5,5,6)$-face. If $d\left(v_{1}\right)=4$, then there is at most one $(4,5,5)$-face and at least one $\left(5,5^{+}, 6\right)$-face in $\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}$, and thus $w^{\prime}(v) \geq w(v)-\frac{1}{2}-\left(\frac{7}{4}+\frac{3}{4}+2 \times 1\right)=0$. Next, assume that $d\left(v_{2}\right)=4$ below. If
$d\left(v_{1}\right)=5$, then $d\left(v_{3}\right)=6$, since $G$ contains no $F_{3}$, and there is no ( $5,5,5$ )-face in $\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}$, since $n_{5}(v) \leq 2$. If $d\left(v_{1}\right)=6$ and $d\left(v_{3}\right)=5$, then $d\left(v_{4}\right)=6$, since $G$ contains no $F_{2}$, and so $\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}$ consists of one ( $4,5,5$ )-face, one $(4,5,6)$, one $(5,5,6)$-face and one $\left(5,5^{+}, 6\right)$-face. If $d\left(v_{1}\right)=6$ and $d\left(v_{3}\right)=6$, then $\left\{f_{1}, f_{2}, f_{3}, f_{4}\right\}$ consists of two ( $4,5,6$ )-faces, one ( $5,5^{+}, 6$ )-face and one $\left(5,5^{+}, 5^{+}\right)$-face. Hence $w^{\prime}(v) \geq w(v)-\frac{1}{2}-\max \left\{\frac{7}{4}+\frac{5}{4}+2 \times \frac{3}{4}, 2 \times \frac{5}{4}+\frac{3}{4}+1\right\} \geq 0$.
(1.2.3) Assume $n_{4}(v)=2$. By the symmetry of $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{5}, v_{4}\right\}$, assume that either $d\left(v_{1}\right)=d\left(v_{3}\right)=4$, or $d\left(v_{1}\right)=d\left(v_{4}\right)=4$, or $d\left(v_{1}\right)=$ $d\left(v_{5}\right)=4$, or $d\left(v_{2}\right)=d\left(v_{4}\right)=4$. Then either $d\left(v_{1}\right)=d\left(v_{4}\right)=4$ or $d\left(v_{1}\right)=$ $d\left(v_{5}\right)=4$, since $G$ contains no $F_{6}$. If $d\left(v_{1}\right)=d\left(v_{4}\right)=4$, then $d\left(v_{2}\right)=d\left(v_{3}\right)=$ $d\left(v_{5}\right)=6$, since $G$ contains no $F_{1}$, and either of $v_{3}, v_{5}$ has no 3-neighbor, since $G$ contains no $F_{7}$. If $d\left(v_{1}\right)=d\left(v_{5}\right)=4$, then $d\left(v_{2}\right)=d\left(v_{3}\right)=d\left(v_{4}\right)=6$, for otherwise $G$ contains $F_{1}$ or $F_{9}$. Hence $w^{\prime}(v) \geq w(v)-\max \left\{\frac{7}{4}+\frac{1}{4}+\frac{6}{4}+\frac{6}{4}, 2 \times \frac{7}{4}+\right.$ $\left.2 \times \frac{2}{4}\right\}=0$ if $\lambda_{4}(v)=0$, and $w^{\prime}(v) \geq w(v)-\frac{1}{2}-\max \left\{3 \times \frac{5}{4}+\frac{2}{4}, 2 \times \frac{5}{4}+2 \times \frac{2}{4}\right\}>0$ if $\lambda_{4}(v)=1$.
(1.3) $\lambda_{3}(v)=5$.

Clearly, $v$ is not a cut vertex. Then $n_{4}(v) \leq 1$, since $G$ contains no $F_{6}$, and each 5 -neighbor of $v$ is incident with at most three 3 -faces by Claim 4-(3). By (R4.2), $v$ gives at most $\frac{7}{4}$ to each ( $4,5,5$ )-face and each ( $4,5,6$ )-face in which the unique 6 -vertex has a 3 -neighbor, 1 to each ( $5,5,5$ )-face, $\frac{3}{4}$ to each $(5,5,6)$ face in which the unique 6 -vertex has a 3 -neighbor, $\frac{2}{4}$ to each $(5,5,6)$-face in which the unique 6 -vertex has no 3 -neighbor and to each ( $5,6,6$ )-face. It is obvious that $w^{\prime}(v) \geq w(v)-5 \times 1=0$ if $n_{4}(v)=0$. Next, assume that $n_{4}(v)=$ 1 , say $d\left(v_{1}\right)=4$. Then $n_{5}(v) \leq 1$, since $G$ contains none of $\left\{F_{2}, F_{3}, F_{13}, F_{14}\right\}$. If $n_{5}(v)=0$, then $\left\{f_{1}, f_{2}, f_{3}, f_{4}, f_{5}\right\}$ consists of two (4,5,6)-faces and three $(5,6,6)$-faces, and so $w^{\prime}(v) \geq w(v)-2 \times \frac{7}{4}-3 \times \frac{2}{4}>0$. If $n_{5}(v)=1$, then by symmetry, assume that $d\left(v_{2}\right)=5$ or $d\left(v_{3}\right)=5$. Since $G$ contains no $F_{4}$ (resp., $F_{8}$ ), $v_{5}$ has no 3 -neighbor if $d\left(v_{2}\right)=5$ (resp., $v_{2}$ has no 3-neighbor if $\left.d\left(v_{3}\right)=5\right)$. Thus $w^{\prime}(v) \geq w(v)-\max \left\{\frac{7}{4}+\frac{3}{4}+\frac{2}{4}+\frac{2}{4}+\frac{6}{4}, \frac{6}{4}+\frac{2}{4}+\frac{3}{4}+\frac{2}{4}+\frac{7}{4}\right\}=0$.
(2) $k=6$.

By Claim 3, $\lambda_{3}(v) \leq 4$. Moreover, when $\lambda_{3}(v)=4, \lambda_{4}(v) \leq 1$ with equality only if $\lambda_{7^{+}}(v)=1$, by Claim 4-(1). We distinguish two cases.
(2.1) $v$ adjacent to no 3 -vertex. Then $v$ is not a 3-master.

If $\lambda_{3}(v)=4$ and $\lambda_{4}(v)=1$, then $\lambda_{7^{+}}(v)=1$, and all vertices of the unique 4-face incident with $v$ are in $N(v) \cup\{v\}$, and so $w^{\prime}(v) \geq w(v)+\left(2-\frac{10}{7}\right)-$ $2 \times 4-\frac{1}{2}>0$ by (R2) and (R5). If $\lambda_{3}(v)=4$ and $\lambda_{4}(v)=0$, then $w^{\prime}(v) \geq$ $w(v)-2 \times 4=0$, since $v$ gives at most 2 to each incident 3-face by (R5.1). If $\lambda_{3}(v) \leq 3$, then every 4 -face incident with $v$ is not a $\left(6,3,6,5^{-}\right)$-face, and thus receives at most $\frac{2}{3}$ from $v$ by (R5.2). So $w^{\prime}(v) \geq w(v)-3 \times 2-3 \times \frac{2}{3}=0$.
(2.2) $v$ adjacent to some 3 -vertices. Then $v$ is a 3 -master of exactly one 3 -neighbor, and so sends $\frac{1}{2}$ to this 3 -neighbor by (R1). By Claims 5 and $6, v$ is incident with at most two ( $3,6,6$ )-faces.
If $\lambda_{3}(v) \leq 2$, then $w^{\prime}(v) \geq w(v)-\frac{1}{2}-\max \left\{2 \times 2+4 \times \frac{3}{4}, 2+5 \times \frac{3}{4}, 6 \times \frac{3}{4}\right\}>0$ by (R5.1) and (R5.2). Assume that $\lambda_{3}(v)=3$. Then $\lambda_{4}(v) \leq 3$. If $\lambda_{4}(v) \leq 2$, then $w^{\prime}(v) \geq w(v)-\frac{1}{2}-\left(2 \times 2+\frac{7}{4}+2 \times \frac{3}{4}\right)>0$ by (R5.1) and (R5.2). If $\lambda_{4}(v)=3$, then the degrees of any two consecutive faces around $v$ are 3 and 4 by Claim 4-(2). Note that $v$ is incident with at least one (3, 6, 6) -face since $v$ adjacent to at least one 3 -vertex. When $v$ is incident with two $(3,6,6)$-faces, it follows from Claim 5 that $v$ is incident with one $(6,6,6)$-face, one $\left(6,6,6,3^{+}\right)$face and two $(3,6,6,6)$-faces, and so $w^{\prime}(v) \geq w(v)-\frac{1}{2}-\left(2 \times 2+\frac{4}{3}+3 \times \frac{2}{3}\right)>0$ by (R5.1) and (R5.2). When $v$ is incident with one (3,6,6)-face, it is incident with at most one $\left(3,6,5^{-}, 6\right)$-face. If $v$ is incident with at least one $(6,6,6)$ face, then $w^{\prime}(v) \geq w(v)-\frac{1}{2}-\left(2+\frac{7}{4}+\frac{4}{3}+\frac{3}{4}+2 \times \frac{2}{3}\right)>0$ by (R5.1) and (R5.2). If $v$ is not incident with a $(6,6,6)$-face, then $v$ is incident with at least one $\left(6,4^{+}, 4^{+}, 4^{+}\right)$-face, thus $w^{\prime}(v) \geq w(v)-\frac{1}{2}-\left(2+2 \times \frac{7}{4}+\frac{3}{4}+\frac{2}{3}+\frac{1}{2}\right)>0$ by (R5.1) and (R5.2).

Assume that $\lambda_{3}(v)=4$. If $\lambda_{4}(v)=0$, then $w^{\prime}(v) \geq w(v)-\frac{1}{2}-(2 \times$ $2+2 \times \frac{7}{4}$ ) $=0$ by (R5.1). If $\lambda_{4}(v)=1$ and the unique 4 -face is incident with no 3 -vertex, then $\lambda_{7^{+}}(v)=1$ by Claim 4-(1), and the unique 4 -face receives $\frac{1}{2}$ from $v$ by (R5.2). It follows from (R2) and (R5) that $w^{\prime}(v) \geq$ $w(v)-\frac{1}{2}+\left(2-\frac{10}{7}\right)-\left(2 \times 2+2 \times \frac{7}{4}+\frac{1}{2}\right)>0$. Hence assume further that $\lambda_{4}(v)=1$ and the unique 4 -face is incident with a 3 -vertex. Then $\lambda_{7^{+}}(v)=1$ by Claim 4(1) again, and $v$ has one of the local structures $(d),(e),(f)$ shown in Fig. 2.2. Recall Claim 2 that there is no even cycle in $G$ such that $d(x)+d(y)=9$ for each edge $x y$ of the cycle.

For structure ( d ), either $d\left(v_{1}\right)=3$ or $d\left(v_{6}\right)=3$, since the 4 -face is incident with a 3 -vertex and $d\left(v_{3}\right) \geq 5$. If $d\left(v_{1}\right)=3$, then $d\left(v_{2}\right)=d\left(v_{3}\right)=6, d\left(v_{4}\right) \neq 3$ and $d\left(v_{6}\right) \neq 3$, and so $w^{\prime}(v) \geq w(v)-\frac{1}{2}+\left(2-\frac{10}{7}\right)-\left(2 \times 2+\frac{4}{3}+\frac{7}{4}+\frac{3}{4}\right)>0$ by (R5); if $d\left(v_{6}\right)=3$, then $d\left(v_{3}\right)=d\left(v_{5}\right)=6, d\left(v_{2}\right) \neq 3, d\left(v_{4}\right) \neq 3$, and so $w^{\prime}(v) \geq w(v)-\frac{1}{2}+\left(2-\frac{10}{7}\right)-\left(2+3 \times \frac{7}{4}+\frac{3}{4}\right)>0$ by (R5).

For structure (e), either $d\left(v_{1}\right)=3$ or $d\left(v_{6}\right)=3$, since the 4 -face is incident with a 3 -vertex and $d\left(v_{3}\right) \geq 5$. If $d\left(v_{1}\right)=3$, then $d\left(v_{2}\right)=d\left(v_{3}\right)=6, d\left(v_{4}\right) \neq 3$ and $d\left(v_{6}\right) \neq 3$, and so $w^{\prime}(v) \geq w(v)-\frac{1}{2}+\left(2-\frac{10}{7}\right)-\max \left\{2 \times 2+2 \times \frac{4}{3}+\frac{3}{4}, 2+\right.$ $\left.\frac{4}{3}+2 \times \frac{7}{4}+\frac{3}{4}\right\}>0$; if $d\left(v_{6}\right)=3$, then $d\left(v_{3}\right)=6$ and $d\left(v_{1}\right) \neq 3, d\left(v_{2}\right) \neq 3$ and $d\left(v_{4}\right) \neq 3$, so $w^{\prime}(v) \geq w(v)-\frac{1}{2}+\left(2-\frac{10}{7}\right)-\max \left\{2+2 \times \frac{7}{4}+\frac{4}{3}+\frac{3}{4}, 4 \times \frac{7}{4}+\frac{3}{4}\right\}>0$.

For structure (f), either $d\left(v_{1}\right)=3$ or $d\left(v_{6}\right)=3$, since the 4-face is incident with a 3-vertex and $d\left(v_{4}\right) \geq 5$. If $d\left(v_{1}\right)=3$, then $d\left(v_{2}\right)=d\left(v_{4}\right)=6$ and $d\left(v_{3}\right) \neq 3, d\left(v_{5}\right) \neq 3$ and $d\left(v_{6}\right) \neq 3$, so $w^{\prime}(v) \geq w(v)-\frac{1}{2}+\left(2-\frac{10}{7}\right)-$ $\left(2+3 \times \frac{7}{4}+\frac{3}{4}\right)>0$ by (R5). If $d\left(v_{6}\right)=3$, then $d\left(v_{4}\right)=6, d\left(v_{1}\right) \neq 3$, $d\left(v_{3}\right) \neq 3$ and $d\left(v_{5}\right) \neq 3$. If $d\left(v_{2}\right)=3$, then $d\left(v_{1}\right)=d\left(v_{3}\right)=6$, so $w^{\prime}(v)=$ $w(v)-\frac{1}{2}+\left(2-\frac{10}{7}\right)-\left(2 \times 2+\frac{4}{3}+\frac{7}{4}+\frac{2}{3}\right)>0$ by (R5); if $d\left(v_{2}\right) \neq 3$, then $w^{\prime}(v) \geq w(v)-\frac{1}{2}+\left(2-\frac{10}{7}\right)-\left(4 \times \frac{7}{4}+\frac{3}{4}\right)>0$ by (R5).

The above analysis shows that all final charges of the vertices and faces are nonnegative, as we claimed. This completes the proof of Theorem 2.3.

### 2.4 Conclusion and future work

In this chapter, we proved Conjecture 1.3 holds for planar graphs with $\Delta \geq 6$ in which each 7-cycle contains no chords.

Recall that earlier results: $\chi^{\prime}(G)=\Delta$ for planar graphs with $\Delta \geq 7$ and $\chi_{\ell}^{\prime}(G)=\Delta$ for planar graphs with $\Delta \geq 12$. This means there is still a large gap for Conjecture 1.2. In the future, we may work on Conjecture 1.2 with $\Delta=11$. And Conjecture 1.3 remains open for $\Delta=5,6,7$. It might be interesting to weaken the conjecture and we may consider whether all planar graphs are edge- $(\Delta+2)$-choosable.

## Chapter 3

## Critical signed edge graphs

In the study of edge colorings of graphs, critical graphs are of particular interest. On one hand, each graph $G$ contains a critical graph $H$ with $\chi^{\prime}(H)=$ $\chi^{\prime}(G)$ as a subgraph. On the other hand, critical graphs have more structure than arbitrary graphs. In this chapter, we study the structure of $\Delta$-critical signed graphs.

### 3.1 Introduction

A graph $G$ or signed graph $(G, \sigma)$ with maximum degree $\Delta$ is said to be $\Delta$-critical (or simply critical) if it is not $\Delta$-edge-colorable, but $G-e$ is $\Delta$-edgecolorable for any edge $e \in E(G)$.

The study of critical graphs was initiated by Vizing. In his paper [62] about edge coloring, he established a result about the neighborhood of an edge in a $\Delta$-critical graph, which known as Vizing's Adjacency Lemma.

Lemma 3.1 (Vizing's Adjacency Lemma [62]). Let $G$ be a $\Delta$-critical graph with $\Delta \geq 2$. Then for each edge $x y, x$ has at least $\max \{2, \Delta-d(y)+1\}$ neighbors of degree $\Delta$ other that $y$.

Many similar adjacency lemmas have been established over the years by various researchers.

Lemma 3.2 (Luo and Zhang [50]). Let $G$ be a $\Delta$-critical graph and $u$ be a 3vertex of $G$. Suppose that the three neighbors of $u$ are all $\Delta$-vertices. If $v \in N(u)$ has a $(\Delta-2)^{-}$-neighbor $w \neq u$, then $u$ has a neighbor distinct from $v$ which has no $(\Delta-2)^{-}$-neighbors other than $u$.

Lemma 3.3 (Zhang [80]). Let $G$ be a $\Delta$-critical graph. If $u v \in E(G)$ and $d(u)+d(v)=\Delta+2$, then
(i) every vertex of $N(N(u, v)) \backslash\{u, v\}$ is of degree at least $\Delta-1$;
(ii) if both $d(u), d(v)<\Delta$, then every vertex of $N(N(u, v)) \backslash\{u, v\}$ is of degree at least $\Delta$.

Lemma 3.4 (Sanders and Zhao [54]). No $\Delta$-critical graph. has distinct vertices $x, y, z$ such that $x$ is adjacent to $y$ and $z, d(z)<2 \Delta-d(x)-d(y)+2$, and $x z$ is in at least $d(x)+d(y)-\Delta-2$ triangles not containing $y$.

These adjacency results have broad applications and are extensively used in edge coloring. It is natural to explore the possible existence and potential applications of analogues of Adjacency Lemmas for signed graph. Cao, Luo, Miao and Zhao extend Lemma 3.1 and Lemma 3.2 to signed graphs with even maximum degree.

Lemma 3.5 (Signed Vizing's Adjacency Lemma [12]). Let $(G, \sigma)$ be a $\Delta$ critical graph with even $\Delta \geq 2$. Then for each edge $x y, x$ has at least $\max \{2, \Delta-$ $d(y)+1\}$ neighbors of degree $\Delta$.

Lemma 3.6 (Cao, Luo, Miao and Zhao [12]). Let $(G, \sigma)$ be a $\Delta$-critical signed graph with even $\Delta$ and $x$ be a 3-vertex of $G$. Suppose that the three neighbors of $x$ are all $\Delta$-vertices. If $y \in N(x)$ has a $(\Delta-2)^{-}$-neighbor $z \neq x$, then $x$ has a neighbor distinct from $y$ which has no $(\Delta-2)^{-}$-neighbors other than $x$.

In this chapter, we establish the following analogue of this statement for signed graphs with maximum degree 8.

Lemma 3.7. Let $(G, \sigma)$ be an 8-critical signed graph with $\Delta(G)=8$ and let $x y \in E(G)$. Then $d(x)+d(y) \geq 10$ and the following statements hold.
(a) If $d(x)+d(y)=10$, then $x$ is adjacent to at least $(8-d(y)+1) 8$-vertices other than $y$.
(b) If $d(x)+d(y)=11$, then $x$ is adjacent to at least $(8-d(y)+1) 7^{+}$-vertices other than $y$.
(c) If $d(x)+d(y)=12$ and $d(x) \in\{7,8\}$, then $x$ is adjacent to at least $(8-d(y)+1) 6^{+}$-vertices other than $y$, and at least three of these vertices are $7^{+}$-vertices.

And we extend Lemma 3.3-(2) to signed graphs with even maximum degree.

Lemma 3.8. Let $G$ be a $\Delta$-critical graph with even $\Delta$, let $u v$ be an edge in $G$. If $d(u)+d(v)=\Delta+2$, and $d(u), d(v)<\Delta$, then every vertex in $N(N(u, v)) \backslash\{u, v\}$ is a $\Delta$-vertex.

In order to present our proof of Lemma 3.7 and Lemma 3.8, we need some additional terminology and notation, and we prove one auxiliary lemmas in the next section.

### 3.2 Signed Kempe chain

The set of all $k$-edge-colorings of a signed graph $(G, \sigma)$ is denoted by $\mathscr{C}^{k}(G, \sigma)$. Let $\varphi \in \mathscr{C}^{k}(G, \sigma)$. For a vertex $u \in V(G)$, define the two color sets

$$
\varphi(u)=\left\{\varphi\left(h_{e}^{u}\right): e \in E(G)\right\} \text { and } \bar{\varphi}(u)=M_{k} \backslash \varphi(u) .
$$

We call $\varphi(u)$ the set of colors present at $u$ and $\bar{\varphi}(u)$ the set of colors missing at $u$. Recalled that the colors of the two half edges on negative edge are the same, we can simply denote the colors of both half edges of $e$ by $\varphi(e)$ for each negative edge $e$.

Kempe chains have been a useful tool in the study of edge coloring of graphs. In [3], Behr extended this concept to signed graphs. We repeat the relevant terminology and notation here.

For $\left\{v_{0}, v_{1}, \ldots, v_{m}\right\} \subseteq V(G)$, and $\left\{e_{1}, \ldots, e_{m}\right\} \subseteq E(G)$, the alternating sequence $v_{0} e_{1} v_{1} \ldots v_{m-1} e_{m} v_{m}$ is called a signed trail if all $e_{i}=v_{i-1} v_{i}$ with sign
$\sigma\left(e_{i}\right)$ for $i \in[1, m]$ are distinct (but vertices may be repeated). See Fig. 3.1 for an example. Since all the edges in the trail are specified by their end vertices, we use the shorthand $\left(v_{0}, v_{1}, \ldots, v_{m}\right)$ for such a trail.

Definition 3.1 (Behr [3]). Let $\varphi$ be a $k$-edge-coloring of a signed graph ( $G, \sigma$ ). A signed trail $T=\left(v_{0}, v_{1}, \ldots, v_{m}\right)$ in $G$ is called an $a / b$-chain at $v_{0}$ with respect to $\varphi$ (See Fig. 3.1) if the following four conditions hold.
(1) $a \in \bar{\varphi}\left(v_{0}\right)$ and $b \in \varphi\left(v_{0}\right)$;
(2) The edge magnitudes alternate between $|a|$ and $|b|$ along $T$ (starting with $|b|)$;
(3) $\left\{\varphi\left(h_{e_{i}}^{v_{i}}\right), \varphi\left(h_{e_{i+1}}^{v_{i}}\right)\right\}=\left\{(-1)^{t_{i}} a,(-1)^{t_{i}} b\right\}$ for $i \in[1, m-1]$, where $t_{i}$ is the number of positive edges that appear on $T$ between $v_{0}$ and $v_{i}$;
(4) The length $m$ of $T$ is maximal.


Figure 3.1: Two signed Kempe chains.

Denote by $P_{v_{0}}(a, b, \varphi)$ the $a / b$-chain starting at $v_{0}$ with respect to $\varphi$. When $a \neq 0$ and $b \neq 0$, let $\varphi / P_{v_{0}}(a, b, \varphi)$ denote the edge $k$-coloring obtained from $\varphi$ by swapping colors $a$ and $b$, and swapping colors $-a$ and $-b$ on $P_{v_{0}}(a, b, \varphi)$. Clearly this swapping does not change the present or missing colors at any vertex except the two ends of the chain.

In [12], Cao et al. gave some basic properties of $a / b$-chain which applied frequently.

Proposition 3.9 (Cao, Luo, Miao and Zhao [12]). Let $\varphi \in \mathscr{C}^{k}(G, \sigma)$ and $a, b \neq 0$ be two colors. Then we have the following:
(1) $\varphi / P_{v}(a, b, \varphi)$ is a proper coloring in $\mathscr{C}^{k}(G, \sigma)$;
(2) For any two vertices $u, v \in V$, if $P_{v}(b, a, \varphi)$ and $P_{v}(-a,-b, \varphi)$ and $P_{u}(a, b, \varphi)$ exist, then
(2.1) $P_{v}(b, a, \varphi)$ and $P_{u}(a, b, \varphi)$ are either identical (the underlying graphs of this two trails are the same subgraph of $G$ ) or are edge-disjoint;
(2.2) $P_{u}(a, b, \varphi)$ and $P_{v}(-a,-b, \varphi)$ are either identical or are edge-disjoint.

Proposition 3.10 (Cao, Luo, Miao and Zhao [12]). Let ( $G, \sigma$ ) be a $\Delta$-critical signed graph with $\Delta \geq 2$. Then $G$ is 2 -connected and $d(u)+d(v) \geq \Delta+2$ for any edge $u v \in E(G)$.

Lemma 3.11. Let $(G, \sigma)$ be a $\Delta$-critical graph and $u v$ be a negative edge in $G$. Let $\varphi \in \mathscr{C}^{\Delta}\left(G-u v,\left.\sigma\right|_{G-u v}\right)$. Let $a \in \bar{\varphi}(u)$ and $b \in \bar{\varphi}(v)$ be two nonzero colors. Then $\bar{\varphi}(u) \cap \bar{\varphi}(v)=\emptyset$ and $P_{u}(a, b, \varphi)=P_{v}(b, a, \varphi)$.

Proof. If $\bar{\varphi}(u) \cap \bar{\varphi}(v) \neq \emptyset$, then $\varphi$ can be extended to a $\Delta$-edge-coloring of $G$ by coloring $u v$ with a color in $\bar{\varphi}(u) \cap \bar{\varphi}(v)$, a contradiction to $(G, \sigma)$ is $\Delta$-critical.

Let $\varphi^{\prime}=\varphi / P_{u}(a, b, \varphi)$. If $P_{u}(a, b, \varphi) \neq P_{v}(b, a, \varphi)$, then $\bar{\varphi}^{\prime}(u) \cap \bar{\varphi}^{\prime}(v) \neq \emptyset$, and so $\varphi^{\prime}$ can be extended to a $\Delta$-edge-coloring of $G$ by coloring $u v$ with a color in $\bar{\varphi}^{\prime}(u) \cap \bar{\varphi}^{\prime}(v)$, a contradiction to $(G, \sigma)$ is $\Delta$-critical.

### 3.3 Proof of Lemma 3.7

Since the relevant case in our proof of Theorem 4.1 in Chapter 4 deals with $\Delta(G)=8$, we restrict ourselves in this section to 8-critical signed graphs and to 8 -edge-colorings. We will prove a number of structural results that will help us to reduce the number of cases in the discharging proof of Chapter 4.

For any $S \subseteq M_{8}$, let $-S=\{-a: a \in S\}$. Obviously, $-S \subseteq M_{8}$. For a fixed vertex $x \in V(G)$, we can ensure that all edges incident with $x$ are negative
by switching at some vertices in $N(x)$. In the remainder of this section, we always assume that the edges incident with $x$ are negative. So for any vertex $y \in N(x)$, we can treat the colors of the two half edges $h_{x y}^{x}$ and $h_{x y}^{y}$ as the color of the edge $x y$.

Let $G$ be an 8 -critical signed graph and let $x$ be this fixed vertex in $G$. Assume that $N(x)=\left\{y_{0}, y_{1}, \ldots, y_{d(x)-1}\right\}$, where $y_{0}=y$ and $\varphi$ is an 8-edgecoloring of $G-x y$. By Lemma 3.11,

$$
\bar{\varphi}(y) \subseteq \varphi(x)=\left\{\varphi\left(x y_{1}\right), \ldots, \varphi\left(x y_{d(x)-1}\right)\right\}
$$

Without loss of generality, assume that $\bar{\varphi}(x)=\left\{a_{1}, a_{2}, \ldots, a_{|\bar{\varphi}(x)|}\right\}, \bar{\varphi}(y)=$ $\left\{b_{1}, b_{2}, \ldots, b_{q}\right\}$ where $q=|\bar{\varphi}(y)|$ and $b_{i}=\varphi\left(x y_{i}\right)$ for each $i \in[1, q]$.

For each $i \in[1, q]$, let $\varphi_{i}$ be obtained from $\varphi$ by uncoloring $x y_{i}$, coloring $x y$ with $\varphi\left(x y_{i}\right)$, and without changing the colors of the other half edges. For each $i \in[0, q]$, let $P_{x}\left(a, b, \varphi_{i}\right)$ be the $a / b$-chain starting at $x$ with respect to $\varphi_{i}$.

Lemma 3.12. Let $G$ be an 8 -critical signed graph, and let $x y$ be an edge of $G$. Assume that $\varphi$ is an 8 -edge-coloring of $G-x y$. For each pair $\{i, j\} \subseteq[1, q]$, the following statements hold.
(1) If $b \in \bar{\varphi}(y) \cap \bar{\varphi}\left(y_{i}\right)$, then
(1.1) there is an index $k \in[1, q] \backslash\{i\}$ such that $b=\varphi\left(x y_{k}\right)$;
(1.2) $P_{x}(a, b, \varphi)=\left(x, y_{k}, \ldots, u, x, y_{i}, \ldots, y\right)$ where $a \in \bar{\varphi}(x), u \in N(x)$, $\left\{\varphi(x u), \varphi\left(x y_{i}\right)\right\}=\{-a,-b\}$.
(2) If $a \in \bar{\varphi}(x)$ and $b \in \varphi(x) \cap \varphi(y) \cap \bar{\varphi}\left(y_{i}\right) \cap \bar{\varphi}\left(y_{j}\right)$, then
(2.1) there is a vertex $u \in N(x) \backslash\left\{y_{1}, \ldots, y_{q}\right\}$ such that $b=\varphi(x u)$;
(2.2) $\varphi\left(x y_{i}\right) \in\{-a,-b\}$ or $\varphi\left(x y_{j}\right) \in\{-a,-b\}$;
(2.3) $\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\} \neq\{-a,-b\}$;
(2.4)

$$
P_{x}\left(a, b, \varphi_{i}\right)= \begin{cases}\left(x, u, \ldots, w, x, y, \ldots, y_{i}\right) & \text { if } \varphi\left(x y_{i}\right) \in\{-a,-b\} \\ \left(x, u, \ldots, w, x, y_{j}, \ldots, y_{i}\right) & \text { if } \varphi\left(x y_{j}\right) \in\{-a,-b\}\end{cases}
$$

$$
\text { where } w \in N(x) \text { and } \varphi(x w)=\{-a,-b\} \backslash\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\}
$$

Proof. (1.1) It can be obtained directly from $\bar{\varphi}(y)=\left\{\varphi\left(x y_{1}\right), \ldots, \varphi\left(x y_{q}\right)\right\}$ and $b \in \bar{\varphi}(y) \cap \bar{\varphi}\left(y_{i}\right)$.
(1.2) Suppose to the contrary that $P_{x}(a, b, \varphi) \neq\left(x, y_{k}, \ldots, u, x, y_{i}, \ldots, y\right)$. By Lemma 3.11, $P_{x}(a, b, \varphi)$ ends at $y$. Since $a \in \bar{\varphi}(x)$, by (1.1) and the definition of $a / b$-chain, $P_{x}(a, b, \varphi)=\left(x, y_{k}, \ldots, y\right)$ or $P_{x}(a, b, \varphi)=$ $\left(x, y_{k}, \ldots, u, x, v, \ldots, y\right)$ where $\{\varphi(x u), \varphi(x v)\}=\{-a,-b\}$ and $u, v$ are distinct neighbors of $x$. Note that $a \in \overline{\varphi_{i}}(x)$ and $b \in \overline{\varphi_{i}}\left(y_{i}\right)=\bar{\varphi}\left(y_{i}\right) \cup\left\{\varphi\left(x y_{i}\right)\right\}$ with respect to $\varphi$. In the following, we will show that $P_{x}\left(a, b, \varphi_{i}\right)$ either does not end at $y_{i}$, or ends at $y_{i}$ but $\varphi_{i}\left(h_{e}^{y_{i}}\right) \neq a$, where $e$ is the last edge of $P_{x}\left(a, b, \varphi_{i}\right)$. This is a contradiction to Lemma 3.11.

If $P_{x}(a, b, \varphi)=\left(x, y_{k}, \ldots, y\right)$ or $P_{x}(a, b, \varphi)=\left(x, y_{k}, \ldots, u, x, v, \ldots, y\right)$ and $y_{i} \notin\{u, v\}$, then $P_{x}\left(a, b, \varphi_{i}\right)$ ends at $y$, a contradiction. If $P_{x}(a, b, \varphi)=$ $\left(x, y_{k}, \ldots, y_{i}, x, v, \ldots, y\right)$, then $P_{x}\left(a, b, \varphi_{i}\right)$ ends at $y_{i}$ since $\varphi\left(x y_{i}\right) \in \overline{\varphi_{i}}\left(y_{i}\right)$, however, $\varphi_{i}\left(h_{e}^{y_{i}}\right) \in\{-a,-b\}$ where $e$ is the last edge of $P_{x}\left(a, b, \varphi_{i}\right)$, a contradiction. This completes the proof of (1).

Before we prove (2) of the lemma, we first make some general observations. For $\ell \in\{i, j\}$, we have $a \in \overline{\varphi_{\ell}}(x), b \in \overline{\varphi_{\ell}}\left(y_{\ell}\right)=\bar{\varphi}\left(y_{\ell}\right) \cup\left\{\varphi\left(x y_{\ell}\right)\right\}$ $\left(b \neq \varphi\left(x y_{\ell}\right)\right)$. By Lemma 3.11, $P_{x}\left(a, b, \varphi_{i}\right)$ ends at $y_{i}$. We will show that $P_{x}\left(a, b, \varphi_{j}\right)$ does not end at $y_{j}$, or ends at $y_{j}$ but $\varphi_{i}\left(h_{e}^{y_{j}}\right) \neq a$, where $e$ is the last edge of $P_{x}\left(a, b, \varphi_{j}\right)$. This is a contradiction to Lemma 3.11.
(2.1) It can be obtained directly from $b \notin \bar{\varphi}(y)=\left\{\varphi\left(x y_{1}\right), \ldots, \varphi\left(x y_{q}\right)\right\}$ and $b \in \varphi(x)$.

By (2.1), $\varphi(x u)=b$ where $u \in N(x) \backslash\left\{y_{1}, \ldots, y_{q}\right\}$.
(2.2) Suppose to the contrary that $\varphi\left(x y_{i}\right) \notin\{-a,-b\}$ and $\varphi\left(x y_{j}\right) \notin$ $\{-a,-b\}$. For each $\ell \in\{i, j\}$, since $\varphi\left(x y_{\ell}\right) \notin\{a,-a,-b\}, x y, x y_{j} \notin E\left(P_{x}(a\right.$, $\left.b, \varphi_{i}\right)$ ). Then $P_{x}\left(a, b, \varphi_{j}\right)$ ends at $y_{i}$ since $b \in \overline{\varphi_{j}}\left(y_{i}\right)=\bar{\varphi}\left(y_{i}\right)$, a contradiction.
(2.3) Suppose to the contrary that $\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\}=\{-a,-b\}$. With respect to $\varphi_{i}, \varphi_{i}(x y) \in\{-a,-b\}$. We claim that $x y \in E\left(P_{x}\left(a, b, \varphi_{i}\right)\right)$. If not, then $P_{x}\left(a, b, \varphi_{i}\right)=\left(x, u, \ldots, y_{i}\right)$, and so $P_{x}\left(a, b, \varphi_{j}\right)$ ends at $y_{i}$, a contradiction. Thus $\{-a,-b\} \subseteq \varphi_{i}(y)=\varphi(y) \cup\left\{\varphi\left(x y_{i}\right)\right\}$. Since $\varphi\left(x y_{j}\right)=$ $\{-a,-b\} \backslash\left\{\gamma_{0}\left(x y_{i}\right)\right\}, \varphi\left(x y_{j}\right) \in \varphi(y)$, a contradiction to $\varphi\left(x y_{j}\right) \in \bar{\varphi}(y)$.
(2.4) Suppose to the contrary that $P_{x}\left(a, b, \varphi_{i}\right) \neq\left(x, u, \ldots, w, x, y, \ldots, y_{i}\right)$ when $\varphi\left(x y_{i}\right) \in\{-a,-b\}$ and $P_{x}\left(a, b, \varphi_{i}\right) \neq\left(x, u, \ldots, w, x, y_{j}, \ldots, y_{i}\right)$ when $\varphi\left(x y_{j}\right) \in\{-a,-b\}$. If $\varphi\left(x y_{i}\right) \in\{-a,-b\}$, then $\varphi\left(x y_{j}\right) \notin\{-a,-b\}$ by (2.3). Since $P_{x}\left(a, b, \varphi_{i}\right) \neq\left(x, u, \ldots, w, x, y, \ldots, y_{i}\right), P_{x}\left(a, b, \varphi_{i}\right)=\left(x, u, \ldots, y_{i}\right)$ or $P_{x}\left(a, b, \varphi_{i}\right)=\left(x, u, \ldots, y, x, w, \ldots, y_{i}\right)$ where $\left\{\varphi_{i}(x y), \varphi_{i}(x w)\right\}=\{-a,-b\}$. For the first case, $P_{x}\left(a, b, \varphi_{j}\right)$ ends at $y_{i}$, a contradiction; for the last case, $P_{x}\left(a, b, \varphi_{j}\right)$ ends at $y$ since $\varphi\left(x y_{i}\right) \in \overline{\varphi_{j}}(y)$, a contradiction.

If $\varphi\left(x y_{j}\right) \in\{-a,-b\}$, then $\varphi\left(x y_{i}\right) \notin\{-a,-b\}$ by (2.3). Since

$$
P_{x}\left(a, b, \varphi_{i}\right) \neq\left(x, u, \ldots, w, x, y_{j}, \ldots, y_{i}\right)
$$

$P_{x}\left(a, b, \varphi_{i}\right)=\left(x, u, \ldots, y_{i}\right)$ or $P_{x}\left(a, b, \varphi_{i}\right)=\left(x, u, \ldots, y_{j}, x, w, \ldots, y_{i}\right)$ where $\left\{\varphi_{i}\left(x y_{j}\right)=\varphi\left(x y_{j}\right), \varphi_{i}(x w)\right\}=\{-a,-b\}$. For the first case, $P_{x}\left(a, b, \varphi_{j}\right)$ ends at $y_{i}$, a contradiction; for the last case, $P_{x}\left(a, b, \varphi_{j}\right)$ ends at $y_{j}$ but $\varphi_{j}\left(h_{e}^{y_{j}}\right) \in$ $\{-a,-b\}$, where $e$ is the last edge of $P_{x}\left(a, b, \varphi_{j}\right)$, a contradiction.

In the following lemma, we show which colors must be present at vertices $y_{1}, \ldots, y_{q}$ with respect to $\varphi$.

Lemma 3.13. Let $G$ be an 8-critical signed graph, and let $x y$ be an edge of $G$. Assume that $\varphi$ is an 8-edge-coloring of $G-x y$ and $P=\varphi(x) \cap \varphi(y)$. For each $i \in[1, q]$, the following statements hold.
(1) If either $d(x)<8$ or $d(x)=8$ and $-\bar{\varphi}(x) \subseteq \varphi(y)$, then

$$
\varphi\left(y_{i}\right) \supseteq \begin{cases}\bar{\varphi}(y) \backslash\left\{-\varphi\left(x y_{i}\right)\right\} & \text { if }-\bar{\varphi}(x) \subseteq P \cap \varphi\left(y_{i}\right) \\ \bar{\varphi}(y) & \text { otherwise }\end{cases}
$$

(2) If $d(x)=8$ and $-\bar{\varphi}(x) \subseteq \bar{\varphi}(y)$, then

$$
\varphi\left(y_{i}\right) \supseteq \begin{cases}\bar{\varphi}(y) \backslash(-P) & \text { if }-\varphi\left(x y_{i}\right) \in \bar{\varphi}(x) \\ \bar{\varphi}(y) & \text { otherwise } .\end{cases}
$$

(3) Let $d(x)=8$ and $-\bar{\varphi}(x)=\left\{-a_{1}\right\} \subseteq \bar{\varphi}(y)$. If $c \in \bar{\varphi}(y) \cap \bar{\varphi}\left(y_{i}\right) \cap(-P)$ where $\varphi\left(x y_{i}\right)=-a_{1}$, then $-c \in \bar{\varphi}\left(y_{i}\right)$.

Proof. (1) Suppose to the contrary that there is a color in $\bar{\varphi}(y) \backslash\left\{-\varphi\left(x y_{i}\right)\right\}$ but not in $\varphi\left(y_{i}\right)$. Without loss of generality, denote this color by $b\left(b \neq-\varphi\left(x y_{i}\right)\right)$. By Lemma 3.12-(1.1), there is an index $k \in[1, q] \backslash\{i\}$ such that $\varphi\left(x y_{k}\right)=b$. If $d_{G}(x) \leq 7$, then $\left\{a_{1}, a_{2}\right\} \subseteq \bar{\varphi}(x)$. There is a color $a \in\left\{a_{1}, a_{2}\right\}$ such that $a \neq-\varphi\left(x y_{i}\right)$. Without loss of generality, assume that $a=a_{1}$. By Lemma 3.12-(1.2), $P_{x}\left(a_{1}, b, \varphi\right)=\left(x, y_{k}, \ldots, u, x, y_{i}, \ldots, y\right)$ where $\left\{\varphi(x u), \varphi\left(x y_{i}\right)\right\}=$ $\left\{-a_{1},-b\right\}$. Since $\varphi\left(x y_{i}\right) \neq-b, \varphi\left(x y_{i}\right)=-a_{1}$, a contradiction to assumption. If $d_{G}(x)=8$, then $\bar{\varphi}(x)=\left\{a_{1}\right\}$. Since $b \neq-\varphi\left(x y_{i}\right)$, by Lemma 3.12-(1.2), $-a_{1}=\varphi\left(x y_{i}\right) \in \bar{\varphi}(y)$, a contradiction to $\left\{-a_{1}\right\}=-\bar{\varphi}(x) \subseteq \varphi(y)$. Hence $\bar{\varphi}(y) \backslash\left\{-\varphi\left(x y_{i}\right)\right\} \subseteq \varphi\left(y_{i}\right)$.

Now we only need to show that the color $b=-\varphi\left(x y_{i}\right) \in \varphi\left(y_{i}\right)$ when $-\bar{\varphi}(x) \nsubseteq \varphi(x) \cap \varphi\left(y_{i}\right) \cap \varphi(y)$. Suppose to the contrary that $b=-\varphi\left(x y_{i}\right) \notin$ $\varphi\left(y_{i}\right)$. If $-\bar{\varphi}(x) \nsubseteq \varphi(x) \cap \varphi\left(y_{i}\right) \cap \varphi(y)$, then there is a color $a \in \bar{\varphi}(x)$ such that $-a \notin \varphi(x)$, or $-a \notin \varphi\left(y_{i}\right)$, or $-a \notin \varphi(y)$. By Lemma 3.12(1.2), $P_{x}(a, b, \varphi)=\left(x, y_{k}, \ldots, u, x, y_{i}, \ldots, y\right)$. If $-a \notin \varphi(x)$ or $-a \notin \varphi\left(y_{i}\right)$, then $P_{x}(a, b, \varphi)=\left(x, y_{k}, \ldots, y\right)$, a contradiction. Note that $a \in \overline{\varphi_{i}}(x)$ and $b \in \overline{\varphi_{i}}\left(y_{i}\right)$ with respect to $\varphi_{i}$. If $-a \notin \varphi(y)$, then $P_{x}\left(a, b, \varphi_{i}\right)$ ends at $y$ since $-a \in \overline{\varphi_{i}}(y)=\bar{\varphi}(y) \backslash\left\{\varphi\left(x y_{i}\right)\right\}$, a contradiction to Lemma 3.11. This completes the proof of (1).
(2) We first note that since $d_{G}(x)=8,-\bar{\varphi}(x)=\left\{-a_{1}\right\}$. When $\varphi\left(x y_{i}\right)=$ $-a_{1}$, suppose to the contrary that there is a color in $\bar{\varphi}(y) \backslash(-P)$ but not in $\varphi\left(y_{i}\right)$. Without loss of generality, denote this color by $b$. By Lemma 3.12(1.1) and (1.2), there is an index $k \in[1, q] \backslash\{i\}$ such that $\varphi\left(x y_{k}\right)=b$ and $P_{x}\left(a_{1}, b, \varphi\right)=\left(x, y_{k}, \ldots, u, x, y_{i}, \ldots, y\right)$ where $\varphi(x u)=-b$. Note that $a_{1} \in$ $\overline{\varphi_{i}}(x)$ and $b \in \overline{\varphi_{i}}\left(y_{i}\right)$ with respect to $\varphi_{i}$. Since $b \notin(-P)$ and $\{b,-b\} \subseteq \varphi(x)$, $-b \in \bar{\varphi}(y)=\overline{\varphi_{i}}(y) \cup\left\{-a_{1}\right\}$. Then $P_{x}\left(a_{1}, b, \varphi_{i}\right)$ ends at $y$, a contradiction to Lemma 3.11.

When $\varphi\left(x y_{i}\right) \neq-a_{1}$, suppose to the contrary that there is a color in $\bar{\varphi}(y)$ but not $\varphi\left(y_{i}\right)$. Without loss of generality, denote this color by $b$. By Lemma 3.12-(1.1) and (1.2), there is an index $k \in[1, q] \backslash\{i\}$ such that $\varphi\left(x y_{k}\right)=b$ and $P_{x}\left(a_{1}, b, \varphi\right)=\left(x, y_{k}, \ldots, u, x, y_{i}, \ldots, y\right)$ where $\varphi(x u)=-a_{1}$ and $\varphi\left(x y_{i}\right)=-b$. Note that $a_{1} \in \overline{\varphi_{i}}(x)$ and $b \in \overline{\varphi_{i}}\left(y_{i}\right)$ with respect to $\varphi_{i}$. Then $P_{x}\left(a_{1}, b, \varphi_{i}\right)$ ends at $y$ since $-a_{1} \in \overline{\varphi_{i}}(y) \cup\{-b\}=\bar{\varphi}(y)$, a contradiction
to Lemma 3.11. This completes the proof of (2).
(3) Suppose to the contrary that $-c \notin \varphi\left(y_{i}\right)$. Note that $\left\{a_{1}\right\}=\bar{\varphi}(x)$ and $c \in \bar{\varphi}(y)$. By Lemma 3.12-(1.1) and (1.2), there is an index $k \in$ $[1, q] \backslash\{i\}$ such that $\varphi\left(x y_{k}\right)=c$ and $P_{x}\left(a_{1}, c, \varphi\right)=\left(x, y_{k}, \ldots, u, x, y_{i}, \ldots, y\right)$. where $\varphi\left(x y_{i}\right)=-a_{1}$ and $\varphi(x u)=-c$. Since $-c \notin \varphi\left(y_{i}\right), P_{x}\left(a_{1}, c, \varphi\right)=$ $\left(x, y_{k}, \ldots, y\right)$, a contradiction. This completes the proof of (3).

Lemma 3.11-3.13 will be used in the proof of the next lemma. The main aim of these four lemmas is to obtain Lemma 3.7, an analogue of Vizing's Adjacency Lemma. This final Lemma 3.7 will provide us with essential structural information on the vertex degrees in an 8-critical signed graph. As such, it will play a key role in our discharging proof of Theorem 4.1 that will be presented in Chapter 4.

Lemma 3.14. Let $G$ be an 8-critical signed graph, and let $x y$ be an edge of $G$ with $d(x)=7$ and $d(y)=5$. Assume that $\varphi$ is an 8 -edge-coloring of $G-x y$ and $\bar{\varphi}(x)=\left\{a_{1}, a_{2}\right\}$. For each pair $\{i, j\} \subseteq[1, q]$, if there is a color $b \in \bar{\varphi}\left(y_{i}\right) \cap \bar{\varphi}\left(y_{j}\right)$, then
(1) $b \in \varphi(x) \cap \varphi(y)$;
(2) $-a_{1} \neq a_{2}$;
(3) $\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\}=\left\{-a_{1},-a_{2}\right\}$;
(4) $-b \in \varphi(y) \cap \varphi\left(y_{i}\right)$.

Proof. Before we prove the four statements of the lemma, we first make some general observations. Note that $\overline{\varphi_{i}}(x)=\overline{\varphi_{j}}(x)=\bar{\varphi}(x)=\left\{a_{1}, a_{2}\right\}$ and $b \in \overline{\varphi_{i}}\left(y_{i}\right) \cap \overline{\varphi_{j}}\left(y_{j}\right)$. By Lemma 3.11, $P_{x}\left(a_{1}, b, \varphi_{i}\right)$ and $P_{x}\left(a_{2}, b, \varphi_{i}\right)$ end at $y_{i}$ with respect to $\varphi_{i}$.

In the following, with respect to $\varphi_{j}$, we consider $P_{x}\left(a_{1}, b, \varphi_{i}\right)$ and $P_{x}\left(a_{2}\right.$, $\left.b, \varphi_{i}\right)$. We will show that there is a color $a \in\left\{a_{1}, a_{2}\right\}$ such that $P_{x}\left(a, b, \varphi_{j}\right)$ does not end at $y_{j}$, or ends at $y_{j}$ but $\varphi_{j}\left(h_{e}^{y_{j}}\right) \neq a$ where $e$ is the last edge of $P_{x}\left(a, b, \varphi_{j}\right)$, a contradiction to Lemma 3.11.
(1) Suppose to the contrary that $b \notin \varphi(x) \cap \varphi(y)$. If $b \notin \varphi(x)$, that is, $b \in \bar{\varphi}(x)$, then coloring $x y$ with $\varphi\left(x y_{i}\right)$ and coloring $x y_{i}$ with $b$, results in an 8-edge-coloring of $G$, a contradiction. If $b \notin \varphi(y)$, that is, $b \in \bar{\varphi}(y)$, then $b=-\varphi\left(x y_{i}\right)$ and $b=-\varphi\left(x y_{j}\right)$ by Lemma 3.13-(1), a contradiction to $i \neq j$. Thus $b \in \varphi(x) \cap \varphi(y)$.

By (1) and Lemma 3.12-(2.1), $b \notin\left\{a_{1}, a_{2}\right\}$ and there is a vertex $u \in$ $N(x) \backslash\left\{y_{1}, \ldots, y_{q}\right\}$ such that $b=\varphi(x u)$.
(2) Suppose to the contrary that $-a_{1}=a_{2}$. Since $-a_{1} \in \varphi(x), P_{x}\left(a_{1}, b\right.$, $\left.\varphi_{i}\right)=\left(x, u, \ldots, y_{i}\right)$. And so $P_{x}\left(a_{1}, b, \varphi_{j}\right)$ ends at $y_{i}$, a contradiction. Thus $-a_{1} \neq a_{2}$.
(3) Suppose to the contrary that $\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\} \neq\left\{-a_{1},-a_{2}\right\}$. Without loss of generality, assume that $\varphi\left(x y_{i}\right) \neq-a_{1}$. We consider two different cases: $\varphi\left(x y_{i}\right)=-a_{2}$ and $\varphi\left(x y_{i}\right) \neq-a_{2}$.

When $\varphi\left(x y_{i}\right)=-a_{2}$, clearly, $\varphi\left(x y_{j}\right) \neq-a_{1}$. By Lemma 3.12-(2.3), $\varphi\left(x y_{j}\right) \neq-b$. It is easy to check that $x y, x y_{j} \notin E\left(P_{x}\left(a_{1}, b, \varphi_{i}\right)\right)$. Then $P_{x}\left(a_{1}, b, \varphi_{j}\right)$ ends at $y_{i}$, a contradiction.

When $\varphi\left(x y_{i}\right) \neq-a_{2}$ and $-b \notin\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\}$, by Lemma 3.12-(2.2) and (2.3), $\varphi\left(x y_{j}\right) \in\left\{-a_{1},-a_{2}\right\}$. Let $-a \in\left\{-a_{1},-a_{2}\right\} \backslash\left\{\varphi\left(x y_{j}\right)\right\}$. It is easy to check that $x y, x y_{j} \notin E\left(P_{x}\left(a, b, \varphi_{i}\right)\right)$. Then $P_{x}\left(a, b, \varphi_{j}\right)$ ends at $y_{i}$, a contradiction.

When $\varphi\left(x y_{i}\right) \neq-a_{2}$ and $-b \in\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\}$, it implies that $-b \in$ $\bar{\varphi}(y)$ and $b \notin\left\{-a_{1},-a_{2}\right\}$. Since $d(y)=5, a_{1} \neq-a_{2}$ and $b \in \varphi(y)$, $\left\{-a_{1},-a_{2}\right\} \nsubseteq \varphi(y)$. Without loss of generality, assume that $-a_{1} \notin \varphi(y)$. If $\varphi\left(x y_{i}\right)=-b$, then $P_{x}\left(a_{1}, b, \varphi_{i}\right)=\left(x, u, \ldots, y_{i}\right)$, a contradiction to Lemma 3.12-(2.4). If $\gamma_{0}\left(x y_{j}\right)=-b$, then $P_{x}\left(a_{1}, b, \varphi_{i}\right)=\left(x, u, \ldots, w, x, y_{j}, \ldots, y_{i}\right)$ by Lemma 3.12-(2.4). And so $P_{x}\left(a_{1}, b, \varphi_{j}\right)$ ends at $y$ since $-a_{1} \notin \varphi_{j}(y)=$ $\varphi(y) \cup\{-b\}$, a contradiction. Thus $\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\}=\left\{-a_{1},-a_{2}\right\}$.
(4) Suppose to the contrary that $-b \notin \varphi(y) \cap \varphi\left(y_{i}\right)$. By (2) and (3), $-a_{1} \neq a_{2}$ and $\left\{\varphi\left(x y_{i}\right), \varphi\left(x y_{j}\right)\right\}=\left\{-a_{1},-a_{2}\right\}$. Since $b \in \varphi(x) \cap \varphi(y), b \notin$ $\left\{-a_{1},-a_{2}\right\}$ and so $-b \notin\left\{a_{1}, a_{2}\right\}$. By Lemma 3.12-(2.4), $P_{x}\left(-\varphi\left(x y_{i}\right), b, \varphi_{i}\right)$ $=\left(x, u, \ldots, w, x, y, \ldots, y_{i}\right)$ where $w \in N(x)$ and $\varphi(x w)=-b$. If $-b \notin \varphi(y)$, then $P_{x}\left(-\varphi\left(x y_{i}\right), b, \varphi_{i}\right)=\left(x, u, \ldots, y_{i}\right)$, a contradiction. If $-b \notin \varphi\left(y_{i}\right)$,
then $P_{x}\left(-\varphi\left(x y_{i}\right), b, \varphi_{j}\right)$ ends at $y_{i}$, a contradiction to Lemma 3.11. Thus $-b \in \varphi(y) \cap \varphi\left(y_{i}\right)$.

Proof of Lemma 3.7 Before we prove the three statements of the lemma, we first make some general observations. Since $G$ is 8 -critical, $G-x y$ has an 8-edge-coloring $\varphi$. Recall that all edges incident with $x$ are negative in $G$. Also recall that $\bar{\varphi}(x)=\left\{a_{1}, a_{2}, \ldots, a_{|\bar{\varphi}(x)|}\right\}, \bar{\varphi}(y)=\left\{b_{1}, b_{2}, \ldots, b_{q}\right\}$ where $|\bar{\varphi}(y)|=q$ and $b_{i}=\varphi\left(x y_{i}\right)$ for each $i \in[1, q]$. Clearly, $q=8-(d(y)-1)=$ $8-d(y)+1$. Let $P=\varphi(x) \cap \varphi(y)$. It follows directly from the definitions that

$$
\begin{equation*}
\bar{\varphi}(x) \cup \bar{\varphi}(y)=\left(M_{8} \backslash \varphi(x)\right) \cup\left(M_{8} \backslash \varphi(y)\right)=M_{8} \backslash P \tag{3.1}
\end{equation*}
$$

By Lemma 3.11, $\bar{\varphi}(y) \subseteq \varphi(x)$, so $\varphi(x) \cup \varphi(y) \supseteq \bar{\varphi}(y) \cup \varphi(y)=M_{8}$. And

$$
\begin{align*}
|P| & =|\varphi(x)|+|\varphi(y)|-|\varphi(x) \cup \varphi(y)|=d(x)+d(y)-2-\left|M_{8}\right|  \tag{3.2}\\
& =d(x)+d(y)-10
\end{align*}
$$

This implies that $d(x)+d(y) \geq 10$.
For each $i \in[1, q]$, when $d(x) \leq 7$ or $d(x)=8$ and $-\bar{\varphi}(x) \subseteq \varphi(y)$, by Lemma 3.11 and Lemma 3.13-(1), $\varphi\left(y_{i}\right) \supseteq \bar{\varphi}(x) \cup \bar{\varphi}(y) \backslash\left\{-\gamma_{0}\left(x y_{i}\right)\right\}$ if $-\bar{\varphi}(x) \subseteq \varphi\left(y_{i}\right) \cap P ; \varphi\left(y_{i}\right) \supseteq \bar{\varphi}(x) \cup \bar{\varphi}(y)$ otherwise. By Eq. (3.1),

$$
\varphi\left(y_{i}\right) \supseteq \begin{cases}M_{8} \backslash P \backslash\left\{-\varphi\left(x y_{i}\right)\right\} & \text { if }-\bar{\varphi}(x) \subseteq \varphi\left(y_{i}\right) \cap P  \tag{3.3}\\ M_{8} \backslash P & \text { otherwise }\end{cases}
$$

When $d(x)=8$ and $-\bar{\varphi}(x)=\left\{-a_{1}\right\} \subseteq \bar{\varphi}(y)$, by Lemma 3.11, Lemma 3.13-(2) and Eq. (3.1),

$$
\varphi\left(y_{i}\right) \supseteq \begin{cases}\bar{\varphi}(x) \cup \bar{\varphi}(y) \backslash(-P)=M_{8} \backslash P \backslash(-P) & \text { if } \varphi\left(x y_{i}\right)=-a_{1}  \tag{3.4}\\ \bar{\varphi}(x) \cup \bar{\varphi}(y)=M_{8} \backslash P & \text { otherwise }\end{cases}
$$

We next prove the three statements of the lemma in the same order.
For (a), since $d(x)+d(y)=10$, we have $|P|=0$ by Eq. (3.2), that is $P=\emptyset$. If $d(x)<8$, then $-\bar{\varphi}(x) \nsubseteq \varphi\left(y_{i}\right) \cap P=\emptyset$. If $d(x)=8$, then $-\bar{\varphi}(x) \subseteq$ $\bar{\varphi}(y)$. For each $i \in[1, q]$, by Eqs. (3.3) and (3.4), $\varphi\left(y_{i}\right) \supseteq M_{8} \backslash P=M_{8}$ or
$\varphi\left(y_{i}\right) \supseteq M_{8} \backslash P \backslash(-P)=M_{8}$. Thus $x$ is adjacent to $q=8-d(y)+18$-vertices (other than $y$ ).

For (b), since $d(x)+d(y)=11$, we have $|P|=1$ by Eq. (3.2). For each $i \in[1, q]$, if $d(x) \neq 8$, then $|\bar{\varphi}(x)| \geq 2$, and so there is a color $a \in \bar{\varphi}(x)$ such that $-a \notin P$, which implies that $-\bar{\varphi}(x) \nsubseteq P$. By Eq. (3.3), $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq$ $\left|M_{8} \backslash P\right|=7$.

When $d(x)=8$ and $-\bar{\varphi}(x)=\left\{-a_{1}\right\} \subseteq \varphi(y)$, if $-a_{1} \notin \varphi\left(y_{i}\right)$, then $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq\left|M_{8} \backslash P\right|=7$; otherwise $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq \mid M_{8} \backslash P \backslash$ $\left\{-\varphi\left(x y_{i}\right)\right\}\left|+\left|\left\{-a_{1}\right\}\right|=7\right.$ by Eq. (3.3).

When $d(x)=8$ and $-\bar{\varphi}(x)=\left\{-a_{1}\right\} \subseteq \bar{\varphi}(y)$, assume that $P=\{c\}$. Clearly, $-c \in \bar{\varphi}(y) \cap(-P)$. By Eq. (3.4), if $\varphi\left(x y_{i}\right) \neq-a_{1}$ or $\varphi\left(x y_{i}\right)=-a_{1}$ and $-P=\{-c\} \subseteq \varphi\left(y_{i}\right)$, then $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq\left|M_{8} \backslash P\right|=7$; if $\varphi\left(x y_{i}\right)=-a_{1}$ and $-c \in \bar{\varphi}\left(y_{i}\right)$, then $c \in \varphi\left(y_{i}\right)$ by Lemma 3.13-(3), and so $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq$ $\left|M_{8} \backslash P \backslash(-P)\right|+|\{c\}|=7$.

Thus in all cases $x$ is adjacent to $(8-d(y)+1) 7^{+}$-vertices.
For (c), since $d(x)+d(y)=12$, by Eq. (3.2), $|P|=2$. We distinguish two cases: $d(x)=7$ and $d(x)=8$.

Case 1. $d(x)=7$ and $d(y)=5$.
Then $\bar{\varphi}(x)=\left\{a_{1}, a_{2}\right\},|\bar{\varphi}(y)|=q=4$. Let $P=\left\{\varphi\left(x z_{1}\right), \varphi\left(x z_{2}\right)\right\}$, where $\left\{z_{1}, z_{2}\right\} \subseteq N_{G}(x) \backslash\left\{y, y_{1}, y_{2}, y_{3}, y_{4}\right\}$. By Lemma 3.11 and $d(y)=5, \varphi(y)=$ $\left\{a_{1}, a_{2}\right\} \cup P$.

By Eq. (3.3), for each $i \in[1,4]$, if $-\bar{\varphi}(x) \subseteq \varphi\left(y_{i}\right) \cap P$, it implies $P=\left\{-a_{1},-a_{2}\right\} \subseteq \varphi\left(y_{i}\right)$. Then $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq\left|\left(M_{8} \backslash P\right) \backslash\left\{-\varphi\left(x y_{i}\right)\right\}\right|+$ $\left|\left\{-a_{1},-a_{2}\right\}\right|=7$.

If $-\bar{\varphi}(x) \nsubseteq \varphi\left(y_{i}\right) \cap P$, then $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq\left|\left(M_{8} \backslash P\right)\right|=6$. Let $J=\{j \in$ $\left.[1,4]: d\left(y_{j}\right)=6\right\}$. We claim that $|J| \leq 1$. If not, without loss of generality, assume that $d\left(y_{1}\right)=d\left(y_{2}\right)=6$, then $\bar{\varphi}\left(y_{1}\right)=\bar{\varphi}\left(y_{2}\right)=P$. By Lemma 3.14(1)~(4), $a_{1} \neq-a_{2},\left\{\varphi\left(x y_{1}\right), \varphi\left(x y_{2}\right)\right\}=\left\{-a_{1},-a_{2}\right\},\left\{\varphi\left(x z_{1}\right),-\varphi\left(x z_{1}\right)\right\} \subseteq$ $\varphi(y)=\left\{a_{1}, a_{2}\right\} \cup P,-\varphi\left(x z_{1}\right) \in \varphi\left(y_{1}\right)$. Since $\varphi\left(x z_{1}\right) \neq a_{1}, a_{2},-\varphi\left(x z_{1}\right) \neq$ $a_{1}, a_{2},-\varphi\left(x z_{1}\right)=\varphi\left(x z_{2}\right)$, a contradiction to $\varphi\left(x z_{2}\right) \in \bar{\varphi}\left(y_{1}\right)$. Thus there are at least three $7^{+}$-vertices in $\left\{y_{1}, \ldots, y_{4}\right\}$.

Case 2. $d(x)=8$ and $d(y)=4$.

Then $q=8-d(y)+1=5$. We distinguish two subcases: $-a_{1} \in \varphi(y)$ and $-a_{1} \in \bar{\varphi}(y)$.

Subcase 2.1. $-\bar{\varphi}(x)=\left\{-a_{1}\right\} \subseteq \varphi(y)$.
Without loss of generality, assume that $\varphi(y)=\left\{a_{1},-a_{1}, c\right\}$. Then $P=$ $\left\{-a_{1}, c\right\}$.

For each $i \in[1,5]$, by Eq. (3.3), if $-a_{1} \notin \varphi\left(y_{i}\right)$ or $-a_{1} \in \varphi\left(y_{i}\right)$ and $-\varphi\left(x y_{i}\right) \in P$, then $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq\left|\left(M_{8} \backslash P\right)\right|=6$; if $-a_{1} \in \varphi\left(y_{i}\right)$ and $-\varphi\left(x y_{i}\right) \notin P$, then $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq\left|\left(M_{8} \backslash P\right) \backslash\left\{-\varphi\left(x y_{i}\right)\right\}\right|+\left|\left\{-a_{1}\right\}\right|=6$.

Let $J=\left\{j \in[1,5]: d\left(y_{j}\right)=6\right\}$. We claim that $|J| \leq 2$. If not, without loss of generality, assume that $d\left(y_{1}\right)=d\left(y_{2}\right)=d\left(y_{3}\right)=6$ and $\varphi\left(x y_{1}\right), \varphi\left(x y_{2}\right) \neq$ $-c$. It is easy check that $c \in \bar{\varphi}\left(y_{1}\right) \cap \bar{\varphi}\left(y_{2}\right) \cap \varphi(x) \cap \varphi(y)$. By Lemma 3.12(2.2), $-a_{1}=\varphi\left(x y_{1}\right) \in \bar{\varphi}(y)$ or $-a_{1}=\varphi\left(x y_{2}\right) \in \bar{\varphi}(y)$, a contradiction to $\left\{-a_{1}\right\} \subseteq \varphi(y)$.

Subcase 2.2. $-\bar{\varphi}(x)=\left\{-a_{1}\right\} \subseteq \bar{\varphi}(y)$.
Without loss of generality, assume that $\varphi(y)=\left\{a_{1}, c_{1}, c_{2}\right\}, \varphi\left(x y_{5}\right)=-a_{1}$, and so $P=\left\{c_{1}, c_{2}\right\}$.

By Eq. (3.4), for vertex $y_{5}$, when $c_{1}=-c_{2}$, it implies that $P=-P$, thus $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq\left|M_{8} \backslash P \backslash(-P)\right|=6$. When $c_{1} \neq-c_{2}$, if there is a color $-c \in(-P) \cap \bar{\varphi}(y) \cap \bar{\varphi}\left(y_{5}\right)$, then $c \in \varphi\left(y_{5}\right)$ by Lemma 3.13-(3). Thus $d\left(y_{1}\right)=\left|M_{8} \backslash P \backslash(-P)\right|+\left|(-P) \cap \varphi\left(y_{5}\right)\right|+\left|-\left((-P) \backslash \varphi\left(y_{5}\right)\right)\right|=6$.

For each $i \in[1,4]$, by Eq. (3.4), $d\left(y_{i}\right)=\left|\varphi\left(y_{i}\right)\right| \geq\left|M_{8} \backslash P\right|=6$.
Let $J=\left\{j \in[1,4]: d\left(y_{j}\right)=6\right\}$. We claim that $|J| \leq 1$. If not, without loss of generality, assume that $d\left(y_{1}\right)=d\left(y_{2}\right)=6$. Clearly, $\bar{\varphi}\left(y_{1}\right)=\bar{\varphi}\left(y_{2}\right)=P=$ $\left\{c_{1}, c_{2}\right\}$. And there is a color $c \in\left\{c_{1}, c_{2}\right\}$ such that $\varphi\left(x y_{1}\right) \neq-c$. By Lemma 3.12-(2.1), there is a vertex $u \in N(x) \backslash\left\{y_{1}, \ldots, y_{q}\right\}$ such that $c=\varphi(x u)$. Since $\varphi\left(x y_{1}\right), \varphi\left(x y_{2}\right) \neq-a_{1}, \varphi\left(x y_{2}\right)=-c$ by Lemma 3.12-(2.2). By Lemma 3.12-(2.4), $P_{x}\left(a_{1}, c, \varphi_{1}\right)=\left(x, u, \ldots, w, x, y_{2}, \ldots, y_{1}\right)$ where $w=y_{5}$. Note that $a_{1} \in \overline{\varphi_{2}}(x)$ and $c \in \overline{\varphi_{2}}\left(y_{2}\right)$. Then $P_{x}\left(a_{1}, c, \varphi_{2}\right)$ ends at $y$ since $-a_{1} \in \overline{\varphi_{2}}(y)=$ $\bar{\varphi}(y) \backslash\{-c\}$, a contradiction to Lemma 3.11. Thus there are at least three $7^{+}$-vertices in $\left\{y_{1}, \ldots, y_{4}\right\}$.

This completes the proof of the lemma.

### 3.4 Proof of Lemma 3.8

Proof. Recall that for each negative edge $x y$, we can treat the colors of the two half edges $h_{x y}^{x}$ and $h_{x y}^{y}$ as the color of edge $x y$. By switching, we assume that edge $u v$ is negative. Let us consider an arbitrary coloring $\varphi \in \mathscr{C}^{\Delta}(G-u v)$. Such a $\varphi$ exists, since $G$ is $\Delta$-critical and $u v$ is a critical negative edge of G. By Proposition 3.11, $\bar{\varphi}(u) \cap \bar{\varphi}(v)=\emptyset$. Since $d(u)+d(v)=\Delta+2$, $\bar{\varphi}(u) \cup \bar{\varphi}(v)=M_{\Delta}$. Note that each vertex $x$ of $G$ satisfies $|\bar{\varphi}(x)|=\Delta-d_{G}(x)+1$ if $x \in\{u, v\}$ and $|\bar{\varphi}(x)|=\Delta-d_{G}(x)$ otherwise.

Now, let us consider an arbitrary vertex in $N(N(u, v)) \backslash\{u, v\}$. If $z \in$ $N(u, v) \backslash\{u, v\}$, then Signed Vizing's Adjacency Lemma (Lemma 3.5) implies that $d_{G}(z)=\Delta$ and we are done. Otherwise, there is an edge $w z \in E(G)$ for some vertex $w \in N_{G}(u, v) \backslash\{u, v\}$. By symmetry and switching at $z$, we may assume that $w \in N_{G}(v)$ and $w z \in E(G)$ is a negative edge where $d_{G}(z)<\Delta$. Thus path ( $u, u v, v, v w, w, w z, z$ ) is a all negative path in $G$. Since $\bar{\varphi}(z) \neq \emptyset$, suppose that $\gamma \in \bar{\varphi}(z)$. Clearly, $\varphi(v w) \neq \varphi(w z) \neq \gamma$. Since $|\bar{\varphi}(u) \cup \bar{\varphi}(v)|=\Delta$, it follows that $\varphi(v w) \in \bar{\varphi}(u)$ and $\gamma, \varphi(w z) \in \bar{\varphi}(u) \cup \bar{\varphi}(v)$. We distinguish two cases with two subcases each, depending on whether $\varphi(w z) \in \bar{\varphi}(v)$ or $\varphi(w z) \in \bar{\varphi}(u)$, as follows.

Case 1. $\varphi(w z) \in \bar{\varphi}(v)$.
Subcase 1.1. $\gamma \in \bar{\varphi}(v)$.
By Proposition 3.11, we know that $P_{u}(\varphi(\nu w), \gamma, \varphi)=P_{v}(\gamma, \varphi(\nu w), \varphi)$ is edge-disjoint from $P_{z}(\gamma, \varphi(v w), \varphi)$. If $w z \in E\left(P_{\nu}(\gamma, \varphi(v w), \varphi)\right)$, then $\varphi(w z)$ $\in\{-\gamma,-\varphi(\nu w)\}$ since $\gamma \in \bar{\varphi}(z)$, and let $\{\alpha\}=\{-\gamma,-\varphi(\nu w)\} \backslash\{\varphi(w z)\}$. Let $\varphi^{\prime}=\varphi / P_{\nu}(\gamma, \varphi(\nu w), \varphi)$. It can check easily that $\gamma \in \bar{\varphi}^{\prime}(u)$. If $w z \notin$ $E\left(P_{\nu}(\gamma, \varphi(\nu w), \varphi)\right)$ or $w z \in E\left(P_{\nu}(\gamma, \varphi(\nu w), \varphi)\right)$ and $\alpha \in \bar{\varphi}(v)$, then we can construct a proper coloring $\varphi^{\prime \prime}$ of $G-w z$ by setting $\varphi^{\prime \prime}(u v)=\varphi^{\prime}(\nu w)$, $\varphi^{\prime \prime}(v w)=\varphi^{\prime}(w z) \in \bar{\varphi}^{\prime}(v)$. And so $\gamma \in \bar{\varphi}^{\prime \prime}(w) \cap \bar{\varphi}^{\prime \prime}(z)$, a contradiction. If $w z \in E\left(P_{\nu}(\gamma, \varphi(\nu w), \varphi)\right)$ and $\alpha \in \varphi(v)$, then $\alpha \in \varphi(u)$. By Proposition 3.9 and 3.11, $P_{u}(\alpha, \varphi(w z), \varphi)=P_{v}(\varphi(w z), \alpha, \varphi)$ is edge-disjoint from $P_{v}(\gamma, \varphi(\nu w), \varphi)$. Let $\varphi^{\prime \prime}=\varphi^{\prime} / P_{v}(\varphi(w z), \alpha, \varphi)$. Then $\varphi^{\prime \prime}(w z)=\varphi^{\prime}(w z) \in \bar{\varphi}^{\prime \prime}(v)$, and so we are back in an earlier subcase with $w z \in E\left(P_{v}(\gamma, \varphi(v w), \varphi)\right)$ and $\alpha \in \bar{\varphi}^{\prime \prime}(v)$ with respect to $\varphi^{\prime \prime}$.

Subcase 1.2. $\gamma \in \bar{\varphi}(u)$.
Since $d_{G}(v)<\Delta,|\bar{\varphi}(v)| \geq 2$. Let $\delta \in \bar{\varphi}(v) \backslash\{\varphi(w z)\}$. Clearly, $\delta \notin$ $\{\varphi(\nu w), \varphi(w z)\}$ and $\gamma \neq \varphi(w z)$. We claim that $\gamma \neq \varphi(\nu w)$. If not, then $P_{\nu}(\varphi(w z), \gamma, \varphi)$ ends at $z$, a contradiction to Proposition 3.11. By Proposition 3.11, $P_{u}(\gamma, \delta, \varphi)=P_{v}(\delta, \gamma, \varphi)$. Let $\varphi^{\prime}=\varphi / P_{v}(\delta, \gamma, \varphi)$. Then $\varphi^{\prime}$ belongs to $\mathscr{C}^{\Delta}(G-u v)$, and $\bar{\varphi}^{\prime}(u)=\bar{\varphi}(u) \backslash\{\gamma\} \cup\{\delta\}, \bar{\varphi}^{\prime}(v)=\bar{\varphi}(v) \backslash\{\delta\} \cup\{\gamma\}$.

We claim that $\{\nu w, w z\} \nsubseteq E\left(P_{\nu}(\delta, \gamma, \varphi)\right)$. If not, then $\{\varphi(\nu w), \varphi(w z)\}=$ $\{-\gamma,-\delta\}$ and $\{-\gamma,-\delta\} \subseteq \varphi(v)$, a contradiction to $\varphi(w z) \in \bar{\varphi}(v)$. When $\nu w \in E\left(P_{\nu}(\delta, \gamma, \varphi)\right)$, it implies that $\varphi(\nu w) \in\{-\gamma,-\delta\} \subseteq \varphi(v)$, and so $\{-\gamma,-\delta\} \subseteq \bar{\varphi}(u)$. When $w z \in E\left(P_{v}(\delta, \gamma, \varphi)\right)$, it implies that $\varphi(w z) \in\{-\gamma$, $-\delta\}$, let $\{\alpha\}=\{-\gamma,-\delta\} \backslash\{\varphi(w z)\}$.

If $v w, w z \notin E\left(P_{v}(\delta, \gamma, \varphi)\right)$, or $v w \in E\left(P_{v}(\delta, \gamma, \varphi)\right)$, or $w z \in E\left(P_{v}(\delta, \gamma, \varphi)\right)$ and $\alpha \in \bar{\varphi}(v)$, then $\gamma \in \bar{\varphi}^{\prime}(v) \cap \bar{\varphi}^{\prime}(z), \varphi^{\prime}(v w) \in \bar{\varphi}^{\prime}(u)$ and $\varphi^{\prime}(w z) \in \bar{\varphi}^{\prime}(v)$. Hence, we are back to Subcase 1.1.

Recall that $\varphi(w z) \in \bar{\varphi}(v)$. If $w z \in E\left(P_{v}(\delta, \gamma, \varphi)\right)$ and $\alpha \in \varphi(v)$, then $\alpha \in \bar{\varphi}(u)$. By Proposition 3.9 and 3.11, $P_{u}(\alpha, \varphi(w z), \varphi)=P_{v}(\varphi(w z), \alpha, \varphi)$ is edge-disjoint from $P_{v}(\delta, \gamma, \varphi)$ and $v w \notin E\left(P_{v}(\varphi(w z), \alpha, \varphi)\right)$. Let $\varphi^{\prime \prime}=$ $\varphi / P_{v}(\varphi(w z), \alpha, \varphi)$. Then we are back in an earlier subcase with $w z \in$ $E\left(P_{v}(\delta, \gamma, \varphi)\right)$ and $\alpha \in \bar{\varphi}^{\prime \prime}(v)$ with respect to $\varphi^{\prime \prime}$.

Case 2. $\varphi(w z) \in \bar{\varphi}(u)$.
Subcase 2.1. $\gamma \in \bar{\varphi}(v)$.
Since $\gamma \in \bar{\varphi}(v)$ and $\varphi(w z) \in \bar{\varphi}(u), P_{u}(\varphi(w z), \gamma, \varphi)=P_{\nu}(\gamma, \varphi(w z), \varphi)$. Clearly, $w z \notin E\left(P_{v}(\gamma, \varphi(w z), \varphi)\right)$ since $\gamma \in \bar{\varphi}(z)$. Let $\varphi^{\prime}=\varphi / P_{v}(\gamma, \varphi(w z), \varphi)$. Then $\varphi^{\prime}$ belongs to $\mathscr{C}^{\Delta}(G-u v), \bar{\varphi}^{\prime}(u)=\bar{\varphi}(u) \backslash\{\varphi(w z)\} \cup\{\gamma\}$ and $\bar{\varphi}^{\prime}(v)=$ $\bar{\varphi}(v) \backslash\{\gamma\} \cup\{\varphi(w z)\}$. If $v w \notin E\left(P_{v}(\gamma, \varphi(w z), \varphi)\right)$, then $\left\{\gamma, \varphi^{\prime}(v w)=\varphi(v w)\right\}$ $\subseteq \bar{\varphi}^{\prime}(u)=\bar{\varphi}(u), \varphi^{\prime}(w z)=\varphi(w z) \in \bar{\varphi}^{\prime}(v)$. Hence, we are back in Subcase 1.2. If $v w \in E\left(P_{v}(\gamma, \varphi(w z), \varphi)\right)$, then $\varphi(\nu w) \in\{-\gamma,-\varphi(w z)\} \subseteq \varphi(v)$, this implies that $\{-\gamma,-\varphi(w z)\} \subseteq \bar{\varphi}(u)$. Then $\varphi^{\prime}(v w) \in \bar{\varphi}^{\prime}(u), \varphi^{\prime}(w z) \in \bar{\varphi}^{\prime}(v)$ and $\gamma \in \bar{\varphi}^{\prime}(u)$, we are also back in Subcase 1.2.

Subcase 2.2. $\gamma \in \bar{\varphi}(u)$.

Since $|\bar{\varphi}(v)| \geq 2$, there is a color at least one color $\delta \in \bar{\varphi}(v)$. Then $\delta \neq \varphi(\nu w)$. Since $\{\varphi(w z), \gamma\} \subseteq \bar{\varphi}(u), \delta \neq \varphi(w z), \gamma$. By Proposition 3.11, $P_{u}(\delta, \gamma, \varphi)=P_{v}(\gamma, \delta, \varphi)$. Let $\varphi^{\prime}=\varphi / P_{\nu}(\gamma, \delta, \varphi)$. Then $\varphi^{\prime}$ belongs to $\mathscr{C}^{\Delta}(G-$ $u v)$.

We first consider $\gamma \neq \varphi(\nu w)$. When $\nu w \in E\left(P_{v}(\delta, \gamma, \varphi)\right)$, it implies that $\varphi(\nu w) \in\{-\gamma,-\delta\} \subseteq \varphi(v)$, and $\{-\gamma,-\delta\} \subseteq \bar{\varphi}(u)$. If $\nu w, w z \notin E\left(P_{\nu}(\delta, \gamma, \varphi)\right)$, or $\nu w \in E\left(P_{\nu}(\delta, \gamma, \varphi)\right)$, then $\left\{\varphi^{\prime}(\nu w), \varphi^{\prime}(w z)\right\} \subseteq \bar{\varphi}^{\prime}(u)$, and $\gamma \in \bar{\varphi}^{\prime}(v)$. In that case we are back in Subcase 2.1. If only $w z \in E\left(P_{v}(\delta, \gamma, \varphi)\right)$, then $\varphi(w z) \in$ $\{-\gamma,-\delta\}$, and $\{\varphi(\nu w)\} \neq\{-\gamma,-\delta\} \backslash\{\varphi(w z)\}$. If $\{-\gamma,-\delta\} \backslash\{\varphi(w z)\} \in$ $\bar{\varphi}(u)$, then $\varphi^{\prime}(w z) \in \bar{\varphi}^{\prime}(u)$, and so we are again back in Subcase 2.1; if $\{-\gamma,-\delta\} \backslash\{\varphi(w z)\} \in \bar{\varphi}(v)$, then we are back in Subcase 1.1.

The subcase in which $\gamma=\varphi(\nu w)$ can be treated in a similar way. We omit the details. This completes the proof of the lemma.

### 3.5 Conclusion and future work

In this chapter, we established two adjacent lemmas about critical signed graphs with even maximum degree.

In the future, we may consider the question was asked by Cao, Luo, Miao and Zhao.

Question 3.1 (Cao, Luo, Miao and Zhao [12]). Is the signed Vizing's Adjacency Lemma true for critical signed graphs with odd maximum degree?

## Chapter 4

## Signed edge coloring of planar graphs

In this chapter, we show that Conjecture 1.4 is true for signed planar graphs with $\Delta \geq 8$, and true for signed planar graphs with $\Delta \geq 6$ and each 6-cycle contains at most one chord by using the lemmas obtained in Chapter 3.

### 4.1 Introduction

In [82], Zhang et al. prove that every signed planar graph $G$ with maximum degree $\Delta$ is $\Delta$-edge-colorable if either $\Delta \geq 10$ or $\Delta \in\{8,9\}$ and $G$ does not contain any adjacent triangles. We improve the above result by applying Lemma 3.7 to prove that every signed planar graph with maximum degree $\Delta \geq 8$ is $\Delta$-edge-colorable.

Theorem 4.1. Every signed planar graph with maximum degree $\Delta \geq 8$ is $\Delta$-edge-colorable.

In [75], Xue and Wu proved that the planar graph $G$ with $\Delta \geq 6$ and any 6 -cycle contains at most one chord is $\Delta$-edge-colorable. We extend this result to signed graphs by applying Lemma 3.5, Lemma 3.6 and Lemma 3.8.

Theorem 4.2. Let $(G, \sigma)$ be a signed planar graph in which each 6 -cycle contains at most one chord. If $\Delta \geq 6$, then $(G, \sigma)$ is $\Delta$-edge-colorable.

Before presenting the proofs of the theorems, we first introduce the known results.

Theorem 4.3 (Tutte [60]). A graph $G$ has a perfect matching if and only if $o(G-S) \leq|S|$ for all $S \subseteq V(G)$, where $o(G-S)$ is the number of odd components of $G-S$.

Proposition 4.4. Let $G$ be a planar graph with order at least three. Then $|E(G)| \leq 3|V(G)|-6$.

For any two sets $X, Y \subseteq V(G)$, denote by $E_{G}(X, Y)$ the set of edges of $G$ joining a vertex of $X$ and a vertex of $Y$, and denote by $\partial_{G}(X)=E_{G}(X, V(G) \backslash X)$ the boundary edge set of $X$, that is, the set of edges with exactly one end in $X$. Recall that $V_{d}(G)\left(V_{d^{+}}(G), V_{d^{-}}(G)\right.$ respectively) is the set of $d$-vertices $\left(d^{+}\right.$vertices, $d^{-}$-vertices, respectively) in $G$. The following result is an immediate corollary of Proposition 4.4.

Corollary 4.5. Let $G$ be a planar graph with maximum degree $\Delta \geq 6$ and $A \subseteq V_{\Delta}(G)$. Then $\left|\partial_{G}(A)\right| \geq 3(\Delta-2)$ if $|A| \geq 3,\left|\partial_{G}(A)\right|=\Delta$ if $|A|=1$, and $\left|\partial_{G}(A)\right| \geq 2(\Delta-1)$ if $|A|=2$.

We obtained the following lemma.
Lemma 4.6. Let $G$ be a planar graph with maximum degree $\Delta \geq 6$. Then $G$ has a matching $M$ such that every $\Delta$-vertex of $G$ is an end of some edge in $M$.

Proof. Assume that $G$ is connected and denote $|V(G)|=n$. Construct an auxiliary graph $G^{\prime}$ from $G$ and a complete graph $K_{n}$ vertex-disjoint from $G$ such that every vertex in $K_{n}$ is adjacent to every vertex in $V_{(\Delta-1)^{-}}(G)$. Then it is sufficient to show that $G^{\prime}$ has a perfect matching. Suppose to the contrary that $G^{\prime}$ has no perfect matching. Then there is a set of vertices $S \subseteq V\left(G^{\prime}\right)$ such that $o\left(G^{\prime}-S\right) \geq|S|+1$ by Theorem 4.3. If $V\left(K_{n}\right) \subseteq S$, then $o\left(G^{\prime}-S\right)=o(G-S) \leq|V(G-S)| \leq|V(G)|=\left|V\left(K_{n}\right)\right| \leq|S|$, a contradiction. Thus $V\left(K_{n}\right) \nsubseteq S$. Let $O_{1}, \ldots, O_{t}$ be the set of odd components of $G^{\prime}-S$
where $t=o\left(G^{\prime}-S\right)$. By the construction of $G^{\prime}$, there is a component $Q$ of $G^{\prime}-S$ such that $Q$ contains all vertices in $\left[V_{(\Delta-1)^{-}}(G) \cup V\left(K_{n}\right)\right] \backslash S$. Without loss of generality, we assume $O_{i} \neq Q$ for each $i=1, \ldots, t-1$ and thus $V\left(O_{i}\right) \subseteq V_{\Delta}(G)$. Hence $\partial_{G^{\prime}}\left(V\left(O_{i}\right)\right)=\partial_{G}\left(V\left(O_{i}\right)\right)$. Since $O_{i}$ is a planar graph and every vertex in $O_{i}$ is a $\Delta$-vertex in $G$, we have $\left|\partial_{G^{\prime}}\left(V\left(O_{i}\right)\right)\right| \geq \Delta$ by Corollary 3.4. Let $S_{1}=S \backslash V\left(K_{n}\right)$. Then for each $i=1, \ldots, t-1$, we further have $\partial_{G^{\prime}}\left(V\left(O_{i}\right)\right)=\partial_{G}\left(V\left(O_{i}\right)\right)=\partial_{G}\left(V\left(O_{i}\right), S_{1}\right)$, and thus $\left|\partial_{G}\left(V\left(O_{i}\right), S_{1}\right)\right| \geq \Delta$. Since $t-1 \geq|S| \geq\left|S_{1}\right|$,

$$
\sum_{i=1}^{t-1}\left|\partial_{G}\left(V\left(O_{i}\right), S_{1}\right)\right| \geq \Delta(t-1) \geq \Delta\left|S_{1}\right|
$$

If $S_{1} \cap V_{(\Delta-1)^{-}}(G) \neq \emptyset$, then $\Delta\left|S_{1}\right| \leq \sum_{i=1}^{t-1}\left|\partial_{G}\left(V\left(O_{i}\right), S_{1}\right)\right| \leq\left|\partial_{G}(S 1)\right| \leq$ $\Delta\left|S_{1}\right|-1$, a contradiction. If $S_{1} \cap V_{(\Delta-1)^{-}}(G)=\emptyset$, then $V(Q) \cap S_{1}=\emptyset$ and $\left|\partial_{G}\left(V(Q) \backslash V\left(K_{n}\right), S_{1}\right)\right| \geq 1$ since $G$ is connected. Thus $\Delta\left|S_{1}\right|+1 \leq$ $\sum_{i=1}^{t-1}\left|\partial_{G}\left(V\left(O_{i}\right), S_{1}\right)\right|+\left|\partial_{G}\left(V(Q) \backslash V\left(K_{n}\right), S_{1}\right)\right| \leq\left|\partial_{G}\left(S_{1}\right)\right| \leq \Delta\left|S_{1}\right|$, a contradiction again. This proves the lemma.

### 4.2 Proof of Theorem 4.1

Suppose to the contrary that Theorem 4.1 is not true. Let $G=(V, E, F)$ be a minimal counterexample, i.e., with $|E|$ as small as possible. This implies that $G$ is connected. Then $\Delta=\Delta(G) \in\{8,9\}$ by Theorem 1.2 , and $G$ is $\Delta$-critical by the minimality of $G$.

Before we start the discharging procedure, we first deal with the case that $\Delta=9$.

If $\Delta=9$, then by Lemma 4.6, there is a matching $M$ in $G$ such that $\Delta(G-M)=8$. By the minimality of $G$, we know that $G-M$ has an 8-edgecoloring $\varphi$ with $\varphi(e) \in\{ \pm 1, \pm 2, \pm 3, \pm 4\}$ for each $e \in E \backslash M$. Thus $G$ has a 9-edge-coloring obtained from $\varphi$ by coloring every edge of $M$ with color 0 , a contradiction.

Thus $\Delta=8$ and $G$ is an 8 -critical signed graph. The following claim follows from Proposition 1.

Claim 1. $d(u)+d(v) \geq 10$ for $u v \in E$.
In the remainder of this section, we will obtain a contradiction by using the discharging method. We assign the initial charge $w: V \cup F \rightarrow \mathbb{Z}$ defined by

$$
\begin{cases}w(v)=3 d(v)-10 & \text { for } v \in V \\ w(f)=2 d(f)-10 & \text { for } f \in F\end{cases}
$$

to the vertices and faces of $G$. Then

$$
\sum_{v \in V}(3 d(v)-10)+\sum_{f \in F}(2 d(f)-10)=-20
$$

In order to reach a contradiction, we redistribute the charges among the vertices and the faces in $G$ by a number of discharging rules. In our discharging, we use the following discharging rules, depending on the degrees of the vertices and faces.
(R1) Every $3^{-}$-vertex $v$ receives $\frac{10-3 d(v)}{d(v)}$ from each of its neighbors.
(R2) Every $k$-vertex $v$ with $k \in\{4,6\}$ sends $\frac{3 d(v)-10}{d(v)}$ to each of its incident faces.
(R3) Every 5-vertex $v$ sends $a$ to each incident face $f$, where

$$
a= \begin{cases}\frac{5}{4} & \text { if } f \text { is a }(5,5,8) \text {-face } \\ \frac{4}{3} & \text { if } f \text { is a }(5,6,6) \text {-face } \\ 1 & \text { if } f \text { is a }\left(5,6,7^{+}\right) \text {-face } \\ \frac{5}{6} & \text { otherwise }\end{cases}
$$

(R4) Every $7^{+}$-vertex $v$ sends $a$ to each incident face $f$, where

$$
a= \begin{cases}2 & \text { if } f \text { is a }\left(3^{-}, 7^{+}, 7^{+}\right) \text {-face; } \\ \frac{13}{6} & \text { if } f \text { is a }(4,6,8) \text {-face; } \\ \frac{7}{4} & \text { if } f \text { is a }\left(4,7^{+}, 7^{+}\right) \text {-face; } \\ \frac{3}{2} & \text { if } f \text { is a }(5,5,8) \text {-face; } \\ \frac{5}{3} & \text { if } f \text { is a }\left(5,6,7^{+}\right) \text {-face; } \\ \frac{19}{12} & \text { if } f \text { is a }\left(5,7^{+}, 7^{+}\right) \text {-face; } \\ \frac{4}{3} & \text { if } f \text { is a }\left(6^{+}, 6^{+}, 7^{+}\right) \text {-face } \\ 1 & \text { otherwise. }\end{cases}
$$

After applying all of the above rules of the discharging process, denote the final charge by $w^{\prime}: V \cup F \rightarrow \mathbb{Z}$. To obtain a contradiction, it is sufficient to prove that $w^{\prime}(z) \geq 0$ for each $z \in V \cup F$.

We do this in a systematic case-by-case way, starting with the faces of $G$. If $d(f) \geq 5$, then $f$ retains its initial charge and it follows that $w^{\prime}(f)=w(f)=$ $2 d(f)-10 \geq 0$.

We next prove that $w^{\prime}(f) \geq 0$ for the other faces of $G$, distinguishing the cases that $d(f)=3$ and $d(f)=4$.
(1.1) $d(f)=3$. Let $f=\left[v_{1} v_{2} v_{3}\right]$ with $d\left(v_{1}\right) \leq d\left(v_{2}\right) \leq d\left(v_{3}\right)$.

First suppose $d\left(v_{1}\right) \leq 3$. Then $d\left(v_{3}\right) \geq d\left(v_{2}\right) \geq 7$ by Claim 1 , thus $w^{\prime}(f)=w(f)+2 \times 2=0$ by (R4.1).

Next suppose $d\left(v_{1}\right)=4$. If $d\left(v_{2}\right)=6$, then $d\left(v_{3}\right)=8$ by Lemma 3.7-(a); if $d\left(v_{2}\right) \geq 7$, then $d\left(v_{3}\right) \geq 7$. Hence in these cases, $w^{\prime}(f) \geq w(f)+\min \left\{\frac{1}{2}+\right.$ $\left.\frac{4}{3}+\frac{13}{6}, \frac{1}{2}+2 \times \frac{7}{4}\right\}=0$ by (R2) and (R4.2) $\sim(R 4.3)$.

Then suppose $d\left(v_{1}\right)=5$. If $d\left(v_{2}\right)=5$, then $d\left(v_{3}\right)=8$ by Lemma 3.7-(a); if $d\left(v_{2}\right)=6$, then $6 \leq d\left(v_{3}\right) \leq 8$; if $d\left(v_{2}\right) \geq 7$, then $d\left(v_{3}\right) \geq 7$. Hence in these cases, $w^{\prime}(f) \geq w(f)+\min \left\{2 \times \frac{5}{4}+\frac{3}{2}, 3 \times \frac{4}{3}, 1+\frac{4}{3}+\frac{5}{3}, \frac{5}{6}+2 \times \frac{19}{12}\right\}=0$ by (R2), (R3) and (R4.4)~(R4.6).

Finally suppose $d\left(v_{1}\right) \geq 6$. Then $d\left(v_{3}\right) \geq d\left(v_{2}\right) \geq 6$, so $w^{\prime}(f)=w(f)+$ $3 \times \frac{4}{3}=0$ by (R2) and (R4.7). This completes the case that $d(f)=3$.
(1.2) $d(f)=4$. If $f$ is incident with a $3^{-}$-vertex, then $f$ is incident with at least two $7^{+}$-vertices by Claim 1 ; otherwise $f$ is incident with four $4^{+}$-vertices.

By (R2), (R3.4) every $k$-vertex ( $4 \leq k \leq 6$ ) sends at least $\frac{1}{2}$ to each incident 4-face, and by (R4.8), every $7^{+}$-vertex sends 1 to each incident 4-face. Hence in these cases, $w^{\prime}(f) \geq w(f)+\min \left\{2 \times 1,4 \times \frac{1}{2}\right\}=0$.

This completes the case that $d(f)=4$ and shows that indeed $w^{\prime}(f) \geq 0$ for each $f \in F$. It remains to prove that $w^{\prime}(v) \geq 0$ for each $v \in V$. This requires a more tedious case distinction.

We start by assuming that $u \in N_{G}(v)$ is a vertex with the smallest degree among all neighbors of $v$ in $G$. Recall that $n_{i}(v)$ (resp., $n_{i^{+}}(v), n_{i^{-}}(v)$ ) be the number of $i$-vertices (resp., $i^{+}$-vertices, $i^{-}$-vertices) adjacent to $v$ in $G$. We deal with the following cases separately, depending on the degree of $v$. By Claim 1 and the assumption that $\Delta=8$, we know that $d(v) \geq 2$.
(2.1) $2 \leq d(v) \leq 3$.

By Claim 1, we know that all the neighbors of $v$ are $7^{+}$-vertices. So, by (R1), $w^{\prime}(v)=w(v)+d(v) \times \frac{10-3 d(v)}{d(v)}=0$.
(2.2) $d(v)=4$ or $d(v)=6$.

It is evident that $v$ is incident with at most $d(v)$ faces. So, by (R2), $w^{\prime}(v)=w(v)-d(v) \times \frac{10-3 d(v)}{d(v)}=0$.
$(2.3) d(v)=5$.
By Claim 1, $d(u) \geq 5$. By (R3), $v$ sends $\frac{5}{4}$ to each ( $5,5,8$ )-face, $\frac{4}{3}$ to each $(5,6,6)$-face, 1 to each $\left(5,6,7^{+}\right)$-face, and $\frac{5}{6}$ to each other $\left(5,7^{+}, 7^{+}\right)$-face and each 4-face. We distinguish a number of cases, depending on the value of $d(u)$, as follows.

If $d(u)=5$, then $n_{8}(v) \geq 8-d(u)+1=4$ by Lemma 3.7-(a). It is easy to check that $v$ is incident with at most two $(5,5,8)$-faces and the other faces incident with $v$ are either $(5,8,8)$-faces or $4^{+}$-faces. Thus $w^{\prime}(v) \geq$ $w(v)-2 \times \frac{5}{4}-3 \times \frac{5}{6}=0$.

If $d(u) \geq 6$, then $n_{7^{+}}(v) \geq 8-d(u)+1=3$ by Lemma 3.7-(b). And so $n_{6}(v)=5-n_{7^{+}}(v) \leq 2$, and $v$ is incident with at most one $(5,6,6)$-face. If $v$ is incident with a $(5,6,6)$-face, then $v$ is incident with at most two $\left(5,6,7^{+}\right)$-faces and the other faces incident with $v$ are either $\left(5,7^{+}, 7^{+}\right)$-faces or $4^{+}$-faces; otherwise $v$ is incident with at most five faces such that any of
them is either a $\left(5,6^{+}, 7^{+}\right)$-face or a $4^{+}$-face. Thus $w^{\prime}(v) \geq w(v)-\max \left\{\frac{4}{3}+\right.$ $\left.2 \times 1+2 \times \frac{5}{6}, 5 \times 1\right\}=0$.
(2.4) $d(v)=7$.

By Claim $1, d(u) \geq 3$. By Lemma 3.7-(a), $v$ is incident with no $(4,6,7)$ face, and no (5,5, 7)-face. By (R1) and (R4), $v$ sends $\frac{1}{3}$ to each neighbor with degree 3 , 2 to each ( $3,7,7^{+}$)-face, $\frac{7}{4}$ to each ( $4,7,7^{+}$)-face, $\frac{5}{3}$ to each $(5,6,7)$-face, $\frac{19}{12}$ to each $\left(5,7,7^{+}\right)$-face, $\frac{4}{3}$ to each $\left(6^{+}, 7,7^{+}\right)$-face, and 1 to each 4-face. We again distinguish a number of cases, depending on the value of $d(u)$, as follows.

If $d(u)=3$, then $n_{8}(v)=d(v)-1=6$ by Lemma 3.7-(a). It is easy to check that $v$ is incident with at most two (3,7,8)-faces and the other faces incident with $v$ are either (7,8,8)-faces or $4^{+}$-faces. Thus $w^{\prime}(v) \geq$ $w(v)-\frac{1}{3}-2 \times 2-5 \times \frac{4}{3}=0$.

If $d(u)=4$, then $n_{7^{+}}(v) \geq 8-d(u)+1=5$ by Lemma 3.7-(b). It is easy to check that $v$ is incident with at most four $\left(4,7,7^{+}\right)$-faces and the other faces incident with $v$ are $\left(7,7^{+}, 7^{+}\right)$-faces or $4^{+}$-faces. Thus $w^{\prime}(v) \geq$ $w(v)-4 \times \frac{7}{4}-3 \times \frac{4}{3}=0$.

If $d(u)=5$, then $n_{6^{+}}(v) \geq 8-d(u)+1=4$ by Lemma 3.7-(c), and there are at least three $7^{+}$-vertices among those $6^{+}$-vertices. It is easy to check that $v$ is incident with a $\left(6^{+}, 6^{+}, 7\right)$-face or a $4^{+}$-face. If $v$ is incident with a $4^{+}$-face or $v$ is incident with at least two $\left(6^{+}, 6^{+}, 7\right)$-faces, then $w^{\prime}(v) \geq$ $w(v)-\max \left\{1+6 \times \frac{5}{3}, 2 \times \frac{4}{3}+5 \times \frac{5}{3}\right\}=0$. So assume that all faces incident with $v$ are 3 -faces, and $v$ is incident with exactly one $\left(6^{+}, 6^{+}, 7\right)$-face. This implies that $n_{5}(v)=3$. Since $n_{7^{+}}(v) \geq 3, v$ is incident with at least four $\left(5,7,7^{+}\right)$-faces, thus $w^{\prime}(v) \geq w(v)-\frac{4}{3}-4 \times \frac{19}{12}-2 \times \frac{5}{3}=0$.

If $d(u) \geq 6$, then $w^{\prime}(v) \geq w(v)-7 \times \frac{4}{3}>0$.
$(2.5) d(v)=8$.
By (R1) and (R4), $v$ sends 2 to each neighbor with degree 2,2 to each $\left(3^{-}, 7^{+}, 8\right)$-face, $\frac{13}{6}$ to each (4, 6, 8)-face, $\frac{7}{4}$ to each $\left(4,7^{+}, 8\right)$-face, $\frac{4}{3}$ to each $\left(6^{+}, 6^{+}, 7^{+}\right)$-face, and 1 to each 4 -face. By Claim 1 , $d(u) \geq 2$. We again distinguish a number of cases, depending on the value of $d(u)$, as follows.

If $d(u)=2$, then $n_{8}(v)=8-d(u)+1=7$ by Lemma 3.7-(a). So $v$ is
incident with at most one $(2,8,8)$-face, at least one $4^{+}$-face, and the other 3faces incident with $v$ are $(8,8,8)$-faces. Thus $w^{\prime}(v) \geq w(v)-2-2-1-6 \times \frac{4}{3}>$ 0 .

If $d(u)=3$, then $n_{7^{+}}(v) \geq 8-d(u)+1=6$ by Lemma 3.7-(b), and so $n_{5^{-}}(v) \leq 2$. It is easy to check that $v$ is incident with at most four $\left(3,7^{+}, 8\right)$ faces and the other faces incident with $v$ are either $\left(7^{+}, 7^{+}, 8\right)$-faces or $4^{+}$faces. Thus $w^{\prime}(v) \geq w(v)-2 \times \frac{1}{3}-4 \times 2-4 \times \frac{4}{3}=0$.

If $d(u)=4$, we consider the faces incident with $v$. If $v$ is incident with no $(4,6,8)$-face, then every face incident with $v$ is a $\left(4,7^{+}, 8\right)$-face, or a $\left(5^{+}, 5^{+}, 8\right)$-face, or a $4^{+}$-face. By (R4), $v$ sends at most $\frac{7}{4}$ to each incident face, thus $w^{\prime}(v) \geq w(v)-8 \times \frac{7}{4}=0$. Next assume that $v$ is incident with a (4, 6, 8)-face. By Lemma 3.7-(c), $n_{5^{-}}(v) \leq 3$ and $n_{6^{+}}(v) \geq 5$, and there are at least three $7^{+}$-vertices among those $6^{+}$-vertices. By Lemma 3.7-(a), for every edge $v w$ with $d(v)+d(w)=10$, every vertex in $(N(v) \cup N(w)) \backslash\{v, w\}$ is an 8 -vertex. So $v$ is incident with at most two $(4,6,8)$-faces for otherwise $n_{7^{+}}(v)=8-6=2<3$. If $v$ is incident with at most seven faces, then $w^{\prime}(v) \geq$ $w(v)-2 \times \frac{13}{6}-5 \times \frac{7}{4}>0$. If $v$ is incident with eight faces, then $v$ is incident with at least two faces such that any one of them is either a $\left(6^{+}, 6^{+}, 8\right)$-face or a $4^{+}$-face since $n_{6^{+}}(v) \geq 5$. Thus $w^{\prime}(v) \geq w(v)-2 \times \frac{13}{6}-2 \times \frac{4}{3}-4 \times \frac{7}{4}=0$.

If $d(u) \geq 5$, by (R4), $v$ sends at most $\frac{5}{3}$ to each incident face, thus $w^{\prime}(v) \geq w(v)-8 \times \frac{5}{3}>0$.

This completes the proof of Theorem 4.1.

### 4.3 Proof of Theorem 4.2

Suppose to the contrary that Theorem 4.2 is not true. Let $G=(V, E, F)$ be a minimal counterexample, i.e., with $|E|$ as small as possible. This implies that $G$ is connected. Then $\Delta=\Delta(G) \in\{6,7\}$ by Theorem 4.1 , and $G$ is $\Delta$-critical by the minimality of $G$.

Recall that $\lambda_{i}(v)\left(\lambda_{i^{+}}(v), \lambda_{i^{-}}(v)\right)$ be the number of $i$-faces $\left(i^{+}\right.$-faces $i^{-}$faces ) of $G$ incident with $v$.

Before we start the discharging procedure, we first deal with the case that $\Delta=7$.

If $\Delta=7$, then by Lemma 4.6 , there is a matching $M$ in $G$ such that $\Delta(G-M)=6$. By the minimality of $G$, we know that $G-M$ has an 6-edge-coloring $\varphi$ with $\varphi(e) \in\{ \pm 1, \pm 2, \pm 3\}$ for each $e \in E \backslash M$. Thus $G$ has a 7-edge-coloring obtained from $\varphi$ by coloring every edge of $M$ with color 0 , a contradiction.

Thus $\Delta=6$ and $G$ is a 6 -critical signed graph. The following claim follows from Proposition 1.

Claim 1. $d(u)+d(v) \geq 8$ for $u v \in E$.
Since any 6 -cycle of $G$ contains at most one chord, we have the following three claims.

Claim 2. If $v$ is a $5^{+}$-vertex of $G$, then $\lambda_{3}(v) \leq\left\lfloor\frac{3}{4} d(v)\right\rfloor$.
Proof. Since $G$ contains no 6-cycles with two chords, $v$ is not incident with four consecutive 3 -faces. So $\lambda_{3}(v) \leq\left\lfloor\frac{3}{4} d(v)\right\rfloor$.

Claim 3. Let $f, f^{\prime}, f^{\prime \prime}$ be three faces incident with $v$ such that $f^{\prime}$ is adjacent to $f$ and $f^{\prime \prime}$. If $d(v) \geq 5, f$ and $f^{\prime \prime}$ are 3 -faces contain no 2 -vertex, then $f^{\prime}$ must be a 3 -face or a $5^{+}$-face.

Proof. Suppose to be contrary that $f^{\prime}$ be a 4 -face. Let $v_{1}, v_{2}, \ldots, v_{d(v)}$ are neighbors of $v$. Since $d(v) \geq 5, f$ and $f^{\prime \prime}$ are not incident. Without loss of generality, suppose that $f=\left[v v_{1} v_{2}\right], f^{\prime \prime}=\left[\nu v_{3} v_{4}\right]$ and $f^{\prime}$ contains edges $v v_{2}, v v_{3}$. Since $d\left(f^{\prime}\right)=4$, there is a vertex $u \in V$ such that $v_{2} u, v_{3} u \in E(G)$. Clearly, $u \notin\left\{v_{2}, v_{3}\right\}$. We claim that $u \notin\left\{v_{1}, v_{4}\right\}$. If not, then $d\left(v_{2}\right)=2$ or $d\left(v_{3}\right)=2$, a contradiction with $f$ and $f^{\prime \prime}$ contain no 2 -vertex. Then 6 -cycle $v v_{1} v_{2} u v_{3} v_{4} v$ contains two chords $v v_{2}, v v_{3}$, a contradiction.

Claim 4. Let $v$ be a 6 -vertex of $G$.
(1) If $\lambda_{3}(v)=4$, then $\lambda_{4}(v)=0$.
(2) If $\lambda_{3}(v)=3$, then $\lambda_{4}(v) \leq 2$.

Proof. Suppose that $N(v)=\left\{v_{1}, v_{2}, \ldots, v_{6}\right\}$.

To prove (1). Since $\lambda_{3}(v)=4$, without loss of generality, suppose that $f_{1}, f_{2}, f_{4}, f_{5}$ or $f_{1}, f_{2}, f_{3}, f_{5}$ are 3 -faces. If $v$ has no 2 -neighbor, then $f_{3}, f_{6}$ or $f_{4}, f_{6}$ are $5^{+}$-faces by Claim 3. Thus assume that $v$ has a 2 -neighbor. By Lemma 3.1, other neighbors of $v$ are 6 -vertices. When $f_{1}, f_{2}, f_{4}, f_{5}$ are 3 -faces, w.l.o.g., assume that $d\left(v_{1}\right)=2$, by Claim $2, f_{3}$ and $f_{4}$ are $4^{+}$-faces, by Claim $3, f_{3}$ is a $5^{+}$-face. If $f_{6}$ is a $5^{+}$-face, then we are done. Otherwise, $f_{6}$ is a 4 -face, that is, $v_{6} v_{2} \in E(G)$. Then 6 -cycle $v_{2} v_{3} v v_{4} v_{5} v_{6} v_{2}$ contains three chords $v v_{2}, v v_{5}, v v_{6}$, a contradiction. The case of $f_{1}, f_{2}, f_{3}, f_{5}$ are 3 -faces is similarly above, we omit it.

To prove (2). Suppose to be contrary that $\lambda_{4}(v)=3$. Since $\lambda_{3}(v)=3$, there are three structures. Without loss of generality, suppose that $f_{1}, f_{3}, f_{5}$ or $f_{1}, f_{2}, f_{4}$ or $f_{1}, f_{2}, f_{3}$ are 3 -faces. Recall that if $n_{2}(v)=1$, then $n_{6}(v)=5$. When $f_{1}, f_{3}, f_{5}$ are 3 -faces, there are at least two 3 -faces contain no 2 -vertex. By Claim 3 , there is at least one $5^{+}$-face in $\left\{f_{2}, f_{4}, f_{6}\right\}$, a contradiction. When $f_{1}, f_{2}, f_{4}$ are 3 -faces, if $d\left(v_{3}\right), d\left(v_{4}\right) \geq 3$, then $f_{3}$ is a $5^{+}$-face, a contradiction. If $d\left(v_{3}\right)=2$, then $v_{2} v_{4} \in E(G)$ since $f_{3}$ is a 4-face. Since $d\left(f_{5}\right)=4$, there is a vertex $u \in V$ such that $v v_{5} u v_{6} v$ is a 4 -face. We Claim that $u \in N(v) \backslash$ $\left\{v_{5}, v_{6}\right\}$. If not, there is a 6 -cycle $v v_{6} u v_{5} v_{4} v_{2} v$ contains two chords $v v_{4}, v v_{5}$, a contradiction. Since $d\left(v_{3}\right)=2, u \neq v_{3}$. If $u=v_{4}$, then $d\left(v_{5}\right)=2$, a contradiction. If $u=v_{2}$, then $d\left(v_{1}\right)=2$ since $d\left(f_{6}\right)=4$, a contradiction. If $u=v_{1}$, then 6 -cycle $v v_{4} v_{5} v_{1} v_{2} v_{3} v$ contains three chords $v v_{1}, v v_{2}, v v_{5}$, a contradiction. The subcase of $d\left(v_{4}\right)=2$ is similar to $d\left(v_{3}\right)=2$, we omit it. The case of $f_{1}, f_{2}, f_{3}$ are 3 -faces is similar to above case, we omit it.

In the remainder of this section, we will obtain a contradiction by using the discharging method.

Let $w(v)=3 d(v)-10$ be the initial charge of each vertex $v$ and $w(f)=$ $2 d(f)-10$ be the initial charge of each face $f$. So $\sum_{x \in V \cup F} w(x)=-20<0$. In the following, we will reassign a new charge denoted by $w^{\prime}(x)$ to each $x \in V \cup F$ according to the discharging rules. We will show that $w^{\prime}(x) \geq 0$ for each $x \in V \cup F$, then we get an obvious contradiction, which completes our proof. The discharging rules are defined as follows.
(R1) Every $3^{-}$-vertex $v$ receives $\frac{10-3 d(v)}{d(v)}$ from each of its neighbors.
(R2) Every 4-vertex $v$ sends $\frac{3 d(v)-10}{d(v)}$ to each of its incident faces.
(R3) Let $v$ be a 5 -vertex.
(R3.1) $v$ sends a to each incident 3-face $f=[v w u]$, where

$$
a= \begin{cases}\frac{7}{4} & \text { if } d(w)=4 \text { and } d(u)=5 \\ \frac{4}{3} & \text { otherwise }\end{cases}
$$

(R3.2) $v$ sends a to each incident 4-face $f$, where

$$
a= \begin{cases}\frac{1}{2} & \text { if } f \text { contains no } 3^{-} \text {-vetex } \\ \frac{2}{3} & \text { otherwise }\end{cases}
$$

(R4) Let $v$ be a 6-vertex.
$(\mathbf{R 4 . 1 )} v$ sends a to each incident 3-face $f=[v w u]$, where

$$
a= \begin{cases}2 & \text { if } d(u)=2, d(w)=6 \\ \frac{8}{3} & \text { if } d(u)=3, d(w)=5 \\ \frac{7}{3} & \text { if } d(u)=3, d(w)=6 \text { and }\left|N_{4^{-}}(v)\right|=1 \\ 2 & \text { if } d(u)=3, d(w)=6 \text { and }\left|N_{4^{-}}(v)\right|=2,\left|N_{4^{-}}(w)\right|=2 \\ \frac{5}{3} & \text { if } d(u)=3, d(w)=6 \text { and }\left|N_{4^{-}}(v)\right|=2,\left|N_{4^{-}}(w)\right|=1 \\ 3 & \text { if } d(u)=4, d(w)=4 \\ \frac{13}{6} & \text { if } d(u)=4, d(w)=5 \\ \frac{7}{4} & \text { if } d(u)=4, d(w)=6 \\ \frac{4}{3} & \text { if } f \text { contains no } 4^{-} \text {-vertex. }\end{cases}
$$

(R4.2) $v$ sends a to each incident 4-face $f$, where

$$
a= \begin{cases}\frac{1}{2} & \text { if } f \text { contains no } 3^{-} \text {-vetex; } \\ 1 & \text { if } f \text { contains two } 3^{-} \text {-vetices; } \\ \frac{3}{4} & \text { if } f \text { contains a 3-vetex and a 4-vertex; } \\ \frac{2}{3} & \text { otherwise. }\end{cases}
$$

Now, let's begin to check $w^{\prime}(x) \geq 0$ for each $x \in V \cup F$. We start with the faces of $G$. If $d(f) \geq 5$, then $f$ retains its initial charge and it follows that
$w^{\prime}(f)=w(f)=2 d(f)-10 \geq 0$.
We next prove that $w^{\prime}(f) \geq 0$ for the other faces of $G$, distinguishing the cases that $d(f)=3$ and $d(f)=4$.
(1.1) $d(f)=3$. Let $f=\left[v_{1} v_{2} v_{3}\right]$ with $d\left(v_{1}\right) \leq d\left(v_{2}\right) \leq d\left(v_{3}\right)$.

First suppose $d\left(v_{1}\right)=2$. Then $d\left(v_{3}\right) \geq d\left(v_{2}\right) \geq 6$ by Claim 1. By (R3.1), each 6-vertex sends 2 to each ( $2,6,6$ )-face, thus $w^{\prime}(f)=w(f)+2 \times 2=0$.

Then suppose $d\left(v_{1}\right)=3$. Then $d\left(v_{2}\right) \geq 5$ and $d\left(v_{3}\right)=6$ by Claim 1 and Lemma 3.5. By (R3.1) and (R4.1), each (3,5,6)-face receives $\frac{4}{3}$ and $\frac{8}{3}$, respectively, from its incident 5-vertex and 6-vertex. Each (3, 6, 6)-face, if one of its incident 6 -vertices adjacent to one $4^{-}$-neighbor, then it receives $\frac{7}{3}$ from that 6 -vertex; if each of its incident 6 -vertices has two $4^{-}$-neighbors, then it receives 2 from each 6-vertex; if one of its incident 6-vertices has two $4^{-}$-neighbors and another 6-vertex has one 4-neighbor, then it receives $\frac{5}{3}$ from the 6-vertex with two 4-neighbors. Thus $w^{\prime}(f)=w(f)+\min \left\{\frac{4}{3}+\frac{8}{3}, \frac{7}{3} \times 2, \frac{7}{3}+\right.$ $\left.\frac{5}{3}, 2 \times 2\right\}=0$.

Next suppose $d\left(v_{1}\right)=4$. If $d\left(v_{2}\right)=4$, then $d\left(v_{3}\right)=6$ by Lemma 3.5; if $d\left(v_{2}\right) \geq 5$, then $d\left(v_{3}\right) \geq 5$. By (R2), each 4 -vertex sends $\frac{1}{2}$ to each of its incident faces. By (R3.1) and (R4.1), each (4,4,6)-face receives 3 from its incident 6 -vertex. Each $(4,5,5)$-face receives $\frac{7}{4}$ from each of its incident 5 -vertices. Each $(4,5,6)$-face receives $\frac{4}{3}$ from its incident 5 -vertex and $\frac{13}{6}$ from its incident 6 -vertex. Finally, each $(4,6,6)$-face receives $\frac{7}{4}$ from each of its incident 6-vertices. Thus $w^{\prime}(f) \geq w(f)+\min \left\{2 \times \frac{1}{2}+3, \frac{1}{2}+2 \times \frac{7}{4}, \frac{1}{2}+\frac{4}{3}+\frac{13}{6}\right\}=$ 0.

Finally suppose $d\left(v_{1}\right) \geq 5$. Then $d\left(v_{3}\right) \geq d\left(v_{2}\right) \geq 5$, so $w^{\prime}(f)=w(f)+$ $3 \times \frac{4}{3}=0$ by (R3.1) and (R4.1). This completes the case that $d(f)=3$.

$$
(1.2) d(f)=4
$$

By Claim 1 and Lemma 3.5, if $f$ is incident with a 2 -vertex, then $f$ is incident with three 6 -vertices; if $f$ is incident with two 3 -vertices, then $f$ is incident with two 6 -vertices; if $f$ is incident with one 3 -vertex and one 4-vertex, then $f$ is incident with two 6-vertices; otherwise $f$ is incident with four $4^{+}$-vertices. By (R2), (R3.2) and (R4.2), $w^{\prime}(f) \geq w(f)+\min \left\{3 \times \frac{2}{3}, 2 \times\right.$ $\left.1, \frac{1}{2}+2 \times \frac{3}{4}, 4 \times \frac{1}{2}\right\}=0$.

This completes the case that $d(f)=4$ and shows that indeed $w^{\prime}(f) \geq 0$ for each $f \in F$. It remains to prove that $w^{\prime}(v) \geq 0$ for each $v \in V$. This requires a more tedious case distinction.

We start by assuming that $u \in N_{G}(v)$ is a vertex with the smallest degree among all neighbors of $v$ in $G$. We deal with the following cases separately, depending on the degree of $v$. By Claim 1 and the assumption that $\Delta=6$, we know that $d(v) \geq 2$.
(2.1) $2 \leq d(v) \leq 3$.

By Claim 1, we know that all the neighbors of $v$ are $5^{+}$-vertices. So, by (R1), $w^{\prime}(v)=w(v)+d(v) \times \frac{10-3 d(v)}{d(v)}=0$.
$(2.2) d(v)=4$.
It is evident that $v$ is incident with at most $d(v)$ faces. So, by (R2), $w^{\prime}(v)=w(v)-d(v) \times \frac{10-3 d(v)}{d(v)}=0$.
$(2.3) d(v)=5$.
By Claim $1, d(u) \geq 3$. By Lemma 3.5, $v$ is incident with no $\left(3,5,3,5^{+}\right)$face. By (R3), v sends $\frac{7}{4}$ to each ( $4,5,5$ )-face, $\frac{4}{3}$ to each other 3 -face, $\frac{1}{2}$ to each 4 -face that does not contain a $3^{-}$-vertex, $\frac{2}{3}$ to each other 4 -face. We distinguish a number of cases, depending on the value of $d(u)$, as follows.

If $d(u)=3$, then $n_{6}(v) \geq 6-d(u)+1=4$ by Lemma 3.5. It is easy to check that $v$ is incident no $(4,5,5)$-face. If $\lambda_{3}(v) \leq 2$, then $w^{\prime}(v) \geq w(v)-\frac{1}{3}-2 \times \frac{4}{3}-$ $3 \times \frac{2}{3}=0$. If $\lambda_{3}(v)=3$, then $\lambda_{4}(v) \leq 1$, thus $w^{\prime}(v) \geq w(v)-\frac{1}{3}-3 \times \frac{4}{3}-\frac{2}{3}=0$.

If $d(u)=4$, then $n_{6}(v) \geq 6-d(u)+1=3$ by Lemma 3.5 , and so $v$ is incident with no $(4,4,5)$-face and at most one $(4,5,5)$-face. If $v$ is incident with no $(4,5,5)$-face, then $w^{\prime}(v) \geq \max \left\{5 \times \frac{2}{3}, \frac{4}{3}+4 \times \frac{2}{3}, 2 \times \frac{4}{3}+3 \times \frac{2}{3}, 3 \times \frac{4}{3}+\right.$ $\left.\frac{2}{3}\right\}>0$. So suppose that $v$ is incident with a $(4,5,5)$-face. If $\lambda_{3}(v)=2$, then $v$ is incident with a $5^{+}$-face or a 4 -face contains no $3^{-}$-vertex; if $\lambda_{3}(v)=3$, then $\lambda_{4}(v) \leq 1$ and $v$ is incident with a 4-face contains no $3^{-}$-vertex. Thus $w^{\prime}(v) \geq \max \left\{5 \times \frac{2}{3}, \frac{7}{4}+4 \times \frac{2}{3}, \frac{7}{4}+\frac{4}{3}+\frac{1}{2}+2 \times \frac{2}{3}, \frac{7}{4}+2 \times \frac{4}{3}+\frac{1}{2}\right\}=0$.

If $d(u)=5$, then $v$ sends at most $\frac{4}{3}$ to each $\left(5,5^{+}, 5^{+}\right)$-face. Thus $w^{\prime}(v) \geq$ $w(v)-\max \left\{5 \times \frac{2}{3}, \frac{4}{3}+4 \times \frac{2}{3}, 2 \times \frac{4}{3}+3 \times \frac{2}{3}, 3 \times \frac{4}{3}+\frac{2}{3}\right\}=0$.
$(2.5) d(v)=6$.

By Claim 1, $d(u) \geq 2$. By Claim 4, if $\lambda_{3}(v)=3$, then $\lambda_{4}(v) \leq 2$; if $\lambda_{3}(v)=4$, then $\lambda_{4}(v)=0$. We again distinguish a number of cases, depending on the value of $d(u)$, as follows.

If $d(u)=2$, then $n_{6}(v)=8-d(u)+1=7$ by Lemma 3.5. So each 3-face incident with $v$ is either a $(2,6,6)$-face or a $(6,6,6)$-face and $v$ is incident with no ( $2,6,4^{-}, 6$ )-face. By (R1) and (R4), $v$ sends 2 to each of its 2-neighbors, 2 to each ( $2,6,6$ )-face, and $\frac{4}{3}$ to each ( $6,6,6$ )-face, at most $\frac{2}{3}$ to each 4 -face. Thus $w^{\prime}(v) \geq w(v)-2-\max \left\{6 \times \frac{2}{3}, 2+5 \times \frac{2}{3}, 2+\frac{4}{3}+4 \times \frac{2}{3}, 2+2 \times \frac{4}{3}+2 \times\right.$ $\left.\frac{2}{3}, 2+3 \times \frac{4}{3}\right\}=0$.

If $d(u)=3$, then $n_{6}(v) \geq 6-d(u)+1=4$ by Lemma 3.5 , and so $n_{5^{-}}(v) \leq 2$. We first consider the case of $n_{3}(v)=2$. By (R4.1), $v$ sends at most 2 to $(3,6,6)$-faces. If $v$ is incident with at most two $(3,6,6)$-faces, then $v$ is incident with at most one $(3,6,3,6)$-face; if $v$ is incident with three $(3,6,6)$ faces, then $v$ is incident with no $(3,6,3,6)$-face. Thus $w^{\prime}(v) \geq w(v)-2 \times \frac{1}{3}-$ $\max \left\{1+5 \times \frac{2}{3}, 2+1+4 \times \frac{2}{3}, 2 \times 2+1+3 \times \frac{2}{3}, 3 \times 2+2 \times \frac{2}{3}, 3 \times 2+\frac{4}{3}\right\}=0$. If $v$ is incident with four $(3,6,6)$-faces, then there is only one structure. W.L.O.G., assume that $f_{1}, f_{2}, f_{4}, f_{5}$ are 3 -faces and $d\left(v_{2}\right)=d\left(v_{5}\right)=3$. By Lemma 3.6, there is at least one 6-vertex $w \in\left\{v_{1}, v_{3}\right\}$ such that $n_{4^{-}}(w) \leq 1$, and at least one 6 -vertex $w^{\prime} \in\left\{v_{4}, v_{6}\right\}$ such that $n_{4^{-}}\left(w^{\prime}\right) \leq 1$. W.L.O.G., suppose that $w=v_{1}$ and $w^{\prime}=v_{4}$. By (4.1), $v$ sends $\frac{5}{3}$ to $f_{1}$ and $f_{4}$, thus $w^{\prime}(v) \geq w(v)-2 \times \frac{1}{3}-2 \times 2-2 \times \frac{5}{3}=0$.

Now we consider the case of $n_{3}(v)=1$ and $n_{4}(v)=1$. Then each 3 -faces incident with $v$ is a $\left(4^{-}, 6,6\right)$-face or a $(6,6,6)$-face and $v$ is incident with at most two $(3,6,6)$-faces. By (R4.1), $v$ sends 2 to each $(3,6,6)$-face, $\frac{7}{4}$ to each $(4,6,6)$-face and at most $\frac{3}{4}$ to each incident 4-face. Thus $w^{\prime}(v) \geq$ $w(v)-\frac{1}{3}-\max \left\{6 \times \frac{3}{4}, 2+5 \times \frac{3}{4}, 2 \times 2+4 \times \frac{3}{4}, 2 \times 2+\frac{7}{4}+2 \times \frac{3}{4}, 2 \times 2+2 \times \frac{7}{4}\right\}>0$.

Last we consider the case of $n_{3}(v)=1$ and $n_{5}(v)=1$. Then $v$ is incident with no $\left(3,6,4^{-}, 6\right)$-face. By (R4.1), $v$ sends $\frac{8}{3}$ to each ( $3,5,6$ )-face, $\frac{7}{3}$ to each $(3,6,6)$-face, and at most $\frac{2}{3}$ to each incident 4 -face. If $v$ is incident with a $(3,5,6)$-face, then each of its incident 3 -faces is either a $(3,6,6)$-face or a $\left(5^{+}, 6,6\right)$-face, and $v$ is incident with at most one $(3,6,6)$-face. Thus $w^{\prime}(v) \geq w(v)-\frac{1}{3}-\frac{8}{3}-\max \left\{5 \times \frac{2}{3}, \frac{7}{3}+4 \times \frac{2}{3}, \frac{7}{3}+\frac{4}{3}+2 \times \frac{2}{3}, \frac{7}{3}+2 \times \frac{4}{3}\right\}=0$.

If $v$ is incident with no $(3,5,6)$-face, then each of its incident 3 -faces is either a $(3,6,6)$-face or a $(6,6,6)$-face, and $v$ is incident with at most two $(3,6,6)$ faces. Thus $w^{\prime}(v) \geq w(v)-\frac{1}{3}-\max \left\{6 \times \frac{2}{3}, \frac{7}{3}+5 \times \frac{2}{3}, 2 \times \frac{7}{3}+4 \times \frac{2}{3}, 2 \times \frac{7}{3}+\right.$ $\left.\frac{4}{3}+2 \times \frac{2}{3}, 2 \times \frac{7}{3}+2 \times \frac{4}{3}\right\}>0$.

If $d(u)=4$, then $n_{6}(v)=6-d(u)+1=3$ by Lemma 3.5. And so $v$ is incident with no (3,6,4,6)-face. By (R4.1) and (R4.2), v sends 3 to each $(4,4,6)$-face, $\frac{13}{6}$ to each $(4,5,6)$-face, $\frac{7}{4}$ to each $(4,6,6)$-face, $\frac{4}{3}$ to each $(6,6,6)$-face and at most $\frac{2}{3}$ to each 4 -face. If $v$ is incident with no $(4,4,6)$-face, then each of its incident 3 -faces is either a $(4,5,6)$-face, or a $(4,6,6)$-face, or a $\left(5^{+}, 6,6\right)$-face. Since $n_{6}(v)=3, v$ is incident with at most two $(4,5,6)$-faces. Moreover, when $\lambda_{3}(v)=4$ and $v$ is incident with two (4,5,6)-faces, $v$ is incident with at least one $(6,6,6)$-face. Thus $w^{\prime}(v) \geq w(v)-\max \left\{6 \times \frac{2}{3}, \frac{13}{6}+\right.$ $\left.5 \times \frac{2}{3}, 2 \times \frac{13}{6}+4 \times \frac{2}{3}, 2 \times \frac{13}{6}+\frac{7}{4}+2 \times \frac{2}{3}, 2 \times \frac{13}{6}+\frac{7}{4}+\frac{4}{3}\right\}>0$. Next assume that $v$ is incident with a $(4,4,6)$-face. By Lemma 3.8, $n_{6}(v)=4$, which implies that $v$ is incident with no $(4,5,6)$-face, each other 3 -face incident with $v$ is a $(4,6,6)$-face or a $(6,6,6)$-face and $v$ is incident with at most two $(4,6,6)$-face. Moreover, when $\lambda_{3}(v)=4, v$ is incident with at least one (6,6,6)-face. Thus $w^{\prime}(v) \geq w(v)-\max \left\{3+5 \times \frac{2}{3}, 3+\frac{7}{4}+4 \times \frac{2}{3}, 3+2 \times \frac{7}{4}+2 \times \frac{2}{3}, 3+2 \times \frac{7}{4}+\frac{4}{3}\right\}>0$.

This completes the proof of Theorem 4.2.

### 4.4 Conclusion and future work

In this chapter, we showed that Conjecture 1.4 holds for signed planar graphs with $\Delta \geq 8$ or $\Delta \geq 6$ and each 6 -cycle contains at most one chord.

In the future, we may give some other sufficient conditions in signed planar graphs with $\Delta=6$ such that Conjecture 1.4 holds. And in the case of edge coloring, we know that $\chi^{\prime}\left(K_{n, n}\right)=n$. However, in the case of signed edge coloring, this doesn't hold when $n$ is even. We are interested in whether it holds when $n$ is odd.

## Chapter 5

## Edge DP-coloring of planar graphs

### 5.1 Introduction

As we mentioned in Chapter 1, edge DP-coloring is a generalization of list edge coloring and $\chi_{D P}^{\prime}(G) \geq \chi_{l}^{\prime}(G)$. In this chapter, we extend some results on list edge coloring to edge DP-coloring.

Let $G$ be a planar graph with maximum degree $\Delta \geq 7$. Borodin [7] showed that $\chi_{\ell}^{\prime}(G)=\Delta$ if $G$ without 3-cycles and Hou, Liu and Cai [30] showed that $\chi_{\ell}^{\prime}(G)=\Delta$ if $G$ without 4-cycles. We partially extend those results to edge DP-coloring.

Theorem 5.1. Let $G$ be a planar graph with maximum degree $\Delta$ such that $G$ has no cycle of length $k$. Then $\chi_{D P}^{\prime}(G)=\Delta$ if either $\Delta \geq 7$ and $k=4$ or $\Delta \geq 8$ and $k=3$.

Borodin [7] confirmed Conjecture 1.3 for planar graphs of maximum degree at least 9 (a simpler proof was later found by Cohen and Havet [14]). We extend this result to edge DP-coloring.

Theorem 5.2. If $G$ is a planar graph with $\Delta \geq 9$, then $\chi_{D P}^{\prime}(G) \leq \Delta+1$.

### 5.2 Main Lemmas

Before presenting the proof of Theorem 5.1 and 5.2, we first introduce some lemmas.

Let $G$ be a graph with an edge list assignment $L$, and $\mathscr{M}_{L}$ be a matching assignment over $L$. Suppose that $H$ is a subgraph of $G$ and $G^{\prime}=G-E(H)$ has an $\mathscr{M}_{L^{\prime}}$-coloring with

$$
\begin{equation*}
\mathscr{M}_{L^{\prime}}=\left\{M_{L, e e^{\prime}} \in \mathscr{M}_{L}: e \sim_{G^{\prime}} e^{\prime}\right\} \tag{5.1}
\end{equation*}
$$

that is there is an independent set $I^{\prime}$ of the $\mathscr{M}_{L^{\prime} \text {-cover }} \tilde{G}^{\prime}$ with $\left|I^{\prime}\right|=\left|E\left(G^{\prime}\right)\right|=$ $|E(G)|-|E(H)|$. Define

$$
\begin{align*}
L^{*}(e) & =L(e) \backslash \bigcup_{e^{\prime} \sim e}\left\{c \in L(e): \exists\left(e^{\prime}, c^{\prime}\right) \in I^{\prime} \text { s.t. }\left(e^{\prime}, c^{\prime}\right)(e, c) \in M_{L, e e^{\prime}} \in \mathscr{M}_{L}\right\} \\
& \forall e \in E(H) \tag{5.2}
\end{align*}
$$

and

$$
\begin{equation*}
M_{L^{*}, e e^{\prime}}=\left\{(e, c)\left(e^{\prime}, c^{\prime}\right) \in M_{L, e e^{\prime}} \in \mathscr{M}_{L}: c \in L^{*}(e), c^{\prime} \in L^{*}\left(e^{\prime}\right)\right\}, \forall e \sim_{H} e^{\prime} \tag{5.3}
\end{equation*}
$$

It is not difficult to check that, if $H$ has an $\mathscr{M}_{L^{*}}$-coloring with $\mathscr{M}_{L^{*}}=\left\{M_{L^{*}, e e^{\prime}}\right.$ : $\left.e \sim_{H} e^{\prime}\right\}$, that is the $\mathscr{M}_{L^{*}}$-cover $\tilde{H}$ has an independent set $I^{*}$ with $\left|I^{*}\right|=|E(H)|$, then $I^{\prime} \cup I^{*}$ is an independent set of the $\mathscr{M}_{L}$-cover $\tilde{G}$ with $\left|I^{\prime} \cup I^{*}\right|=|E(G)|$, and hence $G$ has an $\mathscr{M}_{L}$-coloring. The following result is obtained straightforward.

Lemma 5.3. Let $G$ be a graph with an edge list assignment $L$ and a matching assignment $\mathscr{M}_{L}$, and let $H$ be a subgraph of $G$. If $G-E(H)$ has an $\mathscr{M}_{L^{\prime}}$ coloring and $H$ has an $\mathscr{M}_{L^{*}}$-coloring, then $G$ has an $\mathscr{M}_{L^{-c o l o r i n g, ~ w h e r e ~}} \mathscr{M}_{L^{\prime}}, L^{*}$ and $\mathscr{M}_{L^{*}}$ are defined as above.

By the definition of edge DP-coloring, the following result is straightforward.

Lemma 5.4. Let $P=v_{1} v_{2} \ldots v_{n}(n \geq 3)$ be a path and $L$ be an edge list assignment of $P$. If $\left|L\left(v_{i} v_{i+1}\right)\right| \geq 2$ for $i \in[1, n-2]$ and $\left|L\left(v_{n-1} v_{n}\right)\right| \geq 1$, then $P$ has an $\mathscr{M}_{L}$-coloring for any matching assignment $\mathscr{M}_{L}$.

Lemma 5.5. Let $G$ be a cycle with a pendant edge or a chord, and $L$ be an edge list assignment of $G$ satisfying $|L(u v)| \geq d(u)+d(v)-2$ for every $u v \in E(G)$. Then $G$ has an $\mathscr{M}_{L}$-coloring for any matching assignment $\mathscr{M}_{L}$.

Proof. Let $G=C+e_{0}$, where $C=v_{1} v_{2} \ldots v_{n} v_{1}$ is a cycle and $e_{0}=u v_{1}$ is a pendant edge or a chord of $C$. Fix an arbitrary matching assignment $\mathscr{M}_{L}$.

If $e_{0}=u v_{1}$ is a pendant edge, then $d_{G}(u)=1$. Let $H=u v_{1} v_{2} \ldots v_{n}$ and $G^{\prime}=G-E(H)$. Then $E\left(G^{\prime}\right)=\left\{v_{n} v_{1}\right\}$. Note that $\left|L\left(v_{n} v_{1}\right)\right| \geq 3,\left|L\left(v_{1} v_{2}\right)\right| \geq 3$ and $|L(e)| \geq 2$ for $e \in E(G) \backslash\left\{v_{n} v_{1}, v_{1} v_{2}\right\}$ by the assumption of $L$. Since $\left|L\left(v_{n} v_{1}\right)\right| \geq 3$ and $\left|L\left(e_{0}\right)\right| \geq 2$, there is a member $c \in L\left(v_{n} v_{1}\right)$ such that $\left|L^{*}\left(e_{0}\right)\right| \geq 2$ by Eq. (5.2). By Eq. (5.2) again, $\left|L^{*}(e)\right| \geq|L(e)|-1$ for $e \in$ $\left\{v_{n-1} v_{n}, v_{1} v_{2}\right\}$ and $\left|L^{*}(e)\right|=|L(e)|$ for $e \in E(H) \backslash\left\{v_{n-1} v_{n}, v_{1} v_{2}\right\}$. So $H$ has an $\mathscr{M}_{L^{*}}$-coloring $I^{*}$ by Lemma 5.4, and thus $I^{*} \cup\left\{\left(v_{n} v_{1}, c\right)\right\}$ is an $\mathscr{M}_{L}$-coloring of $G$ by Lemma 5.3.

If $e_{0}=u v_{1}$ is a chord of $C$, then $n \geq 4$. Assume that $u=v_{i}$ with $i \in[3, n-$ 1], and then $\left|L\left(e_{0}\right)\right| \geq 4,|L(e)| \geq 3$ for $e \in\left\{v_{1} v_{2}, v_{1} v_{n}, v_{i} v_{i-1}, v_{i} v_{i+1}\right\}$, and $|L(e)| \geq 2$ for $e \in E\left(G-\left\{v_{1}, v_{i}\right\}\right)$ by the assumption of $L$. Let $H=v_{1} v_{2} \ldots v_{n}$ and $G^{\prime}=G-E(H)$. Then $G^{\prime}=v_{i} v_{1} v_{n}$ is a path of length 2 . Since $\left|L\left(v_{i} v_{1}\right)\right| \geq 4$ and $\left|L\left(v_{1} v_{2}\right)\right| \geq 3$, by Eqs. (5.1) and (5.2), we can pick $c_{1} \in L\left(v_{i} v_{1}\right)$ and then $c_{2} \in L\left(v_{1} v_{n}\right)$ such that $I^{\prime}=\left\{\left(v_{i} v_{1}, c_{1}\right),\left(v_{1} v_{n}, c_{2}\right)\right\}$ is an $\mathscr{M}_{L^{\prime}}$-coloring of $G^{\prime}$ and $\left|L^{*}\left(v_{1} v_{2}\right)\right| \geq 2$. It is easy to check that $\left|L^{*}\left(v_{i} v_{i+1}\right)\right| \geq 2$ for $i \in[2, n-2]$ and
 and thus $I^{*} \cup I^{\prime}$ is an $\mathscr{M}_{L}$-coloring of $G$ by Lemma 5.3.

Lemma 5.6. Let $G=C+\left\{v_{1} v_{2 i}: i \in[2, t-1]\right\}+v_{1} u(t \geq 3)$, where $C=$ $v_{1} v_{2} \ldots v_{2 t} v_{1}$ is a cycle and $v_{1} u$ is a pendant edge. If $L$ is an edge list assignment of $G$ satisfying $\left|L\left(v_{1} u\right)\right| \geq t,\left|L\left(v_{1} v_{2 i}\right)\right| \geq t+1$ for $i \in[1, t],|L(e)| \geq 2$ for other edges $e$ of $G$, then $G$ has an $\mathscr{M}_{L}$-coloring for any matching assignment $\mathscr{M}_{L}$.

Proof. Let $V^{\prime}=\left(N_{G}\left(v_{1}\right) \backslash\left\{v_{2}\right\}\right) \cup\left\{v_{1}\right\}, H=G\left[V^{\prime}\right]$ and $G^{\prime}=G-E(H)$. Then $H=K_{1, t}$ is a star and $G^{\prime}=v_{1} v_{2} \ldots v_{2 t}$ is a path of length $2 t-1$. Since
$\left|L\left(v_{1} u\right)\right| \geq t,\left|L\left(v_{1} v_{2}\right)\right| \geq t+1$ and $\left|L\left(v_{i} v_{i+1}\right)\right| \geq 2$ for $i \in[2,2 t-1]$, by Eqs. (5.1) and (5.2), we can pick $c_{1} \in L\left(v_{1} v_{2}\right)$ and then $c_{i} \in L\left(v_{i} v_{i+1}\right)$ for $i \in[2,2 t-1]$ such that $I^{\prime}=\left\{\left(v_{i} v_{i+1}, c_{i}\right): i \in[1,2 t-1]\right\}$ is an $\mathscr{M}_{L^{\prime}}$-coloring of $G^{\prime}$ and $\left|L^{*}\left(v_{1} u\right)\right| \geq t$. It follows from Eq. (5.2) and the assumption of $L$ that $\left|L^{*}\left(v_{1} v_{2 i}\right)\right| \geq t-2$ for $i \in[2, t-1]$ and $\left|L^{*}\left(v_{1} v_{2 t}\right)\right| \geq t-1$. Thus we can get an $\mathscr{M}_{L^{*}}$-coloring $I^{*}$ of $H=K_{1, t}$ by choosing a member $c_{2 i}^{\prime}$ from $L\left(v_{1} v_{2 i}\right)$ and adding $\left(v_{1} v_{2 i}, c_{2 i}^{\prime}\right)$ to $I^{*}$ in the order $v_{2 i}=v_{4}, v_{6}, \ldots, v_{2 t}, u$. By Lemma 5.3, $I^{\prime} \cup I^{*}$ is an $\mathscr{M}_{L}$-coloring of $G$.

A graph $G$ is minimally non edge DP-k-colorable if it is not edge DP- $k$ colorable, but each of its proper subgraphs is edge DP- $k$-colorable.

Lemma 5.7. Let $G$ be a graph with maximum degree $\Delta \leq k$. If $G$ is minimally non edge DP-k-colorable, then the following statements hold.
(a) G is connected.
(b) $d(u)+d(v) \geq k+2$ for any $u v \in E(G)$.
(c) If $G$ has an even cycle $C=v_{1} v_{2} \ldots v_{2 t} v_{1}$ with $d\left(v_{2 i}\right)=2$ for $i \in[1, t]$, then for $j \in[1, t]$, every vertex in $N_{G}\left(v_{2 j-1}\right) \backslash V(C)$ is a $3^{+}$-vertex of $G$.
(d) If $k>\Delta$, then $G$ has no even cycle $v_{1} v_{2} \ldots v_{2 t} v_{1}$ with $d\left(v_{2 i}\right) \leq k+2-\Delta$ for $i \in[1, t]$.

Proof. Since $G$ is not edge DP- $k$-colorable, there is an edge list assignment $L$ of $G$ with $|L(e)| \geq k$ for every $e \in E(G)$ and a matching assignment $\mathscr{M}_{L}$ such that $G$ has no $\mathscr{M}_{L}$-coloring. By Eq. (5.1) and the minimality of $G$, every proper subgraph $G^{\prime}$ has an $\mathscr{M}_{L^{\prime}}$-coloring. So (a) holds.

We prove (b). Suppose to the contrary that $u v$ is an edge with $d_{G}(u)+$ $d_{G}(v) \leq k+1$. Fix $G^{\prime}=G-u v$ and let $I^{\prime}$ be an $\mathscr{M}_{L^{\prime}}$-coloring of $G^{\prime}$ with $\left|I^{\prime}\right|=\left|E\left(G^{\prime}\right)\right|=|E(G)|-1$. Since $d_{G}(u)+d_{G}(v) \leq k+1$ and $|L(u v)| \geq k$,

$$
\left|L^{*}(u v)\right| \geq|L(u v)|-\left(d_{G}(u)+d_{G}(v)-2\right) \geq 1
$$

by Eq. (5.2). Let $c^{*} \in L^{*}(u v)$. Then $I^{\prime} \cup\left\{\left(u v, c^{*}\right)\right\}$ is an $\mathscr{M}_{L}$-coloring of $G$ by Lemma 5.3, a contradiction.

We prove (c). Suppose to the contrary that there is an odd $i \in[1,2 t]$ such that $v_{i}$ has a neighbor $u$ with $u \notin V(C)$ and $d_{G}(u) \leq 2$. Let $H=C+v_{i} u$ and $G^{\prime}=G-E(H)$. Note that every edge of $H$ is incident with at least one 2vertex since $C=v_{1} v_{2} \ldots v_{2 t} v_{1}$ satisfies that $d_{G}\left(v_{2 i}\right)=2$ for $i \in[1, t]$. For every $x y \in E(H),|L(x y)| \geq \Delta \geq d_{G}(x)+d_{G}(y)-2$ since $\mid L(x y) \geq k \geq \Delta$. Since $G^{\prime}$ is a proper subgraph of $G$, it has an $\mathscr{M}_{L^{\prime}}$-coloring $I^{\prime}$ with $\left|I^{\prime}\right|=\left|E\left(G^{\prime}\right)\right|$ by Eq. (5.1). Thus, for each $x y \in E(H)$, it follows from Eq. (5.2) that

$$
\begin{gathered}
\left|L^{*}(x y)\right| \geq|L(x y)|-d_{G^{\prime}}(x)-d_{G^{\prime}}(y) \geq d_{G}(x)+d_{G}(y)-2-d_{G^{\prime}}(x)- \\
d_{G^{\prime}}(y)=d_{H}(x)+d_{H}(y)-2 .
\end{gathered}
$$

By Lemma 5.5, $H$ has an $\mathscr{M}_{L^{*}}$-coloring $I^{*}$. Hence we get a contradiction that $I^{\prime} \cup I^{*}$ is an $\mathscr{M}_{L}$-coloring of $G$ by Lemma 5.3.

We prove (d). Suppose that $G$ contains an even cycle $C=v_{1} v_{2} \ldots v_{2 t} v_{1}$ with $d\left(v_{2 i}\right) \leq k+2-\Delta$ for $i \in[1, t]$. Further, $d_{G}\left(v_{2 i}\right)=k+2-\Delta$ by (b). Since $k>\Delta, d_{G}\left(v_{2}\right) \geq 3$. Let $u \in N_{G}\left(v_{2}\right) \backslash\left\{v_{1}, v_{3}\right\}$ and $H=C+v_{2} u$. Note that every edge of $H$ has at least an end in $\left\{v_{2 i}: i \in[1, t]\right\}$. For any $x y \in E(H)$, assume that $x \in\left\{v_{2 i}: i \in[1, t]\right\}$ and thus $|L(x y)| \geq k=$ $d_{G}(x)+\Delta-2 \geq d_{G}(x)+d_{G}(y)-2$. Let $I^{\prime}$ be an $\mathscr{M}_{L^{\prime}}$-coloring of the proper subgraph $G^{\prime}=G-E(H)$ by Eq. (5.1). By Eq. (5.2), every edge $x y \in E(H)$ satisfies that

$$
\begin{gathered}
\left|L^{*}(x y)\right| \geq|L(x y)|-d_{G^{\prime}}(x)-d_{G^{\prime}}(y) \geq d_{G}(x)+d_{G}(y)-2-d_{G^{\prime}}(x)- \\
d_{G^{\prime}}(y)=d_{H}(x)+d_{H}(y)-2 .
\end{gathered}
$$

Therefore, $H$ has an $\mathscr{M}_{L^{*}}$-coloring $I^{*}$ by Lemma 5.5, and thus $I^{*} \cup I^{\prime}$ is an $\mathscr{M}_{L}$-coloring of $G$ by Lemma 5.3, a contradiction.

### 5.3 Proofs of Theorems 5.1 and 5.2

Recall that $\lambda_{i}(v)$ be the number of $i$-faces incident with $v$, and $V_{i}$ is the set of $i$-vertices of $G$.

### 5.3.1 Proof of Theorem 5.1

Let $G$ be a counterexample to Theorem 5.1 with $|E(G)|$ minimum. Then there are an edge list assignment $L$ of $G$ with $|L(e)|=\Delta$ for $e \in E(G)$ and a matching assignment $\mathscr{M}_{L}$ such that $G$ has no $\mathscr{M}_{L}$-coloring. By the minimality of $G$, every proper subgraph of $G$ has an $\mathscr{M}_{L}$-coloring, and thus $G$ is minimally non edge DP- $\Delta$-colorable. By (b) and (c) of Lemma 5.7, the following two claims are immediate.

Claim 1. $d(u)+d(v) \geq \Delta+2$ for any $u v \in E(G)$.
Claim 2. If $G$ has an even cycle $C=v_{1} v_{2} \ldots v_{2 t} v_{1}$ with $d_{G}\left(v_{2 i}\right)=2$ for $i \in[1, t]$, then for $j \in[1, t]$, every vertex in $N_{G}\left(v_{2 j-1}\right) \backslash V(C)$ is a $3^{+}$-vertex of $G$.

Let $G_{2}$ be the subgraph of $G$ induced by the edges incident with at least one 2-vertex.

Claim 3. Every component of $G_{2}$ is an even cycle or a tree, and thus $G_{2}$ has a matching, denoted by $M_{2}$, saturating all vertices in $V_{2}(G)$.

Proof. Let $Q$ be a component of $G_{2}$.
Assume that $C$ is a cycle of $Q$. Note that every edge of $G_{2}$ is incident with a 2-vertex in $G$. By Claim 1, every edge of $C$ has an end with degree $\Delta$ in $G$. Since $\Delta \geq 7, C$ is an even cycle. Further, $C=Q$ by Claim 2. This proves the first statement.

In order to prove the second statement, we only need to prove that $Q$ has a matching $M_{Q}$ saturating all vertices in $V(Q) \cap V_{2}(G)$. If $Q$ is an even cycle, then the existence of $M_{Q}$ is obvious. If $Q$ is a tree, we add all pendant edges of $Q$ to $M_{Q}$, delete their ends from $Q$, and obtain $M_{Q}$ by repeating this procedure until $Q$ becomes a single vertex.

For every $u v \in M_{2}$ with $d(u)=2$, we call $v$ the 2-master of $u$. Since $M_{2}$ is a matching and saturates all 2 -vertices of $G$, every 2 -vertex of $G$ has a unique 2 -master and every $\Delta$-vertex of $G$ is the 2 -master of at most one 2 -vertex.

Claim 4. If $G$ contains no 4-cycles, then every vertex $v$ is incident with at $\operatorname{most}\left\lfloor\frac{d_{G}(v)}{2}\right\rfloor 3$-faces.

Proof. Since $G$ contains no 4-cycles, any two 3-cycles are edge-disjoint, and so the claim follows.

Claim 5. If $G$ has an even cycle $C=v_{1} v_{2} \ldots v_{2 t} v_{1}(t \geq 3)$ with $t-2$ chords $\left\{v_{1} v_{2 i}: i \in[2, t-1]\right.$ and for $i \in[1,2 t], d\left(v_{i}\right) \geq \Delta-1$ if $i$ is odd, $d\left(v_{i}\right)=2$ if $i \in\{2,2 t\}$, and $d\left(v_{i}\right)=3$ otherwise, then every vertex in $N_{G}\left(v_{1}\right) \backslash V(C)$ is a $3^{+}$-vertex of $G$.

Proof. Suppose to be contrary that $v_{1}$ has a neighbor $u$ not in $V(C)$ and $d_{G}(u)=2$. Let $H=C+v_{1} u+\left\{v_{1} v_{2 j}: j \in[2, t-1]\right\}$. By the minimality of $G$ and Eq. (5.1), $G^{\prime}=G-E(H)$ admits an $\mathscr{M}_{L^{\prime}}$-coloring $I^{\prime}$. By (5.2), for each $x y \in E(H)$,

$$
\begin{aligned}
\left|L^{*}(x y)\right| & \geq|L(x y)|-d_{G^{\prime}}(x)-d_{G^{\prime}}(y) \\
& =\Delta-\left(d_{G}(x)-d_{H}(x)\right)-\left(d_{G}(y)-d_{H}(y)\right) \\
& = \begin{cases}\Delta-(\Delta-t-1)-1=t & \text { if } x y=v_{1} u ; \\
\Delta-(\Delta-t-1)-0=t+1 & \text { if } x y \in\left\{v_{1} v_{2 j}: j \in[1, t]\right\} ; \\
\Delta-(\Delta-2)-0=2 & \text { otherwise. }\end{cases}
\end{aligned}
$$

 obtain a contradiction that $I^{*} \cup I^{\prime}$ is an $\mathscr{M}_{L}$-coloring of $G$ by Lemma 5.3.

In the rest of this subsection, we complete the proof of Theorem 5.1 by applying discharging method to get a contradiction. We distinguish the following two cases.

Case 1. $\Delta \geq 7$ and $G$ contains no 4-cycles.
Let $w$ be an initial charge on $V(G) \cup F(G)$ satisfying

$$
w(z)=\left\{\begin{aligned}
2 d(z)-6 & \text { if } z \in V(G) \\
d(z)-6 & \text { if } z \in F(G)
\end{aligned}\right.
$$

Then $\sum_{v \in V(G)}(2 d(v)-6)+\sum_{f \in F(G)}(d(f)-6)=-8$.
To redistribute charges among vertices and faces, we design some discharging rules as follows.
(R1) Each 2-vertex receives 2 from its 2-master.
(R2) Let $f=\left[v_{1} v_{2} v_{3}\right]$ with $d\left(v_{1}\right) \leq d\left(v_{2}\right) \leq d\left(v_{3}\right)$.
(R2.1) If $d\left(v_{1}\right) \leq 3$, then $f$ receives $\frac{3}{2}$ from each $(\Delta-1)^{+}$-vertex incident with $f$.
(R2.2) If $d\left(v_{1}\right) \geq 4$, then $f$ receives 1 from each $4^{+}$-vertex incident with $f$.
(R3) If $f$ is a 5 -face, then it receives $\frac{1}{3}$ from each $5^{+}$-vertex incident with $f$.
After the discharging process, let $w^{\prime}(z)$ denote the final charge of every element $z$ in $V(G) \cup F(G)$. Since the rules only move charge around and do not affect the sum,

$$
\sum_{z \in V(G) \cup F(G)} w^{\prime}(z)=\sum_{z \in V(G) \cup F(G)} w(z)=-12 .
$$

We will obtain a contradiction by showing $w^{\prime}(z) \geq 0$ for every $z \in V(G) \cup F(G)$.
Let $z=f \in F(G)$. If $d(f) \geq 6$, then $w^{\prime}(f)=d(f)-6 \geq 0$. We next prove that $w^{\prime}(f) \geq 0$ for the other faces of $G$, distinguishing the cases that $d(f)=3$ and $d(f)=5$.
(1.1) $d(f)=3$.

Let $f=\left[v_{1} v_{2} v_{3}\right]$ with $d\left(v_{1}\right) \leq d\left(v_{2}\right) \leq d\left(v_{3}\right)$. By Claim 1, if $d\left(v_{1}\right) \leq 3$, then $d\left(v_{3}\right) \geq d\left(v_{2}\right) \geq \Delta-1$; if $d\left(v_{1}\right) \geq 4$, then $d\left(v_{3}\right) \geq d\left(v_{2}\right) \geq 5$. Thus it follows from (R2) that $w^{\prime}(f)=3-6+\max \left\{2 \times \frac{3}{2}, 3 \times 1\right\}=0$.
(1.2) $d(f)=5$.

Since $\Delta \geq 7$, by Claim $1, f$ is incident with at least three $5^{+}$-vertices. Thus $w^{\prime}(f) \geq 5-6+3 \times \frac{1}{3}=0$ by (R3).

It remains to prove that $w^{\prime}(v) \geq 0$ for each $v \in V$. Let $d(v)=k$. By Claim 4, if $k \in[5, \Delta]$, then $\lambda_{3}(v) \leq\left\lfloor\frac{k}{2}\right\rfloor$ and $\lambda_{3}(v)+\lambda_{5}(v) \leq d(v)=k$. We distinguish a number of cases, as follows.
(2.1) $k=2$. Then $v$ has a unique 2 -master and so $w^{\prime}(v)=2 \times 2-6+2=0$ by (R1).
(2.2) $k=3$ or 4. If $k=3$, then $w^{\prime}(v)=w(v)=2 \times 3-6=0$. If $k=4$, by Claim 4, $v$ is incident with at most two 3-faces, and so $w^{\prime}(v)=$ $2 \times 4-6-2 \times 1=0$ by (R2.2).
(2.3) $k \in[5, \Delta-2]$. It follows from Claim 1 that the neighbors of $v$ have degree at least 4, and by (R2.2) and (R3), $w^{\prime}(v)=2 k-6-\lambda_{3}(v) \times 1-$ $\lambda_{5}(v) \times \frac{1}{3} \geq 2 k-6-\left\lfloor\frac{k}{2}\right\rfloor \times 1-\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right) \times \frac{1}{3}>0$.
(2.4) $k=\Delta-1$. By (R2) and (R3), $w^{\prime}(v) \geq 2 k-6-\lambda_{3}(v) \times \frac{3}{2}-\lambda_{5}(v) \times \frac{1}{3} \geq$ $2(\Delta-1)-6-\left\lfloor\frac{\Delta-1}{2}\right\rfloor \times \frac{3}{2}-\left(\Delta-1-\left\lfloor\frac{\Delta-1}{2}\right\rfloor\right) \times \frac{1}{3}>0$.
(2.5) $k=\Delta$. Since every $\Delta$-vertex of $G$ is the 2 -master of at most one 2-vertex, it follows from (R1) $\sim(\mathrm{R} 3)$ that $w^{\prime}(v) \geq 2 k-6-2-\lambda_{3}(v) \times \frac{3}{2}-$ $\lambda_{5}(v) \times \frac{1}{3} \geq(2 \Delta-6)-2-\left\lfloor\frac{\Delta}{2}\right\rfloor \times \frac{3}{2}-\left(\Delta-\left\lfloor\frac{\Delta}{2}\right\rfloor\right) \times \frac{1}{3}>0$.

The proof of Case 1 is complete.
Case 2. $\Delta \geq 8$ and $G$ contains no 3-cycles.
Let $w(z)=d(z)-4$ be an initial charge of every $z \in V(G) \cup F(G)$, and define some discharging rules as follows:
(S1) Every 2-vertex receives charge 1 from its 2-master.
(S2) Every $k$-face $f$ sends $\frac{k-4}{\left\lfloor\frac{k}{2}\right\rfloor}$ to each $3^{-}$-vertex incident with $f$.
(S3) Every $k$-vertex $(2 \leq k \leq 3) v$ receives $\frac{1}{k}$ from each of $N_{G}(v)$ if $v$ is incident with no $5^{+}$-faces, and $\frac{1}{2 k}$ from each of $N_{G}(v)$ if $v$ is incident with exactly one $5^{+}$-face.

Similar to Case 1, let $w^{\prime}(z)$ denote the final charge of every member $z$ in $V(G) \cup F(G)$ after the discharging. Since $G$ is a planar graph, by Euler's formula,

$$
\sum_{z \in V(G) \cup F(G)} w^{\prime}(z)=\sum_{z \in V(G) \cup F(G)} w(z)=-8 .
$$

This implies that there is a member $z_{0} \in V(G) \cup F(G)$ such that $w^{\prime}\left(z_{0}\right)<0$. Thus, we only need to prove $w^{\prime}(z) \geq 0$ for every $z \in V(G) \cup F(G)$, and then get a contradiction.

Let $f$ be a $k$-face of $G$. Then $f$ is incident with at most $\left\lfloor\frac{k}{2}\right\rfloor 3^{-}$-vertices by Claim 1. Since $G$ contains no 3 -cycles, $k \geq 4$ and so, by (S2), $w^{\prime}(f) \geq$ $k-4-\left\lfloor\frac{k}{2}\right\rfloor \times \frac{k-4}{\left\lfloor\frac{k}{2}\right\rfloor}=0$.

Let $v$ be a $k$-vertex of $G$. Note that $\frac{k-4}{\left\lfloor\frac{k}{2}\right\rfloor} \geq \frac{1}{2}$ for $k \in[5, \Delta]$.
If $k=2$, then $v$ has a unique 2-master. Thus, $w^{\prime}(v) \geq 2-4+1+\min \{0+$ $\left.2 \times \frac{1}{2}, 1 \times \frac{1}{2}+2 \times \frac{1}{4}, 2 \times \frac{1}{2}+0\right\} \geq 0$ by (S1)~(S3).

If $k=3$, then by (S2) and (S3), we have $w^{\prime}(v) \geq 3-4+\min \{0+3 \times$ $\left.\frac{1}{3}, 1 \times \frac{1}{2}+3 \times \frac{1}{6}, 2 \times \frac{1}{2}+0\right\} \geq 0$.

If $k \in[4, \Delta-2]$, then every neighbor of $v$ is a $4^{+}$-vertex of $G$ by Claim 1, and thus $w^{\prime}(v)=w(v)=k-4 \geq 0$.

If $k=\Delta-1$, then every neighbor of $v$ is a $3^{+}$-vertex of $G$ by Claim 1 , and by (S3), $v$ sends at most $\frac{1}{3}$ to each member in $N_{G}(v)$. So $w^{\prime}(v)=$ $\Delta-1-4-\frac{\Delta-1}{3} \geq 0$ since $\Delta \geq 8$.

Assume that $k=\Delta$ below. Recall that $n_{i}(v)$ (resp., $n_{i^{+}}(v), n_{i^{-}}(v)$ ) be the number of $i$-vertices (resp., $i^{+}$-vertices, $i^{-}$-vertices) adjacent to $v$ in $G$, Then $n_{2}(v)+n_{3}(v) \leq k=\Delta$. We need the facts that $v$ is the 2 -master of at most one 2 -vertex, and that, when $n_{2}(v) \geq 3$, every 4-face incident with $v$ is incident with at most one 2 -vertex by Claims 1 and 2 .

If $n_{2}(v) \in[0,2]$, then, by (S1) and (S3), $w^{\prime}(v) \geq w(v)-1-n_{2}(v) \times \frac{1}{2}-$ $n_{3}(v) \times \frac{1}{3} \geq \Delta-5-2 \times \frac{1}{2}-(\Delta-2) \times \frac{1}{3} \geq 0$ since $\Delta \geq 8$.

If $n_{2}(v) \in\left[3,\left\lfloor\frac{\Delta}{2}\right\rfloor\right]$ and $n_{2}(v)+n_{3}(v)<\Delta$, then by (S1) and (S3), $w^{\prime}(v) \geq$ $w(v)-1-n_{2}(v) \times \frac{1}{2}-n_{3}(v) \times \frac{1}{3} \geq \Delta-5-\left\lfloor\frac{\Delta}{2}\right\rfloor \times \frac{1}{2}-\left(\Delta-1-\left\lfloor\frac{\Delta}{2}\right\rfloor\right) \times \frac{1}{3} \geq 0$ since $\Delta \geq 8$.

If $n_{2}(v) \in\left[3,\left\lfloor\frac{\Delta}{2}\right\rfloor\right]$ and $n_{2}(v)+n_{3}(v)=\Delta$, then it follows from Claim 5 that $\lambda_{4}(v)<\Delta$ and $\lambda_{5}(v) \geq 1$ since Claim 2 implies that every 4-face incident with $v$ is a $(\Delta, 2, \Delta, 3)$ - or $(\Delta, 3, \Delta, 3)$-face. Note that each $5^{+}$-face incident with $v$ contains either two 2 -vertices, or a 2 -vertex and a 3 -vertex, or two 3 vertices of $N_{G}(v)$. Since $\Delta \geq 8$, by (S1) and (S3), $w^{\prime}(v) \geq \Delta-4-1-\max \{2 \times$ $\left.\frac{1}{4}+\frac{n_{2}(v)-2}{2}+\frac{\Delta-n_{2}(v)}{3}, \frac{1}{4}+\frac{n_{2}(v)-1}{2}+\frac{1}{6}+\frac{\Delta-n_{2}(v)-1}{3}, \frac{n_{2}(v)}{2}+2 \times \frac{1}{6}+\frac{\Delta-n_{2}(v)-2}{3}\right\} \geq 0$.

In the final case that $n_{2}(v)=\left\lfloor\frac{\Delta}{2}\right\rfloor+i\left(1 \leq i \leq\left\lceil\frac{\Delta}{2}\right\rceil\right)$, let $N_{2}$ be the set of 2-vertices of $N_{G}(v)$ and $N_{2}^{\prime}$ be the set of vertices of $N_{2}$ incident with at least one $5^{+}$-face. Then $\left|N_{2}\right|=n_{2}(v)=\left\lfloor\frac{\Delta}{2}\right\rfloor+i$. We claim that $\left|N_{2}^{\prime}\right| \geq 2 i$. Suppose to the contrary that $\left|N_{2}^{\prime}\right| \leq 2 i-1$. Then

$$
\begin{aligned}
\left|N_{G}(v) \backslash N_{2}^{\prime}\right|=\Delta-\left|N_{2}^{\prime}\right| & \leq 2\left\lfloor\frac{\Delta}{2}\right\rfloor+1-\left|N_{2}^{\prime}\right| \\
& =2\left(\left|N_{2}\right|-i\right)+1-\left|N_{2}^{\prime}\right| \\
& =2\left|N_{2} \backslash N_{2}^{\prime}\right|+\left(\left|N_{2}^{\prime}\right|-2 i+1\right) \leq 2\left|N_{2} \backslash N_{2}^{\prime}\right|
\end{aligned}
$$

This implies that there are two vertices $u_{1} \in N_{2} \backslash N_{2}^{\prime}$ and $u_{2} \in N_{2} \backslash\left\{u_{1}\right\}$ such that $\left\{v, u_{1}, u_{2}\right\}$ is incident with a 4-face, a contradiction. So the claim holds. Combining the claim with (S1) and (S3), we have $w^{\prime}(v) \geq \Delta-4-1-\left(n_{2}(v)-\right.$ $2 i) \times \frac{1}{2}-2 i \times \frac{1}{4}-\frac{\Delta-n_{2}(v)}{3} \geq 0$ since $\Delta \geq 8$.

The proof of Case 2 is complete and so Theorem 5.1 is true.

### 5.3.2 Proof of Theorem 5.2

Let $G$ be a counterexample to Theorem 5.2 with $|E(G)|$ minimum. Then there are an edge list assignment $L$ of $G$ with $|L(e)|=\Delta+1$ for $e \in E(G)$ and a matching assignment $\mathscr{M}_{L}$ such that $G$ has no $\mathscr{M}_{L}$-coloring. By the minimality of $G, G$ is minimally non edge DP-( $\Delta+1)$-colorable.

The following claim follows from (b) in Lemma 5.7 directly.
Claim 1. $d(u)+d(v) \geq \Delta+3$ for every $u v \in E(G)$.
Claim 2. $\left|V_{\Delta}\right|>2\left|V_{3}\right|$.
Proof. Let $G_{3}$ be the subgraph of $G$ induced by the set of edges incident with a 3-vertex. By Claim 1, for every $u v \in E\left(G_{3}\right)$, one of $\{u, v\}$ is in $V_{3}$ and the other is in $V_{\Delta}$. Hence, $\left|V\left(G_{3}\right)\right| \leq\left|V_{\Delta}\right|+\left|V_{3}\right|$ and $\left|E\left(G_{3}\right)\right|=3\left|V_{3}\right|$. Further, if $C=v_{1} v_{2} \ldots v_{n} v_{1}$ is a cycle of $G_{3}$ with $d_{G}\left(v_{1}\right)=3$, then $n$ is even, $d_{G}\left(v_{i}\right)=3$ for each odd $i \in[1, n]$ and $d_{G}\left(v_{i}\right)=\Delta$ for each even $i \in[1, n]$. By (d) in Lemma 5.7, $G_{3}$ is a forest and so $\left|V\left(G_{3}\right)\right|>\left|E\left(G_{3}\right)\right|$. Thus $\left|V_{\Delta}\right|+\left|V_{3}\right| \geq\left|V\left(G_{3}\right)\right|>\left|E\left(G_{3}\right)\right|=3\left|V_{3}\right|$. The claim is true.

For each $x \in V(G) \cup F(G)$, let $w(x)=d(x)-4$ be the initial charge of $x$. The discharging rules are defined as follows:
(T1) Every $\Delta$-vertex sends $\frac{1}{2}$ to a common pot from which each 3 -vertex receives 1 .
(T2) Let $f=\left[v_{1} v_{2} v_{3}\right]$ be a 3 -face with $d\left(v_{1}\right) \leq d\left(v_{2}\right) \leq d\left(v_{3}\right)$.
(T2.1) If $d\left(v_{1}\right) \leq 4$, then $f$ receives $\frac{1}{2}$ from each of $\left\{v_{2}, v_{3}\right\}$.
(T2.2) If $d\left(v_{1}\right)=5$, then $f$ receives $\frac{1}{5}$ from $v_{1}$ and $\frac{2}{5}$ from each of $\left\{v_{2}, v_{3}\right\}$.
(T2.3) If $d\left(v_{1}\right) \geq 6$, then $f$ receives $\frac{1}{3}$ from each of $\left\{v_{1}, v_{2}, v_{3}\right\}$.
Let $w^{\prime}(x)$ denote the final charge of every element $x$ in $V(G) \cup F(G)$ after the discharging process.

Note that the final charge of the common pot is equal to $\frac{1}{2}\left|V_{\Delta}\right|-\left|V_{3}\right|$ by (T1) and $\frac{1}{2}\left|V_{\Delta}\right|-\left|V_{3}\right|>0$ by Claim 2.

Therefore, similar to Case 2 in the proof of Theorem 5.1, we only need to prove $w^{\prime}(z) \geq 0$ for every $z \in V(G) \cup F(G)$, and then get a contradiction.

We first consider the final charge of each face $f \in F(G)$.
If $d(f) \geq 4$, then $w^{\prime}(f)=w(f)=d(f)-4 \geq 0$. If $d(f)=3$, let $f=\left[v_{1} v_{2} v_{3}\right]$ with $d\left(v_{1}\right) \leq d\left(v_{2}\right) \leq d\left(v_{3}\right)$, then by Claim 1, either $d\left(v_{1}\right) \leq 4$ and $d\left(v_{3}\right) \geq d\left(v_{2}\right) \geq \Delta-1$, or $d\left(v_{1}\right)=5$ and $d\left(v_{3}\right) \geq d\left(v_{2}\right) \geq \Delta-2$, or $d\left(v_{3}\right) \geq$ $d\left(v_{2}\right) \geq d\left(v_{1}\right) \geq 6$. By (T2), $w^{\prime}(f)=3-4+\min \left\{2 \times \frac{1}{2}, \frac{1}{5}+2 \times \frac{2}{5}, 3 \times \frac{1}{3}\right\}=0$.

Now we consider the final charge of each $k$-vertex $v$ of $G$.
If $k=3$, then $w^{\prime}(v)=3-4+1=0$ by (T1). If $k=4$, then $w^{\prime}(v)=w(v)=$ $4-4=0$. Note the assumption that $\Delta \geq 9$. If $k \in[5,6]$, then by Claim 1 , $d(u) \geq \Delta+3-d(v) \geq 6 \geq k$ for each $u \in N_{G}(v)$. By (T2.2) and (T2.3), $v$ sends $\frac{k-4}{k}$ to each 3 -face incident with $v$, and so $w^{\prime}(v) \geq k-4-k \times \frac{k-4}{k}=0$.

If $k \in[7, \Delta-2]$, then every neighbor of $v$ has degree at least 5 by Claim 1 , and $\frac{k-4}{k}>\max \left\{\frac{2}{5}, \frac{1}{3}\right\}$. By (T2.2) and (T2.3) again, $w^{\prime}(v) \geq k-4-k \times \frac{k-4}{k} \geq 0$.

If $k \in[\Delta-1, \Delta]$, together (T1) with (T2.1) $\sim(T 2.3)$, we have $w^{\prime}(v) \geq$ $k-4-(k-\Delta+1) \times \frac{1}{2}-k \times \max \left\{\frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{1}{3}\right\} \geq 0$.

The proof of Theorem 5.2 is complete.

### 5.4 Conclusion and future work

In this chapter, we proved that $\chi_{D P}^{\prime}(G)=\Delta$ if $G$ has no 4-cycles and $\Delta \geq 7$ or $G$ no 3 -cycles and $\Delta \geq 8$. Moreover, we proved that $\chi_{D P}^{\prime}(G) \leq \Delta+1$ if $\Delta \geq 9$.

For the known result in list edge coloring such as: $\chi_{\ell}^{\prime}(G) \leq \Delta+1$ for planar graphs with $\Delta \geq 8$. It's very difficult for us to extend to edge $D P$-coloring since some base results in list edge coloring may not hold in edge DP-coloring. We would like to ask the following question.

Question 5.1. Is $\chi_{D P}^{\prime}(G) \leq \Delta+2$ true for all planar graphs?

## Summary

The results in this thesis all deal with edge colorings of planar graphs, in particular with bounds and exact values of the edge chromatic number with respect to three different variants of edge coloring.

Research related to edge coloring graphs can be generally classified into two types. One direction of research is focused on determining the edge chromatic number of graphs, or bounds for this number. The other direction is focused on studying conditions for the existence of subgraphs with certain coloring characteristics in a graph that has already been edge colored. All our results fall into the first category, and are all restricted to planar graphs, a class of graphs that have received a lot of attention, motivated and inspired by the first studies of graph coloring related to the Four Color Problem. Since then, graph coloring has developed into a rich area, with many celebrated results and a number of conjectures, some of which are still open.

One of the central and classic results regarding edge coloring states that the edge chromatic number of any graph is either equal to its maximum degree or its maximum degree plus one. For generalizations of edge coloring this classic result leads to two natural questions. The first question is whether the edge chromatic number for the new edge coloring concept is equal to the edge chromatic number; the second question is whether this new edge chromatic number is bounded from above by the maximum degree plus one. Both questions are usually posed as conjectures, with the second one clearly weaker than the first one. All our results provide partial answers to these questions, for three different types of generalizations of edge colorings.

In this thesis, we focus on list edge coloring, signed edge coloring, and
edge DP-coloring. With regard to list edge coloring the second question and the associated conjecture are still open for planar graphs with maximum degree at least 5. In Chapter 2, we prove that this conjecture holds for planar graphs with maximum degree at least 6 in which every 7 -cycle is induced. This means that any existing cycles on exactly 7 vertices in the graph have the property that there are no additional edges in the graph between pairs of vertices on the cycle, apart from the cycle edges. Our result improves a result which recently appeared in the literature.

The key idea in our proof is to apply the so-called Combinatorial Nullstellensatz, combined with some recoloring arguments. We use the Combinatorial Nullstellensatz to determine several configurations which cannot appear in an assumed minimal counterexample to our main result. We use recoloring arguments to deal with configurations that are not excluded by the Combinatorial Nullstellensatz.

Our next main result deals with an extension of edge colorings to signed graphs, i.e., graphs in which each edge has an assigned positive or negative signature. The study of edge colorings of signed graphs just started recently, since its introduction by Behr in a paper of 2020. It can be seen as a natural extension of edge coloring for graphs. In his paper, Behr proved that the edge chromatic number of a signed graph is equal to its maximum degree or its maximum degree plus one. In a more recent paper due to Zhang et al., the authors conjecture that the edge chromatic number of a signed planar graph with maximum degree at least 6 is equal to its maximum degree. In Chapter 3, we study the structure of critical signed graphs. We extend some partial adjacency lemmas on edge coloring to signed edge coloring. In Chapter 4, we apply the lemmas obtained in Chapter 3 to prove that the conjecture of Zhang et al. is true for signed planar graphs with maximum degree at least 8, as well as for signed planar graphs with maximum degree at least 6 in which each 6-cycle contains at most one chord.

Our final main results deal with another generalization of edge coloring which is known under the name of edge DP-coloring. There are currently just a few known results about edge DP-coloring and the associated edge chromatic number $\chi_{D P}^{\prime}(G)$. In Chapter 5, we prove three theorems. Let $G$ be a planar graph with maximum degree $\Delta$. We prove that $\chi_{D P}^{\prime}(G)=\Delta$ if
$G$ has no 4-cycles and $\Delta \geq 7$. Moreover, we prove that $\chi_{D P}^{\prime}(G)=\Delta$ if $G$ has no 3 -cycles and $\Delta \geq 8$. In our final result, we prove that $\chi_{D P}^{\prime}(G) \leq \Delta+1$ if $\Delta \geq 9$. The key idea in the proofs lies in the discharging method.

Throughout this thesis, we have determined the list edge chromatic number, signed edge chromatic number and DP-edge chromatic number for classes of planar graphs which are subjected to certain structural conditions. Despite our new contributions, some problems and conjectures remain unresolved. We also present several problems we will consider in future at the end of each chapter. We hope that these problems and open conjectures attract more attention from other researchers.

## Samenvatting

De resultaten in dit proefschrift hebben zonder uitzondering betrekking op lijnkleuringen van planaire grafen, in het bijzonder op grenzen en exacte waarden van het lijnchromatisch getal, voor drie verschillende varianten van lijnkleuring.

Onderzoek op het gebied van lijnkleuring van grafen kan in het algemeen ingedeeld worden in twee typen. Binnen de ene richting richt men het onderzoek op het bepalen van het lijnchromatisch getal van grafen, ofwel op grenzen voor dit getal. De andere richting is gericht op het bepalen van voorwaarden voor het bestaan van deelgrafen met bepaalde kleuringseigenschappen in grafen die een lijnkleuring hebben gekregen. De resultaten van dit proefschrift vallen onder de eerste categorie, en zijn beperkt tot planaire grafen. Dit is een veel bestudeerde klasse van grafen, gemotiveerd en geïnspireerd door de eerste studies betreffende graafkleuring in het kader van het Vierkleurenprobleem. Sinds die tijd heeft het gebied zich sterk ontwikkeld, en is het rijk aan gevierde resultaten en vermoedens, waarvan een aantal nog steeds open is.

Een van de centrale en klassieke resultaten betreffende lijnkleuring is de stelling dat het lijnchromatisch getal van elke graaf gelijk is aan de maximale graad in de graaf of de maximale graad plus éen. Voor generalisaties van lijnkleuring leidt deze klassieke stelling tot twee natuurlijke vragen. De eerste vraag is of het lijnchromatisch getal voor de nieuwe variant van lijnkleuring gelijk is aan het lijnchromatisch getal van klassieke lijnkleuring; de tweede vraag is of het nieuwe lijnchromatisch getal van boven begrensd is door de maximale graad plus één. Beide vragen worden veelal geformuleerd als
vermoedens, waarbij het tweede vermoeden duidelijk zwakker is dan het eerste. Alle resultaten uit dit proefschrift geven gedeeltelijke antwoorden op deze vragen, voor drie verschillende generalisaties van lijnkleuring.

In dit proefschrift ligt de focus op lijst lijnkleuring, gesigneerde lijnkleuring en DP-lijnkleuring.

Wat betreft lijst lijnkleuring zijn de tweede vraag en het bijbehorende vermoeden nog open voor planaire grafen met maximale graad minstens 5. In Hoofdstuk 2 wordt bewezen dat dit vermoeden waar is voor planaire grafen met maximale graad minstens 6 waarin elke 7-cykel geïnduceerd is. Dat laatste wil zeggen dat elke eventueel aanwezige 7-cykel de eigenschap heeft dat er geen andere lijnen in de graaf aanwezig zijn tussen punten van de cykel, naast de lijnen van de cykel zelf. Dit resultaat is een verbetering van een bekend resultaat uit de literatuur. Het sleutelidee in ons bewijs van dit resultaat is het toepassen van de zogenoemde Combinatorial Nullstellensatz, gecombineerd met bepaalde argumenten op het gebied van het herkleuren van lijnen. We gebruiken de Combinatorial Nullstellensatz om een aantal configuraties uit te sluiten in een verondersteld tegenvoorbeeld van ons resultaat. De herkleuringsargumenten worden dan in de overgebleven niet uitgesloten configuraties gebruikt.

Ons volgende hoofdresultaat betreft een uitbreiding van lijnkleuringen naar gesigneerde grafen, dat wil zeggen grafen waarin elke lijn een positieve of negatieve signatuur heeft gekregen. Het onderzoek naar lijnkleuringen van gesigneerde grafen is pas begonnen, met de introductie door Behr in een artikel uit 2020. Deze vorm van lijnkleuring kan gezien worden als een natuurlijke uitbreiding van lijnkleuring naar gesigneerde grafen. In zijn artikel laat Behr zien dat het lijnchromatisch getal van een gesigneerde graaf gelijk is aan de maximale graad of de maximale graad plus één. In een recenter artikel formuleren de auteurs Zhang et al. het vermoeden dat het lijnchromatisch getal van een gesigneerde planaire graaf met maximale graad minstens 6 gelijk is aan de maximale graad van de graaf. In Hoofdstuk 3 bestuderen we de structuur van kritieke gesigneerde grafen. We breiden daarin een aantal structurele resultaten over buurrelaties uit van lijnkleuring naar gesigneerde lijnkleuring. In Hoofdstuk 4 passen we die structurele resultaten toe om te bewijzen dat het vermoeden van Zhang et al. waar is voor gesigneerde
planaire grafen met maximale graad minstens 8, en tevens voor gesigneerde planaire grafen met maximale graad minstens 6 waarin elke 6-cykel hooguit één koorde heeft.

Ons laatste hoofdresultaat betreft een andere generalisatie van lijnkleuring die bekend staat onder de naam DP-lijnkleuring. Er zijn tot nu toe weinig resultaten bekend op het gebied van DP-lijnkleuring en het bijbehorende lijnchromatisch getal $\chi_{D P}^{\prime}(G)$. In Hoofdstuk 5 bewijzen we drie stellingen. Stel hiervoor dat $G$ een planaire graaf is met maximale graad $\Delta$. Allereerst bewijzen we dat $\chi_{D P}^{\prime}(G)=\Delta$ als $G$ geen 4 -cykels heeft en $\Delta \geq 7$ is. Tevens bewijzen we dat $\chi_{D P}^{\prime}(G)=\Delta$ als $G$ geen 3-cykels heeft en $\Delta \geq 8$ is. In ons laatste resultaat laten we zien dat $\chi_{D P}^{\prime}(G) \leq \Delta+1$ als $\Delta \geq 9$. Het basisidee achter alle bewijzen is een bestaande techniek die bekend staat als de "discharging method".

In de technische hoofdstukken van dit proefschrift hebben we het lijnchromatisch getal bepaald voor lijst lijnkleuring, gesigneerde lijnkleuring en DP-lijnkleuring van klassen van planaire grafen die onderhevig zijn aan bepaalde structurele voorwaarden. Niettegenstaande onze nieuwe bijdragen tot dit gebied, blijven een aantal problemen en vermoedens onopgelost. We presenteren ook een aantal problemen die we in de toekomst zullen bespreken aan het einde van elk hoofdstuk. We hopen dat deze problemen en vermoedens de aandacht gaan trekken van andere onderzoekers.

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## Appendix A

## Proof of Lemma 2.6

This appendix contains the analysis for the remaining 13 cases in the proof of Lemma 2.6 that we omitted. We recall the following set-up, where $H$ is one of the graphs $F_{2}-F_{14}$ of Figure 2.1. Suppose $H=F_{i}$ and let the vertices be labeled as in Figure 2.1. Let $\phi$ be an edge- $\left.L\right|_{E(\bar{H})}$-coloring of $\bar{H}$. Denote by $S_{\phi}$ the edge list assignment of $H$ satisfying that, for every $e_{i}=u w \in E(H)$,

$$
S_{\phi}\left(e_{i}\right)=L\left(e_{i}\right) \backslash\left\{\phi(h): h \in E_{\bar{H}}(u) \cup E_{\bar{H}}(w)\right\} .
$$

(1) $H=F_{2}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,3]$, and $x_{4}, x_{5}$ to $v_{1} v_{2}, v_{2} v_{3}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{2}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{2}-x_{3}\right) \cdot\left(x_{1}-x_{4}\right) \cdot\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{4}-x_{5}\right) \cdot\left(x_{3}-x_{5}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 2>1$. We used Mathematica to deduce that Coefficient $\left[P_{F_{2}}, x_{1}^{2} x_{2}^{2} x_{3} x_{4}^{2} x_{5}\right]=-2$.
(2) $H=F_{3}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,3]$, and $x_{4}, x_{5}$ to $v_{1} v_{2}, v_{2} v_{3}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{3}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{2}-x_{3}\right) \cdot\left(x_{1}-x_{4}\right) \cdot\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{4}-x_{5}\right) \cdot\left(x_{3}-x_{5}\right)
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 2>1$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 4>3$, $\left|S_{\phi}\left(\nu v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 3>1$. By Mathematica, Coefficient $\left[P_{F_{3}}, x_{1} x_{2}^{3} x_{3} x_{4}^{2} x_{5}\right]=1$.
(3) $H=F_{4}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,3], x_{4}, x_{5}$ to $v_{1} v_{2}, v_{2} v_{3}$ and $x_{6}$ to $v_{1} u$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{4}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{2}-x_{3}\right) \cdot\left(x_{1}-x_{4}\right)\left(x_{1}-x_{6}\right)\left(x_{4}-x_{6}\right) \cdot\left(x_{2}-x_{4}\right) \\
& \left(x_{2}-x_{5}\right)\left(x_{4}-x_{5}\right) \cdot\left(x_{3}-x_{5}\right)
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 2>1$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 4>3$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{1} u\right)\right| \geq 2>1$. By Mathematica, Coefficient $\left[P_{F_{4}}, x_{1} x_{2}^{3} x_{3} x_{4}^{2} x_{5}^{2} x_{6}\right]=-1$.
(4) $H=F_{5}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,4], x_{5}, x_{6}$ to $v_{1} v_{2}, v_{2} v_{3}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{5}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{3}-x_{4}\right) \cdot\left(x_{1}-x_{5}\right) . \\
& \left(x_{2}-x_{5}\right)\left(x_{2}-x_{6}\right)\left(x_{5}-x_{6}\right) \cdot\left(x_{3}-x_{6}\right) .
\end{aligned}
$$

Obviously, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 4>3,\left|S_{\phi}\left(v v_{2}\right)\right| \geq 3>2,\left|S_{\phi}\left(v v_{3}\right)\right| \geq 4>3$, $\left|S_{\phi}\left(v v_{4}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 2>1$. By Mathematica, Coefficient $\left[P_{F_{5}}, x_{1}^{3} x_{2}^{2} x_{3}^{3} x_{4} x_{5} x_{6}\right]=2$.
(5) $H=F_{6}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,4], x_{5}, x_{6}, x_{7}$ to $v_{1} v_{2}, v_{2} v_{3}, v_{3} v_{4}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{6}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{3}-x_{4}\right) \cdot\left(x_{1}-x_{5}\right) . \\
& \left(x_{2}-x_{5}\right)\left(x_{2}-x_{6}\right)\left(x_{5}-x_{6}\right) \cdot\left(x_{3}-x_{6}\right)\left(x_{3}-x_{7}\right)\left(x_{6}-x_{7}\right) \cdot\left(x_{4}-x_{7}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 4>3,\left|S_{\phi}\left(v v_{2}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 5>4,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v_{3} v_{4}\right)\right| \geq 2>1$. By Mathematica, Coefficient $\left[P_{F_{6}}, x_{1}^{3} x_{2}^{2} x_{3}^{4} x_{4} x_{5} x_{6}^{2} x_{7}\right]=$ -3 .
(6) $H=F_{7}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,4], x_{5}, x_{6}, x_{7}$ to $v_{1} v_{2}, v_{2} v_{3}, u v_{1}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{7}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{3}-x_{4}\right) \cdot\left(x_{1}-x_{5}\right) \\
& \left(x_{1}-x_{7}\right)\left(x_{5}-x_{7}\right) \cdot\left(x_{2}-x_{5}\right)\left(x_{2}-x_{6}\right)\left(x_{5}-x_{6}\right) \cdot\left(x_{3}-x_{6}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 5>4$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 2>1$, $\left|S_{\phi}\left(u v_{1}\right)\right| \geq 2>1$. By Mathematica, Coefficient $\left[P_{F_{7}}, x_{1}^{2} x_{2}^{4} x_{3} x_{4}^{2} x_{5}^{2} x_{6} x_{7}\right]=-1$.
(7) $H=F_{8}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,5], x_{6}, x_{7}, x_{8}, x_{9}$ to $v_{1} v_{2}, v_{2} v_{3}, v_{3} v_{4}$, $u v_{3}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{8}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{1}-x_{5}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right) \cdot\left(x_{1}-x_{6}\right) \cdot\left(x_{2}-x_{6}\right)\left(x_{2}-x_{7}\right)\left(x_{6}-x_{7}\right) . \\
& \left(x_{3}-x_{7}\right)\left(x_{3}-x_{8}\right)\left(x_{3}-x_{9}\right)\left(x_{7}-x_{8}\right)\left(x_{7}-x_{9}\right)\left(x_{8}-x_{9}\right) \cdot\left(x_{4}-x_{8}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 6>5$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 5>3,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 4>3,\left|S_{\phi}\left(v v_{5}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 2>1$, $\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 4>3,\left|S_{\phi}\left(v_{3} v_{4}\right)\right| \geq 2>1,\left|S_{\phi}\left(u v_{3}\right)\right| \geq 3>2$. By Mathematica, Coefficient $\left[P_{F_{8}}, x_{1}^{2} x_{2}^{5} x_{3}^{3} x_{4}^{3} x_{5} x_{6} x_{7}^{3} x_{8} x_{9}^{2}\right]=1$.
(8) $H=F_{9}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,5], x_{6}, x_{7}, x_{8}, x_{9}$ to $v_{1} v_{2}, v_{2} v_{3}, v_{3} v_{4}$, $v_{4} v_{5}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{9}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{1}-x_{5}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right) \cdot\left(x_{1}-x_{6}\right) \cdot\left(x_{2}-x_{6}\right)\left(x_{2}-x_{7}\right)\left(x_{6}-x_{7}\right) . \\
& \left(x_{3}-x_{7}\right)\left(x_{3}-x_{8}\right)\left(x_{7}-x_{8}\right) \cdot\left(x_{4}-x_{8}\right)\left(x_{4}-x_{9}\right)\left(x_{8}-x_{9}\right) \cdot\left(x_{5}-x_{9}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 5>4$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 4>2$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 5>4,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 4>3,\left|S_{\phi}\left(v v_{5}\right)\right| \geq 5>4,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 2>1$, $\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{3} v_{4}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{4} v_{5}\right)\right| \geq 2>1$. By Mathematica, Coefficient $\left[P_{F_{9}}, x_{1}^{4} x_{2}^{2} x_{3}^{4} x_{4}^{3} x_{5}^{4} x_{6} x_{7} x_{8} x_{9}\right]=-2$.
(9) $H=F_{10}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,5], x_{6}, x_{7}, x_{8}, x_{9}$ to $v_{1} v_{2}, v_{2} v_{3}$, $v_{3} v_{4}, v_{4} v_{5}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{10}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{1}-x_{5}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right) \cdot\left(x_{1}-x_{6}\right) \cdot\left(x_{2}-x_{6}\right)\left(x_{2}-x_{7}\right)\left(x_{6}-x_{7}\right) . \\
& \left(x_{3}-x_{7}\right)\left(x_{3}-x_{8}\right)\left(x_{7}-x_{8}\right) \cdot\left(x_{4}-x_{8}\right)\left(x_{4}-x_{9}\right)\left(x_{8}-x_{9}\right) \cdot\left(x_{5}-x_{9}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 5>4$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 4>2$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 5>3,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 5>4,\left|S_{\phi}\left(v v_{5}\right)\right| \geq 4>3,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 2>1$, $\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{3} v_{4}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{4} v_{5}\right)\right| \geq 2>1$. By Mathematica, Coefficient $\left[P_{F_{10}}, x_{1}^{4} x_{2}^{2} x_{3}^{3} x_{4}^{4} x_{5}^{3} x_{6} x_{7} x_{8}^{2} x_{9}\right]=2$.
(10) $H=F_{11}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,5], x_{6}, x_{7}, x_{8}, x_{9}$ to $v_{1} v_{2}, v_{2} v_{3}$, $v_{3} v_{4}, v_{4} v_{5}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{11}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{1}-x_{5}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right) \cdot\left(x_{1}-x_{6}\right) \cdot\left(x_{2}-x_{6}\right)\left(x_{2}-x_{7}\right)\left(x_{6}-x_{7}\right) . \\
& \left(x_{3}-x_{7}\right)\left(x_{3}-x_{8}\right)\left(x_{7}-x_{8}\right) \cdot\left(x_{4}-x_{8}\right)\left(x_{4}-x_{9}\right)\left(x_{8}-x_{9}\right) \cdot\left(x_{5}-x_{9}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 5>4$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 5>2$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 4>3,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 5>4,\left|S_{\phi}\left(v v_{5}\right)\right| \geq 4>3,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{3} v_{4}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{4} v_{5}\right)\right| \geq 2>1$. By Mathematica, Coefficient $\left[P_{F_{11}}, x_{1}^{4} x_{2}^{2} x_{3}^{3} x_{4}^{4} x_{5}^{3} x_{6}^{2} x_{7} x_{8} x_{9}\right]=1$.
(11) $H=F_{12}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,5], x_{6}, x_{7}, x_{8}, x_{9}$ to $v_{1} v_{2}, v_{2} v_{3}$, $v_{3} v_{4}, v_{4} v_{5}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{12}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{1}-x_{5}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right) \cdot\left(x_{1}-x_{6}\right) \cdot\left(x_{2}-x_{6}\right)\left(x_{2}-x_{7}\right)\left(x_{6}-x_{7}\right) . \\
& \left(x_{3}-x_{7}\right)\left(x_{3}-x_{8}\right)\left(x_{7}-x_{8}\right) \cdot\left(x_{4}-x_{8}\right)\left(x_{4}-x_{9}\right)\left(x_{8}-x_{9}\right) \cdot\left(x_{5}-x_{9}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 4>1,\left|S_{\phi}\left(v v_{2}\right)\right| \geq 6>5$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 4>3,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 5>3,\left|S_{\phi}\left(v v_{5}\right)\right| \geq 4>3,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{3} v_{4}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{4} v_{5}\right)\right| \geq 2>1$. By Mathematica, Coefficient $\left[P_{F_{12}}, x_{1} x_{2}^{5} x_{3}^{3} x_{4}^{3} x_{5}^{3} x_{6}^{2} x_{7}^{2} x_{8} x_{9}\right]=-2$.
(12) $H=F_{13}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,5], x_{6}, x_{7}, x_{8}, x_{9}, x_{10}$ to $v_{1} v_{2}$, $v_{2} v_{3}, v_{3} v_{4}, v_{4} v_{5}, v_{1} v_{5}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{13}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{1}-x_{5}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right) \cdot\left(x_{1}-x_{6}\right)\left(x_{1}-x_{10}\right)\left(x_{6}-x_{10}\right) \cdot\left(x_{2}-x_{6}\right) \\
& \left(x_{2}-x_{7}\right)\left(x_{6}-x_{7}\right) \cdot\left(x_{3}-x_{7}\right)\left(x_{3}-x_{8}\right)\left(x_{7}-x_{8}\right) \cdot\left(x_{4}-x_{8}\right)\left(x_{4}-x_{9}\right) \\
& \left(x_{8}-x_{9}\right) \cdot\left(x_{5}-x_{9}\right)\left(x_{5}-x_{10}\right)\left(x_{9}-x_{10}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 6>5$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 5>4$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 4>3,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 5>4,\left|S_{\phi}\left(v v_{5}\right)\right| \geq 4>2,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 4>2$, $\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{3} v_{4}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{4} v_{5}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{1} v_{5}\right)\right| \geq 3>$ 2. By Mathematica, Coefficient $\left[P_{F_{13}}, x_{1}^{5} x_{2}^{4} x_{3}^{3} x_{4}^{4} x_{5}^{2} x_{6}^{2} x_{7} x_{8} x_{9} x_{10}^{2}\right]=2$.
(13) $H=F_{14}$. Assign $x_{i}$ to $v v_{i}$ for $i \in[1,5], x_{6}, x_{7}, x_{8}, x_{9}, x_{10}$ to $v_{1} v_{2}$, $v_{2} v_{3}, v_{3} v_{4}, v_{4} v_{5}, v_{1} v_{5}$. Using Eq. (2.1), we obtain:

$$
\begin{aligned}
P_{F_{14}}= & \left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{1}-x_{4}\right)\left(x_{1}-x_{5}\right)\left(x_{2}-x_{3}\right)\left(x_{2}-x_{4}\right)\left(x_{2}-x_{5}\right) \\
& \left(x_{3}-x_{4}\right)\left(x_{3}-x_{5}\right)\left(x_{4}-x_{5}\right) \cdot\left(x_{1}-x_{6}\right)\left(x_{1}-x_{10}\right)\left(x_{6}-x_{10}\right) \cdot\left(x_{2}-x_{6}\right) \\
& \left(x_{2}-x_{7}\right)\left(x_{6}-x_{7}\right) \cdot\left(x_{3}-x_{7}\right)\left(x_{3}-x_{8}\right)\left(x_{7}-x_{8}\right) \cdot\left(x_{4}-x_{8}\right)\left(x_{4}-x_{9}\right) \\
& \left(x_{8}-x_{9}\right) \cdot\left(x_{5}-x_{9}\right)\left(x_{5}-x_{10}\right)\left(x_{9}-x_{10}\right) .
\end{aligned}
$$

By straightforward calculations, $\left|S_{\phi}\left(v v_{1}\right)\right| \geq 6>5$, $\left|S_{\phi}\left(v v_{2}\right)\right| \geq 4>3$, $\left|S_{\phi}\left(v v_{3}\right)\right| \geq 5>4,\left|S_{\phi}\left(v v_{4}\right)\right| \geq 5>2,\left|S_{\phi}\left(v v_{5}\right)\right| \geq 4>3,\left|S_{\phi}\left(v_{1} v_{2}\right)\right| \geq 3>2$, $\left|S_{\phi}\left(v_{2} v_{3}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{3} v_{4}\right)\right| \geq 3>2,\left|S_{\phi}\left(v_{4} v_{5}\right)\right| \geq 2>1,\left|S_{\phi}\left(v_{1} v_{5}\right)\right| \geq 3>$ 2. By Mathematica, Coefficient $\left[P_{F_{14}}, x_{1}^{5} x_{2}^{3} x_{3}^{4} x_{4}^{2} x_{5}^{3} x_{6}^{2} x_{7} x_{8}^{2} x_{9} x_{10}^{2}\right]=5$.

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greatly enriching my academic journey.
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Li Zhang

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## About the Author

Li Zhang was born on October 20, 1994 in Zizhou County, Shaanxi Province, P.R. China. From 2000 to 2006, Li Zhang completed the primary education in her village. From 2006 to 2012, she completed her junior middle school and senior middle school educations in Zichang County.

In September 2012, she started her study at Faculty of Science, Xi'an University of Technology. After obtaining her Bachelor degree in June 2016, she passed entrance exams to become a master student supervised by Professor You Lu in Northwestern Polytechnical University. In March 2019, she received her Master degree and continued her study as a PhD candidate supervised by Professor Shengui Zhang in Northwestern Polytechnical University.

Starting from December 2021, she visited the group of Formal Methods and Tools, University of Twente as a joint PhD student to perform research on edge coloring theory under the supervision of Professor Hajo Broersma. The research has been sponsored by the China Scholarship Council. The main results obtained from her research during her PhD work have been collected in the current thesis.

